Physics 112 Fall 2017
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Homework 1 Solution

Problem 1. Oski's ice cream parlor

In this problem we enumerate the ways in which we fill a bowl with four scoops of ice cream from a selection of seven different flavors.

a) Here we allow duplication of flavors and care about the order that scoops go in the bowl. We can choose each of m scoops from n flavors. Therefore, there are are $n^m = 7^4 = 2401$ ways to fill the bowl.

b) Here we do not allow duplication of flavors and care about the order that scoops go in the bowl. The first scoop can be selected from n flavors, the second scoop from n-1 flavors, and so on with the last scoop being selected from the remaining (n-m+1) flavors. Therefore, number of scoop combinations is:

$$n(n-1)(n-2)...(n-m+1) = \frac{n!}{(n-m)!} = \frac{7!}{3!} = 840$$

.

c) Here we do not allow duplication of flavors and do not care about the order that scoops go into the bowl. This is similar to part (b) except that we need to account for the fact that there are m! different orders that can result in the same scoops in the bowl. Since we don't care about the order, we need to divide the result of part (b) by m!. Therefore, the number of scoop combinations is:

$$\frac{n!}{(n-m)!m!} = \frac{7!}{4!3!} = 35$$

.

d) Here we allow duplication of flavors and do not care about the order that scoops go into the bowl. As suggested in the hint, you can solve this problem by considering the process of filling the bowl as a series of operations. We start with the first flavor and draw as many scoops as we want and then move on to the next flavor where we again draw as many scoops as we want until we have visited each flavor in turn. We can enumerate the ways of doing this by assigning unique characters to the operations of scooping (0) or moving to the next flavor (1). Examples of possible sequences would be "0110011101" or "1000011111". The number of ways to fill the bowl is then simply the number of unique sequences. For m scoops drawn from n flavors, there are m scoops (0) and n-1 moves (1) in the possible sequences. Therefore, the number of ways to arrange the symbols and the number of unique scoop combinations is:

$$\frac{(n+m-1)!}{(n-1)!m!} = \frac{10!}{6!4!} = 210$$

.

Problem 2. Sharpness of multiplicity function.

The probability $P(n)$ of getting n heads is the number of configurations that have n heads divided by the total number of configurations, or

$$\begin{aligned} P(n) &= \frac{1}{2^N}\binom{N}{n} \\ &= \frac{1}{2^N}\frac{N!}{(N-n)!\,n!} \\ &\approx \frac{1}{2^N}\frac{\sqrt{2\pi N}N^N e^{-N}}{\sqrt{2\pi(N-n)}(N-n)^{(N-n)}e^{-(N-n)}\sqrt{2\pi n}n^n e^{-n}}. \end{aligned} \tag{1}$$

a) The probability of getting 5000 heads and 5000 tails is thus

$$\begin{aligned} P(5000) &\approx \frac{1}{2^{10000}}\frac{\sqrt{2\pi 10000}(10000)^{10000}e^{-10000}}{\sqrt{2\pi 5000}(5000)^{5000}e^{-5000}\sqrt{2\pi 5000}(5000)^{5000}e^{-5000}} \\ &= \frac{1}{\sqrt{2\pi 50}} = 0.00798. \end{aligned}$$

b) The relative probability $P(5100)/P(5000)$ is

$$\begin{aligned} \frac{P(5100)}{P(5000)} &= \frac{\binom{10000}{5100}}{\binom{10000}{5000}} = \frac{5000!5000!}{5100!4900!} \\ &\approx \frac{5000}{\sqrt{5100*4900}}\frac{(5000)^{10000}}{(5100)^{5100}(4900)^{4900}} \\ &= \frac{5000}{\sqrt{5100*4900}}\left(\frac{50}{51}\right)^{5100}\left(\frac{50}{49}\right)^{4900} = 0.135. \end{aligned}$$

c) The relative probability $P(6000)/P(5000)$ is

$$\begin{aligned} \frac{P(6000)}{P(5000)} &= \frac{5000!5000!}{6000!4000!} \\ &\approx \frac{5000}{\sqrt{6000*4000}}\frac{(5000)^{10000}}{(6000)^{6000}(4000)^{4000}} \\ &= \frac{5}{\sqrt{24}}\left(\frac{5}{6}\right)^{6000}\left(\frac{5}{4}\right)^{4000} = 3.6\times 10^{-88}. \end{aligned}$$

d) With 10 coins

$$\begin{aligned} \frac{P(6)}{P(5)} &= \frac{\binom{10}{6}}{\binom{10}{5}} \\ &= \frac{10!}{6!4!}\frac{5!5!}{10!} \\ &= 5/6. \end{aligned}$$

Problem 3. Poisson Distribution.

a) In this part, you are asked to justify the use of the binomial distribution

$$P(n) = \binom{N}{n} p^n (1-p)^{N-n} = \frac{N!}{(N-n)!\,n!}\, p^n (1-p)^{N-n} \tag{2}$$

to describe an the probability that an event P characterized by probability p occurs n times in N trials.

The probability that P occurs in a single trial is p. The probability that P does not occur ("Not P") in a single trial is $(1-p)$. Now consider N trials. The probability that you get n outcomes of event P in N trials requires that you also get $(N-n)$ outcomes of Not P. Since the trials are independent, the probabilities multiply. For a particular microstate (this corresponds to running through N trials once), you get a factor of p for every time you get an event P, and you get a factor of $(1-p)$ every time you get Not P. This explains the factor of $p^n(1-p)^{N-n}$ in $P(n)$ - it gives the probability of a single microstate in which P occurs n times. Now we just have to count the number of such microstates, but this amounts to choosing n trials out of the N total, so the number of such microstates is just $\binom{N}{n}$. Putting this all together gives (**??**).

It's worthwhile to note that if each trial consisted of flipping a coin then we'd have $p = 1/2$, which when plugged into (**??**) just gives (**??**)! Thus, (**??**) generalizes the result of problem 1 to the case where p and Not p are not equally likely.

b) Now we want to show that the binomial distribution becomes the Poisson distribution when $n \ll N$, $p \ll 1$, and $\lambda = Np$. More precisely, we are taking the limits $p \to 0$ and $N \to \infty$ where $\lambda = Np$ stays constant and n is fixed. Using the Stirling approximation for $N!$ and $(N-n)!$ (but not for $n!$) we get

$$\begin{aligned} P(n) &\approx \frac{\sqrt{2\pi N} N^N e^{-N}}{n!\sqrt{2\pi(N-n)}(N-n)^{N-n} e^{-(N-n)}} p^n (1-p)^n \\ &= \frac{e^{-n}(Np)^n (1-p)^{(N-n)} (1-n/N)^{n-1/2}}{n!(1-n/N)^N} \end{aligned} \tag{3}$$

after some algebra. Now, in the limits we're taking we have

$$\begin{aligned} \left(1 - \frac{n}{N}\right)^N &\longrightarrow e^{-n} \\ \left(1 - \frac{n}{N}\right)^{n-1/2} &\longrightarrow 1 \\ (1-p)^{N-n} = \left(1 - \frac{Np}{N}\right)^N (1-p)^{-n} &\longrightarrow e^{-\lambda}, \end{aligned}$$

and substituting these into (**??**) yields

$$P(n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

as desired.

Problem 4. Stirling approximation.

a) Using $N! \approx \sqrt{2\pi N} N^N e^{-N}$ we can rewrite the multiplicity function

$$\begin{aligned} g(N,s) &= \frac{N!}{(N/2+s)!(N/2-s)!} \\ &\approx \frac{\sqrt{2\pi N} N^N}{\sqrt{(2\pi)^2(N/2+s)(N/2-s)}(N/2+s)^{N/2+s}(N/2-s)^{N/2-s}} \\ &= 2^N\sqrt{\frac{2}{\pi N}}\left(1-4\frac{s^2}{N^2}\right)^{-N/2}\frac{(1-2s/N)^{s-1/2}}{(1+2s/N)^{s+1/2}}. \end{aligned}$$

b) To compare the above result with the expression $g(N,s) \approx 2^N\sqrt{\frac{2}{\pi N}}e^{-\frac{2s^2}{N}}$, we have the ratio

$$\frac{\left(1-4\frac{s^2}{N^2}\right)^{-N/2}\frac{(1-2s/N)^{s-1/2}}{(1+2s/N)^{s+1/2}}}{e^{-\frac{2s^2}{N}}}.$$

We get the following numerical values for the ratio:

$$\begin{array}{cccl} N=10, & s=1 & \longrightarrow & 1.01 \quad \text{case}(1), \\ N=1000, & s=100 & \longrightarrow & 0.89 \quad \text{case}(2), \\ N=1000, & s=10 & \longrightarrow & .99 \quad \text{case}(3). \end{array}$$

c) The derivation in part a) only requires Stirling's approximation for the factorials which is valid when $N \gg 1$ and $(N/2-s) \gg 1$. The gaussian approximation we derived in lecture requires additionally that $2s/N \ll 1$ for the Taylor expansion of log. To see the subtle difference we can consider the case N=3000, s=1000: $(N/2-s) = 1000 \gg 1$ so part a) is a valid approximation, but $2s/N = 2/3 \approx 1$ so clearly our Gaussian approximation breaks down. We see in b) that the expressions agree the best for large N and small s (neither approximation should be valid in case (1) so the good agreement may be a coincidence).

Problem 5. The approach to equilibrium.

First we convince ourselves that the equation

$$P(m,s) = \frac{R+m+1}{2R}P(m+1,s-1) + \frac{R-m+1}{2R}P(m-1,s-1) \tag{4}$$

is true. If Rover has m excess fleas at step s then at step $s-1$ Rover either had $m-1$ fleas or $m+1$ fleas. The probability of Rover having $m+1$ fleas at step $s-1$ is $P(m+1,s-1)$, and the probability that one of these fleas is the one that moves at that step is $\frac{R+m+1}{2R}$. Multiplying these gives the first term in (**??**). To get the other term, remember that Rover also has probability $P(m-1,s-1)$ of having $m-1$ fleas at step $s-1$, and to get m fleas in this case one of the fleas on Spot has to move, which has probability $\frac{R-(m-1)}{2R}$. Multiplying these together and adding to the first term gives (**??**).

a) We compute:

$$\begin{aligned} < m(s) > &= \sum_{m=-R}^{R} mP(m,s) \\ &= \sum_{m=-R}^{R} \frac{m(R+m+1)}{2R} P(m+1,s-1) + \sum_{m=-R}^{R} \frac{m(R-m+1)}{2R} P(m-1,s-1) \end{aligned}$$

and defining new dummy indices $m' \equiv m+1$ in the first sum and $m' \equiv m-1$ in the second then gives

$$< m(s) >= \sum_{m'=-R+1}^{R+1} \frac{(m'-1)(R+m')}{2R} P(m',s-1) + \sum_{m'=-R-1}^{R-1} \frac{(m'+1)(R-m')}{2R} P(m',s-1).$$

Our limits of summation look a little screwy, but it's not as bad as it seems. Note that $P(R+1,s-1) = P(-R-1,s-1) = 0$ since $-R \le m \le R$, and so I can omit these terms from the sum. Also note also that I can *add* an $m' = -R$ term to the first sum because that term will actually be zero, and can similarly add an $m' = R$ term to the second sum. Switching dummy indices back to m and adjusting the limits of summation this way gives

$$\begin{aligned} < m(s) > &= \sum_{m=-R}^{R} \left(\frac{(m-1)(R+m)}{2R} + \frac{(m+1)(R-m)}{2R} \right) P(m,s-1) \\ &= \left(\frac{R-1}{R} \right) \sum_{m=-R}^{R} mP(m,s-1) = \left(1 - \frac{1}{R} \right) < m(s-1) > . \end{aligned}$$

Alternatively, we may (sneakily) argue that

$$\begin{aligned} < m(s) > &= < m(s-1) > + 1 \times (\text{ average probability that } m \text{ increases by 1}) \\ & \quad -1 \ \times (\text{average probability that } m \text{ decreases by 1}) \\ &= < m(s-1) > + \frac{R + < m(s-1) >}{2R} - \frac{R - < m(s-1) >}{2R} \\ &= \left(1 - \frac{1}{R} \right) < m(s-1) > . \end{aligned}$$

b)

$$\begin{aligned} < m(s) > &= \left(1 - \frac{1}{R} \right) < m(s-1) >= \left(1 - \frac{1}{R} \right)^2 < m(s-2) > \\ &= \cdots = \left(1 - \frac{1}{R} \right)^s < m(0) >= n \left(1 - \frac{1}{R} \right)^s , \end{aligned}$$

since initially Rover has n excess fleas.

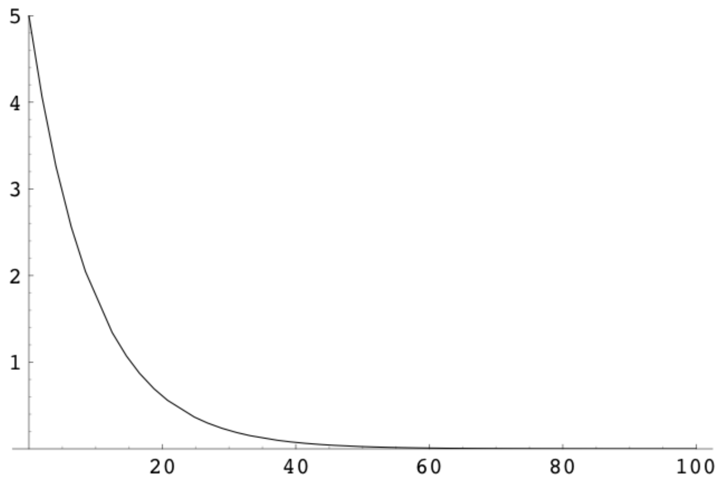

Figure 1: This is the plot for the case $R = 10, n = 5$. The y-axis is $< m(s) >$ and the x-axis is s.

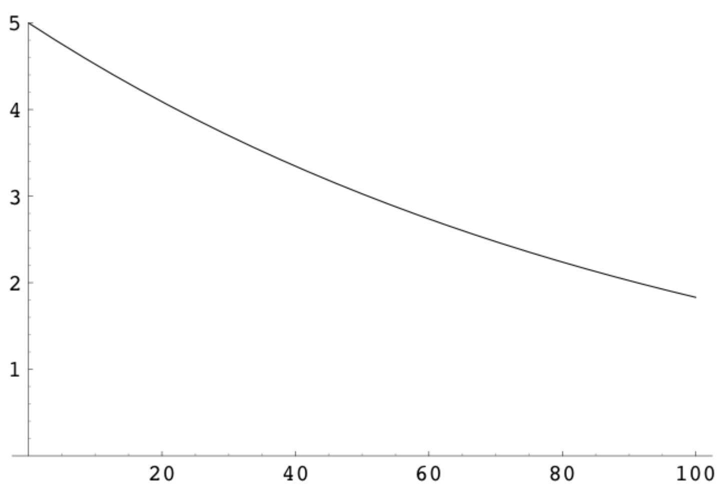

Figure 2: This is the plot for the case $R = 100, n = 5$. The y-axis is $< m(s) >$ and the x-axis is s.

c) See Figures 1 and 2. Clearly it takes more steps to reach equilibrium ($m \approx 0$) when the total number of fleas is increased. This makes sense if you think about the ratio n/R – in reality Figure 2 actually starts much closer to equilibrium (farther down the exponential curve), so it takes longer to drop by the same proportion.

d) Equilibrium occurs when there are the same number of fleas on both dogs: $m \approx 0$. Because fleas are counted in discrete units we can consider the dogs to be in equilibrium when $-1 < (< m(s) >) < 1$. We then have

$$\begin{aligned} < m(s) > &= n\left(1 - \frac{1}{R}\right)^s < 1 \\ &\Rightarrow \log(n) + s\log(1 - \frac{1}{R}) < 0 \\ &\Rightarrow s > -\frac{\log n}{\log(1 - 1/R)} \approx \frac{\log n}{1/R} = R\log n, \end{aligned}$$

where we use that fact that $\log(1 - 1/R)$ is negative.

Problem 6. Diffusion.

a) By symmetry it's clear that $\langle z \rangle = 0$ since it is just as likely to go in the positive z direction as the negative z direction. This can also be seen from the following integral, where using $z = R\cos\theta$ we have

$$\langle z \rangle = \frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} R\cos\theta\sin\theta d\theta d\phi = 0, \tag{5}$$

Here we have kept R as a variable, though of course later we'll set it to 10^{-5} meters.

b) The RMS is given by finding the average value of z^2 and taking its square root. We begin with the mean value of z^2:

$$\begin{aligned} \langle z^2 \rangle &= \frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} R^2\cos^2\theta\sin\theta d\theta d\phi \quad (6) \\ &= \frac{R^2}{4\pi}\left(\int_0^{2\pi} d\phi\right)\left(\int_{-1}^1 u^2 du\right) \quad (7) \\ &= \frac{R^2}{3}, \quad (8) \end{aligned}$$

using in the second line the substitution $u = \cos(\theta)$. We then have

$$z_{\text{rms}} = \sqrt{\langle z^2 \rangle} = \frac{R}{\sqrt{3}} = 5.8 \times 10^{-6}\text{meters} \tag{9}$$

c) Since motion in any direction is equally likely, at all times the average displacement of the molecules in any direction will be zero.

To calculate the standard deviation, we need the RMS value of the displacement after 2 seconds. At this point, a typical molecule will have experienced $N = 2\times10^7$ collisions. We know that on average, the square of the distance that it travels between collisions in the z-direction is $R^2/3$, and we know that it is equally likely to move in the positive or negative z-direction. Therefore, we consider the motion of a molecule in the z-direction to be a one-dimensional random walk, with probability distribution

$$P(N,s) = \frac{g(N,s)}{2^N} = \frac{N!}{(\frac{N}{2}-s)!(\frac{N}{2}+s)!}2^{-N}, \tag{10}$$

where $g(N,s)$ is the multiplicity function, equal to the binomial distribution by analogy to flipping coins/random spins. Here $2s$ is the net distance from the z-axis, and so the displacement squared in the z-direction corresponding to a given state is

$$D_z^2 = (2s)^2 \times \frac{R^2}{3}. \tag{11}$$

Our distribution is heavily weighted towards the region $s \ll N$, and we are dealing with very large $N \gg 1$, so we may use the Gaussian approximation of our probability distribution function:

$$P(N,s) \approx \sqrt{\frac{2}{N\pi}}\exp(-2s^2/N). \tag{12}$$

We may use this to calculate the average z-displacement squared:

$$\begin{aligned} \langle D_z^2(N)\rangle &= \int\left(\frac{4R^2s^2}{3}\right)P(N,s)ds &(13)\\ &= \left(\frac{4R^2}{3}\right)\sqrt{\frac{2}{N\pi}}\int s^2\exp(-2s^2/N)ds &(14)\\ &= \frac{NR^2}{3} &(15)\end{aligned}$$

The standard deviation is then

$$\begin{aligned} \sigma &= \sqrt{\langle D_z^2\rangle - \langle D_z\rangle^2} &(16)\\ &= \sqrt{\frac{NR^2}{3} - 0} &(17)\\ &= \sqrt{\frac{N}{3}}R = 2.6\times10^{-2}\text{meters}. &(18)\end{aligned}$$

(d) We wish to find when 32 percent of the probability lies beyond 6 meters, which corresponds to the 6 meter mark lying at one standard deviation from the origin. We have already found the standard deviation,

$$\sigma = \sqrt{\frac{N}{3}}R \tag{19}$$

Setting this equal to 6 meters, we find

$$N = 3\left(\frac{6}{R}\right)^2 = 1.08\times10^{12}. \tag{20}$$

This corresponds to a time of

$$\frac{(1.08 \times 10^{12})}{10^7 s^{-1}} = 1.08 \times 10^5 \text{ seconds} = 30 \text{ hours}. \tag{21}$$

Problem 7. The meaning of "never".

a) The probability of each character in the sequence being correct is 1/44. So the probability of getting the entire sequence of 10^5 characters is

$$\left(\frac{1}{44}\right)^{100000} = 10^{-164345}. \tag{22}$$

Here we used the trick $\log_{10}(44) = 1.64345$.

b) A single monkey will type $10^{18} \times 10 = 10^{19}$ characters by the end of the universe. A correct version of Hamlet can start at virtually any point in that sequence, so there are approximately 10^{19} possibilities to start Hamlet correctly. Thus the total probability of typing Hamlet is approximately:

$$\begin{aligned} \text{Probability to have monkey Hamlet} &\approx 10^{-164345} \times 10^{19} \text{ tries/monkey} \times 10^{10} \text{ monkeys} \\ &\approx 10^{-164316}. \end{aligned}$$

NOTE: The above argument gives the right answer is but is extremely flawed logically. Say we wanted to know what the probability was of getting the sequence "up-down" in a sequence of 8 coin tosses. The probability of getting "up-down" in 2 coin tosses is 1/4, but if we multiply that by the seven points of the sequence that might be the beginning of "up-down", we get 7/4 which is greater than 1! This is the same logic we applied to the monkey Hamlet. Without getting into too much detail, the problem in the coin toss case is that there are a non-negligible number of configuration where "up-down" occurs more than once, and we have overcounted them. In the case of Hamlet, the number of configurations where Hamlet occurs more than once is so small that it can be neglected. I belabor this point only to warn you that to do this problem rigorously is more complicated than it seems, and should you encounter similar problems in the future you may have to be more careful.