Physics 112 Problem Set 8

Holzapfel, Section 102

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1. (a) Since $\epsilon = \hbar^2 k^2/2m$ and $k = n\pi/L$ from the periodic boundary condition, $\epsilon = \hbar^2\pi^2 n^2/2mL^2$, or

$$n = \frac{L}{\pi\hbar}\sqrt{2m\epsilon}.$$

Then, since n-space is one-dimensional, the total number of electrons is $N = 2n$, since each orbital can hold one spin up and one spin down electron. So

$$N = \frac{2L}{\pi\hbar}\sqrt{2m\epsilon}$$

thus

$$\mathcal{D}(\epsilon) = \frac{dN}{d\epsilon} = \frac{L}{\pi\hbar}\sqrt{\frac{2m}{\epsilon}}.$$

(b) We know $k_x = n_x\pi/L_x$ and $k_y = n_y\pi/L_x$, and $k^2 = k_x^2 + k_y^2$, so $\epsilon = \hbar^2\pi^2 n^2/2mA$ where $n^2 = n_x^2 + n_y^2$ and $A = L_x L_y$. This gives

$$n = \frac{1}{\pi\hbar}\sqrt{2mA\epsilon}.$$

Assuming n-space is circularly symmetric, the total number of electrons is $N = (1/4)\pi n^2$, where the factor of $1/4$ comes from considering only positive quantum numbers. Thus

$$N = \frac{Am}{\pi\hbar^2}\epsilon$$

and

$$\mathcal{D}(\epsilon) = \frac{dN}{d\epsilon} = \frac{Am}{\pi\hbar^2}.$$

2. (a) If $\epsilon \simeq pc$ and $p = (\pi\hbar/L)n$ where $n = \sqrt{n_x^2 + n_y^2 + n_z^2}$, then $\epsilon \simeq (\pi\hbar c/L)n$. Then the Fermi energy ϵ_F occurs at some value for the quantum number n_F, where $\epsilon_F = (\pi\hbar c/L)n_F$.

In the ground state, every possible orbital is filled with two electrons (spin up and spin down) up until the quantum number n_F. So

$$N = 2\left(\frac{1}{8}\right)\int_0^{n_F} 4\pi n^2\, dn = 2\left(\frac{1}{8}\right)\left(\frac{4\pi}{3}\right)n_F^3$$

where the factor of two comes from the spin degeneracy and the factor of 1/8 comes from the fact that we're only considering the positive octant of (spherically symmetric) n-space. So

$$n_F = \left(\frac{3N}{\pi}\right)^{1/3}$$

thus

$$\epsilon_F = \frac{\pi\hbar c}{L}\left(\frac{3N}{\pi}\right)^{1/3} = \pi\hbar c\left(\frac{3N}{\pi V}\right)^{1/3} = \pi\hbar c\left(\frac{3n}{\pi}\right)^{1/3}$$

where $n = N/V$ and $V = L^3$.

(b) By the same reasoning,

$$U_0 = 2\left(\frac{1}{8}\right)\int_0^{n_F} \epsilon(n)\, 4\pi n^2\, dn = \frac{\pi^2\hbar c}{L}\int_0^{n_F} n^3\, dn = \frac{\pi^2\hbar c}{4L} n_F^4$$

and substituting our expression for n_F

$$U_0 = \frac{\pi^2\hbar c}{4L}\left(\frac{3N}{\pi}\right)^{4/3} = \frac{\pi^2\hbar c}{4}\left(\frac{3N}{\pi}\right)\left(\frac{3n}{\pi}\right)^{1/3} = \frac{3}{4}N\epsilon_F.$$

3. (a) ^{3}He is made of two protons and one neutron, so the mass of a single ^{3}He atom is $m = (2\times 1.673 + 1.675)\times 10^{-27} = 5.021\times 10^{-27}$ kg.

   The mass density is given as 0.081 g cm^{-3}, and dividing by the mass, this gives $n = 1.613\times 10^{28}$ m^{-3}.

   So $\epsilon_F = (\hbar^2/2m)(3\pi^2 n)^{2/3} = 6.767\times 10^{-23}$ J $= 0.422$ meV.

   The average velocity is given by $\epsilon_F = (1/2)mv_F^2$, so $v_f = \sqrt{2\epsilon_F/m} = 164$ m/s.

   The Fermi temperature is simply $T_F = \epsilon_F/k_B = 4.90$ K.

   (b) We showed in class that the heat capacity at low temperature is

$$C_{el} = \frac{\pi^2}{2}N\left(\frac{k_B T}{T_F}\right)$$

   which evaluates to $C_{el} = 1.01Nk_bT$, the same order as the experimental value but lower, perhaps because we've failed to account for the heat capacity due to phonon modes.

4. (a) The virial theorem states that for an inverse-square force law (as in the case of gravitational force), $2\langle T\rangle = -\langle U_g\rangle$, where $\langle T\rangle$ is the average thermal kinetic energy.

   We found $U_0 = (3/4)N\epsilon_F = (3/4)N\pi\hbar c(3N/\pi V)^{1/3}$ in the ground state, and we will assume that we are trying to find the minimum possible N, so $\langle T\rangle = U_0$. Approximating the box used to derive the expression for ϵ_F as a sphere,

$$U_0 = \frac{3N\pi\hbar c}{4}\left(\frac{3N}{\pi V}\right)^{1/3} = \frac{3\pi\hbar c}{4R}\left(\frac{9N^4}{4\pi^2}\right)^{1/3}.$$

For a sphere of mass M and radius R, we showed in class that the internal energy due to gravity is $U_g = -3GM^2/5R$. If we assume the whole star is ionized hydrogen, then $M = Nm_H$, so $U_g = -3GN^2m_H^2/5R$.

Thus

$$2\frac{3\pi\hbar c}{4R}\left(\frac{9N^4}{4\pi^2}\right)^{1/3} = \frac{3GN^2m_H^2}{5R}$$

or, rearranging,

$$N = \left(\frac{5\pi\hbar c}{2Gm_H^2}\left(\frac{9}{4\pi^2}\right)^{1/3}\right)^{3/2}.$$

(b) With $m_H = 1.67 \times 10^{-27}$ kg, this yields $N = 2.31 \times 10^{58}$.

5. (a) The magnetic moment is $M = \mu_B(N_{up} - N_{down})/V$. Assuming the gas is in its ground state,

$$N(\epsilon_F) = \frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon_F^{3/2}.$$

Now, the magnetic field perturbs the energy by $\pm\mu_B B$ depending on whether the spin is up or down. So we can calculate the number of electrons in the spin up state (note the additional factor of 1/2 because the previous calculation was for both spin up and spin down electrons)

$$N_{up} = \frac{V}{6\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(\epsilon_F + \mu_B B)^{3/2}$$

and the number in the spin down state is

$$N_{down} = \frac{V}{6\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(\epsilon_F - \mu_B B)^{3/2}.$$

So

$$N_{up} - N_{down} = \frac{V}{6\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon_F^{3/2}\left[\left(1 + \frac{\mu_B B}{\epsilon_F}\right)^{3/2} - \left(1 - \frac{\mu_B B}{\epsilon_F}\right)^{3/2}\right]$$

and since $B \ll \epsilon_F$, $\left(1 \pm \frac{\mu_B B}{\epsilon_F}\right)^{3/2} \approx 1 \pm \frac{3\mu_B B}{2\epsilon_F}$, so

$$N_{up} - N_{down} = \frac{3}{2}\frac{V}{6\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon_F^{3/2}\left(\frac{2\mu_B B}{\epsilon_F}\right) = \mu_B B\mathcal{D}(\epsilon_F) = \mu_0\mu_B\mathcal{D}(\epsilon_F)H.$$

Then

$$\chi = \frac{\mu_0\mu_B^2\mathcal{D}(\epsilon_F)}{V}.$$

(b) We know the density of states at the Fermi energy is $\mathcal{D}(\epsilon_F) = 3N/2\epsilon_F$, so

$$\chi = \frac{3\mu_0\mu_B^2 n}{2\epsilon_F}.$$

The Bohr magneton is $\mu_B = e\hbar/2m_e = 9.274 \times 10^{-24}$ J/T and the permeability of free space is $\mu_0 = 4\pi \times 10^{-7}$ H/m. Given $n = 8.5 \times 10^{28}$ m^{-3} and $\epsilon_F = 7.0$ eV $= 1.1 \times 10^{-18}$ J, this evaluates to $\chi = 1.2 \times 10^{-5}$.

6. (a)

$$N = \int_0^\infty d\epsilon \mathcal{D}(\epsilon) f(\epsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\epsilon^{1/2}}{\exp((\epsilon-\mu)/\tau)+1} d\epsilon.$$

(b) With $\lambda = \exp(\mu/\tau)$,

$$N = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \epsilon^{1/2} \frac{\lambda\exp(-\epsilon/\tau)}{1+\lambda\exp(-\epsilon/\tau)} d\epsilon \simeq \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \epsilon^{1/2}\lambda\exp(-\epsilon/\tau)(1-\lambda\exp(-\epsilon/\tau)) d\epsilon$$

or

$$N \simeq \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \epsilon^{1/2} \exp\left(\frac{\mu-\epsilon}{\tau}\right)\left(1-\exp\left(\frac{\mu-\epsilon}{\tau}\right)\right) d\epsilon.$$

(c) Dividing by V and calculating the integral,

$$\begin{aligned}
n &\simeq \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left[\int_0^\infty \lambda\epsilon^{1/2}\exp(-\epsilon/\tau)d\epsilon - \int_0^\infty \lambda^2\epsilon^{1/2}\exp(-2\epsilon/\tau)d\epsilon\right] \\
n &\simeq \frac{1}{2\pi^2}\left(\frac{2m\tau}{\hbar^2}\right)^{3/2}\left(\lambda - \frac{\lambda^2}{2^{3/2}}\right)\int_0^\infty x^{1/2}\exp(-x)dx \\
n &\simeq 2\left(\frac{m\tau}{2\pi\hbar^2}\right)^{3/2}\left(\lambda - \frac{\lambda^2}{2^{3/2}}\right) \\
\frac{n}{2n_Q} &= \lambda - \frac{\lambda^2}{2^{3/2}}
\end{aligned}$$

where the approximation is to second order in λ, and I've looked up the value of the dimensionless integral.

(d) Starting from

$$\frac{n}{2n_Q} = \lambda\left(1 - \frac{\lambda}{2^{3/2}}\right),$$

we assume $\lambda \ll 1$ and use the approximation $(1-x)^{-1} \approx 1 + x$ for small x to get

$$\lambda = \frac{n}{2n_Q}\left(1 + \frac{\lambda}{2^{3/2}}\right).$$

Then

$$\lambda\left(1-\frac{n}{2^{5/2}n_Q}\right)=\frac{n}{2n_Q}$$

and once again, assuming that $n/n_Q \ll 1$,

$$\lambda=\frac{n}{2n_Q}\left(1+\frac{n}{2^{3/2}n_Q}\right).$$

Then

$$\mu=\tau\log\left(\frac{n}{2n_Q}\right)+\tau\log\left(1+\frac{n}{2^{3/2}n_Q}\right)$$

and since $\log(1+x)\approx x$ for small x,

$$\mu=\tau\left(\log\left(\frac{n}{2n_Q}\right)+\frac{n}{2^{3/2}n_Q}\right).$$

(e) From last week's assignment, we know that $P=2U/3V$. So

$$P=\frac{2}{3V}\int_0^\infty d\epsilon\mathcal{D}(\epsilon)f(\epsilon)\epsilon=\frac{2}{6\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty\frac{\epsilon^{3/2}}{\exp((\epsilon-\mu)/\tau)+1}d\epsilon.$$

(f) Making the same approximations as in (b),

$$PV=\frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left[\int_0^\infty\lambda\epsilon^{3/2}\exp(-\epsilon/\tau)d\epsilon-\int_0^\infty\lambda^2\epsilon^{3/2}\exp(-2\epsilon/\tau)d\epsilon\right]$$
$$PV=\frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left(\lambda-\frac{\lambda^2}{2^{5/2}}\right)\tau^{5/2}\int_0^\infty x^{3/2}\exp(-x)dx$$
$$PV=\frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left(\lambda-\frac{\lambda^2}{2^{5/2}}\right)\tau^{5/2}\frac{3\sqrt{\pi}}{4}$$
$$PV=\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left(\lambda-\frac{\lambda^2}{2^{5/2}}\right)\tau^{5/2}\sqrt{\pi}$$

to second order in λ.

(g) From (d),

$$\lambda=\exp(\mu/\tau)=\frac{n}{2n_Q}\exp\left(\frac{n}{2^{3/2}n_Q}\right)$$

and since $n\ll n_Q$, we can use $\exp(x)\approx 1+x$ to get

$$\lambda=\frac{n}{2n_Q}+\frac{1}{2^{3/2}}\left(\frac{n}{2n_Q}\right)^2.$$

So, substituting this into our result from (f) and keeping only terms to second order in n/n_Q,

$$PV = 2V\tau n_Q \left(\lambda - \frac{1}{2^{5/2}}\lambda^2\right)$$
$$PV = 2V\tau n_Q \left(\frac{n}{2n_Q} + \frac{1}{2^{3/2}}\left(\frac{n}{2n_Q}\right)^2 - \frac{1}{2^{5/2}}\left(\frac{n}{2n_Q}\right)^2\right)$$
$$PV = 2V\tau n_Q \left(\frac{n}{2n_Q} + \frac{1}{2^{5/2}}\left(\frac{n}{2n_Q}\right)^2\right)$$
$$PV = 2V\tau n_Q \left(\frac{n}{2n_Q}\right)\left(1 + \frac{1}{2^{7/2}}\left(\frac{n}{n_Q}\right)\right)$$
$$PV = N\tau \left(1 + \frac{1}{2^{7/2}}\left(\frac{n}{n_Q}\right)\right).$$

This is equivalent to

$$PV = N\tau \left[1 + \frac{N\hbar^3\pi^{3/2}}{4V(m\tau)^{3/2}}\right].$$