Physics 112 Fall 2017
Professor William Holzapfel
Homework 5 Solutions

Problem 1: Seeing in the dark

(a) Treating the filament as a blackbody, we know that total power is given by $P_b = \epsilon\sigma_B T^4 \times A$, so the area is:

$$\begin{aligned} A &= \frac{100\ W}{\epsilon\sigma_B T^4} = \frac{100}{.30 * 5.67 * 10^{-8} * (3500)^4}\ \mathrm{m}^2 \\ &= 3.9 * 10^{-5}\ \mathrm{m}^2 \end{aligned}$$

(b) Treating your eye as a box of photons at temperature $T_b \approx 300K$, the flux incident on your retina from the inside of your eye is $\mathrm{Flux}_b = \sigma_b T_b^4$. The power from the filament spreads out over a sphere of radius D, the distance from the filament to your eye, so $\mathrm{Flux}_f = \sigma_b T_f^4 \frac{A}{4\pi d^2}$. Thus the ratio of the fluxes is

$$\begin{aligned} \frac{\mathrm{Flux}_b}{\mathrm{Flux}_f} &= \frac{\sigma_b T_b^4}{\epsilon\sigma_b T_f^4 \frac{A}{4\pi d^2}} \\ &= \left(\frac{300\ K}{3500\ K}\right)^4 \frac{4\pi\,(10\ \mathrm{m})^2}{0.3 * 3.9 * 10^{-5}\ \mathrm{m}^2} \\ &\approx 5,800. \end{aligned}$$

The reason we can see anything with so much background flux on our retina is our retina is only sensitive to a narrow band of wavelengths in the visible spectrum, and the background flux from our body peaks in the far infrared, contributing very little flux in the visible spectrum.

(c) To figure out how far away we can see an object, we need to consider the flux from the object within the visible spectrum as compared to that of our eye. We can assume the visible band is so narrow we can treat it as infinitesimal $d\omega_v$. Instead of numerically integrating the specific flux over the visible range, we'll assume we can just take the total flux in the visible to be $F_\nu(\omega_v)d\omega_v$, where $F_\nu(\omega_v)$ is the specific flux at the middle of the visible spectrum. The total flux in the visible spectrum from the eye is then

$$\mathrm{Flux}_b = F_\nu(\omega_v)d\omega = \frac{\hbar}{\pi^2 c^2}\frac{\omega_v^3 d\omega}{e^{\hbar\omega_v/\tau_b} - 1}$$

and the total flux from the filament is:

$$\mathrm{Flux}_f = F_\nu(\omega_v)d\omega = \epsilon\frac{\hbar}{\pi^2 c^2}\frac{\omega_v^3}{e^{\hbar\omega_v/\tau_f} - 1} \times \frac{A}{4\pi D^2}.$$

In the limit that we can just barely see the filament these two fluxes should be equal, which gives the relation

$$D = \sqrt{\epsilon\frac{A}{4\pi}\frac{e^{\hbar\omega_v/\tau_b} - 1}{e^{\hbar\omega_v/\tau_f} - 1}}.$$

Now we're told to assume that the peak frequency of the filament is in the visible band, so we can find ω_v by using Wein's displacement law

$$\frac{\hbar\omega_v}{k_B T_f} = 2.82$$

(for the purposes of this problem we don't need to worry about how we're defining 'peak' frequency like we did in problem set 4). Plugging this in the above expression, and using the fact that $e^{2.82\tau_f/\tau_b} \gg 1$ we have:

$$\begin{aligned} D &= \sqrt{\epsilon \frac{A}{4\pi} \frac{e^{2.82\frac{\tau_f}{\tau_b}} - 1}{e^{2.82} - 1}} \\ &\approx \sqrt{\epsilon \frac{A}{4\pi} \frac{e^{2.82\frac{\tau_f}{\tau_b}}}{e^{2.82} - 1}}. \end{aligned}$$

If we plug in for T_f and T_b we get $D \simeq 3200\text{m} = 3.2\text{km}$. Maybe on a dark night out at sea you could see a lightbulb two miles away...

Problem 2: Liquid ^4He

(a) It's tempting to just plug in the the giving quantities into the expression for Debye temperature that we found in lecture. However, looking back at that calculation we realize that we assumed that there were 3 possible polarizations of of phonons; in the case of liquid ^{4}He we're told that there are no transverse phonons, so there is only one polarization (longitudinal) per mode. Looking back at how we derive the Debye temperature, we see that the total number of phonon modes must add up to $3N$. So with one polarization, the expression for the cutoff mode $n_{\max}$ becomes:

$$\begin{aligned} \frac{1}{8}\int_0^{n_{\max}} 4\pi n^2 dn &= 3N \\ \Rightarrow n_{\max} &= \left(\frac{18N}{\pi}\right)^{1/3}. \end{aligned}$$

Following the derivation (p. 105 of Kittel) we see that this extra factor of 3 will appear in the Debye Temperature:

$$\theta = \frac{\hbar v}{k_B}\left(18\pi^2 \frac{N}{V}\right)^{1/3}.$$

Now if we convert the given mass density into number density we can solve for the Debye temperature:

$$\begin{aligned} \frac{N}{V} &= \frac{6.02 * 16^{23}/\text{mole}}{4\ g/\text{mole}^4 He} \frac{0.145\text{ g}}{10^6\text{ m}^3} = 2.18 * 10^{21}\text{ atoms/m}^3 \\ \theta &= \frac{\hbar v}{k_B}\left(18\pi^2 \frac{N}{V}\right)^{1/3} = \frac{1.05 * 10^{-34} * 2.383 * 10^2}{1.38 * 10^{-23}} * \left(18\pi^2 2.18 * 10^{21}\text{ atoms/m}^3\right)^{1/3} \\ &\approx 28.6\ K. \end{aligned}$$

So we expect the Debye model to be a reasonable approximation for $T < 0.6$ K.

(b) Using the expression for heat capacity we've seen for the Debye model (notice that the derivation of specific heat in the Debye model depends on the Debye temperature in the same way, regardless of the number of polarizations) and the Debye temperature we've calculated above:

$$\begin{aligned}
\frac{C_V}{N} &= \frac{12\pi^4}{5} k_B \left(\frac{T}{\theta}\right)^3 \\
&= \frac{12\pi^4}{5} \left(1.38 * 10^{-23}\right) \left(\frac{T}{28.6\ K}\right)^3 \\
&= \left(1.38 * 10^{-25}\ \frac{J}{K \cdot \text{atom}}\right) T^3.
\end{aligned}$$

To compare this to the experimental value we need to convert this to heat capacity per gram by dividing by the molecular weight of ^{4}He ($4\text{g}/6.02 \times 10^{23}$). This gives

$$C_v/\text{g} = \left(2.07 * 10^{-2}\ \frac{\text{J}}{\text{K} \cdot \text{g}}\right) T^3$$

which is within 2% of the experimental value.

Problem 3: The Big Bang

First of all, recall from equation (23) of Ch. 3 that $\sigma \sim V\tau^3$ for a photon gas, so in an isentropic process we know that $V\tau^3$ and hence $\tau V^{1/3}$ are constant.

(a) Now $\tau V^{1/3} = \text{constant} \implies TR = \text{constant}$ where R is the radius of the universe, and so

$$\begin{aligned}
T_i R_i &= T_f R_f \\
\Rightarrow \frac{R_i}{R_f} &= \frac{T_f}{R_i} = \frac{2.7K}{3000K} \approx \frac{1}{1000}.
\end{aligned}$$

Assuming the universe expanded linearly with time $R = at$,

$$\frac{R_i}{R_f} = \frac{at_i}{at_f} = \frac{t_i}{t_f} \approx \frac{1}{1000}.$$

Now we know that the age of the universe is $\sim$ 14 billion years, and the decoupling time (the time when the photons that we see now as the CMB decoupled from the matter) is about 300,000 years after the Big Bang, so this approximation is off by a factor of 50. This is due not only to the fact that the radius of a radiation-dominated expanding universe grows like $t^{1/2}$ (and so is not linear in t!), but also to the fact that about 10,000 years after the Big Bang (before the CMB decoupled) the expansion of the universe began to be dominated by matter instead of radiation, in which case $R \sim t^{2/3}$. Recently, it appears that a strange new form of energy (no one knows what it is, so they call it "dark energy") has begun to dominate the expansion of the universe. Amazingly enough, all of this can be seen in the CMB.

(b) To find the work done during the expansion we need to use the relation $dW = pdV$, the expression for the pressure of a photon gas $p = \frac{1}{3}\frac{U}{V}$, and to solve for dV in our expression above for an isentropic expansion.

$$\begin{aligned}
dW &= pdV = \frac{1}{3}\frac{U}{V} dV = \frac{\pi^2 \tau^4}{45 c^3 \hbar^3} dV = -\left(\tau^3 V\right) \frac{3\pi^2}{45 c^3 \hbar^3} d\tau \\
W &= \frac{\pi^2}{15 c^3 \hbar^3} \left(\tau_i^3 V_i\right) \left(\tau_i - \tau_f\right).
\end{aligned}$$

Problem 4: Melting of Crystalline Solids and the Zero Point Motion

(a) Using equations (2) and (3) from the HW, we have

$$\begin{aligned} < x^2 > &= \sum_k < x_k^2 > \\ &= \sum_k \frac{E_k}{Nm\omega_k^2}. \end{aligned} \tag{1}$$

(b) By quantize, all we mean is to use the result for a quantum harmonic oscillator $E = \hbar\omega(n + 1/2)$, where n is the 'occupation number' of the oscillator. For a specific mode k this then reads

$$E_k = \hbar\omega_k \left(n_k + \frac{1}{2} \right).$$

(c) The sum (**??**) that gives the mean square displacement can be turned into an integral over n, using the same density of states as in the Debye model. The two key points here are that we have a finite number of modes and so must cut off our integral at $n_D = \left(\frac{6N}{\pi}\right)^{1/3}$, and also that the occupation number of each mode is given by the Planck distribution

$$< n_k >= \frac{1}{e^{\hbar\omega_k/\tau} - 1}.$$

Putting this all together and switching mode labels from k to n when we integrate gives

$$\begin{aligned} \langle x^2 \rangle &= \sum_k \frac{E_k}{Nm\omega_k^2} \\ &= \sum_k \frac{\hbar\omega_k}{Nm\omega_k^2} \left(n_k + \frac{1}{2} \right) \\ &= \frac{3}{8} \int_0^{n_m = n_D} 4\pi n^2 dn \left(\frac{\hbar}{Nm\omega_n} \right) \left(\frac{1}{e^{\hbar\omega_n/\tau} - 1} + \frac{1}{2} \right) \\ &\equiv \int_0^{\omega_D} d\omega_n D(\omega_n) \left(\frac{\hbar}{Nm\omega_n} \right) \left(\frac{1}{e^{\hbar\omega_n/\tau} - 1} + \frac{1}{2} \right) \end{aligned} \tag{2}$$

In the last line we wrote this as an integral over ω, where we had to include the density of states defined by

$$D(\omega_n)\, d\omega_n \equiv \frac{3}{8} 4\pi n^2 dn. \tag{3}$$

We know that $\omega_n = n\pi v/L$, and solving this for $n(\omega_n)$ and plugging into the right hand side of (**??**) we obtain:

$$D(\omega_n) = \frac{3\pi}{2} \left(\frac{L}{\pi v} \right)^3 \omega_n^2.$$

Now that we have changed integration variables from n to ω_n, our expression (**??**) for $\langle x^2 \rangle$ becomes:

$$\begin{aligned} \langle x^2 \rangle &= \int_0^{\omega_D} d\omega_n \left[\frac{3\pi}{2} \left(\frac{L}{\pi v} \right)^3 \omega_n^2 \right] \left(\frac{\hbar}{Nm\omega_n} \right) \left(\frac{1}{e^{\hbar\omega_n/\tau} - 1} + \frac{1}{2} \right) \\ &= \frac{3\pi}{2} \left(\frac{L}{\pi v} \right)^3 \left(\frac{\hbar}{Nm} \right) \int_0^{\omega_D} d\omega_n \omega_n \left(\frac{1}{e^{\hbar\omega_n/\tau} - 1} + \frac{1}{2} \right). \end{aligned}$$

(d) At high temperatures $T \gg \Theta = \frac{\hbar v}{k_B}\left(\frac{6\pi^2 N}{V}\right)^{1/3}$ we can approximate the integral:

$$
\begin{aligned}
< x^2 > &= \frac{3\pi}{2}\left(\frac{L}{\pi v}\right)^3\left(\frac{\hbar}{Nm}\right)\int_0^{\omega_D} d\omega_n \omega_n \left(\frac{1}{e^{\hbar\omega_n/\tau}-1}+\frac{1}{2}\right) \\
&\approx \frac{3\pi}{2}\left(\frac{L}{\pi v}\right)^3\left(\frac{\hbar}{Nm}\right)\int_0^{\omega_D} d\omega_n \omega_n \left(\frac{1}{(1+\frac{\hbar\omega_n}{\tau}+\cdots)-1}+\frac{1}{2}\right) \\
&= \frac{3\pi}{2}\left(\frac{L}{\pi v}\right)^3\left(\frac{\hbar}{Nm}\right)\int_0^{\omega_D} d\omega_n \omega_n \left(\frac{\tau}{\hbar\omega_n}+\frac{1}{2}\right) \\
&\approx \frac{3\pi}{2}\left(\frac{L}{\pi v}\right)^3\left(\frac{\hbar}{Nm}\right)\frac{\tau}{\hbar}\omega_D = 9\frac{\hbar^2 T}{mk_B\theta^2}.
\end{aligned}
$$

(e)

$$
\begin{aligned}
x_{\text{rms}} = \sqrt{\langle x^2\rangle} &= \sqrt{9\frac{\hbar^2 T}{mk_B\theta^2}} \approx \frac{1}{10}a \\
\Rightarrow T_m &\approx \frac{mk_B\theta^2}{900\hbar^2}a^2
\end{aligned}
$$

(f) In the low temperature limit, the exponential is very large, so only the zero point energy will contribute to the integral:

$$
\begin{aligned}
< x^2 > &= \frac{3\pi}{2}\left(\frac{L}{\pi v}\right)^3\left(\frac{\hbar}{Nm}\right)\int_0^{\omega_D} d\omega_n \omega_n \left(\frac{1}{e^{\hbar\omega_n/\tau}-1}+\frac{1}{2}\right) \\
&\approx \frac{3\pi}{2}\left(\frac{L}{\pi v}\right)^3\left(\frac{\hbar}{Nm}\right)\frac{1}{2}\frac{1}{2}\omega_D^2 \\
&= \frac{9\hbar^2}{4mk_B\Theta}.
\end{aligned}
$$

(g) For Neon, $\Theta \approx 75K$, so

$$
\langle x^2\rangle = \sqrt{\frac{9\hbar^2}{4mk_B\theta}} \approx 2.7 * 10^{-11} \text{ m}
$$

Lattice spacings are on the order of Angstroms($= 10^{-10}$ m), so we're in the right ballpark.

Problem 5: Adiabatic Demagnetization

(a) We've seen the paramagnet many times before. If we know the one particle partition function Z_1, then we know that for the entire system of N spins the partition function is $Z_N = Z_1^N$ (there's no factor of $N!$ because the spins are all localized on sites of a lattice, and are therefore distinguishable). Thus we calculate:

$$
\begin{aligned}
Z_1 &= e^{mB/\tau} + e^{-mB/\tau} = 2\cosh(mB/\tau) \\
\Rightarrow Z_N &= (2\cosh(mB/\tau))^N.
\end{aligned}
$$

The free energy is then

$$F(\tau) = -\tau \ln Z_N = -\tau N \ln\left(2\cosh(mB/\tau)\right).$$

We can now expand to second order in mB/τ:

$$\begin{aligned} F &\approx -\tau N \ln\left(2 + \frac{m^2B^2}{\tau^2}\right) \approx -\tau N \left(\ln 2 + \frac{m^2B^2}{2\tau^2}\right) \\ \Longrightarrow \sigma_{\text{spins}} &= -\frac{\partial F}{\partial \tau}\bigg|_V = N\left(\ln 2 - \frac{m^2B^2}{2\tau^2}\right). \end{aligned} \tag{4}$$

(b) The above expression (**??**) gives the entropy as a function of B and τ, so in the limit as $B \to 0$ we are holding τ constant. In taking this limit, then, we should imagine the system in contact with a reservoir at temperature τ. Now, as $B \to 0$ the energy cost $2mB$ of increasing entropy by flipping a spin from aligned to anti-aligned becomes very low, and so it becomes entropically favorable for the reservoir to give up energy (and entropy) to the system for this purpose. In the $B \to 0$ limit this will maximize the entropy of the system, giving $\sigma = \ln \binom{N}{N/2} \approx N \ln 2$, which agrees with the $B \to 0$ limit of (**??**).

(c) To find the entropy of the lattice vibrations, we use the expression for energy in the Debye approximation in the low temperature limit

$$U(\tau) \approx \frac{3\pi^4 N \tau^4}{5(k_b\theta)^3}$$

and calculate:

$$\begin{aligned} \frac{\partial \sigma}{\partial \tau}\bigg|_V &= \frac{\partial \sigma}{\partial U}\bigg|_V \frac{\partial U}{\partial \tau}\bigg|_V \\ &= \frac{1}{\tau}\frac{12\pi^4 N\tau^3}{5(k_b\theta)^3} d\tau. \end{aligned}$$

Notice that the Debye temperature $\Theta = \frac{\hbar v}{k_B}\left(\frac{6\pi^2 N}{V}\right)^{1/3}$ is a function of volume but not temperature, so when he hold volume constant it is a constant. Now we can integrate the above expression from $\tau' = 0 \to \tau$ and (using the fact the we expect $\sigma(\tau = 0) = 0$) we have:

$$\sigma_{\text{lattice}} = \frac{4\pi^4 N}{5(k_B\theta)^3}\tau^3.$$

The total entropy is the sum of the entropy of the lattice and the spins:

$$\sigma_{\text{total}} = \sigma_{\text{spins}} + \sigma_{\text{lattice}} = N\left(\ln 2 - \frac{m^2B^2}{2\tau^2}\right) + \frac{4\pi^4 N}{5(k_B\theta)^3}\tau^3.$$

(d) In a reversible process, the total entropy remains constant (if the total entropy increased, then the entropy would decrease in the reverse process, which is forbidden by the Second Law), so when we turn off the magnetic field reversibly, we have the relation:

$$\begin{aligned} \sigma_i &= N\left(\ln 2 - \frac{m^2B^2}{2\tau_i^2}\right) + \frac{4\pi^4 N}{5(k_B\theta)^3}\tau_i^3 = \sigma_f = N\ln 2 + \frac{4\pi^4 N}{5(k_B\theta)^3}\tau_f^3 \\ \Longrightarrow \tau_f &= \left(\tau_i^3 - \frac{15\hbar^3 v^3 m^2 B^2}{4\pi^2 \tau_i^2}\frac{N}{V}\right)^{1/3}. \end{aligned} \tag{5}$$

Thus the temperature will decrease. When we lower the magnetic field, as discussed in part (b), we induce heat flow from the reservoir to the spins. Here, though, the lattice plays the role of the reservoir; it is not infinite, however, and so its temperature will decrease as heat flows out, which is what we see in (**??**). This technique is used all the time in low temperature physics experiments, probably including Prof. Holzapfel's lab.

Problem 6: Black Hole Evaporation

(a) Since Hawking calculated the entropy of a black hole, it's natural to define the temperature of a black hole in the usual way: $\frac{1}{T} \equiv \frac{\partial S}{\partial U}$. Using Hawking's expression for the entropy of a black hole and the relationship between the Schwarzschild radius and the mass $R_s = 2GM/c^2$, we can write the entropy in terms of the energy $U = Mc^2$:

$$\begin{aligned} S &= \frac{k_B c^3 \left(4\pi R_s^2\right)}{4G\hbar} = \frac{4\pi G k_B M^2}{\hbar c} = \frac{4\pi G k_B U^2}{\hbar c^5} \\ \Longrightarrow \frac{1}{T} &\equiv \frac{\partial S}{\partial U} = \frac{8\pi G k_B U}{\hbar c^5} \\ \Longrightarrow T &= \frac{\hbar c^3}{8\pi G k_B M} \end{aligned} \tag{6}$$

By the way, it might seems strange to you that an extensive property of a black hole like entropy is proportional to the *area* of a black hole not its volume (we normally calculate energy or entropy densities, so the total energy or entropy is proportional to volume). This observation raises all kinds of interesting questions about the fundamental nature of entropy and information, and is the central idea behind what is called the *holographic principle*. Raphael Bousso, who is on our faculty, is a pioneer in this emerging field.

(b) Now that we've assigned a temperature to black holes, we might wonder if they radiate thermally like a blackbody. This is exactly what Hawking found (using quantum field theory on a curved spacetime): black holes radiate with a blackbody spectrum according the Stefan Boltzmann law. This should seem strange, considering the usual notion that nothing can escape from a black hole.

If a black hole radiates like a blackbody, then the total power radiated should be the blackbody flux times the Schwarzschild area:

$$\begin{aligned} \text{power radiated} &= -\frac{dU}{dt} = -\frac{dM}{dt}c^2 = \sigma_B T^4 * 4\pi \left(\frac{2GM}{c^2}\right)^2 \\ M^2 dM &= -dt \frac{\hbar c^4}{3 * 5120\pi G^2} \\ \Rightarrow M(t) &= \left(M(0)^3 - t\frac{\hbar c^4}{5120\pi G^2}\right)^{1/3}. \end{aligned}$$

Indeed a black hole will lose mass through its thermal radiation. Notice that as a black hole evaporates, its mass and therefore its area decrease; from Hawking's expression for entropy this means that the entropy of the black hole will decrease as it evaporates. We now have to start worrying about the Second Law of Thermodynamics: is the total entropy decreasing, or is it simply carried away by the thermal radiation? This was at the center of a famous bet that Hawking recently conceded.

(c) Using the above expression, we can calculate the expected lifetime of a black hole of mass $M(0) = 2 \times 10^{11}$ kg:

$$
\begin{aligned}
t_e &= \frac{5120\pi G^2}{\hbar c^4} M(0)^3 = \frac{5120\pi * (6.67 \times 10^{-11}\ \mathrm{N} \cdot \mathrm{m}^2/\mathrm{Kg}^2)^2}{1.05 * 10^{-34}\ J \cdot \mathrm{sec} * (3 * 10^8\ \mathrm{m/sec})^4} (2 \times 10^{11}\ \mathrm{kg})^3 \\
&\approx 7 * 10^{17}\ \mathrm{s} \sim 20 \text{ billion years.}
\end{aligned}
$$

This is slightly longer than the age of the universe, which is about 14 billion years.

(d) For a black hole to evaporate in the presence of background radiation, the flux it radiates must be greater than the flux it absorbs. By Stefan-Boltzmann this means $T_{\mathrm{BH}} > T_b$, and by (**??**) we have

$$
\begin{aligned}
\frac{\hbar c^3}{8\pi G k_B M} &> T_b \\
\implies M &< \frac{\hbar c^3}{8\pi G k_B T_b}.
\end{aligned}
$$

Evaluating this for the CMB at $T = 2.73\mathrm{K}$, we get

$$
\begin{aligned}
M &< \frac{(1.05 * 10^{-34})(3 * 10^8)^3}{8\pi(6.67 * 10^{-11})(1.381 * 10^{-23}) * 2.73} \\
&= 4.5 * 10^{22}\ \mathrm{kg}.
\end{aligned}
$$

The mass of the sun is about $2 * 10^{30}$ kg, so only fairly small black holes will evaporate.