Precession of Perihelion

Course Project for PHY 442: General Relativity

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Abstract

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General Relativity, proposed as the corrected model for the matter's gravitation, had to overcome the shortcomings of the Newtonian model for its adoption. Several of such tests were set up in the weak field regime such that the Newtonian results predicted the reality fairly well; however, they failed in some specialised cases. The precession of Perihelion, the shift in Mercury's orbit, was a well-known phenomenon that this regime could not predict. General Relativity's adoption required it to predict this and several other observed phenomena that the predecessor theories could not predict.

Contents

1 Introduction

Periapsis or Perihelion, the closest point of approach to the sun in a stable orbit, was observed to precess in the plane of its orbit for several planetary bodies in our solar system. There were several attempts to solve this problem under the Newtonian regime, as it was reasoned that the precession occurred solely due to the perturbation of the outer planets. This makes it a multi-body problem, and solving it directly is impossible. The Precession of Perihelion, see image 1, is the precession of Mercury's orbit while staying in the same plane. The closest point of approach takes a different value of the azimuthal angle ϕ compared to the previous approach of the perihelion. The blue dots in the image are the perihelia of Mercury as it orbits the sun.

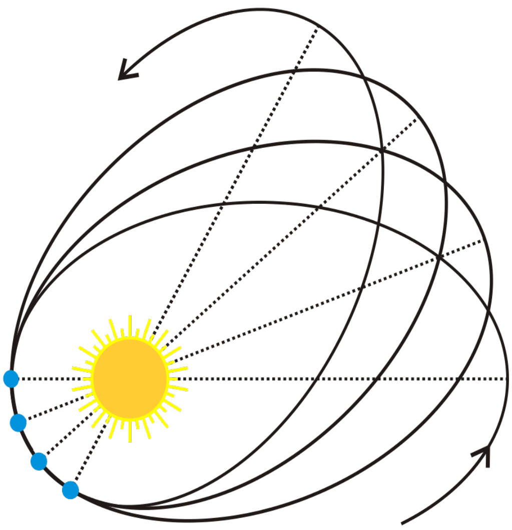

Figure 1: Precession of Perihelion(ref image)

The discrepancy in the Newtonian results was observed due to the difference in the predicted transit time of Mercury in the Earth's sky and the observed transit time. The perturbation of the outer orbits was known to cause the precession of the Perihelion; however, in the 19th century, Urbain Jean Le Verrier, after his Newtonian calculations, concluded that the error of 55 arc minutes in Mercury's transit time could not be resolved using Newtonian Mechanics [1]. The resolution of this error served as evidence for the effectiveness of the General Relativistic model of gravity, as it corrected the gravitational potential, see more in section:3.

2 The Incomplete Newtonian Approach

Multi-body problems in Physics are unsolvable, and extracting any result from a system requires several assumptions that disrupt the physical system being solved. Although these assumptions do not solve the problem entirely, they allow us to predict several exciting phenomena in line with the experimental evidence, depending on how the assumptions are tailored.

Mercury's orbital precession was reasoned using the perturbative effects of the outer planets. This is a reasonable argument, as the outer planets were the only bodies with enough gravitational pull to disrupt the orbit from its static position. However, conventionally this approach would require us to solve the multi-body problem, which would require us to make some clever simplifications that retain our result while making the problem solvable. Following Price and Rush[2], we will assume that outer planets are evenly distributed rings of masses, which, although far from experimental evidence, can produce good approximations in the Newtonian regime.

2.1 The Planetary Rings

To solve the problem in the Newtonian regime, we argue that the evenly distributed rings are a good approximation for two reasons. The first reason is that the planetary bodies follow the regime of classical particles; there is an equal probability of finding them anywhere in their orbit. Secondly, the gravitational pull of the Sun on Mercury is far stronger than that of other planets; consequently, their influence on Mercury at any time would be small. Hence linearly distributed rings set up a solvable problem that can give a reasonable solution, taking the gravitational contributions of outer planets into account. This idea is illustrated in the image below, the grey planet represents mercury, and the rings represent the outer planets.

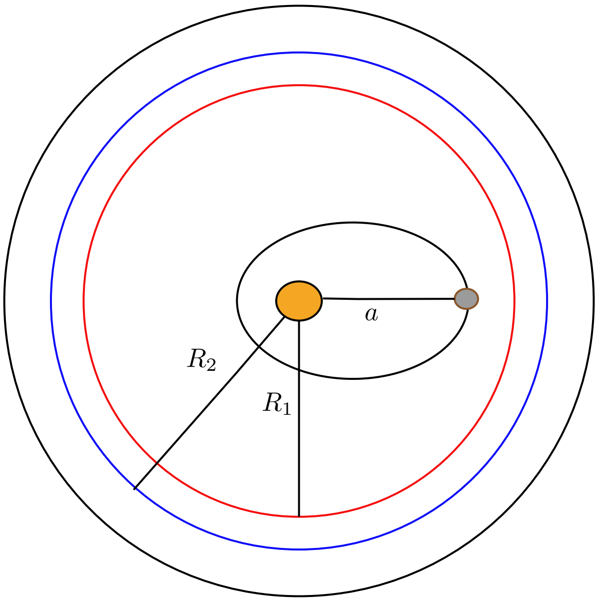

Figure 2: Concentric Planetary rings

We will integrate the Newtonian gravitational pull onto Mercury due to each mass element of these planetary rings to calculate the force due to these rings. The mass of each planet will be linearly distributed along the ring, such that: $\lambda_i = \frac{M_i}{2\pi R_i}$, and we argue that each mass element dm on the ring has a pull equal to :

$$dF_i = GM\frac{Gdm}{r^2}. \tag{2.1}$$

Here r is the distance of the mass element of the i^{th} ring from the planet, and M is the planet's mass. To calculate the contribution of the whole ring as a function of Mercury's radial position, we will take Mercury to be initially at $r = a$, as seen in the image below. From this point, we will measure the distance of each mass element of the outer ring and then integrate it over the whole ring.

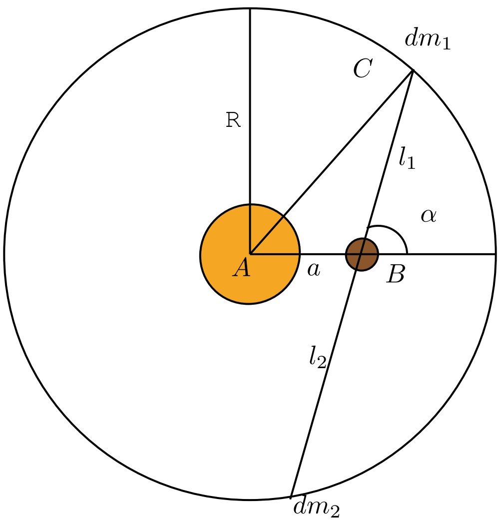

Figure 3: The circular distribution of Planets.

As illustrated in figure 3, dm_1's pull will be counteracted by dm_2, and each mass element will have a counteracting mass due to the circular symmetry of our problem. Using these arguments, (2.1) will become :

$$dF_i = GM\frac{dm_1}{l_1^2} - \frac{dm_2}{l_2^2},$$

upon substituting the mass density: $dm_j = \lambda_i ds_j$, where ds is the arc subtended by l_j and α. Assuming small angular increments in αm the mass element can be rewritten as: $dm_i \approx \lambda l_i d\alpha$, making the equation:

$$dF_i = GM\lambda_i\left(\frac{l_2 - l_1}{l_1 l_2}\right)\hat{l}d\alpha,$$

with force acting in the $\hat{l}$ direction. Additionally, we will argue that because of the circular symmetry of the problem, only the components along the line AB will contribute, and the perpendicular components will cancel. Therefore the force acting will be acting radially :

$$F = GM\lambda\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha\frac{l_2 - l_1}{l_1 l_2}cos(\alpha)\hat{r}. \tag{2.2}$$

We know that l_1 and l_2 will be functions of alpha; substituting these functions and solving the integral in (2.2) will give the force acting on the planet by the external planetary rings. We can see the angle ABC in Fig 3 will be $\pi - \alpha$, and using the law of cosines, we have:

$$R^2 = a^2 + l_1^2 + 2al_1cos(\alpha),$$
$$l_1 = -acos(\alpha) \pm \sqrt{a^2cos^2(\alpha) - a^2 + R^2}.$$

The solution with the negative radical is unphysical as at $\alpha = 0$; we must have $l_1 = R - a$. Applying the same technique on l_2, the following result can be obtained:

$$l_2 = acos(\alpha) + \sqrt{a^2cos^2(\alpha) - a^2 + R^2}.$$

Substituting these in (2.2):

$$F_i = GM\lambda_i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha\frac{2acos^2(\alpha)}{R^2 - a^2} = \frac{MG\lambda_i a\pi}{R_i^2 - a^2}.$$

The relationship above is the force due to the $i'th$ "ring-shaped" exterior planet onto the precessing planet. The total Newtonian Force will be :

$$\mathcal{F}(r) = -\frac{GM_sM}{r^2} + \sum_{i=1}^{8}\frac{MG\lambda_i\pi r}{R_i^2 - r^2}, \tag{2.3}$$

which will lead us to the solutions for the shift in Mercury's orbit.

2.2 The Perturbed Central orce Problem

For stable orbits, we know the moving body is supposed to be bound by a potential, and slight disturbance should not prevent it from retracing the path. Starting with the Lagrangian of motion in polar coordinates[3] :

$$\mathcal{L} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + V(r).$$

The Lagrangian equations of motion reduce our problem to:

$$M(\ddot{r} - r\dot{\phi}^2) = \frac{dV}{dr}.$$

The above equation is known as the radial equation of motion. The $\frac{dV}{dr}$ term is the change in potential in the radial direction, also known as the force. Solving the azimuthal equation gives us the conservation of angular momentum: $L = Mr^2\dot{\phi}$. Substituting this in the radial equation, and replacing the derivative of potential with $\mathcal{F}$, reduces the equation of motion to:

$$M\left(\ddot{r} - \frac{L^2}{Mr^3}\right) = \mathcal{F}(r). \tag{2.4}$$

Here, $\mathcal{F}$ will be (2.3). As discussed, small perturbations in the radial directions for stable orbits should not prevent the body from retracing its path, such that the equations (2.3) and (2.4) should remain valid. At this point, we will make another assumption that locally near a, the particle is performing the circular motion, so in the proximity of a, $\ddot{r} = 0$, which makes (2.4) at a: $\mathcal{F}(a) = -\frac{L^2}{Ma^3}$. Now we will introduce perturbations near a and argue that the planet's radial positions will be oscillations in its radial position about $r = a$ along its elliptical orbit. We will substitute: $r = x + a$, where x is the perturbations about a, in (2.4):

$$\mathcal{F}(r) = \mathcal{F}(x+a) = M\left(\ddot{x} - \frac{L^2}{M(x+a)^3}\right),$$
$$\mathcal{F}(x+a) = M\left(\ddot{x} - \frac{L^2}{Ma^3(1+\frac{x}{a})^3}\right)$$

We will further massage these equations by binomially expanding the denominator of the second equation and keeping the terms up to the first order as $x << a$. Furthermore, we will expand the left-hand side of the equation using the Taylor series about $r = a$ up to the first order, considering the higher order terms will have weaker effects, as x is small.

$$\mathcal{F}(r) = \mathcal{F}(a) + \mathcal{F}'(a)(r-a)$$
$$\mathcal{F}(a) + \mathcal{F}'(a)(r-a) = M\left(\ddot{x} - \frac{L^2}{Ma^3}\left(1 - \frac{3x}{a}\right)\right)$$
$$M\ddot{x} = \mathcal{F}(a) + \frac{L^2}{Ma^3} + \left(\mathcal{F}'(a) - \frac{L^2}{Ma^3}\frac{3}{a}\right)x$$

using the relation $\mathcal{F}(a) = -\frac{L^2}{Ma^3}$, the above equations will reduce to:

$$\ddot{x} = -\frac{1}{M}\left(-\mathcal{F}'(a) - \frac{3}{a}\mathcal{F}(a)\right)x. \tag{2.5}$$

With excessive massage to our equations, we have reduced our problem to a second-order differential equation with oscillatory solutions if the coefficient of x is less than zero. Assuming that the planet is indeed oscillating because it is, we know that the frequency of these oscillations will be the 2π times the root of the coefficient of x; hence the period will be:

$$T = 2\pi\sqrt{\frac{M}{-\mathcal{F}'(a) - \frac{3}{a}\mathcal{F}(a)}}. \tag{2.6}$$

The above expression is a very important result for calculating the precession of Perihelion in the Newtonian regime. Using this, we can calculate the ϕ angle swept in a period, as we expect the planet to have the same distance from the centre after every period.

2.3 Results in the Newtonian Regime

For a precessing orbit, we can expect that in one period, the planet will be at the same radial position from the Sun; however, the angle swept during that period will not be 2π. To calculate that, we will assume that Mercury's radial distance stays close to $r = a$, so the angular momentum remains constant during orbit. This assumption allows us to calculate the angle ϕ swept in 1 period:

$$\phi_T = T\dot{\phi} = 2\pi\sqrt{\frac{M}{-\mathcal{F}'(a) - \frac{3}{a}\mathcal{F}(a)}}\left(\frac{L}{Ma^2}\right). \tag{2.7}$$

Using $\mathcal{F}(a) = -\frac{L^2}{Ma^3}$, in the expression above, we will get:

$$\phi_T = 2\pi\left(3 + a\frac{\mathcal{F}'(a)}{\mathcal{F}(a)}\right)^{-\frac{1}{2}}$$

Plugging in (2.3), and it's derivative in the equation above, we get:

$$\phi_T = 2\pi\left(3 + \frac{-2F_0 + K}{F_0 + \sum_{i=2}^{8} F_i}\right)^{-\frac{1}{2}} = 2\pi\left(\frac{F_0 + 3\sum_{i=2}^{8} F_i + K}{F_0 + \sum_{i=2}^{8} F_i}\right)^{-\frac{1}{2}}$$

Where F_0 is the Newtonian gravity, $K = \sum_{i=2}^{8} \frac{MG\lambda_i \pi a(R_i^2+a^2)}{(R_i^2-a^2)^2}$, and F_i is the force due to the planetary rings as discussed above. We will now divide the numerator and the denominator by F_0 and then further simplify the problem using binomial expansion as $F_0 >> \sum_{i=2}^{8} F_i$. The angle swept in a period subsequently can be written as:

$$\phi_T = 2\pi\left(1 + \frac{3\sum_{i=2}^{8} F_i + K}{F_0}\right)^{-\frac{1}{2}}\left(1 + \frac{\sum_{i=2}^{8} F_i}{F_0}\right)^{\frac{1}{2}}$$

$$\phi_T = 2\pi\left(1 + \frac{\sum_{i=2}^{8} F_i}{2F_0}\right)\left(1 - \frac{3\sum_{i=2}^{8} F_i + K}{2F_0}\right).$$

We can calculate F_0, F_i's and K for each planet and subtract ϕ_T from 2π to get the azimuthal shift from the initial position in the plane of the orbit once it arrives at the same radial position.

Planet	Radius / $(10^{11}m)$	$\Delta\phi_{obs}$ arcsec/cen	$\Delta\phi_{pred}$arcsec/cen
Mercury	0.58	575	532
Venus	1.08	8.4	258
Earth	1.49	5.0	194

The above results[4] for Mercury show an error of 43 arcseconds per century, where arcseconds is $\frac{1}{3600}$ of a degree. Meanwhile, for Earth and Venus, the Newtonian calculation completely breaks down. The error in Mercury's Perihelion is due to the absence of the general relativistic correction that we will discuss in the next section. There were different ideas before general relativity, such as the existence of a dark planet, "Vulcan"; however, the general relativistic correction disproved any such possibility. The problem in the Newtonian predictions of Venus and Earth is that the perturbative model that we discussed in this section does not work well with orbits of low eccentricity as these become highly sensitive to perturbations as we know: $L^2 = ma(1-e^2)$ [5], and plugging this into (2.7) will give a $\Delta\phi$ that is extremely sensitive to perturbations.

3 The General Relativistic Correction

For the adoption of the General Relativistic model of gravity, the theory was required to predict results that removed the discrepancies from the Newtonian calculations. These experiments were mostly related to the weak field limit, where Newton's gravity could explain most of the gravitational effects. The error in Mercury's precession of perihelion also served as a test for General Relativity, and accurate predictions of these results helped General Relativity become the accepted model for gravity.

The general relativistic approach to the residual precession of the perihelion revisits the potential due to the central body, and due to the spherical symmetry in our problem, the metric that describes the spacetime is the Schwarzschild metric. From this point onwards, I will follow d'inverno[5] to calculate the error in Mercury's Perihelion.

3.1 The geodesics of the Schwarzschild metric

Planets in stable orbit are moving along a geodesic. To solve for the orbits in our central mass system with spherically symmetric solutions, we will have to find the geodesics of the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M_s}{r}\right)dt^2 + \left(1 - \frac{2M_s}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2(\theta)d\phi^2. \tag{3.1}$$

As per the variational approach, the geodesics can be solved for by taking the path of the stationary action, which means that we can solve for the geodesic by extremising the action across two events in the space, as the shortest distance between them will be along the geodesic:

$$S = \int_a^b ds = \int_a^b \sqrt{g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta} d\lambda,$$

which gives us the lagrangian:

$$L = \sqrt{g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}.$$

We can solve for the geodesics using the Euler-Lagrange equations on this Lagrangian. However, working with the square of the Lagrangian would be easier, as it would help us to get rid of the square roots. The Euler-Lagrange equations for the Lagrangian above can be written as :

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0.$$

If, instead, we had applied these derivatives on L^2; it would have been equivalent to multiplying the whole expression with $2L$, and considering $L \neq 0$, we would get the same equation with L^2 instead of L. To make the notation cleaner, I will use $\mathcal{L}$ instead of L^2. The new Euler-Lagrange equations will be :

$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0,$$

Where $\mathcal{L}$ represents the squared Lagrangian :

$$\mathcal{L} = -\left(1 - \frac{2M_s}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{2M_s}{r}} + r^2\dot{\theta}^3 + r^2 sin^2(\theta)\dot{\phi}^2. \tag{3.2}$$

For space-like geodesics, we hold the equation above equal to 1; for time-like geodesics, we set it equal to -1; and for null geodesics, it equals zero, as events on these geodesics are separated in such manner. Using the Euler - Lagrange equations, we get the following set of differential equations:

$$\frac{dt}{d\lambda} = \frac{E}{1 - \frac{2GM_s}{r}}$$
$$\frac{d\phi}{d\lambda} = \frac{L}{M_s r^2 sin(\theta)}$$
$$\frac{d(r^2\dot{\theta})}{d\lambda} + r sin^2(\theta)\dot{\phi}^2 = 0$$

The first equation is energy conservation, and the second is the conservation of Angular momentum. For the third equation, if we assume that $\theta(0) = \frac{\pi}{2}$ and $\dot{\theta}(0) = 0$, which we can do as we can take any plane of orbit, and assume that the orbit is stationary in that plane initially, we can set all higher derivatives of θ to zero. Plugging these equations, and $\theta = \frac{\pi}{2}$ in (3.2), and equating with -1 for time-like geodesics, we get:

$$\dot{r}^2 = E^2 - 1 + \frac{2M_s}{r} - \frac{L^2}{r^2} + \frac{2M_s L^2}{r^3}. \tag{3.3}$$

The $\frac{1}{r^3}$ term in this solution is unique to the General Relativistic gravitational potential. We can see that this term will have the most substantial effects at points closer to the central mass as it decays with the inverse cube of the radial coordinate. We can plot the potential, the negative of the right-hand side of the equation, to see the points where we can get stable orbits for massive objects:

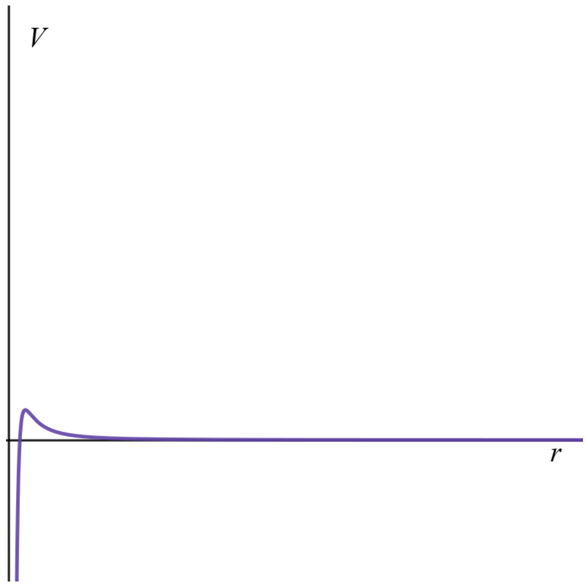

Figure 4: Effective Potential for massive objects

Using the effective potential:

$$V_{eff} = -\frac{M_s}{r} + \frac{L^2}{2r^2} - \frac{M_s L^2}{r^3}, \tag{3.4}$$

we can find the critical points by setting its derivative to zero. We find the stable orbits of this potential to be at $r \geq 6M_s$ and unstable circular orbits at $3M_s < r < 6M_s$.

3.2 The Azimuthal dependence of the orbits

Although we have not completely solved (3.3) for the geodesics, we can still calculate the azimuthal dependence of the radial coordinate. To further simplify our calculations, we will be using the substitution $u = r^{-1}$, and applying the chain rule using the conservation of angular momentum, we will get :

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2 - 1}{L^2} + \frac{2M_s u}{L^2} - u^2 + 2M_s u^3.$$

This equation can be solved by taking the integral; however, the integral would be horrible, one we do not want to solve. To further simplify this equation, we will make the substitution: $\tilde{u} = \frac{L^2 u}{m}$, and take the second derivative with respect to ϕ, which simplifies our problem to a second-order differential equation:

$$\frac{d^2\tilde{u}}{d\phi^2} + \tilde{u} = 1 + \frac{3M_s^2}{L^2}\tilde{u}^2 \tag{3.5}$$

The quadratic term in this differential equation makes it difficult to solve directly, so we will use a perturbative method to solve this problem. We will substitute : $\tilde{u} = \tilde{u}_0 + \frac{3M_s}{L^2}\tilde{u}_1$, and $\epsilon = \frac{3M_s}{L^2}$. We know from the solutions for stable orbits that critical points only occur when: $\frac{L}{M} > \sqrt{12}$, and typically, $L >> M$, making ϵ small. Plugging in these substitutions :

$$\frac{d^2\tilde{u}_0}{d\phi^2} + \epsilon\frac{d\tilde{u}_1}{d\phi^2} + \tilde{u}_0 + \epsilon\tilde{u}_1 = 1 + \epsilon\left(\tilde{u}_0 + \epsilon\tilde{u}_1\right)^2$$

We can solve the equation by breaking it into two separate equations by comparing the coefficients of the terms that are zeroth order in ϵ and first order in ϵ, ignoring the higher order contributions. These allow us to reach the solutions that can approximate our results fairly well, as we are expecting the contributions due to $\frac{1}{r^3}$ term in the (3.3) to be small. The zeroth order differential equation is :

$$\frac{d^2\tilde{u}_0}{d\phi^2} = 1 - \tilde{u}_0. \tag{3.6}$$

This is also known as Binet's equation, which gives the solutions for Kepler's orbits [1]. The solutions of (3.6) will also be those of the conic sections that describe an elliptical orbit; this makes sense as

[1] Goldstein[3] page 93.

the zeroth order perturbation only caters for the terms describing Newtonian gravity. The solution for the elliptical orbit is:

$$\tilde{u}_0 = 1 + ecos(\phi), \tag{3.7}$$

where e is the eccentricity of the ellipse. The differential order with the first-order perturbation terms will be :

$$\frac{d^2\tilde{u}_1}{d\phi^2} + \tilde{u}_1 = \tilde{u_0}^2 = (1 + ecos(\phi))^2. \tag{3.8}$$

Solving this equation would require us to consider the following second-order derivatives:

$$\frac{d^2(\phi sin(\phi))}{d\phi^2} + \phi sin(\phi) = 2cos(\phi),$$

$$\frac{d^2}{d\phi^2}(cos(2\phi)) + cos(2\phi) = -3cos(2\phi).$$

By plugging in the ansatz : $\tilde{u}_1 = A + B\phi sin(\phi) + Ccos(2\phi)$ in (3.8), we will get :

$$2Bcos(\phi) - 3Ccos(2\phi) + A = 1 + 2ecos(\phi) + \frac{e^2}{2}(1 + cos(2\phi)). \tag{3.9}$$

Comparing the coefficients, we will get : $A = 1 + \frac{e^2}{2}$, $B = e$ and, $C = -\frac{e^2}{6}$. Plugging the solutions for $\tilde{u}_0, \tilde{u}_1$, and plugging in the value for ϵ, we have :

$$\tilde{u} = (1 + ecos(\phi)) + \frac{3M_s}{L^2}\left(1 + \frac{e^2}{2} + e\phi sin(\phi) - \frac{e^2}{6}cos(2\phi)\right). \tag{3.10}$$

In the perturbed solution, the most important term is $\phi sin(\phi)$ as it grows with the planet's rotation; the remaining terms are oscillatory and will not have any noticeable contributions to the precession of the Perihelion. Ignoring those terms, the solution becomes:

$$u \approx \frac{M_s}{L^2}\left(1 + ecos(\phi) + \frac{3M_s}{L^2}(e\phi sin(\phi)\right) \tag{3.11}$$

The above equation shows the azimuthal dependence of the radial coordinate, which allows us to predict the radial position of the planet if we know its azimuthal position in its orbit.

3.3 The Residual Precession

With the help of the calculations in section: 3.2, we have derived the radial trajectory of the planet as a function of the azimuthal angle under the general relativistic gravitational potential. Without the loss of generality, we can initially take ϕ as 0 since we expect the perihelion to have shifted azimuthally. We want to find the $\Delta\phi$, such that $u'(2\pi + \Delta\phi)$ will be zero. This is because the perihelion is the closest point of approach, it will have the minimum radial position, and the derivative of u should be zero at that point, and $\Delta\phi$ will be the measure of the precession during one orbit[6].

$$u'(\phi) = \frac{M_s}{L^2}\left(-esin(\phi) + \frac{3M_s}{L^2}esin(\phi) + \frac{3M_s}{L^2}e\phi cos(\phi)\right)$$

plugging in with $2\pi + \Delta\phi$, and equating the equation to zero will give us :

$$sin(2\pi + \Delta\phi) = \frac{3M_s}{L^2}esin(2\pi + \Delta\phi) + \frac{3M_s}{L^2}e(2\pi + \Delta\phi)cos(2\pi + \Delta\phi)$$

Using the small angle approximations, as we are expecting the precession to be slow so $\Delta\phi$ will be small, and the fact that $\frac{3M_s}{L^2}$ is also small, we will get :

$$\Delta\phi = \frac{3M_s}{L^2}\Delta\phi + \frac{3M_s}{L^2}(2\pi + \Delta\phi), \tag{3.12}$$

$$\Delta\phi = \frac{6\pi M_s}{L^2}. \tag{3.13}$$

We have arrived at the final form of our solution. The expression above for $\Delta\phi$ represents the additional angle swept during one orbit as the planet approaches Perihelion again. Using[2]: $L^2 = GM(1-e^2)a$ for elliptical orbits, and restoring the SI units, the Precession can be written as:

$$\Delta\phi = \frac{6\pi GM_s}{c^2(1-e^2)a},$$

Here a is the length of the semi-major axis. The trend with the length of the semi-major axis, as can be seen in the graph below, decays as we move for planets further away from the centre:

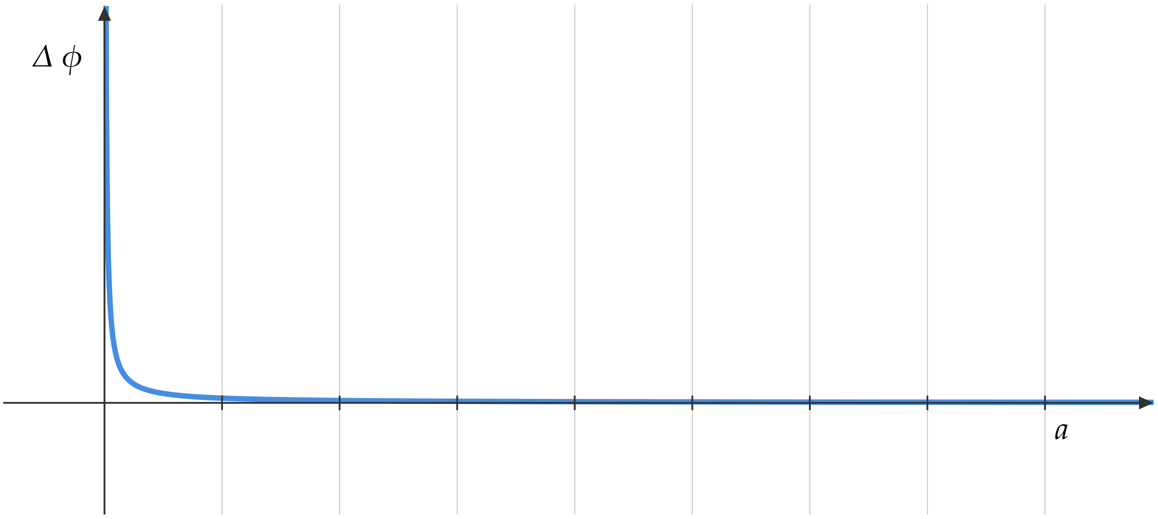

Figure 5: The Advance of Perihelion w.r.t the Semi Major axis

This phenomenon is not restricted to a particular planetary body or a solar system; the Precession can occur in any Mass distribution that sets up the spacetime, like the Schwarzschild metric. The results of these calculations for the residual Precession, as seen in the table below [5], show how the additional term in the general relativistic gravitational potential corrected the errors in the Newtonian correction.

Planet	Radius / $(10^{11}m)$	$\Delta\phi_{obs}$ arcsec/cen	$\Delta\phi_{pred}$arcsec/cen
Mercury	0.58	43.0	43.1
Venus	1.08	8.6	8.6
Earth	1.49	5.0	3.8

The correction term fixed the error in the Newtonian calculations for Mercury's precession, and the observed Precession for Venus and Earth's Precessions also aligned well with the general relativistic predictions.

3.4 Do Light orbits Precess?

In the Newtonian regime, without the concept of Matter curving space-time, the light follows the straight line paths; hence in Newtonian gravity, light does not orbit. In General Relativity, light travels along the null geodesics. Starting from (3.2) and equating it to zero for null geodesics, we can solve for the trajectory of light in an area governed by the Schwarzschild metric.

3.4.1 The null geodesics

Equating the (3.2) to zero will make our $2GM/r$ term zero due to the null path condition. This is a reasonable step as this term describes Newtonian gravity, and as we have discussed before, Newtonian gravity does not act on massless particles. The effective potential for the null geodesic becomes:

$$V_{eff} = \frac{L^2}{2r^2}\left(1 - \frac{2M_s}{r}\right), \tag{3.14}$$

[2]For derivation, see Goldsetein[3] page 95.

When plotted against the radial coordinate:

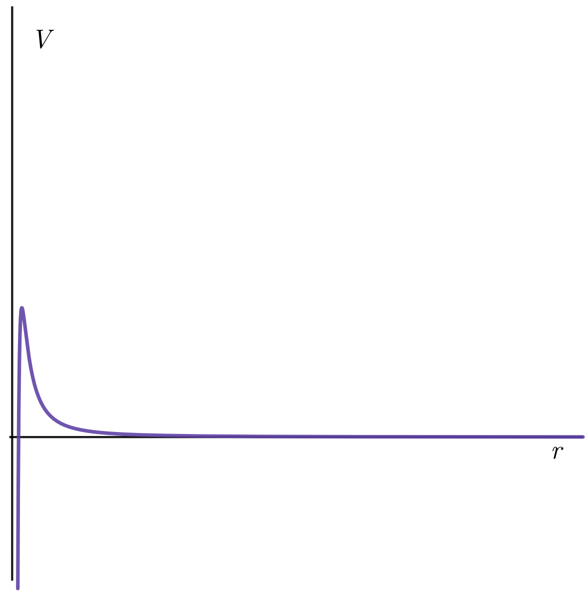

Figure 6: Effective potential on Light

We can see only one critical point on this potential, meaning only one stable orbit exists. Setting $\frac{dv}{dr} = 0$, we can find the only stable light orbit circular at $r = 3M$. We can calculate the effects of the additional $\frac{1}{r^3}$ term on the orbit; however, for the orbit to remain a stable circular orbit, it must stay a fixed distance away from the central Energy distribution. Any noticeable precession in orbit would mean the massless particle will fall into the singularity at $r = 0$ or move infinitely far away. Hence the orbits must be circular so the photons remain a fixed distance from the central distribution. Therefore any degree of the precession of the orbit about the centre will not be noticeable due to the circular symmetry of the orbit. If the precession takes place off-centre, that would require photons to be farther or closer from the central distribution, which would move the light out of its orbit.

4 Conclusion

The precession of Perihelion from the advent of General Relativity was reasoned using the perturbative effects of the outer planets and bodies disrupting the elliptical trajectories of the precessing orbits. It was concluded in the 19th century that Newtonian calculations were insufficient to eradicate the error observed in the precession of Mercury's Perihelion.

The general relativistic solution of the spherically symmetric metric set by a central Energy distribution corrected the gravitational potential, which, as discussed in detail, corrected the error in the Newtonian calculations. Apart from serving as a test for General Relativity, this precession also has other interesting implications, as it is used to reason the long-term climatic cycles described by the Milankovitch cycles[3].

[3] Milankovitch cycles reason the long term climatic cycles of Earth with its long term orbital changes including the Precession of Perihelion, [7] is an interesting read, to begin with.

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