SN

Graph Theory

$n \to$ no. of vertices
$e \to$ no. of edges
$d \to$ degree

1. Self loop provides degree '2'.
2. $\Sigma(\text{degree}) = 2 \times e$
3. No. of odd degree vertices is always even.
4. By default consider graph as simple graph.

5) Max degree in simple graph $= (n-1)$

6) Max no. of edges in simple graph $= n(n-1)/2 = {}^nC_2$

7) Total no. of graphs possible $= 2^{n(n-1)/2}$

8) Graph possible with 'e' no. of edges $= {}^{\frac{n(n-1)}{2}}C_e$

9) ${}^nC_0 + {}^nC_1 + \dots + {}^nC_n = 2^n$

10) $\underset{\text{At least}}{\delta(G)} \le \dfrac{2e}{n} \le \underset{\text{At most}}{\Delta(G)} \le (n-1)$

$\delta \to$ min d
$\Delta \to$ max d

11) From given degree seq, to find no. of edges use theorem $\Sigma d = 2e$ if its graphical seq otherwise do plotting.

12) At least 2 vertex has same degree in a simple graph. $n \ge 02$

13) To check for graphical seq:-
- verify $\max(d) = (n-1)$
- verify no. of odd degree vertices is even
- If $(n-1), (n-1), \dots, 1$ then not graphical
- If all degrees are distinct, " "
- use Havell - Hakimi - keep removing vertex, till its confirm that simple graph can be drawn or not

14) Regular graph : $\delta(G) = 2e/n = \Delta(G)$

15) Complete graph (K_n) : All degrees $= (n-1)$
No of edges $= n(n-1)/2$

16) Cycle graph (C_n) $n \geq 3$: All degree $= 2$ & connected graph
$n = e$ $\qquad C_n \underset{\not\leftarrow}{\rightarrow} n = e$

17) Wheel graph (W_n) $n \geq 4$: $(n-1), 3, 3, \ldots$
$e = 2(n-1)$

18) Bi-partite graph : → can be divided into 2 sets. does not contain odd length cycle ; either contain even length cycle or no cycle.
Max $e = \lfloor n^2/4 \rfloor$ divided into 2 equal halves
↳ exception → 2 isolated vertex is bipartite but
3, 4 " " r not " graph

19) Complete Bipartite graph ($K_{m,n}$) : each vertex of set connected to all vertex of other set
no of vertices $= m+n$; $e = mn$

20) Star graph : ($K_{1,n-1}$) : complete bip. graph with min no. of $e = (n-1)$

21) $G + \bar{G} = K_n$; $e(\bar{G}) = n(n-1)/2 - e(G)$
→ draw edges which r not present in G
Degree seq of $\bar{G} = (n-1-d_1), (n-1-d_2), \ldots$
↳ find only if G is graphical

22) $\overline{K_n}$ = n isolated vertices

23) $K_m \circledast K_n = K_{m+n}$ → each vertex of K_m connected to each vertex of K_n

24) $\overline{K_{1,n-1}}$ = 1 isolated vertex + K_{n-1}

- For a given no. of vertices, self comp. graph is not unique.

→ Hypercube: (Q_n) n is no. of bits
2^n vertices, degree of each vertex = n,
No of edges = $n * 2^{n-1}$

25. $\delta(G) + \delta(\overline{G}) \le (n-1)$
$d(G) + \delta(\overline{G}) = (n-1)$ → In regular graph

26. If degree of each vertex $\ge d$ then, graph contains a cycle of length at least $d+1$.

27. If $e > n^2/4$ then can't be bipartite; contains △

28. Length of longest path in
- $K_{m,n} = 2 * \min(m,n)$
- $Q_n = (2^n - 1)$

29. Hypercube (Q_n): no. of vertex = 2^n
each vertex has degree = n ; $e = n2^{n-1}$

30. Self complementing graph: $G \equiv \overline{G}$
$e = n(n-1)/4$; self comp. graph is possible only if $n \equiv 0$ or $1 \pmod 4$
congruent

31. Cycle graph which is self comp. = C_5

32. Line graph: edge → vertex connect vertices which have something in common
$v_1 \overset{e_1}{—} v_2$ $d(e_1$ in $L(G)) = d(v_1) + d(v_2) - 2$

33.

	repeat V	repeat E
Walk	✓	✓
Trail	✓	X
Path	X	X

34. No. of connected components
K=1 connected K ≥ 2 Disconnected graph

35. - G is disconnected → $\overline{G}$ is connected
- G is connected → $\overline{G}$ may be connected or disconnected.
- At least 1 of G, $\overline{G}$ will be connected.

cut vertex → Articulation point

Ex: $n=7,\ \delta(G)=3$ — K_4 — can't be disconnected

36. If $\delta(G) \geq (n-1)/2$ then graph is connected.

37. Connected graph: $(n-1) \leq e \leq n(n-1)/2$

38. Tree: $e=(n-1)$; minimally connected, no cycle

it does not depend on how vertices r divided into components

39. Disconnected graph: $(n-K) \leq e \leq (n-K)*(n-K+1)/2$

no. of e in forest

40. Disconnected graph has max no. of e when vertices are divided as (1V) (1V) ... (Remaining V complete graph)

Formula gives this (overall) max

If divided randomly then calculate max $e = \sum \frac{n(n-1)}{2}$ of all connected components.

41. If K is not given, max no. of edges is when $K=2$. i.e $(n-2)(n-1)/2$.

- cut edge (Bridge)
- cut set

42. Edge connectivity (Min cut): $\lambda(G)$ removal of min no. of edge will make graph disconnected.

- cut vertex (Articulation point)
- cut vertex set

43. If cut edge exist then cut vertex also exist $(n \geq 3)$, not vice versa.

↳ exception K_2

43. Vertex connectivity: $K(G)$ Removal of min no of vertex will make graph disconnected

44. Q_n: $\lambda = K = n$; Tree: $\lambda = K = 1$

$K_{m,n}$: $\lambda = K = \min(m,n)$

45. Q_n is bipartite; never has odd length cycle

Removal of vertex also removes edges

46. $K(G) \leq \lambda(G) \leq \delta(G) \leq 2e/n \leq \Delta(G) \leq (n-1)$

47. Graph is Euler if and only if degree of all the vertices are even.

- Disconnected graph is not Euler.

→ Euler graph (circuit): closed trail + covers all edges

→ Euler trail/Path/Line: open trail + covers all edges (Semi-Eulerian)

→ Hamilton circuit: closed path + covers all vertices

Single isolated vertex is neither Euler nor Hamilton graph.

48. To check is a graph Euler or not, just check whether all degrees are even or not.

49. Graph contains euler line if and only if graph contains exactly 2 odd vertices. (start & end point)

50. Line graph of every Euler graph is Euler as well as Hamilton.

- Every C_n is Euler graph, K_n is Euler if n is odd, $K_{m,n}$ is Euler if m & n are even, W_n is never Euler.

51. Every C_n, K_n $(n \geq 3)$, W_n $(n \geq 4)$, Cbipartite graph $K_{m,n}$ $(m = n$ & $m, n \geq 2)$ are Hamilton graph.

- Hamiltonian circuit contains Hamiltonian path but not vice versa.
- Traversable graph - graph containing either Euler or Hamiltonian circuit.

→ Covering

- All edges
  - closed trail ↓ Eulerian → Euler ckt
  - open trail ↓ Semi-Eulerian → Euler path/trail
- All vertices
  - closed path ↓ → Hamilton ckt
  - Open path ↓ → Hamilton Path

52. Chromatic no $(\chi(G))$: min no. of colours for proper coloring.
↳ no adjacent vertex has same color

53. If there is triangle, $\chi(G) \geq 3$. (search for $\triangle$ in ques)

54. Every bipartite graph is 2 colorable & vice versa.
Every 2 colorable is either tree or even length cycle which is bipartite.

55. $\chi(K_n) = n$

- No of partitions = Chromatic Number (writing same colored vertices in a set.)

56. Independence no. $(\beta(G))$: No. of vertices present in largest maximal independent set. → set of non adj vertices

$$\alpha(G) \leq \beta(G)$$

NOTE: Tree: $\chi(G) = 2$; isolated vertex $\chi(G) = 1$; C_n $\chi(G) = 2$ if n is even, $= 3$ if n is odd

W_n $\chi(G) = 3$ if n is odd, 4 if n is even

NOTE: Maximal: nothing can be added
Minimal: nothing can be removed.

* Minimally connected graph → Tree

57. Domination no $\gamma(G)$: Min size of minimal dominating set
→ Either person is getting married or his friend
(all vertices either present in dominating set or its adjacent present in set)

58. Every maximal ind. set is minimal dom. set but not vice versa.

59. Matching Number: $M(G)$
Max size of maximal matching set
→ set of not adj edges

60. Empty set is also matching set.

★ 61. $M(C_n) = M(K_n) = M(W_n) = \lfloor n/2 \rfloor$
$M(K_{m,n}) = \min(m, n)$

→ Perfect Maching: Maximal maching set such that it induces degree 1 to all vertex (i.e. sabhi vertex ko hello bole)

62. If perfect matching exists, no. of vertices is even not vice versa.

* 63. No. of perfect matching in $K_{2n} = \dfrac{(2n)!}{2^n n!}$

64. Covering no: size of smallest minimal covering set. → all should get atleast 1 marriage proposal.
→ all vertex should incident on atleast 1 edge in set.

65. Every perfect matching is minimal covering set.

★ 66. Total no. of Hamilton cycles in
$K_n = (n-1)!/2$
$K_{m,m} = m!(m-1)!/2$

67. K_5 is non planar with min no. of vertices
$K_{3,3}$ is " " " " " edges
On removing 1 edge both becomes planar.

NOTE: If graph has $\geq \frac{(n-1)(n-2)}{2}+1$ edges then it's connected.

classmate

Date
Page

68. No. of $\overset{\text{Faces or}}{\text{regions}} = e-n+2$

* 69. Total no. of perfect matching in
$K_{n,n} = n!$ tree = at most 1 $\begin{cases} 0 \text{ if n odd} \\ 1 \text{ if even} \end{cases}$
↳ $K_{m,n}$ has perfect matching iff $m=n$

70. connected graph is Eulerian if and only if its set of edges can be split into disjoint edge cycles.

71. Isomorphism: 2 graphs r isomorphic if there is 1:1 correspondance in vertices, edges and transition function.

72. Planar graph: if graph can be drawn on a plane without intersection of edges.

73. Till $n=5$, $e=9$ all are planar.
K_5 $(n=5, e=10)$ is first non planar graph.

Mathematical Logic

1. Connectives: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$

2. $\neg$ (Negation) is not connective, its modifier.

3. 'AND' hates False, 'OR' loves True

4. $\rightarrow$
   if p then q; if p, q ; q if p ;
   q when p ; q whenever p; q unless $\neg p$

5. $F \rightarrow \_\_\_ = T$; $\_\_\_ \rightarrow T = T$

6. $P \rightarrow Q$
   converse : $Q \rightarrow P$
   Inverse : $\neg P \rightarrow \neg Q$
   contrapositive : $\neg Q \rightarrow \neg P$

7. $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$ (contrapositive)

8. $\leftrightarrow / \equiv$ $\begin{matrix} T & T \\ F & F \end{matrix}$ (iff, if & only if, necessary & sufficient)

*9. GOD: $P \rightarrow Q \equiv \neg P \vee Q$
   $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$

10. Tautology: All True (valid)
    contradiction : All False
    satisfiable : At least 1 true
    contingency : Neither valid nor contradiction

11. No. of non equivalent propositional fu" possible with n variables = 2^{2^n}

p	q	
0	1	T
		F
1	0	

→ Type1 : $(\_\_ \rightarrow \_\_)$

- Try to contradict. If cant be contradicted then its tautology.

- If multiple cases arise, it must be true for all cases.

→ Type2 : Logical equivalent $(\_\_ \equiv \_\_)$

i) $p \wedge T = p$ $\qquad p \vee F = p$

ii) $p \wedge F = F$ $\qquad p \vee T = T$

iii) $p \wedge p = p$ $\qquad p \vee p = p$

iv) $p \wedge q = q \wedge p$ $\qquad p \vee q = q \vee p$

v) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

vi) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$
$p \vee (q \vee r) = (p \vee q) \vee r$

vii) $\neg(p \vee q) = \neg p \wedge \neg q$
$\neg(p \wedge q) = \neg p \vee \neg q$

viii) $p \vee (p \wedge q) = p$ $\qquad p \wedge (p \vee q) = p$

*IMP: Inverse distributive law:
$(p \wedge q) \vee (p \wedge r) = p \wedge (q \vee r)$
Absorption Law:
$p \wedge (p \vee q) = p$ $\qquad p \vee (p \wedge q) = p$
Solve it simply as Boolean expression.

Unsatisfiable → contradiction

NOTE: To prove x is equivalent to y i.e $x \equiv y$, prove
if x is T then y is also T &
if x is F then y is also F

- $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
  $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
  1st var same operator same

- $(p \to q) \wedge (r \to q) \equiv (p \vee r) \to q$
  $(p \to q) \vee (r \to q) \equiv (p \wedge r) \to q$
  2nd var same operator changes

- $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
  $\equiv \neg p \leftrightarrow \neg q$

→ Type 3: inference rule

Put LHS true, if RHS comes to be true, its tautology.

i)
$$\frac{P \to Q \quad P}{Q}$$

(ii)
$$\frac{P \to Q \quad \neg Q}{\neg P}$$

iii)
$$\frac{P \vee Q \quad \neg P}{Q}$$

(iv)
$$\frac{P \to Q \quad Q \to R}{P \to R}$$

v)
$$\frac{P}{P \vee Q}$$

(vi)
$$\frac{P \wedge Q}{P / Q}$$

vii)
$$\frac{P \quad Q}{P \wedge Q}$$

(viii)
$$\frac{P \vee Q \quad \neg Q \vee R}{P \vee R}$$

ix)
$$\frac{P \to Q \quad Q}{P}$$
X Fallacy

* $P_1 \wedge P_2 \wedge P_3 \to C$ To check for satisfiable just take any 1 premise False. Then $F \to \_ =$ True

NOTE: $\forall x(P(x)) = P(1) \land P(2) \land P(3)$
$\exists x(P(x)) = P(1) \lor P(2) \lor P(3)$
$\rightarrow$ $\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$
$\exists x(P(x) \land Q(x)) \rightarrow \exists x P(x) \land \exists x Q(x)$

→ Type 4: Block Diagram

take LHS true & check for RHS

- $\forall x P(x) \rightarrow \exists x (P(x))$
- Negation: $\neg\forall x P(x) = \exists x (\neg P(x))$
  $\neg\exists x P(x) = \forall x (\neg P(x))$

→ Type 5: English → Logical exp conversion

- All $\rightarrow$ Some $\land$

convert it in terms of 'All' and 'some'.

- Not all : $\neg\forall x(\_\_ \rightarrow \_\_)$
  No/None : $\forall x(\_\_ \rightarrow \neg\_\_)$

→ Type 6: Nested Quantifier.

- $\forall x \forall y \equiv \forall y \forall x$
- $\exists x \exists y \equiv \exists y \exists x$

* If nothing (All/some) is given in starting, consider it $\forall$.

- $\forall x \forall y \rightarrow \exists x \exists y$
- $\forall x \exists y \not\rightarrow \exists y \forall x$
  $\exists y \forall x \rightarrow \forall x \exists y$

- ① $\forall x \forall y$ ≡ ② $\forall y \forall x$

$$\begin{array}{ccc} ①\ \forall x \forall y & \equiv & ②\ \forall y \forall x \\ \downarrow & & \downarrow \\ \boxed{\begin{array}{c} \exists y \forall x \\ \downarrow \\ \forall x \exists y \end{array}} & & \boxed{\begin{array}{c} \exists x \forall y \\ \downarrow \\ \forall y \exists x \end{array}} \\ \downarrow & & \downarrow \\ ③\ \exists x \exists y & \equiv & ④\ \exists y \exists x \end{array}$$

1/2 → Everyone
Everyone → 3/4

No relation b/w boxes.

classmate

Date
Page

→ Type 7: Quantifier with inference rule

- $\forall x\, P(x) \rightarrow P(a)$ — $a \rightarrow$ random
  $P(a) \rightarrow \forall x\, (P(x)$ — $a \rightarrow$ "
- $\exists x\, P(x) \rightarrow P(a)$ — a is fix
  $P(a) \rightarrow \exists x\, P(x)$ — a is fix

NOTE:
- Try to contradict $\forall$ by taking a case $T \rightarrow F$
- Try to validate $\exists$ by taking a case $F \rightarrow \_ = T$

NOTE: p is necessary for q: $q \rightarrow p$
p is sufficient for q: $p \rightarrow q$

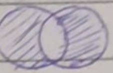
Set Theory

1. N: Natural no. Z: Integer Q: Rational no.
R: Real no. C: complex no
$$N \subseteq Z \subseteq Q \subseteq R \subseteq C$$

2. element $\in$ Set   Set $\subseteq$ Set
{ele}

3. $|\Phi| = 0$ $|\{\Phi\}| = 1$

4. $\Phi \subseteq$ Anything $A \subseteq A$

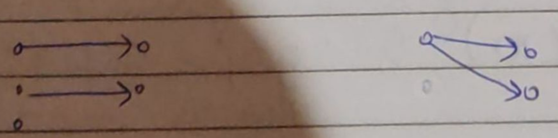
5. $A - B = A - (A \cap B)$
$A \Delta B = (A \cup B) - (A \cap B) = (A - B) \cup (B - A)$ 
↑ symmetric difference

6. Power set $(P(A) / 2^A)$ $|P(A)| = 2^{|A|}$

7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (A \cup B) = A$
$\overline{A \cup B} = \bar{A} \cap \bar{B}$

8. $(A \Delta B) = (C \Delta B) \rightarrow A = C$

Function

1.  Not allowed

2. Range $\subseteq$ codomain

3. Total no. of fu^n = $|RHS|^{|LHS|}$ (= Total diff arrow mapping)

4. 1:1 / Injective : $\forall a \forall b \, (f(a) = f(b) \rightarrow a = b)$
$|LHS| \leq |RHS|$

5. Total no. of 1:1 fu^n = $\begin{cases} {}^{RHS}P_{LHS} & \text{if } |LHS| \le |RHS| \\ 0 & \text{if } |LHS| > |RHS| \end{cases}$

6. <u>Onto fu^n</u> / <u>Surjective fu^n</u>: Range = codomain

$f: A \to B$ $\quad \forall B \; \exists A$ $\quad |LHS| \geq |RHS|$

7. Total no. of onto fu^n = $\begin{cases} \sum_{i=0}^{n} (-1)^i \, {}^nC_i \, (n-i)^m & \text{if } LHS \geq RHS \\ 0 & \text{if } LHS < RHS \end{cases}$

$f: A \to B$ $\quad |A| = m$ $\quad |B| = n$

8. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$

9. <u>1:1 correspondance fu^n</u> / <u>Bijective</u>:

= 1:1 + onto $\quad |LHS| = |RHS|$

10. Total no. bijective fu^n = $n!$

$|LHS| = |RHS| = n$

11. <u>Identity fu^n</u>: $i_A : A \to A$ $\quad$ It is bijective fu^n

12. A fu^n is invertible only if it is bijective.

13. $fog = f(g(x))$ (g pahle) ; $gof = g(f(x))$ (pahle f)

$g: A \to B,\ f: B \to C$ $\quad f: A \to B,\ g: B \to C$

14. $fog \neq gof$ (Not commutative)

15. if f & g $\to$ gof $\quad (f: A \to B,\ g: B \to C)$

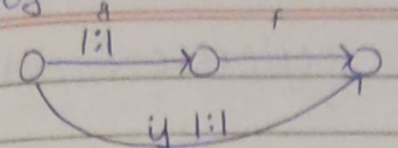
1:1 $\quad$ 1:1

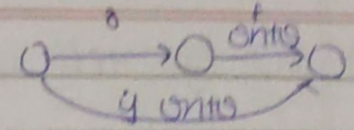
onto $\quad$ onto

1:1 corres. $\quad$ 1:1 correspondance

NOTE: $f(a) = b \Rightarrow f^{-1}(b) = a$
b is image of a
a is preimage of b.

If fog is 1:1 ⇒ g is 1:1 | If fog is onto ⇒ f is onto

16. 



17. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$
$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

Relation

1. Total no. of relation possible $= 2^{mn}$
$|A| = m$, $|B| = n$

2. Symmetric relⁿ: $\forall a \forall b \; aRb \rightarrow bRa$
- Total no. $= 2^{\frac{n(n+1)}{2}}$
- Empty relⁿ is symmetric.

3. Reflexive relⁿ: $\forall a \; aRa$
- Total no. $= 2^{n^2-n}$

4. No. of relⁿ which are reflexive as well as symmetric $= 2^{\frac{n^2-n}{2}}$

5. Antisymmetric relⁿ: $\forall a \forall b \; aRb \wedge bRa \rightarrow a=b$
- Do not allow flipping, if flipping its same ele
- Total no. $= 2^n 3^{\frac{n^2-n}{2}}$

6. No. of relⁿ which are ref & antisym $= 3^{\frac{n^2-n}{2}}$

7. No. of relⁿ which are sym & antisym $= 2^n$

8. Asymmetric relⁿ: $\forall a \forall b \; aRb \rightarrow b \not R a$
- Total no. $= 3^{\frac{n^2-n}{2}}$

→ Rel^n from $Z \times Z$ is like (a, b)
Rel^n " $Z^2 \times Z^2$ " " $((a,b),(c,d))$
Rel^n from $Z^3 \times Z^3$ " " $((a,b,c),(d,e,f))$

classmate
Date
Page

9. No of rel^n which are reflexive & Assy = 0

10. Empty rel^n is sym, asym, & Antisym.

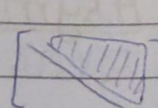
11. Irreflexive: $\forall a \; a \not R a$
- Total no. $= 2^{n^2-n}$

12. Transitive rel^n:
$\forall a \forall b \forall c \; aRb \wedge bRc \rightarrow aRc$
- Pair of same ele does not create any problm.
- If there is flipping i.e. $(a,b) \in R$ & $(b,a) \in R$, R must contain (a,a) & (b,b)

13. R_1, R_2	$R_1 \cup R_2$	$R_1 \cap R_2$
Reflexive	✓	✓
Symm	✓	✓
Antisym.	X	✓
Asym.	X	✓
Transitive	X	✓
Irreflexive	✓	✓
Partial order Rel^n	X	✓

14.

	Total no. of r^n
14. Reflexive: $\forall a \; aRa$	2^{n^2-n}
Sym: $\forall a \forall b \; (aRb \rightarrow bRa)$	$2^{\frac{n(n+1)}{2}}$
Antisym: $\forall a \forall b \; (aRb \wedge bRa \rightarrow a=b)$	$2^n 3^{\frac{n^2-n}{2}}$
Asymm: $\forall a \forall b \; (aRb \rightarrow b \not R a)$	$3^{\frac{n^2-n}{2}}$
Irreflexive: $\forall a \; (a \not R a)$	2^{n^2-n}
Transitive: $\forall a \forall b \forall c \; (aRb \wedge bRc \rightarrow aRc)$	—

NOTE:
- $a \overset{\text{congruent}}{\equiv} b \pmod 4$ means both a & b will give same remainder on dividing by 4.
- No of non-diagonal pairs $= \dfrac{n^2-n}{2}$ 

Total order $\Rightarrow$ Linear order $\Rightarrow$ Chain
rel^n rel^n

NOTE: Remove Reflexive & transitive ordered pairs from Hasse diagram.

15. Partial order rel^n = Reflexive + Anti Sym + Transitive
- If R_1 & R_2 r POR then, $R_1 \cup R_2$ is not necessary POR, $R_1 \cap R_1$ is POR

16. a & b are comparable if (aRb or bRa)

17. Total order rel^n = Partial order rel^n + all ele of set are comparable.

18. POSET = (Set, Relation)
(S, R) is a poset means R is a POR & all ele of set S are not comparable.

19. Every TOSET is a POSET, not vice versa.

20. TOSET Hasse diagram is a straight (verticle) line.

21. Greatest ele: all elem of A $\underset{\text{Related}}{\preceq}$ x (unique)
(A, R) is a poset

22. Least ele: $x \preceq$ all ele of A (unique)

23. (A, R) is a poset & $B \subseteq A$
Upper Bound: all ele of B $\preceq x$ (Not unique)

24. Lower Bound: $x \preceq$ all ele of B (Not unique)

25. Least upper bound (LUB): least of upper bound. (unique)
LUB = $\vee$ = $\cup$ = +

26. GLB: greatest of all lower bounds (unique)
GLB = $\wedge$ = $\cap$ = $\cdot$

27. In $(D_n, /)$ poset
LUB = LCM , GLB = GCD/HCF

Lattice $\rightarrow$ Poset
Linear order relⁿ $\rightarrow$ Lattice

classmate
Date
Page

28. Lattice : poset is lattice if glb & lub exist for all pair of ele
[A, $\vee$, $\wedge$]

29. Lattice satisfies idempotent, commutative, associative & absorption.

30. All Linear order/Total order are always Lattice but not via versa.

31. No. of edges present in total order rel
$(A, R) = \underset{\text{self loop (Reflexive)}}{n} + \underset{\text{edges in Total graph}}{\frac{n(n-1)}{2}}$ $\quad |A| = n$

32. Bounded Lattice : greatest & least ele present

33. Every finite lattice is Bounded Lattice.

34. (A, R) is a poset
B is called sublattice of A, if $B \subseteq A$ & glb, lub of B is same as in A.

35. Complement lattice : every ele has atleast 1 comp.

36. Complement : $a+b=1$, $a \cdot b = 0$
$lub(a,b) = GE$, $glb(a,b) = LE$

37. Distributive lattices
$\forall a \forall b \forall c \begin{cases} a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \end{cases}$

can't say if every ele has at most 1 comp then its dist lattice. Then check by property.

38. Every ele of distr. lattice has at most 1 comp
If any ele is having more than 1 comp its not distributive.

39. Every TOSET is distributive lattice.

40. Boolean algebra = Lattice + complement + Distributive.

41. Sublattice of bounded lattice is also bounded. ✓
" " comp " " " comp ✗

42.

	$(P(A), \subseteq)$	$(D_n, /)$	$(A, \leq)$	TOSET
Lattice	✓	✓	✓	✓
Bounded L.	✓ 1 : A, 0 : Φ	✓ 1 : n, 0 : 1	✗ A is infinite	finite TOSET is bounded
Comp L.	✓	may or may not	✗	✗
Distr. L.	✓	✓	✓	✓
Boolean algebra	✓	may or not	✗	✗

creates partition of set

43. Equivalence Rel^n = Reflexive + Sym + Transitive

$n \leq |R| \leq n^2$

- If (A, R) is equi rel^n & A_1, A_2, A_3 are classes then
$A_1 \cup A_2 \cup A_3 = A$, $A_1 \cap A_2 \cap A_3 = \Phi$

44. Sterling second kind no : $f: A \to B$ $|A| = m$, $|B| = n$

$$\underset{\text{no of partition}}{S(m, \overset{\text{no of ele}}{n})} = \frac{1}{n!} \underbrace{\sum_{i=0}^{n} (-1)^i \, {}^nC_i \, (n-i)^m}_{\text{No. of onto fu}^n}$$

45. Bell no. = No. of equi rel^n = $\sum_{i=1}^{n} S(m, i) = B(\underset{\text{no of ele}}{n})$

B_0 1

B_1 1 2

B_2 2 3 5

* If $|A| = n$
No. of equ. rel^n on set $A = B_n$

→ Composition of rel^n :- composition

$R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$
$R_3 = R_1 \cdot R_2 \subseteq A \times C$

46. If R_1 & R_2 r sym, $R_1 \cdot R_2$ need not be sym.
" ——— Antisym ——— Antisym
——— transitive ——— transitive

→ **Reflexive closure of rel^n R:**
Ref. closure (R^+) of R is min relation containing R and is reflexive.

$$R^+ = R \cup (\text{diagonal ele i.e } aRa)$$

→ **Symmetric closure of rel^n R:**
min symmetric rel^n containing R

$R^+ = R \cup R^{-1}$ 　 if $R = \{(a,b) \mid (a,b) \in R\}$
$R^{-1} = \{(b,a) \mid (b,a) \in R^{-1}\}$

→ **Transitive closure of R:**
min transitive rel^n containing R.

Simultaneously → Product Rule
not → Sum Rule

NOTE:- Every no. can be expressed as $2^k \cdot n$, where $k \geq 0$, n is odd

COMBINATORICS

1. By default consider repeatition allowed.

2. Selection = $^nC_r = \dfrac{n!}{(n-r)!\,r!}$

3. $^nP_r = {}^nC_r * r!$

⇒ **Euler Totient fu^n**

1. $\phi(n)$ = no. of coprimes of n less than n.

2. $\phi(A \times B) = \phi(A) \times \phi(B)$ A & B are coprimes

3. $\phi(p) = p-1$ p is prime no.

4. $\phi(p^a) = p^a\left(1 - \dfrac{1}{p}\right) = \underbrace{p^a}_{\text{Total}} - \underbrace{\dfrac{p^a}{p}}_{\text{noncoprime}}$ p is prime

5. If $n = p_1^a\, p_2^b\, p_3^c$ p_1, p_2, p_3 r prime nos
$\phi(n) = \dfrac{n(p_1-1)(p_2-1)(p_3-1)}{p_1 p_2 p_3}$

6. Circular arrangement = $(n-1)!$

- 'At least' case solve by comb with repeatition → Moving line concept
'At most' " " Generating fu^n.

- $A \cup B = A + B - A \cap B$
$A \cup B \cup C = A + B + C - (A \cap B) - (B \cap C) - (A \cap C) + (A \cap B \cap C)$

$|A_1 \cup A_2 \cup \dots A_n| = \Sigma|A_i| - \Sigma|A_i \cap A_j| + \Sigma|A_i \cap A_j \cap A_k| \dots (-1)^{n+1}|A_1 \cap A_2 \cap \dots A_n|$

→ Pigeon Hole Principle:- If there a n holes & $\geq (n+1)$ pigeons then some of the pigeon holes contain atleast 2 pigeons.

comb with repetition → Distributing n identical objects into m distinct boxes (use line concept) = $\underbrace{{}^{n+(m-1)}}_{\text{total}} C_{(m-1)} \to$ No of lines (each ~~set~~ box contains ≥ 0 ele)

NOTE: Min no. of ele to be taken out of a set containing n ele such that it contains ele x & y where x/y or y/x $\Rightarrow (n/2+1)$

⇒ **Derangement**: no. of arrangements in which all ele are at wrong place.

1. D_n = Total − at least at ele at right place.

2. $D_n = n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots \frac{(-1)^n 1}{n!}\right] = n!\,e^{-1} = n! * 0.368$ (for $n \geq 6$)

3. Prob of getting derangement = 0.368

⇒ **Binomial expansion:**

1. $(a+b)^n = \sum_{i=0}^{n} {}^nC_i\, a^{n-i} b^i$

2. ${}^nC_x = {}^nC_{n-x}$

3. Pascals Δ:

$$\begin{array}{ccccccc} & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ 1 & 4 & & 6 & & 4 & 1 \end{array}$$

For $(a+b)^n$ all terms are +ve.
For $(a-b)^n$ all terms are alternate in sign

4. ${}^nC_x + {}^nC_{x+1} = {}^{n+1}C_{x+1}$

5. ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$

6. ${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots$

7. Extended Binomial coeff: ${}^{-n}C_K = (-1)^K\, {}^{n+K-1}C_K$

8. $\frac{1}{(1-x)} = 1 + x + x^2 + \cdots$ on Differ. $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots$

9. $\frac{1}{(1-ax)} = 1 + ax + (ax)^2 + \cdots$

classmate

Date
Page

10. $\frac{1}{(1+ax)} = 1 - ax + (ax)^2 - (ax)^3 + \cdots$

- Σ infinite GP $= \frac{a}{(1-r)}$

  Σ Finite GP $= \frac{a(1-r^n)}{(1-r)}$

11. $\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

12. Solve 'atleast' ques by combination with repeatition and 'atmost' ques by generating fn

13. Generating fun: $G(x) = \sum_{i=0}^{\infty} a_i x^i$

NOTE: In series.
Numerator me +ve : all terms +ve
-ve : sign alternate
Denominator me -ve ; all terms +ve
+ve : sign alternate

Recurrence Reln

dependent on previous 1 term

1. TYPE 1: $A_n = d A_{n-1}$ $n \geq 1$

   Soln: $A_n = d^n A_0$

dependent on previous 2 terms

2. TYPE 2: Distinct Root (α, β)

   CE: $a_n = \alpha^n C_1 + \beta^n C_2$

* Procedure:
  - write eqn, find roots
  - write charn eqn & put root value
  - ~~Put base condition to~~ get constant (C_1, C_2) value
  - Put C_1, C_2 value back in CE

3. TYPE 3: Same root (α, α)

   CE: $a_n = \alpha^n C_1 + \alpha^n n C_2$

classmate

Date
Page

→ depends on previous 3 terms

4. <u>TYPE 4</u>: 3 roots α, β, γ

   CE: $a_n = \alpha^n C_1 + \beta^n C_2 + \gamma^n C_3$

5. <u>TYPE 5</u>: roots are α, α, β

   CE: $a_n = \alpha^n C_1 + n.\alpha^n C_2 + \beta^n C_3$

6. <u>TYPE 6</u>: roots are α, α, α

   CE: $a_n = \alpha^n C_1 + n\alpha^n C_2 + n^2\alpha^n C_3$

- $a_n - 7a_{n-1} + a_{n-2} = f(n)$

  $f(x) = 0$ Homogeneous Recurrence rel^n

  $f(n) \neq 0$ Non " " "

- Above discussed procedure & 6 types are about homogeneous recurrence rel^n.

7. <u>Procedure for non homogeneous recr rel^n</u>:
- From hom. part form eq & find its roots.
- Find hom. CE as discussed above.
- Write non hom. CE.
- Put non hom. CE in given eq^n to find constant of non hom. CE.
- Total CE = hom. CE + non hom. CE
- Put base condition in Total CE to get constant value of hom. CE.

8. If hom & non hom CE comes same, multiply non hom. CE by 'n'.

<u>NOTE</u>: $a_{n+2} - 4a_{n+1} + 3a_n = 200$

If we take non hom CE, $a_n = A$ then it can't be solved so take non hom CE as $a_n = An$

classmate

Date

Page

NOTE: 10.

Roots of Hom eq^n	$f(n)$	Non hom char eq^n
7, 3	$5(6^n)$	$A(6^n)$
7, 3	$5(7^n)$	$An(7^n)$
7, 3	$5(3^n)$	$An(3^n)$
7, 7	$5(7^n)$	$An^2(7^n)$
3, 3	$5(7^n)$	$A(7^n)$

11. <u>Non-hom. char eq^n:</u>

$f(x)$	Non hom CE
$7(5^n)$	$A(5^n)$
20 ← constant	A
$(n+2)$ or n	$An+B$
$2n^2+n+7$ or n^2	An^2+Bn+C
$(2n+3)(7^n)$	$(An+B)(7^n)$
$(2n^2+n+7)(7^n)$	$(An^2+Bn+C)(7^n)$

NOTE: on problm like $a_{n+2}^2+5a_{n+1}^2+6a_n^2=7n$

let $b_n=a_n^2$ and solve.

GROUP THEORY

1. (Set, operation) eg $(A, *)$ is group if it satisfies following properties:
   - closed : if $a \in A$ & $b \in A$ then $a * b \in A$
   - Associative : $a*(b*c) = (a*b)*c$
   - Identity ele : $a * \underline{e} = a$, $e \in A$
   - Inverse ele : $a * \underline{a^{-1}} = e$, $a^{-1} \in A$

   } e and a^{-1} are unique

NOTE: e is unique for a group and every element has a unique a^{-1}.

2.

Closed	Associative	Identity	Inverse
Quasi group	Semigroup	Monoid	Group

NOTE: ⊕ : addition mod , ⊛ : multiplication mod

3. Finite groups can be represented using Cayley table.

   a opr b

a ↓ \ b →	

4.

	a	b	c
Iden ele	a	b	c

	Identity ele
a	a
b	b
c	c

5. Identity ele is present in all rows & columns if its a group.

6. Subgroup: $(G, +)$ is group, H is called subgrp if $H \subseteq G$ and H is a group.

* Proper subgrp : $H \subset G$

7. Every grp consist of 2 trivial (default) subgrps:
   (i) Identity ele, $H = \{e\}$ (ii) Given set, $H = G$

8. Every subgrp contains identity ele of group.

9. Lagrange theorem: If H is a subgrp of G then $|H|$ divides $|G|$.

10. Abelian group: grp that satisfies commutative property ($ab = ba$ or $(ab)^2 = a^2b^2$ or $a^2 =$ Identity ele) i.e a is inverse of itself

11. Every subgrp of Abelian grp is also abelian grp

→ Exponential:

12. $*$ is some operation
$a^1 = a$
$a^2 = a * a = a^1 * a$
$a^3 = a * a * a = a^2 * a$

13. Exponential tells about subgroup.
If $2^1 = 2$, $2^2 = 4$, $2^3 = 0$, $2^4 = 2$ ← Repeated stop
$\langle 2 \rangle = \{0, 2, 4\}$ (2 belongs to this subgrp)
order $= 3$ (2 is generating 3 elements)

14. order = No. of ele generating.

15. If an ele 'a' generates all the ele of grp on taking exponential, its called generator.

16. Cyclic grp: If a grp contains generator then its called cyclic grp.

17. If 'a' is a generator then 'a^{-1}' is also a generator (both in opposite direction, 1 clockwise & 1 anticlockwise)

no. of coprime less than n

18. No. of generators = Euler-totient fu^n of size of set.

19. Subgrp of cyclic grp is also cyclic

NOTE: Identity ele can't be generator.

20. No. of Abelian grps of order p^k (p is prime)
= No. of partitions of power K.

Ex:- order $= 16 = 2^4$

partitions of $4 = \{4\}, \{1,3\}, \{2,2\}, \{2,1,1\}, \{1,1,1,1\}$

NOTE: If $a^{-1} = 1/a$ then a^{-1} of $a = 0$ does not exist.
if 0 is in set then it does not form grp.

→ Properties of cyclic grp:

- If 'a' is generator of grp then 'a^{-1}' is also a generator.
- If G is a cyclic grp of order 'n' then no. of generators in $G = \phi(n)$.
- * Every grp of prime order is always cyclic
- * Every cyclic grp is an Abelian
- Every subgrp of cyclic grp is also cyclic.