# Event-State Duality: The Enriched Case

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**Abstract.** Enriched categories have been applied in the past to both event-oriented true concurrency models and state-oriented information systems, with no evident relationship between the two. Ordinary Chu spaces expose a natural duality between partially ordered temporal spaces (pomsets, event structures), and partially ordered information systems. Barr and Chu's original definition of Chu spaces however was for the general $V$-enriched case, with ordinary Chu spaces arising for $V = \mathbf{Set}$ (equivalently $V = \mathbf{Pos}$ at least for biextensional Chu spaces). We extend time-information duality to the general enriched case, and apply it to put on a common footing event structures, higher-dimensional automata (HDAs), a cancellation-based approach to branching time, and other models treatable by enriching either event (temporal) space or state (information) space.

## 1 Introduction

Although the present paper is about enrichment in concurrency, the framework facilitating this treatment is the duality of time and information, the title of the author's CONCUR'92 talk [1] at Stonybrook NY exactly a decade ago. The abstract began with

> The states of a computing system bear information and change time, while its events bear time and change information.

The introduction expanded on this by beginning

> The behavior of an automaton is to alternately wait in a *state* and perform a transition or *event*. We may think of the state as bearing information representing the "knowledge" of the automaton when in that state, and the event as modifying that information. At the same time we may think of the event as taking place at a moment in time, and the state as modifying or whiling away time.

This view organizes states and events into two complementary spaces, state spaces or automata and event spaces or schedules, whose distances are respectively information deltas and time deltas. The two spaces are interderivable, and

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are shaped similarly (linearly) by sequential behavior but strikingly differently by concurrency.

This was then and remains today the underlying principle of our event-state symmetric view of behavior, independently of whether sequential or concurrent. The basic framework for this view then was complete semilattices, modified to cater for conflict by replacing bottom by top. Within a month of writing [1], V. Gupta and I [2] had simplified and generalized this framework via Chu spaces [3], which has remained our current view for the past decade [`http://chu.stanford.edu/`].

Yet earlier [4] we had applied categorical enrichment to a unified treatment of ordered time, real time, etc., but at that stage of thinking did not have the notion of information as dual to time. What we did have was a puzzle as to why Girard's Linear Logic (LL) should look so like our Process Specification Language (PSL) except for the self-duality of LL, which PSL lacked. We raised this issue at the end of [4] in the following paragraph.

> A recent noncartesian logic is Girard's linear logic [5]. Like PSL, linear logic distinguishes ordinary and tensor product. Like Boolean logic but unlike PSL, Girard's linear logic is self-dual, giving rise by de Morgan's law to two binary operations dual to the two products. This prompts the following question. Why should self-duality survive the diverging of the two products?

But while [1] answered that puzzle with time-information duality it did so only for ordered time and not for the generalized metrics of [4], leaving this as a loose end.

In between [4] and [1] we introduced the notion of higher dimensional automaton (HDA) [6] as an algebraic topological form of automata theory supporting Papadimitriou's geometric view of concurrency control [7] in terms of higher-dimensional state spaces, in which mutual exclusion takes the form of a hole. At that time we were unable to answer Boris Trakhtenbrot's question after our POPL talk as to how HDAs were related to event spaces, leaving another loose end which we only recently tied up using triadic Chu spaces [8].

We found ourselves subsequently drawn more strongly to the duality puzzle than to HDAs, intuiting that the latter should rest on the former which therefore needed to be understood first. We have since written extensively on this topic [`http://chu.stanford.edu/`], and with V. de Paiva also organized a LICS workshop on Chu spaces. This is not to say that HDAs have gone neglected: thanks to the efforts of a number of researchers, in particular Eric Goubault, HDA theory has since ripened into a relatively popular research area with some dozens of papers, two workshops, and a special issue of MSCS [9]. The upshot has been that both Chu spaces and HDAs have received their fair share of attention during the past several years.

The purpose of this paper is to tie up the remaining loose end involving enrichment, accomplished by passing from ordinary to enriched Chu spaces. This passage provides a common framework with the metric space approaches to both information systems [10] and temporal systems [4], not by unifying them however but by placing them on opposite sides of the duality of states and events. The key idea is to apply the way a metric on $V$ lifts to a metric on a $V$-category to the rows and columns of a matrix to yield separate metrics of time and information.

Enriched Chu spaces greatly simplify this reconciliation to the point of automating it. Ordinary Chu spaces are a sort of halfway-house between universal algebra and category theory. Enriched Chu spaces make the corresponding connection for enriched categories, in the process enriching universal algebra analogously.

Ironically the original definition of Chu spaces [3] was for the enriched case, with ordinary Chu spaces receiving only a passing mention.[1] The first detailed treatment of ordinary Chu spaces was by Lafont and Streicher, and they were subsequently adopted by Gupta and Pratt [2, 11] for the purpose of modeling behavior at a more fundamental level than possible with higher dimensional automata.

## 2 Event-State Duality

Computation is traditionally taught with a focus on states, a point of view that has permeated computer science so thoroughly that event-oriented models are in a distinct minority even at CONCUR. One could imagine a parallel universe in which computer science had focused instead on events, with advocates of a state-oriented perspective in the minority. The situation is rather like the old philosophical problem of the primacy of mind or matter, with science having chosen Hume over Berkeley, matter over mind. Or for that matter the general preference in mathematics of sets over categories.

Just as Russell along with Eccles and Popper advocated a return to the more symmetric view of matter and mind contemplated both by millennia-old yin-yang philosophy and by Descartes in 1647, so does event-state duality take a more symmetric view of events and states, defining them in such a way that each can be understood in terms of the other. The duality of events and states goes hand in hand with that of time and information as the respective metrics on event spaces and state spaces. Event-state duality permits a process to be viewed equally well as a state-based automaton or an event-based schedule. These views are structurally different: automata (or transition systems) are state-based, and branching is disjunctive: the process goes down only one branch. Schedules are event-based, and branching is conjunctive: parallel events all occur. These structural differences notwithstanding, each view fully determines the other, and moreover simply by matrix transposition!

[1] This exclusive attention to the enriched case in the original literature has created the impression in some quarters that Chu spaces are an inaccessibly abstract notion. This does not do justice to the simplicity of ordinary Chu spaces as mere matrices, and moreover matrices over a mere set, unlike the matrices of linear algebra which are over a field.

Event-state duality can be understood in terms of element-predicate duality. The essence of duality is reversal, as with negation of reals which reverses their order while interchanging max and min as well as floor and ceiling. Complementing the elements of a Boolean algebra, De Morgan duality, similarly reverses its order while interchanging the roles of true and false and of conjunction and disjunction. And taking the opposite $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ reverses its morphisms while interchanging limits and colimits, categorical duality. This third example generalizes the first two when made categories by interpreting $a \leq b$ as a morphism from $a$ to $b$.

Element-predicate duality arises as an instance of categorical duality for a category $\mathcal{C}$ as follows. Two objects $g$ and $k$ (not necessarily distinct) are chosen to play the roles of respectively a singleton object and a truth values object. (An obvious and natural choice for the category of sets is $g = 1 = \{0\}$ and $k = 2 = \{0, 1\}$. Less obvious but equally natural choices exist for many popular categories, e.g. for (locally compact) Abelian groups the integers under addition as the free group on one generator and the unit circle of the complex plane under multiplication as its dual.) This choice determines for every object $a$ both the elements of $a$ and the predicates on $a$, as respectively the morphisms from $g$ to $a$, and from $a$ to $k$. When necessary for disambiguation we refer to these as $g$-elements and $k$-predicates. Note that the set $\mathcal{C}(g, k)$ of morphisms from $g$ to $k$ are simultaneously the $g$-elements of $k$ and the $k$-predicates on $g$, constituting the set of truth values which we denote henceforth by $K$. Application of a predicate to an element is accomplished by composition to yield a truth value.

Categorical duality reverses the orientation of the morphisms of $\mathcal{C}$ while interchanging elements and predicates by interchanging the roles of $g$ and $k$: $g$ now serves as the truth values object while $k$ acts as the singleton. In this process *the set of truth values remains unchanged*, since what used to be both $g$-elements of $k$ and $k$-predicates on $g$ have become both $g$-predicates on $k$ and $k$-elements of $g$. That is, the set $K$ is invariant under dualization! Furthermore application of predicate $y$ to element $x$ prior to dualization yields the same truth value as application of $x$ (now a predicate) to $y$ (now an element) after dualization.

A given choice of $g$ and $k$ establishes for every object $a$ of $\mathcal{C}$ a triple $(A, r, X)$ where $A$ is the set $\mathcal{C}(g, a)$ of elements of $a$, $X$ is the set $\mathcal{C}(a, k)$ of predicates on $a$, and $r : A \times X \to K$ is the application function defined by $r(a, x) = x(a)$. Such a triple expresses those aspects of $a$ that can be understood in terms of the interaction of its elements and predicates via application (realized as composition).

A ***Chu space over*** $K$ is any triple $\mathcal{A} = (A, r, X)$ where $A$ and $X$ are sets and $r : A \times X \to K$ is an arbitrary function called the *matrix* of the Chu space. Dualization exchanges $A$ and $X$ and transposes the matrix.

Besides viewing elements and predicates more symmetrically, this element-predicate approach to Chu spaces also views set theory and category theory more symmetrically. From the categorical perspective a Chu space is an object of a category $\mathcal{C}$ together with some[2] incident arrows. From the set perspective a Chu space is a generalized topological space whose set $X$ of "open sets" need not be closed under union or finite intersection, and which allows $|K| > 2$; its morphisms can be understood as simply those functions for which the inverse image of open sets is open, exactly as for topological continuity.

Event-state duality arises as the case of element-predicate duality in which a process $\mathcal{A}$ as an object of a category $\mathcal{C}$ of processes is understood as having events for elements and states for predicates. We call such a process a *schedule*. Dualizing $\mathcal{C}$ as $\mathcal{C}^{\mathrm{op}}$ turns states into elements and events into predicates; we call the objects of $\mathcal{C}^{\mathrm{op}}$ *automata*. Dualizing an object leaves it as the same process with the same events and the same states, changing only which are the elements and which the predicates.

A morphism of Chu spaces $(A, r, X) \to (B, s, Y)$ is a pair of functions $(f : A \to B, g : Y \to X)$ satisfying the ***adjointness*** condition $s(f(a), y) = r(a, g(y))$ for all $a \in A$ and $y \in Y$. Dualizing interchanges $f$ and $g$, thereby reversing the morphism since $f$ and $g$ are oppositely orientated. The category of Chu spaces over $K$ and their morphisms is denoted $\mathbf{Chu}(\mathbf{Set}, K)$.

Define $\hat{r} : A \to K^X$ as $\hat{r}(a)(x) = r(a, x)$ and $\check{r} : X \to K^A$ as $\check{r}(x)(a) = r(a, x)$. We refer to $\hat{r}(a)$ as row $a$ of the matrix $r$, and $\check{r}(x)$ as column $x$.

A Chu space is ***extensional*** when $\check{r} : X \to K^A$ is an injection (no repeated columns in the matrix), ***separable*** when $\hat{r} : A \to K^X$ is an injection (no repeated rows), and ***biextensional*** when both extensional and separable. Biextensional Chu spaces are to Chu spaces as posets are to preordered sets: the former collapses "isomorphic" elements (those lying on a cycle in the case of preordered sets, those indexing equal rows or columns in the case of Chu spaces) to a single element. The subcategory of $\mathbf{Chu}(\mathbf{Set}, K)$ consisting of its biextensional Chu spaces has been denoted $\mathbf{chu}(\mathbf{Set}, K)$, or "little Chu" by Barr. Every morphism $(f, g)$ of little Chu from $\mathcal{A} = (A, r, X)$ to $\mathcal{B} = (B, s, Y)$ is representable as the Chu space $(A, t, Y)$ where $t(a, y) = s(f(a), y) = r(a, g(y))$: the columns of $t$ come from $\mathcal{A}$ and its rows from $\mathcal{B}$, and $t$ uniquely determines $f$ and $g$. $(f, g)$ is reconstructed from $(A, t, Y)$ by taking $f(a)$ to be the location in $\mathcal{B}$ of row $a$ of $t$, and dually $g(y)$ for the location in $\mathcal{A}$ of column $y$ of $t$.

For extensional Chu spaces we can identify each state $x$ with the column $\lambda a.r(a, x)$, a function $A \to K$. This identification permits $r$ to be dropped because we can recover it via $r(a, x) = x(a)$. In this case a Chu space is a pair $(A, X)$ where $X \subseteq K^A$.

## 3 Examples

The table in Figure 1 below is taken from [12]. Each of the four columns (a)-(d) corresponds to a choice of $K$, respectively $\{0, 1\}$, $\{0, \lrcorner\!\ulcorner, 1\}$, $\{0, \times, 1\}$, and

[2] Replacing "some" by "all" when $\mathcal{C}$ is small embeds $\mathcal{C}$ fully in $\mathbf{chu}(\mathbf{Set}, \mathrm{ob}(\mathcal{C}))$ [13]. That is, taking $K$ to be sufficiently large permits the objects of any small category to be represented as Chu spaces in such a way that the morphisms of $\mathcal{C}$ become exactly the Chu morphisms between the representing Chu spaces.

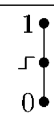
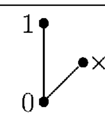
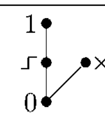
| | (a)  | (b)  | (c)  | (d)  |
|---|---|---|---|---|
| $a\|\|b \stackrel{?}{=} ab + ba$ | a: 0101<br>b: 0011<br>= <br>a: 0101<br>b: 0011 | a: 0⌐10⌐10⌐1<br>b: 000⌐⌐⌐111<br>≠<br>a: 0⌐1010⌐1<br>b: 000⌐⌐111 | a: 0101<br>b: 0011<br>=<br>a: 0101<br>b: 0011 | a: 0⌐10⌐10⌐1<br>b: 000⌐⌐⌐111<br>≠<br>a: 0⌐1010⌐1<br>b: 000⌐⌐111 |
| $a(b + c) \stackrel{?}{=} ab + ac$ | a: 0111<br>b: 0010<br>c: 0001<br>=<br>a: 0111<br>b: 0010<br>c: 0001 | a: 0⌐11111<br>b: 000⌐100<br>c: 00000⌐1<br>=<br>a: 0⌐11111<br>b: 000⌐100<br>c: 00000⌐1 | a: 0111<br>b: 001×<br>c: 00×1<br>≠<br>a: 01111<br>b: 001××<br>c: 0××01 | a: 0⌐11111<br>b: 000⌐1××<br>c: 000××⌐1<br>≠<br>a: 0⌐111⌐111<br>b: 000⌐1××××<br>c: 0××××00⌐1 |

**Fig. 1.** Examples of event-state duality

{0, ⌐, 1, ×}. The elements of $K$ are the possible values of an event in a given state, with 0, ⌐, 1, × corresponding to the respective adverbs *before*, *during*, *after*, and *instead*. In a given state an event has value 0 when it has not yet started, ⌐ when it is happening, 1 when it is finished, and × if it has been canceled. The four corresponding automata, with edges oriented to point upwards, indicate the transitions an event may make. These transitions enrich the structure of $K$ and we shall ignore them for the moment, treating $K$ as merely a set, i.e. focusing only on its points.

The 16 matrices denote the Chu spaces corresponding to the four processes $a||b$, $ab+ba$, $a(b+c)$, and $ab+ac$ for each of the four choices of $K$. Rows represent events and columns states. The first matrix, expressing $a||b$ with $K = \{0, 1\}$, has all possible states for two events, respectively neither $a$ nor $b$ done, $a$ done, $b$ done, and both done. The matrix under it also has four states: nothing done, $a$ done, $a$ and $b$ done, and $a$ and $c$ done. All 16 spaces are biextensional.

The table shows how adding a third value to $K$ serves to distinguish either $a||b$ from $ab+ba$ or $a(b+c)$ from $ab+ac$, but not both, depending on whether the value is ⌐ or ×. To make both distinctions requires both additional values. This is developed in considerably more detail in [12]. The point here is to illustrate and motivate the kinds of primitive structures that might play a role in enrichment-based semantics, in particular as the $V$ we now describe.

## 4 Enrichment

Posets abstract the notion of a set of sets ordered by inclusion: the elements of the inner sets disappear. The ordinary category **Pos** of posets is even more abstract: the elements of the outer sets also disappear, but we still have one straw left to grasp at: the monotone functions between two posets form a mere set. The notion of a poset as a category whose "homsets" are not sets but objects of a certain symmetric monoidal closed category $V$, the chain 2 in the case of posets, is abstract nonsense taken to the max. The defining characteristic of a $V$-category or enriched category is that its morphisms from object $a$ to object $b$ band together not as a set but as an object of $V$, with the characteristic features of a category being reformulated in terms of $V$.

Lawvere [14] has provided an attractive way of conceptually taming $V$-categories, by viewing the objects of $V$ as distances, and $V$-categories as generalized metric spaces satisfying a suitable triangle inequality over such distances. Besides being good pedagogy, this view nicely connects enrichment with the Gauss-Kleene-Floyd-Warshall connection that was emerging in computer science simultaneously with and completely independently of the development of enriched categories [15]. That connection applied semirings to create a common algebraic framework for the Roy-Warshall transitive closure algorithm [16, 17], Floyd's shortest path algorithm [18], Kleene's algorithm for translating nondeterministic finite state automata into regular expressions [19], and even Gauss's algorithm for inverting a matrix. These $O(n^3)$ algorithms can be made instances of the same algorithm parametrized only by choice of semiring, which plays the role for this common algorithm that $V$ plays for enrichment. The uniform expression of this common algorithm in terms of semirings was first described by Robert and Ferland [20].

The ordinary triangle inequality takes the form

$$d(a, b) + d(b, c) \geq d(a, c)$$

where $d(a, b)$ is a nonnegative real constituting the distance from point $a$ to point $b$ of a space. The Fréchet axioms for ordinary metric spaces are the ordinary triangle inequality together with

$$\begin{array}{rcll} d(x,y) & = & d(y,x) & \quad \text{(symmetry)} \\ d(x,x) & = & 0 & \\ d(x,y) \;=\; 0 & \supset & x = y & \end{array}$$

The first step in generalizing the Fréchet axioms is to restate $d(x, x) = 0$ as $d(x, x) \leq 0$, or better yet as $0 \geq d(x, x)$ to give it the same orientation as the triangle inequality. Absent negative distances this rewording changes nothing.

The next step is to drop the second and fourth axioms. Not only do they get in the way of the connection to enrichment, they are not even well motivated in many practical situations. Counterexamples to symmetry abound: the distance between the base and the summit of a mountain measured in climbing days, the distance along an asymmetric toll road measured in dollars, and so on.

If we view points zero distance apart as somehow isomorphic, the fourth axiom identifies isomorphic points. This is analogous to the role of antisymmetry in partially ordering a preordered set (one whose order relation $\leq$ is reflexive and transitive) by identifying points $x, y$ that are isomorphic by virtue of lying on a cycle $x \leq y \leq x$. The practice in surveying of measuring only the horizontal component of distance refutes the fourth axiom in that it makes vertically aligned points isomorphic; the axiom is restored however in the plan view projecting vertical lines to points.

The third step is to view the ordering $\geq$ on ordinary real-valued distances as an instance of $\sqsubseteq$, the generic ordering relation for generalized distances (the reversal is intentional), and real addition as an instance of a commutative monoid operation $x \cdot y$. Here distances are viewed as forming a preordered monoid $(V, \sqsubseteq, \cdot, 1)$, with $\sqsubseteq$ being a reflexive transitive binary relation on $V$ and $\cdot$ and 1 furnishing $V$ with the structure of a commutative monoid. That is, $\cdot$ is associative and commutative, and 1 is the identity for $\cdot$. (Caveat: the generic $\cdot$ and 1 typically revert to + and 0 when the objects of $V$ are numeric.) We also require that $\cdot$ be monotone in each argument with respect to $\sqsubseteq$; the property of being closed actually makes it additive $(s \cdot (t \vee t') = s \cdot t \vee s \cdot t')$.

We can now define the notion of ***generalized metric space*** over a generalized metric $(V, \sqsubseteq, \cdot, 1)$, namely as a set $X$ of points with a metric $d : X^2 \to V$ satisfying the following generalizations of the remaining two Fréchet axioms.

$$\begin{aligned} d(x,y) \cdot d(y,z) &\sqsubseteq d(x,z) \\ 1 &\sqsubseteq d(x,x) \end{aligned}$$

The preorder $\sqsubseteq$ and the monoid $\cdot$ are to be understood as essentially independent structures on $V$, associated with respectively strength and length of distances. Whereas strengths are compared via the preorder, lengths are added via the monoid. Borrowing a convention from 2-categories, we picture length as horizontal (like a road) and strength as vertical (as in a Hasse diagram).

Strength and length are sometimes incorporated into a single number. The canonical example is distance as an upper bound, where $d(a, b) = 3$ means that any trip from $a$ to $b$ traverses *at most* 3 units of length. For example a process presented with different inputs of the same length may follow state trajectories of different lengths. If some path from state $x$ to state $z$ passes through state $y$, and all paths from $x$ to $y$ take at most 5 seconds, and from $y$ to $z$ at most 7 seconds, then any path from $x$ to $z$ via $y$ can take at most 12 seconds. But paths through some other state $w$ may have $d(x, w) = 4$ and $d(w, z) = 9$, preventing us from ruling out the possibility of taking 13 seconds to get from $x$ to $z$. This reasoning shows why the relevant form of the triangle inequality for upper bounds is $d(a, b) + d(b, c) \leq d(a, c)$.

Geodesics constitute *lower* bounds: the length of a geodesic from $a$ to $b$ is a lower bound on the length of an arbitrary path from $a$ to $b$. Ordinary metric spaces are based on geodesic distance and therefore instantiate the generic comparison relation $\sqsubseteq$ with the reverse order $\geq$ on the reals, the appropriate order for lower bounds, with the geodesic triangle inequality being the ordinary $d(a, b) + d(b, c) \geq d(a, c)$.

Distances need not be simple numbers. A natural notion of distance is an interval $[s, t]$ giving a range of possible distances. In this case $[s, t] \sqsubseteq [s', t']$ defined as $s \geq s'$ for the lower bounds and $t \leq t'$ for the upper as per the preceding two paragraphs. But this makes $\sqsubseteq$ reverse inclusion for intervals, the natural information ordering for such applications of intervals as interval arithmetic where strength is measured by the narrowness of the interval. Upper and lower bound distances $t$ can be seen to be the respective special cases $[-\infty, t]$ and $[t, \infty]$ of interval distances. Intervals are added via $[s, t] + [s', t'] = [s + s', t + t']$, with the identity interval being $[0, 0]$.

One last generalization remains, namely of the preorder $\sqsubseteq$ on $V$ to the morphisms of a category structure on $V$. Viewing a preorder as a category with at most one morphism between any two objects, this generalization amounts to dropping the cardinality bound $|V(a,b)| \leq 1$ on homsets of $V$.

The elements of the preorder are now understood as objects. The $\cdot$ operation becomes a functor $V^2 \to V$, and is renamed $\otimes$; likewise 1 becomes an object of $V$ and is renamed $I$ (except in linear logic where it remains 1 and the terminal object is denoted $\top$). The result is a ***symmetric monoidal category*** $V$, a category equipped with a monoid in the form of a tensor product $\otimes$ and tensor unit $I$, whose monoid laws now take the form of suitable natural transformations which join $\otimes$ and $I$ as part of the signature: $\alpha_{stu}$ expressing associativity and $\lambda_s$ and $\rho_s$ expressing the identity laws. These natural transformations themselves obey certain nonobvious *coherence laws*, most notably a pentagonal diagram for $\alpha$ and a triangular diagram for $\lambda$ and $\rho$ [21, §VII-1] [22, §1.1]. When these three natural transformations are identities $V$ is called ***strict monoidal***.

We further require that $V$ be ***closed***, meaning that tensor have a right adjoint in each argument. That is, to each object $s$ of $V$ is associated an isomorphism

$$V(s \otimes t, u) \cong V(t, s \multimap u)$$

natural in $t$ and $u$, that is, an adjunction, with the functor $s\multimap-$ right adjoint to the functor $s \otimes -$. The functor $s\multimap-$ is called the *internal hom*, and serves as a generalized metric on $V$.

We are now ready for the general notion of enriched category or $V$-category.

**Definition.** A ***$V$-category*** $A = (O, d, m, j)$, or ***category enriched in*** $V$, consists of a set $O$ of objects, a function $d : O^2 \to \mathrm{ob}(V)$ constituting the metric, and families of morphisms of $V$, namely compositions $m_{uvw} : d(u,v) \otimes d(v,w) \to d(u,w)$ and identities $j_u : I \to d(u,u)$, such that certain reasonably obvious diagrams commute expressing associativity of composition and the left and right identity laws [22, §1.2].

The class of $V$-categories is itself a category $V$-**Cat** whose morphisms are $V$-functors. The basic constituents of a $V$-functor $F : \mathcal{A} \to \mathcal{B}$ are a function $F_O : \mathrm{ob}(\mathcal{A}) \to \mathrm{ob}(\mathcal{B})$ and an $\mathrm{ob}(\mathcal{A})^2$-indexed family $F_{uv} : d(u,v) \to d(F_O(u), F_O(v))$ of morphisms of $V$ such that certain diagrams commute expressing functoriality [22, §1.2], or [4] for definitions coordinated with this process perspective.

$V$ itself is a $V$-category whose homobjects are given by its internal hom $a \multimap b$. We use this fact later in Plotkin's theorem about $\mathbf{chu}(Q\text{-}\mathbf{Cat}, Q)$.

## 5 Quantales

A large and useful class of monoidal closed categories is given by quantales [23]. A ***quantale with unit*** $(Q, \bigvee, \otimes, 1)$ is a complete-semilattice-ordered monoid, equivalently a one-object[3] $V$-category where $V$ is the category **CSLat** of complete semilattices, equivalently a monoid object in **CSLat**. As such it is a (pose-

[3] Dropping the one-object restriction yields the more general notion of *quantaloid* [24].

tal) monoidal closed cocomplete category, the symmetric case of which is obtained as commutative quantales ($s \otimes t = t \otimes s$). This case provides a suitable $V$ for constructing $V$-categories. A number of quantales (not under that name) applicable to various metrics of computational interest are described in [4].

The finite linearly ordered commutative quantales with bottom 0 and unit 1 form an interesting and useful special case. As shown in [4] there are $2^{n-2}$ such having $n \geq 2$ elements, necessarily satisfying $0 \neq 1$ (if $0 = 1$ then $s = s \otimes 1 = s \otimes 0 = 0$ violating $n \geq 2$).

These have the following picturesque characterization. Pick any chain $(Q, \bigvee)$ with $n$ elements. Holding it vertically with smaller elements lower, grasp the $j$-th element from the top, $1 \leq j \leq n-1$ (so the bottom element is not eligible). Let the portion of the chain above the grasped element fall over (i.e. rotate 180 degrees about the grasped element) so as to dangle down beside the other half of the chain. Arbitrarily interleave the two halves, preserving their respective orders, to give a second linear order on $Q$, while keeping the original bottom at the bottom and the grasped element at the top. The first and second orders on $Q$ have the same bottom element.

We now define $(Q, \bigvee, \otimes, 1)$ as follows. Take $\bigvee$ to be supremum (least upper bound) in the first linear order, so 0 (sup of the empty set) is its bottom (the original bottom). Define $\otimes$ to be infimum (greatest lower bound) in the second linear order, with 1 as its top (the grasped element).

This interleaving can be done in $\binom{n-2}{j-1}$ ways. Summing this over $1 \leq j \leq n-1$ yields the promised result $2^{n-2}$. Exactly one of these ways is a Heyting algebra or cartesian closed poset, namely when $j = 1$, the one case in which the two orders are the same, making $\otimes$ infimum (greatest lower bound) in the first linear order.

The unique such quantale for $n = 2$ is not only a Heyting algebra but a Boolean algebra, unlike the rest. As such it is the quantale to use for $V$ in the analysis of partial orders (more generally preorders) as a $V$-category. For $n = 3$ we have one Heyting algebra and one non-Heyting algebra, denoted 3 and 3' in [4]. 3' is the quantale structure implicit in the "prossets" of [25], which are 3'-categories, and is also the appropriate quantale structure to impose on $K = \{0, \lrcorner\!\ulcorner, 1\}$ for modeling HDAs as triadic Chu spaces [8]. 3 is the quantale structure implicit in Gaifman's treatment of the distinction between incidental and causal order [26], where the schedules he defines amount to $V$-categories for $V = 3$.

Ordinary metric spaces restricted to nonnegative integer distances but supplemented with distance $\infty$ can be understood as the quantale $(\mathbb{N} \cup \{\infty\}, \bigwedge, +, 0)$ (taking the quantale's $\bigvee$ operation to be $\mathbb{N}$'s $\bigwedge$ amounts to reversing the standard order on $\mathbb{N}$).[4] It does not fit the preceding pattern because it is infinite, and because $\otimes$ as integer addition is not idempotent.

The ordinary metric is appropriate for delay between events in sequential computation. When $n$ unit-time events must happen sequentially the time to

[4] Some useful variations: $(\mathbb{Z} \cup \{\infty, -\infty\}, \bigwedge, +, 0)$, $(\mathbb{R}^{\geq 0} \cup \{\infty\}, \bigwedge, +, 0)$, $(\mathbb{R} \cup \{\infty, -\infty\}, \bigwedge, +, 0)$.

pass through the resulting $n+1$ states is $n$, which the ordinary metric arrives at by adding the distance vector between the initial and final state, which consists of $n$ 1's. This remains the case whether the sequentiality is a consequence either of the order being specified, or of the events being performable in any order (but pairwise mutually exclusively, e.g. $ab + ba$), or anything in between (e.g. $abc + cba + cab$, omitting the other three permutations).

While this metric is exactly right for sequential computation, it fails to recognize the performance benefits of concurrency. If instead of addition we take $\otimes$ to be $\bigvee$ in $\mathbb{N}$ ($\bigwedge$ in the quantale's order), we obtain an *ultrametric* space which in this case is also a (complete) Heyting algebra. This metric is appropriate when all events are performed in parallel, giving the expected result that $n$ unit-time events performed in parallel need take only unit time; more generally when the events take different times, their max.

Practical computation is of course somewhere in between these extremes of sequential and concurrent, entailing an appropriate blend of the two metrics to get a satisfactory estimate of running time. We return to this later.

## 6 Enriched Chu Spaces

An ordinary Chu space is a triple $(A, r, X)$ where $A, X$ are sets, i.e. objects of the category **Set**, and $r : A \times X \to K$ is a function, a morphism of **Set**. The passage to the enriched case generalizes **Set** to an arbitrary monoidal closed category $V$ with pullbacks. $A$ and $X$ are objects of $V$, $r$ is a morphism of $V$, and Chu morphisms are pairs $(f : A \to B, g : Y \to B)$ of morphisms of $V$. The adjointness condition is rephrased in the evident way as the diagram

$$\begin{tikzcd} A \otimes Y \arrow[r, "f \otimes Y"] \arrow[d, "A \otimes g"'] & B \otimes Y \arrow[d, "s"] \\ A \otimes X \arrow[r, "r"] & K \end{tikzcd}$$

The enriched case of little $\mathbf{chu}(V, k)$ is defined so that $\hat{r} : A \to K^X$ and $\check{r} : X \to K^A$ are not merely monos but extremal monos. A mono $f$ is ***extremal*** when for all factorizations of $f$ as the composition $g \circ e$ of a morphism $g$ with an epi $e$, $e$ is an isomorphism. In **Set** every mono is extremal, which is why we did not need this distinction for ordinary Chu spaces. In **Pos** extremal monos satisfy not only monotonicity as usual but also its converse $f(x) \leq f(y) \to x \leq y$.

The idea is that the monotonicity of $\hat{r}$ and $\check{r}$ put upper bounds on the structure of $A$ and $X$ respectively, while extremality forces those upper bounds to be attained. This is intended to pin down the structure on each of $A$ and $X$ to exactly that determined by the induced structure on respectively the rows and columns of the matrix.

For example if $K$ is furnished with a binary operation $+$, this operation lifts pointwise to both the rows and columns of $r$, thereby furnishing both $A$ and $X$

with that operation. Any equational properties enjoyed by the operation in $K$, such as associativity or commutativity, lift to these induced operations on $A$ and $X$.

For at least one large and useful class of symmetric monoidal categories this works sufficiently well as to make this induced structure on $A$ and $X$ redundant: $A$ and $X$ can be left discrete and their structure inferred entirely from the matrix, made precise by the following unpublished theorem of G. Plotkin.

**Theorem 1.** *(Plotkin) For any quantale $Q$,* $\mathbf{chu}(Q\text{-}\mathbf{Cat}, Q)$ *is equivalent (in fact isomorphic) to* $\mathbf{chu}(\mathbf{Set}, |Q|)$.

This theorem is remarkable in that alterations to the structure of a given quantale $Q$ leaves the ordinary category $\mathbf{chu}(Q\text{-}\mathbf{Cat}, Q)$ unchanged. What does change is the induced structure on $A$ and $X$, but such changes have no impact on the morphisms! The changes are felt only for $\mathbf{chu}(Q\text{-}\mathbf{Cat}, Q)$ understood as a $Q$-category, where the induced structures on $A$, $X$, $\mathcal{A} \multimap \mathcal{B}$ and so on track changes to the structure on $Q$.

Plotkin's isomorphism in the ordinary case covers many choices of $V$ of practical interest, though certainly not all. In these cases enrichment can be made attractively simple: start with ordinary Chu spaces over $K$, enrich $K$ in the usual manner of furnishing a set with algebraic or relational structure, enrich $A$ from the matrix by reflecting the structure of $K^X$ into $A$ via $\hat{r} : A \to K^X$, and dually for $X$ via $\check{r}$. Call this simplistic approach to enriched Chu spaces ***light enrichment***.

Plotkin's theorem gives enough situations in which light and classical enrichment agree up to category isomorphism that it is natural to ask what the downside might be of substituting light enrichment for classical even when the results are not identical, that is, when the structure on $K$ is not quantalic. Light enrichment is conceptually simple and intuitively appealing. With these advantages in mind we experimentally adopt light enrichment as the preferred method of enrichment for Chu spaces, leaving open the very interesting question of when this can cause problematic discrepancies with classical enrichment in practical situations.

## 7 Time and Information as Induced Metrics

We define time and information as generalized metrics on the spaces of events and states respectively. Classical enrichment provides these directly in the objects $A$ and $X$ when these are drawn from a suitable category of generalized metric spaces. Light enrichment allows the metrics on $A$ and $X$ to be inferred via the matrix $r$ from the structure assigned to $K = \mathrm{ob}(V)$, as follows.

The distance between two vectors (either rows or columns) over $K$ is itself a vector over $K$ determined pointwise from the given vectors via the distance metric on $V$ as the enrichment of $K$, namely its internal hom $s \multimap t$. This vector distance is then converted to a scalar distance by combining the components of the vector with the tensor product of $V$. This calculation of metrics on rows and columns is a core part of the proof of Plotkin's theorem.

For ordered time, the case $V = 2$ where 2 is the 2-element chain understood as a commutative quantale (hence as a symmetric monoidal closed category) makes $V$-Cat the category **Ord** of preordered sets. Here distances $d(u, v)$ are 0 and 1 giving the truth value of $u \leq v$, respectively false and true. Bit vectors $\mathbf{u}, \mathbf{v}$ are compared coordinatewise, with 0 in those positions where $\mathbf{u}$ is 1 and $\mathbf{v}$ is 0, and 1 in all other positions. The tensor product here is conjunction and so the corresponding scalar is 1 just when every bit in the comparison vector is 1: a single counterexample makes it false. For example the comparison of 010011101 with 010010101 first yields the vector distance 111110111, which is then converted to a scalar distance by forming the conjunction of those 9 bits to yield 0, the truth value of $010011101 \leq 010010101$. In this way the order on 2 lifts to a preordering of $A$ and of $X$ (a partial ordering, i.e. satisfying antisymmetry, in the biextensional case).

For real time, the ordinary metric space $(\mathbb{N} \cup \{\infty\}, \bigwedge, +, 0)$ encountered earlier provides a notion of sequential time. If $n$ events happen sequentially their respective durations should be combined with $+$. This is true regardless of what order is specified for them. In particular $ab$, $ba$, $ab + ba$, and $a||b$ all take the same time.

Parallel time recognizes the performance gain possible with $a||b$ over $ab$, $ba$, and $ab + ba$. For $a||b$ the appropriate metric is $(\mathbb{N} \cup \{\infty\}, \bigwedge, \vee, 0)$. If $n$ events happen independently then the time required is that of the slowest, whence their times should be combined with $\vee$. For the other three combinations we retain the sequential metric.

The situation becomes more interesting with arbitrary processes $(A, X)$ whose set $X$ of states is an arbitrary subset of $K^A$. The first question to ask here is whether any reasonable notion of running time is possible for such general processes, independently of whether enrichment is of any use here.

As shown in [12], the presence of $\ulcorner$ in $K$ permits a very simple yet natural notion of distance between any two states $x, y$ of $(A, X)$. Form the power graph $K^A$ derived from any of the four graphs of Figure 1, each understood as being reflexive but not transitive (unless vacuously so as in 1(a) and 1(d)), as proposed in [12]. For any given process $(A, X)$ take its transition system to be the full subgraph of $K^A$ having as vertices the states of $X$, denoted $K^A \cap X$. Define the distance between states to be one less than the usual shortest-path metric in a directed graph, or $\infty$ when there is no path or only infinite paths.

In the absence of value $\ulcorner$ the primitive transition systems for single events are those of Figure 1 (a) and (c). These are automatically transitive by virtue of having only paths of length one, whence the power graph $K^A$ is transitive and hence in fact a power poset. This means that if there is a path from $x$ to $y$ in $K^A$, there is a path of length 0 directly from $x$ to $y$ that survives the removal of any states other than $x$ or $y$; if not then the distance is $\infty$. This is effectively the all-or-nothing poset metric; substituting 1 for 0 and 0 for $\infty$ yields the conventional values for this metric.

Now we could instead take the distance from $x$ to $y$ to be the Hamming metric, which counts the number of events that change in that passage. This metric, which satisfies the ordinary triangle inequality, has two drawbacks. First it does not take the efficiencies of concurrency into account, corresponding instead to the time required to perform the events sequentially. Second, like the all-or-nothing poset metric it is invariant under removal of intermediate states.

Both these drawbacks are overcome with the metric obtained via $K^A$ as above with the graphs $K$ of either Figure 1 (b) or (d), in which an event requires 2 steps, or time 1 after subtracting the 1, to get from 0 to 1, namely via $\ulcorner$. There is a path of length 1 from the initial (all-zero) state to a given final state (all events done or cancelled) if and only if the state in which the events that are done in the final state are all in state $\ulcorner$ (assuming at least one of these) and the rest are 0 or $\times$. Removal of that state constitutes an obstacle, calling for a longer path around that obstacle, provided one exists (otherwise that final state is deemed unreachable).

Taking $K = \{0, \ulcorner, 1\}$, we may illustrate this with the example of three children ($A = 3$) taking turns riding $n$ ponies. For $n = 3$ the relevant process $(A, X)$ is the 3-cube with $|X| = 27$ states, whose cells consist of the $3^3$ triples over $K$, broken down as 8 0-cells or vertices, 12 1-cells or edges, 6 2-cells or faces, and 1 3-cell or interior, the usual organization of the 3-cube. In this case we can pass from 000 via $\ulcorner\ulcorner\ulcorner$ to 111 in two steps, which the subtraction of 1 makes time 1. For $n = 2$, $\ulcorner\ulcorner\ulcorner$ is no longer available and a shortest path (by no means unique) is 000 to $\ulcorner\ulcorner 0$ to $11\ulcorner$ to 111, taking time 2, the time required for two children to ride both ponies and then let the third child have its turn. For $n = 1$ we cannot do better than 000 to $\ulcorner 00$ to $1\ulcorner 0$ to $11\ulcorner$ to 111, or time 3, the completely sequential case.

## 8 Process Algebra

The method of shortest paths in $K^A \cap X$ is a globally defined measure. For programs built up by composition, a more appropriate way to compute running time is by induction on program structure. We repeat here the Chu space definitions of four basic process algebra operations given in [12]. These are based on the following notions of conjunction, disjunction, and termination $\checkmark[A]$ for Chu spaces. Note that these definitions do not assume disjointness of $A$ and $B$, although some applications will force them to be disjoint. For more details see [13].

$$\mathcal{A} \wedge \mathcal{B} \stackrel{\text{def}}{=} (A \cup B, \{z{\in}2^{A \cup B} \mid z{\upharpoonright}A \in X \ \wedge \ z{\upharpoonright}B \in Y\})$$

$$\mathcal{A} \vee \mathcal{B} \stackrel{\text{def}}{=} (A \cup B, \{z{\in}2^{A \cup B} \mid z{\upharpoonright}A \in X \ \vee \ z{\upharpoonright}B \in Y\})$$

$$\checkmark[A] \stackrel{\text{def}}{=} (A, \{1, \times\}^A)$$

We then use these operations to define the following four basic process algebra operations, respectively *concurrence*, *sequence*, *choice*, and *orthocurrence*.

$$\mathcal{A}||\mathcal{B} \stackrel{\text{def}}{=} \mathcal{A} \wedge \mathcal{B}$$

$$\mathcal{A}\mathcal{B} \stackrel{\text{def}}{=} \mathcal{A} \wedge \mathcal{B} \wedge (B{-}A = 0 \ \vee \ \checkmark[A]) \wedge (B = 0 \ \vee \ \checkmark[A{-}B])$$

$$\mathcal{A}+\mathcal{B} \stackrel{\text{def}}{=} (A \cup B = 0) \ \vee \ (\mathcal{A} \not\leq \times \ \wedge \ B{-}A = \times) \ \vee \ (\mathcal{B} \not\leq \times \ \wedge \ A{-}B = \times)$$

$$\mathcal{A} \otimes \mathcal{B} \stackrel{\text{def}}{=} (A \times B, \{z{\in}K^{A\times B} \mid z(\lambda,\forall) \in X \wedge z(\forall,\lambda) \in Y\})$$

Concurrence $\mathcal{A}||\mathcal{B}$ simply collects the events of $\mathcal{A}$ and $\mathcal{B}$ subject to the separate constraints of each (conjunction). Sequence $\mathcal{A}\mathcal{B}$ is the same with the additional constraint that in every state either $\mathcal{B}$ has not yet started or $\mathcal{A}$ has terminated (the nondisjoint case treats $A$ and $B$ even-handedly). Choice $\mathcal{A}+\mathcal{B}$ starts out in the all-zero (initial) state and embarks on one of $\mathcal{A}$ or $\mathcal{B}$ ($\mathcal{A} \not\leq \times$ consists of those states of $\mathcal{A}$ in which at least one event has progressed to $\ulcorner$ or 1) while simultaneously cancelling the unchosen events. Orthocurrence $\mathcal{A}\otimes\mathcal{B}$ forms all states on $A \times B$ that are "bilinear" in both $\mathcal{A}$ and $\mathcal{B}$. (Here $z(\lambda,\forall)$ denotes $\lambda a.z(a,b)$ with $b$ universally quantified in the containing proposition, and dually for $z(\forall,\lambda)$. This makes $z(\lambda,\forall)$ and $z(\forall,\lambda)$ respectively the columns and rows of the $A \times B$ matrix $z$.)

We now specify the distance between states of each of these combinations. For the first three (all but orthocurrence) we are comparing two vectors $x, y$ indexed by the event set $A \cup B$, with no assumption that the event sets $A$ and $B$ are disjoint. We are given the distances from $x{\upharpoonright}A$ to $y{\upharpoonright}A$ and $x{\upharpoonright}B$ to $y{\upharpoonright}B$, call these $s$ and $t$ respectively. For all three operations we combine $s$ and $t$ as $s \otimes t$.

What varies between the operations is the choice of $\otimes$. For $\mathcal{A}||\mathcal{B}$ and $\mathcal{A}+\mathcal{B}$ we take $\otimes$ to be $\wedge$. Thus for $V = 2$, mere reachability, we require that $y{\upharpoonright}A$ be reachable from $x{\upharpoonright}A$ *and* $y{\upharpoonright}B$ from $x{\upharpoonright}B$. For numeric metrics $\wedge$ becomes max. For $\mathcal{A}\mathcal{B}$, in the reachability case we continue to take $\otimes$ to be $\wedge$ (to complete $\mathcal{A}\mathcal{B}$ we must complete both $\mathcal{A}$ and $\mathcal{B}$). In the numeric case however we take $\otimes$ to be $+$.

Having abruptly exhausted the space-time side of this instance of writer-reader duality, we content ourselves with foreshadowing a future expansion of this paper addressing the following concern. Metrics based on enrichment via the simple $V$'s considered here are calculated only from path endpoint information, making them intrinsically oblivious to intermediate obstacles of the kind so adroitly handled by the intransitive $K^A \cap X$ metric. Moreover the metrics computed inductively on process algebra terms employ mixed metrics, which only gets worse with expansions of process algebra to additional operations such as mutual exclusion, asymmetric conflict, etc. Thus the elegant and accurate $K^A{\cap}X$ metric would appear hard to arrive at either via enrichment or inductively, a serious concern for both techniques!

## References


1. Pratt, V.: The duality of time and information. In: Proc. of CONCUR'92. Volume 630 of Lecture Notes in Computer Science., Stonybrook, New York, Springer-Verlag (1992) 237–253
2. Gupta, V., Pratt, V.: Gates accept concurrent behavior. In: Proc. 34th Ann. IEEE Symp. on Foundations of Comp. Sci. (1993) 62–71

3. Barr, M.: $*$-Autonomous categories. Volume 752 of Lecture Notes in Mathematics. Springer-Verlag (1979)
4. Casley, R., Crew, R., Meseguer, J., Pratt, V.: Temporal structures. Math. Structures in Comp. Sci. **1** (1991) 179–213
5. Girard, J. Y.: Linear logic. Theoretical Computer Science **50** (1987) 1–102
6. Pratt, V.: Modeling concurrency with geometry. In: Proc. 18th Ann. ACM Symposium on Principles of Programming Languages. (1991) 311–322
7. Papadimitriou, C.: The Theory of Database Concurrency Control. Computer Science Press (1986)
8. Pratt, V.: Higher dimensional automata revisited. Math. Structures in Comp. Sci. **10** (2000) 525–548
9. Goubault, E.: (ed.) geometry and concurrency. Mathematical Structures in Computer Science, special issue **10** (2000) 409–573 (7 papers)
10. America, P., Rutten, J.: Solving reflexive domain equations in a category of complete metric spaces. Journal of Computer and System Sciences **39** (1989) 343–375
11. Gupta, V.: Chu Spaces: A Model of Concurrency. PhD thesis, Stanford University (1994) Tech. Report, available as `http://boole.stanford.edu/pub/gupthes.ps.gz`.
12. Pratt, V.: Transition and cancellation in concurrency and branching time. Math. Structures in Comp. Sci., special issue on the difference between sequentiality and concurrency, to appear (2002)
13. Pratt, V.: Chu spaces as a semantic bridge between linear logic and mathematics. Theoretical Computer Science (2002) to appear, preprint at `http://boole.stanford.edu/pub/bridge.ps.gz`.
14. Lawvere, W.: Metric spaces, generalized logic, and closed categories. In: Rendiconti del Seminario Matematico e Fisico di Milano, XLIII. Tipografia Fusi, Pavia (1973)
15. Pratt, V.: Enriched categories and the Floyd-Warshall connection. In: Proc. First International Conference on Algebraic Methodology and Software Technology, Iowa City (1989) 177–180
16. Roy, B.: Transitivité et connexité. Comptes Rendues Acad. Sci. **249** (1959) 216–218
17. Warshall, S.: A theorem on Boolean matrices. Journal of the ACM **9** (1962) 11–12
18. Floyd, R.: Algorithm 97: shortest path. Communications of the ACM **5** (1962) 345
19. Kleene, S.: Representation of events in nerve nets and finite automata. In: Automata Studies. Princeton University Press, Princeton, NJ (1956) 3–42
20. Robert, P., Ferland, J.: Généralisation de l'algorithme de Warshall. Revue Française d'Informatique et de Recherche Operationnelle **2** (1968) 13–25
21. Mac Lane, S.: Categories for the Working Mathematician. Springer-Verlag (1971)
22. Kelly, G.: Basic Concepts of Enriched Category Theory: London Math. Soc. Lecture Notes. 64. Cambridge University Press (1982)
23. Rosenthal, K.: Quantales and their applications. Longman Scientific and Technical (1990)
24. Rosenthal, K.: The theory of quantaloids. Longman Scientific and Technical (1996)
25. Gaifman, H., Pratt, V.: Partial order models of concurrency and the computation of functions. In: Proc. 2nd Annual IEEE Symp. on Logic in Computer Science, Ithaca, NY (1987) 72–85
26. Gaifman, H.: Modeling concurrency by partial orders and nonlinear transition systems. In: Proc. REX School/Workshop on Linear Time, Branching Time and Partial Order in Logics and Models for Concurrency, Noordwijkerhout, The Netherlands, Springer-Verlag (1989) 467–488