Gaussian Distribution

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$$N(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

From the definition of expectation of a continuous random value/variable

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

Let's substitute $f(x)$ values as $N(x|\mu,\sigma^2)$

$$E(X) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} x \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx$$

$$\boxed{\int uv = u\int v - \int (u' . \int v)}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi\sigma^2}}\left(\int_{-\infty}^{\infty} x e^{\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)} dx\right)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi\sigma^2}}\left(\int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{(\sqrt{2}\sigma)^2}}\right) dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi\sigma^2}}\left(\int_{-\infty}^{\infty} x e^{-\left(\frac{x-\mu}{\sqrt{2}\cdot\sigma}\right)^2}\right) dx$$

Let us take

$$k = \frac{x-\mu}{\sqrt{2}\,\sigma} \Rightarrow \frac{dk}{dx} = \frac{1}{\sqrt{2}\sigma} \quad (1) \Rightarrow \boxed{dk = \frac{dx}{\sqrt{2}\sigma}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\left(\int_{-\infty}^{\infty} x e^{-k^2}\, dx\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\left(\int_{-\infty}^{\infty} x e^{-k^2}\, (\sqrt{2}\sigma)\, dk\right)$$

$$= \frac{\sqrt{2}\sigma}{\sqrt{2\pi\sigma^2}}\left(\int_{-\infty}^{\infty} x e^{-k^2}\, dk\right)$$

replace 'x' in terms of k

$$k = \frac{x-\mu}{\sqrt{2}\,\sigma} \Rightarrow \sqrt{2}\sigma k = x - \mu$$

$$(x = \sqrt{2}\sigma k + \mu)$$

$$= \frac{\cancel{\sqrt{2}\,\sigma}}{\cancel{\sqrt{2\pi\sigma^2}}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma k + \mu)\, e^{-k^2}\, dk$$

$$= \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\sigma \int_{-\infty}^{\infty} k e^{-k^2} dk \right) + \mu \int_{-\infty}^{\infty} e^{-k^2} dk \Big)$$

From Gaussian Integral

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}}$$

$$= \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\sigma \int_{-\infty}^{\infty} k e^{-k^2} dk \right) + \mu\sqrt{\pi} \Big)$$

$$= \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\,\sigma \left[-\frac{1}{2} \exp(-k^2) \right]_{-\infty}^{\infty} + \mu\sqrt{\pi} \right)$$

$$= \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\sigma \left\{ -\frac{1}{2} \left(e^{\frac{1}{\infty^2}} - e^{\frac{1}{-\infty^2}} \right) \right. \right.$$
$$+ \mu(\sqrt{\pi})$$

$$\frac{1}{\infty} = 0$$

$$\Rightarrow \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\sigma \left[-\frac{1}{2} (1-1) + \mu(\sqrt{\pi}) \right) \right.$$

$$\Rightarrow \frac{\mu \sqrt{\pi}}{\sqrt{\pi}}$$

$$= \mu$$

Hence proved

$$\boxed{E[x] = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2)\, x \, dx = \mu}$$

We know that

$$Var(x) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - (E(x))^2$$

$$\boxed{(E(x^2)) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - Var(x)}$$

$$Var(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$\cancel{= \left(\frac{\sqrt{2}\sigma}{\sigma\sqrt{2\pi}}\right) \int_{-\infty}}$$

Lets use the same technique

$$k = \frac{x-\mu}{\sqrt{2}\sigma}$$

$$\Rightarrow \frac{\sqrt{2}\sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma k + \mu^2) \exp(-k^2) dk - \mu^2)$$

$$\Rightarrow \frac{1}{\sqrt{\pi}} \left(2\sigma^2 \int_{-\infty}^{\infty} k^2 \exp(-k^2)\, dk + (2\sqrt{2}\sigma\mu) \left(\int_{-\infty}^{\infty} k \exp(-k^2)\, dk + \mu^2 \int_{-\infty}^{\infty} \exp(-k^2) dk \right)\right.$$

$$- \mu^2$$

$$\Rightarrow \frac{1}{\sqrt{\pi}} \left(2\sigma^2 \int_{-\infty}^{\infty} k^2 \exp(-k^2) dk + (2\sqrt{2}\sigma\mu) \left[-\tfrac{1}{2} \exp(-k^2)\right]_{-\infty}^{\infty} + \frac{\mu^2}{\sqrt{\pi}}\right)$$

$$- \mu^2$$

$$\Rightarrow \frac{1}{\sqrt{\pi}} \left(2\sigma^2 \int_{-\infty}^{\infty} k^2 \exp(-k^2)\, dk + 2\sqrt{2}\sigma\mu \times 0\right) + \mu^2 - \mu^2$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} k^2 \exp(-k^2) dk$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} k^2 \exp(-k^2)\,dk$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \left(\left[-\frac{k}{2} \exp(-k^2) \right]_{-\infty}^{\infty} + \frac{1}{2} \int_{-\infty}^{\infty} \exp(-k^2) \right.$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \int_{-\infty} \exp(-k^2)\,dk$$

$$= \frac{2\sigma^2\sqrt{\pi}}{2\sqrt{\pi}}$$

$$= \sigma^2$$

$$\boxed{var(x) = \sigma^2}$$

From $var(x) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - (E(x))^2$

$$(E(x))^2 + var(x) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$\boxed{\mu^2 + \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx}$$

Hence proved