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CS 559 HW2

1) Given the bernoulli distribution

$$P(x|\mu) = \mu^{x}(1-\mu)^{1-x} \quad \text{where } \mu \in \{0,1\}$$

Equivalent formulation with $\mu \in \{-1, 1\}$

$$P(x|\mu) = \left(\frac{1-\mu}{2}\right)^{\left(\frac{1-x}{2}\right)} \left(\frac{1+\mu}{2}\right)^{\left(\frac{1+x}{2}\right)}$$

For any Normalized distribution

$$\sum_x P(x|\mu) = 1$$

Substitute $x = +1$

$$P(+1|\mu) = \left(\frac{1-\mu}{2}\right)^{\left(\frac{1-1}{2}\right)} \left(\frac{1+\mu}{2}\right)^{\left(\frac{1+1}{2}\right)}$$

$$= \frac{1+\mu}{2}$$

Sub $x = -1$

$$P(-1|\mu) = \left(\frac{1-\mu}{2}\right)^{\left(\frac{1+1}{2}\right)} \left(\frac{1+\mu}{2}\right)^{\left(\frac{1-1}{2}\right)}$$

$$= \frac{1-\mu}{2}$$

$$\sum_x P(x|\mu) = \frac{1+\mu}{2} + \frac{1-\mu}{2} = 1$$ Hence proved

Mean

We know that mean $= \sum_x x\, p(x|\mu)$

$$\Rightarrow (-1)\left(\frac{1-\mu}{2}\right) + 1\left(\frac{1+\mu}{2}\right) = \frac{\mu-1}{2} + \frac{\mu+1}{2}$$

$$= \mu$$

Variance of distribution can be expressed

as $\sigma^2 = E(x^2) - (E(x))^2$

$$\Rightarrow \sum p(x^2|\mu) - (p(x|\mu))^2$$

$$\Rightarrow \sum p(x^2|\mu) - \mu^2$$

$p(x^2/\mu)$ at $x=1 \rightarrow x^2 p\ (x=1/\mu)$

$p(x^2|\mu)$ at $x=-1 \Rightarrow x^2 p\ (x=-1|\mu)$

$$\Rightarrow \sum p(x^2/\mu) = 1$$

$$\Rightarrow 1-\mu^2$$

mean is μ, variance is $1-\mu^2$

2) Given Guassian random variable x with distribution $N(x|\mu, \Sigma)$

prior distribution $p(\mu) = N(\mu|\mu_0, \Sigma_0)$

posterior distribution $p(\mu|x) = ?$

$$p(\mu) = N(\mu | \mu_0, \Sigma_0)$$

Assuming data points are independent

$$\Rightarrow \prod_{n=1}^{N} N(\mu|\mu_0, \Sigma_0)$$

taking log both sides for convenience

$$\ln p(\mu) = \ln\left(\frac{1}{\sqrt{2\pi\Sigma_0}}\right) + \ln\left(e^{-(\mu-\mu_0)^2/2\Sigma_0}\right)$$

$$= \ln(2\pi\Sigma_0)^{-1/2} - \frac{(\mu-\mu_0)^2}{2\Sigma_0} \times \ln(e)$$

$$= -\frac{1}{2}\ln 2\pi\Sigma_0 - \frac{(\mu-\mu_0)^2}{2\Sigma_0}$$

$$\Rightarrow \sum_{i=1}^{n} \left\{ -\tfrac{1}{2}\ln(2\pi) - \tfrac{1}{2}\ln(\Sigma_0) - \frac{(\mu-\mu_0)}{2\Sigma_0} \right.$$

$$= -\frac{n}{2} \times \ln(2\pi) - n\ln(\Sigma_0) - \sum_{i=1}^{n} \left(\frac{\mu-\mu_0}{2\Sigma_0} \right.$$

Considering more bayesian approach by introducing a prior distribution

$$P(\mu \mid x, y, \mu_0, \Sigma_0) \propto P(y \mid \mu_0, \mu, \Sigma_0) \times P(\mu \mid \Sigma_0)$$

$$\boxed{\begin{aligned} P(\mu/x) &= N(\mu \mid \mu_0, x^{-1}) \\ &= \left(\frac{x}{2\pi}\right)^{(MP1)/2} \exp\left(-\frac{x}{2}\alpha^T\alpha\right) \end{aligned}}$$