Quiz1) **Q1)** Given, $x_1 \sim N(0, I)$, $y_1 \sim \text{Bernoulli}(p)$

The joint liklihood $p(x, y|\omega_1)$ is:

$$p(x, y|\omega_1) = p(x|\omega_1)\, p(y|\omega_1)$$

For $x_1 \sim N(0,1)$: $p(x|\omega_1) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2} x^T x\right)$

log liklihood is - $\ln p(x|\omega_1) = -\frac{d}{2} \ln(2\pi) - \frac{1}{2} x^T x$

For $y_1 \sim \text{Bernoulli}(p)$; $p(y|\omega_1) = p^y (1-p)^{1-y}$

log-liklihood is - $\ln p(y|\omega_1) = y \ln p + (1-y) \ln(1-p)$

Discriminent function:

$$\left[g_1(x_1, y|\omega_1) = -\frac{d}{2} \ln(2\pi) - \frac{1}{2} x^T x + y \ln p + (1-y) \ln(1-p) \right]$$

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it is correct even if first term is not included

Quiz Q2

LRT- likelihood ratio test

$x = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$ ω_1, ω_2 - 2 classes

$\Sigma_1 = I$ $\mu_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\mu_1, \mu_2 \in \mathbb{R}^2$

$P(\omega_1) = P(\omega_2) = 1/2$
(given)

$\Sigma_2 = I$ $\mu_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ $\lambda_{11} = 0$, $\lambda_{12} = 2$
$\lambda_{21} = 1$, $\lambda_{22} = 0$

MVG distribution

$$P(x|\mu, \Sigma) = \frac{1}{\sqrt{2\pi}\, |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$

(2pi)^(d/2)

$\alpha_1 \Rightarrow$ action of choosing/classifying x to class ω_1

$\alpha_2 \Rightarrow$ action of classifying x as ω_2

~~$R(\alpha_1|x)$~~ = . Defining the risk for the actions α_1, α_2

$$R(\alpha_1|x) = \sum_{j=1}^{2} \lambda_{1j} P(\omega_j|x) = \overset{0}{\cancel{\lambda_{11}}} P(\omega_1|x) + \overset{2}{\cancel{\lambda_{12}}} P(\omega_2|x)$$

$$R(\alpha_2|x) = \sum_{j=1}^{2} \lambda_{2j} P(\omega_j|x) = \overset{1}{\cancel{\lambda_{21}}} P(\omega_1|x) + \overset{0}{\cancel{\lambda_{22}}} P(\omega_2|x)$$

$R(\alpha_1|x) = 2P(\omega_2|x)$ $R(\alpha_2|x) = P(\omega_1|x)$

Choose x as ω_1 if $R(\alpha_1|x) < R(\alpha_2|x)$

$$= 2P(\omega_2|x) < P(\omega_1|x)$$

$$= \frac{P(\omega_1|x)}{P(\omega_2|x)} > 2$$

Bayes rule

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$$= \frac{\dfrac{P(x|\omega_1)P(\omega_1)}{\cancel{P(x)}}}{\dfrac{P(x|\omega_2)P(\omega_2)}{\cancel{P(x)}}} > 2$$

$P(\omega_1) = P(\omega_2) = 1/2$

x as ω_1 if $\dfrac{P(x|\omega_1)}{P(x|\omega_2)} > 2$

$x \in w_1$ if $\dfrac{P(x|w_1)}{P(x|w_2)} > 2$

$$P(x|w_1) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2}(x-\mu_1)^T(x-\mu_1)\right)$$

$$P(x|w_2) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-1}{2}(x-\mu_2)^T(x-\mu_2)\right)$$

$\left\{ \Sigma = I,\ |\Sigma| = 1,\ \Sigma^{-1} = I \right\}$

$$\frac{P(x|w_1)}{P(x|w_2)} = \frac{\exp\left(\frac{-1}{2}(x-\mu_1)^T(x-\mu_1)\right)}{\exp\left(\frac{-1}{2}(x-\mu_2)^T(x-\mu_2)\right)}$$

plug in the values

$$= \frac{\exp\left(\frac{-1}{2}\left([0.5 \quad 0.25]\begin{bmatrix}0.5\\0.25\end{bmatrix}\right)\right)}{\exp\left(\frac{-1}{2}\left([-0.5 \quad -0.75]\begin{bmatrix}-0.5\\-0.75\end{bmatrix}\right)\right)}$$

0.75

$$= \frac{\exp\left(\frac{-1}{2}(0.5^2+0.25^2)\right)}{\exp\left(\frac{-1}{2}(0.5^2+0.75^2)\right)} = \frac{e^{-\frac{1}{2}(0.3125)}}{e^{-1/2(0.8125)}}$$

$$= \frac{e^{-0.15625}}{e^{-0.40625}} = \frac{0.8553}{0.6661}$$

$$\boxed{= 1.284}$$

classify w_1 if $\dfrac{P(x|w_1)}{P(x|w_2)} > 2$ $\quad 1.284 > 2$ ✗

Hence we classify $\boxed{x \text{ as } w_2}$

0.25

$$\boxed{x \in w_2}$$

Q.3

(a) $P(\omega_1) = P(\omega_2) = 0.5$

- Bayes rule compares,

$$p(x|\omega_1) P(\omega_1) \quad \text{and} \quad p(x|\omega_2) P(\omega_2)$$

- Decision Boundary condition:

$$p(x|\omega_1) P(\omega_1) = p(x|\omega_2) P(\omega_2)$$

Given $P(\omega_1) = P(\omega_2)$,

we only need to compare $p(x|\omega_1)$ and $p(x|\omega_2)$

- for $x > 0$

$p(x|\omega_1) = \lambda e^{-\lambda x}$ ... positive

$p(x|\omega_2) = 0$ ... support vector for $\omega_2 \leq 0$

$p(x|\omega_1) > p(x|\omega_2)$ $\therefore$ for $x>0$, we choose ω_1.

- for $x < 0$

$p(x|\omega_1) = 0$

$p(x|\omega_2) = \lambda e^{\lambda x}$

$p(x|\omega_1) < p(x|\omega_2)$ $\therefore$ for $x<0$, we choose ω_2.

At $x = 0$

we have,

$$p(x|\omega_1) P(\omega_1) = p(x|\omega_2) P(\omega_2)$$

$\therefore$ Decision Boundary is at $x = 0$ $\quad \begin{cases} x > 0 \rightarrow \omega_1 \\ x < 0 \rightarrow \omega_2 \end{cases}$

0.5

(b) True class ω_2, But classified as ω_1

$$p(\omega_1|\omega_2) = p(x \geq 0|\omega_2)$$

$$p(x \geq 0|\omega_2) = \int_0^\infty p(x|\omega_2) p(\omega_2)\, dx$$

$$= \int_0^\infty 0\, dx \quad \ldots \; p(x|\omega_2) = 0 \text{ for } x > 0.$$

$$= 0$$

0.5

Hence, the probability of misclassifying a sample from ω_2 as ω_1 is zero.

Quiz Q4.

Given, $k(\beta) = \frac{\beta(1-\beta)}{2}(\mu_2-\mu_1)^T[\beta\Sigma_1+(1-\beta)\Sigma_2]^{-1}(\mu_2-\mu_1)$

$$+ \frac{1}{2}\ln\frac{|\beta\Sigma_1+(1-\beta)\Sigma_2|}{|\Sigma_1|^{\beta}|\Sigma_2|^{1-\beta}}$$

$\mu_1 = \mu_2$

$\Sigma_1 = 1, \ \Sigma_2 = 2$

Equal Prior: $P(C_1) = P(C_2) = 0.5$

$\because (\mu_2-\mu_1) = 0 \Rightarrow (\mu_2-\mu_1)^T[\beta\Sigma_1+(1-\beta)\Sigma_2]^{-1}(\mu_2-\mu_1) = 0$

$$\therefore k(\beta) = \frac{1}{2}\ln\frac{|\beta\Sigma_1+(1-\beta)\Sigma_2|}{|\Sigma_1|^{\beta}|\Sigma_2|^{1-\beta}}$$

$$= \frac{1}{2}\ln\frac{|\beta+2-2\beta|}{|1|^{\beta}|2|^{1-\beta}}$$

$$\Rightarrow k(\beta) = \frac{1}{2}\ln\frac{2-\beta}{2^{1-\beta}}$$

0.5

$\therefore$ To find β^*, we diff. above eq^n w.r.t. β,

$$\Rightarrow \frac{d}{d\beta}\left(\frac{1}{2}\ln\frac{2-\beta}{2^{1-\beta}}\right) = 0$$

$$\therefore \beta^* \rightarrow \frac{1}{2}\left(\frac{-1}{2-\beta^*} - (-\ln 2)\right) = 0$$

0.5

$$\Rightarrow \frac{-1}{2-\beta^*} + \ln 2 = 0$$

$$\Rightarrow \beta^* = 2 - \frac{1}{\ln 2}$$

1

$$\approx 0.7726$$

Q5:

Given, $P(\omega_1) = P(\omega_2) = \frac{1}{2}$

$$P(x|\omega_1) = N(0,1) = \frac{1}{\sqrt{2\pi \cdot 1}} e^{-\frac{1}{2}(x-0)^2}$$

$$P(x|\omega_2) = N(1,1) = \frac{1}{\sqrt{2\pi \cdot 1}} e^{-\frac{1}{2}(x-1)^2}$$

For D.B →

$$P(\omega_1|x) = P(\omega_2|x)$$

$$\Rightarrow P(\omega_1) P(x|\omega_1) = P(\omega_2)(P(x|\omega_2))$$

$$\Rightarrow \frac{1}{2} * \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} = \frac{1}{2} * \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2}$$

$$\Rightarrow e^{-\frac{1}{2}x^2} = e^{-\frac{1}{2}(x-1)^2}$$

$$\Rightarrow -x^2 = -(x-1)^2$$

$$\Rightarrow -x^2 = -x^2 + 2x - 1$$

$$\Rightarrow x = \frac{1}{2} = 0.5$$

0.5

So, D.B lies at $x = 0.5$

$$P(\text{error}|x) = \min(P(\omega_1|x), P(\omega_2|x))$$

$$P(\text{error}) = \int_{-\infty}^{+\infty} \min(P(\omega_1|x), P(\omega_2|x))\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{0.5} P(x|\omega_2)\, dx + \frac{1}{2}\int_{0.5}^{\infty} P(x|\omega_1)\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{0.5} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2} dx + \int_{0.5}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

0.5

Now, further breaking down the first expression as $y = x - 1$
$(P(\text{error}|\omega_2))$

$$P(\text{error}|\omega_2) = \frac{1}{2}\int_{-\infty}^{0.5} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}} dx$$

$$= \frac{1}{2}\int_{-\infty}^{-0.5} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

which is equivalent to $\frac{1}{2}\int_{0.5}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2} dy$

$$P(\text{error}) = P(\text{error}|\omega_2) + P(\text{error}|\omega_1)$$

$$= \frac{1}{2}\int_{-\infty}^{-0.5} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy + \frac{1}{2}\int_{0.5}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$= 2 * \frac{1}{2}\int_{0.5}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$$P_e = \int_{0.5}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \quad \text{(Proved)}$$

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