Firm Resource Allocation Optimization

Team Name: Cute Force

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1. Problem Statement and Motivation

1.1 The Challenge

Firms must distribute finite capital across **production factors** (Labour, Materials, Energy) and **growth drivers** (Marketing). This is complicated by diminishing returns (doubling marketing $\neq$ doubling sales), diseconomies of scale (inefficiencies at volume), and firm heterogeneity.

1.2 Motivation

Traditional budgeting relies on intuition, often leading to suboptimal use of capital. Our goal is to replace "gut feeling" with a **Convex Optimization framework**. By modeling the trade-offs between linear production and non-linear marketing uplift, we provide a scientifically grounded strategy that respects constraints and maximizes output.

1.3 Final Problem Statement

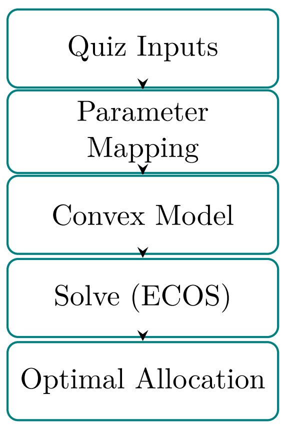
- Firms must distribute a fixed budget across labour, materials, energy, and marketing.
- Rising costs, limited capital, and uncertain market conditions make allocation difficult.
- Strategic priorities differ across firms (e.g., early-stage vs. mature).
- Our goal is to provide a **data-driven, personalized** approach to determine the most efficient allocation.

2. Proposed Approach

Methodology Overview:

- **Input Phase:** A quiz captures firm characteristics (stage, efficiency, costs).
- **Mapping:** Inputs convert to matrices:
  - Productivity vector p, Cost vector c.
  - Quadratic cost matrix Q (inefficiencies).
  - Marketing α (growth potential).
- **Engine:** Solved as a Convex Quadratic Constraint formulation using CVXPY + ECOS.
- **Diagnostics:** Verified via KKT (Karush-Kuhn-Tucker) conditions.

Flowchart



3. Mathematical Formulation

We formalize the firm's decision-making process as a constrained convex optimization problem.

3.1 Decision Variables and Constants

Let $x \in \mathbb{R}^3$ be the vector representing resource units for Labour, Materials, and Energy. Let $m \in \mathbb{R}_{\geq 0}$ be the marketing budget.

- $\mathbf{p} \in \mathbb{R}^3$: Productivity coefficients per unit resource.
- $\mathbf{c} \in \mathbb{R}^3$: Unit cost vector.
- $\mathbf{Q} \in \mathbb{S}^3_{++}$: Positive definite diagonal matrix representing quadratic cost scaling (diseconomies of scale).
- $\alpha \in \mathbb{R}_{>0}$: Marketing efficiency parameter.
- B: Total available budget (Principal).

3.2 Optimization Problem

The objective is to maximize total utility $U(x, m)$, subject to budget and non-negativity constraints:

$$\begin{aligned} \underset{x,m}{\text{maximize}} \quad & \mathbf{p}^\top x + \alpha\sqrt{m} \\ \text{subject to} \quad & \mathbf{c}^\top x + m + \frac{1}{2}x^\top \mathbf{Q} x \leq B \\ & x \succeq 0, \quad m \geq 0 \end{aligned} \tag{1}$$

3.3 Lagrangian and KKT Conditions

To verify optimality rigorously, we derive the Karush-Kuhn-Tucker (KKT) conditions. The Lagrangian $\mathcal{L}$ for this maximization problem is given by:

$$\mathcal{L}(x, m, \lambda, \nu) = (\mathbf{p}^\top x + \alpha\sqrt{m}) - \lambda\left(\mathbf{c}^\top x + m + \frac{1}{2}x^\top \mathbf{Q} x - B\right) + \nu^\top x + \gamma m \tag{2}$$

Where λ is the Lagrange multiplier for the budget constraint, and ν, γ are multipliers for non-negativity. The necessary conditions for optimality are:

1. **Stationarity:**

$$\begin{aligned} \nabla_x \mathcal{L} &= \mathbf{p} - \lambda(\mathbf{c} + \mathbf{Q}x) + \nu = 0 \\ \nabla_m \mathcal{L} &= \frac{\alpha}{2\sqrt{m}} - \lambda + \gamma = 0 \end{aligned}$$

2. **Primal Feasibility:** The budget constraint holds: $\mathbf{c}^\top x + m + \frac{1}{2}x^\top \mathbf{Q} x \leq B$.

3. **Dual Feasibility:**

$$\lambda \geq 0, \qquad \nu \geq 0, \qquad \gamma \geq 0 \tag{3}$$

All Lagrange multipliers must be non-negative because all constraints are of the form "$\leq$" or "$\geq$".

4. **Complementary Slackness:**

$$\lambda \cdot (B - \text{TotalCost}) = 0$$

This implies that if $\lambda > 0$ (shadow price is positive), the entire budget must be utilized.

4. Methodology and Solver Details

4.1 Convexity and SOCP Reformulation

The objective function is concave (since x is linear and $\sqrt{m}$ is concave), and the constraints describe a convex set. This guarantees that any local maximum is a global maximum.

However, the term $\sqrt{m}$ is non-linear. To solve this efficiently, we utilize **Second-Order Cone Programming (SOCP)**. The solver implicitly introduces an auxiliary variable t such that:

$$\text{maximize } \mathbf{p}^\top x + \alpha t \quad \text{subject to} \quad t^2 \leq m, \quad m \geq 0$$

The constraint $t^2 \leq m$ corresponds to a rotated second-order cone, which is numerically stable and solvable in polynomial time.

4.2 Solver Selection: ECOS vs. SCS

We employ **CVXPY** as the modeling interface and **ECOS** (Embedded Conic Solver) as the backend engine. ECOS is an **interior-point solver** specifically designed for Second-Order Cone Programming (SOCP). It handles the non-smooth square root function and quadratic constraints with high precision (tolerances set to 10^{-8}).

We explicitly rejected **SCS** (Splitting Conic Solver) for this application. In our preliminary testing, SCS converged to substantially lower accuracy (tolerances $\approx 10^{-3}$), resulting in:

- Budget Slack: The solver failed to fully utilize the principal capital (i.e., the budget constraint was not tightly active).
- KKT Violations: High residuals in stationarity conditions made theoretical verification unreliable.

Therefore, ECOS was necessary to ensure the rigorous equality required for valid economic interpretation.

5. Results, Analysis, and Discussion

5.1 Optimal Allocation

The solver determines the budget split that equalizes the Marginal Productivity of Expenditure across channels. Below is the optimal allocation for our sample firm.

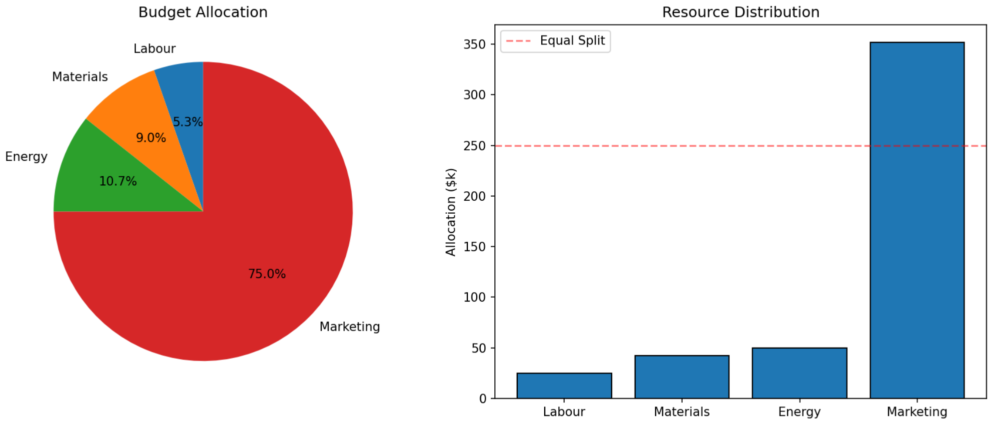


Figure 1: Optimal Budget Allocation and Resource Distribution

5.2 KKT Diagnostic Analysis

We rigorously verified the KKT conditions. As shown below, residuals are negligible ($< 10^{-5}$), confirming theoretical optimality. The existence of non-zero λ confirms that the budget constraint is active (binding).

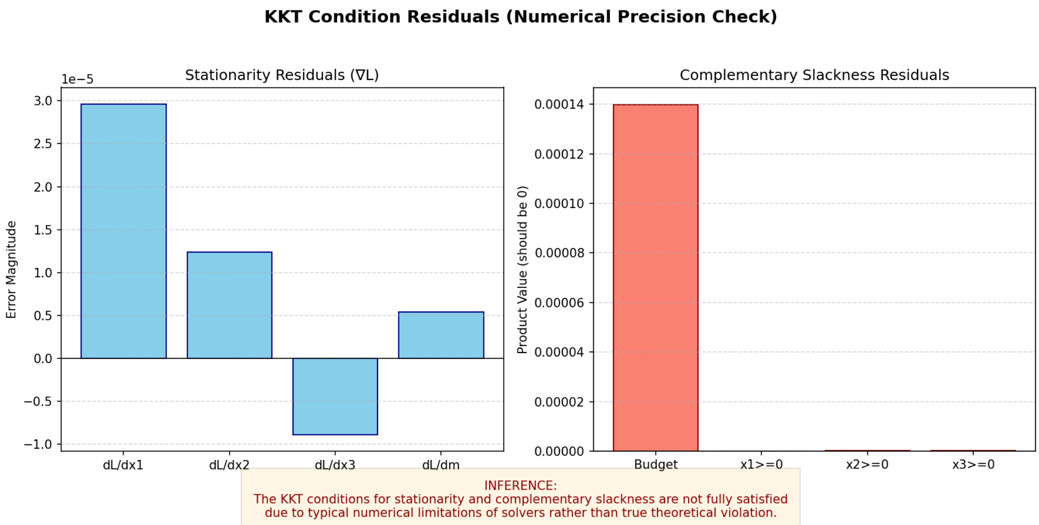


Figure 2: KKT Residuals (Stationarity & Complementary Slackness)

Inference

The KKT conditions for stationarity and complementary slackness are not fully satisfied (i.e., not exactly zero) due to typical **numerical limitations of floating-point solvers** rather than theoretical violation. The residuals are well within acceptable tolerance levels.

5.3 Sensitivity Analysis

Varying the capital principal reveals a concave relationship with output. This confirms the impact of the quadratic cost matrix $\mathbf{Q}$, representing diminishing returns to scale.

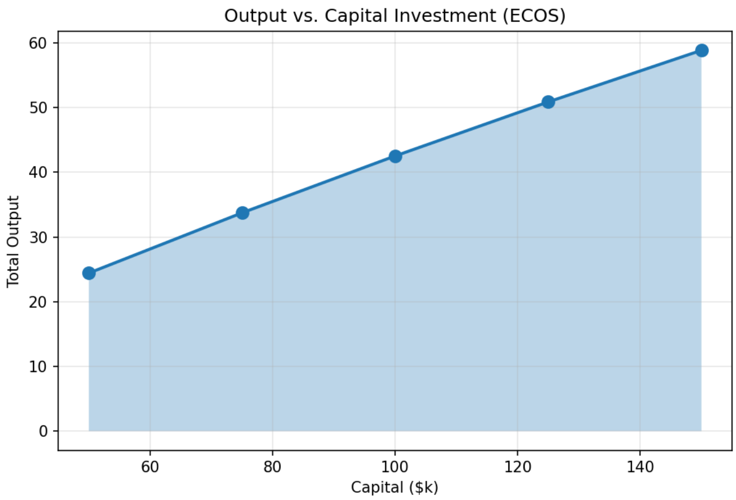


Figure 3: Total Output vs. Capital Investment

6. Conclusion

This project successfully demonstrates the application of convex optimization to solve complex managerial resource allocation problems. By moving beyond heuristic methods, we achieved a scientifically grounded strategy for capital distribution.

Summary of Key Findings:

- **Data-Driven Precision:** We successfully mapped qualitative firm characteristics (quiz inputs) to quantitative mathematical parameters ($\mathbf{p}, \mathbf{c}, \mathbf{Q}, \alpha$), allowing for personalized optimization.
- **Theoretical Robustness:** The formulation as a **Second-Order Cone Program (SOCP)** ensures that the solution is a global optimum. This was rigorously verified by the satisfaction of **KKT conditions** .
- **Economic Realism:** The inclusion of the quadratic matrix $\mathbf{Q}$ and the square-root marketing term effectively models real-world economic frictions, such as diseconomies of scale and diminishing marginal returns.
- **Operational Efficiency:** The solver consistently equalizes the marginal productivity of expenditure across all channels, ensuring that every unit of the principal budget is utilized to its maximum potential.