Physics 112 Fall 2017
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Homework 8 Solutions

Problem 1, Kittel 7.1: Density of Orbitals in one and two dimensions

In one dimension the orbitals are of the form $\psi_n(x) = A\sin(n\pi x/L)$ where n is a positive integer. The energy is

$$\epsilon_n = \frac{\hbar^2}{2m}\left(\frac{\pi n}{L}\right)^2 \tag{1}$$

and the density in n space is simply $2\,dn$, since there is one state per unit interval (the 2 comes from the spin degeneracy). We can invert (**??**) to get

$$\begin{aligned} n(\epsilon) &= \frac{L}{\pi\hbar}\sqrt{2m\epsilon} \\ \Longrightarrow dn &= \frac{L}{2\pi\hbar}\sqrt{\frac{2m}{\epsilon}}d\epsilon. \end{aligned}$$

Since the density of states $D(\epsilon)$ is defined by $D(\epsilon)d\epsilon \equiv 2dn$, we have

$$D(\epsilon) = \frac{L}{\pi\hbar}\sqrt{\frac{2m}{\epsilon}}.$$

(b) Now the orbitals are of the form $\psi_n(x) = A\sin(n_x\pi x/L)\sin(n_y\pi y/L)$, where $n_x, n_y > 0$ and the energy has the same form (**??**), but with $n = \sqrt{n_x^2 + n_y^2}$. The density of states in n-space is

$$\frac{1}{4}\cdot 2\cdot 2\pi n\,dn = \pi n\,dn \tag{2}$$

where the 1/4 is because we're only interested in the first quadrant, the 1/2 is for spin and the $2\pi n dn$ is just the area element in 2D n-space in polar coordinates. As above, we have

$$\begin{aligned} D(\epsilon)d\epsilon &\equiv \pi n\,dn \\ &= \pi\left(\frac{L}{\pi\hbar}\sqrt{2m\epsilon}\right)\cdot\frac{1}{2}\left(\frac{L}{\pi\hbar}\sqrt{\frac{2m}{\epsilon}}\right)d\epsilon \\ &= \frac{mL^2}{\pi\hbar^2}d\epsilon \\ \Longrightarrow D(\epsilon) &= \frac{mL^2}{\pi\hbar^2}. \end{aligned}$$

Problem 2, Kittel 7.2: Energy of a Relativistic Fermi Gas

(a) For a relativistic gas in 3D the orbitals are still of the form

$$\psi_{\vec{n}}(x,y,z) = A\sin(n_x\pi x/L)\sin(n_y\pi y/L)\sin(n_z\pi z/L)$$

but now the energy of the orbitals takes the form

$$\epsilon_n = |\vec{p}|c = \frac{\pi\hbar c}{L}n \tag{3}$$

where $n = \sqrt{n_x^2 + n_y^2 + n_z^2}$. We can still, however, use Kittel's result that in n-space the occupied modes fill up a sphere of radius

$$n_F = \left(\frac{3N}{\pi}\right)^{1/3} \tag{4}$$

since this result is independent of the dispersion relation (??). The Fermi energy is then just

$$\epsilon_F \equiv \epsilon_{n_F} = \frac{\pi\hbar c}{L} n_F = \hbar\pi c\left(\frac{3n}{\pi}\right)^{1/3}.$$

(b) Now using (??) and (??) the ground state energy is easily found:

$$\begin{aligned}
U_0 &= 2 \times \frac{1}{8}\int_0^{n_F} 4\pi n^2 \epsilon_n \, dn \\
&= \frac{\pi^2\hbar c}{L}\int_0^{n_F} n^3 \, dn \\
&= \frac{\pi^2\hbar c}{4L}\left(\frac{3N}{\pi}\right)^{4/3} \\
&= \frac{3}{4}N\epsilon_F.
\end{aligned}$$

Problem 3, Kittel 7.5: Liquid ^3He as a Fermi Gas

(a) The Fermi velocity v_F is defined in terms of the Fermi energy: $\epsilon_F \equiv \frac{1}{2}mv_F^2$. To find the Fermi energy, we need to know the number density; we can find this from the given mass density of ^3He:

$$\begin{aligned}
n &= \frac{\rho_m}{M_{^3He}} = \frac{0.081 \text{ g/cm}^3}{3 \text{ amu}} \\
&= \frac{0.081 \text{ g/cm}^3}{3 \cdot 1.66 \times 10^{-24} g} = 1.63 \times 10^{22}\frac{1}{\text{cm}^3}
\end{aligned}$$

The Fermi velocity is then

$$\begin{aligned}
v_F &= \sqrt{\frac{2\epsilon_f}{m}} = v_F = \sqrt{\frac{2\frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}}{m}} = \frac{\hbar}{m}\left(3\pi^2 n\right)^{1/3} \\
&= \frac{1.06 \times 10^{-27} erg \cdot s}{3 \cdot 1.66 \times 10^{-24} g}\left(3\pi^2 \cdot 1.63 \times 10^{22}\frac{1}{cm^3}\right)^{1/3} \\
&= 1.7 \times 10^4 \text{ cm/s}
\end{aligned} \tag{5}$$

which gives a Fermi energy and temperature of

$$\epsilon_F = \frac{1}{2}mv_F^2 = 7 \times 10^{-23} \text{ J} \tag{6}$$

$$T_F = \epsilon_F / k_b = 5 \text{ K}. \tag{7}$$

(b) The heat capacity of a Fermi gas for $\tau \ll \tau_F$ is given (in conventional units) by

$$C_V = k_b \frac{\pi^2 N}{2} \frac{\tau}{\tau_F} \tag{8}$$

where $\tau_F \equiv \epsilon_F$. Plugging (**??**) into this yields

$$C_V = k_b \cdot \frac{\pi^2}{2} \frac{1}{\tau_F} N k_b T \approx 0.99 N k_b T.$$

This differs from the experimental value by a factor 1/3. The discrepancy is due to the fact that we treated liquid ^3He was a *non-interacting* Fermi gas; in reality a liquid is strongly interacting, and these strong interactions change the effective mass of the ^3He. Because $C_V \sim \frac{1}{\epsilon_F} \sim m$, this suggests that the effective mass of liquid ^3He is about three times that of the mass of a ^3He atom.

Problem 4, Kittel 7.10: Relativistic White Dwarf Stars

(a) The Virial Theorem states that for a system whose potential energy is proportional to the n^{th} power of the displacement $V(r) = Ar^n$, the average kinetic energy is related to the average potential energy by $\langle K \rangle = \frac{n}{2} \langle V \rangle$. So in a gravitational system for which $V(r) = -\frac{GMm}{r}$ we have

$$\langle K \rangle = -\frac{1}{2} \langle V \rangle \, . \tag{9}$$

In problem 2 we found that the kinetic energy of a relativistic Fermi gas (at $T = 0$) is related to the total number of particles:

$$\begin{aligned} < K > = U &= \frac{3}{4} N \epsilon_F \\ &= \frac{3\pi\hbar c}{4} N \left(\frac{3N}{\pi V}\right)^{1/3} \\ &= \frac{3\pi\hbar c}{4} \left(\frac{9}{4\pi^2}\right)^{1/3} \frac{N^{4/3}}{R}. \end{aligned} \tag{10}$$

In lecture (and in problem 7.6) we saw that the gravitational potential energy of a sphere of constant density is:

$$< V > = -\frac{3}{5}\frac{GM^2}{R} = -\frac{3}{5}\frac{GN^2 m_p^2}{R} \tag{11}$$

where we've used the fact that the total mass of the star is related to the number of electrons (which is equal to the number of protons) by $M = N(m_p + m_e) \approx N m_p$. Now combining the virial theorem (**??**) with (**??**) and (**??**) we get

$$\begin{aligned} \frac{3\pi\hbar c}{4} \left(\frac{9}{4\pi^2}\right)^{1/3} \frac{N^{4/3}}{R} &= \frac{1}{2}\frac{3}{5}\frac{GM^2}{R} \\ \Longrightarrow N &= \left(\frac{5}{2}\frac{\hbar c}{G m_p^2}\right)^{3/2} \sqrt{\frac{9\pi}{4}}. \end{aligned}$$

(b) Plugging in the numerical constants, we can calculate the approximate number of atoms needed to form a white dwarf:

$$
\begin{aligned}
N &= \left(\frac{5}{2}\frac{1.05\times10^{-34}\times3\times10^{8}}{6.67\times10^{-11}\times(1.01\times1.66\times10^{-27})^2}\right)^{3/2}\sqrt{\frac{9\pi}{4}} \\
&\approx 2.3\times10^{58}.
\end{aligned}
$$

Problem 5: Magnetic Susceptibility of a Fermi Gas

(a) To start, note that the magnetic susceptibility χ is related in a simple way to the total magnetic moment m:

$$
\chi \equiv \frac{M}{H} = \frac{\mu_0 m}{BV} \tag{12}
$$

where the magnetization $M = \frac{m}{V}$ is the average magnetic moment per unit volume and $B = \mu_0 H$. Thus we just need to find m, which is given by the number of spin up and spin down particles

$$
m_{\text{total}} = \mu_B(N_+ - N_-) \tag{13}
$$

where μ_B is the magnetic moment of a single particle. Taking the B-field to point down, so that the spin up particle has the higher energy, we can write the energy of a spin up/down particle as

$$
\epsilon_\pm = \frac{\hbar^2 k^2}{2m} \pm \mu_B B \equiv \epsilon_k \pm \mu_B B
$$

where $\epsilon_k \equiv \frac{\hbar^2 k^2}{2m}$. The average number of spin up and down particles can then be written as a sum over states weighted by their occupations:

$$
N_+ = \sum_{\vec{k}} f(\epsilon_k + \mu_B B) \tag{14}
$$

$$
N_- = \sum_{\vec{k}} f(\epsilon_k - \mu_B B) \tag{15}
$$

where $f(\epsilon)$ is the Fermi-Dirac distribution with chemical potential $\mu = \epsilon_F$.

We can now turn these sums into integrals over energy as done in the text, and we will get a density of states $D'(\epsilon_k)$. This density of states depends only on ϵ_k and not on $\epsilon_\pm$, since what we're really doing is summing over modes labeled by k, as in (**??**) and (**??**). Furthermore, $D'(\epsilon_k)$ is only *half* the density of states found in the text, because there they multiply by two to take into account spin degeneracy, but in (**??**) and (**??**) each sum is only over a *single* spin state. Putting all this together with (**??**) then gives

$$
m_{\text{total}} = \frac{\mu_B}{2}\int_0^{\epsilon_F} d\epsilon_k D(\epsilon_k) f(\epsilon_k + \mu_B B) - \frac{\mu_B}{2}\int_0^{\epsilon_F} d\epsilon_k D(\epsilon_k) f(\epsilon_k - \mu_B B).
$$

We can now change variables to $\epsilon' = \epsilon_k \pm \mu_B B$ in the first and second integrals respectively. Since $\mu_B B \ll \epsilon_F$ this doesn't significantly alter the limits of integration, so we have

$$
\begin{aligned}
m_{\text{total}} &= \frac{\mu_B}{2}\int_0^{\epsilon_F} d\epsilon' D(\epsilon' - \mu_B B) f(\epsilon') - \frac{\mu_B}{2}\int_0^{\epsilon_F} d\epsilon' D(\epsilon' + \mu_B B) f(\epsilon') \\
&= \frac{\mu_B}{2}\int_0^{\epsilon_F} d\epsilon' f(\epsilon') \left[D(\epsilon' - \mu_B B) - D(\epsilon' + \mu_B B)\right] \\
&\approx \frac{\mu_B}{2}\int_0^{\epsilon_F} d\epsilon' f(\epsilon') \left[-2\mu_B B \frac{\partial D}{\partial \epsilon'}\right]
\end{aligned}
$$

where we Taylor expanded in the last step. Noting that $f(\epsilon) \approx 1$ between 0 and ϵ_F when $T \ll T_F$, we can apply the fundamental theorem of calculus to get

$$m_{\text{total}} = -\mu_B^2 B D(\epsilon_F). \tag{16}$$

Ignoring the minus sign, which is just an artifact of having chosen B to point downward, we can plug this into (**??**) to get

$$\chi = \frac{\mu_0 m}{BV} = \frac{\mu_0 \mu_B^2 D(\epsilon_F)}{V}.$$

(b) Using the density of states in 3D: $D(\epsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\epsilon}$ we can estimate the magnetic susceptibility for copper:

$$\begin{aligned}
\chi &= \frac{\mu_0 \mu_B^2}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\epsilon} \\
&= (4\pi \times 10^{-7})(9.27 \times 10^{-24})^2 \frac{1}{2\pi^2}\left(\frac{2 \times 9.1 \times 10^{-31}}{(1.05 \times 10^{-34})^2}\right)^{3/2}(7.0 \times 1.6 \times 10^{-19}) \\
&\approx 1.23 \times 10^{-5}.
\end{aligned}$$

Problem 6: Corrections to Ideal Gas Behavior

(a) We know in general we can write the total number of particles as an integral over the density of states and occupation:

$$N = \int_0^\infty d\epsilon D(\epsilon) f(\epsilon, \mu, \tau) = \int_0^\infty d\epsilon \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2}\frac{1}{\lambda^{-1}e^{\epsilon/\tau}+1}.$$

(b) Notice that for $\lambda \ll 1$, $\lambda e^{-\epsilon/\tau} \ll 1$ because $e^{-\epsilon\tau} \leq 1$. So we can factor out $\epsilon^{(\mu-\epsilon)}$ from the above equation, and use the expansion: $\frac{1}{1+x} \approx 1 - x$ for small x.

$$\begin{aligned}
N &= \int_0^\infty d\epsilon \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2}\lambda e^{-\epsilon/\tau}\frac{1}{1+\lambda e^{-\epsilon/\tau}} \\
&\approx \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\epsilon \epsilon^{1/2} e^{(\mu-\epsilon)/\tau}\left(1 - e^{(\mu-\epsilon)/\tau}\right)
\end{aligned}$$

(c) We can evaluate the integral in (b)

$$\begin{aligned}
N &= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\epsilon \epsilon^{1/2} e^{(\mu-\epsilon)/\tau}\left(1 - e^{(\mu-\epsilon)/\tau}\right) \\
&= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left(\lambda \int_0^\infty d\epsilon \epsilon^{1/2} e^{-\epsilon/\tau} - \lambda^2 \int_0^\infty d\epsilon \epsilon^{1/2} e^{-2\epsilon/\tau}\right) \\
&= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left(\lambda \tau^{3/2}\int_0^\infty dy y^{1/2} e^{-y} - \lambda^2 (\tau/2)^{3/2}\int_0^\infty dy y^{1/2} e^{-y}\right)
\end{aligned}$$

Now we can use the definition of a gamma function: $\Gamma(n+1) \equiv \int_0^\infty x^n e^{-x} dx$,

$$\begin{aligned} N &= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left(\lambda\tau^{3/2}\Gamma(3/2) - \lambda^2\left(\frac{\tau}{2}\right)^{3/2}\Gamma(3/2)\right) \\ &= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\tau^{3/2}\frac{\sqrt{\pi}}{2}\left(\lambda - \frac{\lambda^2}{2^{3/2}}\right) \\ &\Rightarrow \frac{n}{2n_Q} = \lambda - \frac{\lambda^2}{2^{3/2}} + \cdots \end{aligned}$$

(d) Taking the log of the previous expression, and using the expansion $\ln(1+x) \approx x$, we have, for small λ:

$$\begin{aligned} \ln\frac{n}{2n_Q} &= \mu/\tau + \ln\left(1 - \frac{e^{\mu/\tau}}{2^{3/2}}\right) \approx \mu/\tau - \frac{e^{\mu/\tau}}{2^{3/2}} \\ \mu/\tau &= \ln\frac{n}{2n_Q} + \frac{e^{\mu/\tau}}{2^{3/2}} + \cdots = \frac{n}{2n_Q} + \frac{1}{2^{3/2}}\frac{n}{2n_Q} + \cdots \end{aligned}$$

(e) From the expression $P = \frac{2}{3}\frac{U}{V}$ we can write the pressure in terms of the integral over energy:

$$\begin{aligned} PV &= \frac{2}{3}\int_0^\infty d\epsilon\epsilon D(\epsilon) f(\epsilon,\mu,\tau) \\ &= \frac{2}{3}\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\epsilon\epsilon^{3/2}\frac{1}{\lambda e^{\epsilon/\tau}+1}. \end{aligned}$$

(f) Following the same steps of part b) we have the expression

$$\lambda\tau^{5/2}\int_0^\infty dy y^{3/2}e^{-y} - \lambda^2(\tau/2)^{5/2}\int_0^\infty dy y^{3/2}e^{-y}$$

for the integrand. So we have

$$\begin{aligned} PV &= \frac{2}{3}\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\tau^{5/2}\left(\lambda - \frac{\lambda^2}{2^{5/2}}\right)\Gamma(5/2) \\ &= \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\tau^{5/2}\sqrt{\pi}\left(\lambda - \frac{\lambda^2}{2^{5/2}} + \cdots\right) \end{aligned}$$

using the fact that $\Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$.

(g) First lets look at the expression in the parentheses:

$$\begin{aligned} \lambda - \frac{\lambda^2}{2^{5/2}} &= \left(\lambda - 2\frac{\lambda^2}{2^{5/2}}\right) + \frac{\lambda^2}{2^{5/2}} = \left(\lambda - \frac{\lambda^2}{2^{3/2}}\right) + \frac{\lambda^2}{2^{5/2}} \\ &\approx \frac{n}{2n_Q} + \frac{1}{2^{5/2}}\left(\frac{n}{2n_Q}\right)^2 \end{aligned} \tag{17}$$

Plugging this back into (f) gives us:

$$PV = N\tau\left(1 + \frac{1}{2^{7/2}}\left(\frac{n}{n_Q}\right) + \cdots\right)$$