

EE5609: Matrix Theory

Assignment-4

Sahil Kumar Singh
ES17BTECH11019

1 PROBLEM

Trace the parabola and find its focus.

$$144y^2 - 120xy + 25x^2 + 619x - 272y + 663 = 0 \quad (1.0.1)$$

2 SOLUTION

The general second degree equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.1)$$

where,

$$\mathbf{V} = \begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{u} = \begin{pmatrix} \frac{619}{2} \\ -136 \end{pmatrix} \quad (2.0.3)$$

$$f = 663 \quad (2.0.4)$$

1) Expanding the determinant of \mathbf{V} we observe,

$$\begin{vmatrix} 144 & -60 \\ -60 & 25 \end{vmatrix} = 0 \quad (2.0.5)$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 144 & -60 & \frac{619}{2} \\ -60 & 25 & -136 \\ \frac{619}{2} & -136 & 663 \end{vmatrix} \quad (2.0.6)$$

$$\neq 0 \quad (2.0.7)$$

Hence from (2.0.5) and (2.0.7) we conclude that given equation is an parabola. The characteristic equation of \mathbf{V} is given as follows,

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 144 & 60 \\ 60 & \lambda - 25 \end{vmatrix} = 0 \quad (2.0.8)$$

$$\Rightarrow \lambda^2 - 169\lambda = 0 \quad (2.0.9)$$

Hence the characteristic equation of \mathbf{V} is given by (2.0.9). The roots of (2.0.9) i.e the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 169 \quad (2.0.10)$$

2) For $\lambda_1 = 0$, the eigen vector \mathbf{p} is given by

$$\mathbf{V}\mathbf{p} = 0 \quad (2.0.11)$$

Row reducing \mathbf{V} yields

$$\Rightarrow \begin{pmatrix} -144 & 60 \\ 60 & -25 \end{pmatrix} \xrightarrow[R_2=R_2+5R_1]{R_1=\frac{R_1}{12}} \begin{pmatrix} -12 & 5 \\ 0 & 0 \end{pmatrix} \quad (2.0.12)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{13} \begin{pmatrix} 5 \\ 12 \end{pmatrix} \quad (2.0.13)$$

Similarly,

$$\mathbf{p}_2 = \frac{1}{13} \begin{pmatrix} 12 \\ -5 \end{pmatrix} \quad (2.0.14)$$

Thus, the eigenvector rotation matrix and the eigenvalue matrix are

$$\mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) = \frac{1}{13} \begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix} \quad (2.0.15)$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 169 \end{pmatrix} \quad (2.0.16)$$

The focal length of the parabola is given by

$$\frac{|2\mathbf{u}^T \mathbf{p}_1|}{\lambda_2} = \frac{13}{169} = \frac{1}{13} \quad (2.0.17)$$

and its equation is

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad (2.0.18)$$

where

$$\eta = 2\mathbf{u}^T \mathbf{p}_1 = -13 \quad (2.0.19)$$

and the vertex \mathbf{c} is given by

$$\begin{pmatrix} \mathbf{u}^T + \frac{\eta}{2}\mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (2.0.20)$$

using equations (2.0.3),(2.0.4) and (2.0.13)

$$\begin{pmatrix} 307 & -142 \\ 144 & -60 \\ -60 & 25 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -663 \\ -312 \\ 130 \end{pmatrix} \quad (2.0.21)$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix} 307 & -142 & -663 \\ 144 & -60 & -312 \\ -60 & 25 & 130 \end{pmatrix} \quad (2.0.22)$$

$$R_2 \leftrightarrow \frac{R_2}{12}$$

$$\begin{pmatrix} 307 & -142 & -663 \\ 12 & -5 & -26 \\ -60 & 25 & 130 \end{pmatrix} \quad (2.0.23)$$

$$R_3 \leftrightarrow R_3 + 5R_2$$

$$\begin{pmatrix} 307 & -142 & -663 \\ 12 & -5 & -26 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.24)$$

$$R_1 \leftrightarrow \frac{R_1}{307}$$

$$\begin{pmatrix} 1 & -\frac{142}{307} & -\frac{663}{307} \\ 0 & \frac{307}{169} & -\frac{26}{169} \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.25)$$

$$R_2 \leftrightarrow R_2 - 12R_1$$

$$\begin{pmatrix} 1 & -\frac{142}{307} & -\frac{663}{307} \\ 0 & 1 & -\frac{26}{307} \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.26)$$

$$R_1 \leftrightarrow R_1 + (142/307)R_2$$

$$\begin{pmatrix} 1 & 0 & -29/13 \\ 0 & 1 & -2/13 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.0.27)$$

Thus the vertex \mathbf{c} is:

$$\mathbf{c} = \begin{pmatrix} -29/13 \\ -2/13 \end{pmatrix} \quad (2.0.28)$$

The direction vector of axis of symmetry is

given by :

$$\mathbf{m} = \mathbf{V}\mathbf{c} + \mathbf{u} \quad (2.0.29)$$

$$= \begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix} \begin{pmatrix} -\frac{29}{13} \\ -\frac{2}{13} \end{pmatrix} + \begin{pmatrix} 619 \\ \frac{2}{272} \end{pmatrix} \quad (2.0.30)$$

$$= \begin{pmatrix} 5 \\ -\frac{5}{2} \\ -6 \end{pmatrix} \quad (2.0.31)$$

$$\mathbf{m} = \frac{13}{2} \quad (2.0.32)$$

$$\Rightarrow \frac{\mathbf{m}}{\|\mathbf{m}\|} = \begin{pmatrix} 5 \\ -\frac{13}{12} \\ -\frac{13}{13} \end{pmatrix} \quad (2.0.33)$$

The focus is given by:

$$F = c - \left(\frac{\mathbf{m}}{\|\mathbf{m}\|} \times a \right) \quad (2.0.34)$$

$$= \begin{pmatrix} -\frac{29}{13} \\ -\frac{2}{13} \end{pmatrix} - \left(\begin{pmatrix} 5 \\ -\frac{13}{12} \\ -\frac{13}{13} \end{pmatrix} \times \frac{1}{52} \right) \quad (2.0.35)$$

$$= \begin{pmatrix} -\frac{1503}{676} \\ \frac{23}{169} \end{pmatrix} \quad (2.0.36)$$

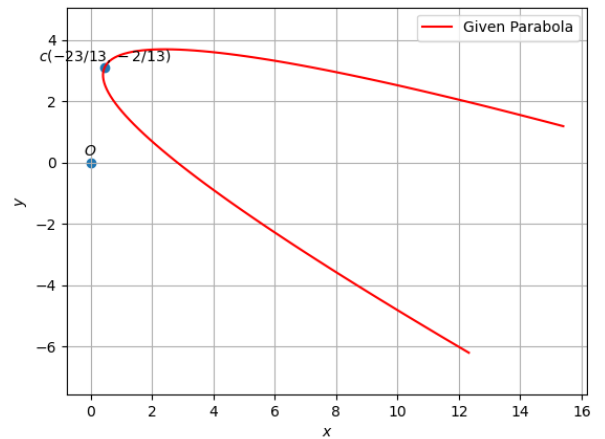


Fig. 2: Traced parabola