AMATH 585
Homework #2
Due Thursday, January 23, 2019
Shabab Ahmed

Homework is due to Canvas by 11:00pm PDT on the due date.

Problem 1.

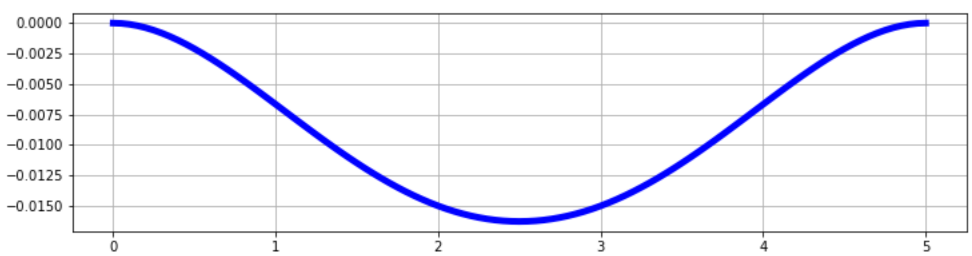
Consider a "rigid" beam of length L that is supported at both ends and sags by a very small amount in the center due to gravity acting on the mass. The small deflection can be modeled by the Euler-Bernoulli beam equation described for example at
`https://en.wikipedia.org/wiki/Euler-Bernoulli_beam_theory`,
which in the simplest case where the beam has constant cross-sectional area and is made of a uniform material takes the simple form

$$u''''(x) = \gamma, \qquad \text{for } 0 \le x \le L,$$

where γ is a constant depending on the material properties. If both ends are embedded in walls that hold their position constant (at $u = 0$, say) and also hold them horizontal at the ends, then the boundary conditions are

$$u(0) = u'(0) = 0, \qquad u(L) = u'(L) = 0.$$

The deflection then looks something like this (with a greatly exagerated vertical scale):



(a) The exact solution to this problem is easy to compute as a quartic function that satisfies the four boundary conditions. Compute this for $L = 5$ and $\gamma = 0.01$, and confirm that it looks like the figure above.

(b) Write a computer program to solve this problem. Use the second-order accurate formula for the fourth derivative that you derived in Homework 1, together with formulas for boundary conditions that preserve the second order accuracy. There is more than one way to do this that would be correct.

Test your program for a series of grids and produce a log-log plot to verify the expected accuracy.

Solution: a) We will compute the exact solution as a quartic function satisfying the four boundary conditions. Then we will plot the solution and compare it with the above plot. We will let $L = 5$ and $\gamma = -0.01$.

$$u'''(x) = \gamma = -0.01 = -\frac{1}{100}$$

Integrating both sides:

$$u'''(x) = -\frac{x}{100} + c_1$$

We will calculate c_1 later using the boundary conditions.

Again, integrating both sides:

$$u''(x) = -\frac{x^2}{200} + c_1 x + c_2$$

$$u'(x) = -\frac{x^3}{600} + c_1 \frac{x^2}{2} + c_2 x + c_3$$

Using the fact that $u'(0) = 0$ we have that $c_3 = 0$. Then, $u'(x) = -\frac{x^3}{600} + c_1 \frac{x^2}{2} + c_2 x$. Now, we make use of the boundary condition $u'(5) = 0$ to get an equation for c_1 and c_2.

$u'(5) = -\frac{5^3}{600} + c_1 \frac{5^2}{2} + 5c_2 = 0$. Then, $\frac{5^2 c_1}{2} + 5c_2 = \frac{5^3}{600}$. So, $\frac{5c_1}{2} + c_2 = \frac{5^2}{600}$ which implies that

$$5c_1 + 2c_2 = \tfrac{1}{12} \ldots (1).$$

We now go back to $u'(x) = -\frac{x^3}{600} + c_1 \frac{x^2}{2} + c_2 x$ and integrate it to get:

$$u(x) = -\frac{x^4}{2400} + c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} + c_4$$

Using the fact that $u(0) = 0$ we get that $c_4 = 0$. The other boundary condition $u(5) = 0$ tells us that $\frac{5^3}{6} c_1 + \frac{5^2}{2} c_2 = \frac{5^4}{2400}$. So, $\frac{5c_1}{6} + \frac{c_2}{2} = \frac{5^2}{2400}$. Therefore, we get:

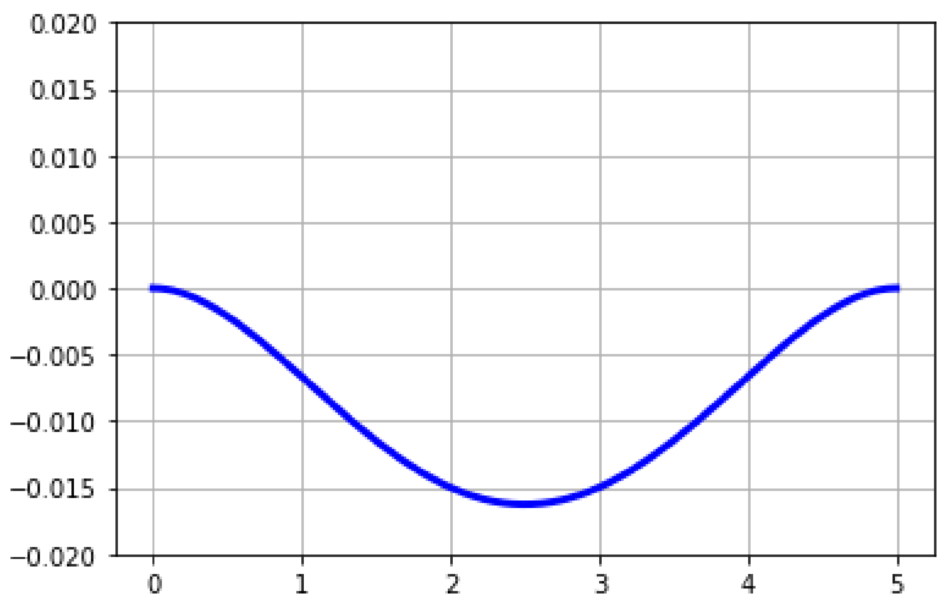
$$5c_2 + 3c_2 = \frac{1}{16} \ldots (2).$$

We have two equations for two unknowns (c_1 and c_2) and we solve for them. Calculating (2) - (1) gives us $c_2 = \frac{1}{16} - \frac{1}{12} = -\frac{1}{48}$. Substituting this value of c_2 into (1) gives us that $5c_1 = \frac{1}{12} + \frac{1}{24}$ and so $5c_1 = \frac{1}{8}$. Therefore, $c_1 = \frac{1}{40}$.

Thus, we have the following solution to $u''''(x) = \gamma = -0.01$ with $u(0) = u'(0) = 0$ and $u(L) = u'(L) = 0$:

$$u(x) = -\frac{x^4}{2400} + \frac{x^3}{240} - \frac{x^2}{96}.$$

We plot this in the Jupyter notebook and get the following:



The plot matches the figure posted above in the question. We can see that at $u(0) = u(5) = 0$ and also in both the figures $u(2) = u(3) = -0.0150$. Both figures have exaggerated vertical scales and thus their similar shapes confirm the exact solution.

b) We use the second-order accurate formula for the fourth derivative derived in Homework 1.

$$u^{(4)}(x_0) \approx \frac{1}{h^4}(u(x_{-2}) - 4u(x_{-1}) + 6u(x_0) - 4u(x_1) + u(x_2)).$$

We aim to solve $u^{(4)}(x) = \gamma = -0.01$ on $x \in [0, 5]$ for boundary conditions $u(0) = u'(0) = 0$ and $u(5) = u'(5) = 0$. These are specific boundary conditions and a specific right hand side. We will start by attempting to solve more general boundary conditions for $x \in [0, L]$, $u(0) = a_0$, $u'(0) = a_1$ and $u(L) = b_0$ ad $u'(L) = b_1$ for a more general right hand side, $f(x)$. Approximating the solution $u(x_j)$ by U_j on m equally spaced grid points we get $D^4 U_j = \frac{1}{h^4}(U_{j-2} - 4U_{j-1} + 6U_j - 4U_{j+1} + U_{j+2})$ and this leads to the following set of algebraic equations:

$$\frac{1}{h^4}(U_{j-2} - 4U_{j-1} + 6U_j - 4U_{j+1} + U_{j+2}) = f(x_j)$$

for $j = 2, 3, \ldots, (m-1)$

We can note that we know the boundary values $U_0 = a_0$ and $U_{m+1} = b_0$. We also have Neumann boundary conditions on both boundaries. We will use one-sided approximations that are second-order accurate for u'. From section 1.2, we have that $u'(x_j) \approx \frac{1}{h}(\frac{3}{2}u(x_j) - 2u_(x_{j-1}) + \frac{1}{2}u(x_{j-2}))$. Similarly, $u'(x_j) \approx -\frac{1}{h}(\frac{3}{2}u(x_j) - 2u_(x_{j+1}) + \frac{1}{2}u(x_{j+2}))$.

Therefore, we have $u'(x_0) = -\frac{1}{h}(\frac{3}{2}u(x_0) - 2u_(x_1) + \frac{1}{2}u(x_2)) = a_1$. So, $\frac{3}{2}u(x_0) - 2u_(x_1) + \frac{1}{2}u(x_2) = -a_1 h$. Again, approximating $u(x_j)$ by U_j leads us to $\frac{3}{2}U_0 - 2U_1 + \frac{1}{2}U_2 = -a_1 h$. Using the fact that $U_0 = a_0$ we have that $-2U_1 + \frac{1}{2}U_2 = -a_1 h - \frac{3}{2}a_0$. We multiply both sides by $\frac{1}{h^4}$ to get $\frac{1}{h^4}(-2U_1 + \frac{1}{2}U_2) = \frac{1}{h^4}(-a_1 h - \frac{3}{2}a_0)$ and this is going to be the first row of our discretization matrix.

Similarly, we use $u'(x_{m+1}) = b_1$ and $u'(x_j) \approx \frac{1}{h}(\frac{3}{2}u(x_j) - 2u_(x_{j-1}) + \frac{1}{2}u(x_{j-2}))$ to get $\frac{3}{2}u(x_{m+1}) - 2u(x_m) + \frac{1}{2}u(x_{m-1}) = b_1 h$. Approximating $u(x_j)$ by U_j and using $U_{m+1} = b_0$ we arrive at $-2U_m + \frac{1}{2}U_{m-1} = b_1 h - \frac{3}{2}b_0$. Again, we multiply both sides by $\frac{1}{h^4}$ to get $\frac{1}{h^4}(-2U_m + \frac{1}{2}U_{m-1}) = \frac{1}{h^4}(b_1 h - \frac{3}{2}b_0)$. This is going to correspond to the last row of our discretization matrix.

We now shift our focus back to $\frac{1}{h^4}(U_{j-2} - 4U_{j-1} + 6U_j - 4U_{j+1} + U_{j+2}) = f(x_j)$ to figure out the other rows of the discretization.

$j = 2:$

$\frac{1}{h^4}(U_0 - 4U_1 + 6U_2 - 4U_3 + U_4) = f(x_2)$

$\frac{1}{h^4}(-4U_1 + 6U_2 - 4U_3 + U_4) = f(x_2) - \frac{a_0}{h^4}$

$j = 3:$

$\frac{1}{h^4}(U_1 - 4U_2 + 6U_3 - 4U_4 + U_5) = f(x_3)$

$\vdots$

$j = m - 2$

$\frac{1}{h^4}(U_{m-4} - 4U_{m-3} + 6U_{m-2} - 4U_{m-1} + U_m) = f(x_{m-2})$

$j = m - 1$

$\frac{1}{h^4}(U_{m-3} - 4U_{m-2} + 6U_{m-1} - 4U_m + U_{m+1}) = f(x_{m-1})$

$\frac{1}{h^4}(U_{m-3} - 4U_{m-2} + 6U_{m-1} - 4U_m) = f(x_{m-1}) - \frac{U_{m+1}}{h^4} = f(x_{m-1}) - \frac{b_0}{h^4}$

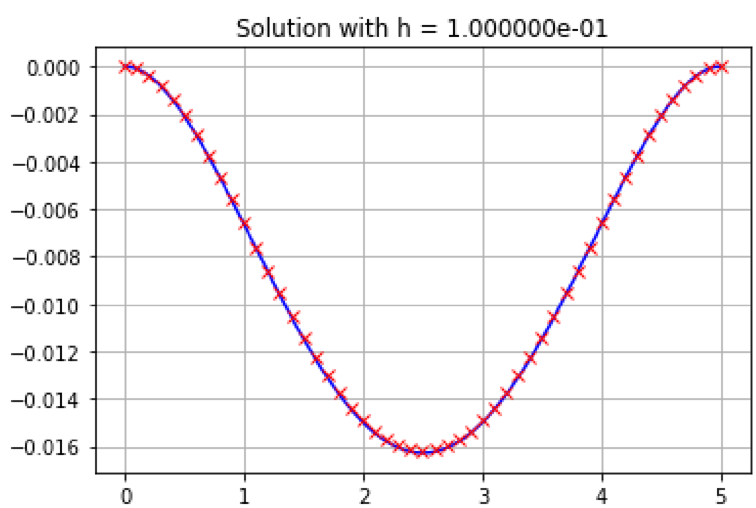
We have a linear system of m equations for the m unknowns $(U_1, U_2, \ldots, U_{m-1}, U_m)$, which can be written in the form $AU = F$, where U is the vector of unknowns $U = [U_1, U_2, \ldots, U_{m-1}, U_m]^T$ and

$$A = \frac{1}{h^4}\begin{bmatrix} -2 & \frac{1}{2} & & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & & \frac{1}{2} & -2 \end{bmatrix}$$

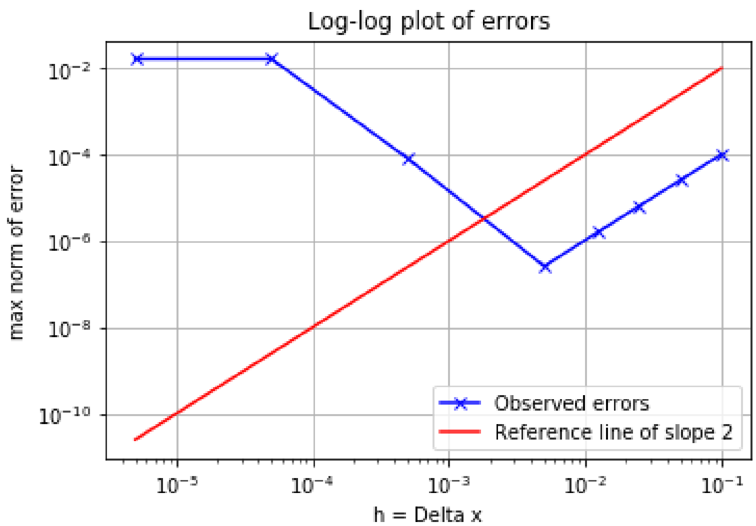
with

$$\text{F} = \begin{bmatrix} \frac{1}{h^4}(-a_1 h - \frac{3}{2}a_0) \\ f(x_2) - \frac{a_0}{h^4} \\ f(x_3) \\ \vdots \\ f(x_{m-2}) \\ f(x_{m-1}) - \frac{b_0}{h^4} \\ \frac{1}{h^4}(b_1 h - \frac{3}{2}b_0) \end{bmatrix}$$

We code this discretization process in the Jupyter notebook. For this specific $f(x) = \gamma = -0.01$ and boundary conditions we get the following plot:



The blue line consists of the exact solution we computed in part a) evaluated on a grid. The red dots are the solutions from the code and as we can see that the two plots match up. This is an indication that our code produces the solution we want but we cannot be sure until we test it for a series of grids. This graph is just for $m = 49$ grid points and so for $h = 10^{-1}$ and we will test the code for $m = [50, 100, 200, 400, 1000, 10000, 100000, 1000000]$ grid points. Then we produce a log-log plot to verify the expected accuracy. We want to check if the error $\approx \mathcal{O}(h^2)$, that is, if the second order accuracy is preserved. Below we have log-log plot of errors that was produced in our Jupyter notebook.



In the figure, we have two lines. The blue line represents the observed errors. The observed errors are calculated by computing the values of the true solution on the grid and then taking its difference with the computed solution on the same grid. So, observed error = |computed solution - actual solution|. Both solutions are evaluated on a grid and so what we get is a vector of errors. We plot the max norm of these vectors on the figure. The reference line is a plot of the expected error. This line has slope 2 because we expect the errors to have this slope for a second-order accurate method. We see that we have the expected shape for the max norm of the observed errors, that is, it has slope 2 until $h = 0.005$. For h smaller than this value, the round off error is not big enough and so the error is $\mathcal{O}(h^2)$ like we expect it to be. But after this particular value of h rounding error takes over. This can be seen by noticing that the max norm of the observed errors increasing for h smaller than than ≈ 0.005.

Problem 2. The problem above doesn't fully test whether the boundary conditions are implemented properly since the values specified are all zero.

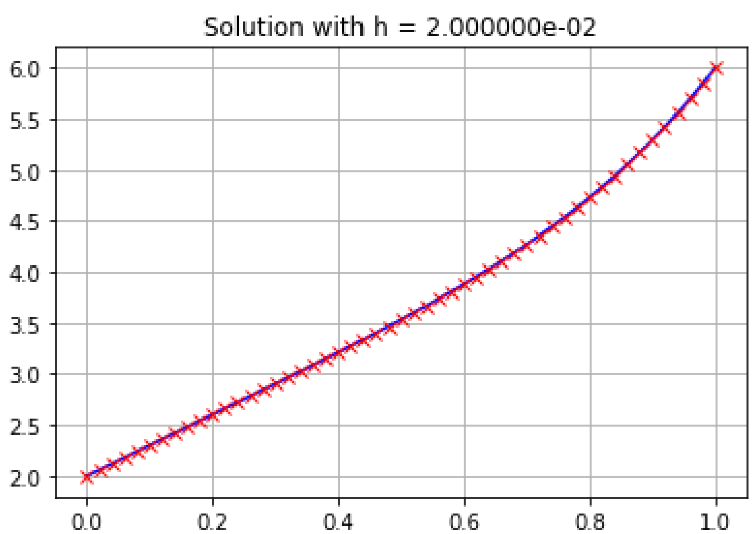
To test your code a bit more, adapt it to solve the problem $u''''(x) = f(x)$ on the interval $0 \leq x \leq 1$ with the function $f(x)$ and boundary conditions on $u(0)$, $u'(0)$, $u(1)$, and $u'(1)$ chosen so that the true solution is $u(x) = 2 + 3x + x^5$. (This is the method of manufactured solutions as discussed in the notebook `BVP1.ipynb`.)

Solution:

Consider $u'''(x) = f(x)$ on the interval $0 \leq x \leq 1$ and boundary conditions $u(0), u'(0), u(1)$ and $u'(1)$. We want the true solution to be $u(x) = 2 + 3x + x^5$. This is the method of manufactured solutions and it lets solve for $f(x)$ and the boundary conditions so that we can plug it back into our code and check whether we get the correct solution.

Since we want $u(x) = 2+3x+x^5$ we need $u(0) = 2$ and $u(1) = 6$. Also, $u'(x) = 3+5x^4$ and so $u'(0) = 3$ and $u'(1) = 8$. To calculate $f(x)$ we compute $u''''(x) = 120x$. Therefore, we are solving $u''''(x) = 120x$ on $x \in [0, 1]$ with boundary conditions $u(0) = 2$, $u(1) = 6$, $u'(0) = 3$ and $u'(1) = 8$. We plug this problem into the code in Jupyter notebook to see if we obtain the right solution.

The following plot contains the computed solution and the true solution. The blue line represents the true solution $u(x) = 2 + 3x + x^5$ and the red marks correspond to the computed solution. We can see that the computed solution matches up with the true solution. However, this is for a particular value of h and so we test the code further by trying it for different values of h.



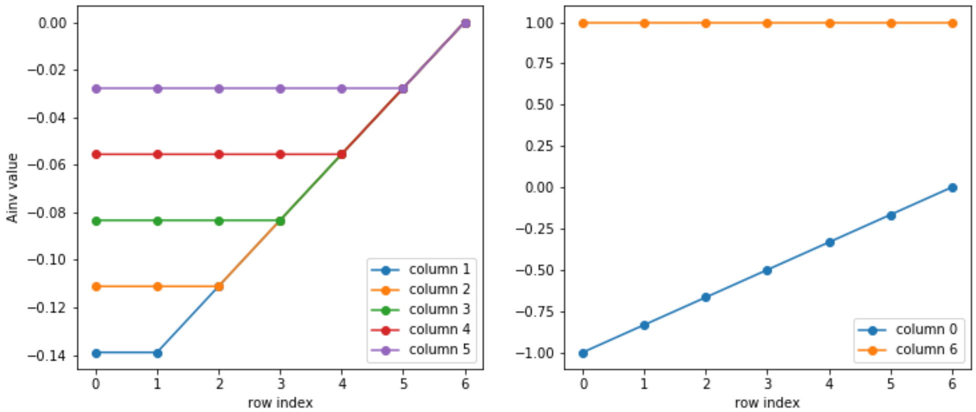
We compute the solution using the code in the Jupyter notebook for different values of h. Then we compute the observed error = | computed solution - actual solution |. Again, these are evaluated on a grid and so what we get is a vector of errors. We take the max norm of these vectors and produce a log-log plot of it against h in the figure below:

The blue line represents the observed errors as mentioned above. The reference line has a slope 2 as we still expect the solution to be second order accurate. We can notice that the observed errors have a slope of 2 until a threshold value of h. This seems to be around $h = 0.0025$. The error then picks up due to rounding error dominating.

Recall from Homework 1 that the L.T.E $= \frac{h^2}{6}u^{(6)}(x_0) + \mathcal{O}(h^4)$. Since $u(x)$ is a polynomial of degree 5, $u^{(6)}(x) = 0$. This should mean that the L.T.E $= 0$. But why is that not the case? This is because we haven't accounted for the fact that we also approximate the Neumann boundary condition using a second order accurate approximation.

Problem 3. The notebook `BVP_stability.ipynb` (visible as a rendered webpage here) shows plots of the columns of A^{-1} that correspond to the discussion of Section 2.11 and Figure 2.1 in the book. Also shown are similar plots for the case of Neumann boundary conditions at the left boundary.

The plot below shows the columns in the simpler case where the matrix from (2.54) is used for the Neumann boundary condition, for the case $m = 5$.



(a) Explain why each of these has the form it does, and give a closed form expression for all elements of A^{-1} in the general case with m interior points (similar to (2.46) for the Dirichlet case).

(b) Using this formula, obtain an upper bound on $\|A^{-1}\|_\infty$ that is independent of h in order to prove stability of this method.

(c) Determine the Green's function for the problem with the Neumann condition at the left boundary, i.e. the function $G(x; \bar{x})$ that solves

$$u''(x) = \delta(x - \bar{x}), \qquad \text{for } 0 \leq x \leq 1,$$

with boundary conditions

$$u'(0) = 0, u(1) = 0.$$

Solution:

a) The matrix A from (2.54) is the following:

$$\frac{1}{h^2}\begin{bmatrix} -h & h & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 2 & -1 & \\ & & & & & 0 & h^2 \end{bmatrix}$$

with $h = \frac{1}{m+1}$.

We can think of the columns of A^{-1} to be solving particular BVPs. The columns can be thought of as approximations to Green's function. So they are solving $u''(x) = \delta(x - \overline{x})$ for the boundary conditions $u'(0) = 0$ and $u(1) = 0$. The first and last columns will have different shapes to the interior columns because of the boundary conditions. The first column is solving $u''(x) = 0$ on the interior points so this should represent a straight line with slope 1. The interior columns correspond to the Green's function for zero boundary conditions. So, these are going to be discrete approximation of $hG(x;x_j)$. Since we know Green's function to be a piecewise linear continuous function it makes sense that the columns of A^{-1} have that form. Similarly, the last column is solving $u''(x) = 0, u'(0) = 0, u(1) = 1$ which gives it the particular shape.

Now, we will attempt to give a closed form expression for all elements of A^{-1}. Let $B = A^{-1}$ and let B_j denote the jth column of B. We know that that $AB = I$. Particularly, $AB_j = e_j$.

Consider,

$$V_1 = \begin{bmatrix} -(m+1)h \\ -(mh) \\ -(m-1)h \\ \vdots \\ -3h \\ -2h \\ -h \\ 0 \end{bmatrix}$$

and then consider AV_1. The last row is going to zero because h^2 x $0 = 0$. All the interior rows are going to be zero as well because the A is an approximation of the second derivative but it has the property that it is exact for linear functions. The $\{2, 3, \ldots, m+1\}$ rows of V_1 are increasing linearly and so the second derivative would be zero. So $(AV_1)_j = 0$ for $j = \{2, 3, \ldots, m+1\}$. Then $(AV_1)_1 = \frac{(h(m+1)h)-(mh)h}{h^2} = \frac{mh+h^2-mh^2}{h^2} = 1$.

So,

$$AV_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This is the first column of the identity matrix and since $AB_j = e_j$ this must be the first column of A^{-1}. This column can be thought of as a discretization of $v(x) = x - 1$. This is going to help us check for the Green's function in part c).

Let

$$V_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Each of the entries are just equal to one and so the first derivative and second derivative would be exact and equal to zero. This means that the first $m+1$ rows of AV_2 will be zero. The last row would be $\frac{h^2}{h^2} = 1$.

Therefore,

$$AV_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

This corresponds to the last column of the identity matrix and so V_2 would be the last column of A^{-1}. This is the discretization of $v(x) = 1$.

Now, we shift our focus to the interior columns. We are going to use the plot to help with this. Consider,

$$v(x) = \begin{cases} x_j - 1 & 0 \le x \le x_j \\ x - 1 & x_j \le x \le 1 \end{cases}$$

and let

$$V_j = \begin{bmatrix} v(x_0) \\ v(x_1) \\ \vdots \\ v(x_{m+1}) \end{bmatrix}$$

Now, we consider AV_j. The last row will again be zero because h^2 x $0 = 0$. The entries are piecewise linear. The first $j-1$ entries are just a constant function with zero slope. So, the first derivative will be zero. Since, A is a second derivative approximation and it is exact for a linear function the first $j-1$ entries of AV_j will be zero. Similarly, $j+1, \ldots, m+1$ entries are linearly increasing and so the corresponding AV_j entries will be zero too due to the second derivative of a linear function being zero. The only nonzero entry would be the the jth entry where there is the kink shown in the picture. The entry would be equal to $\frac{x_j - 2x_j + x_{j+1}}{h^2} = \frac{-jh + (j+1)h}{h^2} = \frac{1}{h}$. So, $AV_j = \frac{1}{h}e_j$. Then, the jth column of A^{-1} would be hV_j. We have all the columns of A^{-1} now.

$$A^{-1} = \begin{bmatrix} x_0 - 1 & h(x_1 - 1) & h(x_2 - 2) & \ldots & 1 \\ x_1 - 1 & h(x_1 - 1) & h(x_2 - 2) & \ldots & 1 \\ x_2 - 1 & h(x_2 - 1) & h(x_2 - 2) & \ldots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ x_{m+1} - 1 & h(x_{m+1} - 1) & h(x_{m+1} - 1) & \ldots & 1 \end{bmatrix}$$

b)$\|A^{-1}\|_\infty = \|B\|_\infty =$ maximum row sum $= \max_i \sum_j |B|_{ij}$

Then, $\sum_j |B|_{ij} = |B_{i0}| + \sum_{j=1}^m |B_{ij}| + |B_{i,m+1}|$.

$|B_{i,m+1}| = 1$ and $|B_{i0}| \le 1$. Also, $\sum_{j=1}^m |B_{ij}| \le mh$ and since $h = \frac{1}{m+1}$ we have that $\sum_{j=1}^m |B_{ij}| \le 1$. Therefore, $\|A^{-1}\|_\infty \le 3$. Thus, by the definition of stability we can say that the method is stable.

c) We are looking for the function that solves $G(x; \overline{x})$ that solves $u''(x) = \delta(x - \overline{x}), \qquad$ for $0 \le x \le 1$, with boundary conditions $u'(0) = 0, u(1) = 0.$.

We will integrate it once using the fact that the derivative of the Heaveside function,

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \ge 0 \end{cases}$$

is equal to $\delta(x)$. We have to be a little careful here because in our problem δ function is centered at $\overline{x}$.

$G(x; \overline{x})$ is the solution to the given problem. So, $G''(x; \overline{x}) = \delta(x - \overline{x})$.

Given that $u'(0) = 0$, $G'(x; \overline{x}) = \int (\delta(x - \overline{x})) = \begin{cases} 0 & 0 \le x \le \overline{x} \\ 1 & \overline{x} < x \le 1 \end{cases}$

.

Integrating again we have: $G(x,\overline{x}) = \begin{cases} c_1 & 0 \leq x \leq \overline{x} \\ c_2 + x & \overline{x} < x \leq 1 \end{cases}$

.

We will now determine the coefficients c_1 and c_2. We know $u(1) = 0$ and so $c_2 + 1 = 0$. Thus $c_2 = -1$. Also, the function is continuous and so $c2 + x = c_1$ at $\overline{x}$. So, $c_1 = \overline{x} - 1$.

Thus, $G(x,\overline{x}) = \begin{cases} \overline{x} - 1 & 0 \leq x \leq \overline{x} \\ x - 1 & \overline{x} < x \leq 1 \end{cases}$

.

We can go back to part a) and compare this with the jth column of the inverse of A. The function we used to discretize is the same as this function. Now, we move onto the finding the $G_0(x)$ and $G_1(x)$. Clearly, $G_1(x) = 1$. This is actually the solution to the problem: $G_1''(x) = 0, G_1'(0) = 0$ and $G_1(1) = 1$.

$G_0(x)$ is the solution to the problem $G_0''(x) = 0, G_0'(0) = 1$ and $G_0(1) = 0$. So, $G_0'(x) = c_1$. Since, $G_0'(0) = 1$ we have that $c_1 = 1$. Then, $G_0(x) = x + c_2$. Given that $G_0(1) = 0$, we have that $c_2 = -1$. Therefore, $G_0(x) = x - 1$. Again, the solutions match up with the discretizations we had in part a).