1. Choose True/False for the following statements. $1 \times 4 = 4$

   (a) Any pure strategy can be expressed as a mixed strategy, but not vice versa. True

   (b) Number of strategies in Mixed Strategy Nash Equilibria (MSNE) can be higher than that of Pure Strategy Nash Equilibria (PSNE). True

   (c) In MSNE, the expected utility of a player i must be same for any strategy that has probability $[0, 1]$ in the equilibrium. False

   (d) In iterated elimination of dominated strategies, the order of elimination does not matter. False

2. Consider a game with mixed strategies is denoted as $\Gamma_{ME} = \langle N, \{\Delta S_i\}, \{U_i\}\rangle$ where N and U_i bear the usual meanings and $\Delta(S_i)$ is the set of all probability distributions on the strategy set S_i.

   (a) Define $\Delta(S_i)$ formally. 1.5

   (b) Define the function U_i. Just mention the domain and co-domain. 1.5

   Solution: See section 7.1 of *Book-Narahari*.

3. Find the MSNE for the *rock-paper-scissors* game: 3

		Player-2		
		Rock	Paper	Scissors
Player-1	Rock	0, 0	−1, 1	1, −1
	Paper	1, −1	0, 0	−1, 1
	Scissors	−1, 1	1, −1	0, 0

Solution: We have to find the Mixed Strategy Nash Equilibrium (MSNE) for the rock-paper-scissors game. Let, Player-1 selects the strategies rock, paper and scissors with probabilities x, y, and $1 - x - y$ respectively, and Player-2 selects the strategies rock, paper and scissors with proabilities p, q, and $1 - p - q$ respectively.

In equilibrium, Player-2 should *mix* her strategies in a proportion such that Player-1 has equal utility for any of her strategies, i.e. Player-1 can be indifferent between strategies.

$$u_1(R) = p.0 + q(-1) + (1 - p - q)(1)$$
$$u_1(P) = p(1) + q.0 + (1 - p - q)(-1)$$
$$u_1(S) = p(-1) + q(1) + 0.(1 - p - q)$$

Setting them equal:

$$u_1(R) = u_1(P) = u_1(S).$$

Similarly, for Player 2:

$$\begin{aligned} u_2(R) &= y(1) + (1-x-y)(-1), \\ u_2(P) &= x(-1) + (1-x-y)(1), \\ u_2(S) &= x(1) + y(-1). \end{aligned}$$

Setting them equal:

$$u_2(R) = u_2(P) = u_2(S).$$

Solving gives:

$$x = y = 1-x-y = \frac{1}{3}, \quad p = q = 1-p-q = \frac{1}{3}.$$

Thus, the MSNE strategy profile is

$$\left[\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right]$$

4. Consider any arbitrary two-player game of the following type (with a, b, c, d any arbitrary real numbers): 3

		Player-2	
		A	B
Player-1	A	$a,\ a$	$b,\ c$
	B	$c,\ b$	$d,\ d$

It is known that the game has a strongly dominant strategy equilibrium. Is it true that the above strongly dominant strategy equilibrium is the only possible mixed strategy equilibrium of the game ? Give reasons in support of your answer.

Solution: Before solving this you should be convinced with the following statements:

- A game with a finite number of players and strategies has at least one MSNE.
- An SDSE is a special case of MSNE in which only one strategy has probability 1 and others have probability 0.

The game has a SDSE. That is, for player-1, one strategy has probability 1 and others have probability 0. Similarly, for player-2, one strategy has probability 1 and others have probability 0. *But, we do not know which strategy has got which probability.* So, we need to consider all possibile **cases** for SDSE in this game.

- Case-1: Player-1: A $(p = 1)$, B $(1-p = 0)$ Player-2: A $(q = 1)$, B $(1-q = 0)$

- Case-2: Player-1: A ($p = 1$), B ($1 - p = 0$) Player-2: A ($q = 0$), B ($1 - q = 1$)
- Case-3: Player-1: A ($p = 0$), B ($1 - p = 1$) Player-2: A ($q = 1$), B ($1 - q = 0$)
- Case-4: Player-1: A ($p = 0$), B ($1 - p = 1$) Player-2: A ($q = 0$), B ($1 - q = 1$)

Now, find the MSNE of the game. Follow the conventional procedure of finding MSNE. Recall, SDSE is also an MSNE. Find the mixture of strategies of player-1 such that player-2 can randomly play any of her strategies.

$$u_2(A) = u_2(B) \tag{1}$$
$$p.a + (1-p).b = p.c + (1-p).d \tag{2}$$

Similarly, find player-2's mixture of strategies such that player-1 can randomly choose any strategy.

$$u_1(A) = u_1(B) \tag{3}$$
$$q.a + (1-q).b = q.c + (1-q).d \tag{4}$$

Now, consider equations (2) and (4) for different cases and find the relations among a, b, c and d.

- Case-1: $[p = 1, q = 1] \longrightarrow a = c$ (from eq. 2) and $a = c$ (from eq. 4) $\longrightarrow a = c$
- Case-2: $[p = 1, q = 0] \longrightarrow a = c$ and $b = d$
- Case-3: $[p = 0, q = 1] \longrightarrow b = d$ and $a = c$
- Case-3: $[p = 0, q = 0] \longrightarrow b = d$ and $b = d$ $\longrightarrow b = d$

If any of the above conditions (for case 1 to 4) hold, then the game will have an SDSE. There can be infinite number of different possibilities that do not satisfy any of the above conditions, e.g. $a > b > c > d$. That is, if the game contains all distinct values for a, b, c and d, then the game will not have MSNE which is not an SDSE.

5. Apply iterative elimination of strongly dominated strategies to the following game. 3

		Player-2		
		X	Y	Z
Player-1	A	2, 3	3, 0	0, 1
	B	0, 0	1, 6	4, 2

Solution: Checking for Strongly Dominated Strategies (Player-1), We compare Player-1's payoffs for each column:

- X: A (2) vs. B (0) $\Rightarrow$ A is better.

- Y: A (3) vs. B (1) $\Rightarrow$ A is better.
- Z: A (0) vs. B (4) $\Rightarrow$ B is better.

Since neither A nor B is strictly worse in all cases, no immediate eliminations for Player-1. Importantly, for Player-1, we cannot find a mix of more than one strategies to dominate another strategy. So, let's check for Player-2.

We compare Player-2's payoffs across strategies X, Y, and Z:

- Z: Strategy Z can be dominated by X when player-1 plays A, and by Y when player-1 plays B. Hence, Z is a candidate for elimination.

We need to find a mixture of the strategies of player-1 such that
if player-2 plays randomly any strategy between X and Y,
then Z will be dominated by the chosen strategy of player-2.

Expected Utility Analysis for Player-2:

- Let player-1 mixes her strategies as A being played with probability p and B with probability $1-p$.

Expected utilities:

$$u_2(X) = 3p + 0(1-p) = 3p \tag{5}$$

$$u_2(Y) = 0p + 6(1-p) = 6 - 6p \tag{6}$$

$$u_2(Z) = 1p + 2(1-p) = p + 2 - 2p = 2 - p \tag{7}$$

To eliminate Z, we can choose randomly between X and Y. So, we have $u_2(X) = u_2(Y)$. Note: we do not equate $u_2(Z)$ because we want to eliminate Z, i.e. ensure Z will never be played.

$$3p = 6 - 6p \tag{8}$$

$$9p = 6 \tag{9}$$

$$p = \frac{2}{3} \tag{10}$$

Thus, $1 - p = 1 - \frac{2}{3} = \frac{1}{3}$.

Of course, you can verify the expected utilities:

$$u_2(X) = 3 \times \frac{2}{3} = 2 \tag{11}$$

$$u_2(Y) = 6 - 6 \times \frac{2}{3} = 6 - 4 = 2 \tag{12}$$

$$u_2(Z) = 2 - p = 2 - \frac{2}{3} = \frac{6}{3} - \frac{2}{3} = \frac{4}{3} \tag{13}$$

Since:

$$u_2(X) = 2 > \frac{4}{3}, \quad u_2(Y) = 2 > \frac{4}{3}$$

Both $u_2(X)$ and $u_2(Y)$ yield a higher payoff than $u_2(Z)$, successfully verifying that Z is strongly dominated and should be eliminated.

Updating the payoff matrix After removing Z, the game reduces to:

	X	Y
A	(2,3)	(3,0)
B	(0,0)	(1,6)

Checking for further eliminations. We compare A vs. B for player-1:

- X: A (2) vs. B (0) $\Rightarrow$ A is better.
- Y: A (3) vs. B (1) $\Rightarrow$ A is better.

Since B is strongly dominated by A, strategy B is eliminated.

	X	Y
A	(2,3)	(3,0)

Similarly, strategy Y is strongly dominated by strategy X of player-2.

	X
A	(2,3)

Now, we are left with only one strategy profile (A, X) and hence this is the equilibrium strategy given that player-1 mixes his strategies A and B with the probability distribution $(\frac{2}{3}, \frac{1}{3})$.

6. Consider a set of charging stations S with a set of available time-slots and units denoted as T and K respectively. V is the set of vehicles that can request dynamically for recharging. Moreover, $\forall j \in S,\ t \in T,\ k \in K \ :\ \ c^j_{t,k+1} \geq c^j_{t,k}$ where c denotes cost of recharging. Given that the allocation for recharging happen at the same time-point when the request is placed, show that "not making a delay" to place recharge request is a dominant strategy for any vehicle $v_i \in V$. 2

Solution: See Theorem-1 of the research paper *Stein2013aamas*.

7. In the research paper for EV charging discussed in the class, all the buyers (EV) and sellers (charging station) send their types (private information) to a centralized agent (i.e., the reservation system). Then the centre decides the allocation of charging station and time-slots for recharging whenever a new request arrives.
   Can you suggest an efficient approach in which the reservation system can be removed but the same purpose can be achieved ? 2

   Solution: You can envision any approach which, you think, will make the reservation system decentralized.

   I will discuss one here. We can think of a *token ring* (brief-idea) based approach for decentralization of the reservation system. We can create a cycle graph connecting the vertices, i.e. charging stations. We can remove the centralized reservation system. In this approach, the recharge requests can directly go to a charging station. When an EV requests for recharging dynamically, it can directly send the request to its most preferable charging station with its preferred payment value (calculated locally). Now, think about the payoff of the charging stations. In the research paper, we saw that the reservation system attempts to maximize the overall profit of the charging stations collectively. In this approach, each charging station individually tries to maximize its own profit. Thus, one charging station may receive multiple requests for recharging at a given time point. Among the requests it receives at a given time point, it will choose the one that maximizes its own profit. It will forward the other requests to the next charging station in the ring. Similar process is executed by the next vertex and so on. Also, note that, at a given time point any charging station may receive request directly from the EVs and forwarded from some other node in the ring of charging stations.