AMath 585
Homework #1
Due Thursday, January 16, 2019
Shabab Ahmed

Homework is due to Canvas by 11:00pm PDT on the due date.

Problem 1. Suppose we wish to approximate the fourth derivative $u^{(4)}(x_0)$ using a 5-point stencil

$$u^{(4)}(x_0) \approx c_{-2}u(x_{-2}) + c_{-1}u(x_{-1}) + c_0u(x_0) + c_1u(x_1) + c_2u(x_2)$$

in the case of equally spaced points, $x_j = x_0 + jh$ for some $h = \Delta x$.

By hand, work out the Vandermonde system of equations to be solved for the coefficients and confirm that the coefficients produced by the `fdcoeffV` function gives coefficients that satisfy this linear system. (Use either the Python or Matlab version, but I suggest you try using Python and the Jupyter notebook.)

Solution: We are trying to find the coefficients $(c_{-2}, c_{-1}, c_0, c_1, c_2)$ in the approximation of the fourth derivative $u^{(4)}(x_0)$ using a 5-point stencil. Given,

$$u^{(4)}(x_0) \approx c_{-2}u(x_{-2}) + c_{-1}u(x_{-1}) + c_0u(x_0) + c_1u(x_1) + c_2u(x_2)$$

we will find the Vandermonde system of equations which will determine c_{-2}, c_{-1}, c_0, c_1 and c_2.

We start off by using the fact that $x_j = x_0 + jh$ and Taylor expanding $u(x_j) = u(x_0 + jh)$ around x_0.

So:

$u(x_{-2}) = u(x_0-2h) = u(x_0)-2h(u'(x_0))+\frac{(2h)^2}{2}u''(x_0)-\frac{(2h)^3}{6}u'''(x_0)+\frac{(2h)^4}{24}u^{(4)}(x_0)-\frac{(2h)^5}{120}u^{(5)}(x_0)+\dots$ (1)

$u(x_{-1}) = u(x_0 - h) = u(x_0) - h(u'(x_0)) + \frac{h^2}{2}u''(x_0) - \frac{h^3}{6}u'''(x_0) + \frac{h^4}{24}u^{(4)}(x_0) - \frac{h^5}{120}u^{(5)}(x_0) + \dots$ (2)

$u(x_0) = u(x_0)$ (3)

$u(x_1) = u(x_0 + h) = u(x_0) + h(u'(x_0)) + \frac{h^2}{2}u''(x_0) + \frac{h^3}{6}u'''(x_0) + \frac{h^4}{24}u^{(4)}(x_0) + \frac{h^5}{120}u^{(5)}(x_0) + \dots$ (4)

$u(x_2) = u(x_0+2h) = u(x_0)+2h(u'(x_0))+\frac{(2h)^2}{2}u''(x_0)+\frac{(2h)^3}{6}u'''(x_0)+\frac{(2h)^4}{24}u^{(4)}(x_0)+\frac{(2h)^5}{120}u^{(5)}(x_0)+\dots$ (5)

Next, we multiply (1) by c_{-2}, (2) by c_{-1}, (3) by c_0, (4) by c_1 and (5) by c_2 in order to find the coefficients of the linear combination we desire. After multiplication, we collect terms to get the following linear combination:

$(c_{-2}+c_{-1}+c_0+c_1+c_2)u(x_0)+(-2c_{-2}-c_{-1}+c_1+2c_2)hu'(x_0)+(2c_{-2}+\frac{1}{2}c_{-1}+\frac{1}{2}c_1+2c_2)h^2u''(x_0)+(-\frac{4}{3}c_{-2}-\frac{1}{6}c_{-1}+\frac{1}{6}c_1+\frac{4}{3}c_2)h^3u'''(x_0)+\frac{2}{3}c_{-2}+\frac{1}{24}c_{-1}+\frac{1}{24}c_1+\frac{2}{3}c_2)h^4u^{(4)}(x_0)$.

We only want the coefficient of $u^{(4)}$ to be non-zero and the others to be zero. We would ideally want more terms to be zeroed out but with 5 unknowns this is the best we can do. Therefore, we have the following equations:

$$c_{-2} + c_{-1} + c_0 + c_1 + c_2 = 0$$
$$-2c_{-2} - c_{-1} + c_1 + 2c_2 = 0$$
$$2c_{-2} + \tfrac{1}{2}c_{-1} + \tfrac{1}{2}c_1 + 2c_2 = 0$$
$$-\tfrac{4}{3}c_{-2} - \tfrac{1}{6}c_{-1} + \tfrac{1}{6}c_1 + \tfrac{4}{3}c_2 = 0$$
$$h^4(\tfrac{2}{3}c_{-2} + \tfrac{1}{24}c_{-1} + \tfrac{1}{24}c_1 + \tfrac{2}{3}c_2) = 1$$

This is the Vandermonde system of equations which leads us to:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & 2 \\ -\frac{4}{3} & -\frac{1}{6} & 0 & \frac{1}{6} & \frac{4}{3} \\ h^4\frac{2}{3} & h^4\frac{1}{24} & 0 & h^4\frac{1}{24} & h^4\frac{2}{3} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix}}_{\mathbf{c}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{f}}$$

We use `fdcoeffV` function to figure out the coefficients. For this function, we make choices of the boundary and x_0 which gives us our h. We choose the values in such a way so that $h = 1$. As we can see from the system above that the coefficients will have a factor of $\frac{1}{h^4}$ and the `fdcoeffV` gives this value. Because h can vary depending on the boundary (a and b), we want to find the part of the coefficients that is independent of h. Since, $h = 1$ we know that $h^4 = 1$ and we get the weights. The coefficients will just be the weights divided by h^4.

The `fdcoeffV` function gives the vector $[1, -4, 6, 4, 1]$ of weights for when $h = 1$.

So, $c = \begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix} = \frac{1}{h^4} \begin{bmatrix} 1 \\ -4 \\ 6 \\ -4 \\ 1 \end{bmatrix}$

We can easily verify that $A * c = f$ for this c. This is done in the accompanying Jupyter notebook.

Therefore,

$$u^{(4)}(x_0) \approx \frac{1}{h^4}(u(x_{-2}) - 4u(x_{-1}) + 6u(x_0) - 4u(x_1) + u(x_2))$$

Problem 2. (a) Determine the order of accuracy and leading term of the asymptotic error for the approximation derived above. In other words, if the function $u(x)$ is sufficiently smooth, show the approximation has an error that is of the form $Ch^p + \mathcal{O}(h^{p+1})$ and determine p and C. The value of C will depend on higher-order derivative(s) of u at x_0.

(b) Test your result by computing the error for $u(x) = \sin(2x)$ at the point $x_0 = 1$ and various choices of h and show this is consistent with what you derived. Produce a log-log plot of the absolute error vs. h and the expected error to show that it has the expected form (similar to the plots produced in the `fdstencil_errors.ipynb` notebook). How good an approximation can you get before rounding error takes over?

Solution:

From Problem 1. we have the following information:

$$u^{(4)}(x_0) \approx \frac{1}{h^4}(u(x_{-2}) - 4u(x_{-1}) + 6u(x_0) - 4u(x_1) + u(x_2))$$

$u(x_{-2}) = u(x_0) - 2h(u'(x_0)) + \frac{(2h)^2}{2}u''(x_0) - \frac{(2h)^3}{6}u'''(x_0) + \frac{(2h)^4}{24}u^{(4)}(x_0) - \frac{(2h)^5}{120}u^{(5)}(x_0) + \frac{64h^6}{720}u^{(6)}(x_0) + \ldots$ (1)

$u(x_{-1}) = u(x_0) - h(u'(x_0)) + \frac{h^2}{2}u''(x_0) - \frac{h^3}{6}u'''(x_0) + \frac{h^4}{24}u^{(4)}(x_0) - \frac{h^5}{120}u^{(5)}(x_0) + \frac{h^6}{720} + \ldots$ (2)

$u(x_0) = u(x_0)$ (3)

$u(x_1) = u(x_0) + h(u'(x_0)) + \frac{h^2}{2}u''(x_0) + \frac{h^3}{6}u'''(x_0) + \frac{h^4}{24}u^{(4)}(x_0) + \frac{h^5}{120}u^{(5)}(x_0) + \frac{h^6}{720}u^{(6)}(x_0) + \ldots$ (4)

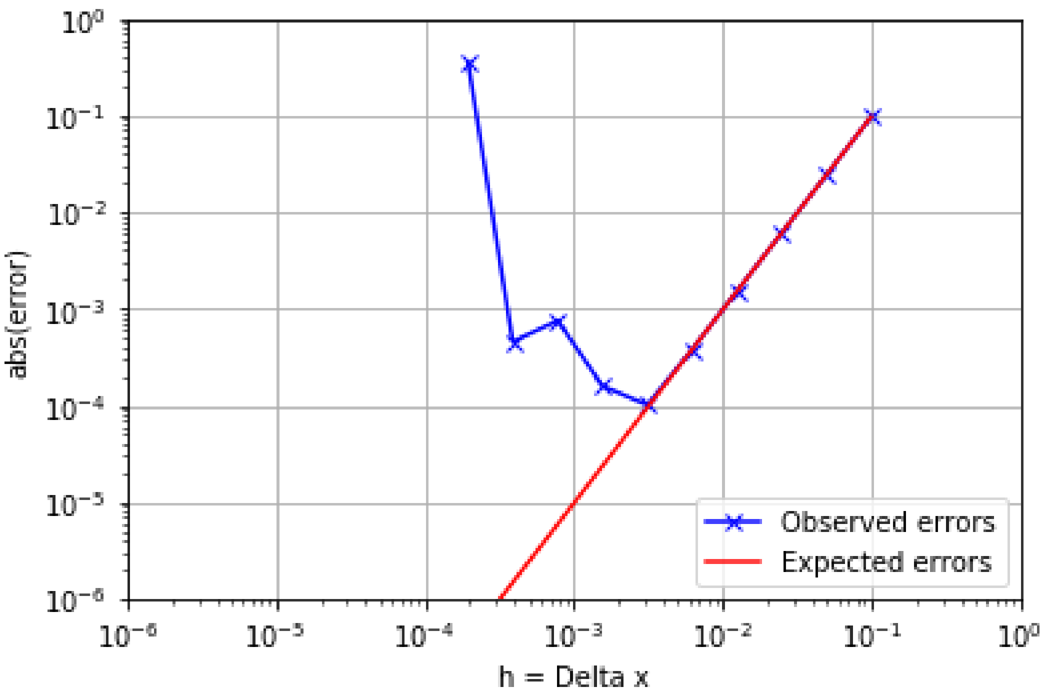
$u(x_2) = u(x_0)+2h(u'(x_0))+\frac{(2h)^2}{2}u''(x_0)+\frac{(2h)^3}{6}u'''(x_0)+\frac{(2h)^4}{24}u^{(4)}(x_0)+\frac{(2h)^5}{120}u^{(5)}(x_0)+\frac{64h^6}{720}u^{(6)}(x_0)+\dots$
(5)

So, using the approximation we get:

$\frac{1}{h^4}(u(x_{-2}) - 4u(x_{-1}) + 6u(x_0) - 4u(x_1) + u(x_2)) = u^{(4)}(x_0) + hu^{(5)}(x_0)(-\frac{32}{120} + \frac{4}{120} + -\frac{4}{120} + \frac{32}{120}) + h^2u^{(6)}(x_0)(\frac{64}{720} - \frac{4}{720} - \frac{4}{720} + \frac{64}{720}) + \mathcal{O}(h^4) = u^{(4)}(x_0) + \frac{h^2}{6}u^{(6)}(x_0) + \mathcal{O}(h^4)$.

We can note that $h^3u^{(7)}$ term will cancel out similar to the $hu^{(5)}$ term. So, this is second order accurate. So the approximation has an error of the form $Ch^p + \mathcal{O}(h^{p+1})$ with $C = \frac{1}{6}$ and $p = 2$. We also that the term added to this is actually $\mathcal{O}(h^{p+2})$.

b) We test the result by computing the error for $u(x) = sin(2x)$ at the point $x_0 = 1$ and various choices of h and this is consistent with what we derived. We produced a log-log plot of the absolute error vs h and the expected error has the expected form. The calculation is included in the Jupyter notebook attached with the homework. The log-log plot is given below:



We see that the best we can do without rounding error taking over is an error $\approx 10^{-4}$. This is when $h = 0.00312500$. After that, the rounding error takes over. Therefore, the best we can do is get the fourth derivative right to three decimal places.

Problem 3. Suppose we want to solve a 2-point boundary value problem of the form

$$u^{(4)}(x) = 3u''(x) + 4u(x) + f(x)$$

for $a \le x \le b$ with prescribed boundary conditions

$$u(a) = \alpha_0, \quad u'(a) = \alpha_1, \qquad u(b) = \beta_0, \quad u'(b) = \beta_1.$$

Set up the linear system of equations that would be solved to find a finite-difference solution to this problem. For this homework you do not need to program this or solve the system, just write out the system in a way that is clear what the banded matrix is, and what system needs to be solved. In particular, write out explicitly at least the first three and last three rows of the matrix and elements of the right-hand side to show how the function values $f(x_j)$ and boundary conditions come into these.

Use a uniform grid $x_j = a + jh$ with $h = (b-a)/(m+1)$, Use the approximation to $u^{(4)}$ from Problem 1 and the standard centered approximation for $u''(x)$. For the boundary conditions on u', use one-sided approximations that are second-order accurate.

Since $u(a)$ and $u(b)$ are both known from the boundary conditions, you can set this up as a system of m equations for the interior unknowns $U_1, \ \ldots, \ U_m$.

Solution:

From problem 1, we have the following approximation to $u^{(4)}(x_j) \approx \frac{1}{h^4}(u(x_{j-2}) - 4u(x_{j-1}) + 6u(x_j) - 4u(x_{j-1}) + u(x_{j-2}))$. Approximating the solution $u(x_j)$ by U_j we get $D^4 U_j = \frac{1}{h^4}(U_{j-2} - 4U_{j-1} + 6U_j - 4U_{j+1} + U_{j+2})$

We also know the following standard centered approximation for $u''(x_j)$: $u''(x_j) \approx \frac{1}{h^2}(u(x_{j-1}) - 2u(x_j) + u(x_{j+1}))$. If we approximate the solution $u(x_j)$ by U_j we get $D^2 U_j = \frac{1}{h^2}(U_{j-1} - 2U_j + U_{j+1})$.

We also rearrange the ODE to isolate $f(x)$ on the right hand side:

$$u^{(4)}(x) - 3u''(x) - 4u(x) = f(x)$$

This leads to the following set of algebraic equations:

$$\frac{1}{h^4}(U_{j-2} - 4U_{j-1} + 6U_j - 4U_{j+1} + U_{j+2}) - \frac{3}{h^2}(U_{j-1} - 2U_j + U_{j+1}) - 4U_j = f(x_j)$$

for $j = 2, 3, \ldots, (m-1)$

We can note that we know the boundary values $U_0 = \alpha_0$ and $U_{m+1} = \beta_0$. We also have Neumann boundary conditions on both boundaries. We will use one-sided approximations that are second-order accurate for u'. From section 1.2, we have that $u'(x_j) \approx \frac{1}{h}(\frac{3}{2}u(x_j) - 2u_(x_{j-1}) + \frac{1}{2}u(x_{j-2}))$. Similarly, $u'(x_j) \approx -\frac{1}{h}(\frac{3}{2}u(x_j) - 2u_(x_{j+1}) + \frac{1}{2}u(x_{j+2}))$.

Therefore, we have $u'(x_0) = -\frac{1}{h}(\frac{3}{2}u(x_0) - 2u_(x_1) + \frac{1}{2}u(x_2)) = \alpha_1$. So, $\frac{3}{2}u(x_0) - 2u_(x_1) + \frac{1}{2}u(x_2) = -\alpha_1 h$. Again, approximating $u(x_j)$ by U_j leads us to $\frac{3}{2}U_0 - 2U_1 + \frac{1}{2}U_2 = -\alpha_1 h$. Using the fact that $U_0 = \alpha_0$ we have that $-2U_1 + \frac{1}{2}U_2 = -\alpha_1 h - \frac{3}{2}\alpha_0$. This is going to correspond to the first row of our discretization matrix.

Similarly, we use $u'(x_{m+1}) = \beta_1$ and $u'(x_j) \approx \frac{1}{h}(\frac{3}{2}u(x_j) - 2u_(x_{j-1}) + \frac{1}{2}u(x_{j-2}))$ to get $\frac{3}{2}u(x_{m+1}) - 2u(x_m) + \frac{1}{2}u(x_{m-1}) = \beta_1 h$. Approximating $u(x_j)$ by U_j and using $U_{m+1} = \beta_0$ we arrive at $-2U_m + \frac{1}{2}U_{m-1} = \beta_1 h + \frac{3}{2}\beta_0$. This is going to correspond to the last row of our discretization matrix.

We now shift our focus back to $\frac{1}{h^4}(U_{j-2} - 4U_{j-1} + 6U_j - 4U_{j+1} + U_{j+2}) - \frac{3}{h^2}(U_{j-1} - 2U_j + U_{j+1}) - 4U_j = f(x_j)$ to figure out the other rows of the discretization.

$j = 2:$

$\frac{1}{h^4}(U_0 - 4U_1 + 6U_2 - 4U_3 + U_4) - \frac{3}{h^2}(U_1 - 2U_2 + U_3) - 4U_2 = f(x_2)$

$\frac{1}{h^4}(-4U_1 + 6U_2 - 4U_3 + U_4) - \frac{3}{h^2}(U_1 - 2U_2 + U_3) - 4U_2 = f(x_2) - \frac{\alpha_0}{h^4}$

$(-\frac{4}{h^4} - \frac{3}{h^2})U_1 + (\frac{6}{h^4} + \frac{6}{h^2} - 4)U_2 + (-\frac{4}{h^4} - \frac{3}{h^2})U_3 + \frac{1}{h^4}U_4 = f(x_2) - \frac{\alpha_0}{h^4}$

$j = 3:$

$\frac{1}{h^4}(U_1 - 4U_2 + 6U_3 - 4U_4 + U_5) - \frac{3}{h^2}(U_2 - 2U_3 + U_4) - 4U_3 = f(x_3)$

$\frac{1}{h^4}U_1 + (-\frac{4}{h^4} - \frac{3}{h^2})U_2 + (\frac{6}{h^4} + \frac{6}{h^2} - 4)U_3 + (-\frac{4}{h^4} - \frac{3}{h^2})U_4 + \frac{1}{h^4}U_5 = f(x_2)$

$j = 4:$

$\frac{1}{h^4}(U_2 - 4U_3 + 6U_4 - 4U_5 + U_6) - \frac{3}{h^2}(U_3 - 2U_4 + U_5) - 4U_4 = f(x_4)$

$\frac{1}{h^4}U_2 + (-\frac{4}{h^4} - \frac{3}{h^2})U_3 + (\frac{6}{h^4} + \frac{6}{h^2} - 4)U_4 + (-\frac{4}{h^4} - \frac{3}{h^2})U_5 + \frac{1}{h^4}U_6 = f(x_4)$

$\vdots$

$j = m-3$

$\frac{1}{h^4}(U_{m-5} - 4U_{m-4} + 6U_{m-3} - 4U_{m-2} + U_{m-1}) - \frac{3}{h^2}(U_{m-4} - 2U_{m-3} + U_{m-2}) - 4U_{m-3} = f(x_{m-3})$

$\frac{1}{h^4}U_{m-5} + (-\frac{4}{h^4} - \frac{3}{h^2})U_{m-4} + (\frac{6}{h^4} + \frac{6}{h^2} - 4)U_{m-3} + (-\frac{4}{h^4} - \frac{3}{h^2})U_{m-2} + \frac{1}{h^4}U_{m-1} = f(x_{m-3})$

$j = m-2$

$\frac{1}{h^4}(U_{m-4} - 4U_{m-3} + 6U_{m-2} - 4U_{m-1} + U_m) - \frac{3}{h^2}(U_{m-3} - 2U_{m-2} + U_{m-1}) - 4U_{m-2} = f(x_{m-2})$

$\frac{1}{h^4}U_{m-4} + (-\frac{4}{h^4} - \frac{3}{h^2})U_{m-3} + (\frac{6}{h^4} + \frac{6}{h^2} - 4)U_{m-2} + (-\frac{4}{h^4} - \frac{3}{h^2})U_{m-1} + \frac{1}{h^4}U_m = f(x_{m-2})$

$j = m-1$

$\frac{1}{h^4}(U_{m-3} - 4U_{m-2} + 6U_{m-1} - 4U_m + U_{m+1}) - \frac{3}{h^2}(U_{m-2} - 2U_{m-1} + U_m) - 4U_{m-1} = f(x_{m-1})$

$\frac{1}{h^4}U_{m-3} + (-\frac{4}{h^4} - \frac{3}{h^2})U_{m-2} + (\frac{6}{h^4} + \frac{6}{h^2} - 4)U_{m-1} + (-\frac{4}{h^4} - \frac{3}{h^2})U_m = f(x_{m-1}) - \frac{1}{h^4}U_{m+1} = f(x_{m-1}) - \frac{\beta_0}{h^4}$

We have a linear system of m equations for the m unknowns $(U_1, U_2, \ldots, U_{m-1}, U_m)$, which can be written in the form $AU = F$, where U is the vector of unknowns $U = [U_1, U_2, \ldots, U_{m-1}, U_m]^T$ and

$$\begin{bmatrix}
-2 & \frac{1}{2} & & & & & & \\
(-\frac{4}{h^4} - \frac{3}{h^2}) & (\frac{6}{h^4} + \frac{6}{h^2} - 4) & (-\frac{4}{h^4} - \frac{3}{h^2}) & (\frac{1}{h^4}) & & & & \\
(\frac{1}{h^4}) & (-\frac{4}{h^4} - \frac{3}{h^2}) & (\frac{6}{h^4} + \frac{6}{h^2} - 4) & (-\frac{4}{h^4} - \frac{3}{h^2}) & (\frac{1}{h^4}) & & & \\
\ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & (\frac{1}{h^4}) & (-\frac{4}{h^4} - \frac{3}{h^2}) & (\frac{6}{h^4} + \frac{6}{h^2} - 4) & (-\frac{4}{h^4} - \frac{3}{h^2}) & (\frac{1}{h^4}) & & \\
 & & (\frac{1}{h^4}) & (-\frac{4}{h^4} - \frac{3}{h^2}) & (\frac{6}{h^4} + \frac{6}{h^2} - 4) & (-\frac{4}{h^4} - \frac{3}{h^2}) & (\frac{1}{h^4}) & \\
 & & & (\frac{1}{h^4}) & (-\frac{4}{h^4} - \frac{3}{h^2}) & (\frac{6}{h^4} + \frac{6}{h^2} - 4) & (-\frac{4}{h^4} - \frac{3}{h^2}) & \\
 & & & & & \frac{1}{2} & -2 &
\end{bmatrix}$$

is the matrix A.

Finally,

$$
\mathrm{F} = \begin{bmatrix} -\alpha_1 h - \frac{3}{2}\alpha_0 \\ f(x_2) - \frac{\alpha_0}{h^4} \\ f(x_3) \\ \vdots \\ f(x_{m-2}) \\ f(x_{m-1}) - \frac{\beta_0}{h^4} \\ \beta_1 h + \frac{3}{2}\beta_0 \end{bmatrix}
$$