Physics 112, Fall 2017, Holzapfel

Problem Set 5 (6 problems). Due Monday, October 9, 5 PM

Problem 1: **Seeing in the Dark**
Consider a light bulb with a tungsten filament at a temperature of $T_f = 3500\,\text{K}$.

(a) If the power input is $100\,\text{W}$ and the emssivity is $\varepsilon = 0.30$, what is the surface area A of the lightbulb filament.

(b) The radiation reaching your retina consists of two components: The 300K blackbody radiation from $\sim 4\pi$ steradians inside the eye, and visible photons arriving in the solid angle filled by the source. Suppose the light bulb is 10 meters away. What is the ratio of the energy flux at your retina from these two sources. Even though the background provides much more flux than the source, the room still gets dark when you turn out the lights. Why is this?

(c) The eyes can be treated as nearly ideal photon detectors operating in the (relatively) narrow band of visible radiation. Even though the number of visible wavelength photons produced by a 300K source is extremely low, it is not zero. Therefore, the $T_b = 300\,\text{K}$ background can present a fundamental limitation to the detection of faint sources. Assume that our eyes are operating in a narrow band at the peak of the lightbulb filament blackbody spectrum and that $T_f >> T_b$. Show that the maximum distance at which one could see a light bulb is:

$$D = \sqrt{\varepsilon \frac{A}{4\pi} \frac{\exp\left(2.82\frac{T_f}{T_b}\right)}{[\exp(2.82) - 1]}}. \tag{1}$$

Compare this result with your experience. How good a job does the human eye do at reaching this fundamental limitation. How could the human eye be changed to improve its performance at this task? (hint: It may be somewhat uncomfortable)

Problem 2: **Liquid ^{4}He**
Kittel 4.14

Problem 3: **The Big Bang**
Kittel, 4.18

Problem 4: **Melting of crystalline solids and the zero point motion**

We start with a perfect crystal i.e. a system of N atoms arranged in a periodic arrangement. As you know, as you turn up the temperature of a solid, the atoms start to vibrate with larger and larger amplitude. Lindemann, in 1910, suggested that one might expect that when this amplitude becomes a significant fraction of the equilibrium spacing between atoms, melting occurs i.e. the crystal undergoes a phase transition from an ordered to a disordered state. In this problem, we're going to use the Debye model discussed in class and in the textbook to derive an equation for the temperature at which this occurs, one which is quite successful at predicting experimental results.

In the second part of the problem, you will find that even at zero-temperature, the atoms in a solid are vibrating with astonishingly large amplitudes as a result of the zero-point motion.

The problem basically consists of calculating the mean square displacement of an atom $< x^2 >$ as a function of temperature. We assume that the modes of the solid are independent of each other (a good approximation). First of all, given the mean square displacement of a particular mode k of the solid $< x_k^2 >$, the mean square displacement of an atom can be written as a sum of this quantity over all modes of the solid:

$$< x^2 >= \sum_k < x_k^2 > \tag{2}$$

Now, for a simple harmonic oscillator, on average the kinetic energy and the potential energy are equal and the total energy is the sum of the two. Thus the total energy is on average twice the potential energy. In the case of a SHO, the total energy in mode k is thus

$$E_k = Nm\omega_k^2 < x_k^2 > \tag{3}$$

Here N is the total number of atoms in the crystal and m is the mass of a single atom.

a) Express the total mean square displacement $< x^2 >$ of an atom in terms of a sum over all modes of the crystal and the variables E_k,N,m and ω_k (this is just simple algebra)

b) Quantize the energies of the vibrational modes by expressing E_k in terms of the number of phonons n_k in mode k. It's important to include the zero-point energy.

c) Convert your sum over modes to an integral over frequencies by using the density of states D(ω). You'll need to make a few changes to the density of states we've already derived for the photon gas.

d) Consider the high temperature limit i.e. $T >> \Theta$ where Θ is the Debye temperature. In this limit, show that

$$< x^2 >= 9\frac{\hbar^2 T}{mk_B\Theta^2} \tag{4}$$

e) Supposing that melting occurs when the rms displacement is one-tenth of the crystalline lattice spacing a, derive an equation for the melting temperature of a solid. This gives very reasonable results for many systems.

f) Finally, consider the zero temperature limit. Show that the zero-point contribution gives rise to a mean-square displacement

$$< x^2 >= \frac{9\hbar^2}{4mk_B\Theta} \tag{5}$$

g) Neon has a Debye temperature of 75 K. Show that the rms displacement is a significant fraction of an Angstrom. This constitutes a huge vibrational amplitude for a system at zero temperature.

Note that your result depends inversely on the mass. Thus for lighter atoms, the zero-point contribution is more significant. Because of this fact, He remains in liquid form all the way down to absolute zero.

Problem 5: **Adiabatic Demagnetization**
This problem describes a very useful technique for cooling systems to low temperatures (into the milli-Kelvin range).

Consider an N-particle spin 1/2 system at temperature τ_i immersed in a magnetic field B. Suppose each particle has magnetic moment m and that the particles are arranged on a lattice in a crystal (and are thus distinguishable) so that they are coupled thermally to the lattice vibrations of the crystal. Neglect any interaction between the spins. Further assume that B is small enough so that mB/τ is small compared to 1.

a) Let's first consider the spin 1/2 system uncoupled to the lattice. Calculate the free energy and the entropy of the N-spin 1/2 system to second order in mB/τ.

b) Does your final result for the entropy agree with what you expect at B=0? Explain.

c) Now consider the solid to which the spin system is coupled. In the Debye approximation, calculate the entropy of the lattice system as a function of temperature, assuming that the temperature is much less than the Debye temperature for the crystal. Thus write down an expression for the entropy of the total system, including the spin and the lattice contributions.

d) Now one slowly and *reversibly* reduces the magnetic field from its inital value B to zero. Calculate the final temperature τ_f of the system. Explain physically why the temperature decreases.

Problem 6: **Black Hole evaporation**

Stepan Hawking (1974,1976) calculated the entropy of a non-roating, uncharged black hole and obtained the following expression:

$$S = \frac{k_b c^3 A}{4G\hbar}, \tag{6}$$

where A is the area of the black hole $A = 4\pi R_S^2$ with R_s the Schwartschild radius, $R_S = 2GM/c^2$. M is the mass of the star and G is Newton's gravitational constant.

(a) Equate the energy of the black hole with its mass $U = Mc^2$ and show that the temperature T of a black hole is given by:

$$T = \frac{\hbar c^3}{8\pi G k_b M}. \tag{7}$$

(b) If black holes radiate according to the Stefan Boltzmann law they lose energy and therefore mass. Show that the Mass of a black hole in empty space as a function of time is given by:

$$M(t) = \left(M_0^3 - \frac{\hbar c^4}{5120\pi G^2} t\right)^{\frac{1}{3}}. \tag{8}$$

(c) For a black hole of $M_0 = 2 \times 10^{11}$ Kg, find its evaporation time and compare it with the age of the Universe $t_U \sim 14$ Gyr.

(d) If the universe is filled with a background of photons at temperature T_b, find an expression for the most massive black hole that can still evaporate. Evaluate this for the present temperature of the Cosmic Microwave Background $T_b = 2.73$ K.