WEAK GRAVITATIONAL LENSING SURVEYS

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Course Project report for
PH 426: Astrophysics
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1 Introduction to Gravitational Lensing

General Relativity predicts that light gets deflected in presence of matter. This was observationally confirmed by Sir Arthur Eddington, and it is regarded as the first experimental test of Albert Einstein's general theory of relativity.
Similar to a glass lens bending light by refraction, a massive object bends light due to its gravitational field and focuses it somewhere(say, the earth). This phenomenon is called gravitational lensing. The more mass the object has, the greater its gravitational field and more the bending of the light rays.

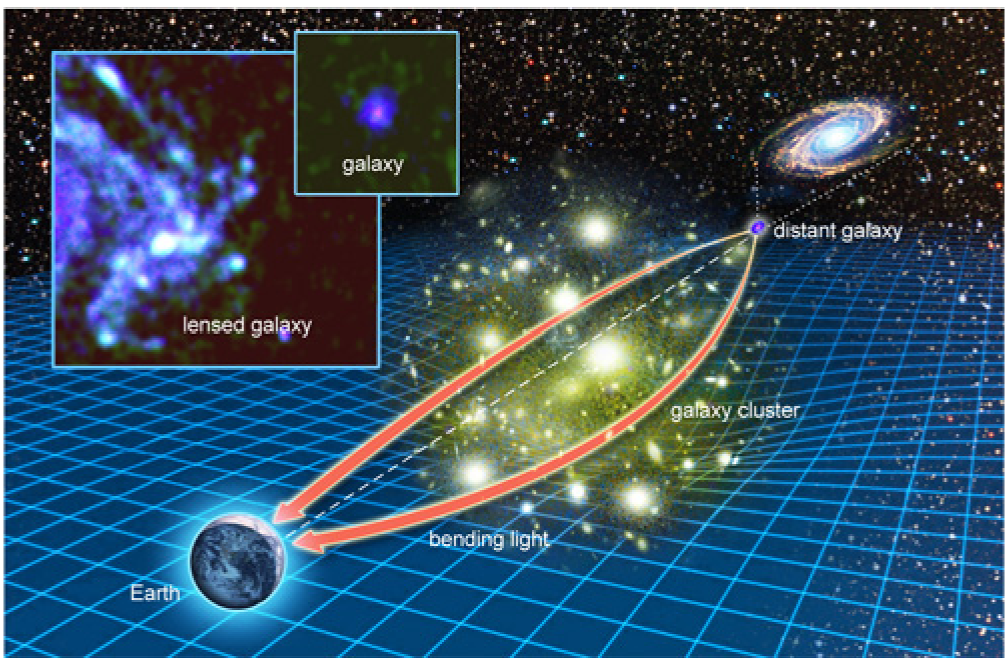

Figure 1: An animated image demonstrating gravitational lensing

Gravitational lensing happens on all scales – the gravitational field of galaxies and clusters of galaxies can lens light, but so can smaller objects such as stars and planets. Even the mass of our own bodies will lens light passing near us a tiny bit, although the effect is too small to ever measure. The kind of lensing that cosmologists are interested in is apparent only on the largest scales – by looking at galaxies and clusters of galaxies because only these would have a large enough mass to produce a detectable bending of light.[1]

1.1 Experimental proof of Gravitational Lensing

The experimental verification of gravity bending light rays was first carried out by British physicist Sir Arthur Eddington on May 29,1919.

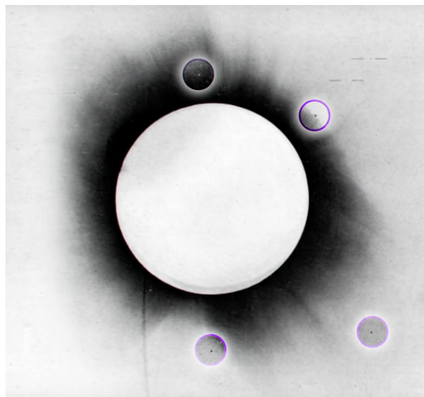

Figure 2: Inverted colour image of photograph taken by Sir Arthur Eddington

The objective was to observe an eclipse and the occurrence of tiny changes in the angular position of nearby stars due to deflection of their light due to suns' gravitational field.This event was particularly important because due to total eclipse we were able to avoid the blinding light of sun and observe just the stars in the near angular vicinity of sun.
The angular shift of the stars during this phenomena were exactly predicted by the calculation in accordance with Einstein's General Theory of Relativity and since then the phenomena of massive heavenly objects being able to converge a beam of light rays was called Gravitational Lensing.[3]

1.2 Strong and Weak Lensing

There are different regimes: strong lensing,weak lensing, and microlensing. The distinction between these regimes depends on the positions of the source, lens and observer, and the mass and shape of the lens (which controls how much light is deflected and where).The most extreme bending of light is when the lens is very massive and the source is close enough to it: in this case light can take different paths to the observer and more than one image of the source will appear.[2] This form of lensing is known as strong lensing and is comparatively rare.

On extra-galactic scales, galaxies, clusters of galaxies, and even the filamentary structure of the cosmic web can act as gravitational lenses. The gravity of those objects causes images high-redshift, background galaxies to be distorted. If the distortions are very small, they cannot be detected on individual galaxies, but only statistically, by averaging over a large number of galaxies, we speak of weak lensing.[4]

1.3 Importance in Cosmology

As gravitational lensing is directly proportional to the mass of the object, it can be majorly attributed to the presence of dark matter as dark matter makes up about 26.8% of the entire Energy in the universe according to Standard Model of Cosmology. It is the dark matter then, which does most of the lensing. Lensing can therefore help astronomers work out exactly how much dark matter there is in the Universe as a whole, and also how it is distributed.

An advantage to using gravitational lensing is that we can neglect other characteristics of the galaxies(formation, emission spectra etc) and only focus on the gravitational aspect. This makes gravitational lensing a very clean and reliable cosmological probe as it relies on few assumptions or approximations.

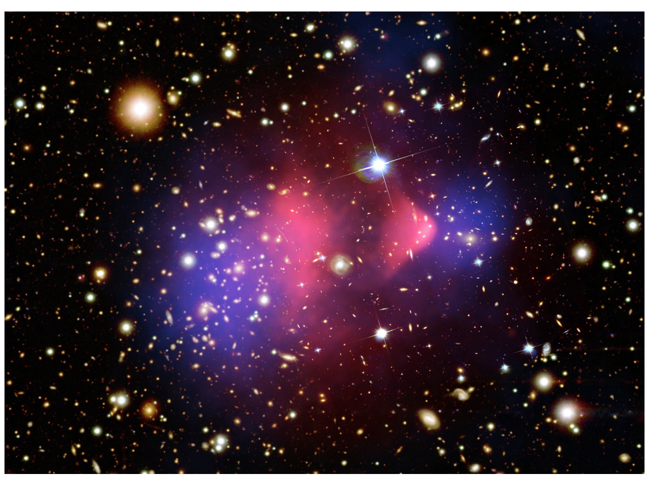

Figure 3: Superposition of X-Ray mapping of visible matter and Lensing Mapping of Total Matter

Gravitational lensing has also been used to verify the existence of Dark Matter itself. In the image above the pink is used to represent X-ray particles and blue represents the mass profile made by lensing. The X-ray particles interact with normal matter and gets scattered but not with dark matter. So, most of the visible matter is in the centre whereas the majority of matter is on the sides.

2 Weak Lensing

The deflection of light is a function of path taken by light and the kind of massive objects in its vicinity of propagation path. Most of the time the light which we receive is red shifted due to the doppler effect on light due to an expanding universe and the source traveling away from the receiver(us).This red shifted light when propagating is distorted by gravitational field of massive object which consists of a network of voids, laments, and halos, when analyzed gives us information about the kind of in homogeneity in space. The larger the amplitude of the in homogeneity of this cosmic web is, the larger the deformations are. The typical distortions of high-red shift galaxies by the cosmic web are on the order of a few percent, much smaller than the width of the intrinsic shape and size distribution. Thus, for an individual galaxy, the lensing effect is not detectable, placing cosmic shear into the regime of weak gravitational lensing. The presence of a tidal field acting as a gravitational lens results in a coherent alignment of galaxy image orientations. This alignment can be measured statistically as a correlation between galaxy shapes.[5]

2.1 Importance in cosmology

The weak lensing distortion includes both a stretching component called "shear" and a dilation component called "convergence". The observed shear field can be used to make maps of the matter in the universe, uncover the mass profiles of galaxies and clusters of galaxies, and even test theoretical models of dark energy.[7]

We shall be discussing in detail about the ellipticity and the rigorous math behind its cosmological applications in the forthcoming weeks.Usually the signal is small, typically inducing an ellipticity of order 1 percent. While this is negligible compared to the intrinsic shape of individual galaxies, it can be measured probabilistic-ally. In the past two decades it has become possible to measure these subtle changes to study the distribution of dark matter in the universe. The first measurements used lensing by galaxy clusters; Recent advancements and interest in this field are due to the fact that there is a lot of information in the redshifted power spectrum form a distant source which has been distorted by gravitational fields in its propagation path. This straightforward interpretation of the signal is rather unique in the tools available for cosmology, and it potentially enables the determination of cosmological parameters with high precision. Lensing measurements are not only sensitive to the geometry, but also provide measures of the growth of large-scale structure that test gravity on cosmological scales. Hence these distortions are a probe into study of dark energy and other variations of theories on gravity.[6].

2.2 Observations

We introduce some cosmological parameters which have been determined by lensing surveys in this section.

1. **Hubble Constant H_o:** It is in some ways the most fundamental cosmological parameter. It is expected that all galaxies follow

   $$v = H_o r$$

   . The Hubble constant is usually parameterized as

   $$Ho = 100hkms^{-1}Mpc^{-1}$$

   . Many other parameters are given in terms of the dimensionless parameter h.[23]

2. **Critical Density ρ_c :** For a given value of Hubble Parameter H. (H_o is the current value of H), there is a unique value of average density that would imply a flat universe. (Curvature $k = 0$)

   $$\rho_c = \frac{3H^2}{8\pi G}$$

   This can be written in terms of h as $\rho_c = 1.8788 \times 10^{-26} h^2 \, \mathrm{kg/m^3}$. Densities are generally given as a dimensionless quantity in units of ρ_c and denoted by Ω

3. **Total Matter Density Ω_m:** This indicates the total matter density and includes baryonic and dark matter contribution. The physical densities of the the individual matter components, $\Omega_i h^2$, are often more useful than the density parameters .

4. **Density Perturbation Amplitude Parameter σ_8:** "The density perturbation amplitude can be specified in many different ways other than the large-scale primordial amplitude, for instance, in terms of its effect on the cosmic microwave background, or by specifying a short-scale quantity, a common choice being the present linear-theory mass dispersion on a scale of $8h^{-1}Mpc$, known as σ_8.

3 Open Questions

We liked the first few open questions and have expanded upon them on the basis of available information:

- **What is the role of weak gravitational lensing in estimation of cosmological constants now that most of the parameters are now known at the per-cent level?**
  Cosmological parameter constraints from the Cosmic Microwave Background (CMB) alone suffer from degeneracies which can only be broken using additional information. Degeneracies in parameters mean that more than one set of parameters can give the same observed result. By measuring the dark-matter density (Ω_m) and the normalisation of its fluctuation amplitude directly(σ_8), gravitational lensing adds constraints which substantially narrow the parameter ranges allowed by the CMB alone.[8]

- **Why are simulations required for interpretation of Weak Lensing Data and how can they be improved?**
  The scales on which weak lensing probes the Large Scale Structure extend deep into the highly non-linear regime. To make analytical predictions of the observations on those scales is very difficult. N-body simulations offer ways to obtain the non-linear power spectrum. Pure dark-matter N- body simulations are essentially solutions for Newton's equations of motion. Higher processing speeds and huge amounts of memory enable us to have very accurate, high-resolution simulations

spanning a wide range of scales and dynamic range.
Baryonic interactions however are not well known in detail. Even the simplistic approximation of the baryonic content as an ideal fluid leads to nonlinear equations, which further make it difficult to include dark matter and baryonic matter together. Additional physics is necessary for such simulations, consisting for example in astrophysical processes like radiative cooling, star formation, supernova feedback, magnetic fields, black hole and AGN feedback, and cosmic rays. Many of those processes are yet to be understood well.[5]

- **How does Weak Lensing effect Cosmic Microwave Background?**
"Observation of cosmic microwave background (CMB) radiation is one of the cleanest probes of cosmology. However lensing of CMB photons by intervening mass clumps can provide additional information about the structure and dynamics of the Universe. Besides, lensing of the CMB can provide information at larger scales and higher redshift than can be reached by any other astronomical observations. Weak lensing of the CMB is responsible for many observable effects which have been studied in extensive detail."[9]

- **How can we be sure that the recovered lensing signal is cosmological in nature, and not dominated by observational distortions?**
Simulated data can be used to test weak lensing analysis. However, these simulations may lack a systematic effect that is present in real data.Fortunately, a number of diagnostic tools can be used to test the reliability of the recovered lensing signal. These tests cannot guarantee whether the recovered signal is free of systematics, but they do often indicate whether systematics are present. The first diagnostic makes use of the fact that the corrected galaxy shapes should not correlate with the (uncorrected) shapes of stars. Another unique diagnostic makes use of the fact that the weak lensing shear arises from a gravitational potential. Consequently, the resulting shear field is expected be curl-free. The observed ellipticity correlation functions can be separated into two independent components, an “E”-mode which is curl-free and a “B”-mode, which is sensitive to the curl of the shear field. Hence, the presence of a significant “B”-mode indicates that residual systematics remain. A diagnostic of the cosmological nature of the lensing signal

is its variation with redshift. Thus, provided the redshift estimates are accurate, this variation can be used to test the cosmological origin of the lensing signal.[7]

- **How does Weak Lensing affect other methods of estimating parameters?**
  The effect on the apparent brightness of sources due to weak lensing can also have unwanted effects : The statistical incoherent lens-induced change of the apparent brightness of (widely separated) "standard candles" – like type Ia supernovae – affects the accuracy of the determination of cosmological parameters.[11]

- **Does the existence of Matter imply the breakdown of known laws and theories of physics like General Relativity?**
  The observation of the cluster merger 1E0657-558, also known as the Bullet Cluster is one of the most significant weak lensing observation pertaining to dark matter. In the collision between those two clusters the dissipationless stellar component and the X-ray emitting plasma are separated.Plasma is expected to be the mass dominant component and not the stellar part. In this observation, however, one sees that the gravitational potential traces the stellar component rather than the plasma implying that major gravitational effects are due to the uncollided part. This is explained by considering a dark matter dominated cluster. The great advantage of this dark matter measurement is that it does not require any assumptions of the dark matter properties. This observation also indicates that the dark matter problem is really due to dark matter and not modifications of gravity; it would be very difficult to modify the laws of gravity in such a way that the gravitational centre do not coincide with the centre of mass of the system. However, such an observation does not rule out modified theories of gravity.[10]

- **What are some of the biggest challenges in weak lensing?**
  Weak lensing surveys needs the observations of a large number of sources as larger the number of sources that is averaged over the smaller the statistical noise (which includes intrinsic alignment) in the measurements. In weak lensing surveys aiming at measuring the mass of a single cluster the region in which the shear can be assumed to be constant, is limited. Hence these measurements requires a high density of sources with measurable shape, which in turn requires deep observations. To

reach a large number density in the observations also the fainter, and more numerous, galaxies needs to be probed. It is difficult to measure the shape of faint galaxies. The observed shape of small galaxies is strongly affected by the atmospheric disturbances, for ground based observations, and also by distortion effects of the telescope. Understanding these effects and correcting them. These effects needs to be understood and corrected for, which is the largest observational challenge in using weak lensing. Also, weak lensing surveys observing large regions of the sky quickly leads to large data sets which can be difficult to handle just as in other astronomical data sets.[10]

- **How can Weak Lensing Surveys be used to find constraints on Modified Gravitational Theories?**
  Explaining the late-time acceleration of the Universe represents a major challenge in modern cosmology, and current interpretations mostly rely on the inclusion of dark energy components and/or modifications to the theory of General Relativity (GR). One important point in such theories is that signs for dark energy can be indistinguishably mapped by modified theories. These two frameworks only really decouple when considering the evolution of matter density fluctuations and of perturbations associated with the metric. In addition, there are various ways in which a modification of gravity on large scales could account for the apparent acceleration. Exploiting this, many observational probes based on large scale structure formation have been proposed to test theories of modified gravity, including galaxy clustering and weak gravitation lensing.[12]

- What are the nature and properties of dark matter?

- Does Dark Matter interact with its surroundings using weak interactions?

- How will the Universe evolve in a large timescale like the next ten to thirteen billion years?

- Can we rule out theories suggesting Cosmological Constant as candidates to explain accelerated expansion of Universe?

- Can the changes in Hubble constant over time by measured using lensing of distant objects?

- What other parameters can be inferred from weak lensing?
- How can extensive weak lensing surveys be used for other astrophysical purposes like detection of minor planets and others?
  The Deep Lens Survey has also been used to identify a significant number of minor planets.[13]

4 Distance Measures

The **comoving distance** (χ) between fundamental observers, i.e. observers that expanding according to Hubble's law, does not change with time, as comoving distance accounts for the expansion of the universe.

Two comoving objects at constant redshift z that are separated by an angle dθ are said to be separated by the distance D(χ)dθ where D(χ) is called the **transverse comoving distance**

5 General Concepts of Lensing

[14]

5.1 General Lens

It can be shown that the deviation due to Gravitational potential is given by

$$\hat{\vec{\alpha}}(b) = \frac{2}{c^2}\int_{-\infty}^{\infty} \vec{\nabla}_{\perp}\Phi dz \tag{1}$$

For a point mass lens,

$$\Phi = -\frac{GM}{R}, |\hat{\vec{\alpha}}| = \frac{4GM}{bc^2} \tag{2}$$

b is the impact parameter as seen in the diagram, and z is the redshift along the line of vision.

The deflection of an array of lenses can be superimposed. If the lenses are discrete mass objects then we'll have to individually sum over their deflection angles. Practically the lens' size is much smaller than the distances between

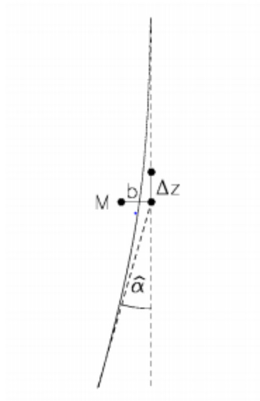

Figure 4: Figure of lensing by a point mass (Introduction to Gravitational Lensing Lecture scripts by Massimo Meneghetti)

the source, lens and the observer. So the deflection happens in a small section of the light path. This lets us approximate the lens as a planar distribution of matter. Within this approximation the lensing matter is fully described by its surface density:

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)dz \tag{3}$$

Here $\vec{\xi}$ is the 2-D vector in the lens plane and ρ is the density. As long as our approximation holds, we can sum over all the deflection angles for the differential masses $\Sigma(\vec{\xi})d^2\xi$.
Summing over these gives us the total deflection angle as:

$$\vec{\alpha}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi'})\Sigma(\vec{\xi'})}{|\vec{\xi} - \vec{\xi'}|} d^2\xi' \tag{4}$$

5.2 Lens Equation

In Fig. 2 we sketch a typical gravitational lens system. A mass concentration is placed at redshift z_L, corresponding to an angular diameter distance D_L. This lens deflects the light rays coming from a source at redshift z_S(or angular distance D_S).

We first define an optical axis, indicated by the dashed line, perpendicular

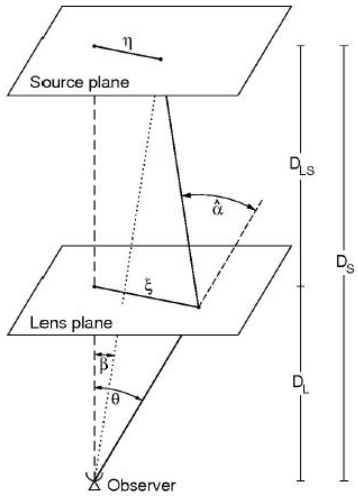

Figure 5: Sketch of a typical gravitational lensing system (Figure from Bartelmann and Schneider, 2001)

to the lens and source planes and passing through the observer. The source lies at a point $\vec{\eta} = \vec{\beta} D_S$ from the optical axis. The impact parameter on the lens plane is given by $\vec{\xi} = \vec{\theta} D_L$.
If $\vec{\theta}$, $\vec{\beta}$ and $\vec{\alpha}$ are sufficiently small, we can approximate arcs as straight lines and obtain the following relation:

$$\vec{\theta} D_S = \vec{\beta} D_S + \hat{\vec{\alpha}} D_{LS} \tag{5}$$

The above equation is called the *lens equation*. We substitute the following dimensionless quantities in the equation:

$$\vec{x} = \frac{\vec{\xi}}{\xi_0} \quad \vec{y} = \frac{\vec{\eta}}{\eta_0} \quad \vec{\alpha}(\vec{x}) = \frac{D_L D_{LS}}{\xi_0 D_S} \hat{\vec{\alpha}}(\xi_0 \vec{x}) \tag{6}$$

Using these substitutions, the equation can be written as $\vec{y} = \vec{x} - \vec{\alpha}(\vec{x})$.

5.3 Lensing Potential

We can also characterize the mass(lens) in terms of an effective potential obtained by projecting the gravitation potential on the lens plane and appropriate rescaling.

$$\hat{\Psi} = \frac{D_{LS}}{D_L D_S} \frac{2}{c^2} \int \Phi(D_L \vec{\theta}, z) dz \tag{7}$$

We make this dimensionless as $\Psi = \frac{D_L^2}{\xi_0^2} \hat{\Psi}$. The lensing potential satisfies the following 2 properties:

1. The gradient of Ψ gives the scaled deflection angle:

$$\nabla_x \Psi(\vec{x}) = \vec{\alpha}(\vec{x}) \tag{8}$$

2. The Laplacian of Ψ gives twice the *convergence*:

$$\nabla_x^2 \Psi(\vec{x}) = 2\kappa(\vec{x}), \kappa(\vec{x}) = \frac{\Sigma(\vec{x})}{\Sigma_{cr}}, \Sigma_{cr} = \frac{c^2}{4\pi G} \frac{D_S}{D_L D_{LS}} \tag{9}$$

This property is derived from the Poisson Equation. Σ_{cr} is called the critical surface density and is used to characterize the lens system and is a function of the angular diameter distances of the lens and source.

We can integrate property 2 to get the lensing potential in terms of the convergence:

$$\Psi(\vec{x}) = \frac{1}{\pi} \int \kappa(\vec{x}') \ln|\vec{x} - \vec{x}'| d^2x' \tag{10}$$

From this we can get the deflection angle as:

$$\vec{\alpha}(\vec{x}) = \frac{1}{\pi} \int \kappa(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|} d^2x' \tag{11}$$

6 Ellipticity

[14] The Relation between source and image can be linearized in terms of Jacobian Matrix:

$$A := \frac{\partial \vec{y}}{\partial \vec{x}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\vec{x})}{\partial x_j}\right) = \left(\delta_{ij} - \Psi_{ij}\right) \tag{12}$$

$$\Psi_{ij} \equiv \frac{\partial^2 \Psi}{\partial x_j \partial x_i} \tag{13}$$

where x_i indicates the i-component of $\vec{x}$ on the lens plane. The above equation shows that the elements of the Jacobian matrix can be written as combinations of the second derivatives of the lensing potential.

$$\gamma_1(\vec{x}) = \frac{1}{2}\Psi_{11} - \Psi_{22} \tag{14}$$

$$\gamma_2(\vec{x}) = \Psi_{12} \tag{15}$$

These are components of a pseudo vector $\vec{\gamma} = (\gamma_1, \gamma_2)$ called shear.

The linearised lens equation tells us the inverse of what we typically want to know from weak gravitational lensing. Since we observe images but cannot access the sources, we need to infer the separation dof image points from the image centre. If the Jacobi matrix A is invertible, we can write

$$d\vec{\theta} = A^{-1}d\vec{\beta} \tag{16}$$

The inverse Jacobi matrix determines how sources are mapped on images.

$$A^{-1} = \frac{1}{det(A)}\begin{pmatrix} 1-\kappa+\gamma_1 & \gamma_2 \\ \gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix} \tag{17}$$

$$\mu = \frac{1}{det(A)} = \frac{1}{(1-\kappa)^2 - (\gamma)^2} \approx 1 + 2\kappa \tag{18}$$

μ represents the increase in solid angle of the image as compared to that of the source, also called the *magnification factor*. Thus, in weak lensing, the magnification of an image is essentially (i.e. to first order) determined by the convergence κ, not by the shear γ.

$$\lambda_{\pm} = \frac{1-\kappa\pm\gamma}{det(A)} = \frac{1}{1-\kappa\mp\gamma} \tag{19}$$

A circular source is deformed by weak gravitational lensing to become an ellipse whose semi-major and semi-minor axes, called a and b, are proportional to the eigenvalues $\lambda_{\pm}$. By its common definition, the *ellipticity* ϵ of such an image is defined as,

$$\epsilon = \frac{a-b}{a+b} = \frac{\gamma}{1-\kappa} = g \tag{20}$$

7 Averaging of image ellipticities

[14]

The sources typically used in observations of weak gravitational lensing are of course not circular, but intrinsically elliptical. In lowest-order approximation, the intrinsic source ellipticity ϵ_s and the ellipticity caused by lensing add. Then,

$$\epsilon \approx g + \epsilon_s \tag{21}$$

One of the most essential assumptions in the interpretation of weak gravitational lensing is that the intrinsic ellipticities cancel out when averaged over sufficiently large samples, $\langle \epsilon_s \rangle = 0$. Thus,

$$\langle \epsilon \rangle \approx \langle g \rangle \tag{22}$$

8 Measurement Process

[11] There are essentially two general approaches to the measurement of image ellipticities. Both of them use the measured surface brightness $I(\vec{\theta})$ of an image. In the first approach, models for the surface brightness of elliptical sources are fit to the image, allowing to read off the model ellipticity once the best fit has been found. In the other, model-free approach, the quadrupole moments Q_{ij} of the surface brightness are measured,

$$Q_{ij} = \int d^2\theta I(\vec{\theta})\theta_i\theta_j, i, j = 1, 2 \tag{23}$$

$$\epsilon_1 = \frac{Q_{11} - Q_{22}}{2N_q} \tag{24}$$

$$\epsilon_2 = \frac{Q_{12}}{N_q} \tag{25}$$

$$N_q := \frac{1}{2}trQ + \sqrt{det(Q)} \tag{26}$$

9 Generalization of Lensing Potential

[11] So far, we have assumed that the lens is geometrically small compared to the overall scale of the lens source observer system. Such a approximation is not appropriate for gravitational lensing by the large scale structures in our Universe. Generalizing the results derived so far to the case of extended lenses, we get (We have assumed the spatial geometry to be flat)

$$\psi = \frac{2}{c^2}\int_0^z \frac{\chi_s - \chi}{\chi_s \chi}\Phi((\chi, \chi)\vec{\theta}) \tag{27}$$

Using Poisson's equation, convergence due to an extended lens is given by

$$\kappa = 4\pi\frac{G}{c^2}\int_0^{\chi_s} d\chi \frac{\chi(\chi_s - \chi)}{\chi_s} a^2 \rho(\chi) \tag{28}$$

where ρ is the fluctuation of mass density around the mean value $\bar{\rho}$

$$\bar{\rho} = \frac{3H_o^2}{8\pi G}\Omega_{m0} a^{-3} = \bar{\rho}_o a^{-3} \tag{29}$$

Introducing the dimensionless density contrast δ as $\rho = \bar{\rho}\delta$, using which convergence is given by

$$\kappa = \frac{3H_o^2\Omega_{m0}}{2c^2}\int_0^{\chi_s} d\chi \frac{\chi(\chi_s - \chi)}{\chi_s}\frac{\delta(\chi)}{a} \tag{30}$$

10 Limber's Approximation and Convergence power Spectrum

It is not possible to consider all the density fluctuations a light ray will encounter. Thus, we use a statistical approach and compute the andular correlation function

$$\langle \kappa(\vec{\theta})\kappa(\vec{\theta} + \vec{\phi})\rangle_{\vec{\theta}} = \xi_\kappa(\phi)$$

Due to isotropy, it can depend only on ϕ and not on the orientation. At many times, we find it easier to use the fourier transform of the correlation function $\xi_\kappa(\phi)$ which we define to be the angular power spectrum as

$$C_\kappa(l) = \int d^2\phi \xi_\kappa(\phi) e^{-i\vec{l}.\vec{\phi}}$$

The vector $\vec{l}$ is the conjugate to $\vec{\phi}$.
Here, we use the limber's approximation which states that if the quantity $x(\vec{\theta})$ defined in the 2 dimensions is a projection of a quantity $y(\vec{r})$ in 3 Dimensions with a function $w(\chi)$ to weigh.

$$x(\vec{\theta}) = \int_o^{\chi_s} d\chi w(y) y(\chi\vec{\theta}, \chi) \tag{31}$$

and $P_y(k)$ is the power spectrum of y at the three dimensional wave number k then we can approximate the angular power spectrum of x to be

$$C_x(l) = \int_0^{\chi_s} d\chi \frac{w^2(\chi)}{\chi^2} P_y(k) \tag{32}$$

Equation [28] thus shows that effective convergence is a projection of density contrast δ with the weight function

$$w(\chi) = \frac{3H_o^2\Omega_{m0}}{2c^2} \frac{\chi(\chi_s - \chi)}{a\chi_s} \tag{33}$$

The power spectrum of the convergence $C_\kappa(l)$ is thus related by a weighted line-of-sight integral to the power spectrum $P_\delta(k)$ of the density contrast.If the density contrast $\delta << 1$, then it can be shown that it grows with proportion to linear growth factor $D_+(a)$ At scales above $8h^{-1}Mpc$ (Typical size of clusters), the power spectrum P_δ as a slowly varying shape function $\mathcal{P}$ times an amplitude σ_8^2

$$\sigma_8^2 = \int_o^\infty \frac{k^2 dk}{2\pi^2} P_\delta(k) W_8^2(k) \tag{34}$$

where W_8 is a filter function to remove out contributions from scales smaller than $8h^{-1}Mpc$. In terms of D_+, density power fluctuations can be written as

$$P_\delta(k) = \sigma_8^2 D_+^2(a)\mathcal{P}(k) \tag{35}$$

Using the above relation, convergence power spectrum can be shown to be

$$C_\kappa(l) = \frac{9}{4}(\frac{H_o}{c})^4 \Omega_{mo}^2 \sigma_8^2 \int_0^{\chi_s} d\chi[\frac{D_+(a)}{a}\frac{\chi(\chi_s-\chi)}{\chi_s}]^2 P(k) \tag{36}$$

In the previous result, $C_\kappa(l)$ can be measured from observations and we can obtain P_δ from measurements too. Thus we can obtain information about $\Omega_{mo}^2\sigma_8^2$ and the linear growth factor $D_+(a)$.
By using Fourier transforms it can be shown that $C_\gamma = C_\kappa$ and C_γ can be obtained from shear measurements.

11 Lensing by Large Scale Structures and Limber's Equation

11.1 Light propagation through an inhomogeneous universe

In unperturbed spacetime, light travels alog null geodesic lines of the symmetric, homogenous and isotropic Friedmann-Lemaitre space-time.
In contrast to the earlier treatment, we have to take into account that lenses can now be of comparable size to the curvature scale of the universe, thus we need to refine the picture of straight light paths which are instantly deflected by sheet-like, thin lenses.
For an unperturbed space-time, the light rays propogate as follows:

$$\frac{d^2\vec{x}}{dw^2} + K\vec{x} = 0 \tag{37}$$

Here $K = (H_0/c^2)(\Omega_0 + \Omega_A - 1)$ is the curvature parameter of the universe. Ω_0 is the density parameter of the universe,

$$\Omega_0 = \left(\frac{3H_0^2}{8piG}\right)^{-1} \rho_0 \tag{38}$$

and Ω_A is the density parameter corresponding to the cosmological constant,

$$\Omega_A = \left(\frac{\Lambda}{3H_0^2}\right). \tag{39}$$

The metric is written as

$$ds^2 = c^2dt^2 - a^2[dw^2 + f_K^2(w)d^2\Omega] \tag{40}$$

Solving the equation results in:

$$\vec{x}(w) = \vec{\theta} f_K(w) \tag{41}$$

For adding perturbations, we use our previous result for the deflection angle:

$$\frac{d^2\vec{x}}{dw^2} = -\frac{2}{c^2}\vec{\nabla}_{\perp\phi} \tag{42}$$

Upon adding this, the propagation equation changes to:

$$\frac{d^2\vec{x}}{dw^2} + K\vec{x} = -\frac{2}{c^2}\vec{\nabla}_{\perp\phi} \tag{43}$$

The general solution to this propagation equation is:

$$\vec{x}(w) = \vec{\theta} f_K(w) - \frac{2}{c^2}\int_0^w dw' f_K(w-w')\vec{\nabla}_{\perp\phi} \tag{44}$$

Like in previous cases, we'll evaluate this integral over the unperturbed path $\vec{\theta} f_K(w)$ and calculate the deflection angle as the difference of $\vec{\theta}$ between the perturbed and unperturbed path.

$$\vec{\alpha} = \frac{2}{c^2}\int_0^w dw' f_K(w-w')\vec{\nabla}_{\perp\phi}[\vec{\theta} f_K(w'), w'] \tag{45}$$

11.2 Effective Convergence

Previously we defined convergence as the half of divergence of $\vec{\alpha}$. Analogously we define an effective convergence for large scale lenses:

$$\kappa_{eff}(\vec{\theta}, w) = \frac{1}{c^2} \int_0^w dw' \frac{f_K(w') f_K(w - w')}{f_K(w)} \nabla^2 \phi(\vec{\theta} f_K(w'), w') \tag{46}$$

We can substitute the poisson equation for $\nabla^2 \phi$. The original looks like:

$$\nabla^2 \phi = 4\pi G \rho \tag{47}$$

Introducing the density contrast δ and replacing $\rho_0 = \rho a^3$, we get:

$$\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}} \qquad \nabla^2 \phi = 4\pi G \bar{\rho} a^{-1} (1 + \delta) \tag{48}$$

Decoupling the potential to a background part not involving δ and a perturbing part involving δ and using the following substitution:

$$\rho_0 = \Omega_0 \frac{3H_0^2}{8\pi G}, \tag{49}$$

we get the effective convergence as follows:

$$\kappa_{eff}(\vec{\theta}, w) = \frac{3\Omega_0}{2} \left(\frac{H_0}{c} \right)^2 \int_0^w dw' \frac{f_K(w') f_K(w - w')}{f_K(w)} \frac{\delta(\vec{\theta} f_K(w'), w')}{a} \tag{50}$$

If instead of a single source, we are given a distribution of sources $G(w)$, the mean effective convergence becomes:

$$\langle \kappa_{eff} \rangle (\vec{\theta}) = \frac{3\Omega_0}{2} \left(\frac{H_0}{c} \right)^2 \int_0^{w_H} dw W(w) f_K(w) \frac{\delta(\vec{\theta} f_K(w), w)}{a(w)} \tag{51}$$

with the effective weight function

$$W(w) = \int_w^{w_H} dw' G(w') \frac{f_K(w' - w)}{f_K(w')} \tag{52}$$

11.3 Limber's Equation

It is impossible to predict exactly which density fluctuations a light ray will find on its way. Concerning the effective convergence, we thus need a statistical approach. We want to compute the correlation function:

$$\left\langle \kappa(\vec{\theta})\kappa(\vec{\theta}+\vec{\phi})\right\rangle = \xi_\kappa(\phi) \tag{53}$$

For convenience, we first deal with the correlation in Fourier space and use the power spectrum instead. We make the calculations for an arbitrary function g in n-dimensional space first and then we'll substitute for convergence.

$$\left\langle g(x)(x+y)\right\rangle = \xi_{gg}(y) \tag{54}$$

We take the fourier transform of $g(x)$ and calculate the correlation in fourier space.

$$\left\langle \hat{g}(k)\hat{g}^*(k')\right\rangle = (2\pi)^n \delta_D^{(n)}(k-k')P_g(k) \tag{55}$$

Here $P_g(k) = \int d^n y \; exp(-iky) \; \xi_{gg}(y)$ is the power spectrum.
Suppose we're given the power spectrum for a 3-D function $\delta(\vec{x})$ and we have to find the power spectrum of a 2-D projection:

$$g(\vec{\theta}) = \int dwq(w)\delta(\vec{\theta}f_K(w), w) \tag{56}$$

Taking the correlation and resolving the expression in terms of the fourier transform of $\delta(\vec{x})$ and putting $\phi = |\vec{\theta} - \vec{\theta'}|$ we get:

$$\xi_{gg}(\phi) = \int q^2(w)dw \int \frac{kdk}{2\pi} P_\delta(k) J_0[f_K(w)\phi k] \tag{57}$$

Using this to calculate the Power Spectrum for g:

$$P_g(l) = \int \frac{q^2(w)}{f_K^2(w)} dw P_\delta\left(\frac{l}{f_K(w)}\right) \tag{58}$$

Now just replace g with K_{eff} with $q(w) = \frac{3H_0^2\Omega_0}{2c^2a} W(w) f_K(w)$ and we get the Power Spectrum for effective convergence as follows:

$$P_\kappa(l) = \left(\frac{3H_0^2\Omega_0}{2c^2a}\right)^2 \int_0^{w_H} W^2(w)dwP_\delta\left(\frac{l}{f_K(w)}\right) \tag{59}$$

This spectrum is useful for calculating all further correlation functions. For example, for convergence:

$$\xi_\kappa(\phi) = \int \frac{ldl}{2\pi} P_\kappa(l) J_0(l\phi) \tag{60}$$

12 Linear Growth Function

In the regime where $|\delta| << 1$, we have the following relation for the linear growth function. It is normalized using the constraint that at a=1, $D_+(a) = 1$ [4]

$$D_+(a) \propto \frac{H(a)}{H_0} \int_0^a \frac{da'}{[\Omega_m/a' + \Omega_\lambda a'^2 - (\Omega_m + \Omega_\lambda - 1)]^{3/2}} \tag{61}$$

$$P(k,z) = \delta^2 \frac{2\pi^2}{k^3} \left(\frac{ck}{hH_{100}}\right)^{3+n} \left(T(k,z)\frac{D_1(z)}{D_1(0)}\right)^2 \tag{62}$$

Here are some of the plots we could code up and plot:

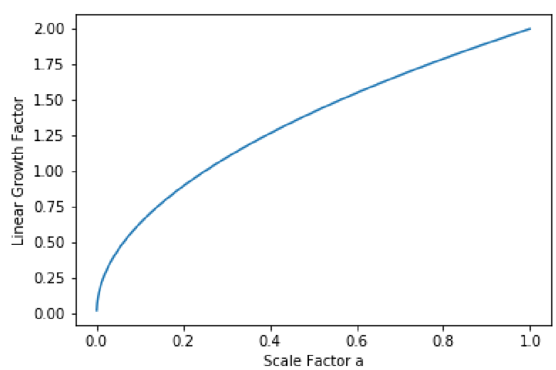

Figure 6: The Unnormalized Linear Growth factor For flat Universe and $\Omega_m = 0$ and $\Omega_\Lambda = 1$

13 Convergence Power Spectrum

Convergence power spectrum for cosmological weak lensing is given by [17]

$$P_\kappa(l) = \frac{9}{4}(\frac{H_o}{c})^4 \Omega_{mo}^2 \int_0^\infty dz \frac{n^2(z)}{a^2(z)} P_\delta(\frac{l}{\chi(z)}, \chi(z)) \tag{63}$$

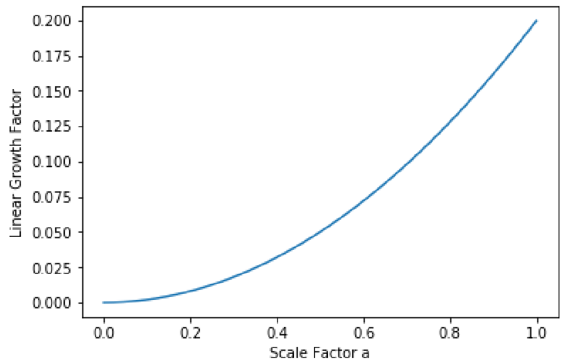

Figure 7: The Unnormalized Linear Growth factor For flat Universe and $\Omega_m = 1$ and $\Omega_\Lambda = 0$

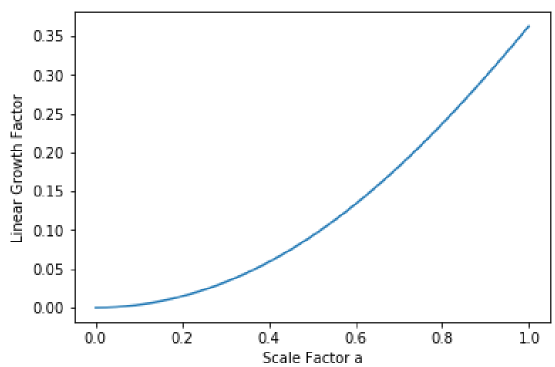

Figure 8: The Unnormalized Linear Growth factor For flat Universe and $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$

where $n(z)$ is a normalized weight function for galaxies assumed to be of the form [18]

$$n(z) \propto (\frac{z}{z_0})^\alpha e^{-(\frac{z}{z_0})^\beta} \tag{64}$$

The Dimensionless Power Spectrum is

$$C(l) = \frac{l(l+1)}{2\pi} P_\kappa(l) \tag{65}$$

14 Code

We have obtained a code used by a PhD student Bikash Dinda and have modified it to our purpose. The final code can be found at https://github.com/kjanyani/Weak_Lensing_Surveys

15 Plots

The newly defined parameters is n_s which is the the index to which k is raised in the primordial Matter Spectrum

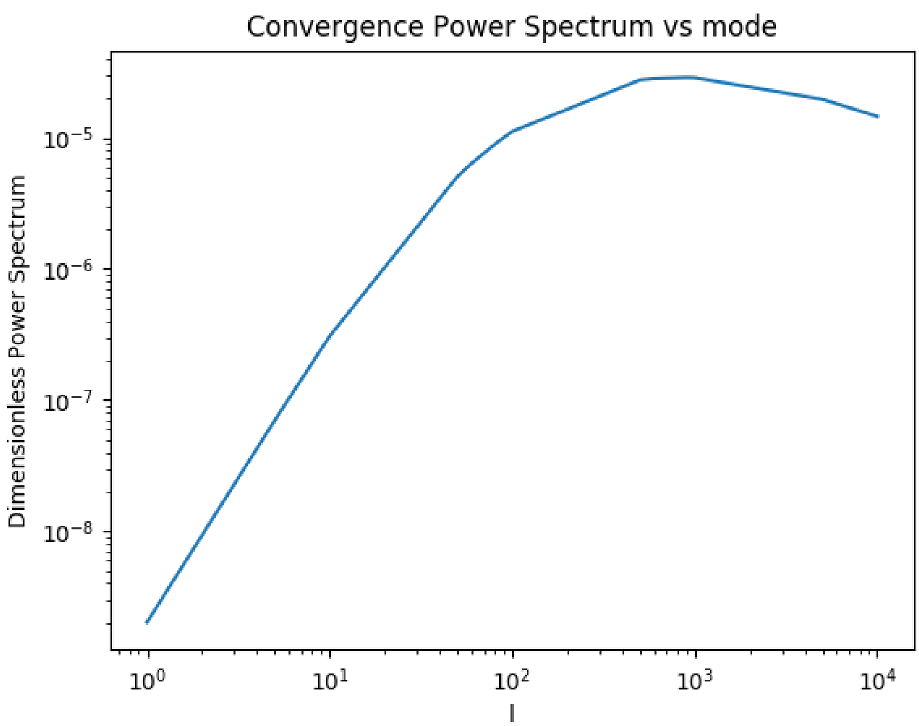

Figure 9: Parameters used are:

$$\Omega_m = 0.3, = 0.65, \Omega_b = 0.044, n_s = 1, \sigma_8 = 1.0, z_0 = 0.5, \alpha = 2, \beta = 1$$

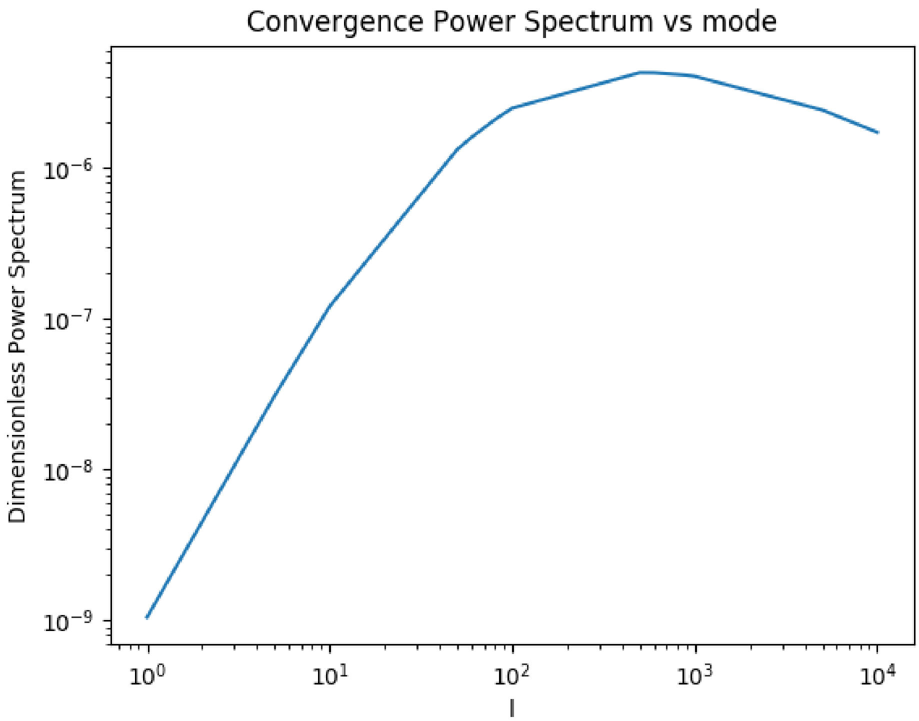

Figure 10: Parameters used are:

$$\Omega_m = 0.3, = 0.65, \Omega_b = 0.044, n_s = 1, \sigma_8 = 0.8, z_0 = 0.5, \alpha = 2, \beta = 1.5$$

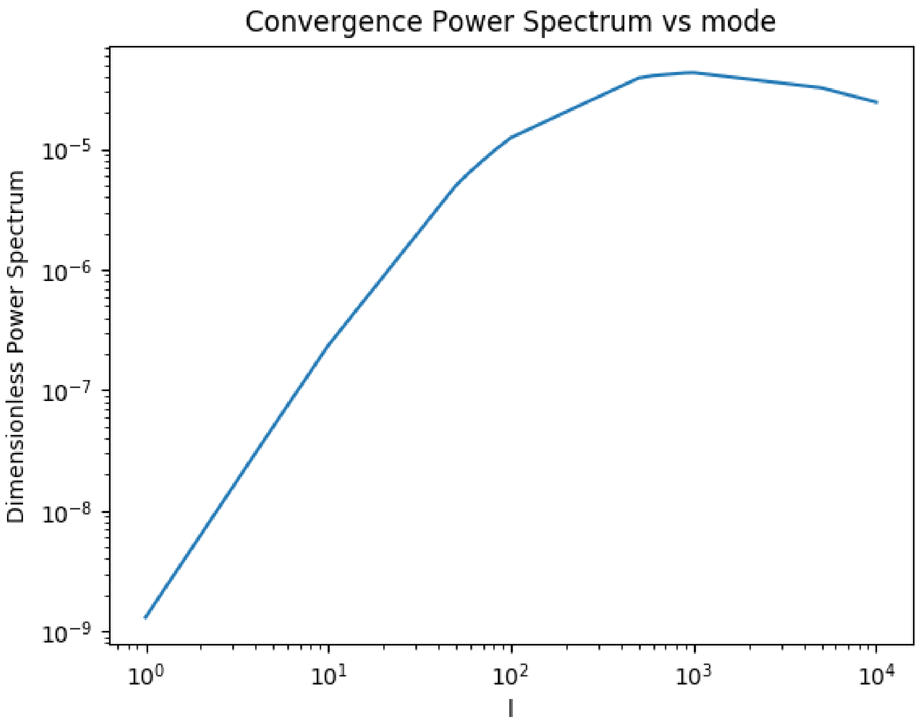

Figure 11: Parameters used are:

$$\Omega_m = 0.3, = 0.7, \Omega_b = 0.044, n_s = 1, \sigma_8 = 0.8, z_0 = 1.0, \alpha = 2, \beta = 1$$

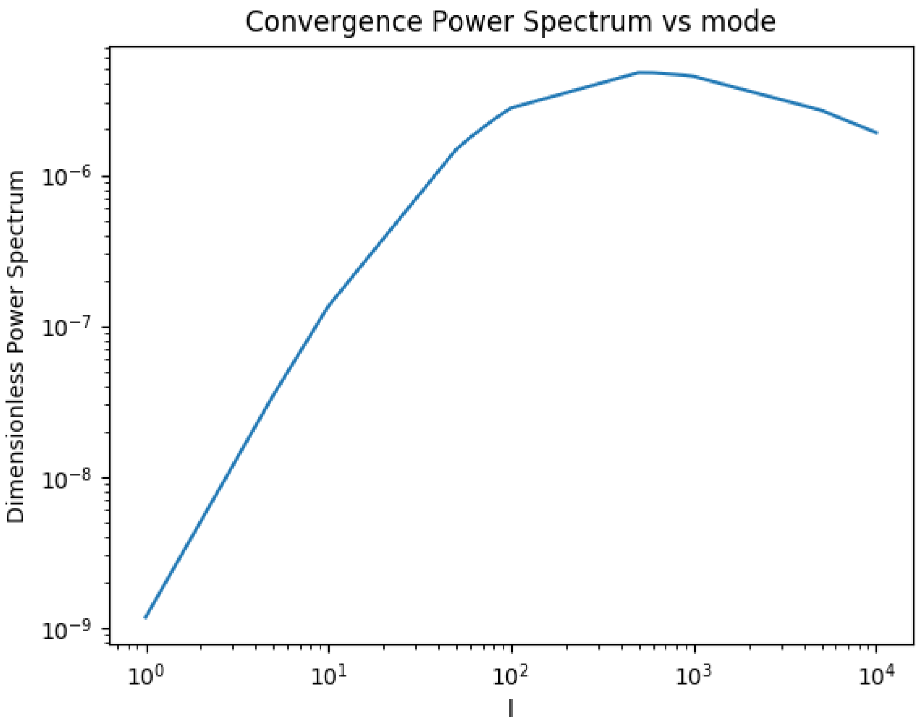

Figure 12: Parameters used are:

$$\Omega_m = 0.3, = 0.7, \Omega_b = 0.044, n_s = 1, \sigma_8 = 0.8, z_0 = 0.5, \alpha = 2, \beta = 1.5$$

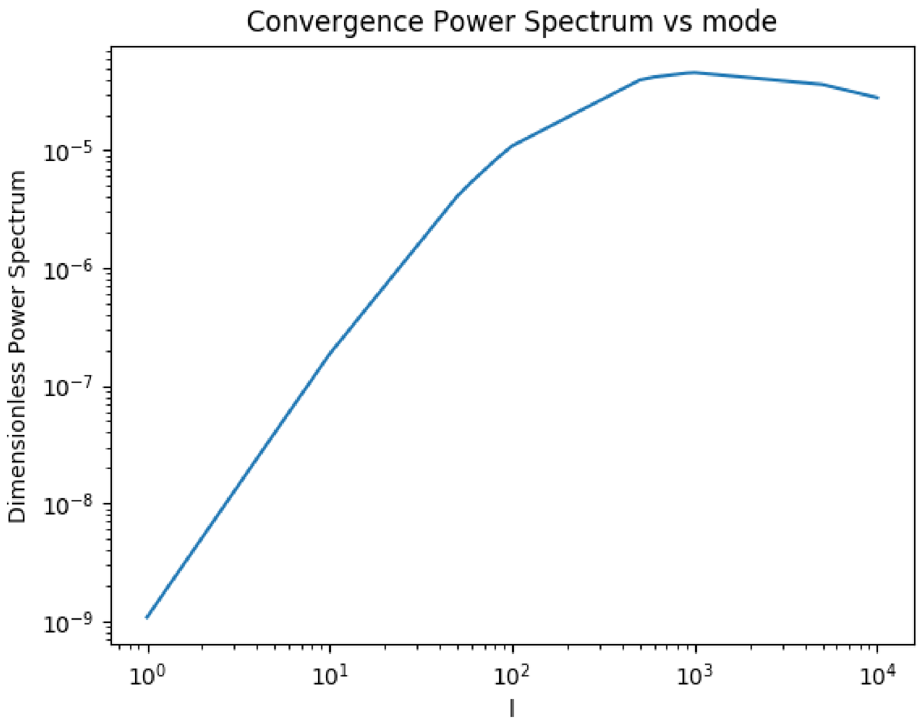

Figure 13: Parameters used are:

$$\Omega_m = 0.5, = 0.7, \Omega_b = 0.044, n_s = 1, \sigma_8 = 0.8, z_0 = 0.5, \alpha = 2, \beta = 1$$

16 Fisher Matrix and its application[21]

16.1 Fisher Matrix for estimating Model Parameters

In this section we'll see how we can calculate the Fisher Matrix without even carrying out the experiment. We'll also mention the Cramer-Rao inequality which helps in putting a minimum bound on the marginal error in our estimate for Model Parameters.
Suppose our data set consists of N numbers, $x_1, x_2, ..., x_N$ which can represent any measurable quantity at N points in a survey. These we arrange in an

N-dimensional vector x. We think of x as a random variable with some probability distribution $L(x, \theta)$, where θ is an M-dimensional vector representing M parameters which have to estimate. These parameters can be anything, say the Hubble constant or mass of a particular star. From now on, θ_0 will be the true values and θ would be the estimated values.
Now the challenge here is to find the best unbiased estimator for θ_α where α is the index of the parameter. The criterion used here is that it should minimize the standard deviation(or variance).
Now we use the Fisher Information Matrix defined as follows:

$$F_{\alpha\beta} = \left\langle \frac{\partial^2 L}{\partial\theta_\alpha \partial\theta_\beta} \right\rangle \tag{66}$$

Here $L = -\ln L$
Another key quantity is the Maximum Likelihood Estimator which is the particular value of θ which maximizes the probability distribution function L and is denoted by θ_{LM}.
Now we state the Cramer-Rao inequality and theorems associated with it:

1. For any unbiased estimator, $\nabla^2\theta_\alpha \geq (F^{-1})^{1/2}_{\alpha\alpha}$

2. If the bound attains the Cramer-Rao bound, it is the ML estimator.

3. The ML estimator is asymptotically the Best Unbiased Estimator.

Now for the last part, we'll calculate the Fisher Matrix for a Gaussian distribution and see how these theorems help us in estimating the parameters. μ and C are the mean vector and the covariance matrix respectively. We also define a data matrix:

$$D = (x - \mu)(x - \mu)^T \tag{67}$$

Using the equation for L and L and neglecting the constant term, we get:

$$L = \ln\det C + (x - \mu)C^{-1}(x - \mu)^T \tag{68}$$

We use the comma notation for the derivatives:

$$C_{,\alpha} = \frac{\partial}{\partial\theta_\alpha} C \tag{69}$$

Using Matrix identities and the the fact that C is a symmetric matrix, we can resolve the Fisher Matrix to:

$$F_{\alpha\beta} = \langle L_{,\alpha\beta} \rangle = \frac{1}{2}\left[C^{-1}C_{,\alpha}C^{-1}C_{,\beta} + C^{-1}M_{\alpha\beta}\right] \tag{70}$$

where $M_{\alpha\beta} = \mu_{,\alpha}\mu_{,\beta}^T + \mu_{,\beta}\mu_{,\alpha}^T$
We can see that the Fisher matrix just depends on the mean and the variance dependence on the parameters. We don't need any data to calculate it if we just know how mean and the covariance matrix varies with the parameters. The Fisher Matrix can give us the minimum bound on the marginal error and then we can design an experiment which will give us the best error on the parameter of interest.

17 Markov Chain

A sequence of random elements X_i of a set is called Markov Chain if the conditional Probability of X_{n+1} given X_1 to X_n depends only on X_n. A chain has stationary transition probabilities if the conditional distribution of X_{n+1} given X_n is independent of n. Markov Chain Monte Carlo (MCMC) methods are dependent on this type of chains.[19]

18 Markov Chain Monte Carlo

MCMC methods are used to solve the sampling problem that is they can approximate the posterior distribution of a parameter of interest by random sampling in a probabilistic space.[20] For Cosmology, the posterior distribution or the target probability density is the likelihoods. The resulting Markov Chain of points samples the posterior, such that the density of points is proportional to the target density (at least asymptotically), so we can estimate all the usual quantities of interest from it (mean, variance, etc). The number of points required to get good estimates is said to scale linearly with the number of parameters, so very quickly becomes much faster than as the number of parameters increases. In cosmology, we are often dealing with around 10-20 parameters, so MCMC has been found to be a very effective tool.[22]

19 References

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23. Andrew Liddle - An Introduction to Cosmology

19.1 Image Sources

Figure 1: Na Kilo Hoku, Newsletter from institute of Astronomy, University of Hawaii
Figure 2: Sir arthur eddington
Figure 3: Website of Professor Catherine Heymans, institute of Astronomy, University of Edinburg

20 Website

The link to our website is
https://astrobuzzph426.wordpress.com/ .
All the assignments will be uploaded on this site.