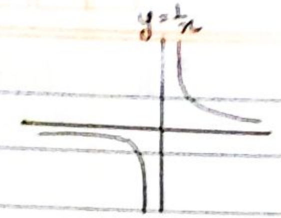
CALCULUS



#LIMITS

1. Indeterminate form → use limits.

$\frac{1}{0} = n \Rightarrow 1 = 0 \cdot n$ → NO sol^n → NO exact sol^n $\frac{0}{0} = n \Rightarrow 0 = n \cdot 0$

2. $1/0$ is undefined, not indeterminate.

3. Indeterminate forms — Technique

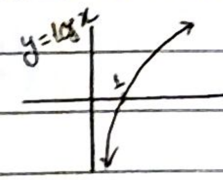
Indeterminate forms	Technique
(i) $0/0$, ∞/∞	L HOspital
(ii) $\infty - \infty$	• if $v \to \infty$, convert as $1/v$ • cond. Rationalization
(iii) ∞^0, 0^0	$v^v \to$ Take log; make sure to take antilog
(iv) $\infty \cdot 0$	convert as $\infty/\infty \to$ L HOspital
(v) 1^∞	use exponential formulae

4. $LHL = \lim_{h \to 0} f(a-h)$ $RHL = \lim_{h \to 0} f(a+h)$

If $LHL = RHL$, then $\lim_{x \to a} f(x)$ exist.

5. MOD fu^n: $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

6. L HOspital: Keep differentiating numerator & denom Jb tk fu^n $0/0$ or ∞/∞ form ka aa rha hai.



7. $\infty \neq \infty$, $\infty + \infty = \infty$

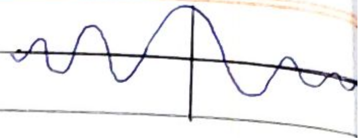
8. $\ln(0) = -\infty$, $\ln(1) = 0$, $\ln(\infty) = +\infty$

$0 = e^{-\infty}$ $1 = e^0$ $\infty = e^{\infty}$

*9. $\lim_{x \to 0} [1 + ax]^{1/x} = e^a$, $\lim_{x \to \infty} [1 + a/x]^x = e^a$

$e^{\infty} = \infty$, $e^{0} = 1$, $e^{-\infty} = 0$

* 10. $\lim_{x \to 0} \frac{\sin x}{x} = 1$, $\lim_{x \to \infty} \frac{\sin x}{x} = 0$



11. $\sin\infty$ / $\cos\infty$ is finite value in range $[-1, 1]$

12. $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\sin 2x = 2\sin x \cos x$

* 13 $\lim_{x \to \infty} x^{1/x} = 1$, $\lim_{x \to 0} x^{x} = 1$

* <u>Properties of Limits</u>:

14. $\lim_{x \to a} [c f(x)] = c \cdot \lim_{x \to a} f(x)$

15. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

16. $\lim_{x \to a} [f(x) g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

17. $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, $\lim_{x \to a} g(x) \neq 0$

18. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$

19. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

* Ans of limit is always some finite value.

* $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\sin^{-1} x}{x} = \lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$

⇒ Derivation Formulae:

1. $\frac{d}{dx} x^n = n x^{n-1}$	2. $\frac{d}{dx} a^x = a^x \log a$	3. $\frac{d}{dx} c = 0$
4. $\frac{d}{dx} e^x = e^x$	5. $\frac{d}{dx} \log_e x = \frac{1}{x}$	6. $\frac{d}{dx} \log_a x = \frac{1}{x \log a}$
7. $\frac{d}{dx} \sin x = \cos x$	8. $\frac{d}{dx} \cos x = -\sin x$	9. $\frac{d}{dx} \tan x = \sec^2 x$
10. $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$	11. $\frac{d}{dx} \sec x = \sec x \tan x$	12. $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$
13. $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	14. $\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$	15. $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$
16. $\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}$	17. $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$	18. $\frac{d}{dx} \operatorname{cosec}^{-1} x = \frac{-1}{x\sqrt{x^2-1}}$
19. $\frac{d}{dx} \sinh x = \cosh x$	20. $\frac{d}{dx} \cosh x = \sinh x$	21. $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$
22. $\frac{d}{dx} \coth x = -\operatorname{cosech}^2 x$	23. $\frac{d}{dx} \operatorname{cosech} x = -\operatorname{cosech} x \cdot \coth x$	24. $\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \cdot \tanh x$

25. $$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\, g'(x) + g(x)\, f'(x)$$

26. $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{[g(x)]^2}$$

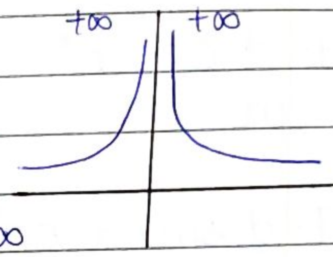
* Let $f(x)$ & $g(x)$ be continuous fuⁿ at 'a' then the following are also continuous:
- $f(x)+g(x)$
- $f(x)-g(x)$
- $f(x)\cdot g(x)$
- $f(x)/g(x)$
- $K\cdot f(x)$
- $f^n(x)$

when there is a sudden jump $\int$ fuⁿ is discontinuous

#CONTINUITY

1. If $LHL = RHL = f(a)$, then $f(x)$ is continuous at $x=a$.

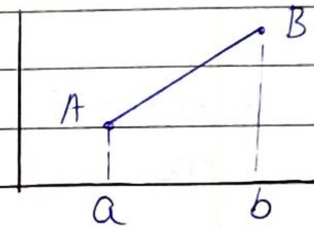
2. $\lim\limits_{x\to 0} \frac{1}{|x|}$



Macrolevel: limit does not exist
Microlevel: Limiting idea (value) $= +\infty$

3. Fuⁿ is discontinuous at values of x which makes fuⁿ undefined i.e. denominator $=0$.

4. At point A, if $RHL = f(a)$ its called Right continuous. LHL does not exist.



At point B, if $LHL = f(b)$ its called Left continuous. RHL does not exist.

⇒ Slope of a line:

1. $\text{Slope} = \frac{dy}{dx} = \frac{y_2-y_1}{x_2-x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{Rise}}{\text{Run}} = \tan\theta$

angle with x axis in anticlockwise direction

2. If θ is acute → slope is +ve
If θ is obtuse → slope is −ve

3. Instantaneous slope:
Slope at point $x=a$
$\Rightarrow f'(a) = \lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$

* Trignometric fu^n are continuous and differentiable.

DIFFERENTIABILITY

1. At point C

$LHD = \lim_{h\to 0} \frac{f(C) - f(C-h)}{h}$; $RHD = \lim_{h\to 0} \frac{f(C+h) - f(C)}{h}$

If LHD = RHD , then fu^n is differentiable at $x = C$.

2. If $f'(C^-) = f'(C^+)$ then $f(x)$ is diff at $x = C$.

3. fu^n is not diff at sharp corner.

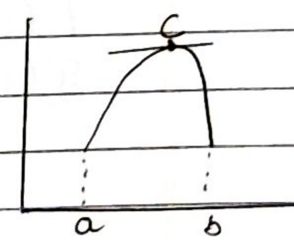
4. MOD fu^n is continuous but not diff at corner points.

5. Continuous $\xrightarrow{\text{May or may not}}$ Differentiable

Continuous $\xleftarrow{\text{Always}}$ Differentiable

MEAN VALUE THEOREM

⇒ Rolle's Theorem :



i) If f is cont. in [a, b]

ii) f is diff. in (a, b)

iii) $f(a) = f(b)$

then there exist atleast 1 point c such that $f'(c) = 0$. $c \in (a, b)$

↳ slope = 0

tangent ‖^{el} to x axis

⇒ Lagrange Mean value theorem:

i) If f is continuous in $[a,b]$
ii) f is diff in (a,b)
iii) $f(a) \neq f(b)$
then there exist atleast 1 point $c \in (a,b)$ whose slope is equal to the average slope of the curve.

$$\underset{\uparrow}{f'(c)} = \frac{f(b)-f(a)}{b-a} \rightarrow \text{Avg slope}$$

Instantaneous slope

1. Rolle's is special case of Lagrange ($f(a)=f(b)$)
2. $\int_a^b f(x)\,dx = f(c)(b-a)$ → Avg value of fu^n

⇒ Cauchy Mean value Theorem:

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

TAYLOR SERIES → use in limits & integration.

1. Approximate any f^n into polynomial fu^n.
2. Taylor Series about $x=0$ → Maclaurin Series
3. By default, fu^n expansion is about $x=0$.

in Taylor expansion
Now put $x=0$ & differentiate and put $x=0$

4. Linear Approximation : $f(x) = A_0 + A_1 x$

5. Quadratic " : $f(x) = A_0 + A_1 x + A_2 x^2$

6. Maclaurin series:
$f(x)$ ka expansion about $x=0$
$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \cdots$$

7. Taylor Series:
$f(x)$ ka expansion about $x=a$
$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \cdots$$

NOTE: i) $f(-x) = -f(x)$ → odd fuⁿ
$f(-x) = f(x)$ → Even fuⁿ

$e^{ix} = \cos x + i\sin x$, $e^{-ix} = \cos x - i\sin x$

ii) $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ $\xrightarrow[i]{\text{Remove}}$ $\sinh x = \frac{e^{x} - e^{-x}}{2}$

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$, $\cosh x = \frac{e^{x} + e^{-x}}{2}$

About $x=0$

odd fuⁿ alternate sign

1) $\sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

even fuⁿ alternate sign

2) $\cos x = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

3) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

4) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

odd fu

5) $\sinh x = \frac{x^1}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

All +ve

6) $\cosh x = \frac{x^0}{1!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

Even fu

7) $(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \cdots$

8) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

9) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

10) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

11) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

12) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

13) $\log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right]$

Finite value → convergent
∞ → divergent
Not a unique value → oscillatory

INFINITE SERIES

1) Sum of infinite GP series $\left(\sum_{n=1}^{\infty} ar^{n-1}\right) = \frac{a}{1-r}$; $|r|<1$

Sum is convergent : $|r|<1$
Divergent : $r \geq 1$ or $r \leq -1$
↳ oscillatory

⇒ P series convergent test:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

if $p>1$: convergent
$p \leq 1$: Divergent

⇒ Ratio Test:

$$\sum_{n=k}^{\infty} U_n$$

$\lim_{n\to\infty} \frac{U_{n+1}}{U_n}$ is <1 : convergent
>1 : Divergent
$=1$: test fails

⇒ Root test:

$\sum_{n=k}^{\infty} (a_n) \Rightarrow \lim_{n\to\infty} (a_n)^{1/n}$

<1 : convergent
>1 : Divergent
$=1$: Test fails

* $\frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2}$ * $\frac{d}{dx}|x| = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}$ for $x=0$, does not exist

* Explicit f^n : $y = f(x)$

* Implicit f^n : $f(x,y) = 0$

* Monotonic f^n : f^n that grows in only 1 direction, either increasing or decreasing

TRICK: Cubic polynomial division

$$x^3 - 10x^2 + 31x - 30 = 0$$ one root is 5.

5	1	-10	31	-30
	↓	5	-25	30
	1	-5	6	0

$\Rightarrow x^2 - 5x + 6 = 0$

$\alpha + \beta + \gamma = \frac{-b}{a}$ $\alpha\beta\gamma = \frac{-d}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

→ 1) Increasing fu^n : $f'(x) \geq 0$ (acute angle)
Strictly incr. fu^n : $f'(x) > 0$

2) Decreasing fu^n : $f'(x) \leq 0$
Strictly decr. fu^n : $f'(x) < 0$

→ Maxima - Minima

(i) Find critical points / stationary / turning point
$f'(x) = 0$ or $f'(x)$ undefined

* 1st der. test : $f'(x)$ changes from + to − max
− to + min

(ii) 2nd der test:
$f''(x) > 0$ minima
$f''(x) < 0$ maxima
$f''(x) = 0$ test fails

NOTE: It's possible that a curve does not have max or min.
Ex:

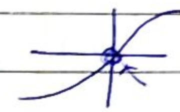
we can't say max is ∞ or min is −∞

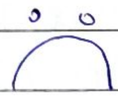
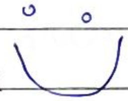
iii) 3rd der test:
$f'''(x) \neq 0$ point of inflextion
$f'''(x) = 0$ fails

$f_x = \frac{\partial f}{\partial x}\Big|_{y = \text{constant}}$ $\quad$ $f_y = \frac{\partial f}{\partial y}\Big|_{x = \text{constant}}$

iv) n^{th} der test is same as 2nd der : if n is even
3rd der : if n is odd.

- For closed interval $[a, b]$, find value at critical points and at a & b then give ans.

- Point of inflextion: change in concavity 

 $f''(x) < 0$ max  $f''(x) > 0$ min

concave down/ convex up $\qquad$ concave up/ convex down

* Partial derivative: $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ in general

when there are more than 1 independent variables

* Homogeneous fn : λ bahar aa jata hai $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

$\rightarrow$ order

Euler hom. fn $\quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n f(x,y)$; if f is homogeneous

Modified Euler hom. fn $\quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\frac{\phi(f)}{\phi'(f)}$; if f is not hom but $\phi(f)$ is hom.

* Total derivative : or implicit eq is given of x, y, z

$u \rightarrow (x, y, z) \rightarrow t$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

* 2 variable Maxima - minima :

(i) Find stationary points from

$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$

(ii) Find $r = f_{xx}$, $t = f_{yy}$, $s = f_{xy}$

$rt - s^2 \big|_{stat.point}$ > 0 max or min

< 0 No max/min → saddle point

(iii) If $rt - s^2 \big|_{sp} > 0$

$f_{xx}\big|_{sp} > 0$ min

$f_{yy}\big|_{sp} < 0$ max

* If $f(x) = a\sin x + b\cos x$

max value $= \sqrt{a^2+b^2}$

min value $= -\sqrt{a^2+b^2}$

* $e^{P(x)}$ is max when $P(x)$ is max.

To find max/min of $e^{P(x)}$ find for $P(x)$.

* If lines are normal, product of slope $= -1$

$\int \frac{f'(x)}{f(x)} dx = \ln(f(x)) + c$

INTEGRATION

① $\int k\,dx = kx + c$

② $\int \frac{1}{x} dx = \ln(x) + c$

③ $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

④ $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

⑤ $\int e^x dx = e^x + c$

⑥ $\int a^x dx = \frac{a^x}{\ln a} + c$

⑦ $\int \ln x\, dx = x\ln x - x + c$

⑧ $\int \log_a x\, dx = x\log_a x - \frac{x}{\log a} + c$

⑨ $\int \sin x = -\cos x + c$

⑩ $\int \cos x = \sin x + c$

⑪ $\int \tan x\, dx = -\ln(\cos x) + c = \ln(\sec x) + c$

⑫ $\int \cot x\, dx = \ln(\sin x) + c$

⑬ $\int \sec x\, dx = \ln(\sec x + \tan x) + c = \log \tan(\pi/4 + x/2) + c$

⑭ $\int \operatorname{cosec} x\, dx = -\ln(\operatorname{cosec} x + \cot x) + c = \log \tan x/2$

⑮ $\int \sec^2 x\, dx = \tan x + c$

⑯ $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$

⑰ $\int \sec x \tan x\, dx = \sec x + c$

⑱ $\int \operatorname{cosec} x \cot x = -\operatorname{cosec} x$

⑱ $\int \sinh x\, dx = \cosh x + c$

㉑ $\int \operatorname{cosech} x\, dx = \ln\left(\tanh \frac{x}{2}\right) + c$

⑲ $\int \cosh x\, dx = \sinh x + c$

㉒ $\int \operatorname{sech} x\, dx = \tan^{-1}(\sinh x) + c$

⑳ $\int \tanh x\, dx = \ln(\cosh x) + c$

㉓ $\int \coth x\, dx = \ln(\sinh x) + c$

$\sin 3x = 3\sin x - 4\sin^3$

Jo diff se aasani se gayab ho jaye wo u

Higher Priority

* Product Rule: $\int uv = u\int v - \int\left(\frac{d}{dx}u \int v\right)$

Highest Priority / Lowest priority

I L A T E

inverse, Log, Algebra, Trig, Exponential

* Hindus: $\int x^n \cdot F(x)\,dx$ — $x^n \to u$, $F(x) \to v$

$n > 0$ +ve integer

$F(x) = \cos ax / \sin ax / e^{ax}$

$= +(u)(\int v) - (u')(\int\int v) + (u'')(\int\int\int v) - \cdots$

do untill it becomes 0

NOTE: If can't solve, differentiate options & verify

$\int f(x)dx = g(x) + C \Rightarrow \frac{d}{dx} g(x) = f(x)$

⇒ Definite Integral:

(i) $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(iii) $\int_a^b f(x)dx = \int_a^b f(t)dt$

(iv) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ $\quad a<c<b$

*KING (v) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ $\quad \underbrace{a-x}_{(UL+LL-x)}$

If have problem in definite integral, use it.

NOTE: when assuming something, make sure to change the limit.

* Walli's theorem:

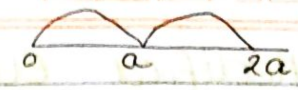
$$\int_0^{\pi/2} \sin^n\theta\, d\theta = \int_0^{\pi/2} \cos^n\theta\, d\theta = \frac{(n-1)}{n}\frac{(n-3)}{(n-2)}\cdots\frac{1}{2} * x$$

If n is even $\rightarrow x = \pi/2$

odd $\rightarrow x = 1$

NOTE: For a continuously increasing f^n $f(x)$

$$\sum_{x=a}^{b} f(x) > \int_a^b f(x)\,dx$$



* If $f(x)$ is periodic

$$\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx \quad , \text{ if } f(na-x) = f(x)$$

* $$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$$

* $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & , \text{ if } f(x) \text{ is even} \\ 0 & , f(x) \text{ is odd} \end{cases}$$

→ Improper integration of 2nd kind:

$\int_{-1}^{1} f(x)\,dx$ if $f(x)$ is undefined at $x=0$

$$= \int_{-1}^{0^-} f(x)\,dx + \int_{0^+}^{1} f(x)\,dx$$

convergent (finite value) ← → Divergent (∞)

→ Improper integ. of 1st kind:

at ∞ it gets indeterminate form $(\infty \cdot 0)$

$\int_a^{\infty}$ or $\int_{-\infty}^{b}$ or $\int_{-\infty}^{\infty}$

$$\lim_{t\to\infty} \int_a^t \qquad \lim_{t\to-\infty} \int_t^b \qquad \lim_{t\to\infty} \int_{-t}^{t}$$

* Euler Gamma fu^n: $\int_0^{\infty} e^{-x} x^n\,dx = \Gamma(n+1) = n!$

- $\Gamma n = (n-1)!$ (use for whole nos)
- $\Gamma(n+1) = n\,\Gamma n = n!$ (" " fractions)
- $\Gamma n\,\Gamma(1-n) = \dfrac{\pi}{\sin(n\pi)}$ $\xRightarrow{\text{Put } n=\frac{1}{2}}$ $\Gamma\tfrac{1}{2} = \sqrt{\pi}$
- $\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \dfrac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma 2m$ (not required)

* Modified Euler Gamma fn:

$$\int_0^\infty e^{-\lambda x} x^n \, dx = \frac{\Gamma(n+1)}{\lambda^{n+1}} = \frac{n!}{\lambda^{n+1}}$$

* Eulers beta function:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}$$

- $\beta(m,n) = \beta(n,m)$
- $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$

* Walli's reduction formula:

$$\int_0^{\pi/2} \sin^m t \cos^n t \, dt = \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots} \times \theta$$

if m & n are even $\rightarrow \theta = \pi/2$

else $\rightarrow \theta = 1$

* Euler:

$$\int_0^{\pi/2} \sin^m t \cos^n t \, dt = \frac{\Gamma\frac{m+1}{2} \, \Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

Double Integration

1) - x = constant, y = variable $\rightarrow$ vertical strip
   - x = variable, y = constant $\rightarrow$ Horizontal strip

2) Jiske limit me variable hai usko phle solve kro.

- x = constant, y = constant Rectangul square
order of integration in general does not matter.

$$\int_0^\infty \int_0^\infty e^{x+y} \, dx\, dy = \int_0^\infty e^x dx \int_0^\infty e^y dy$$

- x & y both limit variable → invalid.

NOTE: $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy \neq \int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx$
becoz fu^n is discontinuous at $(0,0)$

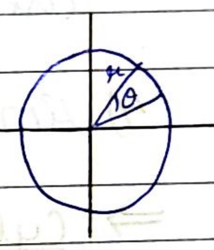
3) If limit not given, area given → take strip and find limit.

4) Area $= \iint dx\,dy = \iint dy\,dx = \int_a^b f(x)\,dx$

limit of either of x, y is variable ↪ $y = f(x)$

NOTE: $\iint$ dependent case unable to solve
↪ change strip
↪ go to polar form

⇒ Polar Form:



$x = r\cos\theta$
$y = r\sin\theta$ $\Big\} \Rightarrow x^2 + y^2 = r^2$
$dx\,dy = r\,dr\,d\theta$
Area $= \iint r\,dr\,d\theta$

- Eq^n of circle: $(x-a)^2 + (y-b)^2 = r^2$
centre $= (a, b)$, radius $= r$

⇒ Jacobian: used for variable conversion

$$\iint_{x\,y} f(x,y)\,dx\,dy = \iint_{u\,v} f(u,v)\,|J|\,du\,dv$$

$x = f_1(u,v)$, $y = f_2(u,v)$

$$|J| = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$\nearrow z = f(x,y)$

- volume = $\iiint dx\,dy\,dz$ = $\iint f(x,y)\,dx\,dy$

* $\iint$ + region is circle → go for polar form

Triple Integration

1) vol. of cuboid/cube = $\int_x \int_y \int_z dx\,dy\,dz$

2) x, y, z all constant → order of integ. does not matter.

$$\int_0^1\int_0^1\int_0^1 dx\,dy\,dz = \int_0^1 dx \int_0^1 dy \int_0^1 dz$$

3) Jiska limit me more number of variables use phle integrate krna.

4) Limit not given, area given → find limit.

⇒ Cylindrical coordinates :

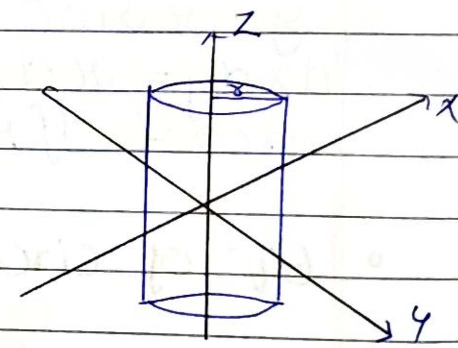
$r \to \rho \qquad \theta \to \phi$

$\rho = 0$ to ∞

$\phi = 0$ to 2π

$z = -\infty$ to ∞

↖ Height



$x = \rho\cos\phi$

$y = \rho\sin\phi$

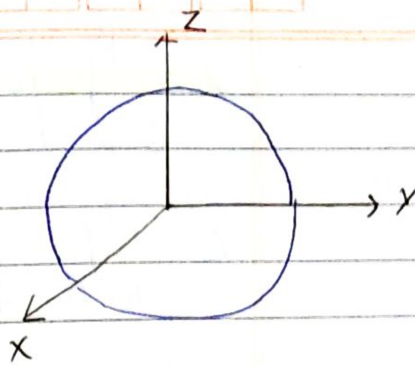
$z = z$

$\Big\} \Rightarrow x^2 + y^2 = \rho^2 \quad ; \quad \tan\phi = \dfrac{y}{x}$

- $\iiint f(x,y,z)\,dx\,dy\,dz = \iiint f(\rho,\phi,z)\,|J|\,d\rho\,d\phi\,dz$
$= \iiint f(\rho,\phi,z)\,\rho\,d\rho\,d\phi\,dz$

- vol of cyl = $\iiint dx\,dy\,dz$ = $\int_z\int_\phi\int_\rho \rho\,d\rho\,d\phi\,dz$

⇒ Spherical coordinates:



$z = r\cos\theta$
$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$

$\theta \rightarrow \angle$ with z axis
$\phi \rightarrow$ angle with x axis

- r: 0 to ∞ θ: 0 to π ϕ: 0 to 2π

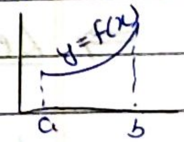
- $x^2 + y^2 + z^2 = r^2$
  $\theta = \cos^{-1}[z/r]$
  $\phi = \tan^{-1}[y/x]$

- volume $= \iiint dx\,dy\,dz = \iiint r^2 \sin\theta \, dr\, d\theta\, d\phi$

Application of Integral

1) Finding length of a curve: $L = \int_{x=a}^{b} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

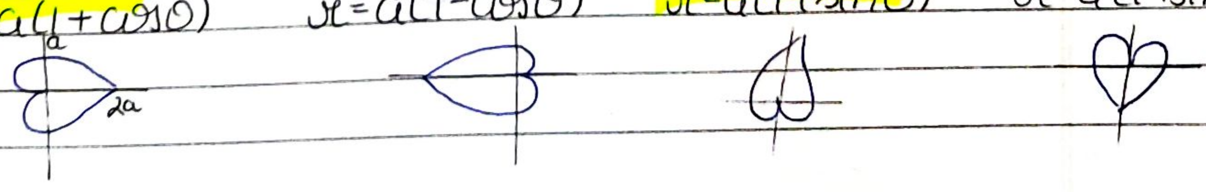
(y is a f^n of x)



$L = \int_{y=a}^{b} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$ (x is a f^n of y)

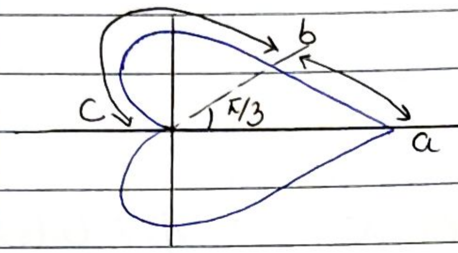
2) Length of cardiod (Heart shape):

$r = a(1+\cos\theta)$ $r = a(1-\cos\theta)$ $r = a(1+\sin\theta)$ $r = a(1-\sin\theta)$



$$L = \int_{\theta=\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad [\, r = f(\theta)\,]$$

$$L = \int_{r=a}^{b} \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}\, dr \qquad [\, \theta = f(r)\,]$$

* 

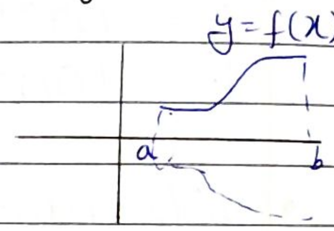
$ab = bc$

3) Finding volume using single integration :

- Revolve about x axis

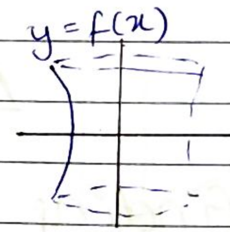
$$V = \int_a^b \pi y^2\, dx$$

$\pi r^2 h$



- Revolve about Y axis

$$V = \int_a^b \pi x^2\, dy$$



⇒ Leibnitz rule : Differ. of definite integral.

$$\frac{d}{dx} \int_{\phi_1(x)}^{\phi_2(x)} f(t)\,dt = \phi_2'(x) f(\phi_2(x)) - \phi_1'(x) f(\phi_1(x))$$

Indefinite integral: $\frac{d}{dx} \int f(x)\,dx = f(x)$