Assignment 5

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Outline

- Question
- Theory
- Solution

Question

Papoulis-Pillai Ch 4 Ex 4-20:

A fair coin is tossed 1000 times. Find the probability p_a that the heads will show 500 times and the probability p_b that the heads will show 510 times.

Theory

DeMoivre-Laplace Theorem:
Suppose a trial is made repeatedly for 'n' number of times, where the probability of success is 'p' and of failure is 'q'. Then the probability 'P' of the trial being successful for exactly 'k' times is given by

$$P(k) = \binom{n}{k} \cdot p^k \cdot q^{n-k} \tag{1}$$

When n is very large and k is in the $\sqrt{npq}$ neighbourhood of np, we can approximate

$$\binom{n}{k} \cdot p^k \cdot q^{n-k} \simeq \frac{1}{\sqrt{2\pi npq}} \cdot e^{\frac{-(k-np)^2}{2npq}} \tag{2}$$

This approximation is known as the DeMoivre-Laplace Theorem.

Solution

(i) Probability that the heads will show 500 times, p_a
As this is a binomial probability distribution and the coin is fair,

$$P(H = 500) = \binom{1000}{500} \cdot (\frac{1}{2})^{(500)} \cdot (\frac{1}{2})^{(1000-500)} \tag{3}$$

As 1000 is a large number, On comparing with the DeMoivre-Laplace Theorem,

$$n = 1000 \tag{4}$$

$$p = \frac{1}{2} \tag{5}$$

$$k = 500 \tag{6}$$

Also, we can see that $np = 500$, $\sqrt{npq} = 5\sqrt{10}$. And $k = 500$ is in the $\sqrt{npq}$ neighbourhood of np.

Solution

On applying the approximation,

$$P(H = 500) = p_a \simeq \frac{1}{\sqrt{2\pi(1000)(\frac{1}{2})(\frac{1}{2})}} \cdot e^{\frac{-(500-(\frac{1000}{2}))^2}{2(1000)(\frac{1}{2})(\frac{1}{2})}} \tag{7}$$

$$\Rightarrow \boxed{p_a \simeq \frac{1}{10\sqrt{5\pi}} = 0.0252} \tag{8}$$

Solution

(ii) Probability that the heads will show 510 times, p_b

$$P(H = 500) = \binom{1000}{510} \cdot (\frac{1}{2})^{(510)} \cdot (\frac{1}{2})^{(1000-510)} \tag{9}$$

Similar to the first part, we can see that $k = 510$ is in the $\sqrt{npq}$ neighbourhood of np.

Solution

So, On applying the approximation

$$P(H = 500) = p_b \simeq \frac{1}{\sqrt{2\pi(1000)(\frac{1}{2})(\frac{1}{2})}} \cdot e^{\frac{-(510-(\frac{1000}{2}))^2}{2(1000)(\frac{1}{2})(\frac{1}{2})}} \tag{10}$$

$$\Rightarrow \boxed{p_b = \frac{e^{\frac{-1}{5}}}{10\sqrt{5\pi}} \simeq 0.0207} \tag{11}$$