Homework 4

Due: 13:10, 12/05, 2024 (in class)

Homework Policy: (READ BEFORE YOU START TO WORK)

- Copying from other students' solution is not allowed. If caught, all involved students get 0 point on that particular homework. Caught twice, you will be asked to drop the course.
- Collaboration is welcome. You can work together with **at most one partner** on the homework problems which you find difficult. However, you should write down your own solution, not just copying from your partner's.
- Your partner should be the same for the entire homework.
- Put your collaborator's name beside the problems that you collaborate on.
- When citing known results from the assigned references, be as clear as possible.

1. (Channel coding with input-output cost constraint) [14]

In this problem we explore channel coding with input and output cost constraint.

a) Consider a DMC $\left(\mathcal{X}, \mathsf{P}_{Y|X}, \mathcal{Y}\right)$. Let $b: \mathcal{X} \times \mathcal{Y} \to [0, \infty)$ be an input-output cost function. Suppose the channel coding has to satisfy the following average cost constraint: for each codeword x^n,

$$\tfrac{1}{n}\textstyle\sum_{i=1}^n \mathsf{E}_{Y_i}[b(x_i, Y_i)] \le \mathsf{B}.$$

Note that Y_i follows distribution $\mathsf{P}_{Y|X}(\cdot|x_i)$.

Argue that the problem is equivalent to another channel coding problem with a properly defined input-only cost function. Show that the capacity-cost function is

$$\mathrm{C}(\mathsf{B}) = \max_{\mathsf{P}_X : \mathsf{E}_{\mathsf{P}_X \mathsf{P}_{Y|X}}[b(X,Y)] \le \mathsf{B}} \mathrm{I}(X;Y). \qquad [8]$$

Hint: Consider an input-only cost function $\tilde{b}(x) := \mathsf{E}[b(x, Y)]$ *and call Theorem 1 of L4.*

b) Using discretization techniques, the above DMC result can be extended to continuous memoryless channels. With the extension (no need to prove it here), let us consider an AWGN channel with *average output power constraint*

$$\tfrac{1}{n}\textstyle\sum_{i=1}^n \mathsf{E}\left[Y_i^2\right] \le \mathsf{B}.$$

where $Y = X + Z$, $Z \perp\!\!\!\perp X$, and $Z \sim \mathrm{N}(0, \sigma^2)$.

Evaluate the channel capacity $\mathrm{C}(\mathsf{B})$. [6]

2. (Differential entropy) [10]

a) Consider a Laplace random variable $X \sim \mathsf{Lap}(\mu, b)$, that is, the probability density function of X is $\mathsf{f}_X(x) = \frac{1}{2b} e^{-\frac{|x-\mu|}{b}}$, $x \in \mathbb{R}$. Compute its differential entropy $\mathrm{h}(X)$. [4]

b) Consider a problem of maximizing differential entropy $\mathrm{h}(X)$ subject to the constraint that $\mathsf{E}[|X|] \le \mathsf{B}$. Find the maximum differential entropy and show that a zero-mean Laplace distributed X attains the maximum value. [6]

3. (Rate distortion function lower bound) [12]

Consider a discrete memoryless source $\{S_i\}$ taking values in a finite alphabet $\mathcal{S}$ with Hamming distortion measure and the reconstruction alphabet $\hat{\mathcal{S}} = \mathcal{S}$.

a) Find $\mathsf{D}_{\max}$. [4]

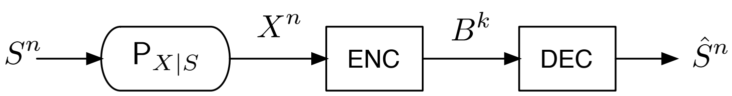
b) Prove that

$$\mathrm{R}(\mathsf{D}) \ge \mathrm{H}(S) - \mathsf{D}\log(|\mathcal{S}| - 1) - \mathsf{H_b}(\mathsf{D}),$$

for $0 \le \mathsf{D} \le \mathsf{D}_{\max}$. When does it hold with equality? [8]

4. (Compression of a remote source) [14]

In standard source coding problems, the encoder is able to observe the source directly. However, in many practical scenarios, the source is *remote* and corrupted by some noise when the encoder observes it. In this problem, we aim to establish the lossy source coding theorem for such *remote* sources.



Consider a DMS $S_i \overset{\text{i.i.d.}}{\sim} \mathsf{P}_S$ for all i, and $\{X_i\}$ be the *corrupted* DMS obtained by passing S_i through a DMC $\mathsf{P}_{X|S}$. The source encoder encodes the noisy observation X^n into bit sequences, and the source decoder would like to produce a reconstruction $\hat{S}^n$ so that the distortion, measured by $d(s, \hat{s})$, is not greater than D.

a) Define

$$\tilde{d}(x, \hat{s}) := \mathsf{E}_{S \sim \mathsf{P}_{S|X}(\cdot|x)}\left[d(S, \hat{s}) \,|\, X = x\right]$$

and consider an alternative lossy source coding problem with X being the source to be reconstructed (instead of S) and $\tilde{d}(\cdot,\cdot)$ being the distortion function. Argue that this lossy source coding problem and the remote lossy source coding problem are **equivalent** in the sense that any (R,D) pair that is achievable in one problem is also achievable in the other and vice versa. [6]

b) Based on Part a), invoke the lossy source coding theorem in our lecture to characterize the rate distortion function of the remote lossy source coding problem. Your characterization should be expressed as a minimization problem of some mutual information term involving X and $\hat{S}$ over $\mathsf{P}_{\hat{S}|X}$ subject to an expected distortion constraint on $d(S,\hat{S})$. [4]

Specify $\mathsf{D}_{\min}$ and $\mathsf{D}_{\max}$. [4]