SML 2025 Mid-Semester Exam Solutions

Q1: Compute Weights for Linear Regression

We are given the dataset:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

The linear regression model is given by:

$$y = mx + c$$

Using the normal equation:

$$W = (X^T X)^{-1} X^T Y$$

Step 1: Compute $X^T X$

$$X^T X = \begin{bmatrix} 10 & 4 \\ 4 & 3 \end{bmatrix}$$

Step 2: Compute $X^T Y$

$$X^T Y = \begin{bmatrix} -7 \\ -2 \end{bmatrix}$$

Step 3: Compute $(X^T X)^{-1}$

$$(X^T X)^{-1} = \frac{1}{14} \begin{bmatrix} 3 & -4 \\ -4 & 10 \end{bmatrix}$$

Step 4: Compute W

$$W = \frac{1}{14} \begin{bmatrix} 3 & -4 \\ -4 & 10 \end{bmatrix} \begin{bmatrix} -7 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{13}{14} \\ \frac{4}{7} \end{bmatrix}$$

Thus, the regression equation is:

$$y = -\frac{13}{14} x + \frac{4}{7}$$

Q2: MAP Estimate for θ in Rayleigh Distribution

Given the likelihood function:

$$p(x|\theta) = \theta x e^{-\frac{\theta x^2}{2}}, \quad x \geq 0$$

Prior:

$$P(\theta) = \lambda e^{-\lambda\theta}, \quad \theta \geq 0$$

The posterior is proportional to:

$$p(\theta|x) \propto P(\theta) \prod_{i=1}^{n} p(x_i|\theta)$$

Taking the log:

$$\log p(\theta|x) = \sum_{i=1}^{n} \left[\log\theta + \log x_i - \frac{\theta x_i^2}{2} \right] - \lambda\theta$$

Solving for θ:

$$\theta_{\text{MAP}} = \frac{n}{\sum x_i^2 + 2\lambda}$$

Q3: First Principal Component

Given:

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Step 1: Centralize Data

$$\mu = \frac{1}{2} \begin{bmatrix} (1+0) \\ (1+1) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

Subtract mean:

$$X_{\text{centered}} = X - \mu = \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0 \end{bmatrix}$$

Step 2: Compute Covariance Matrix

$$\Sigma = \frac{1}{1} X_{\text{centered}} X_{\text{centered}}^T = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}$$

Step 3: Compute Eigenvectors

$$\text{First Principal Component} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Q4: FDA Projection

Given:

$$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X_2 = -X_1$$

Step 1: Compute Class Means

$$\mu_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad \mu_2 = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}$$

Step 2: Compute Scatter Matrices

$$S_1 = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$$S_W = S_1 + S_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Step 3: Compute FDA Projection Vector

$$w = S_W^{-1}(\mu_1 - \mu_2)$$

If inverse doesnt exist, do $(S_W + 0.1 * I)^{-1}$
Step 4: Classify x_{test}

$$x_{\text{test}} = \begin{bmatrix} 0.25 \\ -0.75 \end{bmatrix}$$

If

$$w^T x_{\text{test}} < 0 \Rightarrow x_{\text{test}} \in X_2$$

Q5: FDA Projection After PCA

Applying PCA before FDA, the steps are:
Step 1: Transform data into PCA space

$$Y = U_p^T X$$

Step 2: Compute mean and scatter matrices in PCA space

$$\mu_1' = U_p^T \mu_1, \quad \mu_2' = U_p^T \mu_2$$

$$S_W' = U_p^T S_W U_p, \quad S_B' = U_p^T S_B U_p$$

Step 3: Compute FDA projection in PCA space

$$w' = S_W'^{-1}(\mu_1' - \mu_2')$$

Step 4: Map back to original space

$$w = U_p w' = U_p (U_p^T S_W U_p)^{-1} U_p^T (\mu_1 - \mu_2)$$