Project 2: GPS positioning
Due date: 2023-10-18

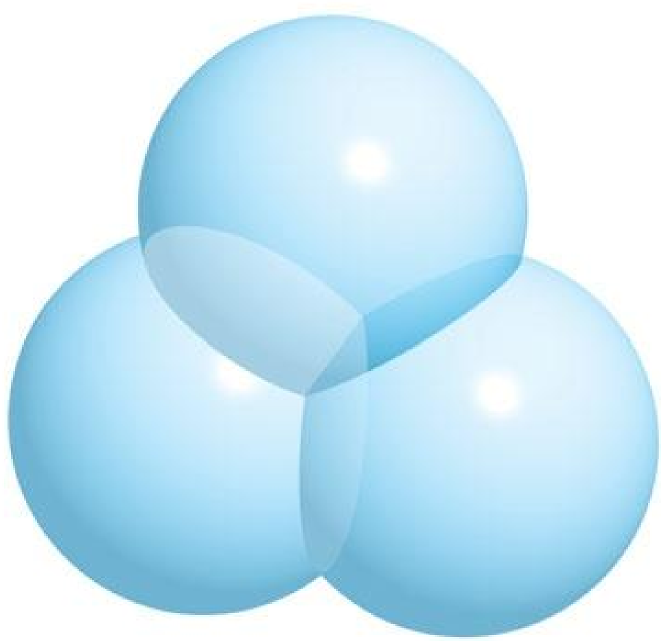
Project

1. GPS fundamentals

- GPS is based on time and the known position of 24 GPS specialized satellites.
- Satellites carry atomic clocks that are synchronized with one another. The satellite locations are also tracked with great precision.
- Satellites continuously transmit data about their current time and location.
- A receiver monitors multiple satellites and solves equations to determine the precise (x, y, z) position of the receiver. (At any time, 5–8 satellites are visible).

2. GPS: detailed description

- At a given instant, the receiver collects signals from 3 satellites and determines their transmission times t_i = the difference between their time of transmission and time of arrival
- The distance of the satellite from the receiver is $r_i = ct_i$, where $c = 299,792.458$ km/sec is the speed of light.
- The receiver is on the surface of a sphere centered at the (known) position of the satellite (A_i, B_i, C_i) with radius r_i.
- Three spheres are known, whose intersection consists of two points. One intersection is the location (x, y, z) of the receiver.



3. GPS equations

- Let (A_i, B_i, C_i) be the location of satellite i.
- Let t_i be the transmission time of satellite i.
- Then the intersection point (x, y, z) of the three spheres satisfies

$$\begin{aligned}\sqrt{(x-A_1)^2+(y-B_1)^2+(z-C_1)^2} &= ct_1\\ \sqrt{(x-A_2)^2+(y-B_2)^2+(z-C_2)^2} &= ct_2\\ \sqrt{(x-A_3)^2+(y-B_3)^2+(z-C_3)^2} &= ct_3.\end{aligned} \tag{1}$$

4. GPS: Time correction

- GPS receiver clocks are much less precise. Thus, there is a drift d between the receiver and satellite clocks.
- To fix this problem, introduce one more equation using a fourth satellite:

$$\begin{aligned}\sqrt{(x-A_1)^2+(y-B_1)^2+(z-C_1)^2} &= c(t_1-d)\\ \sqrt{(x-A_2)^2+(y-B_2)^2+(z-C_2)^2} &= c(t_2-d)\\ \sqrt{(x-A_3)^2+(y-B_3)^2+(z-C_3)^2} &= c(t_3-d)\\ \sqrt{(x-A_4)^2+(y-B_4)^2+(z-C_4)^2} &= c(t_4-d).\end{aligned} \tag{2}$$

- Solve for the unknowns (x, y, z) (position of receiver) and d (time drift of receiver).

5. GPS: Solving the system

- Rewrite the system in a more convenient form and solve for (x, y, z, d)

$$\begin{aligned}(x-A_1)^2+(y-B_1)^2+(z-C_1)^2 &= [c(t_1-d)]^2\\ (x-A_2)^2+(y-B_2)^2+(z-C_2)^2 &= [c(t_2-d)]^2\\ (x-A_3)^2+(y-B_3)^2+(z-C_3)^2 &= [c(t_3-d)]^2\\ (x-A_4)^2+(y-B_4)^2+(z-C_4)^2 &= [c(t_4-d)]^2\end{aligned} \tag{3}$$

- Algebraic solution:
  - Subtract the last three equations from the first, obtaining three linear equations in x, y, z
  - Solve the linear systems for x, y, z and substitute into any of the original equations to obtain a quadratic equation in d
- However, in practice the system is ill-conditioned and can only be solved numerically

6. GPS: Project tasks

(1) **Solve the system by using Multivariate Newtons Method** Find the receiver position (x, y, z) and time correction d for simultaneous satellite positions (A_i, B_i, C_i) equal to $(15600, 7540, 20140)$, $(18760, 2750, 18610)$, $(17610, 14630, 13480)$, $(19170, 610, 18390)$ km, and measured time intervals $t_i = 0.07074, 0.07220, 0.07690, 0.07242$ sec, respectively.

Initial vector $(x_0, y_0, z_0, d_0) = (0, 0, 6370, 0)$.

(2) **Set up a test of the conditioning of the GPS problem.** Define satellite positions (A_i, B_i, C_i) from spherical coordinates (ρ, ϕ_i, θ_i) as

$$\begin{aligned} A_i =& \rho\cos(\phi_i)\cos(\theta_i) \\ B_i =& \rho\cos(\phi_i)\sin(\theta_i) \\ C_i =& \rho\sin(\phi_i) \end{aligned} \tag{4}$$

where $\rho = 26570$ km, $0 \le \phi_i \le \pi/2$ and $0 \le \theta_i \le 2\pi$ for $i = 1, \ldots, 4$ are chosen arbitrarily. The ϕ_i coordinate is restricted so that the four satellites are in the upper hemisphere. Set $x = 0$, $y = 0$, $z = 6370$, $d = 0.0001$, and calculate the corresponding satellite ranges $R_i = \sqrt{A_i^2 + B_i^2 + (C_i - 6370)^2}$ and travel times $t_i = d + R_i/c$, where $c = 299,792.458$ km/sec. Define an error magnification factor as below. The atomic clocks aboard the satellites are correct up to 10^{-8} second. Study the effect of changes in the transmission time of this magnitude.

Let the backward, or input error be the input change. in meters. At the speed of light, $\Delta t_i = 10^{-8}$ second corresponds to $10^{-8}c \approx 3$ meters. Let the forward, or output error be the change in position $\|(\Delta x, \Delta y, \Delta z)\|_\infty$, caused by such a change in t_i, in meters.

$$\text{Error magnification factor} = \frac{\|(\Delta x, \Delta y, \Delta z)\|_\infty}{c\|(\Delta t_1, \ldots \Delta t_4)\|_\infty}$$

$$\text{Condition number} = \text{ maximum error magnification factor for all small } \Delta t_i,\ 10^{-8} \text{ or less}$$

Change each Δt_i by $\Delta t_i = +10^{-8}$ or -10^{-8}, not all the same. Denote the new solution of the equations (2) by $(\bar{x}, \bar{y}, \bar{z}, \bar{d})$.

- Compute $\|(\Delta x, \Delta y, \Delta z)\|_\infty$, and the error magnification factor, by taking different Δt_i's.
- What is the maximum position error found, in meters?
- Estimate the condition number of the problem

(3) **Repeat previous step with a more tightly grouped set of satellites.** Choose all ϕ_i's within 5% of one another and all θ_i's within 5% of one another

- Solve with and without the same input error as in previous step
- Find the maximum position error and error magnification factor
- Compare the conditioning of the GPS problem when the satellites are tightly or loosely bunched

The deliverables consist of a Project Report and a jupyter notebook. Please submit these two files separately through Canvas (do *not* zip them together and submit a zip file).

Project Report

- This is a typed-up report (does not need to be very long, around 10 pages including figures is okay) that carefully discusses and presents your solutions to the project questions. A solution with no explanation may not receive full credit.

- If the activity asks for a numerical solution, always include the numerical solution in the report. (The numerical solution may be a number, a vector, a list of numbers, a table, ...).

- Be sure to use enough significant digits! Remember we are striving for accuracy: why stop at tolerance 10^{-3} if you can easily compute the solution with tolerance 10^{-12}?

- If the activity asks for a plot, always include the plot in the document.

The report should be submitted to the instructor **via Canvas as a PDF** file by the due date.

Code Listing

Your code should be clearly written, consistent with best practices, and **reproducible**. In particular:

- **Write your own numerical code**. The goal of this course is to learn numerical methods. You must program the **numerical methods explained in class** yourselves. Don't use readily-available numeric libraries, such as SciPy's `fsolve` to find roots of an equation.

  Of course, you may use libraries for all other things (e.g. for data manipulation, plotting, etc.). You may also use libraries for standard numerical computations (e.g. standard functions like trigonometric functions, matrix algebra, ...)

- **Reproducibility**. Your code must be **ready to run** and **produce the same results** documented in the project report.

- **Structure your code**. Divide your code into files with meaningful names (e.g. activity1.m, activity2.m, ...). Use **functions** to encapsulate relevant pieces of code. Use meaningful names for constants/variables

- **Comment your code**. Be as verbose as necessary, adding comments profusely whenever code is complex. Here is a pointer to a short, sensible page on commenting code: `https://www.cs.utah.edu/~germain/PPS/Topics/commenting.html`. In particular, add a function header for every numerical method that you write (e.g. Newton's method, Least squares, or Power iteration). File headers and (short) inline comments are also useful. Please read the reference above for more details.

- **Test key functions**. If a function performs some complex task and is very important for your project, test it on a few known examples. The more complex the function, the more exhaustive the testing.

The code should be submitted to the instructor **via Canvas as a jupyter notebook** file by the due date.

CODE OF ETHICS

0.1 Academic Integrity

The submission by a student of any examination, course assignment, or degree requirement is assumed to guarantee that the thoughts and expressions therein not expressly credited to another are literally the student's own. Evidence to the contrary will result in appropriate penalties, described below.

0.2 Cheating on Assignments and/or Exams

Cheating is an affront on academic integrity and ethics. Any instance of dishonesty undermines your work and the work of your classmates and the University.

0.3 Plagiarism

In defining plagiarism, this policy distinguishes between Intentional Misrepresentation (which is deemed to constitute plagiarism) and Misuse of Sources. These are two clear extremes, but this policy also recognizes that there can be a continuum between them.

Intentional Misrepresentation occurs when a student deliberately uses someone else's language, ideas, or other original (not common knowledge) work without acknowledging the source. Examples include but are not limited to when a student submits an Assignment that: a) is downloaded from an Internet source and/or obtained from a paper mill; b) is obtained from someone else (including another student); c) contains part or all of the writings of another person (including another student), without acknowledgment of the source; or d) contains passages that were cut and pasted from an Internet source, without acknowledgement of the source.

Misuse of Sources is the unintentional misappropriation of the language, ideas, and work of others due to a lack of understanding of the conventions of citation and documentation, including paraphrasing, quoting, and the parameters of common knowledge.

Students are responsible for knowing how to quote from, paraphrase, summarize, and cite sources correctly. However, when a student has attempted to acknowledge a source but has not done so fully or completely, the instructor, perhaps in consultation with other faculty, administrators, or an academic integrity panel, may determine that the issue is Misuse of Sources or unsuccessful writing, rather than Intentional Misrepresentation.

0.4 Penalties and Procedures for Violating Academic Integrity Standards

Accordingly, students who act in a dishonest manner by cheating on written exams or plagiarizing are subject to penalty under the procedures stated in the Katz Graduate Catalog. This may include **reduced credit or zero on the assignment, reduced final grade or failing grade**.