this occurs when

$$-150^\circ = -90^\circ - \tan^{-1}\left[\frac{\omega}{\alpha}\right]$$

such that $\omega = \sqrt{3}\,\alpha$.

On substituting for ω in (5.4.2), it is found that an open-loop gain of

$$K = 2\sqrt{3}\,\alpha^2$$

will achieve the desired phase margin.

The value of phase margin selected here is up at the higher end of the values usually chosen. Typical values for reasonable design are 30° to 60° for the phase margin; the gain margin, however, is usually in the range 3 to 12 dB.

5.5 Compensation using the Nyquist plot

Applying compensation to a system by means of the Nyquist plot involves a modification of the original open-loop frequency response characteristics, in order to satisfy some previously specified performance objectives. The objectives can be stated simply in terms of gain and phase margins, and it is this type of requirement which is considered here. In practice, however, it may well be that further desirable performance measurements are given such as bandwidth and pole locations and the Nyquist compensation procedure may well need to be combined with such as Bode analysis and root locus techniques. Indeed this could result in several designs and redesigns being carried out before a final compromise scheme is obtained.

It is considered that the compensation network is included in cascade with the original open-loop system, within a unity feedback loop. This makes it possible to plot firstly the original open-loop system frequency response and secondly the compensated system frequency response as a new open-loop system. The closed-loop stability and relative stability properties of each system can then be witnessed. When feedback compensation is used as well as or instead of cascade compensation, the Nyquist design can be treated either by finding the equivalent sole cascade controller or simply by considering the feedback compensator to be in series with the original plant, such that a Nyquist plot is made of $G(j\omega)H(j\omega)$: this is discussed further in Appendix 4.

5.5.1 Gain compensation

For straightforward and simple performance specifications it is often easiest to meet the requirements by merely adjusting the system open-loop gain K, within a unity feedback loop. In practical terms this will most likely be implemented by means of an amplifier in cascade with the open-loop plant.

Where the plant is of the form $G(j\omega) = KB(j\omega)/A(j\omega)$, if the gain K is increased then this will increase the magnitude of the frequency response plot, without affecting the phase, uniformly over all frequencies. In particular, the frequency at which the phase of $G(j\omega)$ is $\pm 180°$ will not be affected by a variation in K, however the gain of the plot at that frequency and its phase margin will be. Hence stability of a closed-loop system can be ensured by the selection of an appropriate gain value and within limits, both phase and gain margin specifications can be met, as shown in the following example.

Example 5.5.1

Consider the system open-loop transfer function:

$$G(s) = \frac{20(s+0.5)K}{s(s+3)(s+1)}$$

whose frequency response plot for $K = 1$ is considered in detail in Example 5.4.1, and is sketched here again in Fig. 5.23. Using solely the gain $K > 0$, find a value which results in a closed-loop system which has a gain margin of at least 5 dB and a phase margin which lies between $30° < \phi < 50°$.

The original frequency response plot (only positive frequencies are shown in Fig. 5.23) reveals that the -1 point is encircled in a clockwise direction and hence the unity feedback closed-loop system obtained when $K = 1$ is unstable.

While $K = 1$ the gain margin is $20 \log_{10}(3/5) = -4.437$ dB, and in order to obtain a gain margin of at least 5 dB, it is required that the open-loop system gain is $|G(j\omega)| \leqslant 0.5623$ when the phase of $\underline{/G(j\omega)}$ is $\pm 180°$. This value range can be achieved by selecting a gain K such that $0 < K \leqslant 0.3374$.

For a gain $K = 0.3374$, the frequency response plot is equal to unity at a

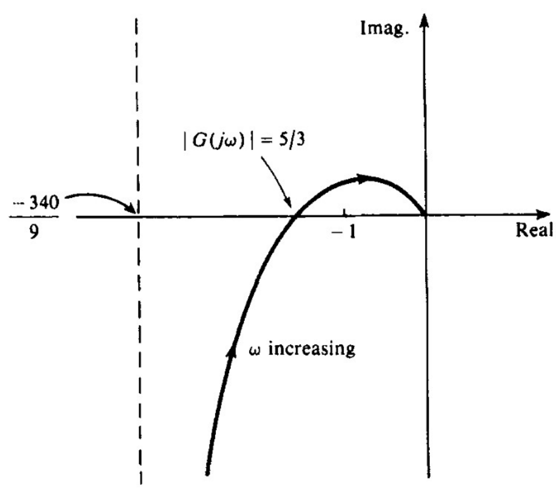

Fig. 5.23 Response plot for Example 5.5.1

frequency found from:

$$|G(j\omega)| = \frac{20(\omega^2+25)^{1/2}K}{\omega(\omega^2+9)^{1/2}(\omega^2+1)^{1/2}} = 1$$

which is satisfied when $\omega = 2.968$ rad/s.

At this frequency the phase $\underline{/G(j\omega)}$ is:

$$\underline{/G(j\omega)} = \tan^{-1}\left[\frac{\omega}{5}\right] - 90^\circ - \tan^{-1}\left[\frac{\omega}{3}\right] - \tan^{-1}[\omega] = -175.38^\circ$$

Hence the phase margin is only 4.62°.

It can subsequently be found (by trial and error for instance) that by employing a gain $K = 0.05$, the frequency response plot is equal to unity at $\omega = 1.086$ rad/s and this results in a phase of $\underline{/G(j\omega)} = -145.01^\circ$, which gives a phase margin of 34.99°. So a phase margin which lies within the desired range has been found by the selection of a suitable gain value, K. Further, with $K = 0.05$ the gain margin is $20\log_{10}(12) = 21.584$ dB, which is certainly more than the required minimum value of 5 dB. Thus, by means of simple gain compensation, the gain and phase margin stipulations have been satisfied.

Unfortunately it is not always possible to satisfy both the gain and phase margin requirements, by only making use of gain compensation. For instance the desire for a gain margin of at least 5 dB but no more than 10 dB could not be achieved simply by gain compensation on the system of Example 5.5.1 if a phase margin within the range $40^\circ < \phi < 50^\circ$ was also required. Hence for a more general type of compensation network it is necessary to employ a slightly more complicated structure, such as the lag/lead compensation considered in the following section.

5.5.2 Lag/lead compensation

The application of a cascade lag/lead compensator is considered in terms of Bode analysis in Section 5.3.3, with a relevant block diagram given in Fig. 5.11. Essentially, the controller adds one pole and one zero to the original open-loop system characteristics, and selection of the positions of these singularities, coupled with a choice of open-loop gain, allows for a great deal more flexibility than is provided by means of gain compensation alone. If the compensator transfer function is given by:

$$D(j\omega) = \frac{j\omega + b}{j\omega + a} \tag{5.5.1}$$

then the frequency response of the cascaded functions $D(j\omega)G(j\omega)$ must be plotted, such that the stability of the unity feedback closed-loop system can be tested. In terms of the Nyquist plot, this means that at each frequency ω, the effect of the cascade compensator is to modify, by a certain amount, the magnitude and phase of the transfer function. The effect of a lead compensator is to make the transfer function phase more positive at each frequency, whereas the effect of a lag compensator is to make the transfer function phase more negative at each frequency.

For a *lag* compensator, the phase angle of the compensator itself is negative in the mid-frequency range, but is 0° at the extremes when $\omega = 0$ and $\omega = \infty$. This can be seen by noting that:

$$\angle D(j\omega) = \tan^{-1}\left[\frac{\omega}{b}\right] - \tan^{-1}\left[\frac{\omega}{a}\right] \tag{5.5.2}$$

and when $a < b$ (for a lag compensator), it follows that the phase will never be positive definite. In fact as

$$|D(j\omega)| = \left[\frac{\omega^2 + b^2}{\omega^2 + a^2}\right]^{1/2} \tag{5.5.3}$$

the frequency response plot for a lag compensator is a semi-circle, as shown in Fig. 5.24. There are no axis crossing points and plot end conditions are found to be $|D(j\omega)| = b/a$ when $\omega = 0$ and $|D(j\omega)| = 1$ when $\omega = \infty$, with the phase equal to 0° in both cases.

So, for any particular frequency value ω, which lies between 0 and ∞, the effect of the lag compensator is

1. To multiply the gain of the original plot by a value which is greater than unity but less than b/a; and
2. To shift the phase by an angle which is less than 0° but greater than $-90°$ (for finite a and b).

When the system open-loop gain $K > 0$ can also be varied, then this will not affect the phase angle in any way, but will multiply the gain at all frequencies by a factor K.

If a phase shift of larger negative characteristics is required, then several lag

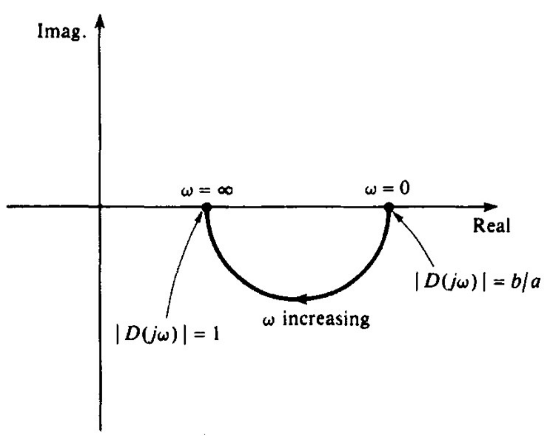

Fig. 5.24 Nyquist plot for a lag compensator

compensators can be used in cascade. Restricting ourselves to a single compensator however, an example of its employment is given by the following.

Example 5.5.2

Consider the open-loop transfer function:

$$G(j\omega) = \frac{20K}{j\omega(j\omega + 2)(j\omega + 1)}$$

A frequency response plot for the system, when $K = 1$, with no compensation applied as shown in Fig. 5.25(a), reveals a negative phase margin of -30.15° (when $\omega = 2.532$ rad/s) and a negative gain margin of -10.46 dB (when $\omega = 1.414$ rad/s). Such phase and gain results show that a unity feedback closed-loop system, with no compensation, would be unstable. Note that gain and phase margin results can be obtained either by exact calculations or as approximate values measured from the sketched Nyquist plot.

A phase lag compensator can be used to obtain a gain margin of at least 8 dB and a phase margin of at least 30°, without affecting the steady-state properties of the system when $K = 1$. This can be achieved by reducing the overall gain at higher frequencies whilst retaining the same value of gain at low frequencies.

The gain margin is found to be -10.46 dB when $\omega = 1.414$ rad/s, for the uncompensated system. If it is ensured that, when the system is compensated, the gain margin will be at least 8 dB, this can be realized by reducing the gain at $\omega = 1.414$ rad/s by approximately 20 dB, on condition that the phase is not also affected to any great extent at that frequency. Such a decrease in gain is obtained by selecting $K = 0.1$. If it is then ensured that $b/a = 10$, this will mean

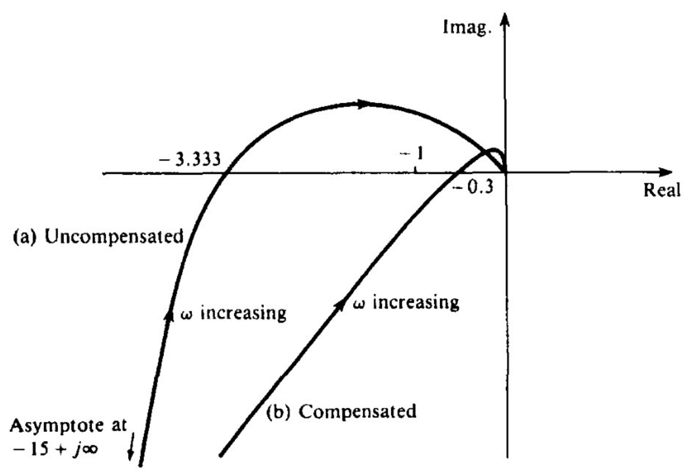

Fig. 5.25 Uncompensated and (b) compensated frequency response plots for the system of Example 5.5.2

that the overall gain at low frequencies will not be affected, i.e.

$$D(j\omega) = \frac{b}{a} = 10 = \frac{1}{K}$$

$$\omega = 0$$

Thus the gain increase, K, is counteracted at low frequencies by the compensator gain, whereas at higher frequencies, no such counteraction takes place. So, at higher frequencies a gain decrease of 20 dB is apparent. This means that, as long as little or no phase change takes place at the frequency at which the overall gain was 20 dB, with $K = 1$, this will become the frequency at which the overall gain is 0 dB, with $K = 0.1$.

Solving

$$|G(j\omega)| = \frac{20}{\omega(\omega^2+4)^{1/2}(\omega^2+1)^{1/2}} = 10 \ [=20 \text{ dB}]$$

$$[K = 1]$$

gives $\omega = 0.749$ rad/s and at this frequency the phase is:

$$\underline{/G(j\omega)} = -90^\circ - \tan^{-1}\left[\frac{\omega}{2}\right] - \tan^{-1}[\omega] = -147.387^\circ$$

such that a phase margin of 32.61° would be obtained, which is above the minimum required.

A choice of $10a = b < 0.0749$ would ensure that almost no phase change, due to the compensator, occurs at $\omega = 0.749$, and hence a choice of $b = 0.05$, $a = 0.005$ will suffice. The resultant effect, obtained by employing a gain $K = 0.1$, along with the lag compensator:

$$D(j\omega) = \frac{j\omega + 0.05}{j\omega + 0.005}$$

is shown in Fig. 5.25(b). The frequency at which the gain is 0 dB (unity magnitude) is now approximately $\omega = 0.75$ rad/s (as opposed to $\omega = 2.532$ rad/s for the uncompensated system). Further, a gain margin of $+9.5$ dB and a phase margin of approximately 32° are obtained, thus meeting the performance requirements. Employment of a lag compensator has resulted in a lower frequency at which the gain is 0 dB, i.e. the bandwidth has been reduced.

Reduction of bandwidth is a property usually obtained by the use of a lag compensator, and can be looked at in another way, in terms of slowing down the system's response. If the steady-state error constant is held at a steady value, as was done in the example, then both the gain and phase margins will normally be increased. Conversely, by holding the gain and phase margins steady, error constant values can be increased by means of a lag compensator.

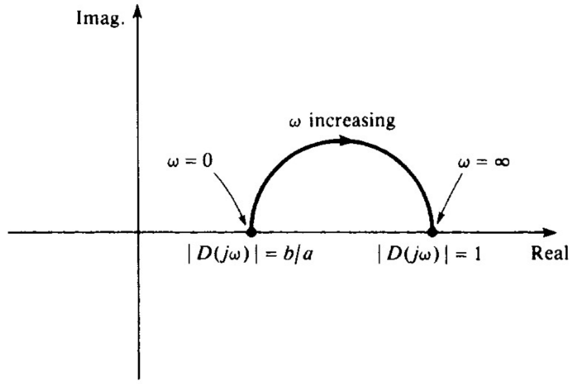

Fig. 5.26 Nyquist plot for a lead compensator

For a *lead* compensator, the phase angle of the compensator itself is positive in the mid-frequency range, but is 0° at the extremes when $\omega = 0$ and $\omega = \infty$. This can be seen by noting that (5.5.2) still holds, however now $b < a$ for a lead compensator. It follows that the phase will never be negative definite. In fact as (5.5.3) also holds, the frequency response plot for a lead compensator is a semi-circle, as shown in Fig. 5.26. There are no axis crossing points and plot end conditions are found to be identical to those for the lag compensator, i.e. $|D(j\omega)| = b/a$ when $\omega = 0$ and $|D(j\omega)| = 1$ when $\omega = \infty$, with the phase equal to 0° in both cases. So for any particular frequency value ω, which lies between 0 and ∞, the effect of a lead compensator is:

1. To multiply the gain of the original plot by a value which is less than unity but greater than b/a; and
2. To shift the phase by an angle which is greater than 0° but less than $+90^\circ$ (for a finite a and b).

When the system open-loop gain $K > 0$ can also be varied, this does not affect the phase angle in any way, but will multiply the gain at all frequencies by a factor K.

If a phase shift of large positive characteristics (more than 90°) is required, then several lead compensators can be used in cascade. Conversely, if a more complicated compensator structure, which is more selective about the frequency ranges amplified, is required, this can often be achieved by cascading a number of lag and lead compensators. For a single lead compensator, the standard design procedure involves firstly selecting the system open-loop gain value, K, in order to meet gain margin requirements (a decrease in gain margin will be required). The ratio $b : a$ can then be found in order to produce necessary steady-state conditions, such as specified error coefficients to relevant input signals. At higher frequencies the overall gain is thereby increased, the tendency being towards a destabilization of the system. The phase shifting properties of the compensator can then be selected, such that they lie in the mid-frequency range and therefore do not affect the specifications already met by the compensator, see Problem 5.5.

Phase lead compensators are usually employed to improve the transient system response, decreasing the phase and gain margins from those corresponding to a sluggish response to those apparent for a specified, acceptable but faster response. In the majority of cases the frequency at which the overall gain is 0 dB, is increased by a phase lead compensator, thus the bandwidth is increased.

5.6 The Nichols chart

Frequency response analysis of a system involves the use of system gain and phase information obtained over a wide range of frequencies, in theory over all frequencies. Bode and Nyquist plots both deal with the same data type, but present it in a distinctly different way. The Nichols chart, another frequency response technique, deals once more with the same data type but presents it in yet another different way.

Much of the procedure employed with a Nichols chart is similar to that used for Bode and Nyquist design methods. The Nichols chart is therefore not considered here in the same depth as the other two methods, which are in many ways easier to understand and use. A common feature with all three methods is that the open-loop system transfer function response is plotted, while results such as relative stability found from the method, pertain to the closed-loop system when connected in unity feedback mode. Where the closed-loop system in question actually employs a feedback term $H(j\omega)$ which is other than unity, then either the equivalent unity feedback system must be found and the effective open-loop transfer function extracted, or the function $G(j\omega)H(j\omega)$ can be plotted. This is discussed in Appendix 4.

Initially, consider solely the system open-loop transfer function, $G(j\omega)$. The gain in dB of this function can be plotted against the corresponding phase angle values over the range of frequencies $\omega = 0$ to $\omega = \infty$, as shown in the following example.

Example 5.6.1

The open-loop system transfer function,

$$G(j\omega) = \frac{20}{j\omega(j\omega + 1)}$$

produces a gain (dB)–v–phase plot as depicted in Fig. 5.27. The gain for this system is:

$$\text{gain (dB)} = 20 \log_{10} \frac{20}{\omega(\omega^2 + 1)^{1/2}}$$

whereas the phase is:

$$\text{phase} = -90^\circ - \tan^{-1}[\omega]$$

Although for the transfer function shown above it is quite straightforward to