SML2025

Practice

Q1. Consider two Cauchy distributions in one dimension, where the probability density function (PDF) for each class is given by:

$$p(x \mid \omega_i) = \frac{1}{\pi b \left(1 + \left(\frac{x-a_i}{b}\right)^2\right)}, \quad i = 1, 2,$$

where:

- a_1 and a_2 are the location parameters ($a_2 > a_1$),
- b is the scale parameter (common for both distributions),
- $P(\omega_1) = P(\omega_2) = 0.5$ (equal priors).

The Neyman-Pearson criterion is applied with a zero-one loss function, where the maximum acceptable error rate for misclassifying a sample from ω_1 as belonging to ω_2 is E_1.

Tasks

1. **Decision Boundary:** Determine the decision boundary in terms of the parameters a_1, a_2, b, and E_1.
2. **Error Rate for ω_2:** For the derived decision boundary, compute the error rate for misclassifying a sample from ω_2 as belonging to ω_1, denoted as $P(\text{error} \mid \omega_2)$.
3. **Overall Error Rate:** Compute the overall error rate $P(\text{error})$ under the zero-one loss assumption:

$$P(\text{error}) = 0.5 \cdot P(\text{error} \mid \omega_1) + 0.5 \cdot P(\text{error} \mid \omega_2).$$

4. **Specific Case:** Apply the results to the specific case where:
   - $b = 1$,
   - $a_1 = -1$,

- $a_2 = 1$,
- $E_1 = 0.1$.

Compute:

(a) The decision boundary.
(b) The error rate $P(\text{error} \mid \omega_2)$.
(c) The overall error rate $P(\text{error})$.

Q2.
Consider a binary classification problem where a random variable x follows one of two Cauchy distributions

$$p(x \mid \omega_i) = \frac{1}{\pi b \left(1 + \left(\frac{x-a_i}{b}\right)^2\right)}, \quad i = 1, 2,$$

and:

- a_1 and a_2 are the location parameters ($a_2 > a_1$),
- b is the common scale parameter for both distributions,
- $P(\omega_1) = P_1$ and $P(\omega_2) = P_2$ are the prior probabilities, with $P_1 + P_2 = 1$.

Tasks

1. Find the discriminant function for each class.
2. Find the decision boundary.
3. **Specific Case:** Apply the results to the following parameters:
   - $a_1 = -1$,
   - $a_2 = 1$,
   - $b = 1$,
   - $P_1 = 0.7$, $P_2 = 0.3$.

   (a) Compute the decision boundary x^*. An analytical form may not be feasbile.
   (b) Determine the class decision for $x = 0$ and $x = 2$.

Q3.
The following are the 2×3 matrices for the three classes:

Data matrix for Class 1 (A)

$$A = \begin{bmatrix} 0.12 & 0.45 & 0.78 \\ 0.34 & 0.56 & 0.91 \end{bmatrix}$$

Data matrix for Class 2 (B)

$$B = \begin{bmatrix} 0.23 & 0.67 & 0.89 \\ 0.45 & 0.12 & 0.34 \end{bmatrix}$$

Data matrix for Class 3 (C)

$$C = \begin{bmatrix} 0.98 & 0.76 & 0.11 \\ 0.33 & 0.44 & 0.55 \end{bmatrix}$$

Assume dimension to be 2. Let the data be distributed according to MVG.

- Compute likelihood for each class.
- Compute the prior.
- Using discriminant analysis, find to which class does a point $x = [0, 0]^\top$ belong to.
- Find the decision boundary between classes A and B.