Physics 112, Fall 2017: Holzapfel
Problem Set 1 (7 problems). Due Friday, September 8, 5 PM

Problem 1: **Combinations**.
Welcome to Oski's Seven Flavors Ice Cream Shop. Your bowl holds four scoops and we want to know how many ways we can fill it from the seven flavors under the following rules:

a) We allow duplication and care about the order that the scoops go in the bowl.

b) We don't allow duplication and care about the order that the scoops go in the bowl.

c) We don't allow duplication and don't care about the order that the scoops go in the bowl.

d) We allow duplication and don't care about the order that the scoops go in the bowl. Hint: this problem can be solved by thinking of filling the bowl from the bins of ice cream as a sequence of operations, scoop (0) or move to next bin (1). One possible sequence would then be "0110011101".

Problem 2: **Sharpness of the multiplicity function**.
Suppose you flip 10000 coins. Let $P(n)$ represent the probability that exactly n coins come up heads.

a) What is the probability, P(5000), of getting 5000 heads and 5000 tails?

b) What is the relative probability P(5100)/P(5000) of getting 5100 heads and 4900 tails?

c) What is the relative probability P(6000)/P(5000) of getting 6000 heads and 4000 tails?

d) Repeat the previous calculation for only 10 coins. What is the relative probability, P(6)/P(5), of getting 6 heads?

Problem 3: **Poisson Distribution**.
The Poisson distribution is a discrete probability distribution that expresses the probability of a number of events occurring in a fixed period of time if these events occur with a known average rate, and are independent of the time since the last event.
a) Show that the probability P(n) that an event characterized by probability p occurs n times in N trails is given by the now familiar binomial distribution.

$$P(n) = \frac{N!}{(n)!(N-n)!} p^n (1-p)^{N-n}. \tag{1}$$

b) Now consider a situation where the probability of an event is low $p << 1$ and the fraction of trial in which an event occurs is low $n << N$. An example would be the emission of alpha particles by a weak radioactive source. Show that the expression found in a) becomes:

$$P(n) = \frac{\lambda^n}{n!} e^{-\lambda}, \tag{2}$$

where $\lambda = Np$ is the mean number of events.

Problem 4: **Stirling approximation**.
Consider a two state system like that discussed in class consisting of N spins with spin excess 2s. The multiplicity function for this system is given by:

$$g(N,s) = \frac{N!}{(N/2+s)!(N/2-s)!}. \tag{3}$$

Apply the Stirling approximation ($N >> 1$) to eliminate the large factorials:

$$N! \approx \sqrt{2\pi N} N^N e^{-N}. \tag{4}$$

a) Show that the multiplicity can be expressed by

$$g(N,s) \approx 2^N \sqrt{\frac{2}{\pi N}} \left(1 - 4\frac{s^2}{N^2}\right)^{-\frac{N}{2}} \frac{(1-2\frac{s}{N})^{s-\frac{1}{2}}}{(1+2\frac{s}{N})^{s+\frac{1}{2}}}. \tag{5}$$

b) Compare this expression with that found in class:

$$g(N,s) \approx 2^N \sqrt{\frac{2}{\pi N}} \exp\left(-\frac{2s^2}{N}\right). \tag{6}$$

Evaluate the ratio of the result from a) over this expression for the specific cases: (N=10, S=1), (N=1000, s=100), and (N=1000, s=10).

c) Explain the relative level of agreement for the two cases considered above. Which expression do you expect to be more accurate? What are the necessary conditions for each of the two approximations to be valid. What is the main limitation of the result found in a).

Problem 5: **The approach to equilibrium**.
This problem is based on a model originally described by P. and T. Ehrenfest (Phys. Zeit. 8 (1907)). It serves as a model for a system approaching the equilibrium state, and describes how irreversibility (and thus a direction of time) emerges from systems which microscopically obey time-symmetric laws. We start with two dogs, Rover and Spot, and 2R fleas, labeled from 1 to 2R. Initially, there are R+n fleas on Rover and R-n fleas on Spot (n$\leq$R). Now at each step s, we pick a random number from (1,2R) and take that numbered flea and cause it to hop from whatever dog it is currently on to the other dog. The question is what happens after many many steps? Hopefully this problem will help you develop some idea of what it means for a system to be in equilibrium.

We define P(m,s) as the probability that after s steps, there are R+m fleas on Rover, where m represents the excess number of fleas on Rover. All fleas are equally likely to jump, and only one will jump per step; the number of fleas on Rover has to increase or decrease by one. Prove to yourself that the probability of finding R+m fleas on Rover in a given step s depends on the probabilities that there are (m+1) or (m-1) fleas on Rover in the previous step in the following way:

$$P(m,s) = \frac{R+m+1}{2R} P(m+1,s-1) + \frac{R-m+1}{2R} P(m-1,s-1). \tag{7}$$

a) Consider the average excess $< m(s) >$ of fleas on Rover as a function of s. As described in class, the mean value of m is given by:

$$< m(s) > = \sum_{m=-R}^{R} m P(m,s) \tag{8}$$

Show that

$$< m(s) > = \left(1 - \frac{1}{R}\right) < m(s-1) > \tag{9}$$

b) Use your results from a) to show that

$$< m(s) >= n\left(1-\frac{1}{R}\right)^s \tag{10}$$

c) Roughly plot $< m(s) >$ as a function of s for $R = 10$ and $R = 100$. Is this what you expect?

d) Find an approximate expression for the number of steps needed to reach equilibrium. (hint: use $x^s = \exp(sln(x))$ and $ln(1+x) \approx x - x^2/2...$)

Problem 6: **Diffusion**.
An ammonia bottle is open briefly in the center of a large room releasing many molecules into the air. The ammonia molecules travel an average of 10^{-5} meters between collisions with other molecules and they collide 10^7 times per second. After each collision they are equally likely to go in any direction.

a) What is the average displacement in one dimension (Example: z) between collisions? Hint: the average of a function f over all solid angles is found from:

$$\bar{f} = (1/4\pi)\int f\, sin\theta\, d\theta\, d\phi. \tag{11}$$

b) What is the root mean square (RMS) of the quantity computed in a).

c) What is the average displacement of the escaped ammonia molecules in the z direction after 2 seconds. What is the standard deviation of this value?

d) If you were standing on the z-axis 6 meters from the bottle, how long would it take until 32% of the ammonia molecules were further from the bottle in the z-direction than you.

Problem 7: **The meaning of "never"**.
Kittel Chapter 2: problem 4