

# Physics 112 Problem Set 4

Holzappel, Section 102

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1. (a) Starting from the spectral intensity per unit wavelength,

$$I_\lambda = \frac{2h\nu^5}{c^3(\exp(h\nu/kT) - 1)},$$

we substitute  $x = h\nu/kT$  and differentiate to find the frequency at which the intensity (per wavelength) peaks

$$0 = \frac{dI_\lambda}{dx} \implies e^x(5 - x) = 5 \implies x = 4.965.$$

This enables us to calculate the temperature of the sun

$$T_S = \frac{h\nu_{peak}}{4.965 \times k} = \frac{hc}{4.965 \times k\lambda_{peak}} = 6037 \text{ K}.$$

Now, for a cone of apex angle  $2\theta$ , the solid angle is

$$\Omega = \int_0^{2\pi} \int_0^\theta \sin \theta \, d\theta d\phi = 2\pi(1 - \cos \theta) \approx \pi\theta^2$$

for  $\theta \ll 1$ . So the proportion of solar flux that reaches the Earth is  $\pi\theta^2/4\pi = \theta^2/4$ , where  $2\theta$  is the angle subtended by the sun,  $0.5^\circ$ .

Since the flux emitted by the sun must be proportional to the fourth power of its temperature (by the Stefan-Boltzmann law) and the flux absorbed by the Earth must likewise be proportional to the fourth power of its temperature, we have

$$T_E^4 = \frac{\theta^2}{4} T_S^4$$

or

$$T_E = \sqrt{\frac{(0.5^\circ/2) \times (\pi/180^\circ)}{2}} T_S = 282 \text{ K}.$$

This is approximately  $9^\circ \text{ C}$ , which seems far too low, especially since the sun is not illuminating the Earth at all hours of the day.

- (b) If we assume that all the radiation from the sun absorbed by the Earth is re-emitted and absorbed by the atmosphere, and that the atmosphere subsequently re-emits this radiation equally in both directions (towards and away from the Earth), then the flux at the surface of the Earth is  $1.5 \times$  the flux emitted by the sun that reaches the Earth, which gives

$$T_E = \left(\frac{3}{2}\right)^{1/4} \sqrt{\frac{(0.5^\circ/2) \times (\pi/180^\circ)}{2}} T_S = 312 \text{ K},$$

or about  $40^\circ \text{ C}$ .

2. (a) The multiplicity is

$$g(N_2) = \frac{N!}{N_2!(N - N_2)!}.$$

- (b) The entropy is

$$\sigma = \log g = \log(N!) - \log(N_2!) - \log((N - N_2)!),$$

and using the Stirling approximation for large  $N$ ,

$$\begin{aligned} \sigma &= N \log N - N - (N_2 \log N_2 - N_2 + (N - N_2) \log(N - N_2) - (N - N_2)) \\ \sigma &= N \log N - N_2 \log N_2 - (N - N_2) \log(N - N_2). \end{aligned}$$

So

$$\frac{\partial \sigma}{\partial N_2} = -\log N_2 - 1 + 1 + \log(N - N_2) = \log\left(\frac{N - N_2}{N_2}\right) = \log\left(\frac{N}{N_2} - 1\right)$$

But  $U = N_2 \Delta$ , so  $\partial \sigma / \partial U = (1/\Delta)(\partial \sigma / \partial N_2)$ , and since  $1/\tau = \partial \sigma / \partial U$ ,

$$\begin{aligned} \frac{1}{\tau} &= \frac{1}{\Delta} \log\left(\frac{N}{N_2} - 1\right) \\ \exp\left(\frac{\Delta}{\tau}\right) + 1 &= \frac{N}{N_2} \\ N_2 &= \frac{N}{1 + \exp(\Delta/\tau)}, \end{aligned}$$

thus

$$U = \frac{N \Delta}{1 + \exp(\Delta/\tau)}.$$

In the limit  $\tau \rightarrow 0$ ,  $U \rightarrow 0$ , as expected. In the limit  $\tau \rightarrow \infty$ ,  $U \rightarrow N \Delta / 2$ , which implies that half the systems are in the excited state.

(c) For the canonical ensemble, the partition function for a single system is

$$Z_1 = \sum_s E_s \exp(-E_s/\tau) = 1 + \exp(-\Delta/\tau).$$

Thus the expected energy for a single system is

$$U_1 = \frac{\Delta \exp(-\Delta/\tau)}{1 + \exp(-\Delta/\tau)} = \frac{\Delta}{1 + \exp(\Delta/\tau)},$$

and the expected energy for  $N$  identical systems is

$$U = \frac{N\Delta}{1 + \exp(\Delta/\tau)},$$

which agrees with the result from the microcanonical ensemble.

3. (a) A single mode, indexed by  $n$ , at frequency  $\omega_n$ , with  $s$  photons has energy  $\epsilon_n = s\hbar\omega_n$ . Thus the partition function for a single mode is

$$Z_n = \sum_{s=0}^{\infty} \exp(-s\hbar\omega_n/\tau) = [1 - \exp(-\hbar\omega_n/\tau)]^{-1}.$$

Then the partition function for the gas with  $n$  modes is

$$Z = \prod_n [1 - \exp(-\hbar\omega_n/\tau)]^{-1}.$$

The Helmholtz free energy is then

$$F = -\tau \log Z = \tau \sum_{n=0}^{\infty} \log [1 - \exp(-\hbar\omega_n/\tau)].$$

To convert a sum over modes into an integral over frequencies, we need the density of photons at each frequency,

$$\rho_{\omega} d\omega = \frac{V\omega^2}{c^3\pi^2} d\omega,$$

thus

$$F = \frac{V\tau}{c^3\pi^2} \int_0^{\infty} \omega^2 \log [1 - \exp(-\hbar\omega/\tau)] d\omega.$$

Substituting  $x = \hbar\omega/\tau$ ,

$$F = \frac{V\tau}{c^3\pi^2} \left(\frac{\tau}{\hbar}\right)^3 \int_0^\infty x^2 \log(1 - e^{-x}) dx = \frac{V\tau^4}{c^3\hbar^3\pi^2} \left[ \frac{1}{3} \log(1 - e^{-x}) x^3 \Big|_0^\infty - \frac{1}{3} \int_0^\infty \frac{x^3 e^{-x}}{1 - e^{-x}} dx \right]$$

$$F = -\frac{V\tau^4}{3c^3\hbar^3\pi^2} \int_0^\infty \frac{x^3}{e^x - 1} dx = -\frac{V\tau^4}{3c^3\hbar^3\pi^2} \frac{\pi^4}{15} = -\frac{V\tau^4\pi^2}{45c^3\hbar^3},$$

where I've looked up the value of the dimensionless integral.

- (b) The radiation pressure is then

$$p = -\frac{\partial F}{\partial V} = \frac{\tau^4\pi^2}{45c^3\hbar^3} = \frac{u}{3},$$

where  $u$  is the energy density of the photon gas,  $u = \tau^4\pi^2/15c^3\hbar^3$ . Compare this to the monatomic ideal gas, where  $p = 2u/3$ . The radiation pressure is greater for the ideal gas than the photon gas.

4. (a) The flux incident on the middle plane is  $J_u + J_l$ , and the flux emitted from the middle plane is  $2J_m$ , so

$$\sigma_B(T_u^4 + T_l^4) = 2\sigma_B T_m^4,$$

or

$$T_m = \left[ \frac{1}{2}(T_u^4 + T_l^4) \right]^{1/4}.$$

Now, the flux between the upper plane and the middle plane is

$$J_{net} = J_u - J_m = \sigma_B T_u^4 - \frac{\sigma_B}{2} (T_u^4 + T_l^4) = \frac{\sigma_B}{2} (T_u^4 - T_l^4),$$

which is half of the flux without the middle plane. Likewise, the flux between the middle plane and the lower plane is

$$J_{net} = J_m - J_l = \frac{\sigma_B}{2} (T_u^4 + T_l^4) - \sigma_B T_l^4 = \frac{\sigma_B}{2} (T_u^4 - T_l^4).$$

- (b) The radiation from the upper surface will bounce back and forth, and with each successive reflection, the flux will be reduced by a factor of  $(1 - \epsilon)$ . However, since the radiation alternates direction as it bounces back and forth, the even-indexed terms will contribute positive flux and the odd-indexed terms will contribute negative flux, so the net flux due to this surface is

$$F_u = \sigma_B T_u^4 [\epsilon - \epsilon(1 - \epsilon) + \epsilon(1 - \epsilon)^2 - \dots] = \epsilon \sigma_B T_u^4 \sum_{n=0}^{\infty} (-1)^n (1 - \epsilon)^n = \left( \frac{\epsilon}{2 - \epsilon} \right) \sigma_B T_u^4.$$

Symmetrically, for the lower surface, only the odd-indexed terms will contribute positive flux. So the net flux due to this surface is

$$F_l = \sigma_B T_l^4 [-\epsilon + \epsilon(1 - \epsilon) - \epsilon(1 - \epsilon)^2 + \dots] = -\epsilon \sigma_B T_l^4 \sum_{n=0}^{\infty} (-1)^n (1 - \epsilon)^n = -\left(\frac{\epsilon}{2 - \epsilon}\right) \sigma_B T_l^4.$$

Thus, the net flux in the middle region is

$$F = \left(\frac{\epsilon}{2 - \epsilon}\right) \sigma_B (T_u^4 - T_l^4).$$

5. (a) Consider photon modes indexed by  $j$ . Each mode has  $s_j$  photons of energy  $\hbar\omega_j$ . Then the total energy is

$$U = \sum_j s_j \hbar\omega_j,$$

thus the pressure is

$$p = -\frac{\partial U}{\partial V} = -\sum_j s_j \hbar \frac{\partial \omega_j}{\partial V}.$$

Consider a cube of side length  $L$ , volume  $L^3$ . Imposing periodic boundary conditions, a half-integer number of wavelengths must fit in the side length, so  $\lambda_j = 2L/j$ , so the (angular) frequency is  $\omega_j = 2\pi c/\lambda_j = j\pi c/L = j\pi c V^{-1/3}$ . Then

$$\frac{\partial \omega_j}{\partial V} = -\frac{j\pi c}{V^{4/3}} = -\frac{\omega_j}{3V}.$$

The pressure is then

$$p = \frac{1}{3V} \sum_j s_j \hbar\omega_j = \frac{U}{3V} = \frac{u}{3},$$

as previously calculated in problem 3(b).

- (b) See 3(c).
6. (a) Starting from the fundamental thermodynamic relation  $dU = \tau d\sigma - p dV$ , we divide by  $d\tau$ . Holding  $V$  constant implies  $dV = 0$ , thus

$$\left(\frac{\partial U}{\partial \tau}\right)_V = \tau \left(\frac{\partial \sigma}{\partial \tau}\right)_V,$$

or  $C_V/\tau = (\partial\sigma/\partial\tau)_V$ .

(b)

$$\left(\frac{\partial \sigma}{\partial V}\right)_\tau = -\frac{\partial^2 F}{\partial \tau \partial V},$$

and

$$\left(\frac{\partial p}{\partial \tau}\right)_V = -\frac{\partial^2 F}{\partial V \partial \tau},$$

thus

$$\left(\frac{\partial\sigma}{\partial V}\right)_\tau = \left(\frac{\partial p}{\partial\tau}\right)_V.$$

(c) Starting from

$$\frac{\partial^2\sigma}{\partial V\partial\tau} = \frac{\partial^2\sigma}{\partial\tau\partial V},$$

we see that

$$\frac{\partial^2\sigma}{\partial\tau\partial V} = \frac{\partial}{\partial\tau} \left(\frac{\partial\sigma}{\partial V}\right)_\tau = \frac{\partial^2 p}{\partial\tau^2} = \frac{1}{3} \frac{\partial^2 u}{\partial\tau^2},$$

since  $p = u/3$ . Also,

$$\frac{\partial^2\sigma}{\partial V\partial\tau} = \frac{\partial}{\partial V} \left(\frac{\partial\sigma}{\partial\tau}\right)_V = \frac{\partial}{\partial V} \left(\frac{C_V}{\tau}\right) = \frac{1}{\tau} \frac{\partial^2 U}{\partial V\partial\tau} = \frac{1}{\tau} \frac{\partial u}{\partial\tau},$$

since  $\partial U/\partial V = u$ . Now  $u$  is independent of volume, so we can write this as a second-order ordinary differential equation

$$\frac{\partial u}{\partial\tau} = \frac{\tau}{3} \frac{\partial^2 u}{\partial\tau^2}.$$

We see that  $u(\tau) \sim \tau^4$  is a solution to this differential equation, as the Stefan-Boltzmann law shows.