SML2025

Assign-1, Deadline Friday 24th Jan 5pm.

Q1 Decision Boundary for Multivariate Gaussian under Linear Transformation

Consider a binary classification problem with two classes, ω_1 and ω_2, where the random variable X is multivariate Gaussian:

$$X \sim \begin{cases} N(\mu_1, \Sigma_1), & \text{if } \omega_1, \\ N(\mu_2, \Sigma_2), & \text{if } \omega_2. \end{cases}$$

Let Y be a linear transformation of X:

$$Y = AX + b,$$

where:

- A is a $k \times d$ matrix,
- b is a k-dimensional bias vector.

The prior probabilities of the classes are $P(\omega_1)$ and $P(\omega_2)$.
Find the decision boundary such that $g_1(Y) = g_2(Y)$. [1]

Q2 Poisson Distribution: Properties and Bayes Classification

The Poisson distribution for a discrete random variable $x = 0, 1, 2, \ldots$ and a real parameter λ is given by:

$$P(x \mid \lambda) = \frac{e^{-\lambda}\lambda^x}{x!}.$$

(a) Prove that the mean of the distribution is $E[x] = \lambda$.

(b) Prove that the variance of the distribution is $\text{Var}[x] = \lambda$.

(c) Consider two equally probable categories, each following a Poisson distribution with differing parameters $\lambda_1 > \lambda_2$. Derive the Bayes classification decision rule for such a scenario. [.5]

(e) Derive the Bayes error rate for this classification problem. Note the decision boundary may be a positive scalar. However, when we plug in the equation for probability of error, the likelihoods only take integers, so you need to round off the decision boundary to the greatest integer smaller than equal to decision boundary. [.5]

Q3 Minimum Risk Discriminant Function for Binary Classification

Under the natural assumption concerning losses, i.e., that $\lambda_{21} > \lambda_{11}$ and $\lambda_{12} > \lambda_{22}$, show that the general minimum risk discriminant function for the independent binary case described in the lecture is given by: [1]

$$g(x) = w^\top x + w_0,$$

where w is unchanged, and

$$w_0 = \sum_{i=1}^{d} \ln \frac{1 - p_i}{1 - q_i} + \ln \frac{P(\omega_1)}{P(\omega_2)} + \ln \frac{\lambda_{21} - \lambda_{11}}{\lambda_{12} - \lambda_{22}}.$$

Q4 Minimizing Bayes risk

Consider a two-class classification problem with arbitrary probability density functions (PDFs) $p(x \mid \omega_1)$ and $p(x \mid \omega_2)$, and prior probabilities $P(\omega_1)$ and $P(\omega_2)$. The decision boundary x^* divides the feature space into two regions:

$$R_1 = \{x \mid x \leq x^*\}, \quad R_2 = \{x \mid x > x^*\}.$$

Suppose the classification problem incorporates a non-symmetric loss matrix:

$$\Lambda = \begin{bmatrix} 0 & \lambda_{12} \\ \lambda_{21} & 0 \end{bmatrix}.$$

Derive the decision rule that minimizes the expected risk (Bayes risk). [1]