CSE 150. Assignment 2 — Solutions *(30 pts total)* *Summer 2016*

1.1 Creative writing (*3 pts — 1pt each*)

Of course there are countless solutions to this problem. Here are some possible examples.

(a) Explaining away:

$X =$ You overslept this morning.
$Z =$ Your bus did not show up.
$Y =$ You are late for school.

$$\begin{aligned} P(X\!=\!1|Y\!=\!1) &> P(X\!=\!1), \\ P(X\!=\!1|Y\!=\!1, Z\!=\!1) &< P(X\!=\!1|Y\!=\!1) \end{aligned}$$

(b) Accumulating evidence:

$X =$ You have a cough.
$Y =$ You are a heavy smoker.
$Z =$ You have a cold.

$$P(X\!=\!1) < P(X\!=\!1|Y\!=\!1) < P(X\!=\!1|Y\!=\!1, Z\!=\!1)$$

(c) Conditional independence:

$X =$ I am carrying an umbrella.
$Y =$ You are carrying an umbrella.
$Z =$ It is raining. (So everyone carries an umbrella.)

$$\begin{aligned} P(X\!=\!1, Y\!=\!1) &\neq P(X\!=\!1)P(Y\!=\!1) \\ P(X\!=\!1, Y\!=\!1|Z\!=\!1) &= P(X\!=\!1|Z\!=\!1)P(Y\!=\!1|Z\!=\!1) \end{aligned}$$

1.2 Probabilistic inference (*12 pts — 2 pts each*)

On these problems it is permissible to re-use results from lecture. However, the full calculations are shown here for completeness.

(a) $\boxed{P(E=1|A=1) = 0.2310}$

From Bayes rule:

$$P(E=1|A=1) \;=\; \frac{P(A=1|E=1)P(E=1)}{P(A=1)}$$

Denominator:

$$\begin{aligned}
P(A=1) &= \sum_{b,e} P(E=e, B=b, A=1) \quad \boxed{\textbf{marginalization}} \\
&= \sum_{b,e} P(E=e)\,P(B=b|E=e)\,P(A=1|B=b,E=e) \quad \boxed{\textbf{product rule}} \\
&= \sum_{b,e} P(E=e)\,P(B=b)\,P(A=1|B=b,E=e) \quad \boxed{\textbf{marginal independence}} \\
&= P(E=0)\,P(B=0)\,P(A=1|E=0,B=0) \;+ \\
&\quad\; P(E=0)\,P(B=1)\,P(A=1|E=0,B=1) \;+ \\
&\quad\; P(E=1)\,P(B=0)\,P(A=1|E=1,B=0) \;+ \\
&\quad\; P(E=1)\,P(B=1)\,P(A=1|E=1,B=1) \\
&= (1-0.002)*((1-0.001)*0.001 + 0.001*0.94) \;+ \\
&\quad\; 0.002*((1-0.001)*0.29 + 0.001*0.95) \\
&= \mathbf{0.002516442}
\end{aligned}$$

First term in numerator:

$$\begin{aligned}
P(A=1|E=1) &= \sum_{b} P(A=1, B=b|E=1) \quad \boxed{\textbf{marginalization}} \\
&= \sum_{b} P(A=1|B=b,E=1)\,P(B=b|E=1) \quad \boxed{\textbf{product rule}} \\
&= \sum_{b} P(A=1|B=b,E=1)\,P(B=b) \quad \boxed{\textbf{marginal independence}} \\
&= P(A=1|E=1,B=1)\,P(B=1) \;+\; P(A=1|E=1,B=0)\,P(B=0) \\
&= 0.95*0.001 + 0.29*(1-0.001) \\
&= \mathbf{0.29066}
\end{aligned}$$

Substituting the results in bold:

$$P(E=1|A=1) \;=\; \frac{0.29066 * 0.002}{0.002516442} \;=\; \mathbf{0.2310}$$

(b) $\boxed{P(E=1|A=1,B=0) = 0.3625}$

From Bayes rule:

$$\begin{aligned}
P(E=1|A=1,B=0) \;&=\; \frac{P(A=1|E=1,B=0)P(E=1|B=0)}{P(A=1|B=0)} \\
&=\; \frac{P(A=1|E=1,B=0)P(E=1)}{P(A=1|B=0)} \quad \boxed{\textbf{marginal independence}}
\end{aligned}$$

Denominator:

$$\begin{aligned}
P(A=1|B=0) \;&=\; \sum_e P(A=1,E=e|B=0) \quad \boxed{\textbf{marginalization}} \\
&=\; \sum_e P(A=1|E=e,B=0)\,P(E=e|B=0) \quad \boxed{\textbf{product rule}} \\
&=\; \sum_e P(A=1|E=e,B=0)\,P(E=e) \quad \boxed{\textbf{marginal independence}} \\
&=\; P(A=1|E=1,B=0)P(E=1) + P(A=1|E=0,B=0)P(E=0) \\
&=\; 0.29 * 0.002 + 0.001 * (1 - 0.002) \\
&=\; \mathbf{0.0016}
\end{aligned}$$

Substituting the result in bold:

$$P(E=1|A=1,B=0) \;=\; \frac{0.29 * 0.002}{0.0016} \;=\; \mathbf{0.3625}$$

Comparing (a) and (b), we find that $P(E{=}1|A{=}1,B{=}0) > P(E{=}1|A{=}1)$.

This agrees with common sense: if we know that a burglar did not trip the alarm, then we are more likely to believe than an earthquake was responsible.

(c) $\boxed{P(A=1|M=1)=0.1501}$

From Bayes rule:

$$P(A=1|M=1) \;=\; \frac{P(M=1|A=1)P(A=1)}{P(M=1)}$$

Denominator:

$$\begin{aligned}
P(M=1) &= \sum_a P(M=1, A=a) \quad \boxed{\textbf{marginalization}} \\
&= \sum_a P(M=1|A=a)\, P(A=a) \quad \boxed{\textbf{product rule}} \\
&= P(M{=}1|A{=}0)\, P(A{=}0) + P(M{=}1|A{=}1)\, P(A{=}1) \\
&= 0.01 * (1 - 0.002516442) + 0.70 * 0.002516442 \quad \boxed{\textbf{using result from part (a)}} \\
&= \mathbf{0.01173634498}
\end{aligned}$$

Substituting into Bayes rule:

$$P(A=1|M=1) \;=\; \frac{0.7 * 0.002516442}{0.01173634498} \;=\; \mathbf{0.1501}$$

(d) $\boxed{P(A=1|M=1, J=0)=0.0182}$

From Bayes rule:

$$\begin{aligned}
P(A=1|M=1, J=0) &= \frac{P(M=1, J=0|A=1)\, P(A=1)}{P(M=1, J=0)} \\
&= \frac{P(M=1|A=1)\, P(J=0|A=1)P(A=1)}{P(M=1, J=0)} \quad \boxed{\textbf{conditional independence}}
\end{aligned}$$

Denominator:

$$\begin{aligned}
P(M=1, J=0) &= \sum_a P(M=1, J=0, A=a) \quad \boxed{\textbf{marginalization}} \\
&= \sum_a P(M=1, J=0|A=a)P(A=a) \quad \boxed{\textbf{product rule}} \\
&= \sum_a P(M=1|A=a)P(J=0|A=a)P(A=a) \quad \boxed{\textbf{conditional independence}} \\
&= P(M{=}1|A{=}0)P(J{=}0|A{=}0)P(A{=}0) \;+\; P(M{=}1|A{=}1)P(J{=}0|A{=}1)P(A{=}1) \\
&= 0.01 * (1 - 0.05) * (1 - 0.002516442) + 0.7 * (1 - 0.9) * 0.002516442 \quad \boxed{\textbf{from part (a)}} \\
&= \mathbf{0.009652244741}
\end{aligned}$$

Substituting into Bayes rule:

$$P(A=1|M=1,J=0) \;=\; \frac{0.7*(1-0.9)*0.002516442}{0.009652244741} \;=\; 0.0182$$

Comparing (c) and (d), we find that $P(A{=}1|M{=}1) > P(A{=}1|M{=}1, J{=}0)$**.**

This agrees with common sense: we're less likely to believe that the alarm has sounded after learning that John has not called.

(e) $\boxed{P(A=1|M=0) = 0.000764}$

From Bayes rule:

$$\begin{aligned}
P(A=1|M=0) &= \frac{P(M=0|A=1)P(A=1)}{P(M=0)} \\
&= \frac{(1-0.7)*0.002516442}{1-0.01173634498} \qquad \boxed{\textbf{using results from parts (a,c)}} \\
&= 0.000764
\end{aligned}$$

(f) $\boxed{P(A=1|M=0,B=1) = 0.826}$

From Bayes rule:

$$\begin{aligned}
P(A=1|M=0,B=1) &= \frac{P(M=0|A=1,B=1)P(A=1|B=1)}{P(M=0|B=1)} \\
&= \frac{P(M=0|A=1)P(A=1|B=1)}{P(M=0|B=1)} \qquad \boxed{\textbf{conditional independence}}
\end{aligned}$$

Second term in numerator:

$$\begin{aligned}
P(A=1|B=1) &= \sum_e P(A=1,E=e|B=1) \qquad \boxed{\textbf{marginalization}} \\
&= \sum_e P(A=1|E=e,B=1)\,P(E=e|B=1) \qquad \boxed{\textbf{product rule}} \\
&= \sum_e P(A=1|E=e,B=1)\,P(E=e) \qquad \boxed{\textbf{marginal independence}} \\
&= P(A=1|E=0,B=1)\,P(E=0) \;+\; P(A=1|E=1,B=1)\,P(E=1) \\
&= 0.94*(1-0.002)+0.95*0.002 \\
&= \mathbf{0.94002}
\end{aligned}$$

Denominator:

$$
\begin{aligned}
P(M=0|B=1) &= \sum_a P(M=0, A=a|B=1) \quad \boxed{\textbf{marginalization}} \\
&= \sum_a P(M=0|A=a, B=1)\, P(A=a|B=1) \quad \boxed{\textbf{product rule}} \\
&= \sum_a P(M=0|A=a)\, P(A=a|B=1) \quad \boxed{\textbf{conditional independence}} \\
&= P(M=0|A=0)\, P(A=0|B=1) \;+\; P(M=0|A=1)\, P(A=1|B=1) \\
&= (1-0.01)*(1-0.94002) \;+\; (1-0.7)*0.94002 \quad \boxed{\textbf{use result from above}} \\
&= \mathbf{0.3413862}
\end{aligned}
$$

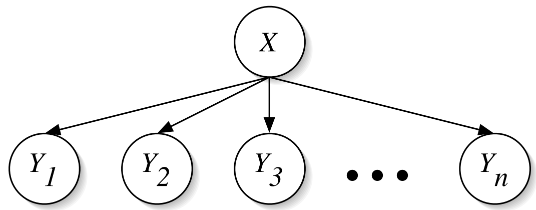
Substituting into Bayes rule:

$$
P(A=1|M=0, B=1) \;=\; \frac{(1-0.7)*0.94002}{0.3413862} \;=\; 0.826
$$

Comparing (e) and (f), we find that $P(A{=}1|M{=}0, B{=}1) \gg P(A{=}1|M{=}0)$**.**

This agrees with common sense: we're much more likely to believe that the alarm has sounded after learning that a burglary has occurred.

1.3 Probabilistic reasoning *(4 pts)*



Conditional probability tables (CPTs):

$$P(X\!=\!1) \;=\; \frac{1}{2}$$

$$P(Y_k\!=\!1|X\!=\!0) \;=\; \frac{1}{2}$$

$$P(Y_k\!=\!1|X\!=\!1) \;=\; \frac{f(k-1)}{f(k)} \quad \textbf{where} \quad f(k) = 2^k + (-1)^k$$

(a) **From Bayes rule:** (1 pt)

$$P(X\!=\!1|Y_1\!=\!1, Y_2\!=\!1, \ldots, Y_k\!=\!1) \;=\; \frac{P(Y_1\!=\!1, Y_2\!=\!1, \ldots, Y_k\!=\!1|X\!=\!1)\; P(X\!=\!1)}{P(Y_1\!=\!1, Y_2\!=\!1, \ldots, Y_k\!=\!1)}$$

$$P(X\!=\!0|Y_1\!=\!1, Y_2\!=\!1, \ldots, Y_k\!=\!1) \;=\; \frac{P(Y_1\!=\!1, Y_2\!=\!1, \ldots, Y_k\!=\!1|X\!=\!0)\; P(X\!=\!0)}{P(Y_1\!=\!1, Y_2\!=\!1, \ldots, Y_k\!=\!1)}$$

Taking the ratio: (1 pt)

$$\begin{aligned} r_k &= \frac{P(X\!=\!1|Y_1\!=\!1, Y_2\!=\!1, \ldots, Y_k\!=\!1)}{P(X\!=\!0|Y_1\!=\!1, Y_2\!=\!1, \ldots, Y_k\!=\!1)} \\ &= \frac{P(Y_1\!=\!1, Y_2\!=\!1, \ldots, Y_k\!=\!1|X\!=\!1)\; P(X\!=\!1)}{P(Y_1\!=\!1, Y_2\!=\!1, \ldots, Y_k\!=\!1|X\!=\!0)\; P(X\!=\!0)} \quad \boxed{\textbf{substituting from above}} \\ &= \frac{P(X\!=\!1)\; \prod_{i=1}^{k} P(Y_i\!=\!1|X\!=\!1)}{P(X\!=\!0)\; \prod_{i=1}^{k} P(Y_i\!=\!1|X\!=\!0)} \quad \boxed{\textbf{conditional independence}} \end{aligned}$$

Substituting the CPTs: (1 pt)

$$
\begin{aligned}
r_k &= \frac{\left(\frac{1}{2}\right)\ (1)\ \prod_{i=2}^{k}\left[\frac{f(i-1)}{f(i)}\right]}{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^k} \\
&= 2^k\left[\frac{f(1)}{f(2)}\cdot\frac{f(2)}{f(3)}\cdots\frac{f(k-1)}{f(k)}\right] \\
&= 2^k\left[\frac{f(1)}{f(k)}\right] \\
&= 2^k\left[\frac{1}{2^k+(-1)^k}\right] \\
&= \frac{1}{1+\left(-\frac{1}{2}\right)^k}
\end{aligned}
$$

(b) **Explanation:** (1 pt)

The ratio r_k is greater than one for odd k and less than one for even k. Thus the diagnosis vacillates: on odd days, the $X=1$ form of the disease seems more likely; on even days, the opposite. In addition, the ratio r_k approaches unity as $k \to \infty$. Thus the diagnosis becomes more uncertain with each day. Note that conditioning on more evidence does not always reduce the amount of uncertainty.

1.4 Hangman *(11 pts)*

(a) **Most and least frequent words** *(1 pt)*

Most common: THREE, SEVEN, EIGHT, WOULD, ABOUT, THEIR, WHICH, FIRST, FIFTY, OTHER, FORTY, YEARS, THERE, SIXTY, STATE.

Least common: BOSAK, CAIXA, MAPCO, OTTIS, TROUP, CCAIR, CLEFT, FABRI, FOAMY, NIAID, PAXON, SERNA, TOCOR, YALOM, BITTY.

Note: there are many words tied for the position of 15th least common.

(b) **Best next guesses and their probabilities** *(5 pts)*

correctly guessed	incorrectly guessed	best next guess ℓ	$P(L_i=\ell \text{ for some } i\in\{1,2,3,4,5\}\|E)$
– – – – –	{}	E	0.5394
– – – – –	{A, O}	E	0.6699
B – – – E	{}	R	0.6229
B – – – E	{R}	A	0.7301
– – H – –	{E, I, M, N, T}	A	0.8753

(c) **Printed source code** *(5 pts)*
