# Homework 4: Space-Time Discretization and Stability

**Due Thursday, November 28th at 1 am.**

In this problem, we will work with the 1D heat equation

$$u_t = \kappa u_{xx}, \quad t \in [0,1], \quad x \in [0,1], \quad \kappa = 0.1.$$

Use periodic boundary conditions[1] $u(t,0) = u(t,1)$, and initial heat distribution

$$u(0,x) = \sin(2\pi x).$$

**Problem 1:**

Implement the forward Euler method to solve the problem. We can write this method in shorthand as

$$D_t u_j^n = \kappa D_x^2 u_j^n$$

and explicitly as

$$\frac{u_j^{n+1} - u_j^n}{\Delta_t} = \kappa \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta_x^2}$$

We suggest you use linear algebra to implement this iteration. That is, define the vector

$$U^n = \begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_m^n \end{bmatrix}$$

and implement the forward Euler iteration as

$$U^{n+1} = U^n + \frac{\kappa \Delta t}{\Delta_x^2} A U^n$$

where $A$ is a discretized operator.

Use $dt = 0.05$ and $dx = 0.1$. Save the solution $u(1,x)$, a column vector of size 10, into `A1.dat`.

What happens if you try to take $dt = dx = 0.05$, or $dt = dx = 0.01$?

**Problem 2:**

Implement the backward Euler method to solve the problem. We can write this method in shorthand as

$$D_t u_j^n = \kappa D_x^2 u_j^{n+1}$$

and explicitly as

$$\frac{u_j^{n+1} - u_j^n}{\Delta_t} = \kappa \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta_x^2}$$

Here, as in the previous problem, use linear algebra to set up the iteration. You will need to solve a linear system in this case, since backward Euler is an implicit method.

Use $dt = 0.05$ and $dx = 0.1$. Save the solution $u(1,x)$, a column vector of size 10, into `A2.dat`.

Use $dt = 0.01$ and $dx = 0.01$. Save the solution $u(1,x)$, a column vector of size 100, into `A3.dat`.

[1] Periodic boundary conditions force simple modifications to discretized spatial operators. Last weeks' homework will help you get through the setup pretty quickly!

**Problem 3:**

Implement the Crank-Nicolson method to solve the problem:

$$D_t u_j^n = \frac{\kappa}{2}\left(D_x^2 u_j^n + D_x^2 u_j^P n + 1\right)$$

and explicitly as

$$\frac{u_j^{n+1} - u_j^n}{\Delta_t} = \frac{\kappa}{2\Delta_x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n + u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right)$$

Here, as in the previous problem, use linear algebra to set up the iteration.

Use $dt = 0.05$ and $dx = 0.1$. Save the solution $u(1, x)$, a column vector of size 10, into `A4.dat`.

Use $dt = 0.01$ and $dx = 0.01$. Save the solution $u(1, x)$, a column vector of size 100, into `A5.dat`.

Make a movie of the time evolution of your heat distributions. Play with how large a step size you can get away with when working with the implicit methods. You don't have to turn anything in for this problem.