Matrix Representation of Complex Variables

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1 Introduction

Complex numbers are generally represented by the formula

$$z = a + ib \mid (a, b) \in \mathbb{R}$$

However, they can also be represented as $2X2$ matrices.

$$z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Using this representation we eliminate the necessity of using i to denote the direction orthogonal to the *real* axis. The *real* and *imaginary* axes on the complex planes are base vectors in an R^2 space. Matrices of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mid (a, b) \in \mathbb{R}$$

form a set over which addition and multiplication is commutative, and division is defined for matrices with a non-zero determinant. The only condition under which the determinant of a matrix of this form becomes 0 is if $a = 0$ and $b = 0$. The determinant is the square of the distance of the complex number from the origin of the complex plane.

Besides addition, multiplication, and division, we can also define exponents over this group.

2 Logarithms of Complex Numbers

Because the group is defined over the set

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix} \mid (x, y) \in \mathbb{R}$$

A complex number raised to the power of another complex number can be denoted by

$$z_1^{z_2} = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix}^{\begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix}}$$

In classical notation a complex number with unit magnitude can be described by

$$e^{i\theta} = \cos\theta + i\sin\theta$$

In matrix representation, this is expressed as

$$\begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}^{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\therefore \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}^{\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Taking the logarithm on both sides,

$$\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} = log(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix})$$

We can make good use of this result while calculating logarithms. For any complex number z:

$$z = re^{i\theta} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}^{\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

We can represent z, r, and θ in terms of a and b.

$$r = \sqrt{a^2 + b^2}$$

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$$

$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$$

$$\therefore z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2+b^2}\begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix}$$

Based on this conversion can derive a general expression for calculating logarithms for complex numbers

$$log(z) = log\left(\sqrt{a^2+b^2}\begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix}\right)$$

$$\therefore log(z) = \frac{1}{2}log(a^2+b^2) + \begin{bmatrix} 0 & -\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right) \\ \sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right) & 0 \end{bmatrix}$$

$$\therefore log(z) = log(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}) = \begin{bmatrix} \frac{1}{2}log(a^2+b^2) & -\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right) \\ \sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right) & \frac{1}{2}log(a^2+b^2) \end{bmatrix}$$

3 Exponents of Complex Numbers

Given this result we can finally calculate exponents for matrices

$$z_1^{z_2} = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix}^{\begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix}}$$

$$\therefore log(z_1^{z_2}) = \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} log(\begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix})$$

$$\therefore log(z_1^{z_2}) = \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix}\begin{bmatrix} \frac{1}{2}log({a_1}^2+{b_1}^2) & -\sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right) \\ \sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right) & \frac{1}{2}log({a_1}^2+{b_1}^2) \end{bmatrix}$$

$$\therefore log(z_1^{z_2}) = \begin{bmatrix} \frac{a_2}{2}log({a_1}^2+{b_1}^2) - b_2\sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right) & -\frac{b_2}{2}log({a_1}^2+{b_1}^2) - a_2\sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right) \\ \frac{b_2}{2}log({a_1}^2+{b_1}^2) + a_2\sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right) & \frac{a_2}{2}log({a_1}^2+{b_1}^2) - b_2\sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right) \end{bmatrix}$$

We decompose this matrix into it's real and imaginary components and write the $z_1^{z_2}$ in the exponential form, such that:

$$z_1^{z_2} = A \cdot B$$

where

$$A = e^{\begin{bmatrix} \frac{a_2}{2} log({a_1}^2 + {b_1}^2) - b_2 \sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right) & 0 \\ 0 & \frac{a_2}{2} log({a_1}^2 + {b_1}^2) - b_2 \sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right) \end{bmatrix}}$$

$$B = e^{\begin{bmatrix} 0 & -\frac{b_2}{2} log({a_1}^2 + {b_1}^2) - a_2 \sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right) \\ \frac{b_2}{2} log({a_1}^2 + {b_1}^2) + a_2 \sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right) & 0 \end{bmatrix}}$$

$$\therefore A = e^{\left(\frac{a_2}{2} log({a_1}^2+{b_1}^2) - b_2 \sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right)\right)}$$

$$\therefore A = \sqrt{\left({a_1}^2 + {b_1}^2\right)^{a_2}} \cdot e^{-b_2 \sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right)}$$

$$\therefore B = e^{\left(\left[\frac{b_2}{2} log({a_1}^2+{b_1}^2) + a_2 \sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right)\right]\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)}$$

$$\therefore B = \left(e^{\frac{b_2}{2} log({a_1}^2+{b_1}^2) + a_2 \sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right)}\right)^{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$

$$\therefore B = \left(\sqrt{\left({a_1}^2 + {b_1}^2\right)^{b_2}} \cdot e^{a_2 \sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right)}\right)^{i}$$

Finally, we get the value of $z_1^{z_2}$

$$\therefore z_1^{z_2} = \sqrt{\left({a_1}^2 + {b_1}^2\right)^{(a_2+ib_2)}} \cdot e^{\left((ia_2 - b_2) \sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right)\right)}$$

$$\therefore z_1^{z_2} = \sqrt{\left({a_1}^2 + {b_1}^2\right)^{(a_2+ib_2)}} \cdot e^{\left(i(a_2 + ib_2) \sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right)\right)}$$

$$\therefore z_1^{z_2} = \sqrt{\left({a_1}^2 + {b_1}^2\right)^{(a_2+ib_2)}} \cdot \left(e^{\left(i\sin^{-1}\left(\frac{b_1}{\sqrt{{a_1}^2+{b_1}^2}}\right)\right)}\right)^{a_2+ib_2}$$

$$\therefore z_1^{z_2} = \sqrt{\left({a_1}^2 + {b_1}^2\right)^{(a_2+ib_2)}} \cdot \left(\frac{a_1 + ib_1}{\sqrt{{a_1}^2 + {b_1}^2}}\right)^{a_2+ib_2}$$

$$\therefore z_1^{z_2} = (a_1 + ib_1)^{a_2+ib_2}$$

4 So, Is there a solution?

We do have a few techniques to find solutions of the form $z = a + ib$ for the value of $z_1^{z_2} = (a_1 + ib_1)^{a_2+ib_2}$. Depending on the values of a_1, b_1, a_2 and b_2:

4.1 $\mathbf{a}_1 = 0$

$$z_1^{z_2} = (ib_1)^{a_2+ib_2}$$

$$\therefore z_1^{z_2} = i^{a_2} \cdot \left(i^i\right)^{b_2} \cdot {b_1}^{a_2} \cdot \left({b_1}^{b_2}\right)^i$$

i^{a_2} may or may not have a real solution

$\left(i^i\right)^{b_2}$ is actually a real number because i^i has a real value (can you guess what it is?)

${b_1}^{a_2}$ is a real number

$\left({b_1}^{b_2}\right)^i$ may or may not have a real solution

Also note that i^i has multiple solutions due to sin and cos being modular transforms.

4.2 $\mathbf{b}_1 = 0$

$$z_1^{z_2} = (a_1)^{a_2+ib_2}$$

$$\therefore z_1^{z_2} = {a_1}^{a_2} \cdot \left({a_1}^{b_2}\right)^i$$

While ${a_1}^{a_2}$ is real $\left({a_1}^{b_2}\right)^i$ may or may not be real.

4.3 $\mathbf{a}_2 = 0$

$$z_1^{z_2} = (a_1 + ib_1)^{ib_2}$$

We can convert the equation to an exponential form

$$z_1^{z_2} = {z_1}^{ib_2} = e^{ib_2 \cdot log(z_1)}$$

$$\therefore z_1^{z_2} = \cos(ib_2 \cdot log(z_1)) + i \sin(ib_2 \cdot log(z_1))$$

Both of these terms may or may not be real. There are cases where $log(z_1)$ cancels out with i or b_2 and the solution becomes easier to calculate.

In any case you can always use the power series expansions of $e, sin(\theta)$ and $log(z)$ to compute the answers to the needed degree.

4.4 $\mathbf{b}_2 = 0$

$$z_1^{z_2} = {z_1}^{a_2}$$
$$\therefore z_1^{z_2} = e^{a_2 \cdot log(z_2)} = e^{i \cdot -a_2 i \cdot log(z_2)}$$
$$\therefore z_1^{z_2} = \cos(a_2 i \cdot log(z_2)) - i \sin(a_2 i \cdot log(z_2))$$

4.5 How computers perform matrix exponentiation

Computers mainly use the power series for functions like $e, sin(\theta)$ and $log(z)$.

$$\therefore {z_1}^{z_2} = e^{z_2 \cdot log(z_1)} = e^{i \cdot -z_2 i \cdot log(z_1)}$$

$$\therefore {z_1}^{z_2} = \cos(z_2 i \cdot log(z_1)) - i \sin(z_2 i \cdot log(z_1))$$

5 Conclusion

We saw how complex numbers can be represented as matrices and how operations like logarithms and exponentiations can be achieved for complex numbers.