ECON 525: Homework 7

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1 Problem 1: Computing an MPE for an Entry/Exit Game

Consider a dynamic game of entry/exit with two players. Time is discrete; $t = 1, 2, \ldots$. At the beginning of each time period, players who stayed in the market in the previous period choose whether to stay in or exit, and players who stayed out of the market in the previous period decide whether to enter or stay out of the market. That is, their choice is binary, i.e., $a_{it} \in \{0, 1\}$ where 0 is for "out of market" and 1 is for "in the market". Let $a_t = (a_{1t}, a_{2t})$: When it enters, a player incurs the cost of entry φ which is assumed to be common across players and across periods. Furthermore, player i receives player- and alternative-specific shocks of $\epsilon_{it} = (\epsilon_{it}(0), \epsilon_{it}(1))$, which is private information and is assumed to be independent across players, alternatives, and periods. A market-level variable, M_t, also affects players' profit, and we assume M_t follows an first-order Markov process, characterized by $f(M_{t+1}|M_t)$. Specifically, we assume that M_t can take two different values, $M_t \in \{m_1, m_2\}$. The transition probability matrix for M is given by

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

where b_{ij} =Pr$(M_{t+1} = m_j | M_t = m_i)$. Let $s_t = (M_t, a_{1t-1}, a_{2t-1})$ and $\epsilon_t = (\epsilon_{1t}, \epsilon_{2t})$ be the vector of common knowledge and private information variables, respectively. When making a decision at time t, player i observes s_t and ϵ_{it}. A player's current profits depend on the realization of M_t, on its own private information ϵ_{it}, and on the vector of players' current decisions $a_{t-1} = (a_{1t-1}, a_{2t-1})$.

Let $\Pi_{it}(a_t, s_t, \epsilon_{it})$ be player i's payoff at t if a set of decisions is given by a_t, the state variable is s_t, and the privately-observed payoff shocks are given by ϵ_{it}. We assume the payoff is additive

$$\Pi_{it}(a_t, s_t, \epsilon_{it}) = \Pi_{it}(a_t, s_t) + \epsilon_{it}(a_{it})$$

and is written as

$$\Pi_{it}(a_t, s_t, \epsilon_{it}) = \begin{cases} \lambda M_t - \delta a_{-i,t} - (1 - a_{i,t-1})\varphi + \epsilon_{it}(1) & \text{if } a_{it} = 1 \\ \epsilon_{it}(0) & \text{if } a_{it} = 0 \end{cases}$$

The timing of the game is as follows. At the beginning of period t, ϵ_t and the market-level variable M_t realize. After observing their private shocks and M_t, players simultaneously make an entry/exit decision. The current period payoff (it depends on entry decisions that have just been made and on the realization of M_t) realizes.

Assume $\lambda = 2, \delta = 2, m_1 = 1, m_2 = 1.5, b_{11} = 0.5, b_{21} = 0.4$ and $\varphi = 1.5$. Set $\beta = 0.95$. Assume that ϵ follows the iid extreme value distribution. Now you are going to compute a symmetric MPE.

1. How many distinct choice probabilities are there?

Solution: We have eight possible states of the form $s_t = (Mt, a_{1t-1}, a_{2t-1})$. They are provided below:

$$
\begin{gathered}
(m_1, 0, 0) \\
(m_1, 0, 1) \\
(m_1, 1, 0) \\
(m_1, 1, 1) \\
(m_2, 0, 0) \\
(m_2, 0, 1) \\
(m_2, 1, 0) \\
(m_2, 1, 1)
\end{gathered}
$$

There are two choices for the firm $0, 1$ and so we should have sixteen choice probabilities. However, if we find the choice probabilities for the choice 0, for example, this completely determines the choice probabilities for choice 1. So, we can say that there are eight distinct choice probabilities, one for each state, since there are only two actions in each state.

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2. Write down the Bellman equation. Then, using the ex-ante (integrated) value functions, write down the problem as a system of linear equations. That is, express the ex-ante value functions as a solution to linear systems.

Solution:

Let $\sigma = \{\sigma_i(s, \epsilon_i)\}$ be a set of strategy functions. A set of conditional choice probabilities $P^\sigma = \left\{P_i^\sigma(a_i|s)\right\}$ associated with σ defined as

$$
P_i^\sigma(a_i|s) = \Pr(\sigma_i(s, \epsilon_i) = a_i|s) = \int \mathbb{1}\{\sigma_i(s, \epsilon_i) = a_i\} g_i(\epsilon_i)\, d\epsilon_i
$$

Let $\Pi_i^\sigma(a_i, s)$ be firm i's expected profit if it chooses a_i and other firm's strategies are given by σ:

$$
\pi^\sigma(a_i, s) = \sum_{a_{-i}} P_{-i}^\sigma(a_{-i}|s) \Pi_i(a_i, a_{-i}, s)
$$

The Bellman equation is the following:

$$
V_i^\sigma(s, \epsilon_i) = \max_{a_i \in \{0,1\}} \left\{ \pi^\sigma(a_i, s) + \epsilon_i(a_i) + \beta \sum_{s' \in S} \left[\int V_i^\sigma(x', \epsilon') g_i(\epsilon_i')\, d\epsilon_i' \right] f_i^\sigma(s'|s, a_i) \right\}
$$

where f_i^σ is the transition probability of s defined as

$$f_i^{\sigma}(s'|s,a_i) = \sum_{a_{-i}\in\{0,1\}} \left(P_{-i}^{\sigma}(a_{-i}|s)\right) f\left(s'|s,a_i,a_{-i}\right).$$

Define:

$$v_i^{\sigma}(a_i,s) = \pi_i^{\sigma}(a_i,s) + \beta \sum_{s'\in S} V_i^{\sigma}(s') f_i^{\sigma}(s'|s,a_i)$$

as the choice-specific value functions. Then, we can write:

$$V_i^{\sigma}(s) = \int \max_{a_i\in\{0,1\}} \left\{v_i^{\sigma}(a_i,s) + \epsilon_i(a_i)\right\} g_i\left(d\epsilon_i\right) \qquad (1)$$

Now, we will write the problem as a system of linear equations. Let P^* be an equilibrium. Then (1) can be written as:

$$V_i^{P^*}(s) = \sum_{a_i\in\{0,1\}} P_i^*(a_i|s)\left[\pi_i^{P^*}(a_i,s) + e_i^{P^*}(a_i,s)\right] + \beta \sum_{s'\in S} V_i^{P^*}(s') f^{P^*}(s'|s)$$

where $f^{P^*}(s'|s)$ is the transition probability of s induced by P^* and

$$e_i^{P^*}(a_i,s) = \mathbb{E}\left(\epsilon_i(a_i)|s, \sigma_i^*(s,\epsilon_i) = a_i\right)$$

This can be written in vector form:

$$\left(I - \beta F^{P^*}\right) V_i^{P^*} = \sum_{a_i\in\{0,1\}} P_i^*(a_i) * \left[\pi_i^{P^*}(a_i) + e_i^{P^*}(a_i)\right]$$

where F^{P^*} is a matrix with transition probabilities $f^{P^*}(s'|s)$, $*$ is the element-by-element product. Let $V_i^{P^*}(s) = \Gamma_i(s;P^*)$ and $\Gamma_i(P^*) = \{\Gamma_i(s;P^*) : s \in S\}$. For any P, we have

$$\Gamma_i(P) = \left(I - \beta F^{P}\right)^{-1} \left\{\sum_{a_i\in\{0,1\}} P_i(a_i) * \left[\pi_i^{P}(a_i) + e_i^{P}(a_i)\right]\right\}$$

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3. Compute a symmetric MPE in the following way. First construct a transition function for any given choice probabilities. Second, for any initial guess of conditional choice probabilities, calculate the ex-ante value functions using the transition function and the system of linear equations that you obtained above. Third, update choice probabilities (i.e, calculate Ψ in our notation we discussed in class).Finally, go back to step 2 and keep iterating. Comment on your results.

Solution: We present the results for one agent only because we consider a symmetric MPE. In particular, we present the results for agent 1. The agent two will have the same values but for a_1 and a_2 flipped, that is, the values will be in a different order. The following table contains the conditional choice probabilities of entering and the value functions at that state:

$s = (a_1, a_2, m)$	P(entry|s)	$V(s)$
(0,0,1)	0.6954	34.3775
(1,0,1)	0.6954	35.0772
(0,1,1)	0.6954	34.3775
(1,1,1)	0.6954	35.0772
(0,0,1.5)	0.7022	34.8488
(1,0,1.5)	0.7022	35.8870
(0,1,1.5	0.7022	34.8488
(1,1,1.5)	0.7022	35.8870

The choice probabilities constituting the symmetric MPE only depends on the value of M_t which is the market-level variable. We are not sure if it makes a lot of sense that the previous choices of agent and its competitors are irrelevant. However, the way choice probability moves with M_t is intuitive. When M_t is higher, the choice probability is higher because it affects the firm's payoffs positively.
