Unit 3
Probability and Random Variable

MCQ (1 Marks)

51. A random variable X has the following probability distribution:

Value of X, x:	0	1	2	3	4	5	6	7	8
P(x):	a	3a	5a	7a	9a	11a	13a	15a	17a

Determine the value of a. (CO3)

A. 1/81
B. 2/81
C. 5/81
D. 1

$81a = 1$

$\boxed{a = \frac{1}{81}}$

52. The distribution function of a random variable X is given by $F(x) = e^{x} - e^{-x}(1+x)$. Find the corresponding density function of random variable X (CO3)

A. $e^{x} + xe^{-x}$
B. $e^{x} - e^{-x}(1+x)$
C. $e^{x} + e^{-x}$
D. xe^{-x}

53. If f(x) = $\begin{cases} 2x, & 0 \le x \le 1 \\ 0, & Otherwise \end{cases}$ find the probability $P\left(\frac{1}{4} \le x \le \frac{1}{2}\right)$. (CO3)

A. 3/16
B. 2/16
C. 9/16
D. N.O.T

54. The diameter, say X of an electric cable, is assumed to be a continuous random variable with p.d.f $f(x)=6x(1-x)$. Compute $P\left(x \le \frac{1}{2} \middle| \frac{1}{3} \le x \le \frac{2}{3}\right)$. (CO3)

A. 11/26
B. 13/26
C. 9/26
D. 17/26.

55. The diameter, say X of an electric cable, is assumed to be a continuous random variable with p.d.f: $f(x) = 6x(1-x)$. Determine the number k such that $P(X < k) = P(X > k)$. (CO3)

A. $\frac{1+\sqrt{3}}{2}$
B. $\frac{1-\sqrt{3}}{2}$
C. $\frac{1\pm\sqrt{3}}{2}$
D. 1/2.

56. Let X and Y be the jointly continuous random variables with joint CDF satisfies the following condition: (CO3)

A. $F_{XY}(-\infty, \infty) = 1$
B. $F_{XY}(x, -\infty) = 1$
C. $F_{XY}(-\infty, y) = 1$
D. None of these

57. Let X be a random variable with probability distribution function

$$f(x) = \begin{cases} 0.2, & \text{for } |x| < 1 \\ 0.1, & \text{for } 1 < |x| < 4 \\ 0 & \text{otherwise} \end{cases}$$

The probability $P\,(0.5 < x < 5)$ is ______ (CO3)

A. 0.3
B. 0.5
C. 0.4
D. 0.8

Very Short Answer Type Question (2 Marks)

58. Define marginal and conditional distribution. (CO3)

Solution: Marginal

Let 'X' and 'Y' continuous random variable if (X, Y) has joint density $f^n f(x,y)$ then marginal dist. f^n is

$\Rightarrow f^n$ of $X = g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$\Rightarrow f^n$ of $Y = h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

Conditional

The conditional event $Y = y_j$ given that $X = x_i$

$P(Y = y_j / X = x_i) = \dfrac{P(X = x_i, Y = y_j)}{P(X = x_i)}$

In case of cont^n random variable

$P(X/Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$

59. A continuous random variable X has a p.d.f. $f(x) = 3x^2, 0 \leq x \leq 1$. Find a such that $P(X \leq a) = P(X > a)$ (CO3)

Solution: $P(X \leq a) = P(X > a)$

$\int_0^a 3x^2\,dx = \int_a^1 3x^2\,dx$

$3\left(\frac{x^3}{3}\right)_0^a = 3\left(\frac{x^3}{3}\right)_a^1$

$3\left(\frac{a^3}{3}\right) = 3\left(\frac{1}{3} - \frac{a^3}{3}\right)$

$a^3 = 1 - a^3$

$2a^3 = 1 \Rightarrow \boxed{a^3 = \frac{1}{2}} \Rightarrow \boxed{a = \left(\frac{1}{2}\right)^{1/3}}$ ** Ans.

60. Write the Statement of the Central Limit Theorem. (CO3)

Solution: It states that the sampling distribution of the mean will always be normal distributed, as long as the sample size is large enough.

61. A die is tossed thrice. A success is getting 1 or 6 on a toss. Find the mean and the variance of the number of successes. (CO3)

Solution: $X = 0, 1, 2, 3$

Probability of success $= \frac{2}{6} = \frac{1}{3}$

" " failure $= 1 - \frac{1}{3} = \frac{2}{3}$

$P(X=0) = 8/27$
$P(X=1) = 12/27$
$P(X=2) = 6/27$
$P(X=3) = 1/27$

X	0	1	2	3
$P(X)$	8/27	12/27	6/27	1/27
$p_i x_i^2$	0	12/27	24/27	9/27

Mean $= \sum p_i x_i = 1 = \mu$

Variance $= \sum p_i x_i^2 - \mu^2$

$= \frac{45}{27} - 1 = \frac{18}{27}$

Short Answer Type Questions (6 Marks)

62. The joint probability distribution of two random variables X and Y is given by: $P(X = 0, Y = 1), P(X = 1, Y = -1) = 1/3$ and $P(X = 1, Y = 1) = 1/3$. Find marginal distributions of X and Y, and the conditional probability distribution of X given Y = 1. (CO3)

Solution:

Y \ X	0	1	
−1	0	1/3	1/3
1	1/3	1/3	2/3
	1/3	2/3	

(i) Marginal dist of X

$P(X=0) = P(X=0, Y=-1) + P(X=0, Y=1)$

$= 1/3$

$P(X=1) = P(X=1, Y=-1) + P(X=1, Y=1)$

$= 2/3$

Marginal dist of Y

$P(Y=-1) = P(X=0, Y=-1) + P(X=1, Y=-1)$

$= 1/3$

$P(Y=1) = P(X=0, Y=1) + P(X=1, Y=1)$

$= 2/3$

(ii) Conditional Probability of X:

$Y = 1$

$P(X = x_i, Y = 1)$

① $P(X=0, Y=1)$

$= \frac{P(X=0, Y=1)}{P(Y=1)}$

$= \frac{1/3}{2/3} = 1/2$

② $P(X=1/Y=1)$

$= \frac{P(X=1, Y=1)}{P(Y=1)}$

$= \frac{\left(\frac{1}{3}\right)}{\left(2/3\right)} = \frac{1}{2}$

63. A petrol pump is supplied with petrol once a day. If its daily volume of sales(x) in thousands of litters is distributed by $f(x) = 5x(1-x)^4, 0 \le x \le 1$, what must be the capacity of its tank in order that the probability that its supply will be exhausted in a given day shall be 0.01? (CO3)

Solution: Let the capacity of tank is 'a'.

$$P(X \geq a) = 0.01$$

$$\int_a^1 5(1-x)^4 dx = 0.01$$

$$\left[\frac{-5(1-x)^5}{5}\right]_a^1 = 0.01$$

$$1-a = \left(\frac{1}{100}\right)^{1/5}$$

$$a = 1-\left(\frac{1}{100}\right)^{1/5} = 0.6019.$$

64. If a random variable X has density function- (CO3)

$$f(x) = \begin{cases} 1/4 & -2 < x < 2 \\ 0 & elsewhere \end{cases}$$

Obtain- $P(|X| > 1)$ and $P(2X + 3 > 5)$

Solution:

$$P(|x| > 1) = P(X < -1 \text{ or } X > 1)$$

$$= \int_{-2}^{-1} \frac{1}{4} dx + \int_1^2 \frac{1}{4} dx = \frac{1}{4}[x]_{-2}^{-1} + \frac{1}{4}[x]_1^2$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

$$P(2x+3>5) \Rightarrow 2x > 2; x > 1$$

$$= \int_1^2 \frac{1}{4} dx = \frac{1}{4}[x]_1^2 = \frac{1}{4}. \text{ Ans.}$$

65. If X and Y are two random variables having the joint density function: (CO3)

$$f(x,y)=\begin{cases}\frac{1}{8}(6-x-y) & 0\le x<2, 2\le y<4\\ 0 & elsewhere\end{cases}$$

Find (I) $P(X<1\cap Y<3)$,
(II) $P(X+Y<3)$
(III) $P(X<1|Y<3)$

Solution:

① $\int_0^1\int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx$

$=\frac{1}{8}\int_0^1\left[6y-xy-\frac{y^2}{2}\right]_2^3 dx$

$=\frac{3}{8}$

② $P(X+Y<3)$

$=\int_0^1\int_2^{3-x}\frac{1}{8}(6-x-y)\,dy\,dx$

$=\frac{1}{8}\int_0^1\left[6y-xy-\frac{y^2}{2}\right]_2^{3-x} dx$

$=\frac{1}{8}\int_0^1\left[6(3-x)-x(3-x)-\frac{(3-x)^2}{2}\right]dx$

$=\frac{1}{16}\int_0^1(x^2-10x+17)\,dx$

$=\frac{37}{54}$

③ $P(X<1|Y<3)$

$=\dfrac{\int_0^1\int_2^3\frac{1}{8}(6-x-y)\,dy\,dx}{\int_0^2}$

66. Five defective bulbs are accidentally mixed with twenty good ones. It is not possible to just look at a bulb and tell whether or not it is defective. Find the probability distribution of the number of defective bulbs, if four bulbs are drawn at random from this lot. (CO3)

Solution: Total defective bulbs = 5
Drawn = 4

$X=0,1,2,3,4$

$P(X=0)=\frac{^{20}C_4}{^{25}C_4}=0.38$

$P(X=1)=\frac{^{20}C_3\times{}^5C_1}{^{25}C_4}=\frac{969}{2530}$

$P(X=2)=\frac{^{20}C_2\times{}^5C_2}{^{25}C_4}=\frac{1140}{2530}$

$P(X=3)=\frac{^{20}C_1\times{}^5C_3}{^{25}C_4}=\frac{40}{2530}$

$P(X=4)=\frac{^5C_4}{^{25}C_4}=\frac{1}{2530}$

X	0	1	2	3	4
P(X)	0.38	0.45	0.15	0.015	$\frac{1}{2530}$

67. If X and Y are two random variables having the joint probability mass function $p(x,y) = \frac{1}{27}(2x+y)$; $x = 0,1,2$; $y = 0,1,2$ Find the conditional distribution of Y for $X = x$. (CO3)

Solution: Conditional Distribution:

for $X=0$; $P(Y=0/X=0) = \frac{P(Y \cap X)}{P(X)} = 0$

$P(Y=1/X=0) = \frac{1}{3}$ and $P(Y=2/X=0) = 2/3$

for $X=1 \Rightarrow P(Y=0/X=1) = \frac{2}{9}$

$P(Y=1/X=1) = 3/9$

$P(Y=2/X=1) = 4/9$

for $X=2 \Rightarrow P(Y=0/X=2) = \frac{4}{15}$

$P(Y=1/X=2) = 5/15$

$P(Y=2/X=2) = \frac{P(x,y)}{P(x)} = \frac{6/27}{(15/27)} = \frac{2}{5}$

X \ Y	0	1	2	Total
0	0	1/27	2/27	3/27
1	2/27	3/27	4/27	9/27
2	4/27	5/27	6/27	15/27
Total	6/27	9/27	12/27	1

68. Suppose that X has pdf:

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & elsewhere \end{cases}$$

Find the pdf of $Y = 3X + 1$. (CO3)

Solution: $Y = 3X+1$; If $X=0$, $Y=1$; $1 < y < 4$

If $X=1$; $Y=4$

Let 'y' be the value in range of Y

$F_Y(y) = P(Y \leq y)$

$= P(3X+1 \leq y)$

$= P\left(X \leq \frac{y-1}{3}\right)$

$$F_Y(y) = \begin{cases} 0 & ; \text{ for } y \leq 1 \\ \frac{1}{3}(y-1) & ; 1 < y < 4 \\ 1 & ; \text{ for } y \geq 4 \end{cases}$$

Now, $F_Y(y) = \frac{d}{dx} F_Y(y)$

$= \frac{d}{dx}\left[\frac{1}{3}(y-1)\right]$

$= \frac{1}{3}(1-0) = \frac{1}{3}$

So, at last

$$F_Y(y) = \begin{cases} \frac{1}{3} & ; 1 < y < 4 \\ 0 & , \text{ elsewhere.} \end{cases}$$

Long Answer Type Questions (10 Marks)

69. Supose the p.d.f of a continuous random variable X has defined as-

$$f(x) = \begin{cases} 1+x, & -1 < x < 0 \\ 1-x, & 0 \leq x < 1 \end{cases}$$

Find the c.d.f $F(x)$. (CO3)

Solution:

$F(x) = -1 < x < 0 = \int_{-1}^{x} (1+x)\,dx = \left[x + \frac{x^2}{2}\right]_{-1}^{x}$

$\Rightarrow x + \frac{x^2}{2} + 1 - \frac{1}{2} \Rightarrow x + \frac{x^2}{2} + \frac{1}{2}$

$0 \leq x < 1 \Rightarrow \int_{-1}^{0} (1+x)\,dx + \int_{0}^{x} (1-x)\,dx$

$= \left[x + \frac{x^2}{2}\right]_{-1}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{x}$

$= 1 - \frac{1}{2} + x - \frac{x^2}{2} \Rightarrow x - \frac{x^2}{2} + \frac{1}{2}$

$F(x) = 1 - \frac{1}{2} + \frac{1}{2} = 1$ 0 $x < -1$

So, $F(x) = \begin{cases} x + \frac{x^2}{2} + \frac{1}{2} ; & -1 < x < 0 \\ x - \frac{x^2}{2} + \frac{1}{2} ; & 0 \leq x < 1 \end{cases}$ Ans.

$x > 1$

70. A random variable X has the following probability mass function: (CO3)

x	0	1	2
$p(x)$	$3c^3$	$4c-10c^2$	$5c-1$

i. Find c.

ii. Evaluate $P(X<2), P(2X+3\geq 5)$ and $P(1<X\leq 2)$.

iii. If $P(X\leq a)<\frac{1}{2}$ find the maximum value of a.

Solution:

(i) $3c^3+4c-10c^2+5c-1=1$

$3c^3-10c^2+9c-1=1$

$3c^3+9c-10c^2-2=0$

$(c-1)(3c^2-7c+2)=0$

$(c-1)(3c-1)(c-1)=0 \Rightarrow \boxed{c=1,1,1/3}$

(ii) $P(X<2)=P(0)+P(1)=3\left(\frac{1}{3}\right)^3+\frac{4}{3}-10\left(\frac{1}{9}\right)$

$=\frac{1}{9}+0.222=0.333\ldots$

$P(2X+3\geq 5)$, $2X+3=5$

$[X=1][X\geq 1]$, $P(X\geq 1)$

$=4\left(\frac{1}{3}\right)-10\left(\frac{1}{3}\right)^2+\frac{5}{3}-1$

$=0.222+0.666\ldots$

$=0.888\ldots$

$P(1<X\leq 2)=P(2)=\frac{5}{3}-1=0.667$

(iii) $P(X\leq a)<\frac{1}{2}$

$P(X\leq 0)=\frac{1}{9}=0.111$

$P(X\leq 1)=0.33$

$P(X\leq 2)=0.999$

So, the max value of 'a' is

$\boxed{a=1}$

for which $P(X\leq 1)<\frac{1}{2}$

satisfied.

71. A random variable X is distributed at random between the values 0 and 1 so that its probability density function is $f(x) = kx^2(1-x^3)$ where k is constant. Find the value of k, find its mean and variance. (CO3)

Solution: A/c to probability density func.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} kx^2(1-x^3)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow 0 + k\left(\frac{x^3}{3} - \frac{x^6}{6}\right) + 0 = 1$$

$$\frac{k}{3} - \frac{k}{6} = 1 \Rightarrow \frac{k}{6} = 1 \Rightarrow k = 6.$$

Mean: $E(x) = \int_0^1 x f(x)\,dx = \int_0^1 x[6x^2(1-x^3)]\,dx$

$$E(x) = 6\left[\int_0^1 x^3\,dx - \int_0^1 x^6\,dx\right] = 6\left[\frac{1}{4} - \frac{1}{7}\right]$$

$$= 0.643$$

Variance: $\sigma_x^2 = E(x^2) - [E(x)]^2$

$$E(x^2) = \int_0^1 x^2 f(x)\,dx = 6\left[\int_0^1 x^4\,dx - \int_0^1 x^7\,dx\right]$$

$$= 6\left[\frac{1}{5} - \frac{1}{8}\right] = 0.45$$

variance $\Rightarrow = 0.45 - (0.643)^2$

$= 0.0365$ Ans.

72. Let X and Y be jointly distributed with pdf- (CO3)

$$f(x,y) = \begin{cases} \frac{1}{4}(1+xy) & |x| < 1, |y| < 1 \\ 0 & elsewhere \end{cases}$$

Show that X and Y are not independent but X^2 and Y^2 are independent.

Solution:

$$f_X(x) = \int_{-1}^{1} \frac{1}{4}(1+xy)\,dy = \left[\frac{1}{4}y + \frac{xy^2}{2}\right]_{-1}^{1} = 1/2$$

$$f_Y(y) = \int_{-1}^{1} \frac{1}{4}(1+xy)\,dy = \left[\frac{1}{4}x + \frac{x^2y}{2}\right]_{-1}^{1} = \frac{1}{2}$$

$$F(X/Y) = \frac{\frac{1}{4}(1+xy)}{f(y)} = \frac{\frac{1}{4}(1+xy)}{\left(\frac{1}{2}\right)} = \frac{1+xy}{2}$$

$$F(Y/X) = \frac{\frac{1}{4}(1+xy)}{f(X)} = \frac{\frac{1}{4}(1+xy)}{\frac{1}{2}} = \frac{1+xy}{2}$$

$$F(X/Y)\cdot F(Y/X) = \frac{(1+xy)^2}{4} \neq f(x,y)$$

So, 'X' & 'Y' are not independent.

(ii) Now; $P(X^2 \leq x) = P(-\sqrt{x} \leq X \leq \sqrt{x}) = \int_{-\sqrt{x}}^{\sqrt{x}} f_X(X)\,dx = \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{2}\,dx = \sqrt{x}$

& $P(Y^2 \leq y) = P(-\sqrt{y} \leq Y \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f_Y(Y)\,dy = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}\,dy = \sqrt{y}$.

Now; $P(X^2 \leq x \cap Y^2 \leq y) = P(-\sqrt{x} < X < \sqrt{x} \cap -\sqrt{y} < Y < \sqrt{y})$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \left[\int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{4}(1+xy)\,dx\right] dy = \frac{1}{4}\int_{-\sqrt{y}}^{\sqrt{y}} \left(x + \frac{x^2y}{2}\right)_{-\sqrt{x}}^{\sqrt{x}} dy$$

$$= \frac{1}{4}\int_{-\sqrt{y}}^{\sqrt{y}} \left[(\sqrt{x} - (-\sqrt{x})) + \left(\frac{x-x}{2}\right)y\right] dy = \frac{1}{4}\int_{-\sqrt{y}}^{\sqrt{y}} 2\sqrt{x}\,dy$$

$$= \frac{1}{2}\sqrt{x}\,[y]_{-\sqrt{y}}^{+\sqrt{y}} = \frac{1}{2}\sqrt{x}\,(2\sqrt{y}) = \sqrt{x}\sqrt{y}$$

$\therefore P(X^2 \leq x \cap Y^2 \leq y) = P(X^2 \leq x)\cdot P(Y^2 \leq y)$

$\therefore X^2$ and Y^2 are independent

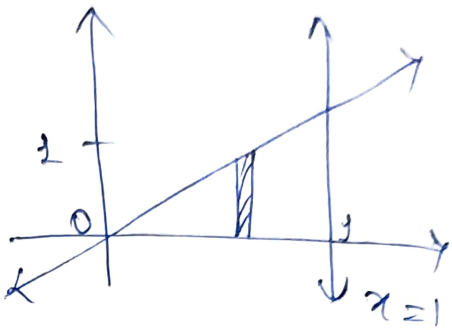
Hence, Proved.

73. The joint probability density function of two-dimensional random variable (X, Y) is given by_ (CO3)

$$f(x,y) = \begin{cases} 2 & 0 \le x < 1, 0 < y < x \\ 0 & elsewhere \end{cases}$$

I. Find marginal density function of X and Y.

II. Find the conditional density function of Y given $X = x$ and conditional distribution of X given $Y = y$.



Solution:

① $f_x(x) = \int_{y=0}^{x} 2\,dy = 2[y]_0^x$

$= 2x$

$f_y(y) = \int_{x=y}^{x=1} 2\,dx = 2[x]_y^1$

$= 2(1-y)$

② $f_{y|x}(Y/X) = \dfrac{f(x,y)}{f_x(x)} = \dfrac{2}{2x} = \dfrac{1}{x}$

$f_{x|y}(X/Y) = \dfrac{f(x,y)}{f_y(y)} = \dfrac{2}{2(1-y)} = \dfrac{1}{1-y}$

74. A random variable X has the following probability distribution: (CO3)

Value of X, x:	0	1	2	3	4	5	6	7
P(x):	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

i. Find k,

ii. Evaluate P(X<6), $P(3 < X \leq 6)$

iii. Find the minimum value of x so that $P(X \leq x) > 1/2$

Solution: A. T. Pmf:

(i) $\Sigma px = 1$

$\Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$10k^2 + 9k - 1 = 0.$

$(10k - 1)(k+1) = 0$

$\boxed{k = \frac{1}{10}}$

(ii) $P(X<6)$

$0 + k + 2k + 2k + 3k + k^2$

$= k^2 + 8k$

$= \frac{1}{100} + \frac{8}{10} = \frac{81}{100}$

$P(3 < X \leq 6)$

$3k + k^2 + 2k^2$

$3k^2 + 3k$

$\frac{3}{100} + \frac{3}{10} = \frac{33}{100}$

(iii) $P(X \leq x) > \frac{1}{2}$

So, $P(X \leq 0) = 0$

at $x = 1$;

$P(X \leq 1) = \frac{1}{10} < \frac{1}{2}$

at $x = 2$;

$P(X \leq 2) = \frac{3}{10} < \frac{1}{2}$

at $x = 3$;

$P(X \leq 3) = \frac{5}{10} = \frac{1}{2}$

$\Rightarrow$ at $x = 4$;

$P(X \leq 4) = \frac{4}{5} > \frac{1}{2}$

$\boxed{x = 4}$

75. Joint distribution of X and Y is given by- (CO3)

$f(x, y) = 4xye^{-(x^2+y^2)}; x \geq 0, y \geq 0.$

Test Whether X and Y are independent. For the above joint distribution, find the conditional density of X given $Y = y$.

Solution: ~~According to probability density func.~~

$$\cancel{\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} kx^2(1-x^3)\,dx = 1}$$

$$\cancel{\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty}}$$

$$g(x) = \int_0^\infty 4xy\, e^{-(x^2+y^2)}\,dy$$

$$= 4x\int_0^\infty y\, e^{-(x^2+y^2)}\,dy = \int_0^\infty 4x\left[y e^{-x^2} \cdot e^{-y^2}\right]dy$$

$$= 4xe^{-x^2}\int_0^\infty ye^{-y^2}\,dy$$

$y^2 = t$

$2y\,dy = dt$

$y\,dy = \frac{dt}{2}$

$$= \frac{4xe^{-x^2}}{2}\left(-e^{-t}\right]_0^\infty = 2xe^{-x^2}$$

$$h(y) = 2ye^{-y^2}$$

$$f(x,y) = g(x)\cdot h(y)$$

$$= 2xe^{-x^2} \times \left(2ye^{-y^2}\right)$$

$$= 4xye^{-(x^2+y^2)}$$

Hence, 'X' and 'Y' are Independent.