Compromise XPs

Olivier Cailloux[1], Ayça Ebru Giritligil[2], Ipek Ozkal Sanver[2], and Remzi Sanver[1]

[1]Université Paris-Dauphine, PSL Research University, CNRS, LAMSADE, 75016 PARIS, FRANCE
[2]Bilgi, ...

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1. Introduction

Goal: investigate empirically the conditions in which people adopt a compromise notion based on minimal inequality of losses, as opposed to more classical compromise notions, in two-persons situations.

2. Basic definitions and notation

We have a non empty set of alternatives $\mathscr{A}$ with $\#\mathscr{A} = m$ and a set $N = \{1, 2\}$ of two individuals. Given $i \in N$, let $\bar{i}$ denote the other individual (and conversely, given $\bar{i} \in N$, let i denote the other individual). The possible profiles are $\mathcal{L}(\mathscr{A})^{\{1,2\}}$. A Social Choice Rule is a function $f : \mathcal{L}(\mathscr{A})^{\{1,2\}} \to \mathscr{P}^*(\mathscr{A})$.

Given $P = \{(i, \succ_i), (\bar{i}, \succ_{\bar{i}})\} \in \mathcal{L}(\mathscr{A})^{\{1,2\}}$ and $x \in \mathscr{A}$, let $\lambda_P(x) : N \to [\![0, m-1]\!]$ associate to each individual $i \in N$ her loss at x, defined as the number of alternatives that are strictly preferred to x in her preference $\succ_i$, namely, $\lambda_P(x)_i = \#\{y \in \mathscr{A} \mid y \succ_i x\}$, where $\{y \in \mathscr{A} \mid y \succ_i x\}$ designate the upper contour set of x. Let $\min \lambda_P(x) = \min_{i \in N} \lambda_P(x)_i$, $\max \lambda_P(x) = \max_{i \in N} \lambda_P(x)_i$ and $\sum \lambda_P(x) = \sum_{i \in N} \lambda_P(x)_i$ designate the minimal value, maximal value and sum of the losses of x. Given a loss vector $l \in [\![0, m-1]\!]^{\{1,2\}}$, define $d(l) = |l_1 - l_2| = \max l - \min l$.

Given $P \in \mathcal{L}(\mathscr{A})^{\{1,2\}}$, the set of Pareto-optimal alternatives is $PO(P) = \{x \in \mathscr{A} \mid \forall y \in \mathscr{A}, \exists i \in N \mid \lambda_P(x) \leq \lambda_P(y)\}$.

Note that $\arg\min_{PO(P)}(d \circ \lambda_P) = \{x \in PO(P) \mid (d \circ \lambda_P)(x) = \min_{y \in PO(P)}(d \circ \lambda_P)(y)\}$ is the least unequal alternatives among the Pareto-optimal ones, according to the measure $d \circ \lambda_P$. Also note that $\min_{\mathscr{A}} \max \lambda_P$ is the threshold that is reached when going from the first (best) rank downwards until some alternative is unanimously considered worth that rank or better.

We study the "min spread" SCR $MS(P) = \arg\min_{PO(P)}(d \circ \lambda_P)$, which picks the least unequal alternatives among the Pareto-optimal ones, according to the differences of losses; the Fallback-Bargaining SCR [Brams and Kilgour, 2001] $FB(P) = \arg\min_{\mathscr{A}} \max \lambda_P$; and the Borda SCR $B(P) = \arg\min \sum \lambda_P$ which selects the alternatives that minimize the sum of losses.

[OC: We should have a look at doi:10.1287/mnsc.2021.4025.]

3. Filtering and classifying profiles

The results that this section refers to are proven in Appendix A.

3.1. Filtering profiles

We are interested in selecting profiles on which FB and MS disagree. As a minimal requirement, we thus request $FB(P) \cap MS(P) = \emptyset$. This implies $\#FB(P) = \#MS(P) = 1$ (Theorem 3).

We also require P to contain no Pareto-dominated alternative with a smaller spread than $\min_{PO(P)}(d \circ \lambda_P)$. This constraint stems from a desire to focus on the subject of our study, and not perturb the subject by letting alternatives that do not distinguish our rules of interest appear possibly attractive. Studying the attractivity of Pareto-dominated alternatives is an interesting goal but requires its own design.

Let $\mathscr{G} = \{P \in \mathcal{L}(\mathscr{A})^{\{1,2\}} \mid FB(P) \cap MS(P) = \emptyset \wedge \forall a \in \mathscr{A} : (d \circ \lambda_P)(a) \geq \min_{PO(P)}(d \circ \lambda_P)\}$ designate the satisfactory profiles. Given $P \in \mathscr{G}$ (thus with $\#FB(P) = \#MS(P) = 1$), we write $\{x\} = FB(P)$ and $\{y\} = MS(P)$.

During a run with a given subject, we want the spread of the Fallback-Bargaining winner, $d(\lambda_P(x))$, to reach high enough values compared to $d(\lambda_P(y))$, in order to ensure a sufficient contrast with min spread. This requires a big enough value for m as $\delta = d(\lambda_P(x)) - d(\lambda_P(y)) \leq \frac{m-5}{2}$ (Corollary 2). We also want m to be small enough for cognitive simplicity. As a compromise, we opt for $m = 13$, which permits to reach $\delta = 4$.

3.2. Classifying profiles

To each profile $P \in \mathscr{G}$ can be associated these properties;:

1. the losses of x and y, $\lambda_P(x)$ and $\lambda_P(y)$;
2. whether $FB(P) = B(P) = \{x\}$, $FB(P) \subset B(P)$, or $FB(P) \cap B(P) = \emptyset$.

Accordingly, we divide $\mathscr{G}$ into classes of profiles and use a representative profile for each such class.

Let $T = \{t \in [\![0, m-1]\!]^4 \mid t_1 < t_2 < t_3 < t_4 \wedge t_1 + t_4 + 1 \leq t_2 + t_3 \wedge t_3 + t_4 \leq m\}$ denote the set of tuples of loss values that satisfy the specified constraints. Given $t \in T$ and $i \in N$, define the class $C_{t,i} = \{P \in \mathscr{G} \mid \lambda_P(x) = \{(i, t_1), (\bar{\imath}, t_3)\} \wedge \lambda_P(y) = \{(i, t_4), (\bar{\imath}, t_2)\}\}$ as the set of profiles where x and y have the specified losses. Define the class $C_t = \bigcup_{i \in N} C_{t,i}$ as the set of profiles where x has the loss values (t_1, t_3) and y has the loss values (t_2, t_4) associated both with individuals $(i, \bar{\imath})$ or both with $(\bar{\imath}, i)$. We consider the classes $\{C_t, t \in T\}$. This set of classes forms a complete cover of $\mathscr{G}$ (Theorem 7).

Within $\mathscr{G}$, the Borda winner always differs from the Min spread winner (Theorem 9); and here are the possible relationships between $B(P)$ and $FB(P) = \{x\}$ (Remark 1):

- if $t_1 = \min \lambda_P(x) = 0$, no profile from $\mathscr{G}$ distinguish the Borda winner and the FB winner, formally, $\forall t \in T, P \in C_t : t_1 = 0 \Rightarrow B(P) = FB(P)$;
- if $t_1 = \min \lambda_P(x) = 1$, $x \in B(P)$, with both $FB(P) = B(P)$ and $FB(P) \subset B(P)$ possible;
- if $t_1 = \min \lambda_P(x) \geq 2$, both $FB(P) = B(P)$ and $FB(P) \cap B(P) = \emptyset$ are possible.

3.3. Enumerating the classes of profiles

Let us now enumerate exhaustively the classes of profiles in $\mathscr{G}$. When $\max \lambda_P(y) = 8$, only the class with $\min \lambda_P(x) = 0$, $\min \lambda_P(y) = 4$, $\max \lambda_P(x) = 5$ exists in $\mathscr{G}$. When $\max \lambda_P(y) \neq 8$, the following constraints exhaust the possibilities (Corollaries 4 and 5).

- $4 \leq \max \lambda_P(y) \leq 7$;
- $2 \leq \min \lambda_P(y) \leq \max \lambda_P(y) - 2$;
- $\max\{\min \lambda_P(y) + 1, \max \lambda_P(y) - \min \lambda_P(y) + 1\} \leq \max \lambda_P(x) \leq \max \lambda_P(y) - 1$;

$\max \lambda_P(y)$	$\min \lambda_P(y)$	$\max \lambda_P(x)$ range	$\min \lambda_P(x)$ range
8	4	$\{5\}$	$\{0\}$
7	5	$\{6\}$	$[0, 3]$
	4	$[5, 6]$	$[0, \max \lambda_P(x) - 4]$
	3	$[5, 6]$	$[0, \max \lambda_P(x) - 5]$
	2	$\{6\}$	$\{0\}$
6	4	$\{5\}$	$[0, 2]$
	3	$[4, 5]$	$[0, \max \lambda_P(x) - 4]$
	2	$\{5\}$	$\{0\}$
5	3	$\{4\}$	$[0, 1]$
	2	$\{4\}$	$\{0\}$
4	2	$\{3\}$	$\{0\}$

Table 1: Possible ranges for $m = 13$.

- $0 \leq \min \lambda_P(x) \leq \max \lambda_P(x) - (\max \lambda_P(y) - \min \lambda_P(y) + 1)$.

Table 1 lists the possible ranges for these variables, and Table 2 lists all possible values.

4. Protocol

4.1. Questions

Here are some questions we want to investigate. Ceteris Paribus...

1. Does high δ favor MS?
   - 0324 (class 15, Example 196) VS 0657 (class 1, Example 182)
2. Does it matter for the FB winner to be top-ranked by a voter?
   - 0324 (class 15, Example 196) VS 1435 with $\{x\} \subset B(P)$ (class 14, Example 213)
3. Does the best position of the FB winner matter?
   - 0324 (class 15, Example 196) VS 3657 with $B(P) = \{x\}$ (class 12, Example 193)
4. Does $FB(P) = B(P)$ favor FB?

id	$\min \lambda_P(x)$	$\min \lambda_P(y)$	$\max \lambda_P(x)$	$\max \lambda_P(y)$	$d(\lambda_P(x))$	$d(\lambda_P(y))$	δ
1	0	5	6	7	6	2	4
2	1	5	6	7	5	2	3
3	0	4	5	6	5	2	3
4	0	4	6	7	6	3	3
5	2	5	6	7	4	2	2
6	1	4	5	6	4	2	2
7	0	3	4	5	4	2	2
8	1	4	6	7	5	3	2
9	0	3	5	6	5	3	2
10	0	3	6	7	6	4	2
11	0	4	5	7	5	3	2
12	3	5	6	7	3	2	1
13	2	4	5	6	3	2	1
14	1	3	4	5	3	2	1
15	0	2	3	4	3	2	1
16	2	4	6	7	4	3	1
17	1	3	5	6	4	3	1
18	0	2	4	5	4	3	1
19	1	3	6	7	5	4	1
20	0	2	5	6	5	4	1
21	0	2	6	7	6	5	1
22	1	4	5	7	4	3	1
23	0	3	4	6	4	3	1
24	0	3	5	7	5	4	1
25	0	4	5	8	5	4	1

Table 2: Possible values for $m = 13$.

- 3657 with $B(P) = \{x\}$ (class 12, Example 193) VS 3657 with $B(P) \cap FB(P) = \emptyset$ (class 12, Example 211)

5. Does $d(\lambda_P(y))$ matter, fixing δ?
   - 0627 (class 21, Example 202) VS 0324 (class 15, Example 196)

6. Do people select Borda rather than FB or MS?
   - 3657 with $B(P) \cap FB(P) = \emptyset$ (class 12, Example 211), 1435 with $\{x\} \subset B(P)$ (class 14, Example 213)

7. Does showing profiles in favor of MS first favor MS choices in other profiles? This will be tested ex-post, the order of presentation being equiprobable across every permutation. How much a profile is MS-favoring could be determined using these criteria, lexicographically.
   - high δ
   - small $\delta^B = \sum \lambda_P(y) - \sum \lambda_P(x)$ (the difference between the Borda score of x and y, or equivalently, the difference between the avg loss of x and y)
   - small $d(\lambda_P(y))$
   - high $\min \lambda_P(x)$

8. Does using the word "compromise" in an abstract scenario trigger a different behavior than using a concrete scenario with no such wording?

As a consequence, we pick the following set of profiles:

1. 0324 (class 15, Example 196),
2. 0657 (class 1, Example 182),
3. 1435 with $\{x\} \subset B(P)$ (class 14, Example 213),
4. 3657 with $B(P) = \{x\}$ (class 12, Example 193),
5. 3657 with $B(P) \cap FB(P) = \emptyset$ (class 12, Example 211),
6. 0627 (class 21, Example 202).

4.2. A proposal for a protocol

We run two settings (each subject is exposed to exactly one of these settings). One is an abstract setting and uses the word uzlaşı for compromise. At end of the experiment we could ask the subject how she defines uzlaşı (free form text). Another setting is a concrete setting that does not use any word related to compromise (the understanding that we need a compromise comes from the description of the situation). Choosing which DVD should be watched (by two friends), or which movie they should go to, or pizza to eat, seems suitable. [OC: We also mentioned: choosing a restaurant for your twins (but this involves children so highlights paternalism), or choosing a honey moon package (but this may introduce gender specific consideration).]

We do not pinpoint x or y: subjects can choose any alternative. (This is still under discussion; the advantage is that it permits to check that they agree with our notion of compromise and are paying close attention, and it lets them pick Borda winners if they want to, so I view it as a more ambitious version which allows to claim that we capture the way people conceive of a compromise; the problem is that we can only do it with sufficient sample as it will increase the noise.)

Pick the classes of profiles indicated above and repeat the fourth one at the end of the sequence, to check for consistency. Pick a random permutation to order the profiles, and pick a random permutation to rename alternatives, for each choice we present.

We ask for how they choose, their rationale, for each choice we present. Subjects answer using free text and must provide some text before continuing.

4.3. An example run

Here is a bunch of six profiles that examplifies a run with a given subject.

$$\begin{array}{ccccccccccccc} b & a & g & m & l & d & j & c & k & h & i & e & f \\ j & d & l & b & c & k & h & i & e & f & a & g & m \end{array}$$

$$\begin{array}{ccccccccccccc} m & h & f & b & d & l & g & a & j & e & c & k & i \\ i & k & c & e & j & a & m & h & f & b & d & l & g \end{array}$$

$$\begin{array}{ccccccccccccc} d & g & l & k & j & f & h & i & a & c & b & e & m \\ a & i & h & f & g & d & c & b & e & m & l & k & j \end{array}$$

$$\begin{array}{ccccccccccccc} a & h & b & e & g & i & d & j & m & l & c & f & k \\ k & f & c & l & m & j & e & g & i & d & b & h & a \end{array}$$

$$\begin{array}{ccccccccccccc} b & j & k & c & d & l & e & i & f & h & a & m & g \\ g & m & a & h & f & i & c & b & j & k & d & l & e \end{array}$$

$$\begin{array}{ccccccccccccc} h & l & f & m & k & c & g & a & d & i & j & e & b \\ i & d & a & j & e & b & h & l & f & m & k & c & g \end{array}$$

4.4. Shuffled profiles

Here is a list of several shuffling of each of our chosen profiles.

Example 1 $(\boldsymbol{\lambda_P(x)} = \{\mathbf{0,3}\}; \boldsymbol{\lambda_P(y)} = \{\mathbf{2,4}\};$ **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & \boxed{e} & f & g & h & i & j & k & l & m \\ g & f & \boxed{e} & \boldsymbol{a} & h & i & j & k & l & m & b & c & d \end{array}$$

$\square$

Example 2 $(\boldsymbol{\lambda_P(x)} = \{\mathbf{0,3}\}; \boldsymbol{\lambda_P(y)} = \{\mathbf{2,4}\};$ **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{d} & b & g & i & \boxed{c} & h & j & m & k & a & f & l & e \\ j & h & \boxed{c} & \boldsymbol{d} & m & k & a & f & l & e & b & g & i \end{array}$$

$\square$

Example 3 $(\boldsymbol{\lambda_P(x)} = \{\mathbf{0,3}\}; \boldsymbol{\lambda_P(y)} = \{\mathbf{2,4}\};$ **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{h} & l & g & m & \boxed{j} & c & e & b & a & i & k & d & f \\ e & c & \boxed{j} & \boldsymbol{h} & b & a & i & k & d & f & l & g & m \end{array}$$

$\square$

Example 4 $(\boldsymbol{\lambda_P(x)} = \{\mathbf{0,3}\}; \boldsymbol{\lambda_P(y)} = \{\mathbf{2,4}\};$ **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{j} & f & b & m & \boxed{a} & c & d & h & g & k & l & e & i \\ d & c & \boxed{a} & \boldsymbol{j} & h & g & k & l & e & i & f & b & m \end{array}$$

$\square$

Example 5 $(\boldsymbol{\lambda_P(x)} = \{\mathbf{0,3}\}; \boldsymbol{\lambda_P(y)} = \{\mathbf{2,4}\};$ **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{m} & d & j & a & \boxed{k} & g & b & e & i & l & c & h & f \\ b & g & \boxed{k} & \boldsymbol{m} & e & i & l & c & h & f & d & j & a \end{array}$$

$\square$

Example 6 $(\boldsymbol{\lambda_P(x)} = \{\mathbf{0,3}\}; \boldsymbol{\lambda_P(y)} = \{\mathbf{2,4}\};$ **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & m & d & \boxed{f} & e & i & h & c & g & k & j & l \\ i & e & \boxed{f} & \boldsymbol{a} & h & c & g & k & j & l & b & m & d \end{array}$$

$\square$

Example 7 ($\boldsymbol{\lambda_P(x) = \{0, 3\}}$; $\boldsymbol{\lambda_P(y) = \{2, 4\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{g} & j & h & i & \boxed{m} & d & b & l & k & a & c & e & f \\ b & d & \boxed{m} & \boldsymbol{g} & l & k & a & c & e & f & j & h & i \end{array} \qquad \square$$

Example 8 ($\boldsymbol{\lambda_P(x) = \{0, 3\}}$; $\boldsymbol{\lambda_P(y) = \{2, 4\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{l} & m & h & d & \boxed{i} & f & b & g & k & j & a & c & e \\ b & f & \boxed{i} & \boldsymbol{l} & g & k & j & a & c & e & m & h & d \end{array} \qquad \square$$

Example 9 ($\boldsymbol{\lambda_P(x) = \{0, 3\}}$; $\boldsymbol{\lambda_P(y) = \{2, 4\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{e} & d & b & i & \boxed{l} & g & k & j & c & f & a & h & m \\ k & g & \boxed{l} & \boldsymbol{e} & j & c & f & a & h & m & d & b & i \end{array} \qquad \square$$

Example 10 ($\boldsymbol{\lambda_P(x) = \{0, 3\}}$; $\boldsymbol{\lambda_P(y) = \{2, 4\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{h} & i & e & f & \boxed{j} & d & l & k & m & b & g & a & c \\ l & d & \boxed{j} & \boldsymbol{h} & k & m & b & g & a & c & i & e & f \end{array} \qquad \square$$

Example 11 ($\boldsymbol{\lambda_P(x) = \{0, 3\}}$; $\boldsymbol{\lambda_P(y) = \{2, 4\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{k} & m & f & j & \boxed{h} & e & g & l & c & i & d & b & a \\ g & e & \boxed{h} & \boldsymbol{k} & l & c & i & d & b & a & m & f & j \end{array} \qquad \square$$

Example 12 ($\boldsymbol{\lambda_P(x) = \{0, 3\}}$; $\boldsymbol{\lambda_P(y) = \{2, 4\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{g} & l & e & m & \boxed{a} & h & d & j & c & f & i & b & k \\ d & h & \boxed{a} & \boldsymbol{g} & j & c & f & i & b & k & l & e & m \end{array} \qquad \square$$

Example 13 ($\boldsymbol{\lambda_P(x) = \{0, 3\}}$; $\boldsymbol{\lambda_P(y) = \{2, 4\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{l} & f & g & i & \boxed{c} & e & k & j & h & b & a & d & m \\ k & e & \boxed{c} & \boldsymbol{l} & j & h & b & a & d & m & f & g & i \end{array} \qquad \square$$

Example 14 ($\boldsymbol{\lambda_P(x) = \{0, 3\}}$; $\boldsymbol{\lambda_P(y) = \{2, 4\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{i} & b & k & e & \boxed{j} & h & f & g & m & l & c & a & d \\ f & h & \boxed{j} & \boldsymbol{i} & g & m & l & c & a & d & b & k & e \end{array} \qquad \square$$

Example 15 ($\boldsymbol{\lambda_P(x) = \{0,3\}}$; $\boldsymbol{\lambda_P(y) = \{2,4\}}$; **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{b} & m & d & g & \boxed{e} & h & c & a & l & f & i & k & j \\ c & h & \boxed{e} & \boldsymbol{b} & a & l & f & i & k & j & m & d & g \end{array} \qquad \square$$

Example 16 ($\boldsymbol{\lambda_P(x) = \{0,3\}}$; $\boldsymbol{\lambda_P(y) = \{2,4\}}$; **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{g} & j & m & e & \boxed{b} & h & k & l & f & c & i & d & a \\ k & h & \boxed{b} & \boldsymbol{g} & l & f & c & i & d & a & j & m & e \end{array} \qquad \square$$

Example 17 ($\boldsymbol{\lambda_P(x) = \{0,3\}}$; $\boldsymbol{\lambda_P(y) = \{2,4\}}$; **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{l} & k & a & c & \boxed{m} & g & h & i & b & d & e & f & j \\ h & g & \boxed{m} & \boldsymbol{l} & i & b & d & e & f & j & k & a & c \end{array} \qquad \square$$

Example 18 ($\boldsymbol{\lambda_P(x) = \{0,3\}}$; $\boldsymbol{\lambda_P(y) = \{2,4\}}$; **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{g} & h & f & m & \boxed{l} & d & i & b & c & e & a & j & k \\ i & d & \boxed{l} & \boldsymbol{g} & b & c & e & a & j & k & h & f & m \end{array} \qquad \square$$

Example 19 ($\boldsymbol{\lambda_P(x) = \{0,3\}}$; $\boldsymbol{\lambda_P(y) = \{2,4\}}$; **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{c} & h & f & b & \boxed{i} & e & a & k & m & l & g & d & j \\ a & e & \boxed{i} & \boldsymbol{c} & k & m & l & g & d & j & h & f & b \end{array} \qquad \square$$

Example 20 ($\boldsymbol{\lambda_P(x) = \{0,3\}}$; $\boldsymbol{\lambda_P(y) = \{2,4\}}$; **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{k} & c & j & l & \boxed{e} & f & i & m & g & h & d & a & b \\ i & f & \boxed{e} & \boldsymbol{k} & m & g & h & d & a & b & c & j & l \end{array} \qquad \square$$

Example 21 ($\boldsymbol{\lambda_P(x) = \{0,3\}}$; $\boldsymbol{\lambda_P(y) = \{2,4\}}$; **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{f} & g & b & k & \boxed{l} & m & a & h & j & d & c & i & e \\ a & m & \boxed{l} & \boldsymbol{f} & h & j & d & c & i & e & g & b & k \end{array} \qquad \square$$

Example 22 ($\boldsymbol{\lambda_P(x) = \{0,3\}}$; $\boldsymbol{\lambda_P(y) = \{2,4\}}$; **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{d} & l & h & k & \boxed{e} & i & m & g & b & c & f & a & j \\ m & i & \boxed{e} & \boldsymbol{d} & g & b & c & f & a & j & l & h & k \end{array} \qquad \square$$

Example 23 ($\boldsymbol{\lambda_P(x)} = \{0,3\}$; $\boldsymbol{\lambda_P(y)} = \{2,4\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{e} & l & a & f & \boxed{h} & d & g & c & m & j & k & b & i \\ g & d & \boxed{h} & \boldsymbol{e} & c & m & j & k & b & i & l & a & f \end{array}$$

□

Example 24 ($\boldsymbol{\lambda_P(x)} = \{0,3\}$; $\boldsymbol{\lambda_P(y)} = \{2,4\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{i} & g & f & c & \boxed{m} & a & d & e & b & h & l & k & j \\ d & a & \boxed{m} & \boldsymbol{i} & e & b & h & l & k & j & g & f & c \end{array}$$

□

Example 25 ($\boldsymbol{\lambda_P(x)} = \{0,3\}$; $\boldsymbol{\lambda_P(y)} = \{2,4\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{h} & g & e & f & \boxed{m} & b & k & c & l & d & i & a & j \\ k & b & \boxed{m} & \boldsymbol{h} & c & l & d & i & a & j & g & e & f \end{array}$$

□

Example 26 ($\boldsymbol{\lambda_P(x)} = \{0,3\}$; $\boldsymbol{\lambda_P(y)} = \{2,4\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & i & h & e & \boxed{m} & f & b & c & g & j & l & k & d \\ b & f & \boxed{m} & \boldsymbol{a} & c & g & j & l & k & d & i & h & e \end{array}$$

□

Example 27 ($\boldsymbol{\lambda_P(x)} = \{0,3\}$; $\boldsymbol{\lambda_P(y)} = \{2,4\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{l} & c & e & b & \boxed{d} & h & m & i & f & j & g & a & k \\ m & h & \boxed{d} & \boldsymbol{l} & i & f & j & g & a & k & c & e & b \end{array}$$

□

Example 28 ($\boldsymbol{\lambda_P(x)} = \{0,3\}$; $\boldsymbol{\lambda_P(y)} = \{2,4\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{l} & f & j & d & \boxed{i} & g & h & m & b & k & c & a & e \\ h & g & \boxed{i} & \boldsymbol{l} & m & b & k & c & a & e & f & j & d \end{array}$$

□

Example 29 ($\boldsymbol{\lambda_P(x)} = \{0,3\}$; $\boldsymbol{\lambda_P(y)} = \{2,4\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{g} & a & i & m & \boxed{h} & b & j & k & f & e & l & d & c \\ j & b & \boxed{h} & \boldsymbol{g} & k & f & e & l & d & c & a & i & m \end{array}$$

□

Example 30 ($\boldsymbol{\lambda_P(x)} = \{0,3\}$; $\boldsymbol{\lambda_P(y)} = \{2,4\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{c} & k & f & j & \boxed{i} & m & l & a & h & e & d & g & b \\ l & m & \boxed{i} & \boldsymbol{c} & a & h & e & d & g & b & k & f & j \end{array}$$

□

Example 31 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; **B**)

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ m & l & k & j & i & \boxed{h} & \boldsymbol{a} & b & c & d & e & f & g \end{array}$$

□

Example 32 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; **B**)

$$\begin{array}{ccccccccccccc} \boldsymbol{j} & i & b & k & m & c & g & \boxed{f} & e & h & a & d & l \\ l & d & a & h & e & \boxed{f} & \boldsymbol{j} & i & b & k & m & c & g \end{array}$$

□

Example 33 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; **B**)

$$\begin{array}{ccccccccccccc} \boldsymbol{h} & i & f & m & b & e & d & \boxed{g} & a & c & j & l & k \\ k & l & j & c & a & \boxed{g} & \boldsymbol{h} & i & f & m & b & e & d \end{array}$$

□

Example 34 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; **B**)

$$\begin{array}{ccccccccccccc} \boldsymbol{h} & b & d & l & f & a & i & \boxed{g} & c & k & m & j & e \\ e & j & m & k & c & \boxed{g} & \boldsymbol{h} & b & d & l & f & a & i \end{array}$$

□

Example 35 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; **B**)

$$\begin{array}{ccccccccccccc} \boldsymbol{b} & a & d & g & h & f & e & \boxed{k} & m & i & c & l & j \\ j & l & c & i & m & \boxed{k} & \boldsymbol{b} & a & d & g & h & f & e \end{array}$$

□

Example 36 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; **B**)

$$\begin{array}{ccccccccccccc} \boldsymbol{d} & a & j & i & h & g & k & \boxed{m} & b & e & l & f & c \\ c & f & l & e & b & \boxed{m} & \boldsymbol{d} & a & j & i & h & g & k \end{array}$$

□

Example 37 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; **B**)

$$\begin{array}{ccccccccccccc} \boldsymbol{j} & h & g & c & a & b & m & \boxed{e} & i & d & f & k & l \\ l & k & f & d & i & \boxed{e} & \boldsymbol{j} & h & g & c & a & b & m \end{array}$$

□

Example 38 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; **B**)

$$\begin{array}{ccccccccccccc} \boldsymbol{m} & f & e & k & a & b & i & \boxed{g} & j & l & h & c & d \\ d & c & h & l & j & \boxed{g} & \boldsymbol{m} & f & e & k & a & b & i \end{array} \qquad \square$$

Example 39 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; **B**)

$$\begin{array}{ccccccccccccc} \boldsymbol{f} & d & j & l & k & h & b & \boxed{a} & i & m & c & g & e \\ e & g & c & m & i & \boxed{a} & \boldsymbol{f} & d & j & l & k & h & b \end{array} \qquad \square$$

Example 40 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; **B**)

$$\begin{array}{ccccccccccccc} \boldsymbol{i} & f & d & m & c & l & h & \boxed{g} & k & a & b & e & j \\ j & e & b & a & k & \boxed{g} & \boldsymbol{i} & f & d & m & c & l & h \end{array} \qquad \square$$

Example 41 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; **B**)

$$\begin{array}{ccccccccccccc} \boldsymbol{j} & a & b & i & d & k & h & \boxed{c} & e & m & l & g & f \\ f & g & l & m & e & \boxed{c} & \boldsymbol{j} & a & b & i & d & k & h \end{array} \qquad \square$$

Example 42 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; **B**)

$$\begin{array}{ccccccccccccc} \boldsymbol{g} & m & h & l & d & j & c & \boxed{k} & b & f & e & i & a \\ a & i & e & f & b & \boxed{k} & \boldsymbol{g} & m & h & l & d & j & c \end{array} \qquad \square$$

Example 43 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; **B**)

$$\begin{array}{ccccccccccccc} \boldsymbol{m} & j & a & g & b & l & e & \boxed{k} & i & h & f & d & c \\ c & d & f & h & i & \boxed{k} & \boldsymbol{m} & j & a & g & b & l & e \end{array} \qquad \square$$

Example 44 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; **B**)

$$\begin{array}{ccccccccccccc} \boldsymbol{g} & j & i & c & m & l & f & \boxed{a} & e & b & k & h & d \\ d & h & k & b & e & \boxed{a} & \boldsymbol{g} & j & i & c & m & l & f \end{array} \qquad \square$$

Example 45 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; **B**)

$$\begin{array}{ccccccccccccc} \boldsymbol{m} & e & a & c & j & l & k & \boxed{f} & b & g & d & h & i \\ i & h & d & g & b & \boxed{f} & \boldsymbol{m} & e & a & c & j & l & k \end{array} \qquad \square$$

Example 46 ($\boldsymbol{\lambda_P(x)} = \{0, 6\}$; $\boldsymbol{\lambda_P(y)} = \{5, 7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{i} & e & h & j & c & k & d & \boxed{a} & l & b & g & m & f \\ f & m & g & b & l & \boxed{a} & \boldsymbol{i} & e & h & j & c & k & d \end{array}$$ □

Example 47 ($\boldsymbol{\lambda_P(x)} = \{0, 6\}$; $\boldsymbol{\lambda_P(y)} = \{5, 7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{g} & b & e & k & m & h & d & \boxed{c} & a & l & j & i & f \\ f & i & j & l & a & \boxed{c} & \boldsymbol{g} & b & e & k & m & h & d \end{array}$$ □

Example 48 ($\boldsymbol{\lambda_P(x)} = \{0, 6\}$; $\boldsymbol{\lambda_P(y)} = \{5, 7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{k} & l & g & f & i & d & e & \boxed{b} & c & h & a & j & m \\ m & j & a & h & c & \boxed{b} & \boldsymbol{k} & l & g & f & i & d & e \end{array}$$ □

Example 49 ($\boldsymbol{\lambda_P(x)} = \{0, 6\}$; $\boldsymbol{\lambda_P(y)} = \{5, 7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{j} & g & m & d & f & k & e & \boxed{i} & b & h & a & c & l \\ l & c & a & h & b & \boxed{i} & \boldsymbol{j} & g & m & d & f & k & e \end{array}$$ □

Example 50 ($\boldsymbol{\lambda_P(x)} = \{0, 6\}$; $\boldsymbol{\lambda_P(y)} = \{5, 7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{m} & j & l & i & k & a & d & \boxed{b} & f & h & e & c & g \\ g & c & e & h & f & \boxed{b} & \boldsymbol{m} & j & l & i & k & a & d \end{array}$$ □

Example 51 ($\boldsymbol{\lambda_P(x)} = \{0, 6\}$; $\boldsymbol{\lambda_P(y)} = \{5, 7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{b} & c & f & j & h & e & l & \boxed{k} & m & a & g & i & d \\ d & i & g & a & m & \boxed{k} & \boldsymbol{b} & c & f & j & h & e & l \end{array}$$ □

Example 52 ($\boldsymbol{\lambda_P(x)} = \{0, 6\}$; $\boldsymbol{\lambda_P(y)} = \{5, 7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{c} & m & a & f & e & i & k & \boxed{j} & d & l & h & b & g \\ g & b & h & l & d & \boxed{j} & \boldsymbol{c} & m & a & f & e & i & k \end{array}$$ □

Example 53 ($\boldsymbol{\lambda_P(x)} = \{0,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{d} & e & a & c & m & i & h & \boxed{k} & l & b & f & j & g \\ g & j & f & b & l & \boxed{k} & \boldsymbol{d} & e & a & c & m & i & h \end{array}$$ □

Example 54 ($\boldsymbol{\lambda_P(x)} = \{0,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{h} & i & m & c & e & l & a & \boxed{k} & b & g & j & f & d \\ d & f & j & g & b & \boxed{k} & \boldsymbol{h} & i & m & c & e & l & a \end{array}$$ □

Example 55 ($\boldsymbol{\lambda_P(x)} = \{0,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{j} & f & a & b & e & g & k & \boxed{m} & i & h & c & d & l \\ l & d & c & h & i & \boxed{m} & \boldsymbol{j} & f & a & b & e & g & k \end{array}$$ □

Example 56 ($\boldsymbol{\lambda_P(x)} = \{0,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{b} & j & i & d & g & h & m & \boxed{e} & l & c & f & k & a \\ a & k & f & c & l & \boxed{e} & \boldsymbol{b} & j & i & d & g & h & m \end{array}$$ □

Example 57 ($\boldsymbol{\lambda_P(x)} = \{0,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{j} & a & c & k & g & f & b & \boxed{i} & m & e & l & h & d \\ d & h & l & e & m & \boxed{i} & \boldsymbol{j} & a & c & k & g & f & b \end{array}$$ □

Example 58 ($\boldsymbol{\lambda_P(x)} = \{0,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & m & j & f & g & c & i & \boxed{k} & d & e & l & h & b \\ b & h & l & e & d & \boxed{k} & \boldsymbol{a} & m & j & f & g & c & i \end{array}$$ □

Example 59 ($\boldsymbol{\lambda_P(x)} = \{0,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{k} & a & c & m & f & l & h & \boxed{g} & d & i & j & e & b \\ b & e & j & i & d & \boxed{g} & \boldsymbol{k} & a & c & m & f & l & h \end{array}$$ □

Example 60 ($\boldsymbol{\lambda_P(x)} = \{0,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{b} & d & k & c & j & f & g & \boxed{a} & e & h & m & i & l \\ l & i & m & h & e & \boxed{a} & \boldsymbol{b} & d & k & c & j & f & g \end{array}$$ □

Example 61 ($\boldsymbol{\lambda}_P(\boldsymbol{x}) = \{1, 4\}$; $\boldsymbol{\lambda}_P(\boldsymbol{y}) = \{3, 5\}$; D)

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & \boxed{f} & g & h & i & j & k & l & m \\ i & h & g & \boxed{f} & \boldsymbol{b} & a & j & k & l & m & c & d & e \end{array}$$

$\square$

Example 62 ($\boldsymbol{\lambda}_P(\boldsymbol{x}) = \{1, 4\}$; $\boldsymbol{\lambda}_P(\boldsymbol{y}) = \{3, 5\}$; D)

$$\begin{array}{ccccccccccccc} c & \boldsymbol{b} & a & j & d & \boxed{g} & m & h & k & l & e & i & f \\ k & h & m & \boxed{g} & \boldsymbol{b} & c & l & e & i & f & a & j & d \end{array}$$

$\square$

Example 63 ($\boldsymbol{\lambda}_P(\boldsymbol{x}) = \{1, 4\}$; $\boldsymbol{\lambda}_P(\boldsymbol{y}) = \{3, 5\}$; D)

$$\begin{array}{ccccccccccccc} d & \boldsymbol{c} & g & m & k & \boxed{l} & e & j & f & i & b & h & a \\ f & j & e & \boxed{l} & \boldsymbol{c} & d & i & b & h & a & g & m & k \end{array}$$

$\square$

Example 64 ($\boldsymbol{\lambda}_P(\boldsymbol{x}) = \{1, 4\}$; $\boldsymbol{\lambda}_P(\boldsymbol{y}) = \{3, 5\}$; D)

$$\begin{array}{ccccccccccccc} k & \boldsymbol{d} & m & h & f & \boxed{b} & i & a & e & c & g & j & l \\ e & a & i & \boxed{b} & \boldsymbol{d} & k & c & g & j & l & m & h & f \end{array}$$

$\square$

Example 65 ($\boldsymbol{\lambda}_P(\boldsymbol{x}) = \{1, 4\}$; $\boldsymbol{\lambda}_P(\boldsymbol{y}) = \{3, 5\}$; D)

$$\begin{array}{ccccccccccccc} e & \boldsymbol{k} & g & d & c & \boxed{m} & b & i & l & a & h & j & f \\ l & i & b & \boxed{m} & \boldsymbol{k} & e & a & h & j & f & g & d & c \end{array}$$

$\square$

Example 66 ($\boldsymbol{\lambda}_P(\boldsymbol{x}) = \{1, 4\}$; $\boldsymbol{\lambda}_P(\boldsymbol{y}) = \{3, 5\}$; D)

$$\begin{array}{ccccccccccccc} l & \boldsymbol{m} & k & d & g & \boxed{i} & a & f & c & j & e & b & h \\ c & f & a & \boxed{i} & \boldsymbol{m} & l & j & e & b & h & k & d & g \end{array}$$

$\square$

Example 67 ($\boldsymbol{\lambda}_P(\boldsymbol{x}) = \{1, 4\}$; $\boldsymbol{\lambda}_P(\boldsymbol{y}) = \{3, 5\}$; D)

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & l & h & g & \boxed{k} & j & i & d & m & e & f & c \\ d & i & j & \boxed{k} & \boldsymbol{b} & a & m & e & f & c & l & h & g \end{array}$$

$\square$

Example 68 ($\boldsymbol{\lambda_P(x) = \{1,4\}; \lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} b & \boldsymbol{f} & c & e & a & \boxed{k} & j & h & g & m & l & i & d \\ g & h & j & \boxed{k} & \boldsymbol{f} & b & m & l & i & d & c & e & a \end{array} \qquad \square$$

Example 69 ($\boldsymbol{\lambda_P(x) = \{1,4\}; \lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} d & \boldsymbol{c} & b & e & h & \boxed{l} & g & f & k & j & a & m & i \\ k & f & g & \boxed{l} & \boldsymbol{c} & d & j & a & m & i & b & e & h \end{array} \qquad \square$$

Example 70 ($\boldsymbol{\lambda_P(x) = \{1,4\}; \lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} j & \boldsymbol{b} & d & a & c & \boxed{m} & l & g & f & e & h & k & i \\ f & g & l & \boxed{m} & \boldsymbol{b} & j & e & h & k & i & d & a & c \end{array} \qquad \square$$

Example 71 ($\boldsymbol{\lambda_P(x) = \{1,4\}; \lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} h & \boldsymbol{j} & c & k & g & \boxed{l} & m & b & f & d & a & e & i \\ f & b & m & \boxed{l} & \boldsymbol{j} & h & d & a & e & i & c & k & g \end{array} \qquad \square$$

Example 72 ($\boldsymbol{\lambda_P(x) = \{1,4\}; \lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} h & \boldsymbol{l} & k & g & b & \boxed{i} & j & f & m & d & a & e & c \\ m & f & j & \boxed{i} & \boldsymbol{l} & h & d & a & e & c & k & g & b \end{array} \qquad \square$$

Example 73 ($\boldsymbol{\lambda_P(x) = \{1,4\}; \lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} a & \boldsymbol{h} & b & k & c & \boxed{j} & m & i & d & e & f & l & g \\ d & i & m & \boxed{j} & \boldsymbol{h} & a & e & f & l & g & b & k & c \end{array} \qquad \square$$

Example 74 ($\boldsymbol{\lambda_P(x) = \{1,4\}; \lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} c & \boldsymbol{h} & g & j & l & \boxed{i} & a & m & k & f & e & d & b \\ k & m & a & \boxed{i} & \boldsymbol{h} & c & f & e & d & b & g & j & l \end{array} \qquad \square$$

Example 75 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} h & \boldsymbol{d} & c & m & b & \boxed{a} & i & l & j & f & k & g & e \\ j & l & i & \boxed{a} & \boldsymbol{d} & h & f & k & g & e & c & m & b \end{array} \qquad \square$$

Example 76 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} b & \boldsymbol{f} & h & i & a & \boxed{k} & g & d & l & c & e & j & m \\ l & d & g & \boxed{k} & \boldsymbol{f} & b & c & e & j & m & h & i & a \end{array} \qquad \square$$

Example 77 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} b & \boldsymbol{l} & i & h & m & \boxed{a} & e & d & f & c & g & j & k \\ f & d & e & \boxed{a} & \boldsymbol{l} & b & c & g & j & k & i & h & m \end{array} \qquad \square$$

Example 78 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} h & \boldsymbol{b} & g & a & f & \boxed{l} & k & e & j & c & m & i & d \\ j & e & k & \boxed{l} & \boldsymbol{b} & h & c & m & i & d & g & a & f \end{array} \qquad \square$$

Example 79 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} j & \boldsymbol{g} & d & m & e & \boxed{a} & l & b & h & k & c & f & i \\ h & b & l & \boxed{a} & \boldsymbol{g} & j & k & c & f & i & d & m & e \end{array} \qquad \square$$

Example 80 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} l & \boldsymbol{e} & m & f & k & \boxed{d} & i & j & b & c & g & h & a \\ b & j & i & \boxed{d} & \boldsymbol{e} & l & c & g & h & a & m & f & k \end{array} \qquad \square$$

Example 81 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} f & \boldsymbol{k} & d & b & a & \boxed{h} & g & m & c & i & l & j & e \\ c & m & g & \boxed{h} & \boldsymbol{k} & f & i & l & j & e & d & b & a \end{array} \qquad \square$$

Example 82 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} j & \boldsymbol{h} & d & m & c & \boxed{k} & e & g & f & b & l & i & a \\ f & g & e & \boxed{k} & \boldsymbol{h} & j & b & l & i & a & d & m & c \end{array} \qquad \square$$

Example 83 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} f & \boldsymbol{a} & g & e & c & \boxed{j} & i & m & h & b & d & k & l \\ h & m & i & \boxed{j} & \boldsymbol{a} & f & b & d & k & l & g & e & c \end{array}$$

□

Example 84 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} d & \boldsymbol{b} & g & f & h & \boxed{a} & c & l & k & j & e & m & i \\ k & l & c & \boxed{a} & \boldsymbol{b} & d & j & e & m & i & g & f & h \end{array}$$

□

Example 85 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} b & \boldsymbol{m} & f & h & g & \boxed{e} & j & i & d & c & l & a & k \\ d & i & j & \boxed{e} & \boldsymbol{m} & b & c & l & a & k & f & h & g \end{array}$$

□

Example 86 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} a & \boldsymbol{l} & e & k & i & \boxed{b} & d & g & f & m & c & j & h \\ f & g & d & \boxed{b} & \boldsymbol{l} & a & m & c & j & h & e & k & i \end{array}$$

□

Example 87 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & d & k & l & \boxed{g} & j & e & h & c & i & m & f \\ h & e & j & \boxed{g} & \boldsymbol{b} & a & c & i & m & f & d & k & l \end{array}$$

□

Example 88 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} e & \boldsymbol{i} & l & g & b & \boxed{d} & c & m & k & f & h & j & a \\ k & m & c & \boxed{d} & \boldsymbol{i} & e & f & h & j & a & l & g & b \end{array}$$

□

Example 89 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} i & \boldsymbol{b} & c & k & l & \boxed{e} & j & f & a & d & m & h & g \\ a & f & j & \boxed{e} & \boldsymbol{b} & i & d & m & h & g & c & k & l \end{array}$$

□

Example 90 ($\boldsymbol{\lambda_P(x) = \{1,4\}}$; $\boldsymbol{\lambda_P(y) = \{3,5\}}$; D)

$$\begin{array}{ccccccccccccc} j & \boldsymbol{d} & l & c & i & \boxed{g} & e & h & a & f & m & k & b \\ a & h & e & \boxed{g} & \boldsymbol{d} & j & f & m & k & b & l & c & i \end{array}$$

□

Example 91 ($\boldsymbol{\lambda_P(x)} = \{3,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} a & b & c & \boldsymbol{d} & e & f & g & \boxed{h} & i & j & k & l & m \\ m & l & k & j & i & \boxed{h} & \boldsymbol{d} & e & f & g & c & b & a \end{array}$$ □

Example 92 ($\boldsymbol{\lambda_P(x)} = \{3,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} i & j & l & \boldsymbol{g} & d & f & b & \boxed{k} & e & m & c & a & h \\ h & a & c & m & e & \boxed{k} & \boldsymbol{g} & d & f & b & l & j & i \end{array}$$ □

Example 93 ($\boldsymbol{\lambda_P(x)} = \{3,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} g & i & h & \boldsymbol{j} & f & m & a & \boxed{e} & k & d & c & l & b \\ b & l & c & d & k & \boxed{e} & \boldsymbol{j} & f & m & a & h & i & g \end{array}$$ □

Example 94 ($\boldsymbol{\lambda_P(x)} = \{3,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} d & i & a & \boldsymbol{g} & f & l & h & \boxed{j} & m & e & c & k & b \\ b & k & c & e & m & \boxed{j} & \boldsymbol{g} & f & l & h & a & i & d \end{array}$$ □

Example 95 ($\boldsymbol{\lambda_P(x)} = \{3,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} e & h & l & \boldsymbol{g} & b & d & a & \boxed{c} & i & f & j & m & k \\ k & m & j & f & i & \boxed{c} & \boldsymbol{g} & b & d & a & l & h & e \end{array}$$ □

Example 96 ($\boldsymbol{\lambda_P(x)} = \{3,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} c & j & g & \boldsymbol{k} & b & l & d & \boxed{e} & f & a & h & i & m \\ m & i & h & a & f & \boxed{e} & \boldsymbol{k} & b & l & d & g & j & c \end{array}$$ □

Example 97 ($\boldsymbol{\lambda_P(x)} = \{3,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} d & l & g & \boldsymbol{f} & h & m & i & \boxed{e} & b & a & c & j & k \\ k & j & c & a & b & \boxed{e} & \boldsymbol{f} & h & m & i & g & l & d \end{array}$$ □

Example 98 ($\boldsymbol{\lambda_P(x)} = \{3,6\}$; $\boldsymbol{\lambda_P(y)} = \{5,7\}$; B)

$$\begin{array}{ccccccccccccc} a & j & d & \boldsymbol{f} & l & m & i & \boxed{k} & h & g & b & c & e \\ e & c & b & g & h & \boxed{k} & \boldsymbol{f} & l & m & i & d & j & a \end{array}$$ □

Example 99 ($\boldsymbol{\lambda_P(x) = \{3, 6\}}$; $\boldsymbol{\lambda_P(y) = \{5, 7\}}$; B)

$$\begin{array}{ccccccccccccc} h & k & c & \boldsymbol{i} & d & l & b & \boxed{e} & a & f & j & m & g \\ g & m & j & f & a & \boxed{e} & \boldsymbol{i} & d & l & b & c & k & h \end{array}$$

$\square$

Example 100 ($\boldsymbol{\lambda_P(x) = \{3, 6\}}$; $\boldsymbol{\lambda_P(y) = \{5, 7\}}$; B)

$$\begin{array}{ccccccccccccc} g & m & i & \boldsymbol{d} & b & e & a & \boxed{c} & l & h & f & j & k \\ k & j & f & h & l & \boxed{c} & \boldsymbol{d} & b & e & a & i & m & g \end{array}$$

$\square$

Example 101 ($\boldsymbol{\lambda_P(x) = \{3, 6\}}$; $\boldsymbol{\lambda_P(y) = \{5, 7\}}$; B)

$$\begin{array}{ccccccccccccc} l & j & m & \boldsymbol{d} & k & c & b & \boxed{h} & i & g & f & e & a \\ a & e & f & g & i & \boxed{h} & \boldsymbol{d} & k & c & b & m & j & l \end{array}$$

$\square$

Example 102 ($\boldsymbol{\lambda_P(x) = \{3, 6\}}$; $\boldsymbol{\lambda_P(y) = \{5, 7\}}$; B)

$$\begin{array}{ccccccccccccc} j & m & c & \boldsymbol{a} & g & l & i & \boxed{b} & f & e & d & h & k \\ k & h & d & e & f & \boxed{b} & \boldsymbol{a} & g & l & i & c & m & j \end{array}$$

$\square$

Example 103 ($\boldsymbol{\lambda_P(x) = \{3, 6\}}$; $\boldsymbol{\lambda_P(y) = \{5, 7\}}$; B)

$$\begin{array}{ccccccccccccc} j & b & f & \boldsymbol{m} & h & a & d & \boxed{i} & k & l & c & g & e \\ e & g & c & l & k & \boxed{i} & \boldsymbol{m} & h & a & d & f & b & j \end{array}$$

$\square$

Example 104 ($\boldsymbol{\lambda_P(x) = \{3, 6\}}$; $\boldsymbol{\lambda_P(y) = \{5, 7\}}$; B)

$$\begin{array}{ccccccccccccc} j & m & g & \boldsymbol{a} & e & d & f & \boxed{i} & l & h & c & b & k \\ k & b & c & h & l & \boxed{i} & \boldsymbol{a} & e & d & f & g & m & j \end{array}$$

$\square$

Example 105 ($\boldsymbol{\lambda_P(x) = \{3, 6\}}$; $\boldsymbol{\lambda_P(y) = \{5, 7\}}$; B)

$$\begin{array}{ccccccccccccc} j & d & f & \boldsymbol{m} & e & b & g & \boxed{k} & l & i & a & h & c \\ c & h & a & i & l & \boxed{k} & \boldsymbol{m} & e & b & g & f & d & j \end{array}$$

$\square$

Example 106 ($\boldsymbol{\lambda_P(x) = \{3, 6\}}$; $\boldsymbol{\lambda_P(y) = \{5, 7\}}$; B)

$$\begin{array}{ccccccccccccc} d & g & i & \boldsymbol{h} & c & l & f & \boxed{a} & j & e & m & k & b \\ b & k & m & e & j & \boxed{a} & \boldsymbol{h} & c & l & f & i & g & d \end{array}$$

$\square$

Example 107 $(\boldsymbol{\lambda_P(x) = \{3,6\}};\ \boldsymbol{\lambda_P(y) = \{5,7\}};\ \mathbf{B})$

$$\begin{array}{ccccccccccccc} a & g & c & \boldsymbol{h} & i & d & f & \boxed{m} & b & j & l & k & e \\ e & k & l & j & b & \boxed{m} & \boldsymbol{h} & i & d & f & c & g & a \end{array}$$ □

Example 108 $(\boldsymbol{\lambda_P(x) = \{3,6\}};\ \boldsymbol{\lambda_P(y) = \{5,7\}};\ \mathbf{B})$

$$\begin{array}{ccccccccccccc} h & c & j & \boldsymbol{f} & b & i & m & \boxed{e} & l & k & g & a & d \\ d & a & g & k & l & \boxed{e} & \boldsymbol{f} & b & i & m & j & c & h \end{array}$$ □

Example 109 $(\boldsymbol{\lambda_P(x) = \{3,6\}};\ \boldsymbol{\lambda_P(y) = \{5,7\}};\ \mathbf{B})$

$$\begin{array}{ccccccccccccc} b & c & m & \boldsymbol{l} & e & j & i & \boxed{a} & d & k & f & g & h \\ h & g & f & k & d & \boxed{a} & \boldsymbol{l} & e & j & i & m & c & b \end{array}$$ □

Example 110 $(\boldsymbol{\lambda_P(x) = \{3,6\}};\ \boldsymbol{\lambda_P(y) = \{5,7\}};\ \mathbf{B})$

$$\begin{array}{ccccccccccccc} m & c & j & \boldsymbol{k} & g & i & l & \boxed{a} & d & f & e & h & b \\ b & h & e & f & d & \boxed{a} & \boldsymbol{k} & g & i & l & j & c & m \end{array}$$ □

Example 111 $(\boldsymbol{\lambda_P(x) = \{3,6\}};\ \boldsymbol{\lambda_P(y) = \{5,7\}};\ \mathbf{B})$

$$\begin{array}{ccccccccccccc} j & b & c & \boldsymbol{g} & f & e & i & \boxed{m} & d & a & l & k & h \\ h & k & l & a & d & \boxed{m} & \boldsymbol{g} & f & e & i & c & b & j \end{array}$$ □

Example 112 $(\boldsymbol{\lambda_P(x) = \{3,6\}};\ \boldsymbol{\lambda_P(y) = \{5,7\}};\ \mathbf{B})$

$$\begin{array}{ccccccccccccc} i & l & k & \boldsymbol{j} & e & f & b & \boxed{a} & c & d & m & g & h \\ h & g & m & d & c & \boxed{a} & \boldsymbol{j} & e & f & b & k & l & i \end{array}$$ □

Example 113 $(\boldsymbol{\lambda_P(x) = \{3,6\}};\ \boldsymbol{\lambda_P(y) = \{5,7\}};\ \mathbf{B})$

$$\begin{array}{ccccccccccccc} l & c & g & \boldsymbol{m} & b & a & f & \boxed{d} & i & e & k & h & j \\ j & h & k & e & i & \boxed{d} & \boldsymbol{m} & b & a & f & g & c & l \end{array}$$ □

Example 114 $(\boldsymbol{\lambda_P(x) = \{3,6\}};\ \boldsymbol{\lambda_P(y) = \{5,7\}};\ \mathbf{B})$

$$\begin{array}{ccccccccccccc} k & m & f & \boldsymbol{j} & c & d & b & \boxed{g} & e & h & l & i & a \\ a & i & l & h & e & \boxed{g} & \boldsymbol{j} & c & d & b & f & m & k \end{array}$$ □

Example 115 ($\boldsymbol{\lambda}_P(\boldsymbol{x}) = \{3,6\}$; $\boldsymbol{\lambda}_P(\boldsymbol{y}) = \{5,7\}$; **B**)

$$\begin{array}{ccccccccccccc} k & f & e & \boldsymbol{m} & c & a & l & \boxed{i} & j & d & g & h & b \\ b & h & g & d & j & \boxed{i} & \boldsymbol{m} & c & a & l & e & f & k \end{array}$$ □

Example 116 ($\boldsymbol{\lambda}_P(\boldsymbol{x}) = \{3,6\}$; $\boldsymbol{\lambda}_P(\boldsymbol{y}) = \{5,7\}$; **B**)

$$\begin{array}{ccccccccccccc} e & l & k & \boldsymbol{j} & d & i & c & \boxed{m} & h & f & b & g & a \\ a & g & b & f & h & \boxed{m} & \boldsymbol{j} & d & i & c & k & l & e \end{array}$$ □

Example 117 ($\boldsymbol{\lambda}_P(\boldsymbol{x}) = \{3,6\}$; $\boldsymbol{\lambda}_P(\boldsymbol{y}) = \{5,7\}$; **B**)

$$\begin{array}{ccccccccccccc} i & f & g & \boldsymbol{a} & l & d & b & \boxed{h} & e & m & k & c & j \\ j & c & k & m & e & \boxed{h} & \boldsymbol{a} & l & d & b & g & f & i \end{array}$$ □

Example 118 ($\boldsymbol{\lambda}_P(\boldsymbol{x}) = \{3,6\}$; $\boldsymbol{\lambda}_P(\boldsymbol{y}) = \{5,7\}$; **B**)

$$\begin{array}{ccccccccccccc} l & h & a & \boldsymbol{d} & j & b & g & \boxed{c} & f & m & k & e & i \\ i & e & k & m & f & \boxed{c} & \boldsymbol{d} & j & b & g & a & h & l \end{array}$$ □

Example 119 ($\boldsymbol{\lambda}_P(\boldsymbol{x}) = \{3,6\}$; $\boldsymbol{\lambda}_P(\boldsymbol{y}) = \{5,7\}$; **B**)

$$\begin{array}{ccccccccccccc} c & b & e & \boldsymbol{i} & d & a & l & \boxed{g} & m & k & h & f & j \\ j & f & h & k & m & \boxed{g} & \boldsymbol{i} & d & a & l & e & b & c \end{array}$$ □

Example 120 ($\boldsymbol{\lambda}_P(\boldsymbol{x}) = \{3,6\}$; $\boldsymbol{\lambda}_P(\boldsymbol{y}) = \{5,7\}$; **B**)

$$\begin{array}{ccccccccccccc} d & g & e & \boldsymbol{h} & i & k & b & \boxed{m} & l & c & j & f & a \\ a & f & j & c & l & \boxed{m} & \boldsymbol{h} & i & k & b & e & g & d \end{array}$$ □

Example 121 ($\boldsymbol{\lambda}_P(\boldsymbol{x}) = \{3,6\}$; $\boldsymbol{\lambda}_P(\boldsymbol{y}) = \{5,7\}$; **D**)

$$\begin{array}{ccccccccccccc} a & b & c & \boldsymbol{d} & e & f & g & \boxed{h} & i & j & k & l & m \\ m & l & k & j & i & \boxed{h} & \boldsymbol{d} & a & b & c & e & f & g \end{array}$$ □

Example 122 ($\boldsymbol{\lambda}_P(\boldsymbol{x}) = \{3,6\}$; $\boldsymbol{\lambda}_P(\boldsymbol{y}) = \{5,7\}$; **D**)

$$\begin{array}{ccccccccccccc} d & j & b & \boldsymbol{g} & m & i & e & \boxed{h} & k & l & a & f & c \\ c & f & a & l & k & \boxed{h} & \boldsymbol{g} & d & j & b & m & i & e \end{array}$$ □

Example 123 ($\boldsymbol{\lambda_P(x) = \{3, 6\}; \lambda_P(y) = \{5, 7\};}$ **D)**

$$\begin{array}{ccccccccccccc} h & e & j & \boldsymbol{a} & b & m & i & \boxed{d} & g & k & f & c & l \\ l & c & f & k & g & \boxed{d} & \boldsymbol{a} & h & e & j & b & m & i \end{array}$$

$\square$

Example 124 ($\boldsymbol{\lambda_P(x) = \{3, 6\}; \lambda_P(y) = \{5, 7\};}$ **D)**

$$\begin{array}{ccccccccccccc} h & m & l & \boldsymbol{i} & j & a & e & \boxed{f} & g & c & b & d & k \\ k & d & b & c & g & \boxed{f} & \boldsymbol{i} & h & m & l & j & a & e \end{array}$$

$\square$

Example 125 ($\boldsymbol{\lambda_P(x) = \{3, 6\}; \lambda_P(y) = \{5, 7\};}$ **D)**

$$\begin{array}{ccccccccccccc} f & h & b & \boldsymbol{g} & c & e & j & \boxed{i} & l & a & m & k & d \\ d & k & m & a & l & \boxed{i} & \boldsymbol{g} & f & h & b & c & e & j \end{array}$$

$\square$

Example 126 ($\boldsymbol{\lambda_P(x) = \{3, 6\}; \lambda_P(y) = \{5, 7\};}$ **D)**

$$\begin{array}{ccccccccccccc} e & g & a & \boldsymbol{f} & b & i & d & \boxed{c} & m & k & h & j & l \\ l & j & h & k & m & \boxed{c} & \boldsymbol{f} & e & g & a & b & i & d \end{array}$$

$\square$

Example 127 ($\boldsymbol{\lambda_P(x) = \{3, 6\}; \lambda_P(y) = \{5, 7\};}$ **D)**

$$\begin{array}{ccccccccccccc} d & g & e & \boldsymbol{a} & k & i & l & \boxed{j} & c & b & h & f & m \\ m & f & h & b & c & \boxed{j} & \boldsymbol{a} & d & g & e & k & i & l \end{array}$$

$\square$

Example 128 ($\boldsymbol{\lambda_P(x) = \{3, 6\}; \lambda_P(y) = \{5, 7\};}$ **D)**

$$\begin{array}{ccccccccccccc} k & h & i & \boldsymbol{b} & f & d & j & \boxed{e} & c & g & a & l & m \\ m & l & a & g & c & \boxed{e} & \boldsymbol{b} & k & h & i & f & d & j \end{array}$$

$\square$

Example 129 ($\boldsymbol{\lambda_P(x) = \{3, 6\}; \lambda_P(y) = \{5, 7\};}$ **D)**

$$\begin{array}{ccccccccccccc} f & b & l & \boldsymbol{j} & c & k & h & \boxed{g} & e & m & i & d & a \\ a & d & i & m & e & \boxed{g} & \boldsymbol{j} & f & b & l & c & k & h \end{array}$$

$\square$

Example 130 ($\boldsymbol{\lambda_P(x) = \{3,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; D)

$$\begin{array}{ccccccccccccc} d & g & f & \boldsymbol{j} & b & k & e & \boxed{i} & a & h & l & m & c \\ c & m & l & h & a & \boxed{i} & \boldsymbol{j} & d & g & f & b & k & e \end{array} \qquad \square$$

Example 131 ($\boldsymbol{\lambda_P(x) = \{3,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; D)

$$\begin{array}{ccccccccccccc} j & c & b & \boldsymbol{l} & f & e & m & \boxed{d} & k & i & g & a & h \\ h & a & g & i & k & \boxed{d} & \boldsymbol{l} & j & c & b & f & e & m \end{array} \qquad \square$$

Example 132 ($\boldsymbol{\lambda_P(x) = \{3,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; D)

$$\begin{array}{ccccccccccccc} e & d & l & \boldsymbol{h} & b & m & f & \boxed{j} & c & a & k & i & g \\ g & i & k & a & c & \boxed{j} & \boldsymbol{h} & e & d & l & b & m & f \end{array} \qquad \square$$

Example 133 ($\boldsymbol{\lambda_P(x) = \{3,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; D)

$$\begin{array}{ccccccccccccc} k & c & g & \boldsymbol{e} & a & d & j & \boxed{i} & f & l & h & b & m \\ m & b & h & l & f & \boxed{i} & \boldsymbol{e} & k & c & g & a & d & j \end{array} \qquad \square$$

Example 134 ($\boldsymbol{\lambda_P(x) = \{3,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; D)

$$\begin{array}{ccccccccccccc} g & h & b & \boldsymbol{j} & k & f & m & \boxed{i} & e & c & l & d & a \\ a & d & l & c & e & \boxed{i} & \boldsymbol{j} & g & h & b & k & f & m \end{array} \qquad \square$$

Example 135 ($\boldsymbol{\lambda_P(x) = \{3,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; D)

$$\begin{array}{ccccccccccccc} d & k & h & \boldsymbol{c} & g & a & m & \boxed{j} & f & l & b & e & i \\ i & e & b & l & f & \boxed{j} & \boldsymbol{c} & d & k & h & g & a & m \end{array} \qquad \square$$

Example 136 ($\boldsymbol{\lambda_P(x) = \{3,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; D)

$$\begin{array}{ccccccccccccc} k & a & d & \boldsymbol{g} & f & h & c & \boxed{e} & b & m & l & j & i \\ i & j & l & m & b & \boxed{e} & \boldsymbol{g} & k & a & d & f & h & c \end{array} \qquad \square$$

Example 137 ($\boldsymbol{\lambda_P(x) = \{3,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; D)

$$\begin{array}{ccccccccccccc} k & i & d & \boldsymbol{g} & m & b & h & \boxed{a} & e & c & j & f & l \\ l & f & j & c & e & \boxed{a} & \boldsymbol{g} & k & i & d & m & b & h \end{array} \qquad \square$$

Example 138 ($\boldsymbol{\lambda_P(x) = \{3,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; D)

$$\begin{array}{ccccccccccccc} i & h & m & \boldsymbol{e} & g & d & k & \boxed{l} & c & f & a & j & b \\ b & j & a & f & c & \boxed{l} & \boldsymbol{e} & i & h & m & g & d & k \end{array} \qquad \square$$

Example 139 ($\boldsymbol{\lambda_P(x) = \{3,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; D)

$$\begin{array}{ccccccccccccc} d & l & m & \boldsymbol{i} & a & f & h & \boxed{e} & b & c & g & j & k \\ k & j & g & c & b & \boxed{e} & \boldsymbol{i} & d & l & m & a & f & h \end{array} \qquad \square$$

Example 140 ($\boldsymbol{\lambda_P(x) = \{3,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; D)

$$\begin{array}{ccccccccccccc} j & k & h & \boldsymbol{b} & d & c & f & \boxed{m} & l & i & e & a & g \\ g & a & e & i & l & \boxed{m} & \boldsymbol{b} & j & k & h & d & c & f \end{array} \qquad \square$$

Example 141 ($\boldsymbol{\lambda_P(x) = \{3,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; D)

$$\begin{array}{ccccccccccccc} h & g & c & \boldsymbol{b} & l & a & j & \boxed{e} & k & m & f & d & i \\ i & d & f & m & k & \boxed{e} & \boldsymbol{b} & h & g & c & l & a & j \end{array} \qquad \square$$

Example 142 ($\boldsymbol{\lambda_P(x) = \{3,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; D)

$$\begin{array}{ccccccccccccc} k & m & e & \boldsymbol{a} & f & j & d & \boxed{g} & c & l & h & b & i \\ i & b & h & l & c & \boxed{g} & \boldsymbol{a} & k & m & e & f & j & d \end{array} \qquad \square$$

Example 143 ($\boldsymbol{\lambda_P(x) = \{3,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; D)

$$\begin{array}{ccccccccccccc} m & f & g & \boldsymbol{b} & j & a & e & \boxed{c} & l & d & h & k & i \\ i & k & h & d & l & \boxed{c} & \boldsymbol{b} & m & f & g & j & a & e \end{array} \qquad \square$$

Example 144 ($\boldsymbol{\lambda_P(x) = \{3,6\}}$; $\boldsymbol{\lambda_P(y) = \{5,7\}}$; D)

$$\begin{array}{ccccccccccccc} f & d & b & \boldsymbol{m} & l & h & j & \boxed{g} & a & c & i & e & k \\ k & e & i & c & a & \boxed{g} & \boldsymbol{m} & f & d & b & l & h & j \end{array} \qquad \square$$

Example 145 $(\boldsymbol{\lambda_P(x)} = \{3,6\}; \boldsymbol{\lambda_P(y)} = \{5,7\}; \text{D})$

$$\begin{array}{ccccccccccccc} h & g & m & \boldsymbol{d} & j & l & e & \boxed{f} & k & i & a & c & b \\ b & c & a & i & k & \boxed{f} & \boldsymbol{d} & h & g & m & j & l & e \end{array}$$ □

Example 146 $(\boldsymbol{\lambda_P(x)} = \{3,6\}; \boldsymbol{\lambda_P(y)} = \{5,7\}; \text{D})$

$$\begin{array}{ccccccccccccc} f & a & g & \boldsymbol{l} & b & d & e & \boxed{k} & j & c & m & h & i \\ i & h & m & c & j & \boxed{k} & \boldsymbol{l} & f & a & g & b & d & e \end{array}$$ □

Example 147 $(\boldsymbol{\lambda_P(x)} = \{3,6\}; \boldsymbol{\lambda_P(y)} = \{5,7\}; \text{D})$

$$\begin{array}{ccccccccccccc} f & h & l & \boldsymbol{j} & b & d & a & \boxed{c} & e & i & m & g & k \\ k & g & m & i & e & \boxed{c} & \boldsymbol{j} & f & h & l & b & d & a \end{array}$$ □

Example 148 $(\boldsymbol{\lambda_P(x)} = \{3,6\}; \boldsymbol{\lambda_P(y)} = \{5,7\}; \text{D})$

$$\begin{array}{ccccccccccccc} k & f & h & \boldsymbol{d} & e & i & a & \boxed{g} & m & l & j & c & b \\ b & c & j & l & m & \boxed{g} & \boldsymbol{d} & k & f & h & e & i & a \end{array}$$ □

Example 149 $(\boldsymbol{\lambda_P(x)} = \{3,6\}; \boldsymbol{\lambda_P(y)} = \{5,7\}; \text{D})$

$$\begin{array}{ccccccccccccc} d & c & m & \boldsymbol{f} & k & g & j & \boxed{b} & e & a & h & l & i \\ i & l & h & a & e & \boxed{b} & \boldsymbol{f} & d & c & m & k & g & j \end{array}$$ □

Example 150 $(\boldsymbol{\lambda_P(x)} = \{3,6\}; \boldsymbol{\lambda_P(y)} = \{5,7\}; \text{D})$

$$\begin{array}{ccccccccccccc} m & h & g & \boldsymbol{k} & e & f & j & \boxed{i} & a & l & b & d & c \\ c & d & b & l & a & \boxed{i} & \boldsymbol{k} & m & h & g & e & f & j \end{array}$$ □

Example 151 $(\boldsymbol{\lambda_P(x)} = \{0,6\}; \boldsymbol{\lambda_P(y)} = \{2,7\}; \text{B})$

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ j & i & \boxed{h} & k & l & m & \boldsymbol{a} & b & c & d & e & f & g \end{array}$$ □

Example 152 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{2,7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{c} & m & a & j & h & g & i & \boxed{l} & f & e & d & k & b \\ e & f & \boxed{l} & d & k & b & \boldsymbol{c} & m & a & j & h & g & i \end{array}$$ □

Example 153 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{2,7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{k} & g & f & j & i & c & m & \boxed{l} & e & b & d & h & a \\ b & e & \boxed{l} & d & h & a & \boldsymbol{k} & g & f & j & i & c & m \end{array}$$ □

Example 154 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{2,7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & k & c & m & d & f & g & \boxed{e} & h & b & i & l & j \\ b & h & \boxed{e} & i & l & j & \boldsymbol{a} & k & c & m & d & f & g \end{array}$$ □

Example 155 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{2,7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{d} & m & f & j & b & c & h & \boxed{k} & a & l & g & e & i \\ l & a & \boxed{k} & g & e & i & \boldsymbol{d} & m & f & j & b & c & h \end{array}$$ □

Example 156 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{2,7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{j} & k & l & c & i & g & a & \boxed{f} & b & h & m & e & d \\ h & b & \boxed{f} & m & e & d & \boldsymbol{j} & k & l & c & i & g & a \end{array}$$ □

Example 157 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{2,7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{f} & j & i & m & k & b & a & \boxed{e} & g & c & l & h & d \\ c & g & \boxed{e} & l & h & d & \boldsymbol{f} & j & i & m & k & b & a \end{array}$$ □

Example 158 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{2,7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{e} & j & i & l & g & a & c & \boxed{h} & m & b & k & f & d \\ b & m & \boxed{h} & k & f & d & \boldsymbol{e} & j & i & l & g & a & c \end{array}$$ □

Example 159 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{2,7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{l} & m & b & f & j & i & a & \boxed{e} & h & d & c & g & k \\ d & h & \boxed{e} & c & g & k & \boldsymbol{l} & m & b & f & j & i & a \end{array}$$ □

Example 160 $(\boldsymbol{\lambda_P(x) = \{0,6\}};\ \boldsymbol{\lambda_P(y) = \{2,7\}};\ \mathbf{B})$

$$\begin{array}{ccccccccccccc} \boldsymbol{c} & e & g & k & a & d & m & \boxed{b} & i & h & j & f & l \\ h & i & \boxed{b} & j & f & l & \boldsymbol{c} & e & g & k & a & d & m \end{array}$$

□

Example 161 $(\boldsymbol{\lambda_P(x) = \{0,6\}};\ \boldsymbol{\lambda_P(y) = \{2,7\}};\ \mathbf{B})$

$$\begin{array}{ccccccccccccc} \boldsymbol{b} & k & d & h & l & a & c & \boxed{e} & m & g & j & f & i \\ g & m & \boxed{e} & j & f & i & \boldsymbol{b} & k & d & h & l & a & c \end{array}$$

□

Example 162 $(\boldsymbol{\lambda_P(x) = \{0,6\}};\ \boldsymbol{\lambda_P(y) = \{2,7\}};\ \mathbf{B})$

$$\begin{array}{ccccccccccccc} \boldsymbol{b} & i & a & k & g & f & c & \boxed{d} & e & h & l & m & j \\ h & e & \boxed{d} & l & m & j & \boldsymbol{b} & i & a & k & g & f & c \end{array}$$

□

Example 163 $(\boldsymbol{\lambda_P(x) = \{0,6\}};\ \boldsymbol{\lambda_P(y) = \{2,7\}};\ \mathbf{B})$

$$\begin{array}{ccccccccccccc} \boldsymbol{e} & a & g & h & j & l & i & \boxed{f} & k & m & c & d & b \\ m & k & \boxed{f} & c & d & b & \boldsymbol{e} & a & g & h & j & l & i \end{array}$$

□

Example 164 $(\boldsymbol{\lambda_P(x) = \{0,6\}};\ \boldsymbol{\lambda_P(y) = \{2,7\}};\ \mathbf{B})$

$$\begin{array}{ccccccccccccc} \boldsymbol{k} & f & e & i & g & c & a & \boxed{d} & b & l & m & j & h \\ l & b & \boxed{d} & m & j & h & \boldsymbol{k} & f & e & i & g & c & a \end{array}$$

□

Example 165 $(\boldsymbol{\lambda_P(x) = \{0,6\}};\ \boldsymbol{\lambda_P(y) = \{2,7\}};\ \mathbf{B})$

$$\begin{array}{ccccccccccccc} \boldsymbol{j} & d & g & m & i & b & l & \boxed{h} & a & c & f & k & e \\ c & a & \boxed{h} & f & k & e & \boldsymbol{j} & d & g & m & i & b & l \end{array}$$

□

Example 166 $(\boldsymbol{\lambda_P(x) = \{0,6\}};\ \boldsymbol{\lambda_P(y) = \{2,7\}};\ \mathbf{B})$

$$\begin{array}{ccccccccccccc} \boldsymbol{f} & c & m & g & e & a & j & \boxed{b} & l & k & h & i & d \\ k & l & \boxed{b} & h & i & d & \boldsymbol{f} & c & m & g & e & a & j \end{array}$$

□

Example 167 ($\boldsymbol{\lambda_P(x) = \{0, 6\}}$; $\boldsymbol{\lambda_P(y) = \{2, 7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{c} & b & j & i & a & g & l & \boxed{h} & f & k & m & e & d \\ k & f & \boxed{h} & m & e & d & \boldsymbol{c} & b & j & i & a & g & l \end{array} \qquad \square$$

Example 168 ($\boldsymbol{\lambda_P(x) = \{0, 6\}}$; $\boldsymbol{\lambda_P(y) = \{2, 7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{f} & l & j & k & b & d & g & \boxed{a} & c & m & e & h & i \\ m & c & \boxed{a} & e & h & i & \boldsymbol{f} & l & j & k & b & d & g \end{array} \qquad \square$$

Example 169 ($\boldsymbol{\lambda_P(x) = \{0, 6\}}$; $\boldsymbol{\lambda_P(y) = \{2, 7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{e} & j & i & b & l & c & m & \boxed{k} & f & g & h & d & a \\ g & f & \boxed{k} & h & d & a & \boldsymbol{e} & j & i & b & l & c & m \end{array} \qquad \square$$

Example 170 ($\boldsymbol{\lambda_P(x) = \{0, 6\}}$; $\boldsymbol{\lambda_P(y) = \{2, 7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{f} & l & m & b & a & j & e & \boxed{c} & h & i & d & g & k \\ i & h & \boxed{c} & d & g & k & \boldsymbol{f} & l & m & b & a & j & e \end{array} \qquad \square$$

Example 171 ($\boldsymbol{\lambda_P(x) = \{0, 6\}}$; $\boldsymbol{\lambda_P(y) = \{2, 7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{c} & b & i & e & j & f & h & \boxed{a} & g & m & d & l & k \\ m & g & \boxed{a} & d & l & k & \boldsymbol{c} & b & i & e & j & f & h \end{array} \qquad \square$$

Example 172 ($\boldsymbol{\lambda_P(x) = \{0, 6\}}$; $\boldsymbol{\lambda_P(y) = \{2, 7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{m} & k & j & d & i & b & g & \boxed{l} & a & h & c & f & e \\ h & a & \boxed{l} & c & f & e & \boldsymbol{m} & k & j & d & i & b & g \end{array} \qquad \square$$

Example 173 ($\boldsymbol{\lambda_P(x) = \{0, 6\}}$; $\boldsymbol{\lambda_P(y) = \{2, 7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{m} & e & h & g & k & a & j & \boxed{l} & b & d & c & i & f \\ d & b & \boxed{l} & c & i & f & \boldsymbol{m} & e & h & g & k & a & j \end{array} \qquad \square$$

Example 174 ($\boldsymbol{\lambda_P(x) = \{0, 6\}}$; $\boldsymbol{\lambda_P(y) = \{2, 7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{f} & j & b & g & e & c & m & \boxed{k} & i & d & a & l & h \\ d & i & \boxed{k} & a & l & h & \boldsymbol{f} & j & b & g & e & c & m \end{array} \qquad \square$$

Example 175 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{2,7\}}$; **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{g} & c & e & f & a & j & m & \boxed{k} & d & i & h & b & l \\ i & d & \boxed{k} & h & b & l & \boldsymbol{g} & c & e & f & a & j & m \end{array}$$

□

Example 176 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{2,7\}}$; **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{f} & d & j & k & c & i & l & \boxed{g} & h & b & a & m & e \\ b & h & \boxed{g} & a & m & e & \boldsymbol{f} & d & j & k & c & i & l \end{array}$$

□

Example 177 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{2,7\}}$; **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{g} & j & c & d & m & i & a & \boxed{h} & l & e & f & b & k \\ e & l & \boxed{h} & f & b & k & \boldsymbol{g} & j & c & d & m & i & a \end{array}$$

□

Example 178 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{2,7\}}$; **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{m} & e & c & f & k & g & l & \boxed{a} & d & b & h & j & i \\ b & d & \boxed{a} & h & j & i & \boldsymbol{m} & e & c & f & k & g & l \end{array}$$

□

Example 179 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{2,7\}}$; **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{f} & c & b & l & j & g & d & \boxed{m} & e & i & a & k & h \\ i & e & \boxed{m} & a & k & h & \boldsymbol{f} & c & b & l & j & g & d \end{array}$$

□

Example 180 ($\boldsymbol{\lambda_P(x) = \{0,6\}}$; $\boldsymbol{\lambda_P(y) = \{2,7\}}$; **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{d} & b & f & g & a & c & i & \boxed{j} & l & k & h & e & m \\ k & l & \boxed{j} & h & e & m & \boldsymbol{d} & b & f & g & a & c & i \end{array}$$

□

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A. Formal statements

A.1. General results

Theorem 1 $\forall P \in \mathcal{L}(\mathscr{A})^{\{1,2\}} : \min\max \lambda_P \leq \frac{m}{2}$. □

Proof Define $A = \{a \in \mathscr{A} \mid 0 \leq \lambda_P(a)_1 \leq \lfloor \frac{m}{2} \rfloor\}$. Define $B = \{a \in \mathscr{A} \mid \lfloor \frac{m}{2} \rfloor < \lambda_P(a)_2 \leq m-1\}$. Observe that $\#A = \lfloor \frac{m}{2} \rfloor + 1$ and $\#B = (m-1) - \lfloor \frac{m}{2} \rfloor = (m - \lfloor \frac{m}{2} \rfloor) - 1 = \lceil \frac{m}{2} \rceil - 1 \leq \lfloor \frac{m}{2} \rfloor$, thus, $\#B < \#A$. Thus, $\exists x \in A \setminus B$. Thus, $\lambda_P(x)_1 \leq \lfloor \frac{m}{2} \rfloor$ (as $x \in A$) and $\lambda_P(x)_2 \leq \lfloor \frac{m}{2} \rfloor$ (as $x \notin B$). It follows that $\max \lambda_P(x) \leq \lfloor \frac{m}{2} \rfloor$. ■

Theorem 2 $\forall P \in \mathcal{L}(\mathscr{A})^{\{1,2\}} : \arg\min\max \lambda_P \subseteq PO(P)$. □

Proof Consider any $x \in \arg\min\max \lambda_P$ and any $y \in \mathscr{A}$.

Observe that $\max \lambda_P(x) \leq \max \lambda_P(y)$, equivalently, $\forall i \in N : \lambda_P(x)_i \leq \max \lambda_P(y)$. Thus, given any $i \in \arg\max \lambda_P(y)$, $\lambda_P(x)_i \leq \lambda_P(y)_i$. ■

A.2. Restraining profiles

Theorem 3 *The following propositions are equivalent:*

1. $FB(P) \cap MS(P) = \emptyset$;
2. $\forall y \in MS(P), x \in FB(P), i \in \arg\max \lambda_P(y) : \min \lambda_P(x) = \lambda_P(x)_i < \min \lambda_P(y) = \lambda_P(y)_{\bar{i}} < \max \lambda_P(x) = \lambda_P(x)_{\bar{i}} < \max \lambda_P(y) = \lambda_P(y)_i$.

Furthermore, they imply:

(i) $\#FB(P) = 1$;

(ii) $\#MS(P) = 1$;

(iii) $\min\max \lambda_P \leq \frac{m-1}{2}$. □

Proof Assuming Item 1, let us prove Item 2. Pick any $x \in FB(P) = \arg\min\max \lambda_P$ and any $y \in MS(P) = \arg\min_{PO(P)}(d \circ \lambda_P)$. Thanks to Item 1, we know that $y \notin FB(P)$ thus $\max \lambda_P(x) < \max \lambda_P(y)$; also, $x \in PO(P)$ (as given by Theorem 2), thus $d(\lambda_P(y)) \leq d(\lambda_P(x))$.

We see that $\min \lambda_P(x) < \min \lambda_P(y)$ as otherwise $\min \lambda_P(y) \leq \min \lambda_P(x) \leq \max \lambda_P(x) < \max \lambda_P(y)$ which yields $d(\lambda_P(x)) < d(\lambda_P(y))$.

Pick any $i \in \arg\max \lambda_P(y)$. We can see that $\min \lambda_P(y) = \lambda_P(y)_{\bar{i}} < \lambda_P(x)_{\bar{i}}$. That is because $\max \lambda_P(x) < \max \lambda_P(y) = \lambda_P(y)_i$ thus $\lambda_P(x)_i \leq \max \lambda_P(x) < \lambda_P(y)_i$ and as $y \in PO(P)$, we need to have $\lambda_P(y)_{\bar{i}} < \lambda_P(x)_{\bar{i}}$.

So far we know that $\min \lambda_P(x) < \lambda_P(y)_{\bar{i}} < \lambda_P(x)_{\bar{i}}$ and $\max \lambda_P(x) < \lambda_P(y)_i$, which also permit to deduce that $\lambda_P(x)_i = \min \lambda_P(x)$ and $\lambda_P(x)_{\bar{i}} = \max \lambda_P(x)$. This establishes Item 2.

In turn, Item 2 implies Item 1 as, picking any $y \in MS(P)$, we see that $y \in FB(P)$ is excluded by picking any $x \in FB(P)$ and using $\max \lambda_P(x) < \max \lambda_P(y)$.

We now turn to the consequences of these claims. Pick any $x \in FB(P)$ and $y \in MS(P)$ and $i \in \arg\max \lambda_P(y)$. Thus, from Item 2, $\max \lambda_P(x) = \lambda_P(x)_{\bar{i}}$.

Considering any $x' \in FB(P)$, from Item 2 again, $\max \lambda_P(x') = \lambda_P(x')_{\bar{i}}$. As $x, x' \in FB(P)$, we also have $\max \lambda_P(x) = \max \lambda_P(x')$. Thus, $\lambda_P(x')_{\bar{i}} = \lambda_P(x)_{\bar{i}}$ and $x = x'$.

A similar argument establishes Item ii.

Define $A = \{a \in \mathscr{A} \mid \max \lambda_P(x) < \lambda_P(a)_i \leq m - 1\}$, $B = \{a \in \mathscr{A} \mid 0 \leq \lambda_P(a)_{\bar{i}} < \max \lambda_P(x)\}$. As $\#FB(P) = 1$ and $x \notin B$, $a \in B \Rightarrow a \notin FB(P)$, whence $a \in B \Rightarrow \max \lambda_P(x) < \max \lambda_P(a) \Rightarrow \max \lambda_P(x) < \lambda_P(a)_i$ (because $a \in B \Rightarrow \lambda_P(a)_{\bar{i}} < \max \lambda_P(x)$), thus $B \subseteq A$. It follows that $\max \lambda_P(x) = \#B \leq \#A = m - 1 - \max \lambda_P(x)$, whence $\max \lambda_P(x) \leq \frac{m-1}{2}$. ■

Comparing Theorem 3 Item iii and Theorem 2, we see that the hypothesis $FB(P) \cap MS(P) = \emptyset$ lead to a slightly lower bound for $\min \max \lambda_P$.

From now on, each time we deal with profiles such that Theorem 3 Item 1 or 2 holds, we will therefore consider, without explicitly invoking Theorem 3, that $\#FB(P) = \#MS(P) = 1$, thus, we will consider these sets as singletons, and similarly for the sets $\arg\min \lambda_P(FB(P))$, $\arg\max \lambda_P(FB(P))$, $\arg\min \lambda_P(MS(P))$ and $\arg\max \lambda_P(MS(P))$.

A.3. Bounds on losses

Theorem 4 *$\forall P \in \mathcal{L}(\mathscr{A})^{\{1,2\}} \mid FB(P) \cap MS(P) = \emptyset$, letting $\{x\} = FB(P)$, $\{y\} = MS(P)$ and $\{i\} = \arg\max \lambda_P(y)$, and defining $t_1 = \min \lambda_P(x)$, $t_2 = \min \lambda_P(y)$, $t_3 = \max \lambda_P(x)$ and $t_4 = \max \lambda_P(y)$:*

1. $t_1 = \lambda_P(x)_i < t_2 = \lambda_P(y)_{\bar{i}} < t_3 = \lambda_P(x)_{\bar{i}} < t_4 = \lambda_P(y)_i$;

2. $t_1 + t_4 + 1 \leq t_2 + t_3$;

3. $t_3 + t_4 \leq m$. □

Proof Item 1 comes directly from Theorem 3 Item 2.

Item 2 holds as $d(\lambda_P(y)) < d(\lambda_P(x))$, equivalently, $\max \lambda_P(y) - \min \lambda_P(y) < \max \lambda_P(x) - \min \lambda_P(x)$, thus, $t_4 - t_2 < t_3 - t_1$.

To prove Item 3, define $A = \{a \in \mathscr{A} \mid 0 \leq \lambda_P(a)_i < t_4\}$ and $B = \{a \in \mathscr{A} \mid t_3 \leq \lambda_P(a)_{\bar{i}} \leq m-1\}$. Pick $a \in A$. As $\lambda_P(a)_i < t_4 = \lambda_P(y)_i$, $a \neq y$. As $y \in PO(P)$, $\lambda_P(y)_i < \lambda_P(a)_i \vee \lambda_P(y)_{\bar{i}} < \lambda_P(a)_{\bar{i}}$, thus $t_2 = \lambda_P(y)_{\bar{i}} < \lambda_P(a)_{\bar{i}}$.

Now assume that $a \in A \cup PO(P)$. As $t_3 = \max \lambda_P(x) = \min \max \lambda_P$, $t_3 \leq \max \lambda_P(a)$. Thus, if $\lambda_P(a)_i < t_3$ then $t_3 \leq \lambda_P(a)_{\bar{i}}$. Otherwise, $t_3 \leq \lambda_P(a)_i$, which also leads to $t_3 \leq \lambda_P(a)_{\bar{i}}$ because otherwise $t_2 < t_3 \leq \lambda_P(a)_i < t_4$ (as $a \in A$) and $t_2 < \lambda_P(a)_{\bar{i}} < t_3 < t_4$, implying $d(\lambda_P(a)) < t_4 - t_2 = d(\lambda_P(y))$, contradicting $y \in MS(P)$. We conclude that $a \in A \cup PO(P) \Rightarrow t_3 \leq \lambda_P(a)_{\bar{i}}$.

Consider now $a \in A$, $a \notin PO(P)$. Thus $\exists a' \in PO(P) \mid \lambda_P(a')_i < \lambda_P(a)_i \wedge \lambda_P(a')_{\bar{i}} < \lambda_P(a)_{\bar{i}}$. Thus, $\lambda_P(a')_i < \lambda_P(a)_i < t_4$, whence $a' \in A$, which implies $t_3 \leq \lambda_P(a')_{\bar{i}}$ and thus, again, $t_3 \leq \lambda_P(a)_{\bar{i}}$.

We obtain $A \subseteq B$. But $\#A = t_4$ and $\#B = m - t_3$, thus establishing Item 3. ■

Let $T = \{t \in [\![0, m-1]\!]^4 \mid t_1 < t_2 < t_3 < t_4 \wedge t_1 + t_4 + 1 \leq t_2 + t_3 \wedge t_3 + t_4 \leq m\}$ be defined as in Section 3.2. We now state a canonical rewriting of these inequalities, and several consequences, that will reveal useful.

Theorem 5 *$\forall t \in [\![0, m-1]\!]^4$, $t \in T$ iff all the following hold:*

1. $0 \leq t_1 < t_2 < t_3 < t_4$ *(thus $t_1 \leq t_2 - 1$, $t_2 \leq t_3 - 1$, $t_3 \leq t_4 - 1$);*
2. $t_1 + t_4 + 1 \leq t_2 + t_3$;
3. $t_3 + t_4 \leq m$.

Equivalently:

4. $0 \leq t_1$;
5. $0 \leq t_3 - t_2 - 1$;
6. $0 \leq t_4 - t_3 - 1$;
7. $0 \leq t_2 - t_1 - t_4 + t_3 - 1$;
8. $0 \leq m - t_3 - t_4$.

In turn, these imply:

9. $t_4 - t_2 \leq t_3 - t_1 - 1$;

10. $0 \le t_2 - t_4 + t_3 - 1$*;*

11. $0 \le t_2 - t_1 - 2$*;*

12. $0 \le t_2 - 2$*;*

13. $0 \le t_4 - t_2 - 2$*;*

14. $0 \le t_1 + t_4 - t_2 - 2$*;*

15. $0 \le t_4 - t_1 - 4$*;*

16. $0 \le t_4 - 4$*;*

17. $0 \le m - 1 - 2t_4 + t_2 - t_1$*;*

18. $0 \le m - 1 - 2t_4 + t_2$*;*

19. $0 \le 2m - 3t_4 - 2$*;*

20. $0 \le m - 2t_3 - 1$ *(that is,* $0 \le \frac{m-1}{2} - t_3$*);*

21. $0 \le \frac{m-3}{2} - t_2$*;*

22. $0 \le t_2 - t_4 + \frac{m-3}{2}$*;*

23. $0 \le m - 1 - t_4 - t_2$*;*

24. $0 \le \frac{m-5}{2} - t_3 + t_1 + t_4 - t_2$*;*

25. $t_1 + t_3 \le m - 4$. □

Proof Items 1 to 3 are together equivalent to $t \in T$ by definition of T. To see that they imply Items 4 to 8, observe that Items 4 to 6 come from Item 1; Item 7 is equivalent to Item 2; and Item 8 is equivalent to Item 3. We now show that Items 4 to 8 together imply Items 9 to 25.

- Item 9 rearranges Item 7.
- Adding Items 4 and 7 yields Item 10.
- Adding Items 6 and 7 yields Item 11.
- Adding Items 4 and 11 yields Item 12.
- Adding Items 5 and 6 yields Item 13.

- Adding Items 4 and 13 yields Item 14.
- Adding Items 11 and 13 yields Item 15.
- Adding Items 4 and 15 yields Item 16.
- Adding Items 7 and 8 yields Item 17.
- Adding Items 4 and 17 yields Item 18.
- Adding Items 17, 4, 5 and 8 yields Item 19.
- Adding Items 6 and 8 yields Item 20.
- Adding Items 5 and 20 yields Item 21.
- Adding Items 4, 7 and 20 yields Item 22.
- Adding Items 5 and 8 yields Item 23.
- Adding Items 14 and 20 yields Item 24.
- Finally, adding Items 8 and 15 yields Item 25.

To end the proof, observe that Items 4 to 8 (which imply Items 9 to 25) imply Items 1 to 3: Items 4, 11, 5 and 6 together yield Item 1; and Items 7 and 8 are respectively equivalent to Items 2 and 3. ■

The following corollary states how to list exhaustively every tuple belonging to t, starting from picking an appropriate value for t_4, then picking an appropriate value for t_2 (whose bounds thus depend on the choice of t_4), then proceeding similarly for t_3 and finally t_1.

Corollary 1 *$\forall t \in [\![0, m-1]\!]^4$, $t \in T$ iff all the following hold:*

(i) $4 \leq t_4 \leq \frac{2m-2}{3}$;

(ii) $\max\{2, t_4 - \frac{m-3}{2}, 2t_4 - m + 1\} \leq t_2 \leq \min\{m - 1 - t_4, t_4 - 2, \frac{m-3}{2}\}$;

(iii) $\max\{t_2 + 1, t_4 - t_2 + 1\} \leq t_3 \leq \min\{\frac{m-1}{2}, t_4 - 1, m - t_4\}$;

(iv) $0 \leq t_1 \leq t_3 - (t_4 - t_2 + 1)$. □

Proof Given $t \in T$, Item i follows from Theorem 5 Items 16 and 19; Item ii from Items 12, 22, 18, 23, 13 and 21; Item iii from Items 5, 10, 20, 6 and 8 and Item iv from Items 4 and 7.

For the backwards direction, Item iv implies Theorem 5 Items 4 and 7 and Item iii implies Items 5, 6 and 8, thus (still using Theorem 5) $t \in T$. ∎

Corollary 2 $\forall P \in \mathcal{L}(\mathscr{A})^{\{1,2\}} \mid FB(P) \cap MS(P) = \emptyset$, *letting* $\{x\} = FB(P)$ *and* $\{y\} = MS(P)$*:*

$$1 \leq d(\lambda_P(x)) - d(\lambda_P(y)) \leq \frac{m-5}{2}.$$ □

Proof Define $t_1 = \min \lambda_P(x)$, $t_2 = \min \lambda_P(y)$, $t_3 = \max \lambda_P(x)$ and $t_4 = \max \lambda_P(y)$ and apply Theorem 4 and Theorem 5 Items 7 and 24 to get $1 \leq t_3 - t_1 - t_4 + t_2 \leq \frac{m-5}{2}$. ∎

Let us write the values of the bounds for $m = 13$, as adopted for our experiment.

Corollary 3 *Given* $m = 13$, $\forall t \in [\![0, m-1]\!]^4$, $t \in T$ *iff all the following hold:*

(i) $4 \leq t_4 \leq 8$*;*

(ii) $\max\{2, t_4 - 5, 2t_4 - 12\} \leq t_2 \leq \min\{12 - t_4, t_4 - 2, 5\}$*;*

(iii) $\max\{t_2 + 1, t_4 - t_2 + 1\} \leq t_3 \leq \min\{6, t_4 - 1, 13 - t_4\}$*;*

(iv) $0 \leq t_1 \leq t_3 - (t_4 - t_2 + 1)$. □

These bounds can be simplified by considering separately the cases $t_4 = 8$ and $t_4 \leq 7$.

Corollary 4 *Given* $m = 13$, $\forall t \in [\![0, m-1]\!]^4 \mid t_4 = 8$, $t \in T$ *iff all the following hold:*

(i) $t_2 = 4$*;*

(ii) $t_3 = 5$*;*

(iii) $t_1 = 0$. □

Corollary 5 *Given* $m = 13$, $\forall t \in [\![0, m-1]\!]^4 \mid t_4 \leq 7$, $t \in T$ *iff all the following hold:*

(i) $4 \leq t_4 \leq 7$;

(ii) $2 \leq t_2 \leq t_4 - 2$;

(iii) $\max\{t_2 + 1, t_4 - t_2 + 1\} \leq t_3 \leq t_4 - 1$;

(iv) $0 \leq t_1 \leq t_3 - (t_4 - t_2 + 1)$. □

A.4. Classifying profiles

Let $C_{t,i} = \{P \in \mathscr{G} \mid \lambda_P(FB(P)) = \{(i, t_1), (\bar{i}, t_3)\} \wedge \lambda_P(MS(P)) = \{(i, t_4), (\bar{i}, t_2)\}\}$ and $C_t = \bigcup_{i \in N} C_{t,i}$ be defined as in Section 3.2. Let $\mathscr{C} = \{C_t \mid t \in T\}$ denote the set of classes.

Theorem 6 *$\forall t \in T, i \in N$:*

- $\exists P_B \in C_{t,i} \mid FB(P_B) = B(P_B)$;
- $t_1 = 1 \Rightarrow \exists P_D \in C_{t,i} \mid FB(P_D) \subset B(P_D)$;
- $t_1 \geq 2 \Rightarrow \exists P_D \in C_{t,i} \mid FB(P_D) \cap B(P_D) = \emptyset$; □

Proof Throughout this proof, item numbers refer to Theorem 5.

Name the alternatives $a_1, \ldots, a_m$. Define $x = a_{t_1+1}$ and $y = a_{t_4+1}$. Define sequences of alternatives

A_1 $(a_{t_4+2}, \ldots, a_{t_2+t_4+1})$;

A_2 $(a_{t_2+t_4+2}, \ldots, a_{t_3+t_4})$ (with $A_2 = \emptyset$ iff $t_3 + t_4 < t_2 + t_4 + 2$);

A_3 $(a_{t_3+t_4+1}, \ldots, a_m)$;

A_4 $(a_1, \ldots, a_{t_1})$;

A_5 $(a_{t_1+2}, \ldots, a_{t_4})$.

Given a sequence A, define A^{-1} as the inverse of A. Define P_B as

i $(a_1, \ldots, a_m) = (A_4, x, A_5, y, A_1, A_2, A_3)$,

$\bar{i}$ $(A_1^{-1}, y, A_2, x, A_3, A_5, A_4^{-1})$

and define P_D as

i $(a_1, \ldots, a_m) = (A_4, x, A_5, y, A_1, A_2, A_3)$,

$\bar{i}$ $(A_1^{-1}, y, A_2, x, A_4, A_3, A_5)$.

Note that P_B and P_D differ only by the position and ordering of A_4 in $\bar{i}$. Therefore, most of the facts we will establish for P_B also hold, and can be established in a similar manner, for P_D. We will therefore reason in parallel for these two profiles in what follows.

Note that $P_B(\bar{i})$ and $P_D(\bar{i})$ are indeed linear orders on $\mathscr{A}$, equivalently, list each alternative of $\mathscr{A}$ exactly once, as they are permutations of the sequence $(A_4, x, A_5, y, A_1, A_2, A_3)$ which lists each alternative of $\mathscr{A}$ exactly once. The latter fact holds because $t_2 + t_4 + 1 \leq t_3 + t_4$, equivalently, $t_2 + 1 \leq t_3$, from Item 1.

Observe that $t_1 = \lambda_{P_B}(x)_i = \lambda_{P_D}(x)_i$ and $t_3 = \#(A_1 \cup \{y\} \cup A_2) = \lambda_{P_B}(x)_{\bar{i}} = \lambda_{P_D}(x)_{\bar{i}}$. It follows that $\min \lambda_{P_B}(x) = \min \lambda_{P_D}(x) = t_1$ and $\max \lambda_{P_B}(x) = \max \lambda_{P_D}(x) = t_3$. Similarly, $t_4 = \lambda_{P_B}(y)_i = \lambda_{P_D}(y)_i$ and $t_2 = \#A_1 = \lambda_{P_B}(y)_{\bar{i}} = \lambda_{P_D}(y)_{\bar{i}}$. It follows that $\min \lambda_{P_B}(y) = \min \lambda_{P_D}(y) = t_2$ and $\max \lambda_{P_B}(y) = \max \lambda_{P_D}(y) = t_4$.

To see that $FB(P_B) = FB(P_D) = \arg\min \max \lambda_{P_B} = \{x\}$, observe that $\max \lambda_{P_B}(x) = \max \lambda_{P_D}(x) = t_3$ and let us show that $\forall z \in \mathscr{A} \setminus \{x\} : \max \lambda_{P_B}(z) > t_3 \wedge \max \lambda_{P_D}(z) > t_3$. Indeed:

- $\forall z \in \{y\} \cup A_1 \cup A_2 \cup A_3, t_3 < t_4 = \lambda_{P_B}(y)_i < \lambda_{P_B}(z)_i \leq \max \lambda_{P_B}(z)$;
- $\forall z \in \{y\} \cup A_1 \cup A_2 \cup A_3, t_3 < t_4 = \lambda_{P_D}(y)_i < \lambda_{P_D}(z)_i \leq \max \lambda_{P_D}(z)$;
- $\forall z \in \{A_4 \cup A_5\}, t_3 = \lambda_{P_B}(x)_{\bar{i}} < \lambda_{P_B}(z)_{\bar{i}} \leq \max \lambda_{P_B}(z)$;
- $\forall z \in \{A_4 \cup A_5\}, t_3 = \lambda_{P_D}(x)_{\bar{i}} < \lambda_{P_D}(z)_{\bar{i}} \leq \max \lambda_{P_D}(z)$.

To show that $MS(P) = MS(P_D) = \arg\min_{PO(P)}(d \circ \lambda_P) = \{y\}$, observe that $y \in PO(P)$, $y \in PO(P_D)$, $(d \circ \lambda_P)(y) = (d \circ \lambda_{P_D})(y) = t_4 - t_2$, and let us show that $\forall z \in \mathscr{A} \setminus \{y\} : (d \circ \lambda_P)(z) > t_4 - t_2 \wedge (d \circ \lambda_{P_D})(z) > t_4 - t_2$.

- $\forall z \in A_1$, $\lambda_{P_B}(z)_{\bar{i}} = \lambda_{P_D}(z)_{\bar{i}} < \lambda_{P_B}(y)_{\bar{i}} < \lambda_{P_B}(y)_i < \lambda_{P_B}(z)_i = \lambda_{P_D}(z)_i$ thus $d(\lambda_{P_B}(z)) = d(\lambda_{P_D}(z)) > d(\lambda_{P_B}(y)) = t_4 - t_2$.
- $\forall z \in A_3$, $d(\lambda_{P_D}(z)) = \max \lambda_{P_D}(z) - \min \lambda_{P_D}(z) = \#(A_4 \cup \{x\} \cup A_5 \cup \{y\} \cup A_1 \cup A_2) - \#(A_1 \cup \{y\} \cup A_2 \cup \{x\} \cup A_4) = \#A_5 = t_4 - t_1 - 1 > t_4 - t_2$ (using $t_1 + 1 < t_1 + 2 \leq t_2$ from Item 11).
- $\forall z \in A_3$, $d(\lambda_{P_B}(z)) = \#(A_4 \cup A_5) \geq \#A_5 > t_4 - t_2$ (from the previous line).

- $\forall z \in A_2$, $d(\lambda_{P_B}(z)) = d(\lambda_{P_D}(z)) > \#A_5 > t_4 - t_2$.

- $\forall z \in A_5$, $d(\lambda_{P_B}(z)) = \max \lambda_{P_B}(z) - \min \lambda_{P_B}(z) = |\#(A_1 \cup \{y\} \cup A_2 \cup \{x\} \cup A_3) - \#(A_4 \cup \{x\})| = |\#(A_1 \cup \{y\} \cup A_2 \cup A_3) - \#A_4| = |m - t_4 - t_1|$. And $m - t_4 - t_1 > m - t_4 - t_1 - 1 \geq t_4 - t_2 > 0$, from Items 17 and 1.

- $\forall z \in A_5$, $d(\lambda_{P_D}(z)) = d(\lambda_{P_B}(z)) + \#A_4 > t_4 - t_2$.

- $d(\lambda_{P_B}(x)) = d(\lambda_{P_D}(x)) = t_3 - t_1 > t_3 - t_1 - 1 \geq t_4 - t_2$ (using Item 9).

- $\forall z \in A_4$, $d(\lambda_{P_B}(z)) = \#(A_1 \cup \{y\} \cup A_2 \cup \{x\} \cup A_3 \cup A_5) - \#A_4 > m - t_4 - t_1 > t_4 - t_2$ (from a previous line).

- $\forall z \in A_4$, $d(\lambda_{P_D}(z)) = \#(A_1 \cup \{y\} \cup A_2 \cup \{x\}) = t_3 + 1 > t_3 \geq t_3 - t_1 > t_4 - t_2$ (from a previous line).

We see that $P_B \in \mathscr{G}$ and $P_D \in \mathscr{G}$ as it has already been established that $\forall z \in \mathscr{A} \setminus \{y\} : (d \circ \lambda_{P_B})(z) > (d \circ \lambda_{P_B})(y) \wedge (d \circ \lambda_{P_D})(z) > (d \circ \lambda_{P_D})(y)$.

Finally, we turn to the Borda winners. Observe that $\sum \lambda_{P_B}(x) = t_1 + t_3$. Let us show first that $\forall z \in \mathscr{A} \setminus (A_4 \cup \{x\}) : \sum \lambda_{P_B}(z) > t_1 + t_3 \wedge \sum \lambda_{P_D}(z) > t_1 + t_3$.

- $\forall z \in A_1$, $\sum \lambda_{P_B}(z) = \sum \lambda_{P_D}(z) = \#(A_4 \cup \{x\} \cup A_5 \cup \{y\} \cup A_1 \setminus \{z\}) = t_4 + 1 + t_2 - 1 = t_2 + t_4 > t_1 + t_3$.

- $\forall z \in \{y\} \cup A_2 \cup A_3$, $\sum \lambda_{P_B}(z) \geq \sum \lambda_{P_B}(y) = t_2 + t_4 > t_1 + t_3$.

- $\forall z \in \{y\} \cup A_2 \cup A_3$, $\sum \lambda_{P_D}(z) \geq \sum \lambda_{P_D}(y) = t_2 + t_4 > t_1 + t_3$.

To see that $B(P_B) = \arg\min \sum \lambda_P = \{x\}$, observe furthermore that $\forall z \in A_4$, $\sum \lambda_{P_B}(z) \geq \#(A_1 \cup \{y\} \cup A_2 \cup \{x\} \cup A_3 \cup A_5 \cup A_4 \setminus \{z\}) = \#\mathscr{A} \setminus \{z\} = m - 1 > m - 4 \geq t_1 + t_3$ (using Item 25).

Only remains to consider $B(P_D)$. We know from above that A_4 and x are the only candidates. Thus, $B(P_D) \subseteq \{a_1, x\}$. (Note that $a_1 \neq x \Leftrightarrow t_1 \geq 1 \Leftrightarrow A_4 \neq \emptyset \Leftrightarrow P_B \neq P_D$.) We see that $\sum \lambda_{P_D}(a_1) - \sum \lambda_{P_D}(x) = (\lambda_{P_D}(a_1)_i - \lambda_{P_D}(x)_i) + (\lambda_{P_D}(a_1)_{\bar{i}} - \lambda_{P_D}(x)_{\bar{i}}) = -\#A_4 + 1 = 1 - t_1$. Thus, if $t_1 = 1$, $B(P_D) = \{a_1, x\}$ and if $t_1 \geq 2$, $B(P_D) = \{a_1\}$. ■

Example 181 Here is an example construction of profiles P_B, P_D as in the proof of Theorem 6, with $m = 13$ and $t_1 = \lambda_P(x)_i = 1$; $t_2 = \lambda_P(y)_{\bar{i}} = 4$; $t_3 = \lambda_P(x)_{\bar{i}} = 5$; $t_4 = \lambda_P(y)_i = 6$; thus $\sum \lambda_P(y) = 10$, $x = \boldsymbol{b}$ and $y = \boxed{g}$ (for $P \in \{P_B, P_D\}$).

A_1 $(a_8, \ldots, a_{11}) = (h, i, j, k)$;

A_2 $(a_{12}, \ldots, a_{11}) = \emptyset$;

A_3 $(a_{12}, \ldots, a_{13}) = (l, m)$;

A_4 $(a_1, \ldots, a_1) = (a)$;

A_5 $(a_3, \ldots, a_6) = (c, d, e, f)$.

The profile P_B is:

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & f & \boxed{g} & h & i & j & k & l & m \\ k & j & i & h & \boxed{g} & \boldsymbol{b} & l & m & c & d & e & f & a. \end{array}$$

Observe that $B(P_B) = \{\boldsymbol{b}\}$.

The profile P_D is:

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & f & \boxed{g} & h & i & j & k & l & m \\ k & j & i & h & \boxed{g} & \boldsymbol{b} & a & l & m & c & d & e & f. \end{array}$$

Observe that $B(P_D) = \{a, \boldsymbol{b}\}$. □

Theorem 7 *The set of classes $\{C_t \mid t \in T\}$ is a complete and disjoint cover of $\mathscr{G}$. Furthermore, given any $t \in T$, no class C_t is empty.* □

Proof First, it is a complete cover, formally, $\mathscr{G} = \bigcup\{C_t \mid t \in T\}$: from Theorem 4, $\forall P \in \mathscr{G}, \exists t \in T \mid P \in C_t$; and conversely, $\forall t \in T, P \in C_t$, it follows from the definitions that $P \in \mathscr{G}$.

Second, that the classes are disjoint, formally, $C_t \cap C_{t'} \neq \emptyset \Rightarrow t = t'$, can be seen as follows. Consider any $P \in C_t$, let $\{x\} = FB(P)$ and $\{y\} = MS(P)$ and pick any $i \mid P \in C_{t,i}$. By definition of $C_{t,i}$, $\lambda_P(x)_i = t_1 < \lambda_P(y)_{\bar{i}} = t_2 < \lambda_P(x)_{\bar{i}} = t_3 < \lambda_P(y)_i = t_4$. It follows that $P \notin C_{t',\bar{i}}$ (as the latter requires $\lambda_P(x)_{\bar{i}} = t'_1 < \lambda_P(x)_i = t'_3$). Therefore, if $P \in C_{t'}$, then $P \in C_{t',i}$, thus $\lambda_P(x)_i = t'_1, \lambda_P(y)_{\bar{i}} = t'_2, \lambda_P(x)_{\bar{i}} = t'_3$ and $\lambda_P(y)_i = t'_4$, whence $t = t'$.

Finally, Theorem 6 proves that no C_t is empty. ■

A.5. Borda

Theorem 8 $\forall P \in \mathcal{L}(\mathscr{A})^{\{1,2\}} : [\exists x \in FB(P) \mid \min \lambda_P(x) = 0] \Rightarrow B(P) \subseteq FB(P)$. □

Proof Consider any $z \in B(P)$. Observe that $\sum \lambda_P(z) \leq \sum \lambda_P(x)$, thus, $\max \lambda_P(z) \leq \max \lambda_P(z) + \min \lambda_P(z) = \sum \lambda_P(z) \leq \sum \lambda_P(x) = \max \lambda_P(x)$. It follows that $z \in FB(P)$. ■

Theorem 9 $\forall P \in \mathcal{L}(\mathscr{A})^{\{1,2\}} : MS(P) \not\subseteq FB(P) \Rightarrow MS(P) \cap B(P) = \emptyset$. □

PROOF Consider any $y \in MS(P)$ and let us show that $y \notin \arg\min \sum \lambda_P = B(P)$. Pick any $x \in FB(P)$ (thus $\max \lambda_P(x) = \min \max \lambda_P$).

As $MS(P) \not\subseteq FB(P)$, $y \notin FB(P)$, thus $\max \lambda_P(x) = \min \max \lambda_P < \max \lambda_P(y)$.

We furthermore deduce that $\min \lambda_P(y) \geq \min \lambda_P(x)$, otherwise, $\min \lambda_P(y) < \min \lambda_P(x) \leq \max \lambda_P(x) = \min \max \lambda_P < \max \lambda_P(y)$ hence $y \notin MS(P) = \arg\min_{PO(P)}(d \circ \lambda_P)$ (using $x \in FB(P) \Rightarrow x \in PO(P)$).

It follows that $\sum \lambda_P(x) < \sum \lambda_P(y)$. ■

Theorem 10 $\forall P \in \mathcal{L}(\mathscr{A})^{\{1,2\}} : [\exists x \in FB(P) \mid \min \lambda_P(x) = 1] \Rightarrow FB(P) \cap B(P) \neq \emptyset$. □

PROOF Considering any $z \notin FB(P)$, let us show that $\sum \lambda_P(x) \leq \sum \lambda_P(z)$ (which excludes $\arg\min \sum \lambda_P \cap FB(P) = \emptyset$). As $x \in FB(P)$, $\max \lambda_P(x) = \min \max \lambda_P$. If $\max \lambda_P(z) = \min \max \lambda_P$ then $z \in FB(P)$, thus, $\max \lambda_P(z) > \max \lambda_P(x)$, equivalently, $\max \lambda_P(x) + 1 \leq \max \lambda_P(z)$. Then $\sum \lambda_P(x) = \min \lambda_P(x) + \max \lambda_P(x) = 1 + \max \lambda_P(x) \leq \max \lambda_P(z) \leq \sum \lambda_P(z)$. ■

Remark 1 Here are the possible relationships between $B(P)$ and $FB(P)$, considering $P \in \mathscr{G}$ (thus $MS(P) \not\subseteq FB(P)$, with $FB(P) = \{x\}$):

- if $\min \lambda_P(x) = 0$, $B(P) = FB(P)$ (Theorem 8);
- if $\min \lambda_P(x) = 1$, $x \in B(P)$ (Theorem 10), with both $FB(P) = B(P)$ and $FB(P) \subset B(P)$ possible (Theorem 6);
- if $\min \lambda_P(x) \geq 2$, both $FB(P) = B(P)$ and $FB(P) \cap B(P) = \emptyset$ are possible (Theorem 6). □

A.6. Tightness results

The following two theorems illustrate why we cannot obtain the results that were obtained above with the hypothesis $FB(P) \cap MS(P) = \emptyset$ by using merely an hypothesis of non inclusion.

Theorem 11 *For any $m \geq 6$, $FB(P) \cap MS(P) \neq \emptyset \not\Rightarrow MS(P) \subseteq FB(P)$. In other words, $\exists P \mid FB(P) \cap MS(P) \neq \emptyset \wedge MS(P) \not\subseteq FB(P)$. In supplement, $\exists P \mid FB(P) \cap \arg\min_{\mathscr{A}}(d \circ \lambda_P) \neq \emptyset \wedge \arg\min_{\mathscr{A}}(d \circ \lambda_P) \not\subseteq FB(P)$.* □

PROOF In the following profile P,

$$\begin{array}{cccccccc} a & b & c & d & e & f & \dots \\ e & d & a & f & b & c & \dots \end{array},$$

$FB(P) = \arg\min\max\lambda_P = \{a\}$ and $MS(P) = \arg\min_{PO(P)}(d \circ \lambda_P) = \arg\min_{\mathscr{A}}(d \circ \lambda_P) = \{a, d\}$. ■

Theorem 12 *For any $m \geq 4$, $FB(P) \cap MS(P) \neq \emptyset \not\Rightarrow FB(P) \subseteq MS(P)$.* □

PROOF In the following profile P,

$$\begin{array}{ccccc} a & c & b & d & \dots \\ d & b & a & c & \dots \end{array},$$

$FB(P) = \arg\min\max\lambda_P = \{a, b\}$ and $MS(P) = \arg\min_{PO(P)}(d \circ \lambda_P) = \arg\min_{\mathscr{A}}(d \circ \lambda_P) = \{b\}$. ■

A.7. Instances B

The profiles in Examples 182 to 206 all satisfy $B(P) = FB(P)$ and are generated using the process described in Theorem 6 and illustrated in Example 181. They are listed in the same order as Table 2.

Example 182 ($\boldsymbol{\lambda_P(x) = \{0, 6\}}$; $\boldsymbol{\lambda_P(y) = \{5, 7\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ m & l & k & j & i & \boxed{h} & \boldsymbol{a} & b & c & d & e & f & g \end{array}$$
□

Example 183 ($\boldsymbol{\lambda_P(x) = \{1, 6\}}$; $\boldsymbol{\lambda_P(y) = \{5, 7\}}$; B)

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ m & l & k & j & i & \boxed{h} & \boldsymbol{b} & c & d & e & f & g & a \end{array}$$
□

Example 184 ($\boldsymbol{\lambda_P(x) = \{0, 5\}}$; $\boldsymbol{\lambda_P(y) = \{4, 6\}}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & e & f & \boxed{g} & h & i & j & k & l & m \\ k & j & i & h & \boxed{g} & \boldsymbol{a} & l & m & b & c & d & e & f \end{array}$$
□

Example 185 $(\boldsymbol{\lambda_P(x)} = \{0,6\}; \boldsymbol{\lambda_P(y)} = \{4,7\};$ **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ l & k & j & i & \boxed{h} & m & \boldsymbol{a} & b & c & d & e & f & g \end{array}$$

□

Example 186 $(\boldsymbol{\lambda_P(x)} = \{2,6\}; \boldsymbol{\lambda_P(y)} = \{5,7\};$ **B)**

$$\begin{array}{ccccccccccccc} a & b & \boldsymbol{c} & d & e & f & g & \boxed{h} & i & j & k & l & m \\ m & l & k & j & i & \boxed{h} & \boldsymbol{c} & d & e & f & g & b & a \end{array}$$

□

Example 187 $(\boldsymbol{\lambda_P(x)} = \{1,5\}; \boldsymbol{\lambda_P(y)} = \{4,6\};$ **B)**

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & f & \boxed{g} & h & i & j & k & l & m \\ k & j & i & h & \boxed{g} & \boldsymbol{b} & l & m & c & d & e & f & a \end{array}$$

□

Example 188 $(\boldsymbol{\lambda_P(x)} = \{0,4\}; \boldsymbol{\lambda_P(y)} = \{3,5\};$ **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & e & \boxed{f} & g & h & i & j & k & l & m \\ i & h & g & \boxed{f} & \boldsymbol{a} & j & k & l & m & b & c & d & e \end{array}$$

□

Example 189 $(\boldsymbol{\lambda_P(x)} = \{1,6\}; \boldsymbol{\lambda_P(y)} = \{4,7\};$ **B)**

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ l & k & j & i & \boxed{h} & m & \boldsymbol{b} & c & d & e & f & g & a \end{array}$$

□

Example 190 $(\boldsymbol{\lambda_P(x)} = \{0,5\}; \boldsymbol{\lambda_P(y)} = \{3,6\};$ **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & e & f & \boxed{g} & h & i & j & k & l & m \\ j & i & h & \boxed{g} & k & \boldsymbol{a} & l & m & b & c & d & e & f \end{array}$$

□

Example 191 $(\boldsymbol{\lambda_P(x)} = \{0,6\}; \boldsymbol{\lambda_P(y)} = \{3,7\};$ **B)**

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ k & j & i & \boxed{h} & l & m & \boldsymbol{a} & b & c & d & e & f & g \end{array}$$

□

Example 192 $(\boldsymbol{\lambda_P(x)} = \{0,5\};\ \boldsymbol{\lambda_P(y)} = \{4,7\};\ \mathrm{B})$

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ l & k & j & i & \boxed{h} & \boldsymbol{a} & m & b & c & d & e & f & g \end{array}$$

□

Example 193 $(\boldsymbol{\lambda_P(x)} = \{3,6\};\ \boldsymbol{\lambda_P(y)} = \{5,7\};\ \mathrm{B})$

$$\begin{array}{ccccccccccccc} a & b & c & \boldsymbol{d} & e & f & g & \boxed{h} & i & j & k & l & m \\ m & l & k & j & i & \boxed{h} & \boldsymbol{d} & e & f & g & c & b & a \end{array}$$

□

Example 194 $(\boldsymbol{\lambda_P(x)} = \{2,5\};\ \boldsymbol{\lambda_P(y)} = \{4,6\};\ \mathrm{B})$

$$\begin{array}{ccccccccccccc} a & b & \boldsymbol{c} & d & e & f & \boxed{g} & h & i & j & k & l & m \\ k & j & i & h & \boxed{g} & \boldsymbol{c} & l & m & d & e & f & b & a \end{array}$$

□

Example 195 $(\boldsymbol{\lambda_P(x)} = \{1,4\};\ \boldsymbol{\lambda_P(y)} = \{3,5\};\ \mathrm{B})$

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & \boxed{f} & g & h & i & j & k & l & m \\ i & h & g & \boxed{f} & \boldsymbol{b} & j & k & l & m & c & d & e & a \end{array}$$

□

Example 196 $(\boldsymbol{\lambda_P(x)} = \{0,3\};\ \boldsymbol{\lambda_P(y)} = \{2,4\};\ \mathrm{B})$

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & \boxed{e} & f & g & h & i & j & k & l & m \\ g & f & \boxed{e} & \boldsymbol{a} & h & i & j & k & l & m & b & c & d \end{array}$$

□

Example 197 $(\boldsymbol{\lambda_P(x)} = \{2,6\};\ \boldsymbol{\lambda_P(y)} = \{4,7\};\ \mathrm{B})$

$$\begin{array}{ccccccccccccc} a & b & \boldsymbol{c} & d & e & f & g & \boxed{h} & i & j & k & l & m \\ l & k & j & i & \boxed{h} & m & \boldsymbol{c} & d & e & f & g & b & a \end{array}$$

□

Example 198 $(\boldsymbol{\lambda_P(x)} = \{1,5\};\ \boldsymbol{\lambda_P(y)} = \{3,6\};\ \mathrm{B})$

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & f & \boxed{g} & h & i & j & k & l & m \\ j & i & h & \boxed{g} & k & \boldsymbol{b} & l & m & c & d & e & f & a \end{array}$$

□

Example 199 ($\lambda_P(x) = \{0,4\}$; $\lambda_P(y) = \{2,5\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & e & \boxed{f} & g & h & i & j & k & l & m \\ h & g & \boxed{f} & i & \boldsymbol{a} & j & k & l & m & b & c & d & e \end{array}$$

□

Example 200 ($\lambda_P(x) = \{1,6\}$; $\lambda_P(y) = \{3,7\}$; B)

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ k & j & i & \boxed{h} & l & m & \boldsymbol{b} & c & d & e & f & g & a \end{array}$$

□

Example 201 ($\lambda_P(x) = \{0,5\}$; $\lambda_P(y) = \{2,6\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & e & f & \boxed{g} & h & i & j & k & l & m \\ i & h & \boxed{g} & j & k & \boldsymbol{a} & l & m & b & c & d & e & f \end{array}$$

□

Example 202 ($\lambda_P(x) = \{0,6\}$; $\lambda_P(y) = \{2,7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ j & i & \boxed{h} & k & l & m & \boldsymbol{a} & b & c & d & e & f & g \end{array}$$

□

Example 203 ($\lambda_P(x) = \{1,5\}$; $\lambda_P(y) = \{4,7\}$; B)

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ l & k & j & i & \boxed{h} & \boldsymbol{b} & m & c & d & e & f & g & a \end{array}$$

□

Example 204 ($\lambda_P(x) = \{0,4\}$; $\lambda_P(y) = \{3,6\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & e & f & \boxed{g} & h & i & j & k & l & m \\ j & i & h & \boxed{g} & \boldsymbol{a} & k & l & m & b & c & d & e & f \end{array}$$

□

Example 205 ($\lambda_P(x) = \{0,5\}$; $\lambda_P(y) = \{3,7\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ k & j & i & \boxed{h} & l & \boldsymbol{a} & m & b & c & d & e & f & g \end{array}$$

□

Example 206 ($\lambda_P(x) = \{0,5\}$; $\lambda_P(y) = \{4,8\}$; B)

$$\begin{array}{ccccccccccccc} \boldsymbol{a} & b & c & d & e & f & g & h & \boxed{i} & j & k & l & m \\ m & l & k & j & \boxed{i} & \boldsymbol{a} & b & c & d & e & f & g & h \end{array}$$

□

A.8. Instances D

The profiles in Examples 207 to 217 all satisfy $B(P) \neq FB(P)$ and are generated using the process described in Theorem 6 and illustrated in Example 181. They are listed in the same order as Table 2, including only those for which satisfying $B(P) \neq FB(P)$ is possible (that is, whenever $\min \lambda_P(FB(P)) \geq 1$).

Example 207 $(\boldsymbol{\lambda_P(x) = \{1,6\}; \lambda_P(y) = \{5,7\};}$ **D)**

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ m & l & k & j & i & \boxed{h} & \boldsymbol{b} & a & c & d & e & f & g \end{array}$$

□

Example 208 $(\boldsymbol{\lambda_P(x) = \{2,6\}; \lambda_P(y) = \{5,7\};}$ **D)**

$$\begin{array}{ccccccccccccc} a & b & \boldsymbol{c} & d & e & f & g & \boxed{h} & i & j & k & l & m \\ m & l & k & j & i & \boxed{h} & \boldsymbol{c} & a & b & d & e & f & g \end{array}$$

□

Example 209 $(\boldsymbol{\lambda_P(x) = \{1,5\}; \lambda_P(y) = \{4,6\};}$ **D)**

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & f & \boxed{g} & h & i & j & k & l & m \\ k & j & i & h & \boxed{g} & \boldsymbol{b} & a & l & m & c & d & e & f \end{array}$$

□

Example 210 $(\boldsymbol{\lambda_P(x) = \{1,6\}; \lambda_P(y) = \{4,7\};}$ **D)**

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ l & k & j & i & \boxed{h} & m & \boldsymbol{b} & a & c & d & e & f & g \end{array}$$

□

Example 211 $(\boldsymbol{\lambda_P(x) = \{3,6\}; \lambda_P(y) = \{5,7\};}$ **D)**

$$\begin{array}{ccccccccccccc} a & b & c & \boldsymbol{d} & e & f & g & \boxed{h} & i & j & k & l & m \\ m & l & k & j & i & \boxed{h} & \boldsymbol{d} & a & b & c & e & f & g \end{array}$$

□

Example 212 $(\boldsymbol{\lambda_P(x) = \{2,5\}; \lambda_P(y) = \{4,6\};}$ **D)**

$$\begin{array}{ccccccccccccc} a & b & \boldsymbol{c} & d & e & f & \boxed{g} & h & i & j & k & l & m \\ k & j & i & h & \boxed{g} & \boldsymbol{c} & a & b & l & m & d & e & f \end{array}$$

□

Example 213 ($\boldsymbol{\lambda_P(x) = \{1, 4\}}$; $\boldsymbol{\lambda_P(y) = \{3, 5\}}$; D)

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & \boxed{f} & g & h & i & j & k & l & m \\ i & h & g & \boxed{f} & \boldsymbol{b} & a & j & k & l & m & c & d & e \end{array}$$

□

Example 214 ($\boldsymbol{\lambda_P(x) = \{2, 6\}}$; $\boldsymbol{\lambda_P(y) = \{4, 7\}}$; D)

$$\begin{array}{ccccccccccccc} a & b & \boldsymbol{c} & d & e & f & g & \boxed{h} & i & j & k & l & m \\ l & k & j & i & \boxed{h} & m & \boldsymbol{c} & a & b & d & e & f & g \end{array}$$

□

Example 215 ($\boldsymbol{\lambda_P(x) = \{1, 5\}}$; $\boldsymbol{\lambda_P(y) = \{3, 6\}}$; D)

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & f & \boxed{g} & h & i & j & k & l & m \\ j & i & h & \boxed{g} & k & \boldsymbol{b} & a & l & m & c & d & e & f \end{array}$$

□

Example 216 ($\boldsymbol{\lambda_P(x) = \{1, 6\}}$; $\boldsymbol{\lambda_P(y) = \{3, 7\}}$; D)

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ k & j & i & \boxed{h} & l & m & \boldsymbol{b} & a & c & d & e & f & g \end{array}$$

□

Example 217 ($\boldsymbol{\lambda_P(x) = \{1, 5\}}$; $\boldsymbol{\lambda_P(y) = \{4, 7\}}$; D)

$$\begin{array}{ccccccccccccc} a & \boldsymbol{b} & c & d & e & f & g & \boxed{h} & i & j & k & l & m \\ l & k & j & i & \boxed{h} & \boldsymbol{b} & a & m & c & d & e & f & g \end{array}$$

□