Physics 112 Fall 2017
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Homework 10 Solutions

Problem 1, Kittel 8.9: Cooling of a Solid to $T = 0$ K

(a) This problem is just like the reversible refrigerator discussed in the text, except for the fact that the temperature T_l of the low temperature reservoir (the solid) changes as we extract heat from it. Since the fridge is reversible the entropy that enters the device from the cold solid is equal to the entropy that leaves into the hot reservoir, and so if T is the temperature of the solid we have

$$\frac{dQ_l}{T} = \frac{dQ_h}{T_i}. \tag{1}$$

Here and below we define all dQs to be positive, as in the text. Now, the first law of thermodynamics (conservation of energy) also gives us

$$dW = dQ_h - dQ_l$$

which when combined with (**??**) gives

$$dW = dQ_l \left(\frac{T_i}{T} - 1 \right). \tag{2}$$

We also have

$$dQ_l = -CdT = -aT^3 dT$$

where the minus sign comes from the fact that dQ_l was defined to be the heat transfer *into* the system, hence *out of* the solid. Putting this together with (**??**) and integrating from the initial temperature of the solid to 0 gives

$$\begin{aligned} W &= \int_{T_i}^{0} dW = \int_0^{T_i} aT^3 \left(\frac{T_i}{T} - 1 \right) dT \\ &= a \int_0^{T_i} (T^2 T_i - T^3)\, dT \\ &= \frac{a}{12} T_i^4. \end{aligned}$$

(b) Now we need to assume that the electronic contribution to the heat capacity ($C_{el} \sim T$) is significant compared to the phonon contribution ($C_{ph} \sim T^3$), so that $C = aT^3 + bT$. We now have

$$dQ_l = -\left(aT_l^3 + bT_l \right) dT_l,$$

so the total work is:

$$\begin{aligned} W &= \int_0^{T_i} (aT^3 = bT) \left(\frac{T_i}{T} - 1 \right) dT \\ &= \frac{a}{12} T_i^4 + \frac{b}{2} T_i^2. \end{aligned}$$

Problem 2, Kittel 8.6: Room Air Conditioner

(a) The work required to remove a infinitesimal amount of the heat dQ_l from the room (the low temperature reservoir) with a reversible air conditioner is

$$dW = \frac{1}{\gamma_C} dQ_l = \frac{T_h - T_l}{T_l} dQ_l.$$

Thus if the power supplied to the air conditioner is P, we have

$$P = \frac{dW}{dt} = \frac{T_h - T_l}{T_l}\frac{dQ_l}{dt}.$$

The rate at which heat is removed from the room is then

$$\frac{dQ_l}{dt} = \frac{T_l}{T_h - T_l} P.$$

We're told that heat flows into the room at a rate $\frac{dQ_{in}}{dt} = A(T_h - T_l)$, and when the system reaches steady state T_l remains constant, so the heat removed by the air conditioner must be equal to the heat absorbed from the outside:

$$\begin{aligned} \frac{dQ_l}{dt} &= \frac{dQ_{in}}{dt} \\ \Longrightarrow \quad \frac{T_l}{T_h - T_l} P &= A\left(T_h - T_l\right). \end{aligned}$$

Solving for T_l, we have:

$$\begin{aligned} & T_l^2 - \left(2T_h + \frac{P}{A}\right) T_l + T_h^2 = 0 \\ \Rightarrow \quad & T_l = T_h + P/(2A) - \sqrt{\left(T_h + P/(2A)\right)^2 - T_h^2}. \end{aligned}$$

where we've picked the minus sign in front of the square root so $T_l < T_h$.

(b) Solving for A in the equation (1) above:

$$A = \frac{T_l}{\left(T_h - T_l\right)^2} P = \frac{280\ K}{\left(20\ K\right)^2}\left(2 \cdot 10^3\ W\right) = 1450 \text{ W/K}$$

Problem 3, Kittel 8.3: Photon Carnot Engine

A Carnot engine is defined by the four-step cycle in figure 8.5 of Kittel:

$1 \to 2$: Isothermal expansion @ $T = T_h$
$2 \to 3$: Isentropic expansion @ $\sigma = \sigma_h$
$3 \to 4$: Isothermal compression @ $T = T_l$
$4 \to 1$: Isentropic compression @ $\sigma = \sigma_l$

For a photon gas, the entropy is related to the temperature by $\sigma(\tau)/V = \frac{4\pi^2}{45\hbar^3 c^3}\tau^3$. We see that for the isothermal steps, $\sigma/V = \text{const}$ and for the isentropic steps $V\tau^3 = \text{const}$.

(a) For $2 \to 3$: $V_2\tau_h^3 = V_3\tau_l^3$ and for $4 \to 1$: $V_4\tau_l^3 = V_1\tau_h^3$, so we have:

$$V_3 = V_2\left(\frac{T_h}{T_l}\right)^3, \qquad V_4 = V_1\left(\frac{T_h}{T_l}\right)^3.$$

(b) The heat absorbed during the first isothermal expansion at $T = T_h$ $(1 \to 2)$:

$$\begin{aligned} Q_h &= \int dQ = \tau_h \int_1^2 d\sigma = \tau_h(\sigma_2 - \sigma_1) = \tau_h \cdot \frac{4\pi^2}{45\hbar^3 c^3}(V_2\tau_2^3 - V_1\tau_1^3) \\ &= \frac{4\pi^2}{45\hbar^3 c^3}(V_2 - V_1)\tau_h^4. \end{aligned} \tag{3}$$

To find the work done we use the first law of thermodynamics, given the internal energy of a photon gas : $\frac{U}{V} = \frac{\pi^2}{15\hbar^3 c^3}\tau^4$

$$\begin{aligned} W_{12} &= -(U_2 - U_1 - Q_{12}) \\ &= -\frac{\pi^2}{15\hbar^3 c^3}\tau_h^4(V_2 - V_1) + \frac{4\pi^2}{45\hbar^3 c^3}(V_2 - V_1)\tau_h^4 \\ &= \frac{\pi^2}{45\hbar^3 c^3}(V_2 - V_1)\tau_h^4 \end{aligned}$$

We see that $W_{12}/Q_h = \frac{1}{4}$, so the work done is not equal to the heat absorbed during the isothermal process. For an ideal gas, the internal energy is only a function of temperature, so for any isothermal process the heat absorbed is equal to the work done by the gas. The internal energy of a photon gas depends on its volume, so its internal energy changes during an isothermal expansion, and some of the heat absorbed goes into changing the internal energy instead of doing work.

(c) No heat flows during a reversible, isentropic expansion: $dQ = \tau d\sigma = 0$ (isentropic $\Rightarrow$ adiabatic for quasi-static processes). What this question is asking is whether or not the work done by the gas during the isentropic expansion is equal to the work done on the gas during the isentropic compression. The work done during the expansion is:

$$W_{23} = -(U_3 - U_2) = -\frac{\pi^2}{15\hbar^3 c^3}(V_3\tau_l^4 - V_2\tau_h^4) = \frac{\pi^2}{15\hbar^3 c^3}V_2\tau_h^3(\tau_h - \tau_l)$$

And the work done during the compression is

$$W_{41} = -(U_1 - U_4) = -\frac{\pi^2}{15\hbar^3 c^3}(V_1\tau_h^4 - V_4\tau_l^4) = -\frac{\pi^2}{15\hbar^3 c^3}V_1\tau_h^3(\tau_h - \tau_l)$$

So the total work done over the two isentropic steps is:

$$W_{23} + W_{41} = \frac{\pi^2}{15\hbar^3 c^3}(V_2 - V_1)\tau_h^3(\tau_h - \tau_l) \neq 0$$

This is in contrast with the ideal gas case; there, the internal energy only depends on temperature, $U_1 = U(T_h) = U_4$ and $U_2 = U(T_h) = U_3$, so $W_{23} + W_{41} = -(U_3 - U_2) - (U_1 - U_4) = 0$.

(d) The total work for an entire cycle is $W_{cyc} = W_{12} + W_{23} + W_{34} + W_{41}$. We already found W_{12}, W_{23}, and W_{41}, so we just need to find W_{34}. First we'll find $Q_l = Q_{34}$:

$$\begin{aligned} Q_l &= \int dQ_l = \tau_l \int_3^4 d\sigma = \tau_l(\sigma_4 - \sigma_3) = \tau_l \cdot \frac{4\pi^2}{45\hbar^3 c^3}(V_4\tau_2^3 - V_3\tau_1^3) \\ &= \frac{4\pi^2}{45\hbar^3 c^3}(V_4 - V_3)\tau_l^4. \end{aligned}$$

Using the First Law:

$$\begin{aligned} W_{34} &= -(U_4 - U_3 - Q_{34}) = -\frac{\pi^2}{15\hbar^3 c^3}\tau_l^4(V_4 - V_3) + \frac{4\pi^2}{45\hbar^3 c^3}(V_4 - V_3)\tau_l^4 \\ &= \frac{\pi^2}{45\hbar^3 c^3}(V_4 - V_3)\tau_l^4. \end{aligned}$$

Now we can use the relations in (a) to write this in terms of V_1 and V_2:

$$W_{34} = -\frac{\pi^2}{45\hbar^3 c^3}(V_2 - V_1)\tau_h^3 \tau_l$$

So the total work is:

$$\begin{aligned}
W_{cyc} &= \frac{\pi^2}{45\hbar^3 c^3}(V_2 - V_1)\tau_h^4 + \frac{\pi^2}{15\hbar^3 c^3}(V_2 - V_1)\tau_h^3(\tau_h - \tau_l) - \frac{\pi^2}{45\hbar^3 c^3}(V_2 - V_1)\tau_h^3 \tau_l \\
&= \frac{\pi^2}{45\hbar^3 c^3}(V_2 - V_1)\tau_h^3\left(\tau_h + 3\left(\tau_h - \tau_l\right) - \tau_l\right) \\
&= \frac{4\pi^2}{45\hbar^3 c^3}(V_2 - V_1)\tau_h^3(\tau_h - \tau_l)
\end{aligned}$$

This along with (**??**) yields

$$\eta = \frac{W}{Q_h} = 1 - \frac{T_l}{T_h}$$

as expected.

Problem 4: Total Available Work

(a) The total work that can be extracted from the two finite size reservoirs is determined by the total heat that flows from the hot reservoir and into the cold reservoir: $W = Q_h - Q_l$. From the definition of the heat capacity $C_P = \left(\frac{dQ}{dt}\right)_p$, and the fact the we are told that C_P is independent of temperature, we have (assuming $T_2 > T_1$)

$$Q_l = \int_{T_1}^{T_f} C_p dT = C_P(T_f - T_1). \tag{4}$$

Remembering that Q_h is defined to be the heat flowing *out* of the hot reservoir, so that $dQ_h = -C_P dT$, we also get $Q_h = C_P(T_2 - T_f)$. The total work is then

$$\begin{aligned}
W &= Q_h - Q_l \\
&= C_P(T_2 - T_f) - C_P(T_f - T_1) \\
&= C_P(T_1 + T_2 - 2T_f) \qquad (5)
\end{aligned}$$

(b) The Second Law of Thermodynamics tells us that $\Delta S_{\text{total}} \geq 0$. We can find the total entropy change in terms of T_1, T_2, and T_f. For the low temperature reservoir:

$$\Delta S_l = \int_{T_1}^{T_f} \frac{dQ_l}{T} = C_P \int_{T_1}^{T_f} \frac{dT}{T} = C_P \ln \frac{T_f}{T_1}$$

Again for the hot reservoir the entropy change is determined by the heat flow *in* to the reservoir, $d\sigma_h = -dQ_h = C_P dT$

$$\Delta S_h = \int_{T_1}^{T_f} \frac{-dQ_h}{T} = C_P \int_{T_2}^{T_f} \frac{dT}{T} = C_P \ln \frac{T_f}{T_2}$$

Using $\Delta S_{\text{total}} \geq 0$ we see:

$$\begin{aligned}
0 \leq \Delta S_{\text{total}} &= C_P \ln \frac{T_f}{T_2} + C_P \ln \frac{T_f}{T_1} \\
&= C_P \ln \left(\frac{T_f^2}{T_1 T_2}\right) \\
\implies T_f &\geq \sqrt{T_1 T_2}.
\end{aligned}$$

(c) From (**??**) we see that the maximum work will be done when the final temperature is a minimum (the reversible case):

$$W_{\max} = C_P(T_1 + T_2 - 2\sqrt{T_1 T_2}).$$

It makes sense that in the most efficient (maximum work) case the common final temperature will be a minimum; in this case we're turning as much of Q_h into work as possible, so the net heat transfer to the cold reservoir is as small as possible, so the cold reservoir's temperature rises slowly and the hot reservoir has to cool down further to meet it.

Problem 5: Maximum Power

First lets make sure we understand this problem. The working substance itself operates between the temperatures T_{hw} and T_{cw}. The heat flows into and out of the engine through thermal resistors from infinite reservoirs at temperatures $T_h > T_{hw}$ and $T_c < T_{cw}$.

(a) The Carnot engine itself is reversible, so the entropy flow into the working substance $d\sigma_{hw} = \frac{dQ_{hw}}{T_h w}$ must be equal to the entropy flow out of the working substance $d\sigma_{cw} = \frac{dQ_{cw}}{T_c w}$. For finite heat transfer this gives

$$\frac{Q_{hw}}{T_{hw}} = \frac{Q_{cw}}{T_{cw}}.$$

Also, the heat flow in and out of the reservoirs must be equal the heat flows in and out of the engine (at steady state, nothing is heating up or cooling down): $Q_h = Q_{hw}$ & $Q_c = Q_{cw}$:

$$\begin{aligned} \frac{Q_h}{T_{hw}} &= \frac{Q_c}{T_{cw}} \\ \Longrightarrow \quad \frac{\Delta t\, K(T_h - T_{hw})}{T_{hw}} &= \frac{\Delta t\, K(T_{cw} - T_c)}{T_{cw}} \end{aligned} \tag{6}$$

This gives the relation

$$T_{cw}(T_h - T_{hw}) = T_{hw}(T_{cw} - T_c). \tag{7}$$

(b) The total work per cycle of the engine is determined by the efficiency: $W = \eta_c Q_h = \left(1 - \frac{T_{cw}}{T_{hw}}\right) Q_h$. Although it's not entirely clear in the problem, let's assume that the absorption and rejection of heat occur sequentially and each take time Δt, so the total time for a cycle is $T_{cyc} = 2\Delta t$. The power output of the engine is then

$$\begin{aligned} P &= \frac{W}{2\Delta t} = \frac{\eta_c Q_h}{2\Delta t} \\ &= \frac{\left(1 - \frac{T_{cw}}{T_{hw}}\right) \Delta t K(T_h - T_{hw})}{2\Delta t} \\ &= \frac{K}{2}\left(1 - \frac{T_{cw}}{T_{hw}}\right)(T_h - T_{hw}). \end{aligned} \tag{8}$$

Solving (**??**) for T_{cw} yields

$$T_{cw} = \frac{T_c T_{hw}}{2T_{hw} - T_h} \tag{9}$$

which we can plug into (**??**) to get

$$P = \frac{K}{2} \frac{(T_h - T_{hw})(T_h - 2T_{hw} + T_c)}{(T_h - 2T_{hw})}.$$

(c) To find the find the maximum power output for fixed T_h and T_c, we differentiate P with repect to T_{hw}:

$$\begin{aligned} \frac{\partial P}{\partial T_{hw}} &= \frac{4T_{hw}^2 - 4T_{hw}T_h - (T_h - T_c)T_h}{(T_h - T_{hw})^2} = 0 \\ \Longrightarrow \quad T_{hw} &= \frac{1}{2}\left(T_h + \sqrt{T_c T_h}\right) \qquad (10) \end{aligned}$$

where we took the positive root in solving the quadratic because (**??**) implies that $2T_{hw} > T_h$, which only holds when we take the positive root. We can now plug (**??**) into (**??**), which after some algebra yields

$$T_{cw} = \frac{1}{2}\left(T_c + \sqrt{T_c T_h}\right). \qquad (11)$$

(d) The efficiency for an engine with maximum power output is:

$$\begin{aligned} \eta &= \frac{T_{hw} - T_{cw}}{T_{hw}} = \frac{T_h - T_c}{T_h + \sqrt{T_c T_h}} \\ &= 1 - \sqrt{\frac{T_c}{T_h}}. \\ &= 1 - \sqrt{\frac{273 + 25}{273 + 600}} \approx 0.42. \end{aligned}$$

Notice that if the engine was optimized for efficiency instead of power, the maximum efficiency it could achieve is $\eta_C = \frac{600-25}{273+600} \approx 0.66$, which is $\sim 50\%$ higher than the efficiency achieved at maximum power output.