Physics 112 Problem Set 1

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1. (a) $7^4 = 2401$
   (b) $\frac{7!}{4!} = 840$
   (c) $\frac{7!}{3!4!} = 140$
   (d) $\frac{10!}{4!6!} = 210$
2. (a) $P(5000) = \frac{10000!}{5000!5000!}(0.5)^{10000} = 0.00798$
   (b) $P(5100)/P(5000) = \frac{5000!5000!}{5100!4900!} = 0.135$
   (c) $P(6000)/P(5000) = \frac{5000!5000!}{6000!4000!} = 3.643 \times 10^{-88}$
   (d) $P(6)/P(5) = \frac{5!5!}{6!4!} = 0.833$
3. (a) The probability of a single sequence of N trials with n events occurring is $p^n(1-p)^{N-n}$, and there are $\binom{N}{n}$ sequences of length N with n events, so the total probability of n events occurring is

$$P(n) = \frac{N!}{n!(N-n)!}p^n(1-p)^{N-n}.$$

   (b) With $\lambda = Np$, we have

$$P(n) = \frac{N(n-1)(n-2)\dots(N-n+1)}{n!}\left(\frac{\lambda}{N}\right)^n\left(1-\frac{\lambda}{N}\right)^{N-n}$$
$$P(n) = \frac{N^n + \mathcal{O}(N^{n-1})}{n!}\left(\frac{\lambda}{N}\right)^n\left(1-\frac{\lambda}{N}\right)^{N-n}.$$

For large n, this is approximately

$$P(n) \approx \frac{\lambda^n}{n!}\left(1-\frac{\lambda}{N}\right)^{N-n}.$$

Take the limit as $n \to \infty$, and recall $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$. Then

$$P(n) \approx \frac{\lambda^n}{n!}e^{-\lambda}.$$

4. (a)

$$
\begin{aligned}
g(N,s) &= \frac{N!}{(N/2+s)!(N/2-s)!} \\
g(N,s) &= \frac{\sqrt{2\pi N}N^N e^{-N}}{\sqrt{2\pi(N/2+s)}\,(N/2+s)^{N/2+s}\,e^{-N/2-s}\sqrt{2\pi(N/2-s)}\,(N/2-s)^{N/2-s}\,e^{-N/2+s}} \\
g(N,s) &= \frac{\sqrt{N}N^N}{\sqrt{2\pi}} \frac{1}{N/2\sqrt{(1+2s/N)(1-2s/N)}} \frac{1}{[N/2(1+2s/N)]^{N/2+s}} \frac{1}{[N/2(1-2s/N)]^{N/2-s}} \\
g(N,s) &= \frac{\sqrt{2}N^N}{\sqrt{\pi N}} \frac{1}{(N/2)^N} \frac{1}{[(1+2s/N)(1-2s/N)]^{N/2}} \frac{1}{(1+2s/N)^{1/2+s}} \frac{1}{(1-2s/N)^{1/2-s}} \\
g(N,s) &= 2^N\sqrt{\frac{2}{\pi N}}\left(1-4\frac{s^2}{N^2}\right)^{-N/2} \frac{(1-2s/N)^{s-1/2}}{(1+2s/N)^{s+1/2}}
\end{aligned}
$$

(b) Call the expression from (a) $g_1(N,s)$ and the expression from (b) $g_2(N,s)$. Then

- $g_1(10,1)/g_2(10,1) = 1.019$
- $g_1(1000,100)/g_2(1000,100) = 0.891$
- $g_1(1000,10)/g_2(1000,10) = 1.000$.

(c) These are both approximations to the true multiplicity function, so it's expected that they give similar results. I expect the first function, g_1, to be more accurate, since it only used the assumption that $N \gg 1$, from the Stirling approximation. The function derived in class, g_2, also used this assumption, but added the additional assumption that $s \ll N$, from the Taylor expansion of $\ln(1-2s/N)$. This is why we see better agreement between the two functions for $g(1000,10)$ than for $g(1000,100)$, because the $s \ll N$ assumption is more accurate in the first case. The main limitation of the function in (a), then, is just the requirement that N is large.

5. (a)

$$
\begin{aligned}
\langle m(s)\rangle &= \sum_{m=-R}^{R} mP(m,s) \\
&= \sum_{m=-R}^{R} m\left[\frac{R+m+1}{2R}P(m+1,s-1)+\frac{R-m+1}{2R}P(m-1,s-1)\right].
\end{aligned}
$$

Reindexing the sums,

$$
\langle m(s)\rangle = \sum_{m'=-R+1}^{R+1}(m'-1)\left[\frac{R+m'}{2R}P(m',s-1)\right] + \sum_{m'=-R-1}^{R-1}(m'+1)\left[\frac{R-m'}{2R}P(m',s-1)\right].
$$

We know $-R \le m' \le R$, since there are only $2R$ fleas, so the $m' = R+1$ and $m' = -R-1$ terms are unphysical, and we drop them. Furthermore, we can add the term indexed by $m = -R$ to the first sum and the term indexed by $m' = R$ to the second sum, since the value of these terms is zero. So

$$
\begin{aligned}
\langle m(s)\rangle &= \sum_{m'=-R}^{R}(m'-1)\left[\frac{R+m'}{2R}P(m',s-1)\right] + \sum_{m'=-R}^{R}(m'+1)\left[\frac{R-m'}{2R}P(m',s-1)\right] \\
&= \left[\frac{1}{2}\langle m(s-1)\rangle + \frac{1}{2R}\langle m^2(s-1)\rangle - \frac{1}{2} - \frac{1}{2R}\langle m(s-1)\rangle\right] + \\
&\left[\frac{1}{2}\langle m(s-1)\rangle - \frac{1}{2R}\langle m^2(s-1)\rangle + \frac{1}{2} - \frac{1}{2R}\langle m(s-1)\rangle\right] \\
&= \left(1-\frac{1}{R}\right)\langle m(s-1)\rangle.
\end{aligned}
$$

(b) We will prove this by induction. At the beginning, the excess number of fleas on Rover is n, so $\langle m(0)\rangle = n = n(1-1/R)^0$. So the base case holds.

Assume $\langle m(k)\rangle = n(1-1/R)^k$. Then from the recurrence relation in (a), $\langle m(k+1)\rangle = (1-1/R)\times n(1-1/R)^k = n(1-1/R)^{k+1}$.

So, by induction, $\langle m(s)\rangle = n(1-1/R)^s$ for all integers $s \geq 0$.

Figure 1: $\langle m(s)\rangle$ for $R = 10$

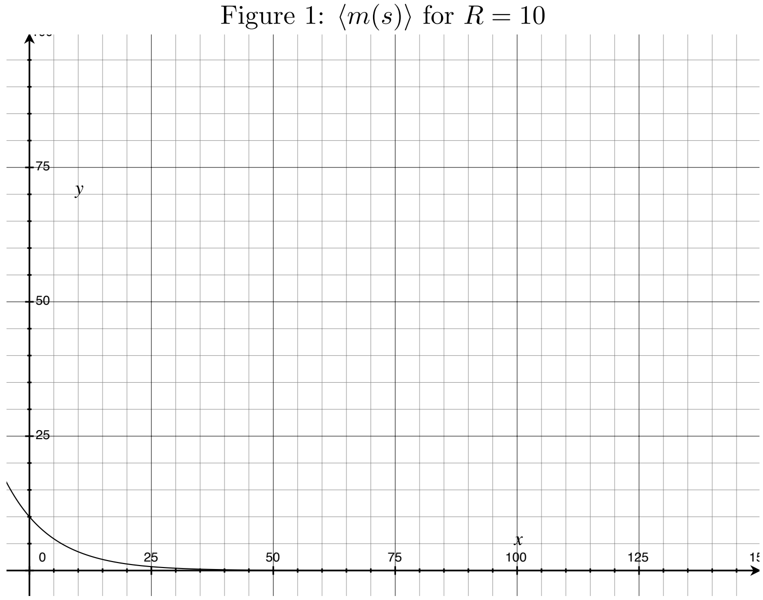
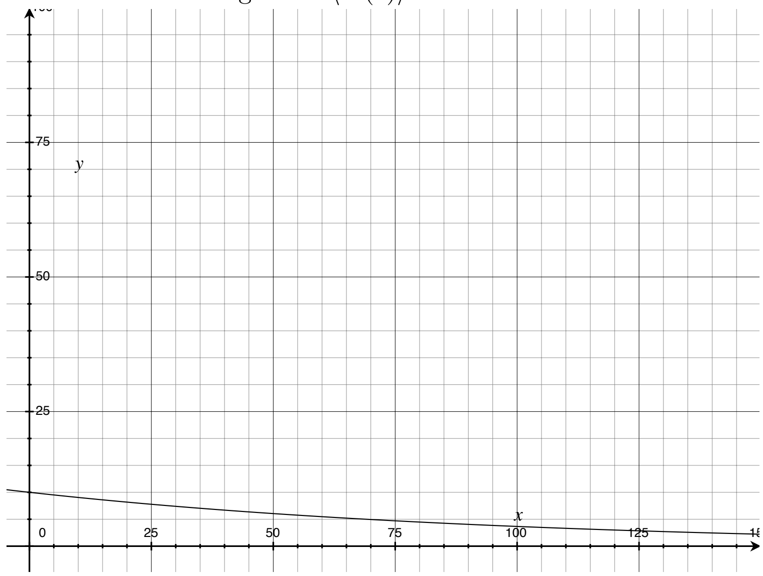


Figure 2: $\langle m(s)\rangle$ for $R = 100$



(c) This behavior is roughly expected – we expect the fleas to be equalized between the two dogs ($\langle m(s)\rangle \to 0$ as $s \to \infty$), and the more fleas they are (larger R), the longer this will take.

(d) For large R, $\langle m(s)\rangle = n\exp(s\ln(1-1/R)) \approx n\exp(-s/R)$. Making the analogy to RC circuits, the characteristic time scale of this exponential decay is R. That is, it takes about R steps for the flea excess to fall off appreciably (to $1/e$). The larger the number of fleas, the longer it will take to reach equilibrium, and this relationship is roughly linear in R.

6. (a) The average path length is 10^{-5} m, and the projection of a path of this length in the z direction is $10^{-5}\cos\theta$. So the average displacement in the z direction, integrating over all solid angles, is

$$\langle z\rangle = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} 10^{-5}\cos\theta\sin\theta\; d\phi d\theta = 0,$$

as expected, since the particles are equally likely to travel in the $+z$ and $-z$ directions.

(b) The same argument applies, so

$$\langle z^2\rangle = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}(10^{-5}\cos\theta)^2\sin\theta\; d\phi d\theta = 3.333\times 10^{-11},$$

thus $\sqrt{\langle z^2\rangle} = 5.774\times 10^{-6}$ m.

(c) In 2 seconds, the ammonia molecules experience $N = 2\times 10^7$ collisions. So we can think of the displacement in the z direction as the sum of 2×10^7 independent random variables, each with mean $\langle z\rangle = 0$ and standard deviation $\sigma = \sqrt{\langle z^2\rangle - \langle z\rangle^2} = 5.774\times 10^{-6}$ m.

So, the average displacement after 2 seconds is $N\langle z\rangle = 0$. The standard deviation is $\sqrt{N}\sigma = \sqrt{2\times 10^7}(5.774\times 10^{-6}) = 0.0258$ m.

(d) The standard deviation after t seconds is $\sqrt{t\times 10^7}(5.774\times 10^{-6})$. When the standard deviation of z is 6 m, approximately 32% of the molecules are further than 6 m from the origin (assuming a Gaussian distribution, which is reasonable given the large number of molecules). This occurs at

$$t = \left(\frac{6}{\sqrt{10^7}(5.774 \times 10^{-6})} \right)^2 = 108000 \text{ s},$$

or 300 hours.

7. (a) $P = \left(\frac{1}{44}\right)^{100000}$, so $\log_{10} P = -10000 \log_{10} 44 = -164345$, thus $P = 10^{-164345}$.

   (b) Typing 10 keys per second, each monkey has typed $10^{18} \times 10 = 10^{19}$ keys since the beginning of the universe. Each key (approximately) represents the start of another 10^5 character sequence. So the 10^{10} monkeys have had $10^{10} \times 10^{19} = 10^{29}$ opportunities to type Hamlet. The probability of any one random sequence resulting in Hamlet is $10^{-164345}$, from (a), so the probability that Hamlet has been typed is $10^{29} \times 10^{-164345} = 10^{-164316}$ (which is still extremely small).