Physics 112, Fall 2017: Holzapfel

Problem Set 4 (6 problems). Due Friday, September 29, 5PM

Problem 1: **Temperature of the Earth & Greenhouse Effect**
As seen from the earth, the sun subtends an angle of $\sim$1/2 degree. The spectral intensity I_λ of the observed radiation from the sun peaks at a wavelength of 480 nm. Assume that both the earth and the sun are perfect blackbodies. Also, assume the the earth maintains a constant temperature over its surface throughout the day/night cycle. Hint: Note that $I_\lambda \neq I_\nu$.

(a) Neglecting the atmosphere, calculate the surface temperature of the earth? Does this temperature seem low or high when compared with your experience.

(b) The greenhouse effect occurs because gasses present in the earths atmosphere such as CO_2 and water vapor are transparent to optical radiation and are opaque to infrared radiation. Light from the sun is absorbed by the earth and then re-emitted in the IR. Rather than escaping as in part (a), this radiation is absorbed by the atmosphere and then re-emitted isotropically. Model the atmosphere by a thin shell at a height h above the earth's surface such that $h << R$, where R is the radius of the earth. Assume that the atmosphere absorbs 100% of the IR radiation. Repeat your calculation for the temperature of the earth in this simple model. Again, compare the temperature you find with what you have observed and comment on the causes of any possible discrepancies.

Problem 2: **The Canonical vs. the Microcanonical Ensemble**

So far in class we've developed two different ways of solving problems in statistical mechanics. The first way considers the system under study as an isolated system of fixed total energy U. One calculates the multiplicity function g and then uses the fundamental theorem of statistical mechanics to find the most probable macrostate as a function of temperature. Often one considers an ensemble of such systems. This is called the "Microcanonical Ensemble". In the microcanonical ensemble, the probability of each microstate is the same.

The second method considers the system under study as coupled to a heat reservoir so that energy is allowed to be exchanged between the two. An ensemble of such systems is called the "Canonical Ensemble". In the canonical ensemble, the probability of each microstate is not the same, and is given by the Boltzmann factor, normalized by the partition function.

This problem is meant to emphasize that we really do have two, consistent ways of solving the same problems. Consider N two-level systems with energies 0 and Δ (Δ is the "energy gap"). The total energy is U.

a) Calculate the multiplicity function g. So that everyone uses the same notation, let N_2 be the number of particles in the excited state.

b) Use g to find the dependence of the total energy on the temperature U(τ). Consider the limits for τ small and large to check if your result makes sense.

c) Now calculate U(τ) in the canonical ensemble. Do your results agree?

Note that there are some subtleties to this problem. We'll come back to this when we consider the effects of distinguishability and indistinguishability in statistical mechanics, later in the semester.

Problem 3: **Free energy of a photon gas**
(a) Kittel 4.7

b) Use your result from part (a) to calculate the pressure of a photon gas as a function of the energy density. Compare your result to that for a monotomic ideal gas.

Problem 4: **Heat shields**
(a) Kittel Problem 4.8

(b) Consider the case in which the surfaces are partially reflective and both absorb and emit radiation with emissivity ε. Show that the net energy flux density between two surfaces with no intervening heat shields is:

$$J_U = \left(\frac{\varepsilon}{2-\varepsilon}\right)\sigma_B(T_u^4 - T_l^4). \tag{1}$$

(Hint: You will have to sum an infinite series to find the absorbed power as the radiation bounces back and forth between the reflecting surfaces)
Using your result form part (a), this can be generalized to the case of N intervening heat shields:

$$J_U = \left(\frac{\varepsilon}{2-\varepsilon}\right)\frac{1}{N+1}\sigma_B(T_u^4 - T_l^4). \tag{2}$$

Filling the space between two bodies you wish to thermally isolate with layers of aluminum coated mylar (called "super-insulation") is a common technique in cryogenics. For achievable values of $N = 10$ and $\varepsilon = 0.05$, the net flux density is reduced by a factor of ~ 400.

Problem 5: **Pressure of Thermal Radiation**
(a) Kittel 4.6

b) Use your result from part (a) to calculate the pressure of a photon gas as a function of the energy density. Compare your result to that for a monotomic ideal gas.

Problem 6: **Classical derivation of Stefan-Boltzmann Law**
We noted in class that the temperature dependence of the energy density of a black body was already known before Planck introduced the method of quantization. In this problem the goal is to derive this temperature dependence, making only a few assumptions. You'll also get some practice manipulating some of the thermodynamic variables we've been discussing.

We make two assumptions. First, we assume that the energy density $u(\tau)=U/V$ is independent of volume. Second, we assume the relation between the pressure and the energy density derived in problem 5.

a) First of all, show that

$$\left(\frac{\partial\sigma}{\partial\tau}\right)_V = \frac{C_v}{\tau} \tag{3}$$

b) We found in class that

$$\sigma = -\left(\frac{\partial F}{\partial\tau}\right)_V \tag{4}$$

and

$$p = -\left(\frac{\partial F}{\partial V}\right)_\tau \tag{5}$$

Now mathematically it must be the case that

$$\frac{\partial^2 F}{\partial V \partial \tau} = \frac{\partial^2 F}{\partial \tau \partial V} \tag{6}$$

Show that this implies that

$$\left(\frac{\partial \sigma}{\partial V}\right)_\tau = \left(\frac{\partial p}{\partial \tau}\right)_V \tag{7}$$

c) It must also be the case that

$$\frac{\partial^2 \sigma}{\partial V \partial \tau} = \frac{\partial^2 \sigma}{\partial \tau \partial V} \tag{8}$$

Show that this gives a differential equation for u which has solution $u \sim \tau^4$, as expected.