The man who knew ∞
Life Story of Srinivasa Ramanujan

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1 Early Life

Generally acknowledged as India's greatest mathematician, Srinivasa Ramanujan (see figure 1) was born on 22 December 1887 in Erode, located in the southern Indian state of Tamil Nadu.

Figure 1: Photo of Ramanujan taken in his late twenties

He began to focus on mathematics at an early age, and, at the age of about fifteen, borroweda copy of G.S. Carr's *Synopsis of Pure and Applied Mathematics* [3], which served as his primary source of learning mathematics. At about the time Ramanujan entered college, he began to record his mathematical discoveries in notebooks. Living in poverty with no mean of financial support, suffering at times from serious illnesses and working in isolation, Ramanujan devoted his efforts to mathematics and continued to record his discoveries withou proofs in noteboks for the next six years.

2 His Notebooks

To provide a feeling for Ramanujans notebooks, we reproduce here (see figure 2) a page from his third notebook, Chapter XVIII [1]. There is no particular reason for choosing this page except that it contains an entry which has become a 'folklore' which illustrates the attachment Ramanujan had to numbers prompting Littlewood to state that to Ramanujan every number is a *personal friend*.

There are two parts shown in this page (see figure 2). The first part is concerning the geometrical construction of a square whose area is equal to that of a given circle. Here he gives a geometrical construction for finding the length of the side of a square whose area equals that of a circle. He also reproduced this and another geometrical construction for π in his later paper on "Modular equations and approximations to π". In this paper he deduced several formulae for π such as:

$$\frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right) = 3.14159265380... \quad (1)$$

$$\pi = 12\sqrt{190}\log 2\sqrt{2}+\sqrt{10})(3+\sqrt{10}) \quad (2)$$

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!(1103+26390n)}{(n!)^4 396^{4n}} \quad (3)$$

The expression in equation (1) gives the value of π accurate to 9 decimal places and the one equation in (2) has an accuracy of 18 decimal places. Ramanujan asserted that equation (3) is the most intriguing one because it is a very "*rapidly convergent*" series. In 1986, two computer scientists used a version of Ramanujan's formula to calculate π to 17 million places and found that the formula converges on the exact value with far greater efficiency than any previous method. This success proved that Ramanujan's insight was correct.

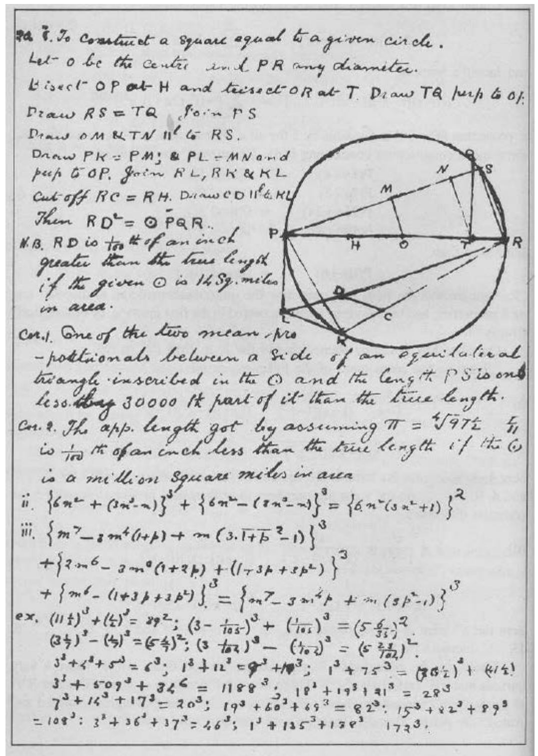

Figure 2: Ramanujan's handwritten page from his "Notebook".

3 Elementary Mathematics

Many of Ramanujan's discoveries can be appreciated by those with only a knowledge of high school algrebra. Chapter 2 in the second notebook, the unorganized portions of the second and third notebooks [2], and the problems that Ramanujan submitted to the *Journal of Indian Mathematical Society* [5] are excellent sources for these gems.

Those familiar with the famous texicab number 1729 might correctly surmise that Ramanujan enjoyed finding equal sums of powers. This, for example, $a + b + c = 0$ [2, p. 96],

$$2(ab + ac + bc)^4 = a^4(b - c)^4 + b^4(a - c)^4 + c^4(a - b)^4.$$

In fact, in his third notebook [6, p. 385], Ramanujan recorded similar formulas for $2(ab+ac+bc)^{2n}$, when $n \in 1,2,3,4$,and wrote "and so on" to indicate that he possessed a general procedure for finding such formulas.

Did you know that [2, pp. 109, 39]

$$2\sin\left(\frac{\pi}{18}\right) = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\cdots}}}}$$

where the sequence of signs -,+,+,... has period 3, or that

$$(\cos 40^\circ)^{1/3} + (\cos 80^\circ)^{1/3} - (\cos 20^\circ)^{1/3} = \sqrt[3]{\tfrac{3}{2}(\sqrt[3]{9}-2)}$$

Ramanujan was fond of stating intriguing formulas such as those above, but as in most instances, they are special cases of more general theorems that he established.

4 Number Theory

We cite only one theorem in the notebooks from the theory of numbers, as reformulated by K.S.Williams [2,p.71].

Theorem 1. *Let $a, b, A, B \in Z^+$ be such that*

$$gcd(a,b) = 1 = \text{gdc}(A,B), \qquad \text{ab} \neq \text{square}.$$

Suppose that every time prime $p \equiv B$ (mod A) with gcd(p,2ab)=1 is expressible in the form $ax^2 - by^2$ for some integers x and y. Then every prime q such that $q \equiv -B$ (mod A) and gcd(q,2ab)=1 is expressible in the form $bX^2 - aY^2$ for some integers X and Y.

5 Infinite Series

Once the following problem was posed by Ramanujan: evaluate the following series for n¿3,

$(\frac{1}{1^n}) + \frac{1}{2}(\frac{1}{3^n}) + \frac{1\cdot 3}{2\cdot 4}(\frac{1}{5^n}) + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}(\frac{1}{7^n}) + \cdots$.

In his attempt to sum the series he arrived at the following results: Let

$$\begin{gathered} f(p) = (\tfrac{1}{1+p}) + \tfrac{1}{2}(\tfrac{1}{3+p}) + \tfrac{1\cdot 3}{2\cdot 4}(\tfrac{1}{5+p}) + \cdots \\ = \int_0^1 x^p (1 + \tfrac{1}{2}x^2 + \tfrac{1\cdot 3}{2\cdot 4}x^4 + \cdots) dx \\ = \int_0^1 \tfrac{x^p}{\sqrt{1-x^2}} dx = \tfrac{1}{2} \int_0^1 x^{\frac{1}{2}(p-1)} (1-x)^{\frac{-1}{2}} dx \\ = \tfrac{1}{2} \tfrac{\Gamma(\frac{p+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p+2}{2}} = \tfrac{\pi^{\frac{1}{2}}}{2} \tfrac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p+2}{2})^2} \end{gathered}$$

But

$$\Gamma(\tfrac{p+1}{2}) = \tfrac{\pi^{\frac{1}{2}}}{2^p} \tfrac{\Gamma(p+1)}{\Gamma(\frac{p+2}{2})}$$

Therefore

$$f(p) = \tfrac{\pi}{2^{p+1}} \tfrac{\Gamma(p+1)}{\Gamma(\frac{p+2}{2})^2}$$

From this he deduced the following identities:

$$\begin{gathered} 1 + \tfrac{1}{2}(\tfrac{1}{3}) + \tfrac{1\cdot 3}{2\cdot 4}(\tfrac{1}{5}) + \cdots = \tfrac{\pi}{2}, \\ 1 + \tfrac{1}{2}(\tfrac{1}{3^2}) + \tfrac{1\cdot 3}{2\cdot 4}(\tfrac{1}{5^2}) + \cdots = \tfrac{\pi}{2}(\log 2), \\ 1 + \tfrac{1}{2}(\tfrac{1}{3^3}) + \tfrac{1\cdot 3}{2\cdot 4}(\tfrac{1}{5^3}) + \cdots = \tfrac{\pi^3}{48} + \tfrac{\pi}{4}(\log 2)^2. \end{gathered}$$

6 Partition Fractions

Another area of Number Theory in which Ramanujan had worked extensively ans is well-known for his contributions is the theory of **partition fractions**. The partition function $p(n)$ gives the number of ways of writing an integer n as a sum of smaller positive integers.Thus 4 can e written in five different ways, as

$$4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1.$$

Number	1	2	3	4	5	6	7	8	9	10
p(n)	1	2	3	5	7	11	15	22	30	42

Table 1: Partition Numbers.

The partition number $p(4)$ is therefore 5. The partition numbers of the numbers 1-10 are given in Table 1.

7 Conclusion

Srinivasa Ramanujan (1887-1920) hailed as an all-time great mathematician, like Euler, Gauss or Jacobi, for his natural genius, as left behind 4000 original theories, despite his lack of formal education and a short life span. As for his place in the world of Mathematics, we quote Bruce C Berndt:

”Paul Erdoc has passed on to us Hardy’s personal ratings of mathematicians. Suppose that we rate mathematicians on the basis of pure talent on a scale from 0 to 100, Hardy gave himself a score of 25, Littlewood 30, Hilbert 80 and Ramanujan 100”.

If you want to know more about Srinivasa Ramanujan [7], then you may want to read Robert Kanigel’s book[4].

References

1. Bruce C. Berndt, *Ramanuan’s Notebooks, Part III*, Springer-Verlag, New York (1991).

2. Bruce C. Berndt, *Ramanujan’s Notebooks, Part IV*, Springer-Verlag, New York (1994.

3. George S Carr, *A Synopsi of Elementary Results in Pure and Applied Mathmatics (2 Volume set)*, Cambridge University Pres (2013).

4. Robrt Kenigel, *The Man who knew Infinity: A life of the Genius Ramanujan*, fifth edition, Washington Square Press (1992).

5. Srinivasa Ramanujan, *Collected Papers*, American Mathematical Society, Providence (2000)

6. Srinivasa Ramanujan, *Notebooks (2 Volumes)*, Tata Institute of Fundamental Research, Mumbai (1957).

7. Other useful links are given at ¡http.//people.rit.edu/axasma/ramanujan.html¿