Assignment 7

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Question

Papoulis-Pillai Ch 6 Ex 4-20:
The random variables x and y are independent with exponential densities

$$f_x(x) = \alpha e^{-\alpha x} u(x) \qquad\qquad f_y(y) = \beta e^{-\beta y} u(y) \tag{1}$$

Find the densities of the following random variables:

a)$2x + y$ **b)**$x - y$ **c)**$\frac{x}{y}$ **d)**$max(x, y)$ **e)**$min(x, y)$

Solution

a) From the given density of x, density of 2x will be equal to $\frac{1}{2}\ f_x(\frac{x}{2})$,
Hence if $z = 2x + y$, then

$$f_z(z) = \int_0^z \frac{\alpha}{2} e^{-\alpha x/2} \beta e^{-\beta(z-x)}\, dx$$

On integration,

$$f_z(z) = \frac{\alpha\beta}{\alpha - 2\beta}(e^{\beta z} - e^{-\alpha z/2})u(z)$$

Solution

b) From the given density of y, density of -y will be equal to $f_y(-y)$.
Hence if z = x - y, then

$$f_z(z) = f_x(z) * f_y(-z) \quad = \alpha\beta \begin{cases} \int_z^\infty e^{-\alpha x} \cdot e^{-\beta(x-z)}\, dx = \frac{\alpha\beta}{\alpha+\beta} \cdot e^{-\alpha z}, z > 0 \\ \\ \int_z^\infty e^{-\alpha x} \cdot e^{-\beta(x-z)}\, dx = \frac{\alpha\beta}{\alpha+\beta} \cdot e^{\beta z}, z < 0 \end{cases}$$

Solution

c) z = x/y
Let,

$$w = y \qquad\qquad x = wz$$

Now we can write $f_z(z)$ as

$$f_z(z) = \alpha\beta \int_0^{\infty} w \cdot e^{-\alpha z w} \cdot e^{-\beta w}\, dw$$
$$= \frac{\alpha\beta}{(\alpha z + \beta)^2} u(z)$$

Solution

d) z = max(x,y)
Here,

$$F_z(z) = F_{xy}(z, z) = F_x(z) \cdot F_y(z) \tag{2}$$

On differentiating,

$$f_z(z) = f_x(z) \cdot F_y(z) + F_x(z) \cdot f_y(z)$$

$$= [\alpha e^{-\alpha z}(1 - e^{-\beta z}) + \beta e^{-\beta z}(1 - e^{-\alpha z})]u(z)$$

Solution

e) z = min(x,y)
Here,

$$F_z(z) = F_x(z) + F_y(z) - F_x(z) \cdot F_y(z) \tag{3}$$

On differentiating,

$$f_z(z) = f_x(z) \cdot [1 - F_y(z)] + f_y(z) \cdot [1 - F_x(z)]$$

$$= (\alpha + \beta) e^{-(\alpha+\beta)z} u(z)$$