# Physics 112 Fall 2017
# Professor William Holzapfel
# Homework 4 Solutions

## Problem 1: Temperature of the Earth and the Greenhouse Effect

(a) We know that at equilibrium, the earth must absorb as much power as it emits: if it absorbed more, it's temperature would increase, and if it emitted more, it's temperature would decrease. We can treat the earth as a blackbody, so we have the simple relationship which determines the temperature:

$$P_E^{emit} = \text{Area} \times \text{Flux} = \left(4\pi R_E^2\right) \sigma_b T_E^4 = P_E^{abs}.$$

Clearly the power absorbed by the earth is some fraction of the power emitted by the sun. The earth will absorb the solar radiation over an area equal to it's cross sectional area $A_e^{cs} = \pi R_E^2$. Treating the sun as a blackbody, the total power it emits is: $4\pi R_S^2 \sigma_b T_s^4$. This power spreads out isotropically over a sphere as it travels radially away from the sun; the solar flux at a given distance from the sun $d$ is Flux = Power/Area = $\sigma_b T_S^4 \times \frac{4\pi R_S^2}{4\pi d^2}$. So the total power absorbed by the earth is:

$$P_E^{abs} = A_E^{cs} \times \text{Flux}_{\text{sun}} = \left(\pi R_E^2\right) \sigma_b T_S^4 \times \frac{4\pi R_S^2}{4\pi d^2} = P_E^{emit} = \left(4\pi R_E^2\right) \sigma_b T_E^4$$

where $d$ is the distance from the earth to the sun. We can solve for the temperature of the earth in terms of the temperature of the sun and the ratio $(2R_s)/d = \tan(\theta) \approx \theta$ where $\theta \approx 1/2^o = \frac{1}{2}\frac{2\pi \text{ rad}}{360^o} \approx 9 * 10^{-3}$ rad is the angle the sun subtends as viewed from the earth. (Remember that the angle that the Sun subtends is given by the ratio of the *diameter* of the sun to the distance to the sun; that's where the factor of 2 comes from). Solving for $T_E$ then gives

$$T_E = T_S\sqrt{\frac{1}{4}\frac{2R_S}{d}} = T_S\sqrt{\frac{1}{4}\theta}.$$

Now we need to figure out what the temperature of the sun is. Assuming the sun is perfect blackbody, we can figure out its temperature just by *looking* at it; the peak wavelength (i.e. the color the sun looks) uniquely determines its temperature. Now, as mentioned in lecture, we have to be a little careful about how we define peak wavelength. We can define the peak wavelength as the peak of the spectral intensity in terms of frequency, $I_\nu$, or as the peak of $I_\lambda$, the peak of the spectral intensity as a function of wavelength. $I_\nu$ and $I_\lambda$ are different functions, and therefore have maximums at different places, so we can't simply find the maximum of one of the two and convert between $\lambda$ and $\nu$ (doing this would change what we mean by peak wavelength).

$I_\nu$ and $I_\lambda$ are related by the total flux:

$$\begin{aligned}
\text{Total Flux} &= \int\int I_\nu \cos\theta d\Omega d\nu = \int\int I_\lambda \cos\theta d\Omega d\lambda \\
\Rightarrow \int I_\nu d\nu &= \int \frac{2h\nu^3}{c^2\left(e^{\frac{h\nu}{\tau}} - 1\right)} d\nu = \int \frac{2h\left(\frac{c}{\lambda}\right)^3}{c^2\left(e^{\frac{h\left(\frac{c}{\lambda}\right)}{\tau}} - 1\right)} d\left(\frac{c}{\lambda}\right) = \int I_\lambda d\lambda \\
\Rightarrow I_\lambda &= \frac{2hc^2\lambda^{-5}}{e^{\frac{hc}{\lambda\tau}} - 1}.
\end{aligned}$$

By defining $x \equiv \frac{hc}{\lambda\tau}$, we can find the maximum of $I_\lambda$:

$$
\begin{aligned}
\frac{d}{d\lambda} I_\lambda &= 0 \\
\Longrightarrow \frac{d}{dx}\frac{x^5}{e^x - 1} &= 0 \\
\Longrightarrow \frac{5x^4}{e^x - 1} + \frac{(-1)x^5 e^x}{(e^x - 1)^2} &= 0 \\
\Longrightarrow 5\left(e^x - 1\right) - xe^x &= 0 \\
\Longrightarrow e^x(5 - x) - 5 &= 0.
\end{aligned}
$$

This can be solved numerically: $x_{max} = 4.97 = \frac{hc}{\lambda k_b T} \Rightarrow T_S = \frac{hc}{4.97\lambda k_b}$. When we plug in $\lambda_{max} = 480$ nm we get $T_S \approx 6000K$. Now we can solve for the temperature of the earth:

$$
T_E = T_S\sqrt{\frac{1}{4}\theta} = (6000K)\sqrt{\frac{1}{4}9 * 10^{-3}} \approx 285K
$$

$12^o$ C seems a little a colder than the average temperature of the earth actually is (although it's pretty good considering how 'simple' a calculation it was!). Let's see what happens when we include the greenhouse effect in part b).

(b) The idea is that if we assume the atmosphere is completely transparent to visible radiation but completely absorbs IR radiation, then we can assume that all of the solar radiation (which is mostly in the visible and UV) will pass through it, while all of the earth's radiation (which is mostly in the IR) will be absorbed by the atmosphere. You might worry that if the atmosphere is transparent to certain frequencies then we can't treat it like a blackbody. While this is a valid concern, for our approximation we'll treat it like a blackbody when it's emitting to simplify things. Because the atmosphere is absorbing the Earth's radiation and re-emits some (half) of it back to the earth, the earth now is absorbing more power in equilibrium than without the atmosphere. The two equations we have in equilibrium equate the power emitted to the power absorbed for both the earth and the atmosphere:

$$
\begin{aligned}
P_A^{emit} &= \left(2 * 4\pi R_A^2\right)\sigma_b T_A^4 = P_A^{abs} = P_E^{emit} = \left(4\pi R_E^2\right)\sigma_b T_E^4 \\
P_E^{emit} &= \left(4\pi R_E^2\right)\sigma T_E^4 = P_E^{abs} = \left(4\pi R_A^2\right)\sigma_b T_A^4 + \left(\pi R_E^2\right)\sigma_b T_S^4 * \frac{4\pi R_S^2}{4\pi d^2}
\end{aligned}
$$

The factor of 2 in the $P_A^{emit}$ comes from the fact the atmosphere emits both down towards the earth and out into space, so the total surface area of the atmosphere is $2 * 4\pi R_A^2$. In the limit that the height of the atmosphere is negligible $h << R_E$ we see that $2T_A^4 = T_E^4$ so $T_A = 2^{-1/4} T_E$. The second equation reduces to:

$$
\begin{aligned}
4T_E^4 &= 4 * \frac{1}{2}T_E^4 + T_S^4 * \frac{R_S^2}{d^2} \\
\Rightarrow T_E &= 2^{1/4}\sqrt{\frac{\theta}{4}}T_s.
\end{aligned}
$$

So with this model of the greenhouse effect, the earth's temperature is raised by a factor of $2^{1/4} \approx 1.19$, and we get $T_E \approx 339K = 66^\circ$ C. If this seems too high, you're right; we've neglected the fact that the Earth reflects 30% of the incoming radiation from the sun. Adding in this correction brings the temperature down to a more reasonable $29^\circ$ C.

## Problem 2: The Canonical vs. the Microcanonical Ensemble

(a) This is identical to the paramagnet; there are two energy levels for each spin/system, so the multiplicity is:

$$g = \frac{N!}{(N-N_2)!N_2!}.$$

(b) We've done this before; to introduce the temperature, find the entropy in terms of the energy, and then solve for the energy in terms of temperature:

$$\begin{aligned}
\sigma &= \ln g = \ln N! - \ln N_2! - \ln(N-N_2)! \\
&\approx N\ln N - N_2 \ln N_2 - \Big(N - N_2\Big)\ln(N-N_2) \\
&= N\ln N - \frac{U}{\triangle}\ln\left(\frac{U}{\triangle}\right) - \left(N - \frac{U}{\triangle}\right)\ln\left(N - \frac{U}{\triangle}\right)
\end{aligned}$$

and so

$$\frac{1}{\tau} = \frac{\partial\sigma}{\pi U}\Big|_N = \frac{1}{\triangle}\ln\left(-1+\frac{\triangle N}{U}\right).$$

Solving for $U$ then gives

$$U = \frac{N\triangle}{1+e^{\triangle/\tau}} \quad\longrightarrow\quad \begin{cases} 1 & \text{as } \tau\to 0 \\ \frac{N\triangle}{2} & \text{as } \tau\to\infty \end{cases}$$

In the limit $\tau \to 0$, the energy goes to zero as we expect. At high temperatures the energy goes to $N\Delta/2$, i.e. half the particles are in the excited state. Just like in the case of the paramagnet, we know that at high temperatures entropy is maximized, and the state of highest entropy has half the particles in the excited state and half in the ground state.

(c) In the canonical ensemble, we need to calculate the partition function $Z$ and calculate the energy from derivatives of $Z$:

$$\begin{aligned}
Z_1 &= e^{0/\tau} + e^{-\triangle/\tau} \\
\Longrightarrow Z_N &= \frac{Z_1^N}{N!} = \frac{1}{N!}\left(1+e^{-\triangle/\tau}\right)^N \\
\Longrightarrow \ln Z_N &= N\ln\left(1+e^{-\triangle/\tau}\right) - \ln N! \\
\Longrightarrow U &= \tau^2\frac{\partial}{\partial\tau}\ln Z_N = \frac{N\triangle}{1+e^{\triangle/\tau}}.
\end{aligned}$$

This agrees with what we got in part (b). Alternatively, we could note that for a single two-level system the average energy is

$$\begin{aligned}
< U >_1 &= \frac{1}{Z_1}(0+\Delta e^{-\Delta/\tau}) \\
&= \frac{\Delta}{1+e^{\Delta\tau}}
\end{aligned}$$

and so for $N$ particles the average energy is

$$< U >_N = N < U >_1 = \frac{N\Delta}{1+e^{\Delta\tau}}$$

as expected.

## Problem 3: Free energy of a photon gas

(a) We've found the partition function of a single mode in lecture: $Z_n = \frac{1}{1-e^{-\hbar\omega_n/\tau}}$. The total parition function of all modes is then just the product of the partition functions of each mode:

$$Z = \prod_n Z_n = \prod_n \frac{1}{1 - e^{-\hbar\omega_n/\tau}}$$

The free energy is then simply:

$$F = -\tau \ln Z = \tau \sum_n \ln\left(1 - e^{-\hbar\omega_n/\tau}\right)$$

The possible modes are $\omega_n = n\pi c/L$, and we can convert the integral to a sum:

$$\begin{aligned}
F &= \tau \sum_n \ln\left(1 - e^{-\hbar\omega_n/\tau}\right) = 2\tau \int d^3n \ln\left(1 - e^{\frac{-\hbar\pi c}{\tau}n}\right) \\
&= 2\tau \int_0^\infty \frac{1}{8} n^2 dn 4\pi \ln\left(1 - e^{\frac{-\hbar\pi c}{\tau}n}\right) = -\pi\tau \int_0^\infty \frac{n^3}{3} \frac{\hbar\pi c}{L\tau} \frac{e^{\frac{-\hbar\pi c}{\tau}n}}{1 - e^{\frac{-\hbar\pi c}{\tau}n}} dn \\
&= -\frac{\pi}{3}\tau \left(\frac{L\tau}{\hbar\pi c}\right)^3 \int_0^\infty \frac{x^3}{e^x - 1} dx \\
&= -\frac{\pi^2 V \tau^4}{45 c^3 \hbar^3}.
\end{aligned}$$

We've multiplied by a factor of 2 for the two possible polarizations per frequency, and the 1/8 in the volume element selects only the positive values of $n$. When integrating by parts in the third line we've omitted the term that is zero at both limits.

(b) The pressure is then simply:

$$p = -\left.\frac{\partial F}{\partial V}\right|_\tau = \frac{\pi^2 \tau^4}{45 c^3 \hbar^3} = \frac{1}{3}\frac{U}{V}.$$

For a monatomic ideal gas we have $pV = N\tau$ and $U = \frac{3}{2}N\tau$ so $p = \frac{2}{3}\frac{U}{V}$. The pressure of a photon gas is half of that of an ideal gas of the same energy density.

## Problem 4: Heat Shields

(a) In equilibrium, the total power radiated by the middle plane must be equal to the power it absorbs. It absorbs and emits power from both sides:

$$P_m^{rad} = 2A\sigma_b T_u^4 = P_m^{abs} = P_u^{rad} + P_l^{rad} = A\sigma_b T_u^4 + A\sigma_b T_l^4$$

where $A$ is the area of one side of the planes. So the temperature of the middle plane is:

$$T_m = \left( \frac{T_u^4}{2} + \frac{T_l^4}{2} \right)^{1/4}.$$

Let's assume the $u$ plane is on the left and the $l$ plane is on the right. We'll define the energy flux to be positive if it's going to the right. The net energy fluxes between $u$ and $m$, $J_U^{um}$ and between $m$ and $l$, $J_U^{ml}$ are:

$$\begin{aligned} J_U^{um} &= \frac{P_u^{rad} - P_m^{rad}}{A} = \sigma_b \left(T_u^4 - T_m^4\right) = \sigma_B \left( T_u^4 - \left[ \frac{T_u^4}{2} + \frac{T_l^4}{2} \right] \right) \\ J_U^{ml} &= \frac{P_m^{rad} - P_l^{rad}}{A} = \sigma_b \left(T_m^4 - T_l^4\right) = \sigma_B \left( \left[ \frac{T_u^4}{2} + \frac{T_l^4}{2} \right] - T_l^4 \right) \end{aligned}$$

We see that in equilibrium, the two energy fluxes are equal, and are half of $J_U^i = \sigma_B \left(T_u^4 - T_l^4\right)$ the flux without the middle plane :

$$J_U^{um} = J_U^{ml} = J_U^f = \frac{1}{2}\sigma_B \left(T_u^4 - T_l^4\right) = \frac{1}{2} J_U^i$$

So the heat *shield* cuts the net power radiating from the hot to the cold surface in half.

(b) By definition, an object with emissivity $\epsilon$ will emit a flux $J_U = \epsilon \sigma_b T^4$, and absorb $\epsilon$ times the incident power, $P_a = \epsilon P_i$. Because we know that the reflection ($r$) and absorption ($a$) coefficients are related by $r + a = 1$, then the flux reflected from the non-black object with $a = \epsilon$ is $J_U^r = (1 - \epsilon) J_U^i$. Now that power will reflect off the planes, to determine the total flux we must consider not only the flux directly propagating from the $u$ plane to the $l$ plane and vice versa $\epsilon \sigma_B \left(T_u^4 - T_l^4\right)$, but also flux from the $u$ plate that is refleted off the $l$ plate and flux from the $l$ plate that is reflected off the $u$ plate: $(-1)(1-\epsilon)\epsilon\sigma_B \left(T_u^4 - T_l^4\right)$. Each subsequent reflection will give an addition factor of $1 - \epsilon$ for the reflection coefficient and a factor of $-1$ because the flux is changing directions. The total flux will thus be an infinite series of all possible reflections:

$$\begin{aligned} J_U &= \epsilon\sigma_B \left(T_u^4 - T_l^4\right) + (-1)(1-\epsilon)\epsilon\sigma_B \left(T_u^4 - T_l^4\right) + (-1)^2(1-\epsilon)^2\epsilon\sigma_B \left(T_u^4 - T_l^4\right) + \cdots \\ &= \left(1 - (1-\epsilon) + (1-\epsilon)^2 + \cdots\right) \epsilon\sigma_B \left(T_u^4 - T_l^4\right) = \sum_{n=0}^{\infty} (\epsilon - 1)^n \epsilon\sigma_B \left(T_u^4 - T_l^4\right) \\ &= \frac{1}{1 - (\epsilon - 1)} \epsilon\sigma_B \left(T_u^4 - T_l^4\right) = \frac{\epsilon}{2 - \epsilon} \sigma_B \left(T_u^4 - T_l^4\right) \end{aligned}$$

## Problem 5: Pressure of Thermal Radiation

a) The energy of a photon gas $U = \sum_n < \epsilon_n > = \sum_n \frac{\hbar\omega_n}{e^{\hbar\omega_n/\tau} - 1}$ can be expressed in terms of the average number of photons in each mode, $s_n = \frac{1}{e^{\hbar\omega_n/\tau} - 1}$:

$$U = \sum_n s_n \hbar\omega_n \tag{1}$$

so the pressure is given by:

$$p = -\frac{\partial U}{\partial V}\bigg|_\sigma = -\sum_n \left( \frac{\partial s_n}{\partial V}\bigg|_\sigma \hbar\omega_n + s_n \hbar \frac{\partial \omega_n}{\partial V}\bigg|_\sigma \right) \tag{2}$$

Now entropy is given by $\sigma = -\frac{\partial F}{\partial \tau}\Big|_V$ so using the result from problem 3, $F = -\frac{\pi^2 V \tau^4}{45c^3\hbar^3}$ we have:

$$\sigma = -\left( \frac{\partial}{\partial \tau} \frac{-\pi^2 V \tau^4}{45c^3\hbar^3} \right)_V = \frac{4\pi^2 V \tau^3}{45c^3\hbar^3}$$

So constant entropy means that $V^{1/3}\tau = L\tau =$ constant. Now lets look at the first term in (2):

$$\frac{\partial s_n}{\partial V}\bigg|_\sigma = \frac{-e^{\hbar\omega_n/\tau}}{\left(e^{\hbar\omega_n/\tau} - 1\right)^2} \left( \frac{\partial}{\partial V} \hbar\omega_n/\tau \right)_\sigma \tag{3}$$

and using $\omega_n = n\pi c/L$,

$$\left( \frac{\partial}{\partial V} \hbar\omega_n/\tau \right)_\sigma = n\pi\hbar c \left( \frac{\partial}{\partial V} \frac{1}{L\tau} \right)_\sigma = n\pi\hbar c \left( \frac{\partial}{\partial V} \text{constant} \right) = 0.$$

So at constant entropy, the average number of photons in a mode doesn't change. This is what we expect from the adiabatic theorem in quantum mechanics: if the gas is expanded adiabatically (i.e. at constant entropy) the occupation of the energy levels will remain constant as the energy levels are slowly changed. Now we see that (2) reduces to the desired result:

$$p = -\sum_n s_n \hbar \frac{\partial \omega_n}{\partial V}\bigg|_\sigma .$$

b) $\frac{\partial \omega_n}{\partial V}\Big|_\sigma$ is easily calculated:

$$\frac{\partial \omega_n}{\partial V}\bigg|_\sigma = \left( \frac{\partial}{\partial V} \frac{n\pi c}{L} \right)_\sigma = n\pi c \frac{\partial V^{-1/3}}{\partial V}\bigg|_\sigma = -\frac{1}{3} n\pi c V^{-2/3} = -\frac{\omega_n}{3V}$$

c) Finally we have the well known result for a photon gas:

$$\begin{aligned} p &= -\sum_n s_n \hbar \frac{\partial \omega_n}{\partial V}\bigg|_\sigma = -\sum_n s_n \hbar \left( -\frac{\omega_n}{3V} \right) = \frac{1}{3V} \sum_n s_n \hbar\omega_n \\ &= \frac{1}{3}\frac{U}{V}. \end{aligned}$$

d) We can write the radiation pressure in terms of the temperature:

$$p_r = \frac{1}{3}\frac{U}{V} = \frac{1}{3}\frac{\pi^2}{15\hbar^3 c^3}(k_B T)^4 = \frac{4}{3c}\sigma_B T^4.$$

The kinetic pressure is given by the ideal gas law:

$$p_k = \frac{N k_B T}{V} = n k_B T.$$

The temperature at which the radiation pressure is equal to the kinetic pressure is then

$$T_{eq} = \left(\frac{3ck_B n}{4\sigma_B}\right)^{1/3}.$$

Using $n \approx 1\ \mathrm{mol/cm}^3 = 6 \times 10^{29}/\mathrm{m}^3$, we get

$$T_{eq} \approx 3 \times 10^7 K.$$

## Problem 6: Classical Derivation of Stefan-Boltzmann Law

(a) Using the definition of free energy $F = U - \tau\sigma$ and the relation $\sigma = -\frac{\partial F}{\partial \tau}\Big|_V$ we can get the specific heat in terms of entropy:

$$\begin{aligned}
\frac{\partial F}{\partial \tau}\Big|_V &= \frac{\partial U}{\partial \tau}\Big|_V - \sigma - \tau\frac{\partial \sigma}{\partial \tau}\Big|_V \\
\Longrightarrow \tau\frac{\partial \sigma}{\partial \tau}\Big|_V &= \frac{\partial U}{\partial \tau}\Big|_V = C_V \\
\Longrightarrow \frac{\partial \sigma}{\partial \tau}\Big|_V &= \frac{C_V}{\tau}.
\end{aligned} \tag{4}$$

(b) Using $\sigma = -\frac{\partial F}{\partial \tau}\Big|_V$ and $p = -\frac{\partial F}{\partial V}\Big|_\tau$ we get:

$$\begin{aligned}
\frac{\partial^2 F}{\partial V \partial \tau} &= \frac{\partial}{\partial V}\Big|_\tau \left(\frac{\partial F}{\partial \tau}\Big|_V\right) = -\frac{\partial \sigma}{\partial V}\Big|_\tau \\
\frac{\partial^2 F}{\partial \tau \partial V} &= \frac{\partial}{\partial \tau}\Big|_V \left(\frac{\partial F}{\partial V}\Big|_\tau\right) = -\frac{\partial p}{\partial \tau}\Big|_V \\
&\Rightarrow \frac{\partial \sigma}{\partial V}\Big|_\tau = \frac{\partial p}{\partial \tau}\Big|_V.
\end{aligned}$$

(c) Let look at both second derivatives separately:

$$\frac{\partial^2 \sigma}{\partial V \partial \tau} = \frac{\partial}{\partial V}\left(\frac{\partial \sigma}{\partial \tau}\right) = \frac{\partial}{\partial V}\left(\frac{1}{\tau}\frac{\partial U}{\partial \tau}\right) = \frac{\partial}{\partial V}\left(\frac{1}{\tau}\frac{\partial u(\tau)V}{\partial \tau}\right) = \frac{1}{\tau}\frac{\partial u(\tau)}{\partial \tau} \tag{5}$$

$$\frac{\partial^2 \sigma}{\partial \tau \partial V} = \frac{\partial}{\partial \tau}\left(\frac{\partial p}{\partial \tau}\right) = \frac{\partial^2}{\partial \tau^2}\left(\frac{1}{3}u(\tau)\right) \tag{6}$$

Equating the two, we have:

$$\frac{1}{3}\frac{\partial^2}{\partial\tau^2}\left(u(\tau)\right) = \frac{1}{\tau}\frac{\partial u(\tau)}{\partial\tau}.$$

We can easily check that this equation is solved for $u \sim \tau^4$.