CSE 667 (DMMRS) Assignment-3 Full marks: 20 **Deadline: 24-04-2025**

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1. Suppose, X and Y are the beliefs of two different robots. $H(X)$ represents entropy of X; this is alternatively written as $H(P_X)$ where P_X is the probability distribution of X. Also, P_{XY} is the joint distribution of X and Y. For your reference, Bayes rule can be written as $P_{XY}(x, y) = P_{X|Y}(x \mid y).P_Y(y)$. 4+3

   (a) This is important to quantify how much information is revealed by belief Y about the belief X. This is called *Conditional Entropy*. This gives a notion of uncertainty remaining in X given Y, i.e. $H(X \mid Y)$. We can write it as $H(X \mid Y) \equiv H(P_{XY} \mid P_Y) := \sum_{y_i} P_Y(y_i).H(P_{X|Y=y_i})$.

   RHS of the above expression is the average entropy of the conditional distribution. Show that:

   $$H(X \mid Y) = H(X, Y) - H(Y)$$

   (b) Prove that $H(X \mid Y) = 0$ if and only if $X = g(Y)$ for some function g. Here, $H(.)$ represents entropy, X and Y are the beliefs of two different robots.

2. In Q-1, we have defined the residual uncertainty (i.e. conditional entropy) of X given Y. In this question, we will look at the *information revealed* by Y about X; this term is called the *mutual information* $I(X; Y)$ between X and Y.
   Note that X, Y, $H(.)$, P_X, P_Y, P_{XY} are the notations already explained in Q-1. 2+2+3

   (a) Write the mathematical expression for $I(X; Y)$ in terms of uncertainty of X and conditional entropy between X and Y.

   (b) Prove that the information revealed by Y about X is same as the information revealed by X about Y.

   (c) *KL-divergence* gives the measure of difference (or, distance) between two probability distributions. Show that:

   $$I(X; Y) = KL(P_{XY} \mid\mid P_X P_Y)$$

3. Consider a mobile robot in a workspace $S = \{s_1, s_2, \ldots, s_n\}$. Belief of the robot at current time t is given by $P(X \mid \xi_t, a_t)$ where X is the random variable representing the environment and ξ_t is the history of states and actions until time t. Mathematically, $\xi_t = \langle s_1, a_1, \ldots, s_t \rangle$. An effective exploration strategy is to select an action that gives maximum information about the environment, i.e. maximizes the reduction in uncertainty about the environment. From the current state s_t the next state distribution is denoted as S_{t+1}. 3+3

   (a) Write down the mathematical expression for selecting the action that maximally reduces uncertainty in X between time $t + 1$ and t.

   (b) Express the above term using *KL-divergence*.