Physics 112, Spring 2017: Holzapfel
Problem Set 8 (6 Problems). Friday, Novermber 3, 5 PM

Problem 1: **Density of orbitals in one and two dimensions**
Kittel 7.1

Problem 2: **Energy of a Relativistic Fermi Gas**
Kittel 7.2

Problem 3: **Liquid ^{3}He as a Fermi gas**
Kittel 7.5

Problem 4: **Relativistic White Dwarf Stars**
Kittel 7.10

Problem 5: **Magnetic Susceptibility of a Fermi Gas**
An ideal gas of N spin 1/2 fermions is constrained in a volume V. A small, constant magnetic induction field $B = \mu_0 H$ is applied along the direction so that the energies are

$$\varepsilon = \frac{\hbar^2 k^2}{2m} \pm \mu_B B$$

with the minus sign if the spins are parallel to the field.
a) Show that the magnetic susceptibility is

$$\chi = \frac{M}{H} = \frac{\mu_0 \mu_B^2 D(\varepsilon_F)}{V}$$

where $D(\varepsilon_F)$ is the density of states in energy at the Fermi level.
b) Use the fact that $D(\varepsilon) \sim V\sqrt{\varepsilon}$ to estimate χ for copper, given that $\varepsilon_F = 7.0\,\mathrm{eV}$ and the density of free electrons is $n = 8.5 \times 10^{28}\mathrm{m}^{-3}$.

Problem 6: **Corrections to ideal gas behavior**
In this problem, the goal is to start from the classical limit and derive the first order quantum mechanical corrections for a gas of fermions. The final result of the problem will be the discovery of deviations from the ideal gas law PV=Nτ. You should find that:

$$PV = N\tau \left[1 + \frac{N\hbar^3 \pi^{3/2}}{4V(m\tau)^{3/2}} + \cdots \right]. \tag{1}$$

Thus the inclusion of quantum mechanical exchange effects slightly increases the pressure from the classical value. In other words, for the case of fermions, there exists an effective repulsion between particles as a result of the Pauli exclusion principle.

a) Write down an exact integral expression for N, the number of particles in the system, using the density of states and the Fermi-Dirac distribution function.

b) In the classical limit, we've argued that the density n=N/V is much less than the quantum concentration n_Q. Equivalently, $\lambda = e^{\mu/\tau} << 1$. In this limit, approximate your expression for N to second order in λ. You should find that

$$N \sim \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \varepsilon^{1/2} e^{\frac{\mu-\varepsilon}{\tau}}(1-e^{\frac{\mu-\varepsilon}{\tau}})d\varepsilon. \tag{2}$$

c) Use this result to show that

$$\frac{n}{2n_Q} = \lambda - \frac{\lambda^2}{2^{3/2}} + \cdots \tag{3}$$

You might want to make use of the fact that $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$.

Note that neglecting the term of order λ^2 just gives the result $\lambda = n/2n_Q$ or equivalently,

$$\mu = \tau \log\left(\frac{n}{2n_Q}\right) \tag{4}$$

This is the classical result for a monatomic ideal gas, except for the factor of two. The factor of two is a result of the fact that we've included the two possible spin states for a spin 1/2 particle in the density of states. Kittel discusses how to include internal degrees of freedom such as spin on pages 169-171 (see in particular equation (54)).

d) You showed above that taking only first order terms reproduces the classical result for a monatomic (spin 1/2) ideal gas. Now include the second order term and show that

$$\mu = \tau\left(\log\frac{n}{2n_Q} + \frac{1}{2^{3/2}}\frac{n}{2n_Q} + \cdots\right) \tag{5}$$

Thus, as expected, the chemical potential is a slightly less negative number than the classical result predicts.

e) Now onto the equation of state (the ideal gas law). Write down an exact integral for the pressure of a Fermi gas. You might find it useful to use a result from last week's homework assignment.

f) Make the same approximations as above to show that

$$PV = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\tau^{5/2}\sqrt{\pi}\left[\lambda - \frac{1}{2^{5/2}}\lambda^2 + \cdots\right] \tag{6}$$

Note that $\Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$.

g) Finally, using the results derived in part d, show that this becomes the ideal gas law with the correction term stated in equation 3. Note that this result may also be written as

$$PV = N\tau\left[1 + \frac{1}{2^{7/2}}\left(\frac{n}{n_Q}\right)\cdots\right] \tag{7}$$

thus explicitly demonstrating the key importance of the ratio of the interparticle spacing to the DeBroglie wavelength in determining how important quantum mechanical effects really are.