Study of BTZ Blackhole and Green's Function

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Report

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Chapter 1

General Relativity and Preliminaries

1.1 Null Hypersurfaces

A hyper-surface σ is described by setting a function of the coordinates of an n dimensional manifold to a constant.

$$f(x) = c \tag{1.1}$$

where c is a constant.
The vector field $\xi_\mu = g(x)\nabla_\mu f$ (g(x) is arbitrary non-zero function)is normal to every vector belonging to the tangent space of the point in consideration. There are only three possible cases of hypersurfaces:

1. Spacelike: ξ_μ is timelike.
2. Timelike: ξ_μ is spacelike.
3. Null: ξ_μ is null.

Let χ be a null hypersuface with normal field τ_μ. We see that $\tau_\mu \tau^\mu = 0$ and thus τ_μ is also a tangent vector. Therefore,

$$\tau^\mu = \frac{dx^\mu(\lambda)}{d\lambda} \tag{1.2}$$

for some curve $x^\mu(\lambda)$. We can prove that these curves x^μ are geodesics and these geodesics for which the corresponding tangent vectors are also normal to the surface are called generators of the null hypersurface χ.

References for this subsection: Sean Carroll [1] Appendix D, P.K. Townsend Lecture Notes [2] Section 2.3.5

1.2 Event Horizon

Event horizon can be described as a hypersurface separating those spacetime points that are connected to infinity by a timelike path from those that are not. It is important to note that structure and location of event horizon depends on the entire manifold and given a general manifold, there is no straightforward way to check whether an event horizon exists or not. An event horizon by definition is a null hypersurface. The surface r=2M describes an event horizon for Schwarzchild blackhole.

A similar concept is that of an Ergosphere which is the hypersurface beyond which no stationary path is timelike. Rotating Kerr Blackholes have an ergosphere which is different than the event horizon. Once inside the ergosphere, an observer necessarily has to "revolve around the blackhole center". We will show the existence of these hypersurfaces for the BTZ blackhole.

References for this subsection: Sean Carroll [1] Chapter 6. Event Horizon Wikipedia page. [3]

1.3 Killing Horizon and Surface Gravity

Consider a killing vector field τ^μ, if the vector field becomes null on the hypersurface σ, then σ is called the killing Horizon for the killing field τ^μ. We find that under general conditions found an event horizon is a killing vector for some killing field χ^μ and furthermore if the spacetime is stationary and axisymmetric with rotational Killing Vector field $R^\mu = (\partial_\phi)^\mu$, then the killing vector field for which the event horizon is killing horizon will be $\chi^\mu = (\partial_t)^\mu + \Omega R^\mu$, where we can identify Ω as the "Angular Velocity of the blackhole".

We define surface gravity for a Killing Horizon by the equation evaluated at the horizon,

$$\chi^\mu \nabla_\mu \chi^\nu = -\kappa \chi^\nu$$

The surface gravity κ is named so because it is the acceleration measured at infinity of an object at the horizon for a stationary black hole. However, we need to first suitably normalize χ^ν such that $\kappa \geq 0$ and the surface acceleration interpretation holds. Such a normalization is done by making sure $(\partial_t)^\mu (\partial_t)_\mu \to -1$ as $r \to \infty$.

References for this subsection: Sean Carroll [1] Chapter 6

Chapter 2

BTZ Black Hole

Discussion in this chapter mainly follows from S.Carlip [4] and two BTZ Papers [5], [6] . We will now summarize the Black Hole Solution in (2+1) Dimensions as was discovered by Máximo Bañados, Claudio Teitelboim, and Jorge Zanelli and hence named BTZ Blackhole. The discovery was important as the properties of BTZ Blackholes are very similar to the conventional (3+1) Dimensional Kerr blackhole and simplifies Quantum gravity based calculation.

2.1 Metric

We can verify that the metric given satisfies the Einstein's Equation for 2+1 dimensions for cosmological constant $\Lambda = -\frac{1}{l^2}$

$$ds^2 = -(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2})dt^2 + \frac{1}{(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2})}dr^2 + r^2(d\phi - \frac{J}{2r^2}dt)^2 \tag{2.1}$$

We can identify as M as the mass and J as the Angular Momentum of the system. As of now, we have not identified this system as a blackhole. But, nevertheless let us look at some of the features. The metric is stationary and has killing vectors ∂_t and ∂_ϕ We see that $g^{rr} = 0$ at

$$r_\pm^2 = \frac{Ml^2}{2}[1 \pm \sqrt{1 - \frac{J^2}{M^2l^2}}]$$

if $|J| < Ml$ while g_{tt} vanishes at $r_{erg} = M^{\frac{1}{2}}l$

2.2 Physical Properties

The metric (2.1) doesn't have any curvature singularity as it has (2+1) dimensions and a vacuum solution to Einstein's equations (with cosmological constant). Then why is it a blackhole? The singularity exists at $r = 0$ and is one in the causal structure. We will discuss the singularity in global geometry. In this section, we show some other properties about the horizons of this blackhole which are analogous to the Kerr Blackhole's corresponding Horizon.

Consider timelike curves with $r < r_{erg}$, we have $\frac{ds^2}{d\tau^2} < 0$ and thus,

$$(M - \frac{r^2}{l^2})(\frac{dt}{d\tau})^2 + \frac{1}{-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}}(\frac{dr}{d\tau})^2 + r^2(\frac{d\phi}{d\tau})^2 - J(\frac{d\phi}{d\tau})(\frac{dt}{d\tau}) < 0 \tag{2.2}$$

Now, if $r > r_+$, then the first three terms of (2.2) are positive and so is $\frac{dt}{d\tau}$. Thus,

$$J(\frac{d\phi}{d\tau}) > 0 \tag{2.3}$$

This shows that timelike curves cannot remain stationary inside the hypersurface $r = r_{erg}$ and the observer must rotate in the same direction as that of blackhole. This property is similar to that of Kerr Blackholes.

To explore the properties of $r = r_+$, we transform our coordinates to Eddingston-Finkelstein like coordinates,

$$dv = dt + \frac{dr}{-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}}, d\phi' = d\phi - \frac{\frac{-J}{2r^2}}{-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}} dr$$

to get

$$ds^2 = -(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2})dv^2 + 2dvdr + r^2(d\phi' - \frac{-J}{2r^2}dv)^2 \tag{2.4}$$

In these coordinates, normal to the surface $r = r_+$ is

$$\Gamma_\mu = f(x^\nu)(0, 1, 0), \Gamma^\mu = f(x^\nu)(1, 0, \frac{J}{2r_+^2})$$

where $f(x^\nu)$ is an arbitrary non-zero function of coordinates. We see that $\Gamma_\mu \Gamma^\mu = 0$ and thus, $r = r_+$ is a null surface. Now, consider any null geodesic at $r = r_+$ it satusfies $(\frac{ds}{d\lambda})^2 = 0$ for an affine parameter λ and hence,

$$(\frac{dv}{d\lambda})(\frac{dr}{d\lambda}) = -\frac{r_+^2}{2}(\frac{d\phi'}{d\lambda} - \frac{J}{2r_+^2}\frac{dv}{d\lambda})^2 \leq 0 \tag{2.5}$$

Hence, we see that any light ray at $r = r_+$ needs to either stay on surface or go inside towards the singularity. With this, we conclude that the event horizon of the blackhole is the surface $r = r_+$. Also, the null generators of the surface are geodesics satisfying

$$r = r_+, \frac{d\phi'}{d\lambda} - \frac{J}{2r_+^2}\frac{dv}{d\lambda} = 0$$

Finally, we see that

$$\chi = \partial_v + \frac{J}{2r_+^2}\partial_{\phi'}$$

is a killing vector field and is normal to the surface. Hence the event horizon is the killing horizon for this killing field. The corresponding surface gravity can be calculated to be

$$\kappa = \frac{r_+^2 - r_-^2}{l^2 r_+}$$

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2.3 Global Geometry and Green's Functions

We aim to find Green's function for scalar fields in BTZ Background and understand the nature of Singularity

2.3.1 Quotient Space Construction

As stated above, we can verify that the metric given (2.1) satisfies the Einstein's Equation for 2+1 dimensions for cosmological constant $\Lambda = -\frac{1}{l^2}$ and the Ricci scalar is $R = -\frac{6}{l^2}$. Each point in this metric has a neighborhood locally isometric to the AdS^3 metric. Let us first look at BTZ as a quotient space of the covering space of AdS^3 denoted by $\widetilde{AdS}^3$ with some group of isometries.

We will first try to explain what is meant by $\widetilde{AdS}^3$ and then try to explain it's purpose here. AdS^3 space can be inherited by an embedding in $\mathbb{R}^{2,2}$. It is defined by the following equations.

$$ds^2 = -dT_1{}^2 - dT_2{}^2 + dX_1{}^2 + dX_2{}^2 \tag{2.6}$$

$$-T_1{}^2 - T_2{}^2 + X_1{}^2 + X_2{}^2 = -l^2 \tag{2.7}$$

Let's do a co-ordinate transformation defined by the following equations

$$\begin{aligned}
T_1 &= l\cosh\mu\sin\lambda \\
T_2 &= l\cosh\mu\cos\lambda \\
X_1 &= l\sinh\mu\sin\theta \\
X_2 &= l\sinh\mu\cos\theta \\
\mu &\in \mathbb{R}; 0 \leq \lambda \leq 2\pi; 0 \leq \theta \leq 2\pi
\end{aligned} \tag{2.8}$$

to get,

$$ds^2 = l^2(-\cosh^2\mu d\lambda^2 + d\mu^2 + \sinh^2\mu d\theta^2) \tag{2.9}$$

λ is an angle but the timelike co-ordinate in this form of the metric. This implies the existence of closed timelike curves i.e. spacetime trajectories that start and return to the spacetime point (time-travel xD). We open this periodicity in the λ co-ordinate and extend its range. $\lambda \in \mathbb{R}$. This new space is called the covering space of AdS^3 i.e. $\widetilde{AdS}^3$.

Now we will show how the BTZ metric can be obtained by the metric of $\widetilde{AdS}^3$. Consider the set of transformations given at Section 2 of [4]. We just state one of these transformations for completeness.

for $r \geq r_+$,

$$\begin{aligned}
X_1 &= l\sqrt{\alpha}sinh(\frac{r_+\phi}{l} - \frac{r_-t}{l^2}) \\
X_2 &= l\sqrt{\alpha-1}cosh(\frac{r_+t}{l^2} - \frac{r_-\phi}{l}) \\
T_1 &= l\sqrt{\alpha}cosh(\frac{r_+\phi}{l} - \frac{r_+t}{l^2}) \\
T_2 &= l\sqrt{\alpha-1}sinh(\frac{r_+t}{l^2} - \frac{r_-\phi}{l})
\end{aligned} \tag{2.10}$$

with

$$\alpha = \frac{r^2 - r_-^2}{r_+^2 - r_-^2}; M = \frac{r_+^2 - r_-^2}{l^2}; J = \frac{2r_+r_-}{l}; t, \phi \in \mathbb{R} \tag{2.11}$$

We recover the metric (2.1) but with ϕ not as an angular variable. Now $\widetilde{AdS}^3$ has $SO(2,2)$ and $SL(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ symmetry as these are the symmetries for AdS^3. To apply an element of $SL(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, we first write an point in the AdS^3 in the form of

$$X = \frac{1}{l}\begin{bmatrix} T_1 + X_1 & T_2 + X_2 \\ -T_2 + X_2 & T_1 - X_1 \end{bmatrix}$$

with $det(X) = 1$ and make the transformation $X \to \rho_L X \rho_R$ with $\rho_L, \rho_R \in SL(2, \mathbb{R})$. Now with

$$\rho_L = \begin{pmatrix} e^{\frac{\pi(r_+ - r_-)}{l}} & 0 \\ 0 & e^{\frac{-\pi(r_+ - r_-)}{l}} \end{pmatrix}, \rho_R = \begin{pmatrix} e^{\frac{\pi(r_+ + r_-)}{l}} & 0 \\ 0 & e^{\frac{-\pi(r_+ + r_-)}{l}} \end{pmatrix} \tag{2.12}$$

We have the transformation $\phi \to \phi + 2\pi$. Hence, to get the BTZ Spacetime, we identify ϕ with $\phi + 2\pi$ or wrap together the ϕ coordinate. Hence, we see that the BTZ Spacetime is isometric to $\widetilde{AdS}/ < \rho_L, \rho_R >$.

This completes our objective, now about the singularity, we simply state that in transformations (2.10) if we consider the region $r \leq 0$, we will have closed timelike curves. Hence there is singularity of casual structure at $r = 0$.

2.3.2 Method of Images

Let $G_{BTZ}(x, x')$ denote a Green's function in BTZ Background while $G_{\widetilde{AdS}}(x, x')$ denotes one in $\widetilde{AdS}$ Background. We know use method of images to state that

$$G_{BTZ}(x, x') = \sum_n G_{\widetilde{AdS}}(x, x'_n) \tag{2.13}$$

where $x' = (r, t, \phi)$ and $x'_n = (r, t, \phi + 2\pi n)$ In some sense, A position x' in BTZ Background is equivalent to infinite x'_n in the $\tilde{AdS}$ background.

We can have some generality in (2.13) by having the field

$$\phi(x'_n) = e^{-in\delta}\phi(x')$$

to get,

$$G_{BTZ}(x, x') = \sum_n e^{-in\delta} G_{\widetilde{AdS}}(x, x'_n) \tag{2.14}$$

Fields where $\delta = \pi$ are called twisted fields. Now, (2.13) can be used to reduce the problem of finding the green's function in BTZ Background to a much simpler background of $\tilde{AdS}$. Some green's functions corresponding to different boundary conditions are given in S. Carlip Review Paper[4].

Chapter 3

Introduction to QFT

This section follows from D.Tong Lecture notes [7] and Peskin and Schroeder [8].

3.1 Causality and Propogators

Let us ask the question that suppose we have created a particle at a spacetime point y and then what is the probability amplitude of finding the particle at another spacetime point x? or what is the amplitude for particle to propagate from y to x? This quantity is $\langle 0| \phi(y)\phi(x) |0\rangle$. We can evaluate this quantity using the commutation relations and expanding the scalar fields in terms of creation and destruction operators to get,

$$D(x-y) = \langle 0| \phi(y)\phi(x) |0\rangle = \int d^3p \frac{e^{-ip.(x-y)}}{(2\pi)^3 2E_p} \tag{3.1}$$

We can use this expression to show that the amplitude decays exponentially outside the light cone but is still non-zero. However, the real question that we should ask is that whether a measurement can affect another measurement outside the light cone? For this, we evaluate

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x)$$

Now, $D(x-y)$ is Lorentz invariant quantity as can be seen from (3.1). If $(x-y)^2 < 0$, then there exists a continuous Lorentz transformation from $(x-y)$ to $(y-x)$ and hence, $D(x-y) = D(y-x)$ whenever $(x-y)^2 < 0$ or outside the light cone. If $[\phi(x), \phi(y)] = 0$, we can show that any other commutator involving the scalar field will be 0. Hence, we conclude that no measurement in the Klein-Gordon Theory can affect the other outside the light cone.

We define the Feynman Propagator to be

$$\triangle_F(x-y) = \langle 0| T(\phi(x)\phi(y)) |0\rangle = \begin{cases} \text{D(x-y) if } x^0 > y^0 \\ \text{D(y-x) if } y^0 > x^0 \end{cases} \tag{3.2}$$

where T is for time ordering which places operators evaluated at later times to the left.

We present a very convenient way of writing the Feynman propagator.

$$\triangle_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip.(x-y)} \tag{3.3}$$

where the integral over p^0 has to be taken in a special contour as shown in the figure below We note that

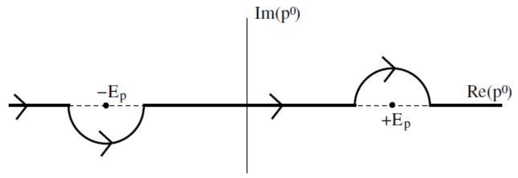

Figure 3.1: Contour for Feynman Propagator, Figure Credits: D. Tong Lecture Notes

two poles for the functions in the p^0 plane are $p^0 = +E_p$ and $p^0 = -E_p$ as

$$\frac{1}{p^2 - m^2} = \frac{1}{(p^0)^2 - E_p^2} \tag{3.4}$$

and the corresponding residue for pole at $\pm E_p = \pm\frac{1}{2E_p}$. When $x^0 < y^0$, we close the contour in the upper half anticlockwise (leading to a negative sign), where $p^0 \to +i\infty$ so that the integrand vanishes and hence just the residue at $-E_p$ is important to give

$$\triangle_F(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip.(y-x)}}{2E_p} = D(y-x) \tag{3.5}$$

and similarly for the other case. We can also write the propagator as

$$\triangle_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip.(x-y)} \tag{3.6}$$

with $\epsilon > 0$ and infinitesimal and the contour of p^0 along the real p^0 axis.

We can also see that the propagator is a Green's function for Klein Gordon equation.

$$\begin{aligned}(\partial_t^2 - \nabla^2 + m^2)\triangle_F &= \int \frac{d^4p}{(2\pi)^4} \frac{i(-p^2 + m^2)}{p^2 - m^2} e^{-ip.(x-y)} \\ &= -i \int \frac{d^4p}{(2\pi)^4} e^{-ip.(x-y)} \\ &= -i\delta^4(x-y)\end{aligned} \tag{3.7}$$

We can now use different contours from the 4 possible along the poles of p^0 integral to get different green's function for the Klein Gordon equation. One of them is the retarded Green's function defined by the contour: which has the property that

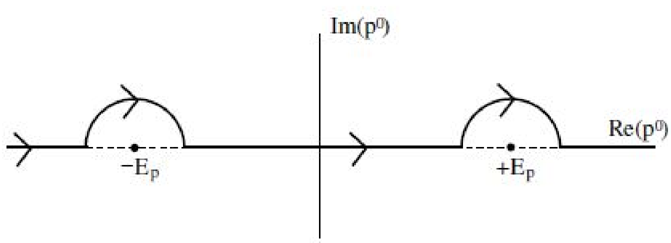

Figure 3.2: Contour for Retarded Green's Function

$$\triangle_R(x-y) = \begin{cases} \text{D(x-y)-D(y-x) if } x^0 > y^0 \\ 0 \qquad\qquad\qquad\quad \text{if } y^0 > x^0 \end{cases} \tag{3.8}$$

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