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EXERCISE-2

Task 2.1

Given the joint density function

$$f_{xy}(x,y) = \begin{cases} 2 & \text{for } x \geq 0 \text{ and } x+y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Calculate $f_x(x)$, $f_y(y)$, $F_{xy}(x,y)$, $F_x(x)$, $F_y(y)$

Solution

1). Calculation of $f_x(x)$ (Pdf):

Since we know that-

$$f_x(x) = \int_{-\infty}^{+\infty} f_{xy}(x,y)\,dy$$

breaking down the above limits we get-

$$f_x(x) = \int_{-\infty}^{0} 0 \cdot dy + \int_{0}^{1-x} 2 \cdot dy + \int_{1-x}^{+\infty} 0 \cdot dy \quad \text{for } 0 \leq x \leq 1$$

$$f_x(x) = 2(1-x)$$

$$f_x(x) = \begin{cases} 2(1-x) & \text{for } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

2). Calculation of $f_y(y)$ (Pdf):-

Again, we know that -

$$f_y(y) = \int_{-\infty}^{+\infty} f_{xy}(x,y)\, dx$$

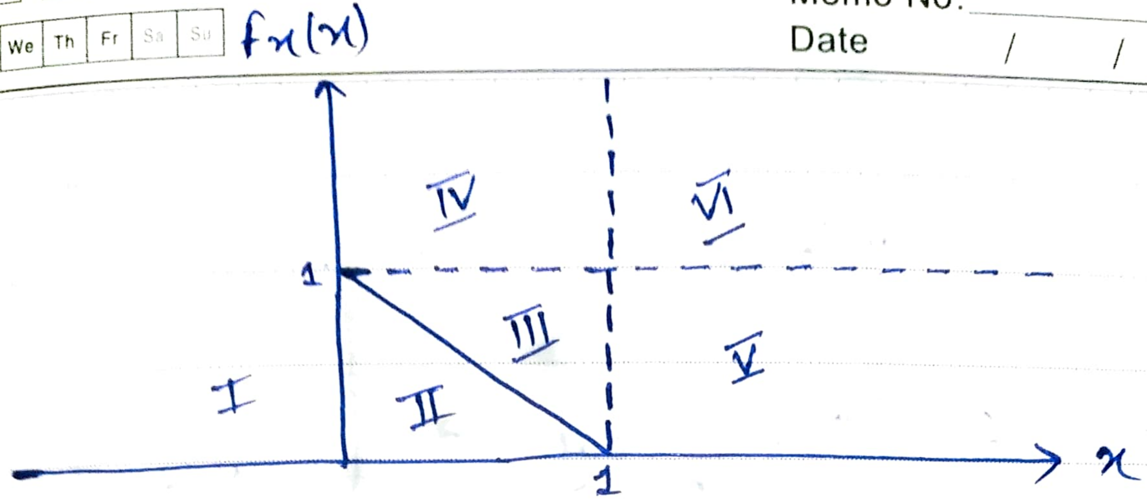
breaking down the above limits we get-

$$f_y(y) = \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{1-y} 2 \cdot dx + \int_{1-y}^{+\infty} 0 \cdot dx \quad \text{for } 0 \leq y \leq 1$$

$$f_y(y) = 2(1-y)$$

$$f_y(y) = \begin{cases} 2(1-y) & \text{for } 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

3). Calculation of $F_{xy}(x,y)$ (cdf / joint cdf)-



a) $x \leq 0$ and $y \leq 0$

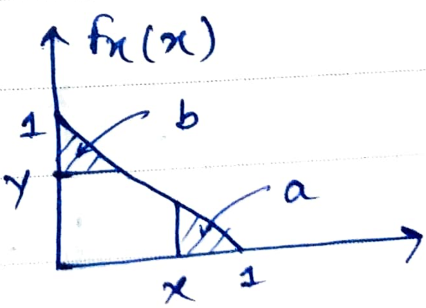
$$f_{xy}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} 0 \cdot dy\,dx = 0$$

b) $x \geq 0$; $y \geq 0$; $x + y \leq 1$

$$f_{xy}(x,y) = \int_{0}^{x}\int_{0}^{y} 2 \cdot dy\,dx = \int_{0}^{x} 2y \cdot dx = 2xy$$

c) $x \leq 1$, $y \leq 1$, $x + y \geq 1$

$$f_{xy}(x,y) = 1 - \iint_{a} 2\,dx\,dy - \iint_{b} 2\,dy\,dx$$



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$$F_{xy}(x,y) = 1 - \int_y^1 \int_0^{1-y} 2\,dx\,dy - \int_x^1 \int_0^{1-x} 2\,dy\,dx$$

$$f_{xy}(x,y) = 1 - \int_y^1 [2x]_0^{1-y} dy - \int_x^1 [2y]_0^{1-x} dx$$

$$f_{xy}(x,y) = 1 - \left[2y - \frac{2y^2}{2}\right]_y^1 - \left[2x - \frac{2x^2}{2}\right]_x^1$$

$$f_{xy}(x,y) = 1 - (2 - 1 - 2y + y^2) - (2 - 1 - 2x + x^2)$$

$$f_{xy}(x,y) = 1 - (1-y)^2 - (1-x^2)$$

d) $0 \leq x \leq 1$, $y \geq 1$

$$f_{xy}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{xy}(x,y)\,dy\,dx$$

$$= \int_0^x \int_0^{1-x} 2\,dy\,dx$$

$$f_{xy}(x,y) = \int_0^x [2y]_0^{1-x} dx = \int_0^x (2-2x)\,dx$$

$$f_{xy}(x,y) = \left[2x - \frac{2x^2}{2}\right]_0^x$$

$$f_{xy}(x,y) = [2x - x^2] = 1-(1-x)^2$$

e) $0 \leq y \leq 1, \quad x \geq 1$

$$f_{xy}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{xy}(x,y)\,dy\,dx$$

$$f_{xy}(x,y) = \int_0^{1-y}\int_0^{y} 2\,dy\,dx = \int_0^{1-y} [2y]_0^y\,dx$$

$$f_{xy}(x,y) = \int_0^{1-y} [2y-0]\,dx = [2yx]_0^{1-y}$$

$$= 2y - 2y^2$$

$$= 1-(1-y)^2$$

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f) $y \geq 1,\ x \geq 1$

$$F_{xy}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{xy}(x,y)\,dy\,dx = 1$$

Because cdf is a probability for a random variable for a variable.

$$F_{xy}(x,y) = \begin{cases} 0 & \text{for } x \leq 0 \text{ and } y \leq 0 \\ & \text{for } x \leq 0 \text{ or } y \leq 0 \\ 2xy & \text{for } x \geq 0,\ y \geq 0,\ x+y \leq 1 \\ 1-(1-y)^2-(1-x)^2 & \text{for } x \leq 1,\ y \leq 1,\ x+y \geq 1 \\ 1-(1-x)^2 & \text{for } 0 \leq x \leq 1,\ y \geq 1 \\ 1-(1-y)^2 & \text{for } 0 \leq y \leq 1,\ x \geq 1 \\ 1 & \text{for } y \geq 1,\ x \geq 1 \end{cases}$$

4.) Calculation of $F_x(x)$

$$F_x(x) = F_{xy}(x, +\infty)$$

$$F_x(x,y) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1-(1-x)^2 & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

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5.) calculation of $f_y(y)$

$$f_y(y) = F_{xy}(+\infty, y)$$

$$F_y(x,y) = \begin{cases} 0 & \text{for } y \leq 0 \\ 1-(1-y)^2 & \text{for } 0 \leq y \leq 1 \\ 1 & \text{for } y \geq 1 \end{cases}$$

Task-2.2

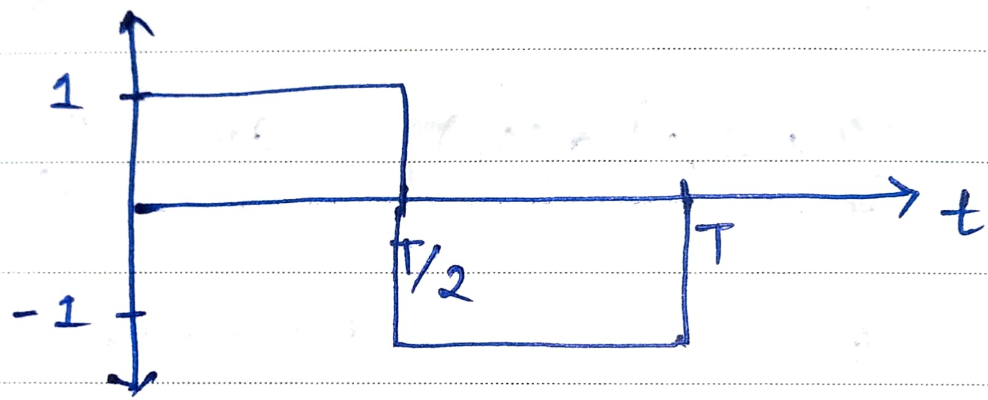
Let $a(\xi)$ be a random variable taking the value 0 with probability $p_0 = 1/4$ & the value 1 with probability $p_1 = 3/4$. A random process $x(\xi,t)$ is defined as -

$$x(\xi,t) = \begin{cases} 1 - \frac{4}{T} t\, a(\xi) & \text{for } 0 \leq t \leq T/2 \\ -1 + \left(\frac{4}{T}t - 2\right) a(\xi) & \text{for } \frac{T}{2} \leq t < T \\ 0 & \text{otherwise} \end{cases}$$

a) Sketch all distinct pattern functions of the random process $X(\xi,t)$.

Sol^n- for $a(\xi) = 0$, $P_0 = 1/4$

$$x(\xi, t) = \begin{cases} 1 & \text{for } 0 \leq t \leq T/2 \\ -1 & \text{for } T/2 \leq t < T \\ 0 & \text{otherwise} \end{cases}$$

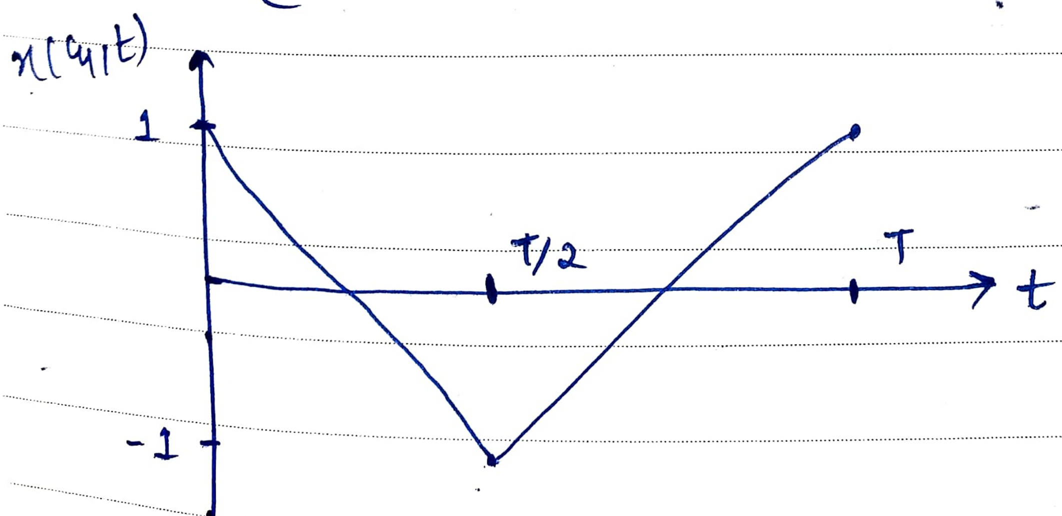


for $a(\xi) = 1$, $P_1 = 3/4$

$$x(q_1,t) = \begin{cases} 1 - \dfrac{4t}{T} & \text{for } 0 \leq t < T/2 \\ -3 + \dfrac{4t}{T} & \text{for } T/2 \leq t < T \\ 0 & \text{otherwise} \end{cases}$$

So taking the values of t at different time interval in above equation

$$x(q_1,t) = \begin{cases} 1 & \text{when } t = 0 \\ -1 & \text{when } t = T/2 \\ 1 & \text{when } t = T \end{cases}$$



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b) calculate the mean $m_x^{(1)}(t)$.

Sol^n since we know that -

$$m_x^{(1)} = x \cdot pdf(x) \quad \text{———} \quad (1)$$

Mean

$$m_x^{(1)} = \begin{cases} 1 \cdot \frac{1}{4} + \left(1 - \frac{4t}{T}\right)\frac{3}{4} & \text{for } 0 \leq t < T/2 \\ -1 \cdot \frac{1}{4} + \left(-3 + \frac{4t}{T}\right)\frac{3}{4} & \text{for } T/2 \leq t < T \\ 0 & \text{otherwise} \end{cases}$$

$$m_x^{(1)} = \begin{cases} 1 - \frac{3t}{T} & \text{for } 0 \leq t < T/2 \\ -\frac{5}{2} + \frac{3t}{T} & \text{for } T/2 \leq t < T \\ 0 & \text{otherwise} \end{cases}$$

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e) Autocorrelation $S_{xx}(\tau) = ?$

<u>case 1 :-</u>

when $t_1, t_2 < T/2$

$$S_{xx}(\tau) = (1)(1)\cdot\frac{1}{4} + \left(1-\frac{4t_1}{T}\right)\left(1-\frac{4t_2}{T}\right)\left(\frac{3}{4}\right)$$

$$S_{xx}(\tau) = \frac{1}{4} + \frac{3}{4}\left[\left(1-\frac{4t_1}{T}\right)\left(1-\frac{4t_2}{T}\right)\right]$$

<u>case 2 :-</u>

when $T/2 \leq t_1, t_2 < T$

$$S_{xx}(\tau) = (-1)(-1)\cdot\frac{1}{4} + \left(\frac{4t_1}{T}-3\right)\left(\frac{4t_2}{T}-3\right)\left(\frac{3}{4}\right)$$

$$S_{xx}(\tau) = \frac{1}{4} + \frac{3}{4}\left(\frac{4t_1}{T}-3\right)\left(\frac{4t_2}{T}-3\right)$$

<u>case 3 :-</u>

when $0 \leq t_1 < T/2$ & $T/2 \leq t_2 < T$

$$S_{xx}(\tau) = (1)(-1)\left(\tfrac{1}{4}\right) + \left(1 - \frac{4t_1}{T}\right)\left(\frac{4t_2}{T} - 3\right)\left(\tfrac{3}{4}\right).$$

$$S_{xx}(\tau) = -\frac{1}{4} + \frac{3}{4}\left(1 - \frac{4t_1}{T}\right)\left(\frac{4t_2}{T} - 3\right)$$

<u>Case 4 :-</u>

when $T/2 \leq t_1 < T$ and $0 \leq t_2 < T/2$

$$S_{xx}(\tau) = (-1)(1)\tfrac{1}{4} + \left(\frac{4t_1}{T} - 3\right)\left(1 - \frac{4t_2}{T}\right)\left(\tfrac{3}{4}\right)$$

$$S_{xx}(\tau) = -\frac{1}{4} + \frac{3}{4}\left(\frac{4t_1}{T} - 3\right)\left(1 - \frac{4t_2}{T}\right)$$

(d) Calculate the Variance $\sigma_x^2(t)$?

<u>Sol^n</u> since we know that -

$$\sigma_x^2 = m_x^{(2)} - \left(m_x^{(1)}\right)^2$$

$$m_x^{(2)} \begin{cases} (1)\cdot\frac{1}{4} + \frac{3}{4}\left(1 - \frac{4t}{T}\right)^2 & 0 \leq t < T/2 \\ (1)\cdot\frac{1}{4} + \frac{3}{4}\left(\frac{4t}{T} - 3\right)^2 & T/2 \leq t < T \end{cases}$$

$$m_x^{(2)} = \begin{cases} \frac{1}{4} + \frac{3}{4}\left(1 - \frac{4t}{T}\right)^2 & 0 \leq t < T/2 \\ \frac{1}{4} + \frac{3}{4}\left(\frac{4t}{T} - 3\right)^2 & T/2 \leq t < T \end{cases}$$

So, Now

^{2} = for $0 \leq t < T/2$

$$\sigma_x^2 = m_x^{(2)} - \left(m_x^{(1)}\right)^2$$

$$= \frac{1}{4} + \frac{3}{4}\left(1 - \frac{4t}{T}\right)^2 - \left(1 - \frac{3t}{T}\right)^2$$

$$= \frac{1}{4} + \frac{3}{4}\left(1 + \frac{16t^2}{T^2} - \frac{8t}{T}\right) - \left(1 + \frac{9t^2}{T^2} - \frac{6t}{T}\right)$$

$$= \frac{1}{4} + \frac{3}{4} + \frac{12t^2}{T^2} - \frac{6t}{T} - 1 - \frac{9t^2}{T^2} + \frac{6t}{T}$$

$$= 1 + \frac{3t^2}{T^2} - 1$$

$$= \frac{3t^2}{T^2}$$

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for $T/2 \leq t < T$

$$\sigma_x^2 = m_x^{(2)} - \left(m_x^{(1)}\right)^2$$

$$= \frac{1}{4} + \frac{3}{4}\left(\frac{4t}{T} - 3\right)^2 - \left(\frac{-5}{2} + \frac{3t}{T}\right)^2$$

$$= \frac{1}{4} + \frac{3}{4}\left(\frac{16t^2}{T^2} + 9 - \frac{24t}{T}\right) - \left(\frac{9t^2}{T^2} + \frac{25}{4} - \frac{15t}{T}\right)$$

$$= \frac{1}{4} + \frac{12t^2}{T^2} + \frac{27}{4} - \frac{18t}{T} - \frac{9t^2}{T^2} - \frac{25}{4} + \frac{15t}{T}$$

$$= \frac{3}{4} + \frac{3t^2}{T^2} - \frac{3t}{T}$$

$$\therefore \sigma_x^2 = \begin{cases} \dfrac{3t^2}{T^2} & \text{for } 0 \leq t < T/2 \\ 3/4 + \dfrac{3t^2}{T^2} - \dfrac{3t}{T} & \text{for } T/2 \leq t < T \\ 0 & \text{otherwise} \end{cases}$$