EE5609: Matrix Theory Assignment-4

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1 Problem

Trace the parabola and find its focus.

$$144y^2 - 120xy + 25x^2 + 619x - 272y + 663 = 0 \tag{1.0.1}$$

2 Solution

The general second degree equation can be expressed as follows,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.1}$$

where,

$$\mathbf{V} = \begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} \frac{619}{2} \\ -136 \end{pmatrix} \tag{2.0.3}$$

$$f = 663 \tag{2.0.4}$$

1) Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 144 & -60 \\ -60 & 25 \end{vmatrix} = 0 \tag{2.0.5}$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 144 & -60 & \frac{619}{2} \\ -60 & 25 & -136 \\ \frac{619}{2} & -136 & 663 \end{vmatrix} \tag{2.0.6}$$

$$\neq 0 \tag{2.0.7}$$

Hence from (2.0.5) and (2.0.7) we conclude that given equation is an parabola. The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 144 & 60 \\ 60 & \lambda - 25 \end{vmatrix} = 0 \tag{2.0.8}$$

$$\implies \lambda^2 - 169\lambda = 0 \tag{2.0.9}$$

Hence the characteristic equation of $\mathbf{V}$ is given by (2.0.9). The roots of (2.0.9) i.e the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 169 \tag{2.0.10}$$

2) For $\lambda_1 = 0$, the eigen vector $\mathbf{p}$ is given by

$$\mathbf{V}\mathbf{p} = 0 \tag{2.0.11}$$

Row reducing $\mathbf{V}$ yields

$$\implies \begin{pmatrix} -144 & 60 \\ 60 & -25 \end{pmatrix} \xleftrightarrow[R_2 = R_2 + 5R_1]{R_1 = \frac{R_1}{12}} \begin{pmatrix} -12 & 5 \\ 0 & 0 \end{pmatrix} \tag{2.0.12}$$

$$\implies \mathbf{p}_1 = \frac{1}{13}\begin{pmatrix} 5 \\ 12 \end{pmatrix} \tag{2.0.13}$$

Similarly,

$$\mathbf{p}_2 = \frac{1}{13}\begin{pmatrix} 12 \\ -5 \end{pmatrix} \tag{2.0.14}$$

Thus, the eigenvector rotation matrix and the eigenvalue matrix are

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix} \tag{2.0.15}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 169 \end{pmatrix} \tag{2.0.16}$$

The focal length of the parabola is given by

$$\frac{\left|2\mathbf{u}^T\mathbf{p_1}\right|}{\lambda_2} = \frac{13}{169} = \frac{1}{13} \tag{2.0.17}$$

and its equation is

$$\mathbf{y^T D y} = -\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{2.0.18}$$

where

$$\eta = 2\mathbf{u}^T\mathbf{p_1} = -13 \tag{2.0.19}$$

and the vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} \mathbf{u}^{\mathbf{T}} + \frac{\eta}{2}\mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.20}$$

using equations (2.0.3),(2.0.4) and (2.0.13)

$$\begin{pmatrix} 307 & -142 \\ 144 & -60 \\ -60 & 25 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -663 \\ -312 \\ 130 \end{pmatrix} \tag{2.0.21}$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix} 307 & -142 & -663 \\ 144 & -60 & -312 \\ -60 & 25 & 130 \end{pmatrix} \tag{2.0.22}$$

$$R_2 \leftrightarrow \frac{R_2}{12}$$

$$\begin{pmatrix} 307 & -142 & -663 \\ 12 & -5 & -26 \\ -60 & 25 & 130 \end{pmatrix} \tag{2.0.23}$$

$$R_3 \leftrightarrow R_3 + 5R_2$$

$$\begin{pmatrix} 307 & -142 & -663 \\ 12 & -5 & -26 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.24}$$

$$R_1 \leftrightarrow \frac{R_1}{307}$$

$$\begin{pmatrix} 1 & \frac{-142}{307} & \frac{-663}{307} \\ 0 & \frac{169}{307} & \frac{-26}{307} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.25}$$

$$R_2 \leftrightarrow R_2 - 12R_1$$

$$\begin{pmatrix} 1 & \frac{-142}{307} & \frac{-663}{307} \\ 0 & 1 & \frac{-26}{307} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.26}$$

$$R_1 \leftrightarrow R_1 + (142/307)R_2$$

$$\begin{pmatrix} 1 & 0 & -29/13 \\ 0 & 1 & -2/13 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.27}$$

Thus the vertex $\mathbf{c}$ is:

$$\mathbf{c} = \begin{pmatrix} -29/13 \\ -2/13 \end{pmatrix} \tag{2.0.28}$$

The direction vector of axis of symmetry is given by :

$$\mathbf{m} = \mathbf{V}\mathbf{c} + \mathbf{u} \tag{2.0.29}$$

$$= \begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix} \begin{pmatrix} -\frac{29}{13} \\ -\frac{2}{13} \end{pmatrix} + \begin{pmatrix} \frac{619}{2} \\ -\frac{272}{2} \end{pmatrix} \tag{2.0.30}$$

$$= \begin{pmatrix} -\frac{5}{2} \\ -6 \end{pmatrix} \tag{2.0.31}$$

$$\mathbf{m} = \frac{13}{2} \tag{2.0.32}$$

$$\implies \frac{\mathbf{m}}{\|\mathbf{m}\|} = \begin{pmatrix} -\frac{5}{13} \\ -\frac{12}{13} \end{pmatrix} \tag{2.0.33}$$

The focus is given by:

$$\mathbf{F} = \mathbf{c} - \left(\frac{\mathbf{m}}{\|\mathbf{m}\|} \times a \right) \tag{2.0.34}$$

$$= \begin{pmatrix} -\frac{29}{13} \\ -\frac{2}{13} \end{pmatrix} - \left(\begin{pmatrix} -\frac{5}{13} \\ -\frac{12}{13} \end{pmatrix} \times \frac{1}{52} \right) \tag{2.0.35}$$

$$= \begin{pmatrix} -\frac{1503}{676} \\ -\frac{23}{169} \end{pmatrix} \tag{2.0.36}$$

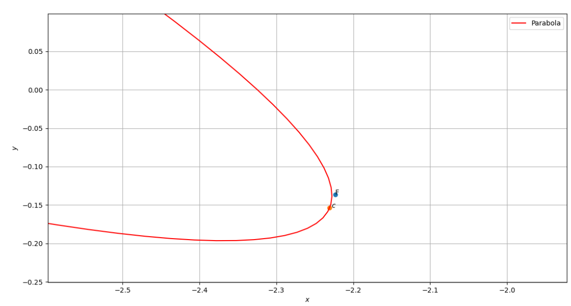

Fig. 2: Traced parabola