016 - *p*-values

EPIB 607

Sahir Rai Bhatnagar
Department of Epidemiology, Biostatistics, and Occupational Health
McGill University

`sahir.bhatnagar@mcgill.ca`

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p-values

The prosecutor’s fallacy

Relationship between *p*-value and CI

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Summary

p-values and statistical tests

Definition 1 (*p*-value).

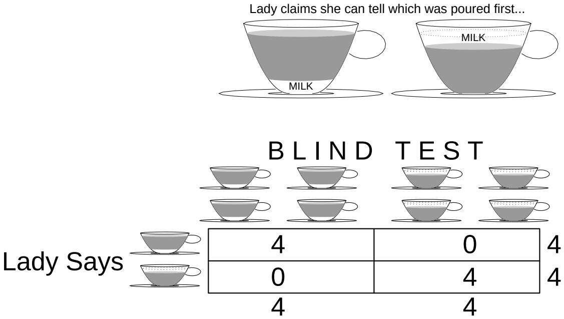
A **probability concerning the observed data**, calculated under a **Null Hypothesis** assumption, i.e., assuming that the only factor operating is sampling or measurement variation.

Use To assess the evidence provided by the sample data in relation to a pre-specified claim or 'hypothesis' concerning some parameter(s) or data-generating process.

Basis As with a confidence interval, it makes use of the concept of a *distribution*.

Caution A *p*-value is NOT the probability that the null 'hypothesis' is true

Example 1 – from *Design of Experiments*, by R.A. Fisher



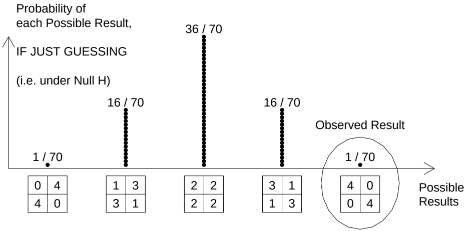
4	0	4
0	4	4
4	4	

Null Hypothesis (H_{null}): she can not tell them apart, i.e., just guessing.

Alternative Hypothesis (H_{alt}): she can.

The evidence provided by the test

- Rank possible test results by degree of evidence against H_{null}.
- "p-value" is the probability, calculated under null hypothesis, of observing a result as extreme as, or more extreme than, the one that was obtained/observed.



In this example, observed result is the most extreme, so

$$P_{value} = \text{Prob[correctly identifying all 4, IF merely guessing]} = 1/70 = 0.014.$$

- Interpretation of such data often rather simplistic, as if these *data alone* should *decide*: i.e. if $P_{value} < 0.05$, we ~~'reject' H~~$_{null}$; if $P_{value} > 0.05$, we don't (or worse, we ~~'accept' H~~$_{null}$). Avoid such simplistic 'conclusions'.

p-value via the Normal (Gaussian) distribution.

- When judging extremeness of a sample mean or proportion (or difference between 2 sample means or proportions) calculated from an amount of information that is sufficient for the Central Limit Theorem to apply, one can use Gaussian distribution to readily obtain the p-value.
- Calculate how many standard errors of the statistic, $SE_{statistic}$, the statistic is from where null hypothesis states true value should be. This "number of SE's" is in this situation referred to as a 'Z_{value}.'

  $$Z_{value} = \frac{statistic - \text{its expected value under } H_{null}}{SE_{statistic}}.$$

  p-value can then be obtained by determining what % of values in a Normal distribution are as extreme or more extreme than this Z_{value}.
- If n is small enough that value of $SE_{statistic}$, is itself subject to some uncertainty, one would instead refer the "number of SE's" to a more appropriate reference distribution, such as Student's t- distribution.

More about the *p*-value

- The *p*-value is a **probability concerning data**, **conditional on the Null Hypothesis being true**.
- **Naive (and not so naive) end-users sometimes interpret the *p*-value as the probability that Null Hypothesis is true**, *conditional on – i.e. given – the data.*

$$\begin{aligned} p_{value} &= P(\text{this or more extreme data}|H_0) \\ &\neq P(H_0|\text{this or more extreme data}). \end{aligned}$$

- Statistical tests are often coded as statistically significant or not according to whether results are extreme or not with respect to a reference (null) distribution. But a test result is just one piece of data, and needs to be considered *along with rest of evidence* before coming to a 'conclusion.'
- **Likewise with statistical 'tests': the *p*-value is just one more piece of *evidence*, hardly enough to 'conclude' anything**.

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The prosecutor's fallacy [1]

- The case of Troy Brown – a man convicted of the rape of a 9-year-old girl. The evidence of Brown's guilt, excluding DNA, was both circumstantial and equivocal.

- However, the jury's guilty verdict was influenced at least in part by the prosecution's claim that only one in 3 million random people would have the same DNA profile as the rapist, and hence there was only a 0.000033% chance that Brown was innocent.

- Upon appeal of the case, the defence argued that the conclusions drawn from the statistics cited by the prosecution were incorrect, and were an example of "the prosecutor's fallacy". The Supreme Court, writing in its decision on the Brown case, described the fallacy as:

  > *the assumption that the random match probability is the same as the probability that the defendant was not the source of the DNA sample. ... ("Let P equal the probability of a match, given the evidence genotype. The fallacy is to say that P is also the probability that the DNA at the crime scene came from someone other than the defendant"). ... It is further error to equate source probability with probability of guilt*

[1] The Bayesian flip Correcting the prosecutor's fallacy. Significance. August 2015.

The prosecutor's fallacy [2]

- Restating this both more succinctly, and in terms better suited to a statistically literate readership, the prosecutor's fallacy is to calculate P(evidence | innocence) and interpret it as P(innocence | evidence).

- It may be true that if the accused were innocent, there is only one chance in 3 million of a DNA match. But the DNA match does not necessarily imply that there is only one chance in 3 million of the accused being innocent.

- Stated more generally, the prosecutor's fallacy is

$$P(A|B) = P(B|A)$$

- We know, from Bayes' rule, that

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

[2] The Bayesian flip Correcting the prosecutor's fallacy. Significance. August 2015.

The Bayesian Flip

- In many investigations we may be presented with P(data | theory), but what we would really like to know is P(theory | data): the probability that our theory is correct, given what we have observed
- To move from P(data | theory) to P(theory | data), we need to do the Bayesian flip.
- Every year in the United States 38 million women are tested for breast cancer with mammograms. Of these, 140 000 have cancer. Mammograms have been determined to be 90% accurate for women with breast cancer.
- This figure was calculated by tallying all of the women who were eventually determined to have breast cancer and looking back to see if their initial mammograms were positive, thus:

  $$P(+mammogram|cancer) = 0.90$$

  and, using a similar empirical investigation,

  $$P(+mammogram|nocancer) = 0.10$$

The Bayesian Flip

- It is important to know that a test is both powerful and has a relatively low rate of false positives. But when one is faced with a positive mammogram result, these are hardly useful. We administer a mammogram because we do not know whether or not someone has cancer.
- What we want to know is

$$P(cancer| + mammogram)$$

- This probability is a fraction that has as its numerator the number of women annually diagnosed with breast cancer via mammograms, or 140 000, and as its denominator the number of positive mammograms (including both true cancer cases and false positives):

$$\begin{aligned} P(cancer| + mammogram) &= \frac{True\,positives}{(True\,positives + False\,positives)} \\ &= 140000/(140000 + 0.1\times 38\ million) \\ &= 140000/(140000 + 3800000) \\ &= 140000/3940000 = 0.036 = 3.6\% \end{aligned}$$

The Bayesian Flip

- Thus, if an asymptomatic woman receives the dreadful news that her mammogram has come back positive, more than 96% of the time it is a false positive – she is fine. The dramatic difference between the 90% statistical power of the test and its 3.6% accuracy demonstrates the importance of not confusing the former with the latter; we must do the Bayesian flip.

The defence attorney's fallacy: the O. J. Simpson trial

- Alan Dershowitz, an advisor to Simpson's defence attorneys, claimed that Simpson's previous accusation of spousal abuse was not particularly relevant. The evidence was that only about one in 2500 men who batter their significant others (wives, girlfriends) go on to kill them.
- The more relevant concern: if a previously battered woman has been murdered, what is the probability that her batterer committed the crime? Clearly, the previous accusations of battery would be considered relevant evidence if we knew, for example, that one in 3 murdered women were murdered by their batterers.
- Let
  - B: woman battered by her husband, boyfriend, or lover
  - M: represents the event "woman murdered", and by extension,
  - M, B: denotes "woman murdered by her batterer".
- Our goal is to compare $P(M, B|M)$ to $P(M, B|B)$.

The defence attorney's fallacy: the O. J. Simpson trial

- Using Bayes' rule, we have

$$P(M,B|M) = \frac{P(M|M,B)P(M,B)}{P(M)}$$

- In 1992, the population of women in the United States was approximately 125 million. That year, 4936 women were murdered. So, one marginal probability, $P(M) = 4936/125000000 = 0.00004$.
- Approximately 3.5 million women are battered every year, so we estimate $P(B) = 0.028$
- That same year 1432 women were murdered by their previous batterers, so the marginal probability of that event is $P(M,B) = 1432/125000000 = 0.00001$

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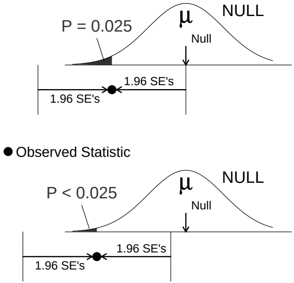
(Intimate) Relationship between *p*-value and CI



- (Upper graph) If upper limit of 95% CI *just touches* null value, then the 2 sided *p*-value is 0.05 (or 1 sided *p*-value is 0.025).
- (Lower graph) If upper limit *excludes* null value, then the 2 sided *p*-value is less than 0.05 (or 1 sided *p*-value is less than 0.025).
- (Graph not shown) If CI *includes* null value, then the 2-sided *p*-value is greater than (the conventional) 0.05, and thus observed statistic is “not statistically significantly different” from hypothesized null value.

Don’t be overly-impressed by *p*-values

- *p*-values and ‘significance tests’ widely misunderstood and misused.
- Very large or very small *n*’s can influence what is or is not ‘statistically significant.’
- Use CI’s instead.
- *Pre study* power calculations (the chance that results will be ‘statistically significant’, as a function of the true underlying difference) of some help.
- *post-study* (i.e., *after the data have ‘spoken’*), a CI is much more relevant, as it focuses on magnitude & precision, not on a probability calculated under H_{null}.

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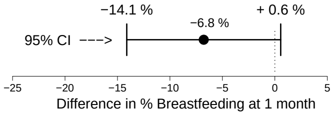
Do infant formula samples ↓ dur$^{n.}$ of breastfeeding?[3]

Randomized Clinical Trial (RCT) which withheld free formula samples [given by baby-food companies to breast-feeding mothers leaving Montreal General Hospital with their newborn infants] from a random half of those studied.

	Mothers			
At 1 month	given sample	not given sample	Total	**Conclusion...**
Still Breast feeding	175 (***77%***)	182 (***84%***)	357 (80.4%)	P=0.07. So, ... the difference is “Not Statistically Significant” at 0.05 level
Not Breast feeding	52	35	87	
Total	227	217	444	



[3] Bergevin Y, Dougherty C, Kramer MS. Lancet. 1983 1(8334):1148-51

Messages

- no matter whether the *p*-value is "statistically significant" or not, always look at the location and width of the confidence interval. it gives you a better and more complete indication of the magnitude of the effect and of the precision with which it was measured.

- this is an example of an **inconclusive negative** study, since it has **insufficient precision** ("resolving power") **to distinguish** between two important possibilities – **no harm**, and what authoroties would consider a **substantial harm: a reduction of 10 percentage points** in breastfeeding rates .

- "**statistically** significant" and "**clinically**-" (or "**public health**-") significant are different concepts.

- (message from 1st author:) plan to have **enough statistical power**. his study had only 50% power to detect a difference of 10 percentage points)

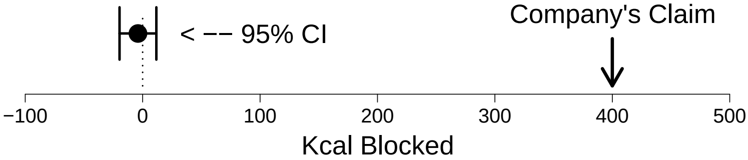
Do starch blockers really block calorie absorption?

Starch blockers – their effect on calorie absorption from a high-starch meal. Bo-Linn GW. et al New Eng J Med. 307(23):1413-6, 1982 Dec 2

- Known for more than 25 years that certain plant foods, e.g., kidney beans & wheat, contain a substance that inhibits activity of salivary and pancreatic amylase.
- More recently, this antiamylase has been purified and marketed for use in weight control under generic name "starch blockers."
- Although this approach to weight control is highly popular, it has never been shown whether starch-blocker tablets actually reduce absorption of calories from starch.
- Using a one-day calorie-balance technique and a high starch (100 g) meal (spaghetti, tomato sauce, and bread), we measured excretion of fecal calories after $n = 5$ normal subjects in a cross-over trial had taken either placebo or starch-blocker tablets.
- If the starch-blocker tablets had prevented the digestion of starch, fecal calorie excretion should have increased by 400 kcal.

Do starch blockers really block calorie absorption?

- However, fecal calorie excretion was same on the 2 test days (mean ± S.E.M., 80 ± 4 as compared with 78 ± 2).



- We conclude that starch blocker tablets do not inhibit the digestion and absorption of starch calories in human beings.
- EFFECT IS MINISCULE (AND ESTIMATE QUITE PRECISE) AND VERY FAR FROM COMPANY'S CLAIM !!!
- A '**DEFINITIVELY NEGATIVE**' STUDY.

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SUMMARY - 1

- Confidence intervals preferable to *p*-values, since they are expressed in terms of (comparative) parameter of interest; they allow us to judge magnitude and its precision, and help us in ‘ruling in / out’ certain parameter values.

- A ‘statistically significant’ difference does not necessarily imply a clinically important difference.

- A ‘not-statistically-significant’ difference does not necessarily imply that we have ruled out a clinically important difference.

SUMMARY - 2

- Precise estimates distinguish b/w that which – if it were true – would be important and that which – if it were true – would not. 'n' an important determinant of precision.
- A lab value in upper 1% of reference distribution (of values derived from people without known diseases/conditions) does not mean that there is a 1% chance that person in whom it was measured is healthy; i.e., it doesn't mean that there's a 99% chance that the person in whom it was measured does have some disease/condition.
- Likewise, p-value $\neq$ probability that null hypothesis is true.
- The fact that

  $$Prob[the\ data \mid Healthy] \text{ is small [or large]}$$

  does not necessarily mean that

  $$Prob[Healthy \mid the\ data] \text{ is small [or large]}$$

SUMMARY - 3

- Ultimately, p-values, CI's and other evidence from a study need to be combined with other information bearing on parameter or process.

- Don't treat any one study as last word on the topic.