Worst-Case Analyses

I. KNOWN RESULTS

A. Worst-Case Analyses (Upper Bounds)

The literature concerning worst-case point sets for *minimum Steiner trees* started with Chung and Graham [1]. They consider the following question: what is the greatest length $s(n)$ (resp. $s^*(n)$) a minimum Steiner tree (resp. *rectilinear Steiner minimum tree*) for a set of n points contained in a unit square can have? Equations (1) and (2) are theorems proved in [1]. Equation (3) is conjectured in [1] but not proved.

$$s(n) < 0.995n^{1/2}. \tag{1}$$

$$s^*(n) \leq n^{1/2} + 1 + o(1). \tag{2}$$

$$s^*(n) \leq n^{1/2} + 1. \tag{3}$$

Snyder [2] used the above results to generalize for d dimensions. He proved that there exists a positive constant β_d depending on the dimension $d \geq 2$ such that as $n \to \infty$,

$$s^*(n) \sim \beta_d \cdot n^{(d-1)/d}. \tag{4}$$

Equation (5) gives us upper and lower bounds for the rectilinear constant β_d. For all $d \geq 1$ [2],

$$1 \leq \beta_d \leq d \cdot 4^{(1-d)/d}. \tag{5}$$

[3] deals with worst-case arrangements of points for problems in *combinatorial optimization*; it is shown that these points are *equidistributed*. It proves that if $\{S_L^{(n)} : 1 \leq n < \infty\}$ is a sequence of worst-case point sets for the function L, where L is the minimum spanning tree, the minimum

matching, or the rectilinear minimum Steiner tree, then, for any rectangle $R \subset [0,1]^2$,

$$\lim_{x \to \infty} \frac{1}{n} |S_L^{(n)} \cap R| = Area(R). \tag{6}$$

B. Zero- and Bounded-Skew Tree

Zhu and Dai proved that the asymptotic bound for total wire length of the clock tree with n sinks which are independent and uniformly distributed, in probability, in $[0,1]^2$, is $O(\sqrt{n})$ [4].

For the ZST problem in general metric spaces Charikar et al. [6], give an approximation algorithm with a performance guarantee of $2e$. They then give a constant-factor approximation algorithm for the bounded skew clock routing problem in general metric spaces. For the planar ZST problem (L_1 metric), they give an $(8/ln2)$-approximation algorithm and a constant factor approximation algorithm for the planar embeddable bounded skew clock routing problem.

Improving on and using results from [6] Zelikovsky and Mandoiu obtain algorithms with improved approximation factors of 4 and 14, using a new approach based on zero-skew "stretching" of spanning trees [7]. For the planar problem (L_1 metric) their algorithms find zero- and bounded-skew trees of length at most 3 and 9 times the optimum.

II. My Work

The question that arises is whether ZSTs are in any way "different" from other functionals of geometric point set. This work was undertaken with this question in mind.

A. Equidistributed sinks

[3] showed that worst-case arrangements of points for problems in combinatorial optimization (viz. TSP, minimum Steiner tree, RSMT etc.) are equidistributed. This leads to the study of cost of *minimum* ZST[1] for equidistributed sinks (point sets).

[1] Henceforth, minimum ZST (minimum ZST cost) will refer to the ZST (cost of ZST) for a fixed sink distribution, which has minimum cost over all *binary* topologies.

A.1 1-D

I ran a program over all possible binary tree topologies to check which topology gave minimum ZST for $n = 2^k$ sinks equally distributed for k=1,2,3 in $[0, 1]$. In all the cases the symmetric topology gave the minimum ZST cost. **Can we generalize this for all n??**

Therefore, calculating recursively the expression for the cost of minimum ZST ($L(k)$) with $n = 2^k$ equidistributed sinks in $[0, 1]$ is,

$$L(k) = \frac{2^k - 2}{2^k - 1} * L(k-1) + \frac{2^{k-1}}{2^k - 1}, \tag{7}$$

with $L(1) = 1$ (2 sinks at 0 and 1). $L(k)$ is plotted vs. k and n in Figure 1. $L(k)$ vs. k is linear for large k. Therefore, the asymptotic bound for cost of minimum ZST for $n = 2^k$ equidistributed sinks in $[0, 1]$ is $O(log(n))$.

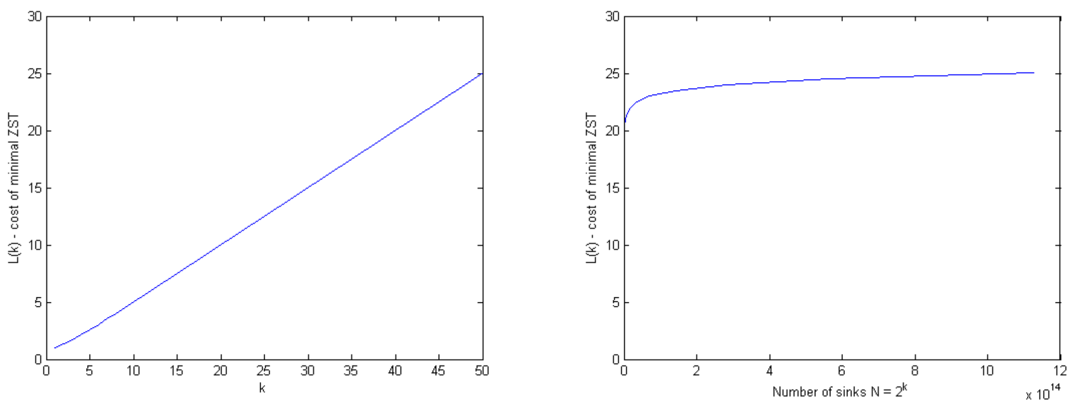

Fig. 1. Cost of ZST for equidistributed sinks in 1-D (a) vs. k (b) vs. n

A.2 2-D

In this case, $n = 4^k$ sinks are equally distributed in $[0, 1]^2$. The symmetric H-tree topology will give minimum ZST cost. The expression for $L(k)$ is (with $L(1) = 3$),

$$L(k) = \frac{2^k - 2}{2^k - 1} * L(k-1) + \mathbf{3}\left(\frac{\mathbf{4^{k-1}}}{\mathbf{2^k - 1}}\right). \tag{8}$$

The asymptotic bound for cost of minimum ZST for $n = 4^k$ equidistributed sinks in $[0, 1]^2$ is $O(\sqrt{n})$.

A.3 Manhattan arc

$n = 2^k$ sinks are equally distributed on a single Manhattan arc. The end points of the arc are at $(0, a)$ and $(a, 0)$. The root of the ZST is at the origin. The minimum ZST cost in this case is given by,

$$L(k) = 2a + \frac{2^k}{2^k - 1} \cdot \left[k - 2 + \left(\frac{1}{2}\right)^{k-1} \right] \cdot a. \tag{9}$$

Since $L(k)$ vs. k is linear for large k, the asymptotic bound for cost of minimum ZST for $n = 2^k$ equidistributed sinks on Manhattan arc with endpoints $(0, 1)$ and $(1, 0)$ is also $O(log(n))$. Note that the ZST in this case is also a *Rectilinear Steiner Aborescence* (RSA) tree. This fact will be used in Section B.2.

A.4 3-D

In this case, $n = 8^k$ sinks are equally distributed in $[0, 1]^3$. The planar H tree can be generalized to the three-dimensional structure via adding line segments on the direction perpendicular to the H tree plane [8]. The expression for $L(k)$ is (with $L(1) = 7$),

$$L(k) = \frac{2^k - 2}{2^k - 1} * L(k-1) + \mathbf{7}\left(\frac{\mathbf{8^{k-1}}}{\mathbf{2^k - 1}}\right). \tag{10}$$

The asymptotic bound for cost of minimum ZST for $n = 8^k$ equidistributed sinks in $[0, 1]^3$ is $O(n^{2/3})$.

B. Worst-case analysis

B.1 1-D

I designed a linear program to compute the worst-case sink configuration for n sinks in [0,1]. An example input file to CPLEX is shown in Figure 2. The results obtained are displayed in Table

I.

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Problem name: LP

Maximize
M  \ Auxiliary variable (indicating worst case ZST cost)

Subject to
\ M is upper bounded by the costs due to all possible topologies
\ This will ensure optimal ZST cost
M +0x0 +1x1 -1x2 +1x0 -0.5x1 -0.5x2<=0
M +1x0 -1x1 +0x2 +0.5x0 +0.5x1 -1x2<=0
\ Constraints on sink locations
x0-x1<=0
x1-x2<=0

Bounds
0<=x0<=1
0<=x1<=1
0<=x2<=1
end
```

Fig. 2. Sample input to CPLEX for $n = 3$.

Observations and notes

- For $n = 6, 7, 12$ the worst case sink locations are not following the pattern followed for all other n.
- The tree topologies which give optimal ZST cost for worst case sink configurations are shown in Figures 3 through 8. Since sink configurations are symmetric (except for $n = 5$) the mirror image of each unbalanced topology is not shown.

B.2 Manhattan arc

For sinks distributed in a diamond region, the worst case sink distribution is when the sinks are distributed around the boundary with the source at the center [4]. Consider a special case of this described in Section A.3. Note that the ZST in this case is also an RSA tree (refer Section A.3). Modifying the proof of Theorem 6 in [9], I prove Theorem 1.

TABLE I
RESULTS OF LINEAR PROGRAM FOR 1-D

n	Cost of Minimum ZST	Sink locations in [0,1]	Pattern
2	1	{0,1}	✓
3	1.25	$\{0, \frac{2}{4}, 1\}$	✓
4	1.4	$\{0, \frac{2}{5}, \frac{3}{5}, 1\}$	✓
5	1.500	$\{0, 0.2, 0.4, 0.6, 1\}$, $\{0, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, 1\}$	✓
6	1.611	$\{0, 0.278, 0.444, 0.556, 0.722, 1\}$	✗
7	1.700	$\{0, 0.25, 0.4, 0.5, 0.6, 0.75, 1\}$	✗
8	$\frac{16}{9}$	$\{0, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}, 1\}$	✓
9	1.85	$\{0, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, 1\}$	✓
10	$\frac{21}{11}$	$\{0, \frac{2}{11}, \frac{3}{11}, \frac{4}{11}, \frac{5}{11}, \frac{6}{11}, \frac{7}{11}, \frac{8}{11}, \frac{9}{11}, 1\}$	✓
11	1.958	$\{0, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}, \frac{7}{12}, \frac{8}{12}, \frac{9}{12}, \frac{10}{12}, 1\}$	✓
12	2.01	$\{0, 0.156, 0.234, , 0.312, \ldots, 0.688, 0.766, 0.844, 1\}$	✗

Theorem 1:

$$L(n) \leq \sqrt{n} + 2, \tag{11}$$

where $L(n)$ is the cost of worst case ZST for n sinks on a single Manhattan arc,

Proof: Construct an RSA A which contains the y-axis, the x axis, t equally spaced horizontal line segments (denoted by H) from the x-axis to the horizontal line $y = \frac{t-1}{t}$ with endpoints on the y axis and the Manhattan arc, and a vertical path from each sink to the line just below it. Now $l(A)$ is clearly bounded by $t/2 + 2 + n/t$. Consider an infinite class C of arborescences obtained from A by sliding the horizontal line segments H upward, but keeping their spacings, until the top one coincides with the hoizontal line $y = 1$. For each arborescence in C we always connect each sink by a vertical path to the line just below it. Consider a random arborescence in C. Since the length of the vertical path from a sink is uniformly distributed over the range $[0, 1/t)$, its expected length over all arborescences in C is $1/2t$. Therefore an arborescence in C has average length $t/2 + 2 + n/2t$. There must thus exist an arborescence $A' \in C$ with at most that length. Minimizing over t, we obtain $t = \sqrt{n/4}$ and the length is $2 + \sqrt{n}$. ■

C. "Charikar-like" Lower Bounds

The worst-case growth rates for the "Charikar-like" lower bound (henceforth referred to as just lower bound in this section) is analyzed. Consider n points distributed in a Manhattan cicle of radius $1/2$. The following analysis is carried out after rotating the Manhattan plane by 45 degrees and stretching by $\sqrt{2}$ (i.e. in the "Max. norm"/L_∞ plane). Therefore, our region of interest is the square $[0, 1]^2$, Manhattan balls become squares, etc.

1. $2m^2$ points can force m^2 squares of edge $l = 1/m$ (radius $r = 1/2m$) to cover all the points (packed cover). (This is a conservative estimate. If we can force a packed cover by fewer points our lower bound estimate will increase.)
2. The lower bound (LB) for n points is lower bounded by the shaded area in Figure 9. This is represented by Equation (12).

$$
\begin{aligned}
LB &= \int_0^{1/2} N(r)\,\mathrm{d}r. \text{ (from [6])} \\
&\geq \sum_{m=1}^{\sqrt{n/2}} \frac{1}{2m} \cdot (m^2 - (m-1)^2). \qquad (12) \\
&= \sum_{m=1}^{\sqrt{n/2}} 1 - \frac{1}{2m}. \\
&= \sqrt{\frac{n}{2}} - \sum_{m=1}^{\sqrt{n/2}} \frac{1}{2m}. \\
&\geq \sqrt{\frac{n}{2}} - \frac{1}{2} \cdot log(\sqrt{n/2}) - \frac{1}{2}. \\
&\in \Theta(\sqrt{n}). \qquad (13)
\end{aligned}
$$

Therefore, the "Charikar-like" lower bound is atleast $\Theta(\sqrt{n})$ (Equation (13)).

REFERENCES

[1] F.R.K. Chung and R.L. Graham, “On Steiner Trees for Bounded Point Sets”, *Geometriae Dedicata* 11, 1981, pp. 353-361.

[2] T.L. Snyder, “Worst-Case Minimum Rectilinear Steiner Trees in All Dimensions”, *Discrete and Computational Geometry* 8, 1992, pp. 73-92.

[3] T.L. Snyder and J.M. Steele, “Equidistribution of Point Sets for the Traveling Salesman and Related Problems”, *Proc. Fourth Annual ACM-SIAM Symposium on Discrete Algorithms*, 1993, pp. 462-466.

[4] Q. Zhu and W. W.-M. Dai, “Perfect-balance planar clock routing with minimal path-length”, *Technical Report, Computer Engineering Board of Studies, U.C. Santa Cruz*, UCSC-CRL-92-12, April 1992.

[5] K. Boese and A. Kahng, “Zero-skew clock routing trees with minimum wirelength”, *Proc. IEEE International ASIC Conf.*, 1992, pp. 17-21.

[6] M. Charikar, J. Kleinberg, R. Kumar, S. Rajagopalan, A. Sahai, and A. Tomkins, “Minimizing wirelength in zero and bounded skew clock trees”, *Proc. 10th ACM-SIAM Symposium on Discrete Algorithms*, 1999, pp. 177-184.

[7] A.Z. Zelikovsky and I.I. Mandoiu, “Practical approximation algorithms for zero- and bounded-skew trees”, *Proc. 12th ACM-SIAM Annual Symposium on Discrete Algorithms*, 2001, pp. 407-416.

[8] B. Hou, H. Xie, W. Wen, and P. Sheng, “Three-dimensional metallic fractals and their photonic crystal characteristics”, *Phys. Rev. B 77*, 2008, 125113.

[9] S.K. Rao, P. Sadayappan, F.K. Hwang, and P.W. Shor, “Rectilinear Steiner Arborescence Problem”, *Algorithmica* 7 (1992), 277-288.

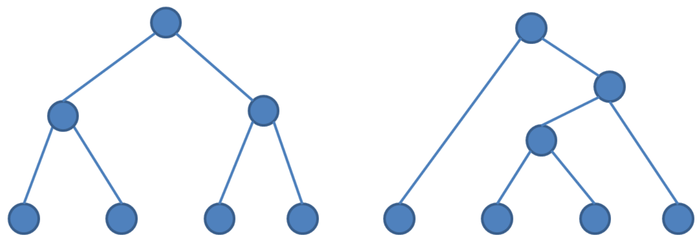

Fig. 3. Topologies for $n = 4$

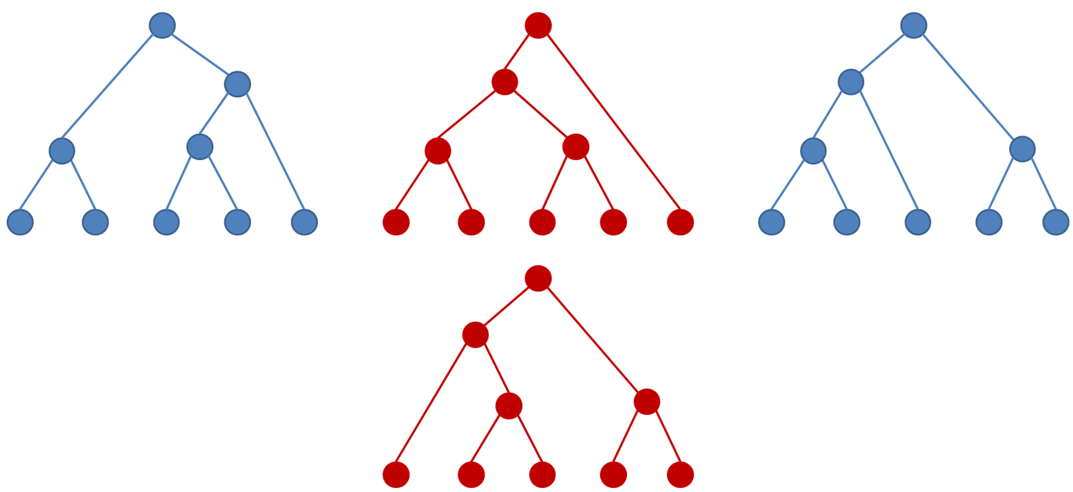

Fig. 4. Topologies for $n = 5$

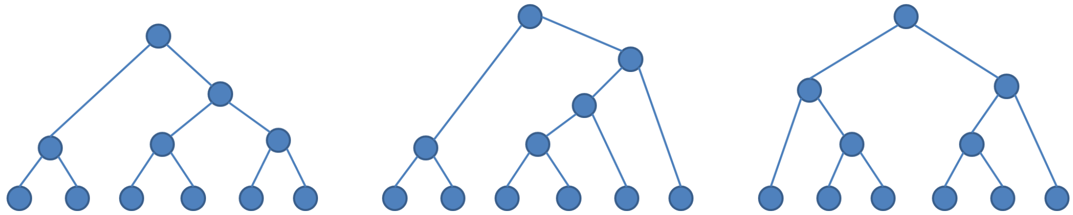

Fig. 5. Topologies for $n = 6$

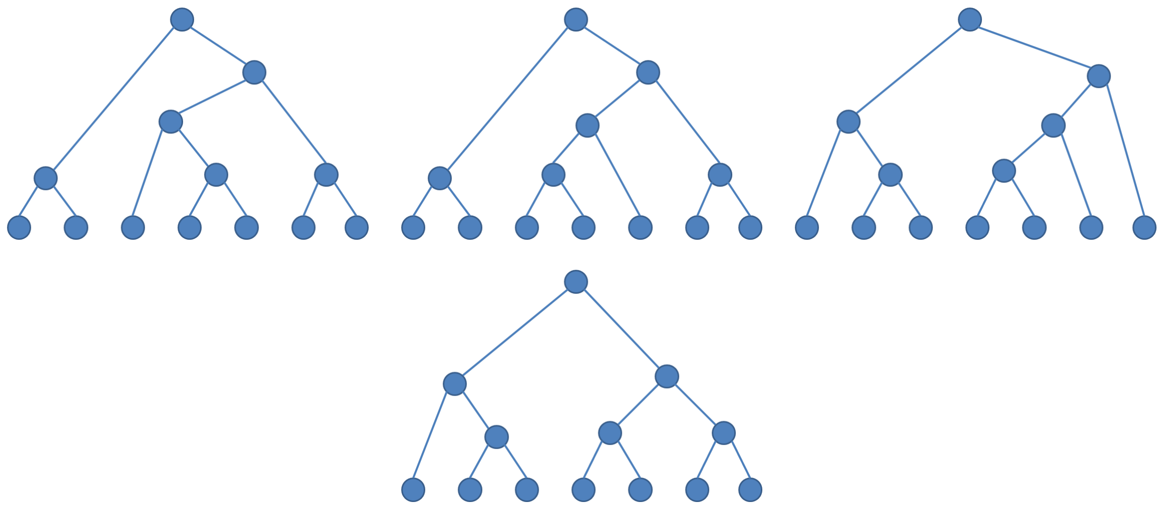

Fig. 6. Topologies for $n = 7$

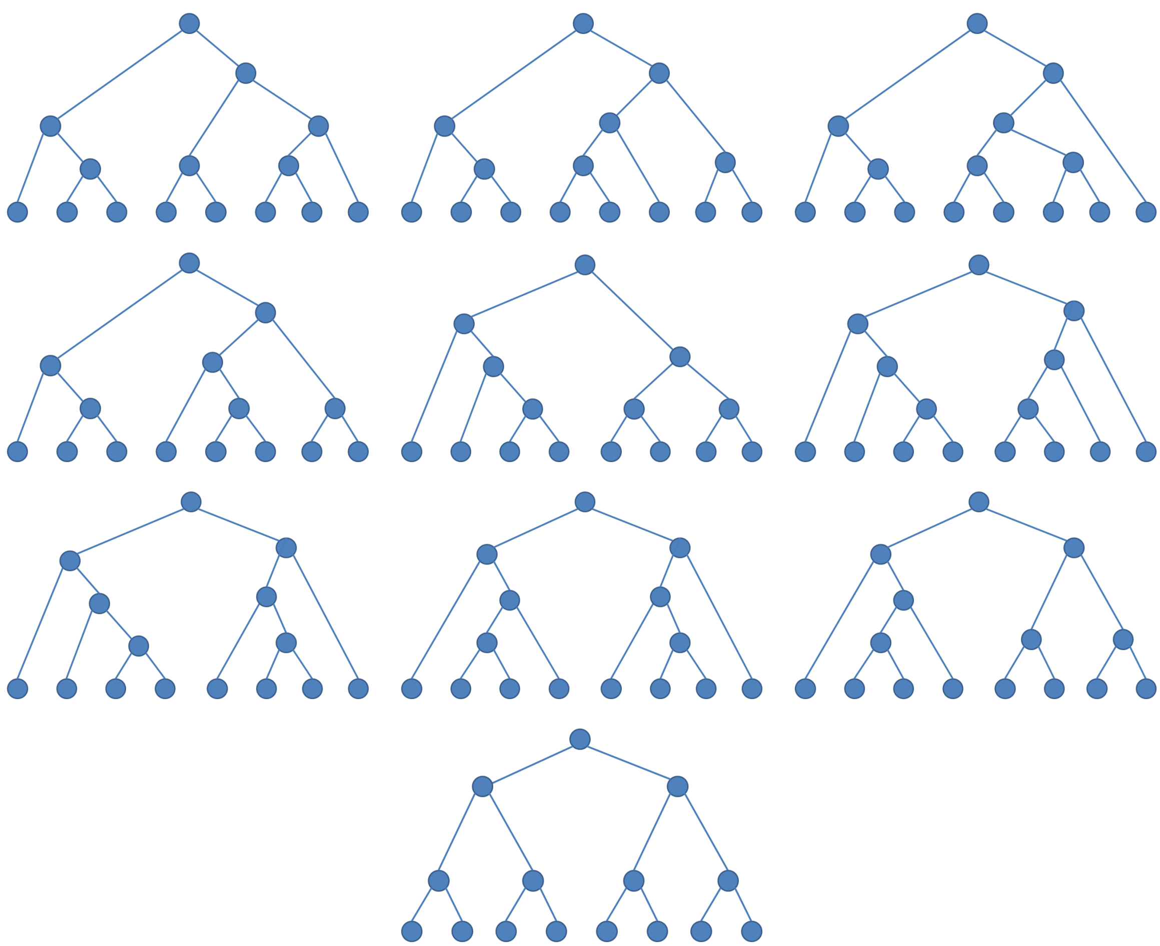

Fig. 7. Topologies for $n = 8$

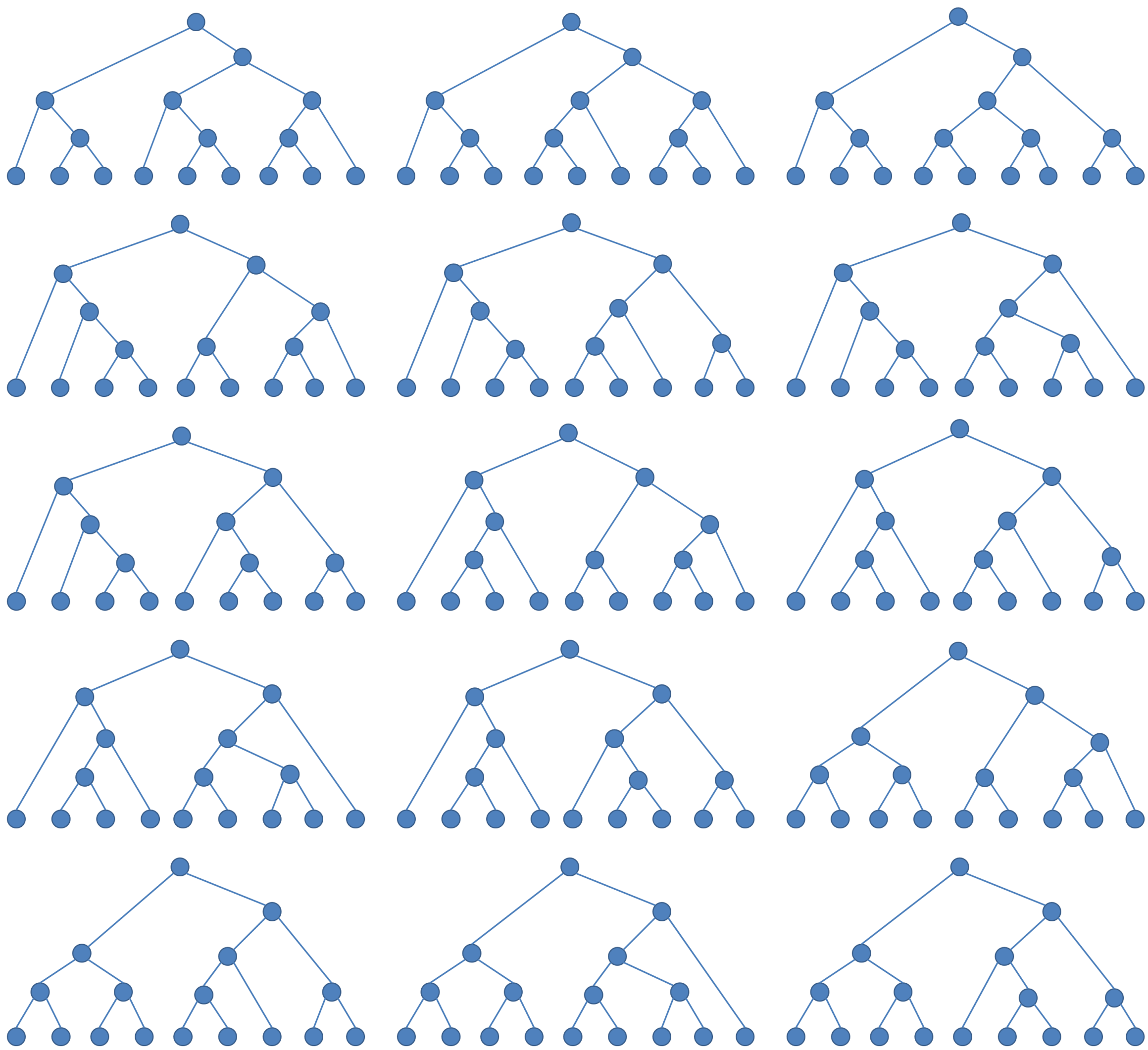

Fig. 8. Topologies for $n = 9$

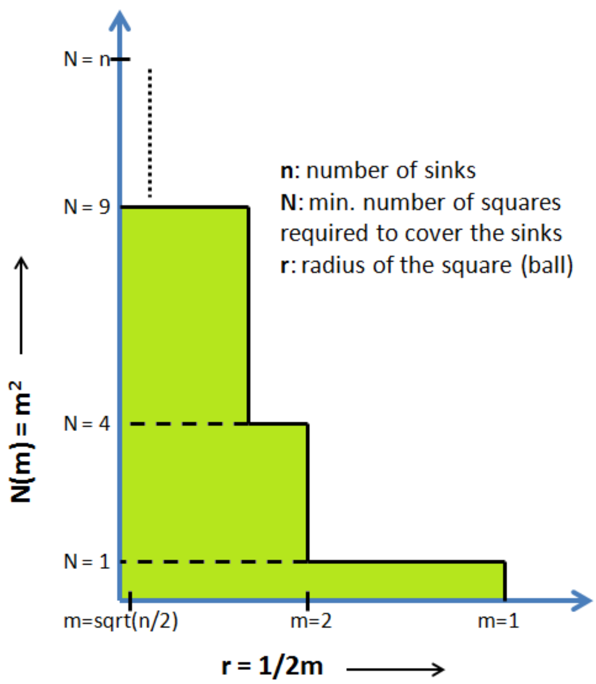

Fig. 9. Lower bound for n sinks in Manhattan circle of radius 1/2.