SML2025

Practice

Q1. Use the Bayesian decision rule to classify a fish as sea bass or salmon. The likelihoods are given as:

$$P(x \mid \omega_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}$$

$$P(x \mid \omega_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-3)^2}{2}}$$

The priors are:

$$P(\omega_1) = 0.5, \quad P(\omega_2) = 0.5.$$

The posterior probabilities for each class can be computed using Bayes' theorem:

$$P(\omega_j \mid x) = \frac{P(x \mid \omega_j) P(\omega_j)}{P(x)},$$

where

$$P(x) = P(x \mid \omega_1) P(\omega_1) + P(x \mid \omega_2) P(\omega_2).$$

Tasks

1. Compute the posterior probabilities $P(\omega_1 \mid x)$ and $P(\omega_2 \mid x)$. 2. Derive the Bayesian decision boundary, where:

$$P(\omega_1 \mid x) = P(\omega_2 \mid x).$$

3. Solve for the decision boundary in terms of x.

Q2

You are given the following set of 3D sample observations:

$$\mathbf{X} = \begin{bmatrix} 1.2 & 2.4 & 3.1 \\ 0.8 & 2.1 & 2.9 \\ 1.5 & 2.5 & 3.3 \\ 1.1 & 2.3 & 3.0 \\ 0.9 & 2.0 & 2.8 \end{bmatrix}$$

Each row represents a sample in 3D space, with the columns corresponding to the three features x_1, x_2, and x_3.

Tasks

1. **Compute the Mean Vector**
Calculate the mean vector for each feature:

$$\mu = \begin{bmatrix} \mu_{x_1} \\ \mu_{x_2} \\ \mu_{x_3} \end{bmatrix}$$

2. **Center the Data**
Subtract the mean vector from each sample to obtain the centered data matrix $\mathbf{X}_c$.

3. **Compute the Covariance Matrix**
Using the centered data matrix $\mathbf{X}_c$, calculate the covariance matrix Σ using the formula:

$$\Sigma = \frac{1}{n-1}\mathbf{X}_c^\top \mathbf{X}_c$$

where n is the number of samples.

4. **Interpret the Covariance Matrix**
Discuss the meaning of the diagonal and off-diagonal elements of the covariance matrix. What do these values indicate about the relationships between the features?

Q3.

Let $\mathbf{x}$ be a random variable that follows a Multivariate Gaussian Distribution with a mean vector μ and a covariance matrix Σ. The probability density function (PDF) is given by:

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^\top \Sigma^{-1}(\mathbf{x}-\mu)\right),$$

where:

- $\mathbf{x} \in \mathbb{R}^d$ is the point of interest,
- $\mu \in \mathbb{R}^d$ is the mean vector,
- $\Sigma \in \mathbb{R}^{d\times d}$ is the covariance matrix,
- $|\Sigma|$ is the determinant of Σ.

Given:

$$\mathbf{x} = \begin{bmatrix} 1.5 \\ 2.0 \end{bmatrix}, \quad \mu = \begin{bmatrix} 1.0 \\ 2.5 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}.$$

Tasks:

1. **Compute the Mahalanobis Distance**
Evaluate the quadratic form:

$$(\mathbf{x} - \mu)^\top \Sigma^{-1} (\mathbf{x} - \mu).$$

2. **Calculate the Determinant of Σ**
Compute $|\Sigma|$, the determinant of the covariance matrix.
3. **Compute the Likelihood**
Using the given PDF, calculate the likelihood $p(\mathbf{x})$.

Q4. Find if the covariance matrix is Positive Semidefinite (PSD).

Q5. Find var of y, when $y = ax + b$, a, b are constants and x is scalar with mean μ and variance σ^2. Find covariance of $\mathbf{y}$, when $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$, a, b are constants and $\mathbf{x}$ is vector with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$.