Physics 112 Problem Set 10

Holzapfel, Section 102

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1. (a) For heat capacity $C = aT^3$ (assuming it is measured in J/K), $dQ = aT^3 dT$ (at constant volume). Then, since this solid is the low-temperature reservoir of the refrigerator system, $dQ_l = -aT_l^3 dT_l$, where the negative sign comes from the fact that dT_l is negative (the low-temperature reservoir is cooling) and we want to represent the heat flow into the system as a positive quantity. For a refrigerator operating in the reversible limit, $Q_l + W = Q_h$, and $dQ_l/T_l = dQ_h/T_h$, so

$$dW = dQ_h - dQ_l = dQ_l\left(\frac{T_h}{T_l} - 1\right) = -aT_l^3 dT_l\left(\frac{T_h}{T_l} - 1\right)$$
$$dW = -a(T_h T_l^2 - T_l^3)dT_l$$

Now the total work (electrical energy) required to cool the solid from T_h to 0 is

$$W = \int_{T_h}^{0} -a(T_h T_l^2 - T_l^3)\, dT_l = -a\left(\frac{1}{3}T_h T_l^3 - \frac{1}{4}T_l^4\right)\Bigg|_{T_h}^{0} = \frac{a}{12}T_h^4.$$

(b) If the heat capacity is $C = aT^3 + bT$, the expression for the differential work becomes

$$dW = -(aT_l^3 + bT_l)dT_l\left(\frac{T_h}{T_l} - 1\right)$$
$$dW = -a(T_h T_l^2 - T_l^3)dT_l - b(T_h - T_l)dT_l$$

So the energy required to cool the solid to absolute zero is

$$W = \frac{a}{12}T_h^4 - \int_{T_h}^{0} b(T_h - T_l)dT_l = \frac{a}{12}T_h^4 + \frac{b}{2}T_h^2.$$

2. (a) For a refrigerator operating in the reversible limit, $W = Q_l(T_h - T_l)/T_l$, so $P = dW/dt = (dQ_l/dt)(T_h - T_l)/T_l$. Since the room is gaining heat from the outdoors, $dQ_l/dt = A(T_h - T_l)$, thus

$$P = \frac{A}{T_l}(T_h - T_l)^2$$
$$\frac{P}{A}T_l = T_h^2 - 2T_hT_l + T_l^2$$
$$T_l^2 - 2\left(T_h + \frac{P}{2A}\right)T_l + T_h^2 = 0$$

which is a quadratic equation whose solution is

$$T_l = T_h + \frac{P}{2A} - \left[\left(T_h + \frac{P}{2A}\right)^2 - T_h^2\right]^{1/2}.$$

(b) From the previous calculation, we see the heat loss coefficient is given by

$$A = \frac{PT_l}{(T_h - T_l)^2}$$

which, for $P = 2$ kW, $T_h = 310$ K and $T_l = 290$ K, is $A = 1450$ W/K.

3. (a) Recall that for blackbody radiation (which we modeled as a photon gas), the energy per unit volume was given by $U/V = a\tau^4$. Then $dU = 4aV\tau^3 d\tau$, and since $d\sigma = dU/\tau$, $\sigma = (4a/3)V\tau^3 + C$ for constants a, C.

This means that for an isoentropic process in a photon gas, the quantity $V\tau^3$ must be constant. For a Carnot cycle, $2 \to 3$ and $4 \to 1$ are isoentropic, so $V_2\tau_h^3 = V_3\tau_l^3$, so $V_3 = V_2(\tau_h/\tau_l)^3$, and $V_4\tau_l^3 = V_1\tau_h^3$, so $V_4 = V_1(\tau_h/\tau_l)^3$.

(b) The heat taken up by the photon gas in the isothermal expansion is

$$Q_h = \tau_h(\sigma_2 - \sigma_1) = \frac{4a}{3}\tau_h^4(V_2 - V_1).$$

Recall that for a photon gas, $p = U/3V = (a/3)\tau^4$. Thus

$$W = \int_{V_1}^{V_2} p dV = \frac{a}{3}\tau^4 \int_{V_1}^{V_2} dV = \frac{a}{3}\tau_h^4(V_2 - V_1),$$

which does not equal Q_h.

For an ideal gas expanding isothermally, $U = (3/2)N\tau$, thus $\Delta U = 0$ and $Q = W$ by conservation of energy.

(c) For the two isoentropic processes, $Q = 0$, so $W = -\Delta U$.

So for $2 \to 3$

$$W_{2\to3} = -a(V_3\tau_l^4 - V_2\tau_h^4) = -aV_2\tau_h^3(\tau_l - \tau_h),$$

and for $4 \to 1$,

$$W_{4\to 1} = -a(V_1\tau_h^4 - V_4\tau_l^4) = -aV_1\tau_h^3(\tau_h - \tau_l).$$

So the total work done by both isoentropic processes is

$$W = W_{2\to 3} + W_{4\to 1} = -a\tau_h^3(\tau_l - \tau_h)(V_2 - V_1) \neq 0,$$

unlike the ideal gas, where the work done by these two processes canceled.

(d) From (b),

$$W_{1\to 2} = \frac{a}{3}\tau_h^4(V_2 - V_1),$$

and similarly,

$$W_{3\to 4} = \frac{a}{3}\tau_l^4(V_4 - V_3) = \frac{a}{3}\tau_h^3\tau_l(V_2 - V_1)$$

so the total work done by the gas is

$$\begin{aligned} W &= W_{1\to 2} + W_{2\to 3} + W_{3\to 4} + W_{4\to 1} \\ W &= -a\tau_h^3(\tau_l - \tau_h)(V_2 - V_1) + \frac{a}{3}\tau_h^3(\tau_h - \tau_l)(V_2 - V_1) \\ W &= \frac{4a}{3}\tau_h^3(V_2 - V_1)(\tau_h - \tau_l). \end{aligned}$$

From (a), the heat taken up at the high temperature is

$$Q_h = \frac{4a}{3}\tau_h^4(V_2 - V_1),$$

so the energy conversion efficiency is

$$\eta = \frac{W}{Q_h} = \frac{\tau_h - \tau_l}{\tau_h},$$

which is just the Carnot efficiency η_C.

4. (a) $W = Q_1 - Q_2$. Assume (without loss of generality) that $T_1 > T_f > T_2$, and note that we want to represent all heat flows as positive quantities. So $Q_1 = C_p(T_1 - T_f)$ and $Q_2 = C_p(T_f - T_2)$. Thus $W = C_p(T_1 + T_2 - 2T_f)$.

(b) The differential entropy is $dS = dQ/T = C_p dT/T$, so the total change in entropy is

$$\Delta S = \int_{T_1}^{T_f} \frac{C_p}{T}\, dT + \int_{T_1}^{T_f} \frac{C_p}{T}\, dT = C_p \log(T_f/T_1) + C_p \log(T_f/T_2) = C_p \log\left(\frac{T_f^2}{T_1 T_2}\right).$$

Since the total entropy must increase, we have $\Delta S \geq 0$, or

$$T_f \geq \sqrt{T_1 T_2}.$$

(c) It follows that the total work is bounded by

$$W \le C_p(T_1 + T_2 - 2\sqrt{T_1 T_2}),$$

and the maximum work obtainable by the engine occurs when the two quantities are equal.

5. (a) If all the entropy is created during the two heat transfer processes, to ensure that there is no buildup of entropy in the engine, we must have

$$\frac{Q_h}{T_{hw}} = \frac{Q_c}{T_{cw}},$$

and since the rates of heat transfer are equivalent for the low and high temperature reservoirs, we also have

$$\frac{Q_h}{T_h - T_{hw}} = \frac{Q_c}{T_{cw} - T_c},$$

so

$$\frac{T_{hw}}{T_h - T_{hw}} = \frac{T_{cw}}{T_{cw} - T_c}.$$

Solving for T_{cw}, we find

$$T_{cw} = \frac{T_c T_{hw}}{2T_{hw} - T_h}.$$

(b) The work done by the engine is

$$W = Q_h - Q_c = K\Delta t(T_h - T_{hw} + T_c - T_{cw})$$

and since each of the two isothermal process takes time Δt and the adiabatic processes occur almost instantaneously, the power output is

$$P = \frac{W}{2\Delta t} = \frac{K}{2}(T_h - T_{hw} + T_c - T_{cw}),$$

or, substituting the expression from (a),

$$P = \frac{K}{2}\left(T_h - T_{hw} + T_c - \frac{T_c T_{hw}}{2T_{hw} - T_h}\right).$$

(c) Maximizing the power with respect to T_{hw}, the first-order condition is

$$\frac{\partial P}{\partial T_{hw}} = \frac{K}{2}\left(-1 - \frac{T_c(2T_{hw} - T_h) - 2T_c T_{hw}}{(2T_{hw} - T_h)^2}\right) = 0$$

which gives

$$T_c T_h = (2T_{hw} - T_h)^2$$

or

$$T_{hw} = \frac{1}{2}(T_h - \sqrt{T_h T_c}).$$

Substituting this into the expression for T_{cw},

$$\begin{aligned}
T_{cw} &= \frac{T_c(T_h - \sqrt{T_h T_c})}{2(T_h - \sqrt{T_h T_c} - t_h)} \\
T_{cw} &= \frac{T_c\sqrt{T_h T_c} - T_c T_h}{2\sqrt{T_h T_c}} \\
T_{cw} &= \frac{1}{2}(T_c - \sqrt{T_h T_c}).
\end{aligned}$$

(d) The efficiency is then

$$\begin{aligned}
\eta &= \frac{W}{Q_h} = 1 - \frac{Q_c}{Q_h} = 1 - \frac{T_{cw}}{T_{hw}} \\
\eta &= 1 - \frac{T_c - \sqrt{T_c T_h}}{T_h - \sqrt{T_c T_h}} \\
\eta &= 1 - \frac{T_c T_h + \sqrt{T_c T_h}(T_h - T_c) - T_c T_h}{T_h(T_h - T_c)} \\
\eta &= 1 - \sqrt{\frac{T_c}{T_h}}.
\end{aligned}$$

(e) For $T_h = 600°\text{C} = 873$ K and $T_c = 25°\text{C} = 298$ K, the efficiency is $\eta = 0.42$. The Carnot efficiency is $\eta_C = 1 - (T_c/T_h) = 0.66$. So the efficiency of the typical coal power plant is closer to the more realistic efficiency calculated in this problem than to the Carnot efficiency.