

Physics 112 Problem Set 6

Holzapfel, Section 102

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1. The centrifugal force on an atom of the gas is $F_c(r) = M\omega^2 r$, so the potential energy due to the rotation is $U_c(r) = -M\omega^2 r^2/2$. Let $\mu_{ext}(r) = U_c$, and note that $\mu_{int}(r) = \tau \log(n(r)/n_Q)$ for an ideal gas. Then, $\mu(r) = \mu_{ext} + \mu_{int}$ needs to be constant in the cylinder, so

$$\begin{aligned}\mu(0) &= \mu(r) \\ \tau \log\left(\frac{n(0)}{n_Q}\right) &= -\frac{M\omega^2 r^2}{2} + \tau \log\left(\frac{n(r)}{n_Q}\right) \\ \log\left(\frac{n(r)}{n(0)}\right) &= \frac{M\omega^2 r^2}{2\tau} \\ n(r) &= n(0) \exp\left(\frac{M\omega^2 r^2}{2\tau}\right).\end{aligned}$$

2. The potential difference is given by

$$\begin{aligned}\mu_{inside} - \mu_{outside} \\ \tau \log\left(\frac{n_{inside}}{n_Q}\right) - \tau \log\left(\frac{n_{outside}}{n_Q}\right) \\ \tau \log\left(\frac{n_{inside}}{n_{outside}}\right).\end{aligned}$$

At 300 K, this is approximately

$$k \times (300 \text{ K}) \times \log(10^4) = 3.8 \times 10^{-20} \text{ J} = 0.24 \text{ eV}$$

and since each K^+ atom has charge e , this corresponds to an energy difference of 0.24 V.

3. (a) If 90% of the sites are occupied by oxygen, then the probability of a single Hb site being occupied by oxygen is 0.9, so

$$\begin{aligned}0.9 &= \frac{\lambda(O_2) \exp(-\epsilon_A/\tau)}{1 + \lambda(O_2) \exp(-\epsilon_A/\tau)} \\ 0.9 &= 0.1 \lambda(O_2) \exp(-\epsilon_A/\tau) \\ \epsilon_A &= \tau \log\left(\frac{\lambda(O_2)}{9}\right)\end{aligned}$$

which, for $T = 37 + 273 = 310 \text{ K}$ and $\lambda(O_2) = 10^{-5}$, evaluates to $\epsilon_A = -5.87 \times 10^{-20} \text{ J} = -0.366 \text{ eV}$.

(b) Now, we admit carbon monoxide, and if 10% of the sites are occupied by oxygen, then

$$\begin{aligned}
0.1 &= \frac{\lambda(\text{O}_2) \exp(-\epsilon_A/\tau)}{1 + \lambda(\text{O}_2) \exp(-\epsilon_A/\tau) + \lambda(\text{CO}) \exp(-\epsilon_B/\tau)} \\
0.1 + 0.1\lambda(\text{CO}) \exp(-\epsilon_B/\tau) &= 0.9\lambda(\text{O}_2) \exp(-\epsilon_A/\tau) \\
\exp(-\epsilon_B/\tau) &= \frac{9\lambda(\text{O}_2) \exp(-\epsilon_A/\tau) - 1}{\lambda(\text{CO})} \\
\epsilon_B &= \tau \log \left(\frac{\lambda(\text{CO})}{9\lambda(\text{O}_2) \exp(-\epsilon_A/\tau) - 1} \right)
\end{aligned}$$

which, for $T = 310$ K, $\lambda(\text{CO}) = 10^{-7}$ and $\lambda(\text{O}_2)$ and ϵ_A as previously, evaluates to $\epsilon_B = -8.78 \times 10^{-20}$ J = -0.548 eV.

4. (a) If

$$\mathcal{Z} = \sum_{ASN} \exp \left(\frac{N\mu - \epsilon}{\tau} \right)$$

then

$$\frac{\partial^2 \mathcal{Z}}{\partial \mu^2} = \sum_{ASN} \frac{N^2}{\tau^2} \exp \left(\frac{N\mu - \epsilon}{\tau} \right).$$

So

$$\frac{\tau^2}{\mathcal{Z}} \frac{\partial^2 \mathcal{Z}}{\partial \mu^2} = \frac{\sum_{ASN} N^2 \exp((N\mu - \epsilon)/\tau)}{\mathcal{Z}} = \langle N^2 \rangle.$$

(b) If

$$\langle N \rangle = \frac{\tau}{\mathcal{Z}} \left(\frac{\partial \mathcal{Z}}{\partial \mu} \right)_{\tau, V}$$

then

$$\tau \frac{\partial \langle N \rangle}{\partial \mu} = \tau^2 \left(-\frac{1}{\mathcal{Z}^2} \left(\frac{\partial \mathcal{Z}}{\partial \mu} \right)_{\tau, V}^2 + \frac{1}{\mathcal{Z}} \left(\frac{\partial^2 \mathcal{Z}}{\partial \mu^2} \right)_{\tau, V} \right) = \langle (\Delta N)^2 \rangle.$$

5. The water vapor at the roots must be in diffusive equilibrium with the water vapor at the uppermost leaves. Note that $\mu_{ext} = Mgh$ and $\mu_{int} = \tau \log(n/n_Q)$. So

$$\begin{aligned}
\mu(0) &= \mu(h) \\
\tau \log(n(0)/n_Q) &= \tau \log(n(h)/n_Q) + Mgh \\
\tau \log(n_0/rn_0) &= Mgh \\
h &= -\frac{kT}{Mg} \log r.
\end{aligned}$$

For $T = 298$ K, $r = 0.9$, $M = 18$ amu = 2.99×10^{-26} kg, this evaluates to $h = 1480$ m.

6. Starting from the Sackur-Tetrode equation

$$\sigma = N \log \left(\frac{V}{N^{5/2}} \left(\frac{4\pi m U}{3\hbar^2} \right)^{3/2} \right) + \frac{5}{2} N$$

and differentiating with respect to N ,

$$\begin{aligned} \left(\frac{\partial \sigma}{\partial N} \right)_{U,V} &= \log \left(\frac{V}{N^{5/2}} \left(\frac{4\pi m U}{3\hbar^2} \right)^{3/2} \right) + N \frac{N^{5/2}}{V} \left(\frac{3\hbar^2}{4\pi m U} \right)^{3/2} V \left(\frac{4\pi m U}{3\hbar^2} \right)^{3/2} \left(\frac{-5}{2} \right)^{-7/2} + \frac{5}{2} \\ \left(\frac{\partial \sigma}{\partial N} \right)_{U,V} &= \log \left(\frac{1}{n} \left(\frac{4\pi m U}{3\hbar^2 N} \right)^{3/2} \right) - \frac{5}{2} + \frac{5}{2}. \end{aligned}$$

For an ideal gas, $U = 3N\tau/2$, so

$$\left(\frac{\partial \sigma}{\partial N} \right)_{U,V} = \log \left(\frac{1}{n} \left(\frac{2\pi m \tau}{\hbar^2} \right)^{3/2} \right) = \log \left(\frac{n_Q}{n} \right)$$

and

$$\mu = -\tau \left(\frac{\partial \sigma}{\partial N} \right)_{U,V} = \tau \log \left(\frac{n}{n_Q} \right).$$