

CSE 21
Practice Final Exam
Summer Academy 2016

1. **Sorting and Searching.** Give the number of comparisons that will be performed by each sorting algorithm if the input list of length n happens to be of the form $n, 1, 2, \dots, n-3, n-2, n-1$ (i.e., sorted except the largest element is first). On the real exam, you would be given pseudocode for the algorithms, though it is a very good idea to be comfortable with how the algorithms work to save time on the exam. For now, you can refer to the textbook for pseudocode.

- (a) MinSort (SelectionSort)
- (b) BubbleSort
- (c) InsertionSort

2. **Order notation.**

For each of the following pairs of functions f and g , is $f \in O(g)$? Is $f \in \Omega(g)$? Is $f \in \Theta(g)$?

- (a) $f(n) = n, g(n) = 2^{\lfloor \log n \rfloor}$
- (b) $f(n) = \log(n^4), g(n) = (\log n)^4$.
- (c) $f(n) = n^2, g(n) = 1000n^2 + 2^{100}$
- (d) $f(n) = 2^{2n}, g(n) = 2^n$.
- (e) $f(n) = n!, g(n) = n^n$.

3. **Iterative algorithms.**

In the following problem, a polynomial $p(x) = a_0 + a_1x^1 + a_2x^2 + \dots + a_{n-1}x^{n-1}$ of degree $n-1$ is specified by a list of its n coefficients $A = a_0, \dots, a_{n-1}$. We want to evaluate $p(v)$ for some real number v . Here is an iterative algorithm to do so:

Evaluate(A : list of n coefficients, v : a real number)

- 1. $y := 0$
- 2. **for** $i := 1$ to n
- 3. $y := y * v + a_{n-i}$
- 4. **return** y

- (a) Prove the following loop invariant for this algorithm: After i times through the **for** loop,

$$y = a_{n-i} + a_{n-i+1}v + a_{n-i+2}v^2 + \dots + a_{n-1}v^{i-1}.$$

- (b) Conclude from the loop invariant that the algorithm is correct.
- (c) Describe the running time of this algorithm in Θ notation, assuming that arithmetic operations take constant time. Justify your answer.

4. Recursive algorithms.

In the following problem, a polynomial $p(x) = a_0 + a_1x^1 + a_2x^2 + \cdots + a_{n-1}x^{n-1}$ of degree $n - 1$ is specified by a list of its n coefficients $A = a_0, \dots, a_{n-1}$. We want to evaluate $p(v)$ for some real number v . Here is a recursive algorithm to do so:

RecEvaluate(A : list of n coefficients, v : a real number)

1. **if** $n = 1$ **then**
2. **return** a_0
3. $B := a_1, a_2, \dots, a_{n-1}$
4. $y := \text{RecEvaluate}(B, v)$
5. $y := y * v + a_0$
6. **return** y

- (a) Prove by induction on n that $\text{RecEvaluate}(A, v)$ returns $p(v)$.
- (b) Give a recurrence for the time this algorithm takes on an input of size n , assuming that arithmetic operations take constant time. Explain your reasoning.
- (c) Solve this recurrence to determine the running time of this algorithm in Θ notation.

5. Best and worst case.

- (a) Describe, in English and in pseudocode, an algorithm to determine whether a bitstring of length n contains the substring 101. For example, the string 1101 contains the substring 101 but the string 1001 does not.
- (b) Which bitstrings of length n would be best-case inputs to your algorithm? Describe the best-case time of your algorithm in Θ notation.
- (c) Which bitstrings of length n would be worst-case inputs to your algorithm? Describe the worst-case time of your algorithm in Θ notation.

6. Representing problems as graphs.

You have a system of n variables representing real numbers, X_1, \dots, X_n . You are given a list of inequalities of the form $X_i < X_j$ for some pairs i and j . You want to know whether you can deduce with certainty from the given information that $X_1 < X_n$.

For example, say n is 4. A possible input is the list of inequalities $X_1 < X_2, X_1 < X_3, X_4 < X_3$ and $X_3 < X_2$. Does it follow that $X_1 < X_4$?

- (a) Give a description of a directed graph that would help solve this problem. Be sure to define both the vertices and edges in terms of the variables and known inequalities.
- (b) Draw the graph you described for the example above. Does $X_1 < X_4$ follow? Why or why not?
- (c) Say which algorithm from lecture we could use on such a graph to determine whether $X_1 < X_n$ follows from the known inequalities.

7. DAGs and trees.

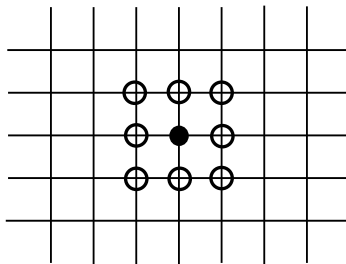
For each statement, either prove that it is true or give a counterexample to show that it is false.

- (a) Every tree on n nodes has exactly $n - 1$ edges.
- (b) Every graph with exactly $n - 1$ edges is a tree.

8. **Counting.**

- (a) How many 5-card hands can be formed from an ordinary deck of 52 cards if exactly two suits are present in the hand?
- (b) In any bit string, the longest consecutive run length is the maximum number of consecutive 1's or consecutive 0's in the string. For example, in the string 1101000111, the longest consecutive run length is 3. How many bit strings of length 10 have a longest consecutive run length of 6?
- (c) A software company assigns its summer interns to one of three divisions: design, implementation, and testing. In how many ways can a group of ten interns be assigned to these divisions if each division needs at least one intern?

9. **Encoding and decoding.** Suppose you are standing on the coordinate plane at the origin $(0,0)$. You will take a walk for n steps, each step moving one unit horizontally, vertically, or diagonally from your current position. The figure uses open circles to show where your next step can be, if your current position is on the shaded circle. You are allowed to revisit points on your walk.



- (a) How many walks of n steps are possible?
- (b) How many bits (0's and 1's) are required to represent a walk of n steps? Simplify your answer.
- (c) Describe how to encode a walk of n steps using the number of bits you gave in part (b). Illustrate your description with an example.

10. **Probability.**

- (a) I have 10 shirts, 6 pairs of pants, and 3 jackets. Every day I dress at random, picking one of each category. What is the probability that today I am wearing at least one garment I was wearing yesterday?
- (b) A permutation of size n is a rearrangement of the numbers $\{1, 2, \dots, n\}$ in any order. A *rise* in a permutation occurs when a larger number immediately follows a smaller one. For example, if $n = 5$, the permutation 1 3 2 4 5 has three rises. What is the expected number of rises in a permutation of size n ?
- (c) There are n teams in a sports league. Over the course of a season, each team plays every other team exactly once. If the outcome of each game is a fair random coin flip, and there are no ties, what is the probability that some team wins all of its games?
- (d) Suppose that one person in 10,000 people has a rare genetic disease. There is an excellent test for the disease; 99.9% of people with the disease test positive and only 0.02% who do not have the disease test positive. What is the probability that someone who tests positive has the genetic disease? What is the probability that someone who tests negative does not have the disease?