Logic Synthesis & Verification, Fall 2024

National Taiwan University

Reference Solution for Problem Set 3

1 [Cofactor and Generalized Cofactor]

(a) Let $xf_x \oplus (\neg x)f_{\neg x} = h$. Then

$$\begin{aligned} h_x &= (1f_x) \oplus 0 \\ &= f_x, \text{ and} \end{aligned}$$

$$\begin{aligned} h_{\neg x} &= 0 \oplus (1f_{\neg x}) \\ &= f_{\neg x}. \end{aligned}$$

Therefore,

$$\begin{aligned} xf_x \oplus (\neg x)f_{\neg x} &= xh_x + (\neg x)h_{\neg x} \\ &= xf_x + (\neg x)f_{\neg x} \\ &= f. \end{aligned}$$

(b)

$$\begin{aligned} & g \wedge co(f,g) \vee \neg g \wedge co(f, \neg g) \\ =\ & ((g, 0, \neg g) \wedge (fg, \neg g, (\neg f)g)) \vee ((\neg g, 0, g) \wedge (f(\neg g), g, (\neg f)(\neg g))) \\ =\ & (fg, 0, (\neg f)g \vee (\neg g)) \vee (f(\neg g), 0, g \vee (\neg f)(\neg g)) \\ =\ & (f, 0, \neg f) \\ =\ & f. \end{aligned}$$

(c)

$$\begin{aligned} co(f,h) \oplus co(g,h) &= (f \wedge h, \neg h, \neg f \wedge h) \oplus (g \wedge h, \neg h, \neg g \wedge h) \\ &= (f \wedge h \oplus g \wedge h, \neg h, \neg f \wedge h \oplus \neg g \wedge h) \\ &= ((f \oplus g) \wedge h, \neg h, \neg(f \oplus g) \wedge h) \\ &= co(f \oplus g, h) \end{aligned}$$

(d)

$$\begin{aligned} co(\neg f, g) &= (\neg f \wedge g, \neg g, f \wedge g) \\ \neg co(f,g) &= \neg(f \wedge g, \neg g, \neg f \wedge g) \\ &= (\neg f \wedge g, \neg g, f \wedge g) \end{aligned}$$

Therefore,

$$co(\neg f, g) = \neg co(f, g)$$

Note. Operations on incompletely specified functions can be derived from operations on don't cares. Take the $\wedge$ operation for example. Suppose $F = (f_F, d_F, r_F)$ and $G = (f_G, d_G, r_G)$ are incompletely specified functions. Then $H = F \wedge G$ is also an incompletely specified function. Let $H = (f_H, d_H, r_H)$. According to the truth table of 3-valued logic, $x \wedge y = 1$ if and only if $x = y = 1$, so $f_H = f_F \wedge f_G$. Similarly, $x \wedge y = 0$ if and only if $x = 0$ or $y = 0$, so $r_H = r_F \vee r_G$. Finally, the value of $x \wedge y$ cannot be decided for the rest of the conditions, so $d_H = (d_F \wedge f_G) \vee (f_F \wedge d_G) \vee (d_F \wedge d_G)$.
Following the same idea, here are some operations on incompletely specified functions.

$$
\begin{aligned}
(f_F, d_F, r_F) \wedge (f_G, d_G, r_G) &= (f_F f_G &&, d_F f_G \vee f_F d_G \vee d_F d_G &&, r_F \vee r_G) \\
(f_F, d_F, r_F) \vee (f_G, d_G, r_G) &= (f_F \vee f_G &&, d_F r_G \vee r_F d_G \vee d_F d_G &&, r_F r_G) \\
\neg(f_F, d_F, r_F) &= (r_F &&, d_F &&, f_F)
\end{aligned}
$$

2 [Operation on Cube Lists]

1. Trying to add the cube $(100 - - - 0)$:
   $(100 - - - 0)$ is not orthogonal to the first cube $(-000 - - -)$ and third cube $(-0 - 10 - 0)$.
   Split on 4th variable results in $(1000 - -0), (1001 - -0)$.
   And we can find that $(1000 - -0) \subseteq (-000 - - -)$; therefore, we don't need to add the cube $(1000 - -0)$.

   Trying to add the cube $(1001 - -0)$:
   $(1001 - -0)$ is not orthogonal to the third cube $(-0 - 10 - 0)$.
   Split on 5th variable results in $(10010 - 0), (10011 - 0)$.
   And we can find that $(10010 - 0) \subseteq (-0 - 10 - 0)$; therefore, we don't need to add the cube $(10010 - 0)$.

   Trying to add the cube $(10011 - 0)$:
   $(10011 - 0)$ is orthogonal to the cube lists.
   Now the cube list becomes

$$
\begin{pmatrix}
- & 0 & 0 & 0 & - & - & - \\
0 & 1 & - & - & 1 & 1 & 0 \\
- & 0 & - & 1 & 0 & - & 0 \\
1 & 0 & 0 & 1 & 1 & - & 0
\end{pmatrix}.
$$

3 [Symmetric Functions]

(a) Here we show the process to derive the necessary and sufficient condition of f to be S_1-symmetric on variables x_1 and x_2. By definition, if f is S_1-symmetric on variables x_1 and x_2, then $f(x_1, x_2, x_3, \ldots) = f(x_2, x_1, x_3, \ldots)$.

For the left-hand side, we can expand $f(x_1, x_2, x_3)$ as

$$\begin{aligned} f(x_1, x_2, x_3, \ldots) = &\quad \overline{x_1} \cdot \overline{x_2} \cdot f(0, 0, x_3, \ldots) + \overline{x_1} \cdot x_2 \cdot f(0, 1, x_3, \ldots) \\ &+ x_1 \cdot \overline{x_2} \cdot f(1, 0, x_3, \ldots) + x_1 \cdot x_2 \cdot f(1, 1, x_3, \ldots). \end{aligned}$$

For the right-hand side, we can expand $f(x_2, x_1, x_3, \ldots)$ as

$$\begin{aligned} f(x_2, x_1, x_3, \ldots) = &\quad \overline{x_1} \cdot \overline{x_2} \cdot f(0, 0, x_3, \ldots) + \overline{x_1} \cdot x_2 \cdot f(1, 0, x_3, \ldots) \\ &+ x_1 \cdot \overline{x_2} \cdot f(0, 1, x_3, \ldots) + x_1 \cdot x_2 \cdot f(1, 1, x_3, \ldots). \end{aligned}$$

Note that this form can be obtained by grouping minterms in the minterm canonical form into four groups according to x_1 and x_2, so this form is also canonical. Therefore, $f(x_1, x_2, x_3, \ldots) = f(x_2, x_1, x_3, \ldots)$ if and only if $f(0, 1, x_3, \ldots) = f(1, 0, x_3, \ldots)$. In other words, $f_{x_1 \cdot \overline{x_2}} = f_{\overline{x_1} \cdot x_2}$.

Following the same idea, the necessary and sufficient condition of f to be S_i-symmetric on variables x_1 and x_2 are as follows.
S_1: $f_{x_1 \cdot \overline{x_2}} = f_{\overline{x_1} \cdot x_2}$
S_2: $f_{\overline{x_1} \cdot \overline{x_2}} = f_{x_1 \cdot x_2}$
S_3: f can never be S_3-symmetric
S_4: f can never be S_4-symmetric
S_5: $f_{\overline{x_1} \cdot \overline{x_2}} = f_{\overline{x_i} \cdot x_2} = f_{x_1 \cdot \overline{x_2}} = f_{x_1 \cdot x_2}$ (i.e., f does not depend on x_1 and x_2)
S_6: $f_{\overline{x_1} \cdot \overline{x_2}} = f_{\overline{x_i} \cdot x_2} = f_{x_1 \cdot \overline{x_2}} = f_{x_1 \cdot x_2}$ (i.e., f does not depend on x_1 and x_2)
S_7: $f_{\overline{x_1} \cdot \overline{x_2}} = (\neg f_{\overline{x_1} \cdot x_2}) = (\neg f_{x_1 \cdot \overline{x_2}}) = f_{x_1 \cdot x_2}$
S_8: $f_{\overline{x_1} \cdot \overline{x_2}} = (\neg f_{\overline{x_1} \cdot x_2}) = (\neg f_{x_1 \cdot \overline{x_2}}) = f_{x_1 \cdot x_2}$

(b) Only S_2 does not satisfy transitivity.

- S_1 satisfies transitivity. The proof is as follows.
  Suppose f is S_1-symmetric on (x_1, x_2) and (x_2, x_3), then

$$\begin{aligned} f_{x_1 \cdot \overline{x_3}} &= \overline{x_2} \cdot f_{x_1 \cdot \overline{x_2} \cdot \overline{x_3}} + x_2 \cdot f_{x_1 \cdot x_2 \cdot \overline{x_3}} \\ &= \overline{x_2} \cdot f_{x_1 \cdot \overline{x_2} \cdot \overline{x_3}} + x_2 \cdot f_{x_1 \cdot \overline{x_2} \cdot x_3} && (\because f_{x_2 \cdot \overline{x_3}} = f_{\overline{x_2} \cdot x_3}) \\ &= \overline{x_2} \cdot f_{\overline{x_1} \cdot x_2 \cdot \overline{x_3}} + x_2 \cdot f_{\overline{x_1} \cdot x_2 \cdot x_3} && (\because f_{x_1 \cdot \overline{x_2}} = f_{\overline{x_1} \cdot x_2}) \\ &= \overline{x_2} \cdot f_{\overline{x_1} \cdot \overline{x_2} \cdot x_3} + x_2 \cdot f_{\overline{x_1} \cdot x_2 \cdot x_3} && (\because f_{x_2 \cdot \overline{x_3}} = f_{\overline{x_2} \cdot x_3}) \\ &= f_{\overline{x_1} \cdot x_3}, \end{aligned}$$

  so f is also S_1-symmetric on (x_1, x_3).
- S_2 does not satisfy transitivity. Here is a counterexample.

$$f = x_1 x_2' + x_2' x_3 + x_1 x_3,$$

  where f is S_2-symmetric on (x_1, x_2) and (x_2, x_3), but $f_{\overline{x_1} \cdot \overline{x_3}} = 0 \neq 1 = f_{x_1 \cdot x_3}$, so f is not S_2-symmetric on (x_1, x_3).
- S_5 and S_6 satisfies transitivity. The proof is as follows.
  Suppose f is $S_{5(6)}$-symmetric on (x_1, x_2) and (x_2, x_3), then f does not depend on x_1, x_2 and x_3. Therefore, f is also $S_{5(6)}$-symmetric on (x_1, x_3).

- S_7 and S_8 satisfies transitivity. The proof is as follows.
  Suppose f is $S_{7(8)}$-symmetric on (x_1, x_2) and (x_2, x_3), then

$$\begin{aligned}
f_{\overline{x_1}\cdot\overline{x_3}} &= \overline{x_2}\cdot f_{\overline{x_1}\cdot\overline{x_2}\cdot\overline{x_3}} + x_2 \cdot f_{\overline{x_1}\cdot x_2\cdot\overline{x_3}} \\
&= \overline{x_2}\cdot(\neg f_{x_1\cdot\overline{x_2}\cdot\overline{x_3}}) + x_2 \cdot f_{\overline{x_1}\cdot x_2\cdot\overline{x_3}} && (\because f_{\overline{x_1}\cdot\overline{x_2}} = \neg f_{x_1\cdot\overline{x_2}}) \\
&= \overline{x_2}\cdot(\neg f_{x_1\cdot\overline{x_2}\cdot\overline{x_3}}) + x_2 \cdot (\neg f_{x_1\cdot x_2\cdot\overline{x_3}}) && (\because f_{\overline{x_1}\cdot x_2} = \neg f_{x_1\cdot x_2}) \\
&= \neg f_{x_1\cdot\overline{x_3}}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
f_{\overline{x_1}\cdot\overline{x_3}} &= \overline{x_2}\cdot f_{\overline{x_1}\cdot\overline{x_2}\cdot\overline{x_3}} + x_2 \cdot f_{\overline{x_1}\cdot x_2\cdot\overline{x_3}} \\
&= \overline{x_2}\cdot(\neg f_{\overline{x_1}\cdot\overline{x_2}\cdot x_3}) + x_2 \cdot f_{\overline{x_1}\cdot x_2\cdot\overline{x_3}} && (\because f_{\overline{x_2}\cdot\overline{x_3}} = \neg f_{\overline{x_2}\cdot x_3}) \\
&= \overline{x_2}\cdot(\neg f_{\overline{x_1}\cdot\overline{x_2}\cdot x_3}) + x_2 \cdot (\neg f_{\overline{x_1}\cdot x_2\cdot x_3}) && (\because f_{x_2\cdot\overline{x_3}} = \neg f_{x_2\cdot x_3}) \\
&= \neg f_{\overline{x_1}\cdot x_3}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
f_{\overline{x_1}\cdot\overline{x_3}} &= \overline{x_2}\cdot f_{\overline{x_1}\cdot\overline{x_2}\cdot\overline{x_3}} + x_2 \cdot f_{\overline{x_1}\cdot x_2\cdot\overline{x_3}} \\
&= \overline{x_2}\cdot(\neg f_{\overline{x_1}\cdot\overline{x_2}\cdot x_3}) + x_2 \cdot f_{\overline{x_1}\cdot x_2\cdot\overline{x_3}} && (\because f_{\overline{x_2}\cdot\overline{x_3}} = \neg f_{\overline{x_2}\cdot x_3}) \\
&= \overline{x_2}\cdot(\neg f_{\overline{x_1}\cdot\overline{x_2}\cdot x_3}) + x_2 \cdot (\neg f_{\overline{x_1}\cdot x_2\cdot x_3}) && (\because f_{x_2\cdot\overline{x_3}} = \neg f_{x_2\cdot x_3}) \\
&= \overline{x_2}\cdot(f_{x_1\cdot\overline{x_2}\cdot x_3}) + x_2 \cdot (\neg f_{\overline{x_1}\cdot x_2\cdot x_3}) && (\because f_{\overline{x_1}\cdot\overline{x_2}} = \neg f_{x_1\cdot\overline{x_2}}) \\
&= \overline{x_2}\cdot(f_{x_1\cdot\overline{x_2}\cdot x_3}) + x_2 \cdot (f_{x_1\cdot x_2\cdot x_3}) && (\because f_{\overline{x_1}\cdot x_2} = \neg f_{x_1\cdot x_2}) \\
&= f_{x_1\cdot x_3},
\end{aligned}$$

  so f is also $S_{7(8)}$-symmetric on (x_1, x_3).

4 [Unate Functions]

(a) True. A prime cover of a unate function must be a unate cover.
Let the unate function be f, and the prime cover be F. If F is not unate, by definition, there exist a variable v and two cubes $c_1, c_2 \in F$ such that $v \in c_1$ and $\overline{v} \in c_2$. Consider the two possibilities.
  (1) If f is positive unate in v:
  By definition, $f_{\overline{v}} \subseteq f_v$. Since f contain $c_2 = \overline{v}\cdot c_2 = \overline{v}\cdot (c_2)_{\overline{v}}$, f should also contain $v \cdot (c_2)_{\overline{v}}$. Therefore, f contains $(c_2)_{\overline{v}}$, and the literal $\overline{v}$ in c_2 can be removed without affecting the functionality of F. However, F should be a prime, so $(F\backslash\{c_2\}) \cup \{(c_2)_{\overline{v}}\} \neq f$, which leads to a contradiction.
  (2) If f is negative unate in v:
  By definition, $f_v \subseteq f_{\overline{v}}$. Since f contain $c_1 = v\cdot c_1 = v\cdot (c_1)_v$, f should also contain $\overline{v} \cdot (c_1)_v$. Therefore, f contains $(c_1)_v$, and the literal v in c_1 can be removed without affecting the functionality of F. However, F should be a prime, so $(F\backslash\{c_1\}) \cup \{(c_1)_v\} \neq f$, which leads to a contradiction.
Therefore, F must be a unate cover.

(b) False. An irredundant prime cover of a unate function must be a minimum sum-of-products expression. We can prove this by contradiction. Assume that an irredundant prime cover of a unate function is not a minimum sum-of-products expressions.

Lemma 1. *For each variable x, only one of its literal (x or x') exists in all primes.*

Proof. We can prove this by contradiction. By definition, if a function f is unate, either $f_x \subseteq f_{x'}$ or $f_{x'} \subseteq f_x$.
Assume $x \in c_1$ and $x' \in c_2$ for some variable x and some prime c_1, c_2. There are 3 cases:
(a) $c_1 \setminus x \subseteq c_2 \setminus x'$: Then $c_1 \setminus x$ is also an implicant, c_1 is not a prime.
(b) $c_2 \setminus x' \subseteq c_1 \setminus x$: Then $c_2 \setminus x'$ is also an implicant, c_2 is not a prime.
(c) None of the above case: Then neither $f_x \subseteq f_{x'}$ nor $f_{x'} \subseteq f_x$. f is not a unate function.
Since there is contradiction in all cases, the assumption does not hold. There is no variable with both of its literal exists in some primes. Q.E.D.

Lemma 2. *All primes in a unate function must be essential primes*

Proof. Let c be a prime in an unate function f. For each variable x that is not in c, if x is in c' for some other prime c', we add the opposite literal to c. By lemma 1, we can always find such literal in a unate function. The resulting cube is not contained by any other primes. Therefore, c is an essential prime. Q.E.D.

A prime cover must include at least all the essential primes. By **??**, all the primes are essential primes. Therefore, An (irredundant) prime cover must be minimum.

5 [Threshold and Unate Functions]

x_1	x_2	x_3	f	$\sum_{i=1}^{3} w_i x_i$
0	0	0	0	0
0	0	1	0	w_3
0	1	0	0	w_2
0	1	1	1	$w_2 + w_3$
1	0	0	0	w_1
1	0	1	1	$w_1 + w_3$
1	1	0	1	$w_1 + w_2$
1	1	1	1	$w_1 + w_2 + w_3$

(a) We want that $0 < T, w_1 < T, w_2 < T, w_3 < T$ and $w_1 + w_2 \geq T, w_1 + w_3 \geq T, w_2 + w_3 \geq T, w_1 + w_2 + w_3 \geq T$.
And it is easy to find an arguement for threshould function $w_1 = w_2 = w_3 = 1$ and $T = 2$ to build the Boolean function f.

(b) Without loss of generality, let the threshold function f be defined on variables $\{x_1, \ldots, x_n\}$, where $w_i \geq 0$ for $1 \leq i \leq m$, and $w_i < 0$ for $m < i \leq n$. Let $s(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_i$. Then $f(x_1, \ldots, x_n) = 1$ if and only if $s(x_1, \ldots, x_n) \geq T$.

Consider an arbitrary input assignment $(a_1, \ldots, a_n)$. For any $1 \leq i \leq m$, we have

$$\begin{aligned}
&f(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots a_n) = 1 \\
\Rightarrow &s(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots a_n) \geq T \\
\Rightarrow &s(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots a_n) = s(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots a_n) + w_i \geq T \ \ (\because w_i \geq 0) \\
\Rightarrow &f(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots a_n) = 1.
\end{aligned}$$

Since the above relation holds for any input assignment, we can conclude

$$f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots x_n) \rightarrow f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots x_n).$$

Therefore, $f_{\overline{x_i}} \subseteq f_{x_i}$, indicating that f is positive unate in x_i $\forall 1 \leq i \leq m$. Similarly, for any $m < i \leq n$, we have

$$\begin{aligned}
&f(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots a_n) = 1 \\
\Rightarrow &s(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots a_n) \geq T \\
\Rightarrow &s(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots a_n) = s(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots a_n) - w_i \geq T \ \ (\because w_i < 0) \\
\Rightarrow &f(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots a_n) = 1.
\end{aligned}$$

Since the above relation holds for any input assignment, we can conclude

$$f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots x_n) \rightarrow f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots x_n).$$

Therefore, $f_{x_i} \subseteq f_{\overline{x_i}}$, indicating that f is negative unate in x_i $\forall m < i \leq n$. Since f is unate in every variable, f must be a unate function.

(c) Consider the function $f = x_1x_2 + x_3x_4$ is a unate function. We then show that f is not a threshold function.

Suppose there exsits some threshold function $g = f$. That is

$$g(x_1, x_2, x_3, x_4) = \begin{cases} 1, \text{ if} \sum_{i=1}^{4} w_i x_i \geq T \\ 0, \text{ otherwise} \end{cases}$$

The weights of g must satisfy the following constraints

$$\begin{aligned}
g(1,1,0,0) = 1 \quad & w_1 + w_2 \geq T \\
g(0,0,1,1) = 1 \quad & w_3 + w_4 \geq T \\
g(0,1,0,1) = 0 \quad & w_2 + w_4 < T \\
g(1,0,1,0) = 0 \quad & w_1 + w_3 < T
\end{aligned}$$

There can not exist some w_i which satisfy the above constraints. Therefore, there does not exist such a threshold function g that equivalent to the unate function f.

6 [Unate Recurisve Paradigm: Prime Generation]

The derivation are shown in fig. 1 and fig. 2. When merging primes, we need to check all the following 3 types of cubes:

1. xq, where q is a prime of f_x and $q \not\subset f_{x'}$
2. $x'r$, where r is a prime of $f_{x'}$ and $r \not\subset f_x$
3. qr, where q is a prime of f_x and r is a prime of $f_{x'}$

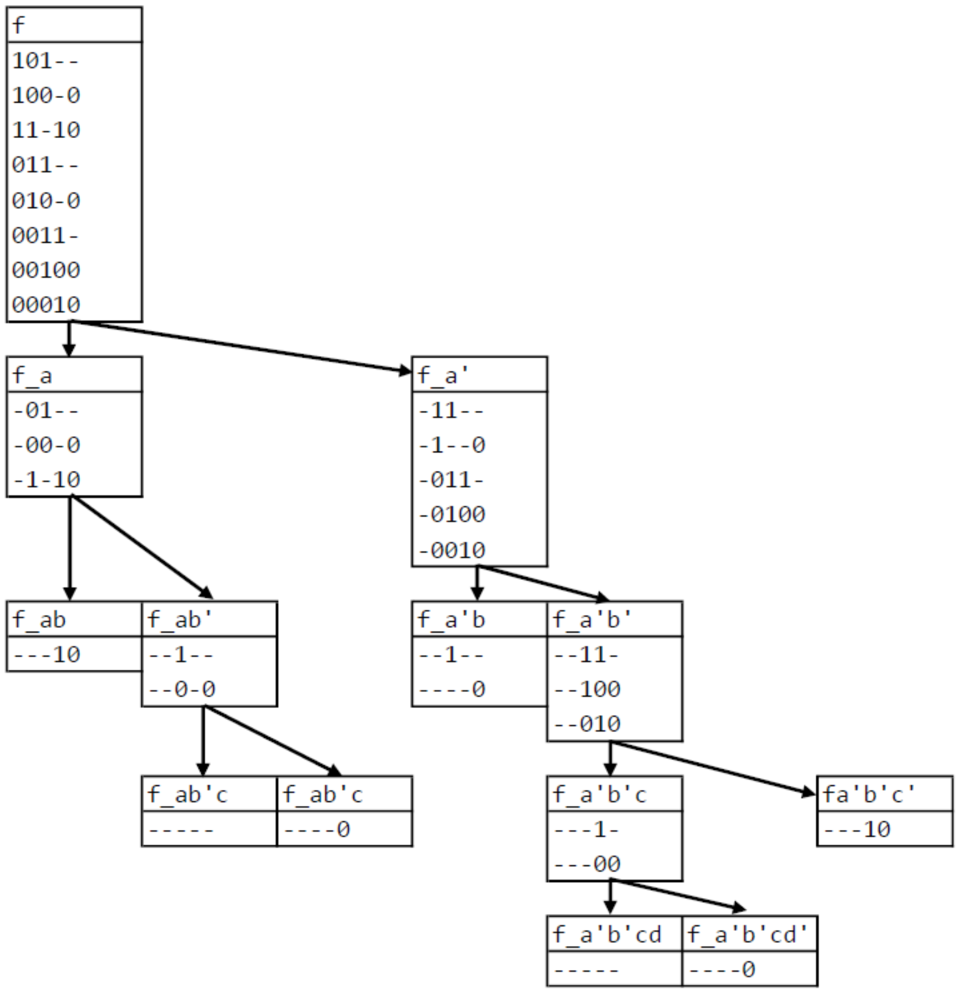

Fig. 1. Cofactoring function f to unsate leaves.

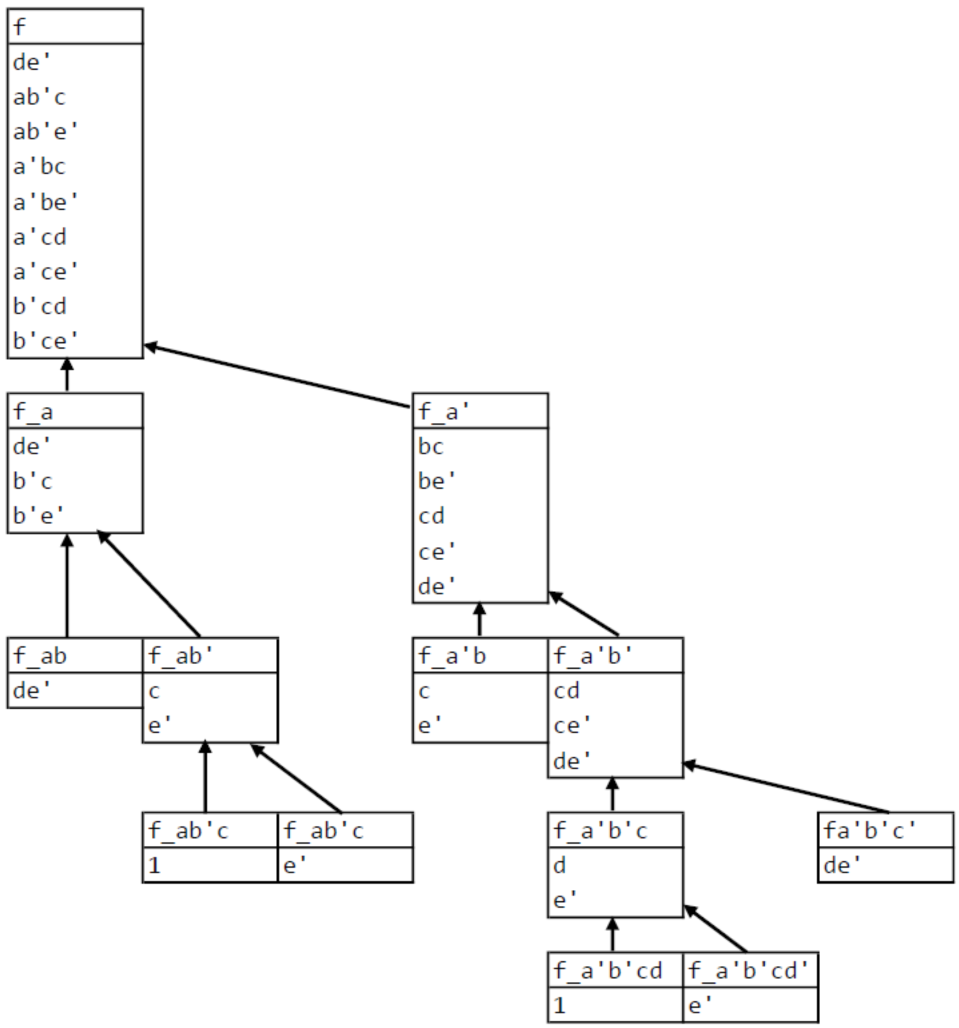

Fig. 2. Merging primes from unate leaves back to f to generate all primes of f.

7 [Quine-McCluskey]

(a) The derivation of all prime implicants for the multi-output cover are shown in fig. 3.

abcd	fg		abcd	fg		abcd	fg	
0001	-0	v	0-01	10	P1	01--	01	P6
0010	10	v	-010	10	P2	-1-0	01	P7
0100	11	v	010-	11	P3	1--0	01	P8
1000	01	v	01-0	01	v			
			-100	01	v	-11-	01	P9
0101	11	v	10-0	01	v			
0110	01	v	1-00	01	v			
1010	11	v						
1100	01	v	01-1	01	v			
			-101	10	P4			
0111	01	v	011-	01	v			
1101	-0	v	-110	01	v			
1110	1-	v	1-10	11	P5			
			11-0	01	v			
1111	01	v						
			-111	01	v			
			111-	01	v			

Fig. 3. Prime generation by pairwise minterm mergin for problem 7(a).

(b) The Boolean matrix for column covering are shown in fig. 4.

abcd fg	P1	P2	P3	P4	P5	P6	P7	P8	P9
0010 10		1							
0100 10			1						
0101 10	1		1	1					
1010 10		1			1				
1110 10					1				
0100 01			1			1	1		
0101 01			1			1			
0110 01						1	1		1
0111 01						1			1
1000 01								1	
1010 01					1			1	
1100 01							1	1	
1111 01									1

Fig. 4. The Boolean matrix for problem 7(b).

(c) As shown in fig. 4, P_2, P_3, P_5, P_8, P_9 are essential primes. After removing the rows covered by these essential primes, the remaing matrix is empty. There is no cyclic core.

(d) The minimum column covering is P_2, P_3, P_5, P_8, P_9. The minimum multi-output cover is

$$F = b'cd' + a'bc' + acd'$$
$$G = a'bc' + ad' + bc$$