EE5609: Matrix Theory
Assignment-8

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1 Problem

1) Consider a Markov chain with transition probability matrix P given by

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \tag{1.0.1}$$

For any two states i and j.Let $P_{ij}^{(n)}$ denote the n step transition probability of going form i to j. Identify Correct statements.

$$i) \lim_{n\to\infty} P_{11}^{(n)} = \frac{2}{9} \tag{1.0.2}$$

$$ii) \lim_{n\to\infty} P_{21}^{(n)} = 0 \tag{1.0.3}$$

$$iii) \lim_{n\to\infty} P_{32}^{(n)} = \frac{1}{3} \tag{1.0.4}$$

$$iv) \lim_{n\to\infty} P_{13}^{(n)} = \frac{1}{3} \tag{1.0.5}$$

2 solution

Using Theorem : If a finite Markov Chain is irreducible and aperiodic then there is a unique stationary distribution

$$\lim_{k\to\infty} P^{(k)} = Ik \tag{2.0.1}$$

where I represents a column vector with each entry as 1. For 3 states we have :

$$\lim_{n\to\infty} P^{(n)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \tag{2.0.2}$$

$$= \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \tag{2.0.3}$$

$$\implies \lim_{n\to\infty} \begin{pmatrix} P_{11}^{(n)} & P_{12}^{(n)} & P_{13}^{(n)} \\ P_{21}^{(n)} & P_{22}^{(n)} & P_{23}^{(n)} \\ P_{31}^{(n)} & P_{32}^{(n)} & P_{33}^{(n)} \end{pmatrix} \tag{2.0.4}$$

$$= \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \tag{2.0.5}$$

A stationary distribution π is a row vector whose entries are non-negative and sums up to 1, is unchanged by the operation of transition matrix p on it and so it is defined by $\pi P = \pi$.

Let

$$\pi = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$$

$$\implies \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$$

$$\implies \begin{pmatrix} \frac{\pi_1}{2} + \frac{\pi_3}{3} & \frac{\pi_1}{2} + \frac{\pi_2}{2} + \frac{\pi_3}{3} & \frac{\pi_2}{2} + \frac{\pi_3}{3} \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$$

$$\implies \begin{pmatrix} \frac{\pi_1}{2} + \frac{\pi_3}{3} & \frac{\pi_1}{2} + \frac{\pi_2}{2} + \frac{\pi_3}{3} & \frac{\pi_2}{2} + \frac{\pi_3}{3} \end{pmatrix} - \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} = 0$$

$$\implies \begin{pmatrix} \frac{\pi_1}{2} + \frac{\pi_3}{3} - \pi_1 & \frac{\pi_1}{2} + \frac{\pi_2}{2} + \frac{\pi_3}{3} - \pi_2 & \frac{\pi_2}{2} + \frac{\pi_3}{3} - \pi_3 \end{pmatrix} = 0$$

$$\implies \begin{pmatrix} -3\pi_1 + 2\pi_3 & 3\pi_1 - 3\pi_2 + 2\pi_3 & 3\pi_2 - 4\pi_3 \end{pmatrix} = 0 \tag{2.0.6}$$

$$-3\pi_1 + 2\pi_3 = 0 \tag{2.0.7}$$

$$3\pi_1 - 3\pi_2 + 2\pi_3 = 0 \tag{2.0.8}$$

$$3\pi_2 - 4\pi_3 = 0 \;\;\&\;\; \pi_1 + \pi_2 + \pi_3 = 1 \tag{2.0.9}$$

These equations can be transformed into the matrix :

$$\begin{pmatrix} -3 & 0 & 2 \\ 3 & -3 & 2 \\ 0 & 3 & -4 \\ 1 & 1 & 1 \end{pmatrix} \pi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\implies \xleftrightarrow{R_2 = R_1 + R_2} \begin{pmatrix} -3 & 0 & 2 \\ 0 & -3 & 4 \\ 0 & 3 & -4 \\ 1 & 1 & 1 \end{pmatrix} \pi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\implies \xleftrightarrow{R_1 = \frac{R_1}{-3}} \begin{pmatrix} 1 & 0 & \frac{-2}{3} \\ 0 & -3 & 4 \\ 0 & 3 & -4 \\ 1 & 1 & 1 \end{pmatrix} \pi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\implies \xleftrightarrow{R_2 = \frac{R_2}{-3}} \begin{pmatrix} 1 & 0 & \frac{-2}{3} \\ 0 & 1 & \frac{-4}{3} \\ 0 & 3 & -4 \\ 1 & 1 & 1 \end{pmatrix} \pi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\implies \xleftrightarrow{R_3 = R_3 - 3R_2} \begin{pmatrix} 1 & 0 & \frac{-2}{3} \\ 0 & 1 & \frac{-4}{3} \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \pi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\implies \xleftrightarrow{R_3 = R_4, R_4 = R_3} \begin{pmatrix} 1 & 0 & \frac{-2}{3} \\ 0 & 1 & \frac{-4}{3} \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \pi = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\implies \xleftrightarrow{R_3 = R_3 - R_1} \begin{pmatrix} 1 & 0 & \frac{-2}{3} \\ 0 & 1 & \frac{-4}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 \end{pmatrix} \pi = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\implies \xleftrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} 1 & 0 & \frac{-2}{3} \\ 0 & 1 & \frac{-4}{3} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \pi = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{3} \\ 0 \end{pmatrix} \tag{2.0.10}$$

$$\Longrightarrow \xleftrightarrow{R_2=R_2+\frac{4}{3}R_3} \begin{pmatrix} 1 & 0 & \frac{-2}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \pi = \begin{pmatrix} 0 \\ \frac{4}{9} \\ \frac{1}{3} \\ 0 \end{pmatrix}$$

$$\Longrightarrow \xleftrightarrow{R_1=R_1+\frac{2}{3}R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \pi = \begin{pmatrix} \frac{2}{9} \\ \frac{4}{9} \\ \frac{1}{3} \\ 0 \end{pmatrix} \tag{2.0.11}$$

$$\pi = \begin{pmatrix} \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \end{pmatrix} \tag{2.0.12}$$

$$\Longrightarrow \lim_{n\to\infty} \begin{pmatrix} P_{11}^{(n)} & P_{12}^{(n)} & P_{13}^{(n)} \\ P_{21}^{(n)} & P_{22}^{(n)} & P_{23}^{(n)} \\ P_{31}^{(n)} & P_{32}^{(n)} & P_{33}^{(n)} \end{pmatrix} = \begin{pmatrix} \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \end{pmatrix} \tag{2.0.13}$$

$$P_{11}^{(n)} = \frac{2}{9} \tag{2.0.14}$$

$$P_{13}^{(n)} = \frac{1}{3} \tag{2.0.15}$$

Hence , Options (i) and (iv) are correct.