Assignment-3

Problem 1

Suppose, X and Y are the beliefs of two different robots. H(X) represents entropy of X; this is alternatively written as H(PX) where PX is the probability distribution of X. Also, PXY is the joint distribution of X and Y. For your reference, Bayes rule can be written as $PXY(x, y) = PX|Y(x \mid y).PY(y)$.

Part (a)

This is important to quantify how much information is revealed by belief Y about the belief X. This is called Conditional Entropy. This gives a notion of uncertainty remaining in X given Y, i.e. $H(X \mid Y)$. We can write it as $H(X \mid Y) \equiv H(PXY \mid PY) := \sum yi\ PY(yi).H(PX|Y=yi)$.

RHS of the above expression is the average entropy of the conditional distribution. Show that: $H(X \mid Y) = H(X, Y) - H(Y)$

Solution:

Starting with the definition of conditional entropy: $H(X \mid Y) = \sum yi\ PY(yi) \cdot H(PX|Y=yi)$

Expanding $H(PX|Y=yi)$: $H(PX|Y=yi) = -\sum xj\ P(xj \mid yi) \log P(xj \mid yi)$

Substituting this into the conditional entropy definition: $H(X \mid Y) = \sum yi\ PY(yi) \cdot [-\sum xj\ P(xj \mid yi) \log P(xj \mid yi)] = -\sum yi \sum xj\ PY(yi) \cdot P(xj \mid yi) \log P(xj \mid yi)$

Using Bayes' rule, we know that $P(xj \mid yi) \cdot PY(yi) = PXY(xj, yi)$, so: $H(X \mid Y) = -\sum yi \sum xj\ PXY(xj, yi) \log P(xj \mid yi)$

Now, the joint entropy H(X, Y) is defined as: $H(X, Y) = -\sum yi \sum xj\ PXY(xj, yi) \log PXY(xj, yi)$

And the entropy of Y is: $H(Y) = -\sum yi\ PY(yi) \log PY(yi)$

Working with the joint entropy expression: $H(X, Y) = -\sum yi \sum xj\ PXY(xj, yi) \log PXY(xj, yi) = -\sum yi \sum xj\ PXY(xj, yi) \log [P(xj \mid yi) \cdot PY(yi)] = -\sum yi \sum xj\ PXY(xj, yi) [\log P(xj \mid yi) + \log PY(yi)] = -\sum yi \sum xj\ PXY(xj, yi) \log P(xj \mid yi) - \sum yi \sum xj\ PXY(xj, yi) \log PY(yi)$

For the second term, notice that $\sum xj\ PXY(xj, yi) = PY(yi)$, so: $-\sum yi \sum xj\ PXY(xj, yi) \log PY(yi) = -\sum yi\ PY(yi) \log PY(yi) = H(Y)$

Therefore: $H(X, Y) = -\sum yi \sum xj\ PXY(xj, yi) \log P(xj \mid yi) + H(Y)$

Rearranging: $H(X, Y) - H(Y) = -\sum yi \sum xj\ PXY(xj, yi) \log P(xj \mid yi) = H(X \mid Y)$

Thus, we've proven that: $H(X \mid Y) = H(X, Y) - H(Y)$

Part (b)

Prove that $H(X \mid Y) = 0$ if and only if $X = g(Y)$ for some function g. Here, H(.) represents entropy, X and Y are the beliefs of two different robots.

Solution:

First, let's prove the forward direction: If $X = g(Y)$ for some function g, then $H(X \mid Y) = 0$.

If $X = g(Y)$, then knowing Y completely determines X. This means that for any specific value y_i of Y, the conditional probability distribution $P(X \mid Y = y_i)$ is deterministic - it assigns probability 1 to $X = g(y_i)$ and 0 to all other values.

For a deterministic distribution, the entropy is zero. Therefore: $H(P_{X|Y=y_i}) = 0$ for all y_i

Since $H(X \mid Y) = \sum_{y_i} P_Y(y_i) \cdot H(P_{X|Y=y_i})$, and all terms in the sum are zero, we have $H(X \mid Y) = 0$.

Now for the reverse direction: If $H(X \mid Y) = 0$, then $X = g(Y)$ for some function g.

Given $H(X \mid Y) = 0$: $\sum_{y_i} P_Y(y_i) \cdot H(P_{X|Y=y_i}) = 0$

Since $P_Y(y_i) \geq 0$ and $H(P_{X|Y=y_i}) \geq 0$ (entropy is always non-negative), the only way this sum equals zero is if $H(P_{X|Y=y_i}) = 0$ for every y_i with $P_Y(y_i) > 0$.

A probability distribution has zero entropy if and only if it's deterministic (assigns probability 1 to one outcome and 0 to all others). So for each y_i with positive probability, the conditional distribution $P(X \mid Y = y_i)$ must be deterministic.

This means for each y_i, there exists exactly one value of X, let's call it $g(y_i)$, such that $P(X = g(y_i) \mid Y = y_i) = 1$.

Therefore, we can define a function g such that $X = g(Y)$ with probability 1, which means X is a function of Y.

Thus, we've proven that $H(X \mid Y) = 0$ if and only if $X = g(Y)$ for some function g.

Problem 2

In Q-1, we have defined the residual uncertainty (i.e. conditional entropy) of X given Y. In this question, we will look at the information revealed by Y about X; this term is called the mutual information $I(X; Y)$ between X and Y. Note that X, Y, H(.), P_X, P_Y, P_{XY} are the notations already explained in Q-1.

Part (a)

Write the mathematical expression for $I(X; Y)$ in terms of uncertainty of X and conditional entropy between X and Y.

Solution:

The mutual information $I(X; Y)$ represents how much information Y provides about X. Intuitively, it's the reduction in uncertainty about X when we learn Y.

Therefore, $I(X; Y)$ can be expressed as: $I(X; Y) = H(X) - H(X \mid Y)$

Where:

- $H(X)$ is the entropy (uncertainty) of X
- $H(X \mid Y)$ is the conditional entropy (remaining uncertainty of X given Y)

This formula directly captures the concept that mutual information equals the initial uncertainty minus the remaining uncertainty after knowing Y.

Part (b)

Prove that the information revealed by Y about X is same as the information revealed by X about Y.

Solution:

I need to prove that $I(X; Y) = I(Y; X)$.

Starting with $I(X; Y) = H(X) - H(X \mid Y)$

Similarly, $I(Y; X) = H(Y) - H(Y \mid X)$

From part (a) of Question 1, we know that $H(X \mid Y) = H(X, Y) - H(Y)$

Similarly, $H(Y \mid X) = H(X, Y) - H(X)$

Substituting these expressions: $I(X; Y) = H(X) - [H(X, Y) - H(Y)] = H(X) + H(Y) - H(X, Y)$

And: $I(Y; X) = H(Y) - [H(X, Y) - H(X)] = H(Y) + H(X) - H(X, Y)$

We can see that both expressions are identical: $I(X; Y) = I(Y; X) = H(X) + H(Y) - H(X, Y)$

Thus, the information revealed by Y about X is indeed the same as the information revealed by X about Y.

Part (c)

KL-divergence gives the measure of difference (or, distance) between two probability distributions. Show that: $I(X; Y) = KL(PXY \mid\mid PXPY)$

Solution:

The KL-divergence between distributions P and Q is defined as: $KL(P \mid\mid Q) = \sum P(x) \log(P(x)/Q(x))$

Let's compute $KL(P_{XY} || P_X P_Y)$:

$KL(P_{XY} || P_X P_Y) = \sum_{x,y} P_{XY}(x,y) \log(P_{XY}(x,y)/(P_X(x)P_Y(y))) = \sum_{x,y} P_{XY}(x,y) \log(P_{XY}(x,y)) - \sum_{x,y} P_{XY}(x,y) \log(P_X(x)P_Y(y)) = \sum_{x,y} P_{XY}(x,y) \log(P_{XY}(x,y)) - \sum_{x,y} P_{XY}(x,y) [\log(P_X(x)) + \log(P_Y(y))]$

Looking at the second term: $\sum_{x,y} P_{XY}(x,y) \log(P_X(x)) = \sum_x \log(P_X(x)) \sum_y P_{XY}(x,y) = \sum_x \log(P_X(x)) P_X(x) = \sum_x P_X(x) \log(P_X(x))$

Similarly: $\sum_{x,y} P_{XY}(x,y) \log(P_Y(y)) = \sum_y P_Y(y) \log(P_Y(y))$

Substituting these back: $KL(P_{XY} || P_X P_Y) = \sum_{x,y} P_{XY}(x,y) \log(P_{XY}(x,y)) - \sum_x P_X(x) \log(P_X(x)) - \sum_y P_Y(y) \log(P_Y(y)) = -H(X,Y) + H(X) + H(Y)$

From part (b), we know that: $I(X; Y) = H(X) + H(Y) - H(X, Y)$

Therefore: $KL(P_{XY} || P_X P_Y) = I(X; Y)$

Thus, the mutual information $I(X; Y)$ equals the KL-divergence between the joint distribution P_{XY} and the product of marginal distributions $P_X P_Y$.

Problem 3

Consider a mobile robot in a workspace $S = \{s_1, s_2, ..., s_n\}$. Belief of the robot at current time t is given by $P(X | \xi_t, a_t)$ where X is the random variable representing the environment and ξ_t is the history of states and actions until time t. Mathematically, $\xi_t = \langle s_1, a_1, ..., s_t \rangle$. An effective exploration strategy is to select an action that gives maximum information about the environment, i.e. maximizes the reduction in uncertainty about the environment. From the current state s_t the next state distribution is denoted as S_{t+1}.

Solution

Part (a): Mathematical expression for selecting the action that maximally reduces uncertainty in X between time t + 1 and t.

The uncertainty about the environment X at time t is represented by the entropy of the current belief: $H(X | \xi_t, a_t)$

After taking action a_t and transitioning to a new state s_{t+1}, the uncertainty becomes: $H(X | \xi_t, a_t, s_{t+1})$

The reduction in uncertainty after observing the new state is: $H(X | \xi_t, a_t) - H(X | \xi_t, a_t, s_{t+1})$

Since the next state is random (governed by distribution S_{t+1}), we consider the expected reduction in uncertainty over all possible next states:

Expected reduction = $H(X \mid \xi_t, a_t) - E[H(X \mid \xi_t, a_t, S_{t+1})] = H(X \mid \xi_t, a_t) - \sum_{s_{t+1}} P(s_{t+1} \mid s_t, a_t) \cdot H(X \mid \xi_t, a_t, s_{t+1})$

This expected reduction in uncertainty is precisely the conditional mutual information between X and S_{t+1}, given the history ξ_t and action a_t:

$I(X; S_{t+1} \mid \xi_t, a_t) = H(X \mid \xi_t, a_t) - H(X \mid S_{t+1}, \xi_t, a_t)$

Therefore, the optimal action selection criterion is:

$a_t^* = \text{argmax}[a_t]\ I(X; S_{t+1} \mid \xi_t, a_t)$

This means we should choose the action that maximizes the mutual information between the environment X and the next state S_{t+1}, conditioned on our current history and action.

Part (b): Express the above term using KL-divergence.

The conditional mutual information can be expressed using KL-divergence as follows:

$I(X; S_{t+1} \mid \xi_t, a_t) = \sum_{s_{t+1}} P(s_{t+1} \mid s_t, a_t) \cdot KL(P(X \mid \xi_t, a_t, s_{t+1}) \mid\mid P(X \mid \xi_t, a_t))$

Where the KL-divergence term is:

$KL(P(X \mid \xi_t, a_t, s_{t+1}) \mid\mid P(X \mid \xi_t, a_t)) = \sum_x P(x \mid \xi_t, a_t, s_{t+1}) \cdot \log(P(x \mid \xi_t, a_t, s_{t+1}) / P(x \mid \xi_t, a_t))$

Therefore, our exploration strategy can be written as:

$a_t^* = \text{argmax}[a_t] \sum_{s_{t+1}} P(s_{t+1} \mid s_t, a_t) \cdot KL(P(X \mid \xi_t, a_t, s_{t+1}) \mid\mid P(X \mid \xi_t, a_t))$

This formulation quantifies the expected "distance" between the posterior belief after observing s_{t+1} and the prior belief before the observation, averaged over all possible next states. The action that maximizes this expected KL-divergence will provide the most informative observations about the environment.