EE5609: Matrix Theory Assignment-6

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1 Problem

Find the centre of parabola using SVD and verify through least square method.

$$144y^2 - 120xy + 25x^2 + 619x - 272y + 663 = 0 \tag{1.0.1}$$

2 Solution

The general second degree equation can be expressed as follows,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.1}$$

where,

$$\mathbf{V} = \begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} \frac{619}{2} \\ -136 \end{pmatrix} \tag{2.0.3}$$

$$f = 663 \tag{2.0.4}$$

1) Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 144 & -60 \\ -60 & 25 \end{vmatrix} = 0 \tag{2.0.5}$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 144 & -60 & \frac{619}{2} \\ -60 & 25 & -136 \\ \frac{619}{2} & -136 & 663 \end{vmatrix} \tag{2.0.6}$$

$$\neq 0 \tag{2.0.7}$$

Hence from (2.0.5) and (2.0.7) we conclude that given equation is an parabola. The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 144 & 60 \\ 60 & \lambda - 25 \end{vmatrix} = 0 \tag{2.0.8}$$

$$\implies \lambda^2 - 169\lambda = 0 \tag{2.0.9}$$

Hence the characteristic equation of $\mathbf{V}$ is given by (2.0.9). The roots of (2.0.9) i.e the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 169 \tag{2.0.10}$$

2) For $\lambda_1 = 0$, the eigen vector $\mathbf{p}$ is given by

$$\mathbf{V}\mathbf{p} = 0 \tag{2.0.11}$$

Row reducing $\mathbf{V}$ yields

$$\implies \begin{pmatrix} -144 & 60 \\ 60 & -25 \end{pmatrix} \xleftrightarrow[R_2 = R_2 + 5R_1]{R_1 = \frac{R_1}{12}} \begin{pmatrix} -12 & 5 \\ 0 & 0 \end{pmatrix} \tag{2.0.12}$$

$$\implies \mathbf{p}_1 = \frac{1}{13}\begin{pmatrix} 5 \\ 12 \end{pmatrix} \tag{2.0.13}$$

Similarly,

$$\mathbf{p}_2 = \frac{1}{13}\begin{pmatrix} 12 \\ -5 \end{pmatrix} \tag{2.0.14}$$

Thus, the eigenvector rotation matrix and the eigenvalue matrix are

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix} \tag{2.0.15}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 169 \end{pmatrix} \tag{2.0.16}$$

The focal length of the parabola is given by

$$\frac{\left|2\mathbf{u}^T\mathbf{p_1}\right|}{\lambda_2} = \frac{13}{169} = \frac{1}{13} \tag{2.0.17}$$

and its equation is

$$\mathbf{y^T D y} = -\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{2.0.18}$$

where

$$\eta = 2\mathbf{u}^T\mathbf{p_1} = -13 \tag{2.0.19}$$

and the vertex $\mathbf{c}$ is given by

$$\begin{pmatrix}\mathbf{u^T} + \frac{\eta}{2}\mathbf{p_1^T} \\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f \\ \frac{\eta}{2}\mathbf{p_1} - \mathbf{u}\end{pmatrix} \tag{2.0.20}$$

using equations (2.0.3),(2.0.4) and (2.0.13)

$$\begin{pmatrix}307 & -142 \\ 144 & -60 \\ -60 & 25\end{pmatrix}\mathbf{c} = \begin{pmatrix}-663 \\ -312 \\ 130\end{pmatrix} \tag{2.0.21}$$

2.1 *Singular Value Decomposition:*

$$\mathbf{Mc} = \mathbf{b} \tag{2.1.1}$$

where

$$\mathbf{M} = \begin{pmatrix}307 & -142 \\ 144 & -60 \\ -60 & 25\end{pmatrix}, b = \begin{pmatrix}-663 \\ -312 \\ 130\end{pmatrix} \tag{2.1.2}$$

To solve (2.1.1), we perform singular value decomposition on $\mathbf{M}$ given as

$$\mathbf{M} = \mathbf{USV^T} \tag{2.1.3}$$

Substituting the value of $\mathbf{M}$ from (2.1.3) in (2.1.1), we get

$$\mathbf{USV^Tc} = \mathbf{b} \tag{2.1.4}$$

$$\implies \mathbf{c} = \mathbf{VS_+U^Tb} \tag{2.1.5}$$

where, $\mathbf{S_+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigen-vectors of $\mathbf{MM^T}$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M^TM}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M^TM}$. First calculating the eigenvectors corresponding to $\mathbf{M^TM}$.
Using Python, $\mathbf{U}, \mathbf{S}$ and $\mathbf{V}$ for $\mathbf{M}$ is given by:

$$\mathbf{U} = \begin{pmatrix}-\frac{8946}{10000} & \frac{4468}{10000} & 0 \\ \frac{-4124}{10000} & \frac{-8258}{10000} & \frac{3846}{10000} \\ \frac{1718}{10000} & \frac{3441}{10000} & \frac{9231}{10000}\end{pmatrix} \tag{2.1.6}$$

$$\mathbf{S} = \begin{pmatrix}\frac{3780744}{10000} & 0 \\ 0 & \frac{58110}{10000} \\ 0 & 0\end{pmatrix} \tag{2.1.7}$$

$$\mathbf{V} = \begin{pmatrix}\frac{-9108}{10000} & \frac{-4128}{10000} \\ \frac{4128}{10000} & \frac{-9108}{10000}\end{pmatrix} \tag{2.1.8}$$

Now, More-Pen-Rose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S_+} = \begin{pmatrix}\frac{10000}{3780744} & 0 & 0 \\ 0 & \frac{10000}{58110} & 0\end{pmatrix} \tag{2.1.9}$$

Hence, we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \begin{pmatrix}-\frac{8946}{10000} & \frac{4468}{10000} & 0 \\ \frac{-4124}{10000} & \frac{-8258}{10000} & \frac{3846}{10000} \\ \frac{1718}{10000} & \frac{3441}{10000} & \frac{9231}{10000}\end{pmatrix}\begin{pmatrix}\frac{3780744}{10000} & 0 \\ 0 & \frac{58110}{10000} \\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{-9108}{10000} & \frac{4128}{10000} \\ \frac{-4128}{10000} & \frac{-9108}{10000}\end{pmatrix} \tag{2.1.10}$$

From (2.1.2) and (2.1.6)

$$\mathbf{U^Tb} = \begin{pmatrix}-\frac{8946}{10000} & \frac{-4124}{10000} & 0 \\ \frac{-4468}{10000} & \frac{-8258}{10000} & \frac{3846}{10000} \\ \frac{1718}{10000} & \frac{3441}{10000} & \frac{9231}{10000}\end{pmatrix}\begin{pmatrix}-663 \\ -312 \\ 130\end{pmatrix} = \begin{pmatrix}\frac{7441606}{10000} \\ \frac{61658}{10000} \\ 0\end{pmatrix} \tag{2.1.11}$$

From (2.1.8) and (2.1.9)

$$\mathbf{VS_+} = \begin{pmatrix}\frac{-9108}{10000} & \frac{-4128}{10000} \\ \frac{4128}{10000} & \frac{-9108}{10000}\end{pmatrix}\begin{pmatrix}\frac{10000}{3780744} & 0 & 0 \\ 0 & \frac{10000}{58110} & 0\end{pmatrix} = \begin{pmatrix}\frac{-24}{10000} & \frac{-710}{10000} & 0 \\ \frac{11}{10000} & \frac{-1567}{10000} & 0\end{pmatrix} \tag{2.1.12}$$

Substitute (2.1.12) and (2.1.11) in (2.1.5) we get,

$$\mathbf{c} = \begin{pmatrix}\frac{-24}{10000} & \frac{-710}{10000} & 0 \\ \frac{11}{10000} & \frac{-1567}{10000} & 0\end{pmatrix}\begin{pmatrix}\frac{7441606}{10000} \\ \frac{61658}{10000} \\ 0\end{pmatrix} \tag{2.1.13}$$

$$\implies \boxed{\mathbf{c} = \begin{pmatrix}\frac{-22308}{10000} \\ \frac{-1538}{10000}\end{pmatrix} = \begin{pmatrix}-2.23 \\ -0.15\end{pmatrix}} \tag{2.1.14}$$

2.2 *Least Square Verification*

Now, verify our solution using,

$$\mathbf{M^TMc} = \mathbf{M^Tb} \tag{2.2.1}$$

From (2.1.2),

$$\begin{pmatrix}307 & 144 & -60 \\ -142 & -60 & 25\end{pmatrix}\begin{pmatrix}307 & -142 \\ 144 & -60 \\ -60 & 25\end{pmatrix}\mathbf{c} \tag{2.2.2}$$

$$= \begin{pmatrix}307 & 144 & -60 \\ -142 & -60 & 25\end{pmatrix}\begin{pmatrix}-663 \\ -312 \\ 130\end{pmatrix} \tag{2.2.3}$$

$$\implies \begin{pmatrix}118585 & -53734 \\ -53734 & 24389\end{pmatrix}\mathbf{c} = \begin{pmatrix}-256269 \\ 116116\end{pmatrix} \tag{2.2.4}$$

Solving the augmented matrix, we get

$$\begin{pmatrix}118585 & -53734 & -256269\\ -53734 & 24389 & 116116\end{pmatrix} \xleftrightarrow{R_1=\frac{R_1}{118585}} \begin{pmatrix}1 & \frac{-53734}{118585} & \frac{-256269}{118585}\\ -53734 & 24389 & 116116\end{pmatrix} \tag{2.2.5}$$

$$\xleftrightarrow{R_2=R_2+53734R_1} \begin{pmatrix}1 & \frac{-53734}{118585} & \frac{-256269}{118585}\\ 0 & \frac{4826809}{118585} & \frac{-742586}{118585}\end{pmatrix} \tag{2.2.6}$$

$$\xleftrightarrow{R_2=\frac{118585}{4826809}R_2} \begin{pmatrix}1 & \frac{-53734}{118585} & \frac{-256269}{118585}\\ 0 & 1 & \frac{-742586}{4826809}\end{pmatrix} \tag{2.2.7}$$

$$\xleftrightarrow{R_1=R_1+\frac{53734}{118585}R_2} \begin{pmatrix}1 & 0 & \frac{-11970539}{5723871}\\ 0 & 1 & \frac{-742586}{4826809}\end{pmatrix} \tag{2.2.8}$$

Thus,

$$\implies \boxed{\mathbf{c} = \begin{pmatrix}\frac{-11970539}{5723871}\\ \frac{-742586}{4826809}\end{pmatrix} = \begin{pmatrix}-2.23\\ -0.15\end{pmatrix}} \tag{2.2.9}$$

Hence, verified the result from SVD.