Question: Bias of the MAP Estimate for a Gaussian Mean

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) samples from a normal distribution:

$$X_i \sim \mathcal{N}(\mu, \sigma^2), \quad i = 1, 2, \ldots, n$$

where μ is the unknown mean and σ^2 is known. Assume we place a normal prior on μ:

$$\mu \sim \mathcal{N}(\mu_0, \tau^2).$$

1. Derive the Maximum A Posteriori (MAP) estimate of μ.
2. Compute the expectation $E[\hat{\mu}_{MAP}]$ and determine whether the MAP estimate is an unbiased estimator of μ.

Question: MAP Estimate for a Gaussian Mean with Uniform Prior

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) samples from a normal distribution:

$$X_i \sim \mathcal{N}(\mu, \sigma^2), \quad i = 1, 2, \ldots, n$$

where σ^2 is known, and μ is the unknown mean.
We assume a **uniform prior** on μ:

$$\mu \sim U(-1, 1)$$

1. Write down the likelihood function $P(X|\mu)$ for the given data.
2. Derive the Maximum A Posteriori (MAP) estimate of μ.
3. If the prior were $U(-a, a)$, how would the MAP estimate change?

Question: Bias of Mean Estimator Based on Min and Max

Let D=$\{X_1, X_2, \ldots, X_n\}$ be independent and identically distributed (i.i.d.) samples from a normal distribution:

$$X_i \sim \mathcal{N}(\mu, \sigma^2), \quad i = 1, 2, \ldots, n$$

We estimate the mean using:

$$\hat{\mu} = \frac{\min(D) + \max(D)}{2}$$

Find if $\hat{\mu}$ is biased or unbiased estimator for the true mean μ.

Question: MAP Estimate for the Mean of a Multivariate Gaussian with a Multivariate Bernoulli Prior

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) samples from a multivariate normal distribution:

$$X_i \sim \mathcal{N}(\mu, \Sigma), \quad i = 1, 2, \ldots, n$$

where:

- μ is the unknown mean vector of dimension d.
- Σ is the known covariance matrix of size $d \times d$.

We assume a multivariate Bernoulli prior on μ, where each component μ_j follows an independent Bernoulli distribution:

$$P(\mu) = \prod_{j=1}^{d} p_j^{\mu_j} (1 - p_j)^{1-\mu_j}, \quad \mu_j \in \{0, 1\}.$$

Compute the unnormalized (without evidence) posterior distribution $P(\mu|X)$ and derive the MAP estimate $\hat{\mu}_{MAP}$.