Entangled Pair Creation

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Entanglement

Quantum entanglement is a groundbreaking property of multi particle systems. It has raised questions about information transmission, that defy the constraints of the relativistic regimes. To begin with, an entangled state is the one that **cannot** be created as a product of 2 **single particle states** i.e.

$$|\text{entangled pair}\rangle \neq |\psi\rangle_1 |\psi\rangle_2.$$

For the sake of this document, I will not go into the EPR paradox and it's conclusion and we will jump to the Bell states and how to create (one of) them inside the lab.

The interesting part however is that in entangled system, measurement on one of the particles gives information about the second particle as well.

We will now look at the **Bell states** [1] :

$$\begin{aligned} |\beta_{00}\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \\ |\beta_{01}\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \\ |\beta_{10}\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \\ |\beta_{01}\rangle &= \frac{|01\rangle - |10\rangle}{\sqrt{2}} \end{aligned}$$

We can proved the states being completely mixed by taking the trace of ρ^2 - the density matrix of the states. It can also be seen that if we any one of the bell states, measuring one of the qubits/particles, instantly gives us the information about the second qubit, challenging the preconceived notion of **Local Realism**. However, that discussion can be left for later, the purpose of this document is to understand how to create an entangled pair.

One thing is clear from the notion of the entangled pair, they are created simultaneously as separately created systems would not have such a correlation/could be written as the product state.

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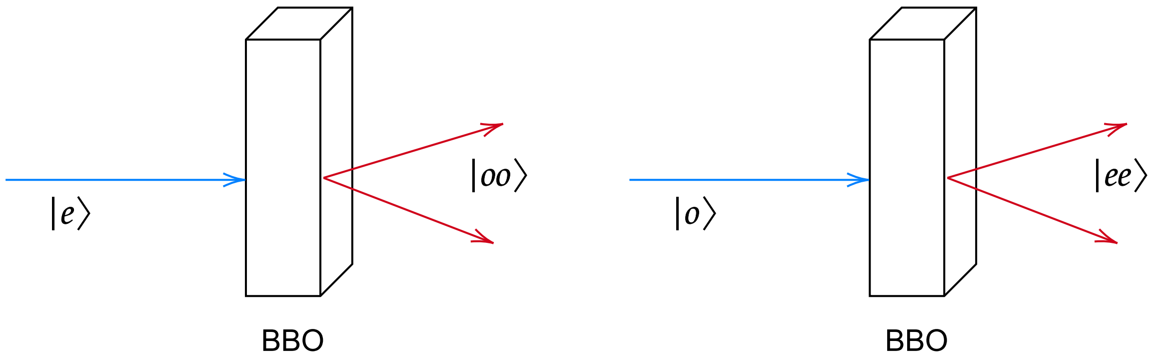
Polarization entangled Photons

Type 1 SPDC scheme

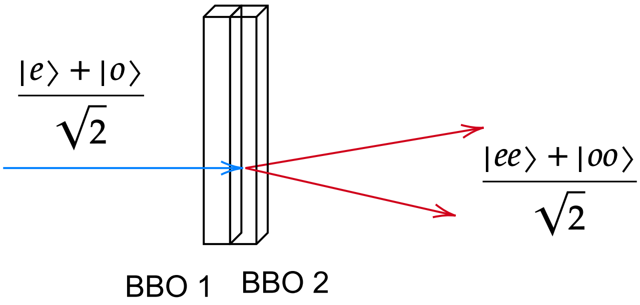
The bell states can belong to an 2 particle system that has 2 levels. You can read below the phase matching in SPDC, allows us to tune the polarization state of the signal and idler photons due to their birefringence. This part relies on [2] In Type 1 phase matching, the signal and idler beams the same polarization, orthogonal to the pump beams. We can tune that using the optical axis alignment of the crystal.



So here's the plan, if we stick 2 thin enough type 1 SPDC crystals such as the BBO, and supply it with pump beam in the state:

$$|\psi\rangle = \frac{|e\rangle + |o\rangle}{\sqrt{2}}.$$

The $|e\rangle$ will be down converted to $|oo\rangle$ and $|o\rangle$ is down converted to $|ee\rangle$, as the crystals are aligned orthogonally to each other. Of course, the modes will be different wrt pump beams, however the polarizations of the pump, signal, and idler beams are of the orthogonal nature. The schematic for this can be seen below:



Lab setup for making the $|\beta_{00}\rangle$ state

To get entangled photons, we must be able to detect single photons and hence make measurements of the polarization, and then test for entanglement based on CHSH inequalities. For that we will begin with a coherent light source. This part is also based on the experimental setup made by the Physlab at LUMS as described by [2] Every laser that we choose will begin with the light being in a mixed state, we will have to polarize it into the state:

$$|\psi\rangle = A|o\rangle + Be^{i\phi}|e\rangle.$$

Let's say we begin with a completely mixed state :

$$\rho = \frac{|e\rangle\langle e| + |o\rangle\langle o|}{2}.$$

.
We will place a linearly polarize to the diagonal state that has the matrix:

$$J_p(\theta) = \cos(\theta)^2|e\rangle\langle e| + \sin(\theta)^2|o\rangle\langle o|) + \sin(\theta)\cos(\theta)(|e\rangle\langle o| + |o\rangle\langle e|).$$

if we set the angle of the polarizer to to $\pi/4$ and apply it to the density matrix, our state will be :

$$\rho' = J_P(\pi/4)\rho = \frac{1}{2}(|e\rangle\langle e| + |e\rangle\langle o| + |o\rangle|e\rangle + |o\rangle\langle o|).$$

Which can be written as a product state :

$$\rho' = |\psi\rangle\langle\psi|,$$

which means that the density matrix represents the state $|\psi\rangle$.

Now that we have prepared our polarized state, we must set up the two SPDC crystals such that they down convert light and give us the entangled state.

Phase matching

The renowned BBO crystal is used in various experiments for down conversion. We will describe the phase matching conditions for the Type 1 SPDC (you can read the next sections for the types of phase matching).
The two BBO's stacked together have optical axes perpendicular as discussed above. The phase matching conditions in the components of the signal and the idler beams is :

$$n_p\omega_p = n_s\omega_s cos(\theta_s) + n_i\omega_i cos(\theta_i),$$
$$n_s\omega_s sin(\theta_s) + n_i\omega_i sin(\theta_i) = 0.$$

The down converted photons can be multimodal, but we will only consider the ones that come out of the horizontal plane with down converted beams of half the frequency as the pump beams i.e. $\omega_s = \omega_i = \omega_p/2$, $n_s = n_i$, and $\theta_s = \theta_i$.

The BBO is a uniaxial birefringent crystal with ordinary polarization perpendicular to the optical axis and extraordinary parallel to the optical axis of the crystal. We can align the optical axis such that the axes align with the lab frame. The BBO has the refractive index :

$$n = \left(A + \frac{B}{(\lambda^2 + C)} + D\lambda^2\right)^{1/2}.$$

λ is the wavelength in μm and the constants for the ordinary and extraordinary coefficients depending on the birefringence:

- $A_o = 2.7359$
- $B_o = 0.01878\mu m^2$
- $C_o = -0.01878\mu m^2$
- $D_o = -0.01354\mu m^{-2}$
- $A_e = 2.3753$
- $B_e = 0.01224\mu m^2$
- $C_o = -0.01667\mu m^2$
- $D_o = -0.01516\mu m^{-2}$.
  The effective refractive index when light is at an angle from the perpendicular axis is :

  $$n_{eff}(\theta_m) = \left(\frac{cos\theta_m^2}{n_o^2} + \frac{sin\theta_m^2}{n_e^2}\right)^{-1/2}.$$

  We can work out the θ_m using snell's law as outside the crystal, Snell's law Page would be : $\sin(\theta_l) = n_s \sin\theta_s$, which can be used to find the phase matching angle θ_m.

Here is the python snipped that works out the refractive indices and the group index to calculate the phase delay:

```
import numpy as np
class smellierUniaxial:

  ''' This is the type 1 scheme uniaxial crystal model.

  This works to find out the refractive indices based on

   the polarization and the wavelength of the signal,

  pump, and the idler beams.

  Uniaxial smellier model with 4-term form per axis (common BBO
variant).



    n^2(λ) = A + B/(λ^2 + C) + D * λ^2   with λ in micrometers
(µm)



    Provide (A,B,C,D) for ordinary and extraordinary axes. You can
subclass for a

    different functional form if you prefer. '''

  def __init__(self, Ao, Bo, Co, Do, Ae, Be, Ce, De, name: str =
"material"):

      ''' initializes the crystal, with the crystal constants'''


                                                              Page
      self.Ao, self.Bo, self.Co, self.Do = Ao, Bo, Co, Do
```

```
        self.Ae, self.Be, self.Ce, self.De = Ae, Be, Ce, De

        self.name = name

    def n_o(self, lam_um: np.ndarray | float) -> np.ndarray | float:

        ''' calculates the ordinary refractive indices based

        on the constants defined in the constructor.

        n^2(λ) = A + B/(λ^2 + C) + D * λ^2

        inputs:

        - lam_um : wavelenght of the beam in micro meters'''



        lam2 = np.asarray(lam_um) ** 2

        n2 = self.Ao + self.Bo / (lam2 - self.Co) - self.Do * lam2

        return np.sqrt(n2)



    def n_e(self, lam_um: np.ndarray | float) -> np.ndarray | float:

        ''' calculates the ordinary refractive indices based

        on the constants defined in the constructor.

        n^2(λ) = A + B/(λ^2 + C) + D * λ^2

        inputs:

        - lam_um : wavelenght of the beam in micro meters'''

        lam2 = np.asarray(lam_um) ** 2
```

```
        n2 = self.Ae + self.Be / (lam2 - self.Ce) - self.De * lam2
        return np.sqrt(n2)


    def n_eff_extra(self, lam_um: np.ndarray | float, theta: np.ndarray | float) -> np.ndarray | float:
        """Angle-dependent extraordinary index for a uniaxial crystal.
        1/n_eff^2 = cos^2(theta)/n_e(λ)^2 + sin^2(theta)/n_o(λ)^2
        where theta = angle between k and optic axis (inside crystal).
        inputs:
        - lam_um : wavelenght of the beam in micro meters
        - theta : angle between k and optic axis (inside crystal).
        output:
        - n_eff : The effective refractive index for oblique polarization.
        """
        ne = self.n_e(lam_um)
        no = self.n_o(lam_um)
        c2 = np.cos(theta) ** 2
        s2 = 1.0 - c2
        inv2 = c2 / (ne ** 2) + s2 / (no ** 2)
        return 1.0 / np.sqrt(inv2)
```

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```
  def group_index(self, lam_um: float, pol: str, theta: float |
None = None) -> float:

        """Group index n_g = n − λ dn/dλ. Determines group
velocity v_g = c/n_g.

        - For 'o': uses ordinary index.

        - For 'e': uses effective extraordinary index at given
theta.

        Important: group indices control relative delays of
pump/signal/idler pulses.

        The derivative is calculated using the mathod of
differences.

        """

        dl = 1e-4  # µm step for numeric derivative

        if pol == 'o':

            n0 = self.n_o(lam_um)

            n_p = self.n_o(lam_um + dl)

            n_m = self.n_o(lam_um - dl)

        elif pol == 'e':

            if theta is None:

                raise ValueError("theta required for extraordinary
group index")

            n0 = self.n_eff_extra(lam_um, theta)
```

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```
        n_p = self.n_eff_extra(lam_um + dl, theta)

        n_m = self.n_eff_extra(lam_um - dl, theta)

    else:

        raise ValueError("pol must be 'o' or 'e'")

    dn_dlam = (n_p - n_m) / (2 * dl)

    return float(n0 - lam_um * dn_dlam)
```

Photon counting and Entanglement testing

Once we have satisfied the phase matching conditions, we have a convincing hunch that the SPDC is spewing out an entangled pair of photons at an instance. We have shown below that the coincidence count of the down converted photons is close to one for SPDC. Using that we will see how we can show that we have an entangled pair. For this we will have to use some photon detectors, and work on the photon statistics to show that the pair we have produced is entangles. For this part, again we will refer to [2] for the methods of verifying the Freedman's test compliance of the system.

Freedman's inequality

Freedman's inequality is a test for local realism. It sets up constraints based on the hidden variables encompassing locality of the system. It's a simpler alternative to the CHSH inequality [3] The setup begins with six numbers with the following constraints:

$$\begin{gathered} 0 \leq x_1, x_2 \leq X, \\ 0 \leq y_1, y_2 \leq Y, \\ U = x_1 y_1 + x_2 y_2 + x_2 y_1 - x_1 y_2 - Y x_2 - X y_1. \end{gathered}$$

Then we can show that :

$$-XY \leq U \leq 0.$$

We will prove the upper bound case by case:

$$\textbf{Case 1 } x_1 \geq x_2 \textbf{ :}$$
$$U = (x_1 - X)y_1 + (y_1 - Y)x_2 + (x_2 - x_1)y_2$$
$$\text{Here we can see that parentheses are all negative.}$$

$$\textbf{Case 2 } x_2 \geq x_1\textbf{:}$$
$$U = x_1(y_1 - y_2) + (x_1 - X)y_1 - x_2(Y - y_2)$$
$$\leq x_1(y_1 - y_2) + (x_1 - X)y_1 - x_1(Y - y_2).$$
$$\text{The first two terms on the right of inequality are negative,}$$
$$\text{and the third term is a small positive being subtracted.}$$

We can show the lower bound, we factor :

$$U + XY = (X - x_2)(Y - y_1) + x_1y_1 + y_2(x_2 - x_1),$$
$$= (X - x_2)(Y - y_1) + x_2y_2 + x_1(y_1 - y_2),$$
$$= (X - x_2)(Y - y_1) + x_2y_1 + (y_2 - y_1)(x_2 - x_1).$$

We can examine these case by case to figure the lower bound $-XY \leq U$.

Now drawing from the typical Alice - Bob thought experimentation, if we have two photons, one going to Alice, the other going to bob, the probability of them being detected together (coincidental counting) is :

$$P(a, b) = \frac{N(a, b)}{N_{tot}}.$$

We assume that each photon pair has a probability of detection: $p_{12}(\lambda, a, b)$ of detection, where λ is a hidden variable which predetermines the likelihood of photon detection. Let's consider a special case : p_{12} is either 1, 0 such that each photon pair comes with an already decided fate with complete certainty. This is known as realism that a measurement has a definite value even if it isn't measured. If $\rho(\lambda)$ is the distribution for the hidden variable λ then we have :

$$P(a, b) = \int \rho(\lambda)p_{12}(\lambda, a, b)d\lambda.$$

We assume that the 2 systems are mutually exclusive:

$$p_12(\lambda, a, b) = p_1(\lambda, a)p_2(\lambda, b).$$

This assumes locality, that the measurement of one system does not affect the other system. Let $p_1(\lambda, \infty)$ the probability of detecting one photon when the polarizer is removed, then:

$$0 \leq p_1(\lambda, a) \leq p_1(\lambda, \infty),$$
$$0 \leq p_1(\lambda, a') \leq p_1(\lambda, \infty),$$
$$\text{similarly:}$$
$$0 \leq p_2(\lambda, b) \leq p_2(\lambda, \infty),$$
$$0 \leq p_2(\lambda, b') \leq p_2(\lambda, \infty).$$

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Applying the lemma defined above:

$$\begin{gathered}-p_1(\lambda,\infty)p_2(\lambda,\infty)\leq p_1(\lambda,a)p_2(\lambda,b)-p_1(\lambda,a)p_2(\lambda,b')\\ +p_1(\lambda,a')p_2(\lambda,b)+p_1(\lambda,a')p_2(\lambda,b')\\ -p_1(\lambda,a')p_2(\lambda,\infty)-p_1(\lambda,\infty)p_2(\lambda,b)\leq 0.\end{gathered}$$

Plugging this into the integral. we have :

$$-P(\infty,\infty)\leq P(a,b)-P(a,b')+P(a',b)+P(a',b')-P(a'.\infty)-P(\infty,b)\leq 0.$$

Now we will further assume that the coincidence counts will only depend on the polarization difference(rotational invariance) between the 2 beams : $\phi=|a-b|$. The angles are then chosen such that :

$$|a-b|=|a'-b|=|a'-b'|=\frac{|a-b'|}{3}=\phi,$$

simplifying the probabilities as :

$$-P(\infty,\infty)\leq 3P(\phi)-P(3\phi)-P(a',\infty)-P(\infty,b')\leq 0.$$

We also know that $P(\phi)=P(\phi+\pi)$, This must hold true for all angles, we will now choose $\phi=67.5^\circ$. The inequality becomes:

$$-P(\infty,\infty)\leq 3P(67.5)-P(22.5)-P(a',\infty)-P(\infty,b')\leq 0.$$

For $\phi=22.5^\circ$, we have:

$$-P(\infty,\infty)\leq 3P(22.5)-P(67.5)-P(a',\infty)-P(\infty,b')\leq 0.$$

We will multiply the first equation with -1 , and add the two together, yielding:

$$-P(\infty,\infty)\leq 4P(22.5)-4P(67.5)\leq P(\infty,\infty).$$

We can rewrite the result above as :

$$-N_0\leq 4N(22.5)-4N(67.5)\leq N_0.$$

we can further simplify, however this is the Freedman inequality to :

$$\delta=\left|\frac{N(22.5)-N(67.5)}{N_0}\right|-\frac{1}{4}\leq 0$$

Failure to satisfy, this inequality, our system will show a violation of local realism as imposed by the Freedman's inequality.

Freedman's test on Qubits

Let's begin with the first bell state :

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$$|\psi\rangle = \frac{1}{\sqrt{2}}(|e\rangle_A|e\rangle_B + |o\rangle_A|o\rangle_B)$$

We have photons A going towards detector A(Alice), and photons B going to detector B(Bob). The joint probability of photons polarized along θ_A are detected by Alice and photons polarized along θ_B are detected by Bob is:

$$P_{ideal}(\theta_A, \theta_B) = |_A\langle\theta_A|_B\langle\theta_B||\psi\rangle|^2.$$

where:

$$|\theta_A\rangle = \cos\theta_A|e\rangle + \sin\theta_A|o\rangle.$$

Then the expression for the probability above becomes:

$$P_{ideal}(\phi) = P_{ideal}(\theta_A, \theta_B) = \frac{1}{2}(\cos(\theta_A)\cos(\theta_b) + sin(\theta_A)\sin(\theta_B))^2,$$
$$P_{ideal}(\phi) = P_{ideal}(\theta_A, \theta_B) = \frac{1}{2}\cos^2(\theta_A - \theta_B) = \frac{1}{2}\cos^2(\phi)$$

if ϵ_a is the transmittance of the polarizer to Alice and ϵ_b to Bon, the actual probability of coincidental detection will be :

$$P_{actual}(\phi) = \frac{N(\phi)}{N_0} = \frac{1}{2}\epsilon_a\epsilon_b\cos^2\phi.$$

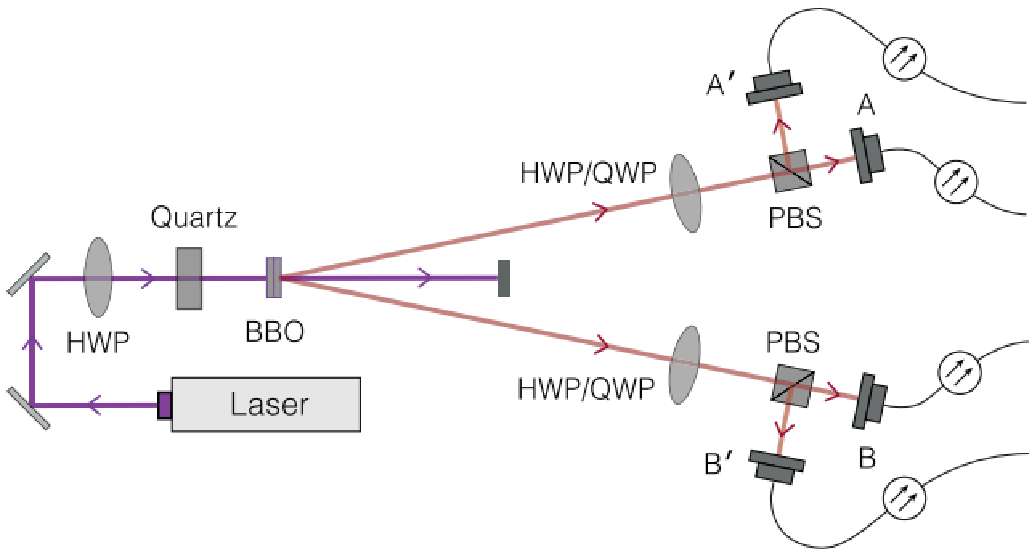
Setting the polarization angle difference $\phi = 22.5$, we have :

$$\delta = \left|\frac{N(22.5) - N(67.5)}{N_0}\right| - \frac{1}{4} = \frac{\epsilon_a\epsilon_b}{2\sqrt{2}} - \frac{1}{4} \leq 0$$

The lower bound for this case yields the transmittance to be $\epsilon_a\epsilon_b = 0.71$ $\epsilon_a\epsilon_b = 0.71, \sqrt{\epsilon_a\epsilon_b} = 0.84$, which implies that the polarizer transmittance must have a geometric mean of at least 0.84.

Experimental setup

In the path of the beam, a HWP is placed to give the diagonal state to the pump beam, in order to get the entangled state from the SPDC. A quartz plate is also introduced to minimize the phase shift that is going to arise as a result of the BBO's phase delay. The stacked up BBO crystal setup will be used as discussed above in the following setup [2] :



Now we must choose the polarizers such that their transmittance allow us to achieve a geometric mean of at least 0.84 to show the violation of the Freedman's inequality.

LAB GUIDE

[2]

Required Apparatus:

- 405 nm Continuous LASER (MDL-III-405-50 mW, CNI).
- Linear Polarizers.
- HWPs
- QWPs
- Quartz
- BBO birefringent SPDC crystals
- PBS
- Beam collimators
- SPCM (Single Photon Counting Modules)
- FPGA for processing the inputs from the SPCM
- Lab Mounts
- Irises.

Preparing the state

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The light from the pump source at 402nm comes in a completely mixed state :

$$\rho_p = \frac{I_2}{2}.$$

The state must be prepared in the lab in the diagonal state so that the type 1 phase matching is able to make an extraordinary pair when the polarization of the pump is ordinary(perpendicular to the optical axis) and ordinary pair when the pump is parallel to the optical axis of the second crystal.

Jones calculus of the Retarders:

The retarder used in the experiment are the **HWP**(Half wave plate), **QWP** (Quarter Waveplate) and, the Quartz. The first two also have a rotational properties based on their orientation. The jones operation if the lab axes align with $|H\rangle$ and $|V\rangle$ is :

$$J = R(-\theta)\mathrm{diag}(1, e^{i\delta})R(\theta).$$

Where $R(\theta)$ is the rotational matrix,θ above the horizontal axis is the fast axis. The δ is the phase delay between the fast and the slow axis. The $\delta_{HWP} = \pi$, and $\delta_{QWP} = \frac{\pi}{2}$. For the quartz crystal, the phase delay depends on the wavelength of the pump :

$$\delta_{\text{Quartz}} = \frac{2\pi \Delta n L}{\lambda}.$$

Where, the Δn is the birefringence of the material, L is the material thickness.

To make the diagonal state from the mixed state, We will now linearly polarize light to let's say the $|H\rangle\langle H|$ with probability p.
The normalized polarized state will be :

$$\rho_p = p|H\rangle\langle H| + \frac{1-p}{2} I_2$$

The diagonal Polarized state can be made using the HWP with angle set at $\theta = \frac{\pi}{4}$
.

$$J_{HWP} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$
$$= \cos 2\theta |H\rangle\langle H| + \sin 2\theta(|H\rangle\langle V| + |V\rangle\langle H|) - \cos 2\theta |V\rangle\langle V|.$$

The post operation state will be : $J_{HWP}\rho_p J_{HWP}^{\dagger}$;however the HWP is a unitary retarder hence the post op mixed state will be returned as it is : Page

$$
\begin{aligned}
\rho_p' &= p\begin{bmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{bmatrix} + \frac{1-p}{2}I_2 \\
&= p\begin{bmatrix}\cos^2 2\theta & \cos 2\theta \sin 2\theta \\ \sin 2\theta \cos 2\theta & sin^2 2\theta\end{bmatrix} + \frac{1-p}{2}I_2 \\
&= \cos^2 2\theta |H\rangle\langle H| + \cos 2\theta \sin 2\theta (|H\rangle\langle V| + |V\rangle\langle H|) + \sin^2 2\theta |V\rangle\langle V| + \frac{1-p}{2}I_2.
\end{aligned}
$$

For $\theta = \frac{\pi}{8}$, we will have

$$
\begin{aligned}
\rho_p' &= \frac{p}{2}(|H\rangle + |V\rangle)(\langle H| + \langle V|) + \frac{1-p}{2}I_2 \\
&= p|D\rangle\langle D| + \frac{1-p}{2}I_2,
\end{aligned}
$$

where $|D\rangle = \frac{|H\rangle+|V\rangle}{\sqrt{2}}$.

The **Quartz** Crystal is added for a specific purpose, that is to cater for the phase delay added by the BBO, a negative of that phase delay is added preemptively, so that the BBO's phase delay factor is catered by it. The thickness of the crystal, and the birefringence choose the phase delay of the crystal. Since BBO is a *negative* birefringent crystal, a positive birefringent crystal is added to cater to the phase delay exactly.

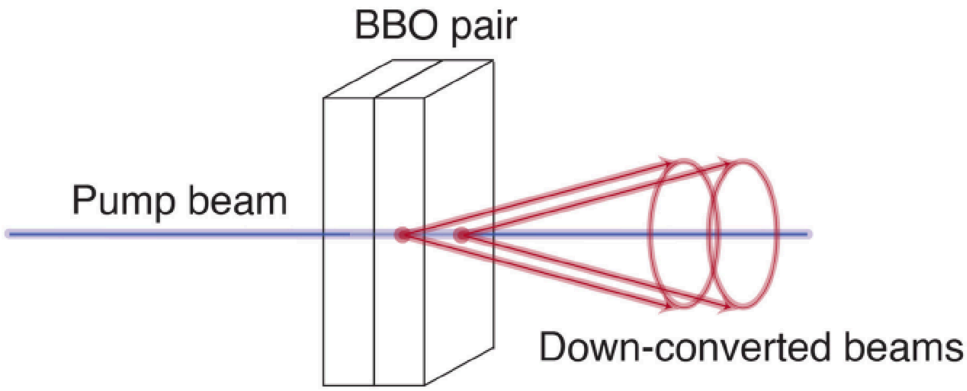
SPDC pair creation.

The SPDC are discussed in detail below, however, in this part we will be discussing the Type 1 SPDC's operation on the polarization eigenstates. In the KWIAT model, there are 2 crystals that are used, it down converts the ordinary pump into the extraordinary signal and idler beams. In this experiment, we will carefully chose the optical axes of the BBO crystals so that we are able to achieve the entangle pair.

We have three objectives in this down conversion process:

- Down convert $|V\rangle \rightarrow |HH\rangle$.
- Down convert $|H\rangle \rightarrow |VV\rangle$.
- Align the 2 crystals such that the 2 polarizations come out as one beam, to have information erasure of the pump state.
  The extraordinary polarization aligns with the optical axis of the crystal. If we align optical axis of the crystal along the length of the optical bread board, we will achieve the down conversion of the the vertical polarization state as per the lab axis. The other crystal will be perpendicular to the breadboard, such that the polarization along the breadth $|H\rangle$ aligns with the optical axis. Page

Important Note: The 2 crystals must be aligned such that the down converted polarizations are collinear in the signal and idler beams. Otherwise, we would have 2 separate systems.



The Extraordinary photons as per crystal 1 will be $|V\rangle$. Hence $\langle V|\rho_p|V\rangle$ will be the amplitude for $|HH\rangle\langle HH|$. Similarly, the extraordinary crystals as per the second crystal is the $|H\rangle$, hence the amplitude of $|VV\rangle\langle VV|$, will be $\langle H|\rho_p|H\rangle$. The density matrix is Hermitian, so the amplitude for $|HH\rangle\langle VV|$ will be $\langle V|\rho_p|H\rangle$, and there will be the conjugate terms as well. The state post down conversion will be :

$$\rho_d = \langle V|\rho_p|V\rangle|HH\rangle\langle HH| + \langle H|\rho_p|H\rangle|VV\rangle\langle VV|$$
$$+\kappa e^{i\phi}\langle V|\rho_p|H\rangle|HH\rangle\langle VV| + \kappa e^{-i\phi}\langle V|\rho_p|H\rangle|VV\rangle\langle HH|$$
$$= \frac{1}{2}(|HH\rangle\langle HH| + |VV\rangle\langle VV|) + \kappa\frac{p}{2}(e^{i\phi}|HH\rangle\langle VV| + e^{-i\phi}|H\rangle|VV\rangle\langle HH|).$$

As we can see, P decided how much of an entangled state we have and the phase decides if we have the first bell state or the second bell state. If we keep ϕ = 0, we will have the first bell state, and if we keep the $\phi = \pi$, we will have the second bell state, this can be adjusted using the quartz crystal alignment. The κ is the coherence term. Of course, there will be noise inflicted on the state that we must inculcate to make this closer to the reality.

The state with noise added will be :

$$\rho_{dw} = (1-w)\rho_d + w\frac{I_4}{4}$$

Important Note:
Another thing to bear in mind is that the down converted will be 810 nm, out of the visible spectrum. After the down conversion, we will require auxiliary equipment to find out the ray.

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Ray Finding

To find the down converted beams, we will take the following few steps:

- Calculate the angle based on the Conservation of momenta.
- Block the Pump radiation and all other radiations to have only the detection of the light.
- Add filters to concentrate the light we want to detect.

Measurements

Now we will prepare the state after the down conversion to direct the photons to the detectors based on the polarizations. We are going to use HWP, QWP to rotate the polarization as per our measurement basis. We can change our state into the diagonal antidiagonal basis e.t.c

The PBS - Polarizing Beam splitter will then split the beam based on the orthogonal polarizations based on its axial alignment. The alignment will be changed measure the polarizations of the amplitude basis. The settings for both of the equipment will be in tandem for the signal and idler beams.

The output will then be detected into 4 SPMC modules that will give us the photon counts. that will be processed by the FPGA module.

Violation of Local Realism

We can measure the violation of local realism, by measuring the coincidence counts and verifying them with the Local Realism. There are several experiments that can be designed to see the violation of the CHSH or the freedman's inequalities.

Without explaining the formulation of the CHSH inequalities, I would state them here :

$$S = E(\alpha_1, \beta_1) - E(\alpha_1, \beta_2) + E(\alpha_2, \beta_1) + E(\alpha_2, \beta_2).$$

Where α is detected by Detector A, α' with Detector A'. Similarly β is at Detector B and, β' at Detector B'.

The $E(\alpha, \beta)$ is defined as:

$$E(\alpha, \beta) = P(\alpha, \beta) + P(\alpha', \beta') - P(\alpha, \beta') - P(\alpha', \beta').$$

The Local realism assumptions from which CHSH inequalities arise, affix $|S| \leq 2$. For all values of α and β .

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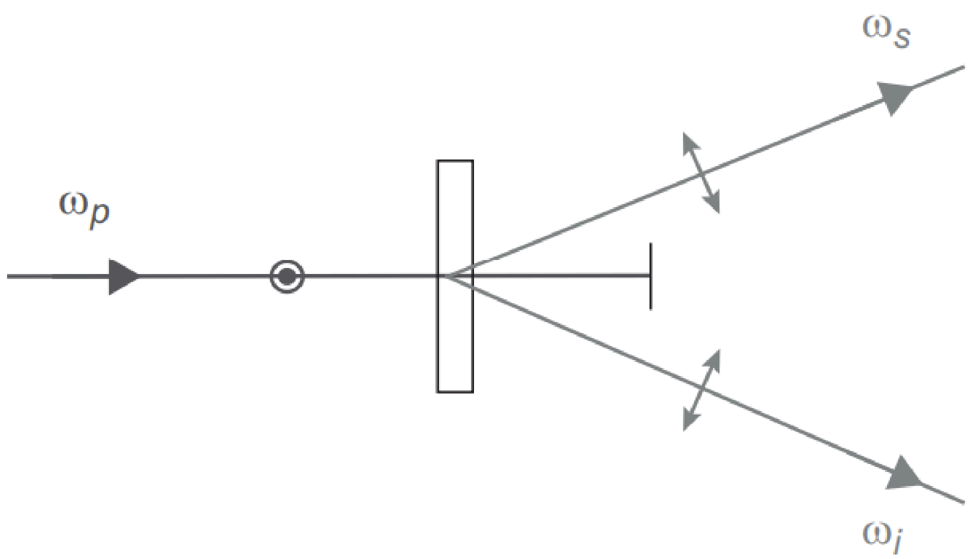
We can chose $\alpha_1 = 0.0$, $\alpha_2 = \frac{\pi}{4}$, $\beta_1 = \frac{\pi}{8}$,and $\beta_2 = \frac{3\pi}{8}$ using the HWP, we can figure out the values at which we should see the maximum violation of the CHSH inequalities. The primes will be the orthogonal polarization sent to the other detector. I am leaving the theoretical calculation as an activity for the reader.

Spontaneous Parametric Down Conversion (SPDC)

In spontaneous parametric down conversion, a photon of a certain frequency is down converted to 2 photons, usually half of the energy. This is a Non Linear process $A(v+u) \neq Av + Au$ unlike absorption, reflection, polarization e.t.c. In the SPDC setup, a pump beam of a shorter wavelength is supplied to the crystal. The 2 output beams are the signal and the idler beams, usually scaled up by a factor of 2 [4] .
As can be seen below:



The pump frequency ω_p splits into the signal and idler beams. SPDC is still constrained by the conservation of momentum and the conservation of energy. The signal, idler, and pump frequencies/ momenta are related by these constraints:

$$\begin{gathered}\text{Conservation of Energy:}\\ \hbar\omega_p = \hbar\omega_s + \hbar\omega_i\\ \omega_p = \omega_s + \omega_i\\ \text{Conservation of Momentum}\\ \hbar \boldsymbol{k}_p = \hbar \boldsymbol{k}_s + \hbar \boldsymbol{k}_i\\ \boldsymbol{k}_p = \boldsymbol{k}_s + \boldsymbol{k}_i\end{gathered}$$

The pump beams follow the dispersion relationship: $k_p = \frac{n_p\omega_p}{c}$ at the incidence. The refractive index is a function of frequency $n \to n(\omega)$, hence if the signal and the idler waves are collinear, it implies that the refractive index of all three beams is roughly the same. However the down converted beams are halved and will Page have a significant change in the refractive. Due to second order polarization, a phase mismatch occurs in the signal and the idler beams.

Phase Mismatch in SPDC

In Non Linear media, the response to the external electric field to the molecular displacement is significant to square of electric field.:

$$P = \epsilon_0[\chi^{(1)}E + \chi^{(2)}E^2 + \dots].$$

The second order term:

$$P^{(2)} = \epsilon_0\chi^{(2)}E^2,$$

is significant in non linear materials. The signal and idler fields generation is a direct result of this second order terms and grow as :

$$\vec{E}_s, \vec{E}_i \propto \int_0^L P^{(2)}(z')e^{i\Delta k z'}dz',$$

where the exponential term with $\Delta k = k_p - k_s - k_i$ accounts for the phase mismatch in the pump and the signal/idler beams. This integral leads to constructive interference only when $\Delta K \approx 0$.

To cater to this problem a birefringent crystal is used for different refractive indices. They are cleverly oriented using the propagation angle w.r.t the optical axis of the birefringent SPDC crystal through various schemes that give us the famously known **SPDC types**:

- **Type 0 Phase matching in SPDC**:
  All three beams are in the same polarizations (extraordinary polarization, as it is tunable using the optical axis of the crystal)
  Usually, the extraordinary polarization/refractive index is used. The refractive index is a function of the angle of incidence [5] .
  The refractive index, n depends on the angle of propagation, and the frequency of light. The phase matching condition would be cleverly set and the crystal will be rotated so that:

  $$n_e(\theta_p, \omega_p) = n_e(\theta_i, \omega_i) = n_e(\theta_s, \omega_s),$$

  which allows for the same refractive index, and same polarizations. The only caveat is that the refractive index of the extraordinary polarization is dependent on the angle of propagation. Choosing extraordinary polarization for all three beams restricts us to find a **sweet spot** where we can find the same refractive index for the 3 beams, and it maybe very difficult to do so.

- **Type 1 Phase matching in SPDC**:
  This is the more widely used in SPDC configurations, uses fixed refractive indices and polarizations parallel to each other for the signal and the idler beams. The crystal is cut in such a way that the down converted beams have a component perpendicular to the optical axis which will give us beams in ordinary polarizations that will have a fixed refractive index. The optical axis of the crystal is moved so that the the refractive index of the the beams can be matched.

  The following scheme is used for the type 1 SPDC:

  $$n_e(\omega_p.\theta) = n_o(\omega_s) = n_o(\omega_i)$$

  The optical axis or the angle of incidence of the pump beams is rotated so that the refractive index of the 3 beam is matched along with the phase.
- **Type 2 Phase matching in SPDC**:
  In type 2 phase matching, the pump beam matches the polarization of one of the 2 signal beams: e $\rightarrow$ e+o or
  e $\rightarrow$ o+e, with down converted beams having orthogonal polarizations [6] .
  Here, we do not solve the
  $\omega_p = 2\omega_s$ condition, and it also works for asymmetrically down converted beams.
  Here the condition to be fulfilled, for the optical axis to be chosen is :

  $$n_e(\omega_p, \theta_p) = n_o(\omega_s) + n_e(\omega_i, \theta_i)$$

  The optical axis for the pump beam is now shifted such that it equals the refractive indices of the 2 beams. This ensures that the 2 down converted beams are not destructively interfered by themselves or the pump beams. This also naturally creates an entangled pair [7] .

Mechanics of the SPDC

For this part, we will look at the generation of signal and idler beams in the SPDC, the photon statistics and the rate of down conversion. Furthermore, the quantum nature of the signal and idler beams. This will be extensive and will include rigorous mathematics based on [8] . We will begin from he Hamiltonian inside the SPDC and reason the certain assumptions. We will not be going over the canonical second quantization of the field, as the discussion on that will be quiet extensive.

To start off, In the down conversion process, The total energy would depend on the total number of the signal, idle, and the pump beams. The assumptions is that the down converted beams are much weaker compared to the intense pump beam, hence it can be treated classically The annihilation operator for the pump beam can be written as $a_0 = v_0 e^{i\omega t}$, where v_o is a constant amplitude of the beam. This follows from the assumption $\langle n_1(t)\rangle, \langle n_2(t)\rangle << \langle n_0(t)\rangle = |v_0|^2$. Additionally, the energy of the system would depend on the creation of the down converted particles, and the annihilation of the pump beam. Additionally, since it is still possible physically, the Hermitian conjugate or the reversal of this process must also be included in this Hamiltonian of the system.

We begin with :

$$\hat{H} = \sum_{i=0}^{2} \hbar\omega_i (n_i + \frac{1}{2}) + \hbar g[\hat{a_1}^\dagger \hat{a_2}^\dagger \hat{a_0} + \hat{a_1}\hat{a_2}\hat{a_0}^\dagger].$$

We will now impose the condition of the classical treatment of the pump beam:

$$\hat{H} = \sum_{i=1}^{2} \hbar\omega (n_i + \frac{1}{2}) + \hbar g[\hat{a_1}^\dagger \hat{a_2}^\dagger v_o e^{-i\omega_0 t} + a_1 a_2 v_0^* e^{i\omega_0 t}].$$

It is convenient to check the commutation relationship of the operator $\hat{n_1} - \hat{n_2}$ with the Hamiltonian to see the time evolution of this operator:

$$\begin{gathered} i\hbar[\dot{\hat{n_1}} - \dot{\hat{n_2}}] = [\hat{n_1} - \hat{n_2}, H] \propto [\hat{n_1} - \hat{n_2}, \hat{n_1}] + [\hat{n_1} - \hat{n_2}, \hat{n_2}] \\ +[\hat{n_1} - \hat{n_2}, \hat{a_1}^\dagger \hat{a_2}^\dagger v_o e^{i\omega t}] + [\hat{n_1} - \hat{n_2}, a_1 a_2 v_0^* e^{-i\omega t}], \end{gathered}$$

$$\begin{gathered} = \hat{n_1}\hat{n_1} - \hat{n_2}\hat{n_1} - \hat{n_1}\hat{n_1} + \hat{n_1}\hat{n_2} + \hat{n_1}\hat{n_2} - \hat{n_2}\hat{n_2} - \hat{n_2}\hat{n_1} + \hat{n_2}\hat{n_2} \\ +\hat{n_1}\hat{a_1}^\dagger \hat{a_2}^\dagger v_o e^{i\omega t} - \hat{n_2}\hat{a_1}^\dagger \hat{a_2}^\dagger v_o e^{i\omega t} - \hat{a_1}^\dagger \hat{a_2}^\dagger v_o e^{i\omega t}\hat{n_1} + \hat{a_1}^\dagger \hat{a_2}^\dagger v_o e^{i\omega t}\hat{n_2} \\ +\hat{n_1} a_1 a_2 v_0^* e^{-i\omega t} - \hat{n_2} a_1 a_2 v_0^* e^{-i\omega t} - a_1 a_2 v_0^* e^{-i\omega t}\hat{n_1} + a_1 a_2 v_0^* e^{-i\omega t}\hat{n_2}, \end{gathered}$$

$$\begin{gathered} \text{Using the decomposition of the number operator } \rightarrow n_i = a_i^\dagger a_i \\ = \hat{a_1}^\dagger \hat{a_1}\hat{a_1}^\dagger \hat{a_2}^\dagger v_o e^{i\omega t} - \hat{a_2}^\dagger \hat{a_2}\hat{a_1}^\dagger \hat{a_2}^\dagger v_o e^{i\omega t} - \hat{a_1}^\dagger \hat{a_2}^\dagger \hat{a_1}^\dagger \hat{a_1} v_o e^{i\omega t} + \hat{a_1}^\dagger \hat{a_2}^\dagger \hat{a_2}^\dagger \hat{a_2} v_o e^{i\omega t} \\ +\hat{a_1}^\dagger \hat{a_1} a_1 a_2 v_0^* e^{-i\omega t} - \hat{a_2}^\dagger \hat{a_2} a_1 a_2 v_0^* e^{-i\omega t} - a_1 a_2 v_0^* e^{-i\omega t}\hat{a_1}^\dagger \hat{a_1} + a_1 a_2 v_0^* e^{-i\omega t}\hat{a_2}^\dagger \hat{a_2}, \end{gathered}$$

$$\begin{gathered} \text{Using the bosonic commutation relationship } \rightarrow [a_i, a_i^\dagger] = 1 \text{ and } [a_i, a_j] = 0, \\ = \hat{a_1}^\dagger \hat{a_1}\hat{a_1}^\dagger \hat{a_2}^\dagger v_o e^{i\omega t} - \hat{a_2}^\dagger \hat{a_2}\hat{a_1}^\dagger \hat{a_2}^\dagger v_o e^{i\omega t} - \hat{a_1}^\dagger \hat{a_2}^\dagger (\hat{a_1}\hat{a_1}^\dagger - 1) v_o e^{i\omega t} + \hat{a_1}^\dagger \hat{a_2}^\dagger (\hat{a_2}\hat{a_2}^\dagger - 1) v_o \\ +\hat{a_1}^\dagger \hat{a_1} a_1 a_2 v_0^* e^{-i\omega t} - \hat{a_2}^\dagger \hat{a_2} a_1 a_2 v_0^* e^{-i\omega t} - a_1 a_2 v_0^* e^{-i\omega t}(\hat{a_1}\hat{a_1}^\dagger - 1) + a_1 a_2 v_0^* e^{-i\omega t}(\hat{a_2}\hat{a_2}^\dagger - 1 \end{gathered}$$

$$= a_1^\dagger a_2^\dagger v_0 e^{i\omega t} - a_1^\dagger a_2^\dagger v_0 e^{i\omega t} + a_1 a_2 v_0^* e^{-i\omega t} - a_1 a_2 v_0^* e^{-i\omega t} = 0.$$

This is a particularly helpful result that proves that at any moment in time the rate of change of the signal and idler particles is the same, showing the pair creation of the signal and idler photons.

Time evolution of the ladder operators

The Hamiltonian mechanics is a very convenient tool over here to figure out the photon statistics, that can eventually prove the non classical nature (deviation from the classical model of the SPDC) To work on the photon statistics, it is convenient to move to the Heisenberg picture of Quantum mechanics and evolve the operators in time so that we are able to work on the time evolution of the photon statistics.

$$a_i(t) = U_H^\dagger a_i U_H$$
$$a_1(t) = e^{\frac{i}{\hbar}\hat{H}t} a_1 e^{-\frac{i}{\hbar}\hat{H}t}$$
$$a_2(t) = e^{\frac{i}{\hbar}\hat{H}t} a_2 e^{-\frac{i}{\hbar}\hat{H}t}$$

from the Heisenberg's equation of motion, we can figure out the time evolution of the operators above as the Hamiltonian in the Schrodinger picture is time independent.

$$\dot{a_i} = \frac{i}{\hbar}[H_H, a_i] + \partial_t a_i.$$

We know that the ladder operators in the Schrodinger picture don't have explicit time dependence, we can rewrite the equation above as:

$$\dot{a_i} = \frac{i}{\hbar}[H, a_i].$$

We can breakdown the commutation relationships with the previously stated Hamiltonian:

- The commutation for the free part:

$$[n_i, a_j] = -\delta_{ij} a_j,$$
$$[n_1, a_1] = -a_1, [n_2, a_2] = -a_2$$

- The commutation $[a_i, a_j^\dagger] = \delta_{ij}$, for the down conversion part gives us

$$\begin{aligned} [a_1^\dagger a_2^\dagger, a_1] &= a_1^\dagger a_2^\dagger a_1 - a_1 a_1^\dagger a_2^\dagger \\ &= a_1^\dagger a_1 a_2^\dagger - (1 + a_1^\dagger a_1) a_2^\dagger, \\ [a_1^\dagger a_2^\dagger, a_1] &= -a_2^\dagger. \\ \text{Genenralizing for the second annihilator:} \\ [a_1^\dagger a_2^\dagger, a_2] &= -a_1^\dagger \end{aligned}$$

- The commutation with the Hermitian conjugate interaction part

$$[a_1 a_2, a_1] = 0 = [a_1 a_2, a_2]$$

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Now we can expand and write the commutators:

$$[H, a_1] = -\hbar\omega_1 a_1 - \hbar g v_o e^{-i\omega_0 t} a_2^\dagger,$$
$$[H, a_2] = -\hbar\omega_2 a_2 - \hbar g v_o e^{-i\omega_0 t} a_1^\dagger.$$

Using this, we will simplify the Heisenberg of the equations of motion :

$$\dot{a_1} = -i\omega_1 a_1 - igv_o e^{-i\omega_0 t} a_2^\dagger$$
$$\dot{a_2} = -i\omega_2 a_2 - igv_o e^{-i\omega_0 t} a_1^\dagger$$

It is often convenient to have slowly varying versions of these operators by multiplying the conjugate of the free oscillation term:

$$\mathcal{A}_i = a_i e^{i\omega t}$$

This is particularly helpful when looking at the time derivatives of the operators and takes away the free rotation of the operators and focuses on the interesting interaction parts/ time evolution other than natural oscillation of the operators.

This gives us :

$$\dot{\mathcal{A}}_1 = \dot{a_1} e^{i\omega_1 t} + i\omega_1 a_1 e^{i\omega_1 t}$$
$$\dot{\mathcal{A}}_1 = -i\omega_1 a_1 e^{i\omega_1 t} - igv_o e^{i(-\omega_0+\omega_1)t} a_2^\dagger + i\omega_1 a_1 e^{i\omega_1 t}$$
$$\dot{\mathcal{A}}_1 = -igv_o e^{i(-\omega_0+\omega_2)t} a_2^\dagger = -igv_0 e^{i(\omega_1+\omega_2-\omega_0)t} \mathcal{A}_2^\dagger$$
$$\text{for} : \omega_0 = \omega_1 + \omega_2:$$
$$\dot{\mathcal{A}}_1 = -igv_0 \mathcal{A}_2^\dagger$$
$$\text{similarly for } \mathcal{A}_2 :$$
$$\dot{\mathcal{A}}_2 = -igv_0 \mathcal{A}_1^\dagger$$

We can differentiate the 2 equations twice to get:

$$\ddot{\mathcal{A}}_1 = -igv_0 \dot{\mathcal{A}}_2^\dagger = -igv_0(igv_0^*)\mathcal{A}_1 = g^2|v_0|^2 \mathcal{A}_1(t)$$
$$\ddot{A}_2 = g^2|v_0|^2 \mathcal{A}_2.$$

The general solutions of these equations are of the form:

$$\mathcal{A}_1(t) = \mathcal{C}_1(0)e^{kt} + D_1(0)e^{-kt},$$

We have a few constraints that will help us find the solution:

$$\mathcal{A}_1(0) = \mathcal{C}_1 + \mathcal{D}_1.$$

The fist derivative gives us :

$$\dot{\mathcal{A}}_1 = k\mathcal{C}_1(0)e^{kt} + -kD_1(0)e^{-kt} = k(\mathcal{C}_1 - \mathcal{D}_1).$$

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Using the 2 equations, we can find the constants of the equation:

$$\mathcal{C}_1 = \frac{1}{2}\left(\mathcal{A}_1 + \frac{1}{k}\dot{\mathcal{A}}_1\right), \mathcal{D}_1 = \frac{1}{2}\left(\mathcal{A}_1 - \frac{1}{k}\dot{\mathcal{A}}_1\right).$$

Plugging this back in, we get :

$$\mathcal{A}_1(t) = \mathcal{A}_1(0)cosh(kt) + \frac{\dot{\mathcal{A}}_1(0)}{k}sinh(kt).$$

Taking the second derivative to find k:

$$\mathcal{A}_1(t) = k^2\left(\mathcal{A}_1(0)cosh(kt) + \frac{\dot{\mathcal{A}}_1(0)}{k}sinh(kt)\right).$$

which gives us : $k = g|v_0|$.

Using $v_0 = |v_0|e^{i\theta}$, and substituting $\dot{\mathcal{A}}_1(0) = -igv_0\mathcal{A}_2^\dagger(0)$:

$$\mathcal{A}_1(t) = \mathcal{A}_1(0)cosh(g|v_0|t) - ie^{i\theta}\mathcal{A}_2{}^\dagger(0)sinh(g|v_0|t)$$
$$\mathcal{A}_2(t) = \mathcal{A}_2(0)cosh(g|v_0|t) - ie^{i\theta}\mathcal{A}_1{}^\dagger(0)sinh(g|v_0|t)$$

We have arrived to the solutions for the slowly evolving time evolution of the ladder operators. This will be particularly helpful for us to work photon statistics, and their time evolutions.

Photon Statistics

Let's begin with the vacuum state, where there are no excitations i.e there are existence of the photons/phonons in the system. The vacuum state for signal photons $|vac\rangle_1$ is such that $a_1|vac\rangle_1 = 0$. Let's prepare our system form the vacuum state and then we will work the r^{th} moment of the expectation of the number of signal and idler photons.

$$\langle \hat{n_1}^{(r)}(t)\rangle =_{1,2} \langle vac|a_1^\dagger(t)a_1(t)|vac\rangle_{1,2} = \langle vac|\mathcal{A}_1^\dagger(t)^r\mathcal{A}_1(t)^r|vac\rangle_{1,2}$$

$$= \langle vac|[\mathcal{A}_1{}^\dagger(0)cosh(g|v_0|t) + ie^{-i\theta}\mathcal{A}_2(0)sinh(g|v_0|t)]^r$$
$$[\mathcal{A}_1(0)cosh(g|v_0|t) - ie^{i\theta}\mathcal{A}_2{}^\dagger(0)sinh(g|v_0|t)]^r|vac\rangle_{1,2}$$

$$=_{1,2} \langle vac|[\mathcal{A}_1{}^\dagger(0)cosh(g|v_0|t)]^r[\mathcal{A}_1(0)cosh(g|v_0|t)]^r|vac\rangle_{1,2}$$
$$-_{1,2}\langle vac|[\mathcal{A}_1{}^\dagger(0)cosh(g|v_0|t)[ie^{i\theta}\mathcal{A}_2{}^\dagger(0)sinh(g|v_0|t)]^r|vac\rangle_{1,2}$$
$$+_{1,2}\langle vac|[ie^{-i\theta}\mathcal{A}_2(0)sinh(g|v_0|t)]^r[\mathcal{A}_1(0)cosh(g|v_0|t)]^r|vac\rangle_{1,2}$$
$$-_{1,2}\langle vac|[ie^{-i\theta}\mathcal{A}_2(0)sinh(g|v_0|t)]^r[ie^{i\theta}\mathcal{A}_2{}^\dagger(0)sinh(g|v_0|t)]^r|vac\rangle_{1,2}.$$

We know that $\langle vac|a^\dagger = a|vac\rangle = 0$, hence the first 3 terms are zero.

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$$\langle \hat{n}_1(t)^r \rangle = |(-ie^{i\theta})^r sinh^r(g|v_0|t)\mathcal{A}_2{}^\dagger(0)^r|vac\rangle_{1,2}|^2$$
$$= sinh^{2r}(g|v_0|t)_{1,2}\langle vac|\mathcal{A}_2{}^r(0)\mathcal{A}_2{}^{\dagger r}(0)|vac\rangle_{1,2}$$

Using the commutator $[A_2, A_2^\dagger] = 1$,
we will try to resolve the ladder operators into number operators

$$\mathcal{A}_2{}^r\mathcal{A}_2^{\dagger^r} = \mathcal{A}_2{}^{r-1}\mathcal{A}_2\mathcal{A}_2^\dagger\mathcal{A}_2^{\dagger^{r-1}} = \mathcal{A}_2{}^{r-1}[\mathcal{A}_2^\dagger\mathcal{A}_2 + 1]\mathcal{A}_2^{\dagger^{r-1}}$$
$$= \mathcal{A}_2{}^{r-1}\mathcal{A}_2^\dagger\mathcal{A}_2\mathcal{A}_2^{\dagger^{r-1}} + \mathcal{A}_2{}^{r-1}\mathcal{A}_2^{\dagger^{r-1}}$$
$$= \mathcal{A}_2{}^{r-2}\mathcal{A}_2^\dagger\mathcal{A}_2{}^2\mathcal{A}_2^{\dagger^{r-1}} + 2\mathcal{A}_2{}^{r-1}\mathcal{A}_2^{\dagger^{r-1}}$$

From here we can deduce that:

$$\mathcal{A}_2{}^r\mathcal{A}_2^{\dagger^r} = \mathcal{A}_2^{r-j}\mathcal{A}_2^\dagger\mathcal{A}_2^j\mathcal{A}_2^{r-1} + j\mathcal{A}_2^{r-1}\mathcal{A}_2^{\dagger r}$$

we can set $j = r$:

$$\mathcal{A}_2{}^r\mathcal{A}_2^{\dagger^r} = \mathcal{A}_2^\dagger\mathcal{A}_2\mathcal{A}_2^{r-1}\mathcal{A}_2^{r-1} + r\mathcal{A}_2^{r-1}\mathcal{A}_2^{\dagger r}$$
$$= (n_2 + r)(\mathcal{A}_2^{r-1}\mathcal{A}_2^{\dagger r-1})$$

We can apply the same treatment to the remaining ladder operators part, to get:

$$\mathcal{A}_2{}^r\mathcal{A}_2^{\dagger^r} = (n_2 + r)(n_2 + r - 1)(n_2 + r - 2)\dots(n_2)$$

We will plug this back in the inner product,
Since we are in the vacuum state,
the operation of the number operators will give us the null state.

$$\langle \hat{n}_1(t)^r \rangle = sinh^{2r}(g|v_0|t)_{1,2}\langle vac|(n_2 + r)(n_2 + r - 1)(n_2 + r - 2)\dots(n_2)|vac\rangle_{1,2}$$
$$\langle \hat{n}_1(t)^r \rangle = sinh^{2r}(g|v_0|t)r!$$

The identical result follows for $\langle \hat{n_2} \rangle$ too using the same steps. For the expectation value of the number operator i.e. $r = 1$, the value increases quadratically if $g|v_0|t << 1$, but once it's order exceeds 1, it grows exponentially. However, the time t is the interaction time is of a lower order as the photon travels for a small amount of time in the crystal.

Another important quantity that comes out is the cross correlation that will tell us how much of a correlation is there between the signal and idler photons:

$$\langle\hat{n}_1(t)\hat{n}_2(t)\rangle =_{1,2} \langle vac|\mathcal{A}_1^\dagger(t)\mathcal{A}_2^\dagger(t)\mathcal{A}_1(t)\mathcal{A}_2(t)|vac\rangle_{1,2}$$
$$=_{1,2} \langle vac|[\mathcal{A}_1{}^\dagger(0)cosh(g|v_0|t) + ie^{-i\theta}\mathcal{A}_2(0)sinh(g|v_0|t)]$$
$$[\mathcal{A}_2^\dagger(0)cosh(g|v_0|t) + ie^{-i\theta}\mathcal{A}_1(0)sinh(g|v_0|t)]$$
$$[\mathcal{A}_2(0)cosh(g|v_0|t) - ie^{i\theta}\mathcal{A}_1{}^\dagger(0)sinh(g|v_0|t)]$$
$$[\mathcal{A}_1(0)cosh(g|v_0|t) - ie^{i\theta}\mathcal{A}_2^\dagger(0)sinh(g|v_0|t)]|vac\rangle_{1,2}$$

$$\text{skippping the tedious algebraic manipulation:}$$
$$= |[-ie^{-i\theta}(\mathcal{A}_2^\dagger(0)\mathcal{A}_2(0) + 1)cosh(g|v_0|t)sinh(g|v_0|t)$$
$$-e^{2i\theta}\mathcal{A}_1^\dagger(0)\mathcal{A}_2^\dagger(0)sinh^2(g|v_0|t])|vac\rangle_{1,2}|^2$$
$$= cosh^2(g|v_0|t)sinh^2(g|v_0|t) + sinh^4(g|v_0|t)$$
$$= sinh^2(g|v_0|t)[1 + 2sinh^2(g|v_0|t)$$
$$= \langle n_j(t)\rangle + 2\langle n_j(t)\rangle^2 \qquad j = 1,2$$
$$= \langle n_j(t)\rangle + \langle n_j(t)^2\rangle \qquad j = 1,2$$

We can find the cross correlation of the photon number fluctuations by :

$$\langle\Delta\hat{n}_1\Delta\hat{n}_2\rangle = \langle\hat{n}_1\hat{n}_2\rangle - \langle\hat{n}_1\rangle\langle\hat{n}_2\rangle = \langle\hat{n}_j\rangle(1 + \langle\hat{n}_j\rangle)$$

For the normalized correlation (in statistics known as the Pearson correlation coefficient):

$$\sigma_{1,2} = \frac{\langle\Delta\hat{n}_1\Delta\hat{n}_2\rangle}{[\langle\Delta\hat{n}_1{}^2\rangle\langle\Delta\hat{n}_2{}^2\rangle]^{\frac{1}{2}}}$$

$$= \frac{\langle\hat{n}_j\rangle(1 + \langle\hat{n}_j\rangle)}{[\langle\hat{n}_1\rangle(1 + \langle\hat{n}_1\rangle)\langle\hat{n}_2\rangle(1 + \langle\hat{n}_2\rangle)]^{\frac{1}{2}}} = 1$$

This shows the complete correlation of the signal and idler photons. One way to interpret this result when $\langle\hat{n}_j\rangle << 1$, is :

$$\langle\hat{n}_1(t)\hat{n}_2(t) \approx \langle\hat{n}_j(t)\rangle.$$

This right side of this result can be interpreted as the probability of the of a signal or the idler photon being detected by a perfect detector. The left side of the result can be interpreted ass the joint probability of of detecting both a signal and an idler photon. This interpretation holds only when $\langle n_j(t)\rangle \geq 1$. This is an idealistic treatment of course, more than one modes are generated in reality and a multimodal treatment is needed to solve for those problems.

Multimode Treatment

Will be added later.

Page

Entanglement of the down converted Pair

Will be added later.

Page

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