Differential Geometry, Group theory and Physics

Project Report

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August-November 2018

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Chapter 1

Introduction

The abstract formalism of differential geometry generalizes many of the theorems and operations first seen in multi-variable calculus. I was fascinated by the elegant features of differential forms which can reproduce familiar Maxwell's equations in a simpler fashion. The nice interplay of group theory symmetries and the conservation laws in field theory indicates that the tools of group theory are vital in the study of many physical systems. These aspects of Physics were the driving force to understand the formal mathematical definitions which will be the theme of the next two chapters. In the concluding chapter, we will briefly highlight some of the interesting connections to Chern-Simons theory.

This report is written keeping a student like me who might be unaware of these ideas in mind. It is meant to be an introduction which is mathematically accurate but not too rigorous.

Chapter 2

Vector fields

This discussion is meant to put forth the important properties and functions of the objects related to vector fields follwing the book by J.C. Baez - Gauge fields, Knots and Gravity. [1]

2.1 Directional derivative

We are all familiar with imagining a vector field as an object consisting of arrows defined for every point on the manifold. A manifold is essentially a space that locally looks like a Riemannian space and can be defined by patching together multiple charts from this manifold to $\mathbb{R}^n$ with the property that the different charts are the same in the regions where their domains in the manifold overlap.

Given a field of arrows , one can differentiate a function f in the direction of the arrows i.e in the direction of the vector field v defined on $\mathbb{R}^n$.

$$vf = v^\mu \partial_\mu f \tag{2.1}$$

Here vf = directional directive of f in v direction and the Einstein summation convention is used where indices being repeated as a subscript and a superscript mean that they are summed over. vf is also equal to the familiar notation $\nabla f.v$. Since (2.1) is independent of the function f we can write

$$v = v^\mu \partial \mu \tag{2.2}$$

∂_μ is the partial derivative $\partial / \partial x^\mu$. The left side of (2.2) is a vector field and the right side is an operator that takes a derivative in the direction of the vector field. This doesn't make sense at first glance but this is actually how we intend to define v.

Before we define a vector field, let us define $C^\infty(M)$ as the set of functions (smooth and real valued) on a manifold M where C^∞ stands for the functions having 'infinitely many continuous derivatives'. $C^\infty(M)$ is an 'algebra' over real number closed under point-wise addition and multiplication. A vector field is then essentially a map defined from $C^\infty(M) \to C^\infty(M)$ that gives another real valued function defined on the manifold. Vect(M) is the set of all vector fields on M.

Interestingly, when the manifold is $\mathbb{R}^n$ itself, ∂_μ form a basis for Vect($\mathbb{R}^n$). Every vector field can be written as $v = v^\mu \partial_\mu$ where v^μ are the components of the vector field. Now we return to 'arrows on a manifold' description of the vector field. These arrows are actually called as tangent vectors that help us take directional derivative at the point to which it is attached.

$$v(f)(p) = v_p(f) \tag{2.3}$$

where $v_p(f)$ is the tangent vector at point P of the vector field v. It helps to remember that a tangent vector at $P \in$ M is a map from $C^\infty(M) \to \mathbb{R}$. T_pM is the set of all tangent vectors at P. Adding or comparing tangent vectors makes no sense unless they belong to the same tangent space. This can be done sloppily only in $\mathbb{R}^n$ because ∂_μ form a basis at every point in the manifold.

2.2 Covariant and Contravariant nature

Now, let us define a map between two manifolds ϕ.

$$\phi : M \to N \tag{2.4}$$

where M and N are manifolds. ϕ allows us to go 'forward' from M to N while the ϕ^* operation allows us to transport functions defined on N to functions on M 'backwards' via an action called the 'pullback'.

$$\phi^* f = f o \phi \tag{2.5}$$

Incidentally, quantities like functions having this opposite behaviour i.e. which can be 'pulled back' will be called contravariant quantities.

Similarly, we can 'push' tangent vectors forward from M to N which is why they are called covariant as opposed to contravariant. A tangent vector $v_p \in T_pM$ can be 'pushed' forwards by ϕ to give a tangent vector $\phi_* v \in T_{\phi(p)}N$. This is defined by

$$(\phi_* v)(f) = v(\phi^* f) \tag{2.6}$$

We have used a subscript asterisk for pushforwards and a superscript for pullbacks. In physics, people prefer to remember simple rules like if an object has an index as a subscript it is covariant and vice versa, which is correct. They prefer to think of a vector field $v = v^\mu \partial_\mu$ in terms of its components v^μ and hence call it as contravariant as opposed to a mathematician calling it as covariant as it can be pushed forward.

Chapter 3

Differential forms

This discussion is meant to put forth the important properties and functions of the objects related to vector fields follwing the book by J.C. Baez - Gauge fields, Knots and Gravity. [1]

3.1 Introduction

Most of the fields that we work with in Physics like the electric field, the magnetic field, the tensors that we see in General relativity, etc. - all of these are examples of differential forms. Also the different operators that we see in multivariable calculus like the gradient, curl and the divergence can all be thought as different aspects of single operator *d* that acts on differential forms. The fundamental theorem of calculus, Stokes' theorem and Gauss' theorem are all special cases of a single Stokes' theorem about differential forms. So while they are somewhat abstract, differential forms are a powerful unifying notion.

3.2 1-forms

Our goal is to generalize the concepts of gradient of a function to functions on arbitrary manifolds. The directional derivative of a function f on $\mathbb{R}^n$ in the direction of v is just the dot product of ∇f with v as seen in (2.1). The gradient of a function on $\mathbb{R}^n$ is a vector field, but it is not the same on a manifold. Dot product isn't easily defined on a manifold, dot product needs to be defined with the help of a metric. In fact, d of a smooth function on a manifold is a 1-form.

Identifying the fact that ∇f is independent of the vector field v with which it is 'dotted' with, and that it is essentially a map from Vect$(M) \to C^\infty(M)$, we define 1-form w as a map that takes in a vector field v and returns a function $w(v)$. Also, we define $\Omega^1(M)$ as the space of all 1-forms on M.

For any function f, df is a trivial 1-form defined such that

$$df(v) = vf \tag{3.1}$$

This one form is called the exterior derivative of f. d is a map from $C^\infty(M) \to \Omega^1(M)$. An important property that d satisfies is the Leibniz rule also known more commonly as product rule of differentiation.

$$d(fg) = (df)g + f(dg) \tag{3.2}$$

Since on $\mathbb{R}^n$, ∂_μ form a basis for Vect($\mathbb{R}^n$), dx^μ form a basis for 1-forms on $\mathbb{R}^n$ for $\Omega^1(\mathbb{R}^n)$. This implies that every one form df can be written as

$$df = \partial \mu f dx^\mu \tag{3.3}$$

Vector field $\to$ Tangent vector at every point
1-form $\to$ Cotangent vector at every point

w_p is a map from $T_p(M) \to \mathbb{R}$ defined such that

$$w_p(v_p) = w(v)(p) \tag{3.4}$$

Just like tangent vectors can be interpreted as a field of arrows defined at every point, cotangent vectors also have a physical interpretation which is interesting. Cotangent vector at a point can be thought of as a set of planes defined at every point that map the tangent vectors at the point to a real number which is equal to the number of planes that the tangent vector cuts. Also, cotangent vectors can be pulled back due to their contravariant nature. We could also pullback the 1-form completely.

$T_p^*(M)$ is the space of all cotangent vectors at p. If w is a cotangent vector at $\phi(x)$, we can define the pullback $\phi^* w$ acting on a tangent vector in $T_x(M)$ where ϕ is the map from $M \to N$. Summarising,

$$\begin{aligned} \phi &: M \to N \\ \phi_* &: T_p M \to T_p N \\ \phi^* &: T_p^* N \to T_p^* M \end{aligned} \tag{3.5}$$

3.3 p-forms

If you have ever found the cross product a little strange and restrictive, you are certainly asking the right questions. Why do we need such an arbitrary definition to make sense of cross product of two vectors? We will try to learn more about defining products for getting higher order differential forms.

Let us define a new generalized cross product or the wedge product $\wedge$ between vectors[1] in a vector space V to form a new bigger space ΛV consisting of products of vectors taken any number of times. The wedge product is also called the exterior product with the associated

[1] We don't necessarily mean 'vectors' as discussed until now, it means the elements of the vector space on which the exterior algebra is defined in this case

algebra ΛV called as the exterior algebra. One important property that we want the new wedge product to satisfy is antisymmetry similar to conventional cross product.

$$u \wedge v = -\, v \wedge u \tag{3.6}$$

Wedge product is also associative and hence doesn't need brackets in its definition as in conventional vector product where $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$. This new wedge product has value equal to the 'volume' of the n dimensional parallelopiped formed by the n vectors. Another important point is that we choose to define exterior product for 1-forms and not vectors. Let's look at an interesting example where dx, dy, dz form a basis for V as in $\Omega^1(\mathbb{R}^3)$. Then,

$$\begin{aligned}
&u = u_x dx + u_y dy + u_z dz \\
&v = v_x dx + v_y dy + v_z dz \\
&u \wedge v = (u_x dx + u_y dy + u_z dz) \wedge (v_x dx + v_y dy + v_z dz) \\
&= (u_x v_y - u_y v_x) dx \wedge dy + (u_y v_z - u_z v_y) dy \wedge dz + (u_z v_x - u_x v_z) dz \wedge dx
\end{aligned} \tag{3.7}$$

Observe how similar this is to the cross product. Also, $u \wedge v \wedge w$ is similar to

$$\vec{u}.(\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}).\vec{w} = [\vec{u}, \vec{v}, \vec{w}]$$

which gives the volume of the parallelopiped formed by these three vectors.
ΛV can be written as a direct sum of $\Lambda^i V$ with i summed over all non-negative integers. $\Lambda^i V$ is defined as the space formed by linear combination of wedge product of i vectors taken at a time i.e. objects of the form.

$$v_1 \wedge v_2 \wedge \cdots \wedge v_i \tag{3.8}$$

Dimension of $\Lambda^i V$ space is

$$\begin{cases} \frac{n!}{i!(n-i)!} & \text{for} \quad 0 \leq i \leq n \\ 0 & \text{for} \quad i > n \end{cases}$$

and thus the dimension of ΛV in 2^n.

ΛV is a graded commutative algebra that is if $w \in \Lambda^p V$ and $u \in \Lambda^q V$, then

$$w \wedge u = (-1)^{pq} u \wedge w \tag{3.9}$$

Now, let's get back to example (3.7). We can make the wedge product look exactly like the cross product. This requires something called as 'the Hodge star' operator which is a map from $\Lambda^2 V \to \Lambda V$. In this case,

$$\begin{aligned}
\star &: dx \wedge dy \to dz \\
\star &: dy \wedge dz \to dx \\
\star &: dz \wedge dx \to dy
\end{aligned} \tag{3.10}$$

Thus the cross product is equivalent to taking the wedge product and then applying the Hodge star operator. It could have just as easily been defined so that it becomes 'the Left Hand rule'.

Interestingly, this can be done only in 3 dimensions. Only in 3 dimensions is the dimension of $\Lambda^2 V$ equal to the dimension of V.

Now that we know the basics of an exterior algebra. Let us generalize the definitions for 1-forms on a manifold. Earlier, we defined the exterior algebra 'over $\mathbb{R}^n$' i.e. the p-fold products (3.8) were multiplied with real numbers during the linear combination. Now we will define $\Omega(M)$ as the exterior algebra formed by the 1-forms $\in \Omega^1(M)$ 'over $C^\infty(M)$'. Functions are in fact also called 0-forms and belong to $\Omega^0(M)$. Objects in $\Omega(M)$ are called as '**differential forms**'. This can also be used to explain why differential forms and functions have the same contravariant nature. The process looks like this

$$
\begin{gathered}
\text{Vectors} \rightarrow \text{1-forms} \\
\mathbb{R} \rightarrow C^\infty(M) \\
V \rightarrow M \\
\Lambda V \rightarrow \Omega(M)
\end{gathered}
$$

Also $f \wedge w$ where f is a 0-form/function and w is a p-form is defined as the standard multiplication fw.

3.4 Exterior derivative

Let us revisit the exterior derivative now that we know more about differential forms. The exterior derivative is an operator that takes a p form to give a $p+1$ form. This is the operator which mysteriously takes the form of curl, gradient or divergence depending on the object on which it acts.

$$d : \Omega^p(M) \rightarrow \Omega^{p+1}(M) \tag{3.11}$$

The important properties of the exterior derivative are as follows

$$d(u \wedge v) = du \wedge v + (-1)^p u \wedge dv \tag{3.12}$$

$$
\begin{aligned}
d(d) &= 0 \\
d^2 &= 0
\end{aligned} \tag{3.13}
$$

where u is a p form and v is some differential form.
Let's take the exterior derivative of a 1-form u on $\mathbb{R}^3$ and apply properties (3.3), (3.12) and (3.13).

$$
\begin{aligned}
&u = u_x dx + u_y dy + u_z dz \\
&u = u_x \wedge dx + u_y \wedge dy + u_z \wedge dz \\
&du = du_x \wedge dx + du_y \wedge dy + du_z \wedge dz \\
&du = (\partial_x u_y - \partial_y u_x) dx \wedge dy + (\partial_y u_z - \partial_z u_y) dy \wedge dz + (\partial_z u_x - \partial_z u_x) dz \wedge dx \\
&\star\, du = (\partial_x u_y - \partial_y u_x) dz + (\partial_y u_z - \partial_z u_y) dx + (\partial_z u_x - \partial_z u_x) dy
\end{aligned} \tag{3.14}
$$

The fourth step is the exterior derivative of the 1-form and then we have used (3.10) in step 5 which can be done only in $\mathbb{R}^3$. Thus the exterior derivative of a 2-form is essentially the 'curl' of the 1-form.
Similarly the exterior derivative of a 2-form on $\mathbb{R}^3$ is just a divergence in disguise. All it needs in the last step is a map from $dx \wedge dy \wedge dz \in \Omega^3(M) \to \mathbb{R}$. Again observe that they have the same dimension number equal to 1 if $M = \mathbb{R}^3$. In short, the exterior derivative becomes under special cases the following familiar operators:

- Gradient, $d : \Omega^0(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$
- Curl, $d : \Omega^1(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$
- Divergence, $d : \Omega^2(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3)$

The above results also help us understand that (3.13) on $\mathbb{R}^3$ splits into the two familiar results.

$$\nabla.(\nabla \times \vec{v}) = 0 \tag{3.15}$$

$$\nabla \times (\nabla f) = 0 \tag{3.16}$$

3.5 Stokes' theorem

p-forms can be integrated over p-dimensional manifolds with a boundary. The generalized form of Stokes' theorem says that under certain conditions for an n-dimensional manifold with w, an $(n-1)$-form defined on the manifold,

$$\int_M dw = \int_{\partial M} w \tag{3.17}$$

We can easily see that for $\mathbb{R}^3$ using the results listed above it leads to familiar 'Stokes's theorem'

$$\oint_C \vec{F}.d\vec{l} = \iint_A (\nabla \times \vec{F}).(d\vec{S}) \tag{3.18}$$

where C is the curve enclosing the area A.

The Divergence theorem and the Fundamental theorem of calculus also result from the generalized Stokes' theorem.

Chapter 4

Conclusions and future directions

In this section, I will present the problem that I was working on later as I referred to a paper by Tudor Dimofte and Sergei Gukov titled 'Quantum Field Theory and the Volume Conjecture' [2].

The volume conjecture in Knot theory states that for a hyperbolic knot K in S^3 the asymptotic growth of the colored Jones polynomial of K is related to the hyperbolic volume of the knot complement S^3/K. The volume referred to here is not the volume that we are familiar with, instead the hyperbolic volume. This is the original statement of the Volume conjecture in Knot theory. The paper tries to explain how it is related to Chern-Simons theory.

4.1 Knots

Mathematically a knot is an embedding of a circle S^1 in $\mathbb{R}^3$ or often S^3. This essentially means

$$\exists \quad \phi : S^1 \rightarrow M \in \mathbb{R}^3 \tag{4.1}$$

where M is a sub-manifold of $\mathbb{R}^3$ and the function ϕ is smooth and invertible, also called a diffeomorphism. ϕ lets us treat S^1 as a subset of $\mathbb{R}^3$.

Knot complement is the set all the points of $\mathbb{R}^3$ not contained in the knot. Hyperbolic knot means that the knot complement has a Riemannian metric with constant negative curvature. Crossing number is the minimum number of crossings that a knot has in its projection.

Two knots are considered equivalent if one can essentially be pulled and twisted to form the other without opening and closing the loop. The unknot is the simplest knot which is a simple closed loop or the knot with 0 crossing number. Knots are distinguished using knot invariants i.e. quantities that are invariant under 'twisting and pulling', one of them is the knot polynomial. Another one is the hyperbolic volume.

4.2 Knot polynomials

Knot polynomials are defined for a knot projection using a skein relation. If two knot diagrams or projections have different knot polynomials, they represent different knots. Though, the converse of the above statement is not true. A skein relation along with a definition for the unknot is used to give a simple recursive definition for knot polynomials, Jones polynomial is one such knot polynomial. It gives a simple relation between the values of a knot polynomial relating 'a right crossing', 'a left crossing' and the 'no crossing' keeping the rest of the knot same. The complicated knot diagram is solved one crossing at a time recursively until we are left with the unknot which is defined.

The 'coloring' referred to previously means writing the knot polynomial of a knot for a particular representation of $SU(2)$ group. The classical representation corresponds to writing it for the 2 dimensional representation of $SU(2)$ group. Thus we get the colored Jones polynomial for the knot using a skein relation and 'coloring' rules.

4.3 Quantizing the simple harmonic oscillator

The paper [2] discusses geometric quantization of the simple harmonic oscillator potential in one dimension. This is the problem referred to earlier. Geometric quantization as a process isn't completely clear to me yet.

The process of geometric quantization begins with a classical phase space M which is $\mathbb{R}^2 = (\mathrm{x}, \mathrm{p})$ in this example, as a manifold equipped with a closed 2-form $w = dp \wedge dx$. Closed refers to the fact that $dw = 0$ where d is the exterior derivative. Pre-quantization is the first step in this quantization which constructs a quantum Hilbert space H using the previous two. It transforms Poisson brackets on the classical side into commutators on the quantum side.

$$\begin{array}{ccc} (M, \omega) & \rightsquigarrow & \mathcal{H} \ (= \text{Hilbert space}) \\ & & \circlearrowright \\ \text{alg. of functions on } \mathcal{M} & \rightsquigarrow & \text{alg. of operators on } \mathcal{H} \\ f & \mapsto & \mathcal{O}_f : \mathcal{H} \to \mathcal{H}\,. \end{array}$$

Figure 4.1: Quantization [2]

The process of quantization also explains how classical trajectories of a physical system are associated to quantum states in H. A classical trajectory is described by a Lagrangian submanifold $L \subset M$. Let θ be 1-form that satisfies $w = d\theta$. Then the Lagrangian L is called quantizable if for any closed cycle γ

$$\oint_\gamma \theta \in 2\pi\hbar\mathbb{Z} \tag{4.2}$$

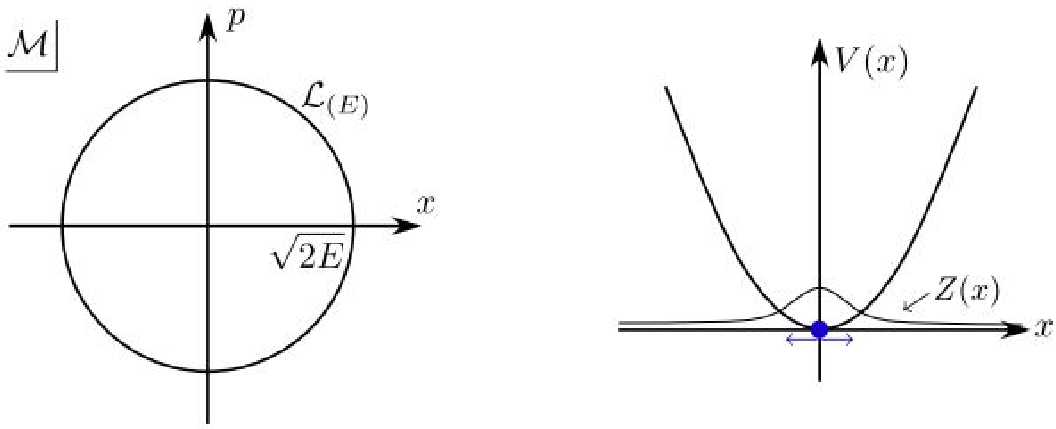

Figure 4.2: The harmonic oscillator: potential $V(x) = \frac{1}{2}kx^2$ in physical space, phase space M, a classical trajectory L_E in phase space, and the ground state quantum wavefunction $Z(x)$. [2]

The total (potential + kinetic) energy of the particle at any moment of time is given by the Hamiltonian

$$H = \frac{1}{2}x^2 + \frac{1}{2}p^2 \tag{4.3}$$

A classical trajectory with energy $H = E$ is just a circle of radius $\sqrt{2E}$ in phase space. This defines a Lagrangian submanifold in M.

$$\begin{aligned} \oint_{\mathcal{L}} \theta &= \oint_{S^1} \theta \\ &= \iint_A w \\ &= \iint_A dp \wedge dx \\ &= \pi(\sqrt{2E})^2 \\ &= 2\pi E \end{aligned} \tag{4.4}$$

The restriction (4.2) on L quantizes the energy. It implies that $2\pi E$ should be equal to $2\pi\hbar\mathbb{Z}$, Accounting for some corrections it leads to

$$E = \hbar\left(n + \frac{1}{2}\right) \tag{4.5}$$

4.4 Future work

The aim was to understand the quantization process and wave function definitions and apply them to solve the simple harmonic oscillator. The paper shows the quantization part with hints for calculating the wavefunctions but I couldn't complete the problem. I hope to keep working on it.

The hyperbolic volume of the knot is related to the Chern-Simons functional on the manifold. Also, there are other exciting relations of the previous section's approach to Cherns-Simons theory, knots and volume conjecture which I hope to pursue in future.

-The End-

Bibliography

[1] John C. Baez and Javier P. Munian. *Gauge fields, Knots, and Gravity.* 1994.

[2] Tudor Dimofte and Sergei Gukov. Quantum field theory and the volume conjecture. *arXiv:1003.4808*, 2010.