

Physics 112 Problem Set 9

Holzappel, Section 102

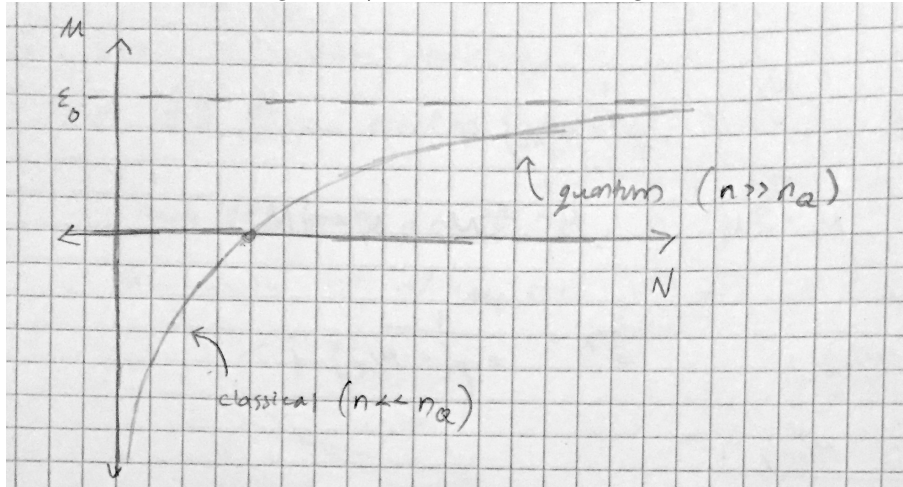
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1. The chemical potential of both the Fermi and Bose gases in the classical regime ($n \ll n_Q$) is given by $\mu = \tau \log(n_Q/n) \propto \log(1/N)$ for fixed T, V .

The chemical potential of a Bose gas in the quantum regime ($n \gg n_Q$) is given by $\mu = \epsilon_0 - (\tau/N)$, so it asymptotically approaches the ground state energy ϵ_0 as $N \rightarrow \infty$.

Figure 1: μ versus N for a Bose gas

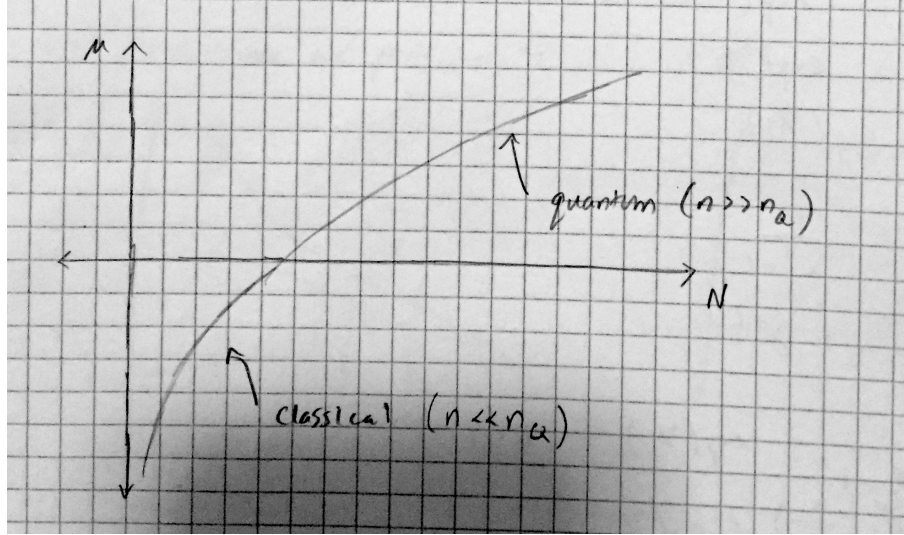


For the Fermi gas, we can find the chemical potential with the integral

$$N = \int_0^{\infty} d\epsilon \mathcal{D}(\epsilon) f(\epsilon, \tau, \mu) \propto \int_0^{\infty} d\epsilon \frac{\epsilon^{1/2}}{\exp((\epsilon - \mu)/\tau)} = \int_0^{\mu} \epsilon^{1/2} d\epsilon \propto \mu^{3/2},$$

to leading order (by way of a Sommerfeld expansion). So the chemical potential of a Fermi gas in the quantum regime is given by $\mu \propto N^{2/3}$, so it increases without bound as $N \rightarrow \infty$.

Figure 2: μ versus N for a Fermi gas



2. If there are N_0 atoms in the ground state and N_1 particles in the first excited state, $N_0 = 2N_1$ and $N_0 + N_1 = N$, then

$$N_0 = \frac{2}{3}N = (\exp(-\mu/\tau) - 1)^{-1}$$

and

$$N_1 = \frac{1}{3}N = (\exp(\epsilon - \mu/\tau) - 1)^{-1}.$$

Then, taking $\lambda = \exp(\mu/\tau)$,

$$\begin{aligned} 2 &= \frac{\lambda^{-1} \exp(\epsilon/\tau) - 1}{\lambda^{-1} - 1} \\ 2\lambda^{-1} - 1 &= \lambda^{-1} \exp(\epsilon/\tau) \\ 2 - \lambda &= \exp(\epsilon/\tau). \end{aligned}$$

But

$$\begin{aligned} \frac{2}{3}N &= \frac{1}{\lambda^{-1} - 1} \\ \frac{3}{2N} + 1 &= \lambda^{-1}, \end{aligned}$$

so

$$2 - \frac{1}{1 + (3/2N)} = \exp(\epsilon/\tau).$$

But $N \gg 1$, so using the approximation $(1 + x)^{-1} \approx 1 - x$ for small x ,

$$\begin{aligned}
2 - \left(1 - \frac{3}{2N}\right) &= \exp(\epsilon/\tau) \\
1 + \frac{3}{2N} &= \exp(\epsilon/\tau) \\
\log\left(1 + \frac{3}{2N}\right) &= \frac{\epsilon}{\tau}.
\end{aligned}$$

Once again, in the limit $N \gg 1$, we can use the approximation $\log(1+x) \approx x$ for small x , so

$$\begin{aligned}
\frac{3}{2N} &= \frac{\epsilon}{\tau} \\
\tau &= \frac{2N\epsilon}{3}.
\end{aligned}$$

3. (a) The energy levels are given by $\epsilon_k = \hbar^2 k^2 / 2m$, and imposing periodic boundary conditions, we see that $k_x = (\pi/L_x)n_x$ (and k_y and k_z are symmetric), so

$$\epsilon_{n_x, n_y, n_z} = \frac{\hbar^2}{2m} \left(\frac{\pi}{V^{1/3}} \right)^2 (n_x^2 + n_y^2 + n_z^2)$$

where we have assumed that $L_x = L_y = L_z = V^{1/3}$. Then, the energy of the lowest orbital, where $n_x = n_y = n_z = 1$, is

$$\epsilon_0 = \frac{3\hbar^2}{2m} \left(\frac{\pi}{V^{1/3}} \right)^2.$$

Since we have $V = 10^{-15} \text{ m}^3$ and $m = 87 \text{ u}$ for Rb-87, this gives $\epsilon_0 = 7.11 \times 10^{-14} \text{ eV}$.

- (b) The Einstein temperature is given by

$$\tau_E = \frac{2\pi\hbar^2}{m} \left(\frac{N}{2.61V} \right)^{2/3}$$

which for $N = 10^4$ and m and V as previously, gives $\tau_E = 7.39 \times 10^{-12} \text{ eV}$.

- (c) If $T = 0.9T_E \implies \tau = 0.9\tau_E = 6.65 \times 10^{-12} \text{ eV}$, then the number of particles in the ground state is

$$N_0 = N - N_e(\tau) = N - 2.612V \left(\frac{m\tau}{2\pi\hbar^2} \right)^{3/2} = 1.46 \times 10^3.$$

The chemical potential is $\mu = \epsilon_0 - (\tau/N_0) = 6.66 \times 10^{-14} \text{ eV}$, or $4.56 \times 10^{-14} \text{ eV}$ less than the ground state energy.

The number of particles in each of the threefold-degenerate first excited states, which have energy

$$\epsilon_1 = \frac{3\hbar^2}{m} \left(\frac{\pi}{V^{1/3}} \right)^2 = 1.42 \times 10^{-13} \text{ eV}$$

is

$$N_1 = \frac{1}{\exp((\epsilon_1 - \mu)/\tau) - 1} = 99.$$

(d) For $N = 10^6$, $\tau_E = 1.59 \times 10^{-10}$ eV. So $\tau = 0.9\tau_E = 1.43 \times 10^{-10}$ eV.

Thus $N_0 = N - N_e(\tau) = 1.48 \times 10^5$.

Then $\mu = \epsilon_0 - (\tau/N_0) = 7.11 \times 10^{-14}$, or 9.66×10^{-17} less than the ground state energy.

And the number of particles in each of the first excited states is 2.0×10^3 .

We see there are relatively more particles in the ground state as the number of particles N increases because the chemical potential is increasingly closer to the ground state energy. So when $\Delta\epsilon \gg \mu \implies \epsilon_1 \gg (\tau/N)$, the number of particles in the ground state is much greater than the number of particles in the first excited state.

4. (a) For a one-dimensional system, imposing periodic boundary conditions on a line of length L such that $k = \pi n/L$

$$\epsilon(n) = \frac{\hbar^2}{2m} k^2 = \frac{\hbar^2}{2m} \frac{\pi^2}{L^2} n^2 \implies n = \frac{2mL^2}{\hbar^2 \pi^2} \epsilon^{1/2},$$

thus

$$\mathcal{D}(\epsilon) = \frac{dn}{d\epsilon} = \frac{mL^2}{\hbar^2 \pi^2} \epsilon^{-1/2}.$$

So the total number of particles in excited orbitals can be approximated by the integral

$$N_\epsilon = \int_0^\infty f(\epsilon, \tau) dn = \int_0^\infty \mathcal{D}(\epsilon) f(\epsilon, \tau) d\epsilon$$

where $f(\epsilon, \tau) = (\exp((\epsilon - \mu)/\tau) - 1)^{-1} = (\lambda^{-1} \exp(\epsilon/\tau) - 1)^{-1}$ is the Bose-Einstein occupancy function. Thus

$$N_\epsilon = \frac{2mL^2}{\hbar^2 \pi^2} \int_0^\infty \frac{\epsilon^{-1/2}}{\lambda^{-1} \exp(\epsilon/\tau) - 1} d\epsilon.$$

For $\lambda = 1$, this becomes

$$N_\epsilon = \frac{2mL^2}{\hbar^2 \pi^2} \tau^{1/2} \int_0^\infty \frac{x^{-1/2}}{\exp(x) - 1} dx,$$

which does not converge.

- (b) For the integral to converge, the density of states must contain a factor of ϵ with an exponent ≥ 0 . For a d -dimensional system, if we impose symmetry in k -space,

$$D(\epsilon) = \frac{d^d n}{d\epsilon} \propto \epsilon^{-1/2} n^{d-1} dn \propto \epsilon^{(d/2)-1}.$$

So the condition for convergence is $(d/2)-1 \geq 0$, or $d \geq 2$. If the system is at least two-dimensional, there will be an abrupt Bose-Einstein transition.