# ECON 525: Homework 2

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## 1 Problem 1:

Consider an infinite-horizon Ramsey model given by

$$\max_{\{K_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t)$$

subject to

$$F(K_t) \geq C_t + I_t$$
$$K_0 = K$$

where

$$I_t = K_{t+1} - (1-\delta)K_t$$

and non-negativity constraints on $C_t$ and $K_{t+1}$.

(a) Assume $u(C_t) = \ln(C_t)$ and $F(K_t) = AK_t^{\alpha}$ with $\beta = 0.6$, $A = 20, \alpha = 0.3$, and $\delta = 0.5$. Write the Bellman function and compute the value function using value function iterations. Plot $V(K)$ over the range $K \in [0, 12]$.

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**Solution:**

We will plug $u(C_t) = \ln(C_t)$ and $F(K_t) = AK_t^{\alpha}$ into the problem. We will substitute for the parameter values at the end. The problem then becomes:

$$\max_{\{K_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(C_t)$$

subject to

$$\text{(i)}\ \ C_t \leq AK_t^\alpha - I_t$$
$$\text{(ii)}\ \ K_0 = K$$

Since $\ln(C_t)$ is monotonically increasing, we know (i) binds. Also, we can notice that $MU(0) = \infty$ and so $C_t > 0$ which means that $K_t > 0$. Thus:

$$\begin{aligned}
C_t &= AK_t^\alpha - I_t \\
C_t &= AK_t^\alpha - K_{t+1} + (1-\delta)K_t
\end{aligned}$$

Plugging this in:

$$\max_{\{K_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t) = \max_{\{K_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(C_t) = \max_{\{K_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(AK_t^\alpha - K_{t+1} + (1-\delta)K_t)$$

We can rewrite the above as the following:

$$\begin{aligned}
V(K_0) &= \max_{\{K_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(AK_t^\alpha - K_{t+1} + (1-\delta)K_t) \\
&= \max_{K} \ln(AK^\alpha - K_1 + (1-\delta)K) + \max_{\{K_t\}_{t=2}^{\infty}} \sum_{t=1}^{\infty} \beta^t \ln(AK_t^\alpha - K_{t+1} + (1-\delta)K_t)
\end{aligned}$$

Let $t = s+1$. Then, we can rewrite the second term in the above expression as the following:

$$\begin{aligned}
\max_{\{K_t\}_{t=2}^{\infty}} \sum_{t=1}^{\infty} \beta^t \ln(AK_t^\alpha - K_{t+1} + (1-\delta)K_t) &= \max_{\{K_{s+1}\}_{s=1}^{\infty}} \sum_{s=0}^{\infty} \beta^{s+1} \ln(AK_{s+1}^\alpha - K_{s+2} + (1-\delta)K_{s+1}) \\
&= \beta \max_{\{K_{s+1}\}_{s=1}^{\infty}} \sum_{s=0}^{\infty} \beta^{s} \ln(AK_{s+1}^\alpha - K_{s+2} + (1-\delta)K_{s+1}) \\
&= \beta V(K_1)
\end{aligned}$$

Therefore:

$$V(K) = \max_{K} \ln(AK^\alpha - K_1 + (1-\delta)K) + \beta V(K_1)$$

Finally, we can write the Bellman equation more generally:

$$\boxed{V(K_t) = \max_{K_t} \ln(AK_t^\alpha - K_{t+1} + (1-\delta)K_t) + \beta V(K_{t+1})}$$

The Bellman equation after plugging in the parameter values is:

$$\boxed{V(K_t) = \max_{K_t} \ln(20K_t^{0.3} - K_{t+1} + (0.5)K_t) + 0.6V(K_{t+1})}$$

This is using the repetitive nature of the problem as the problem faced at time 0 is the same problem faced at time $t$. The plot of $V(K)$ over the range $K \in [0, 12]$ is given below:

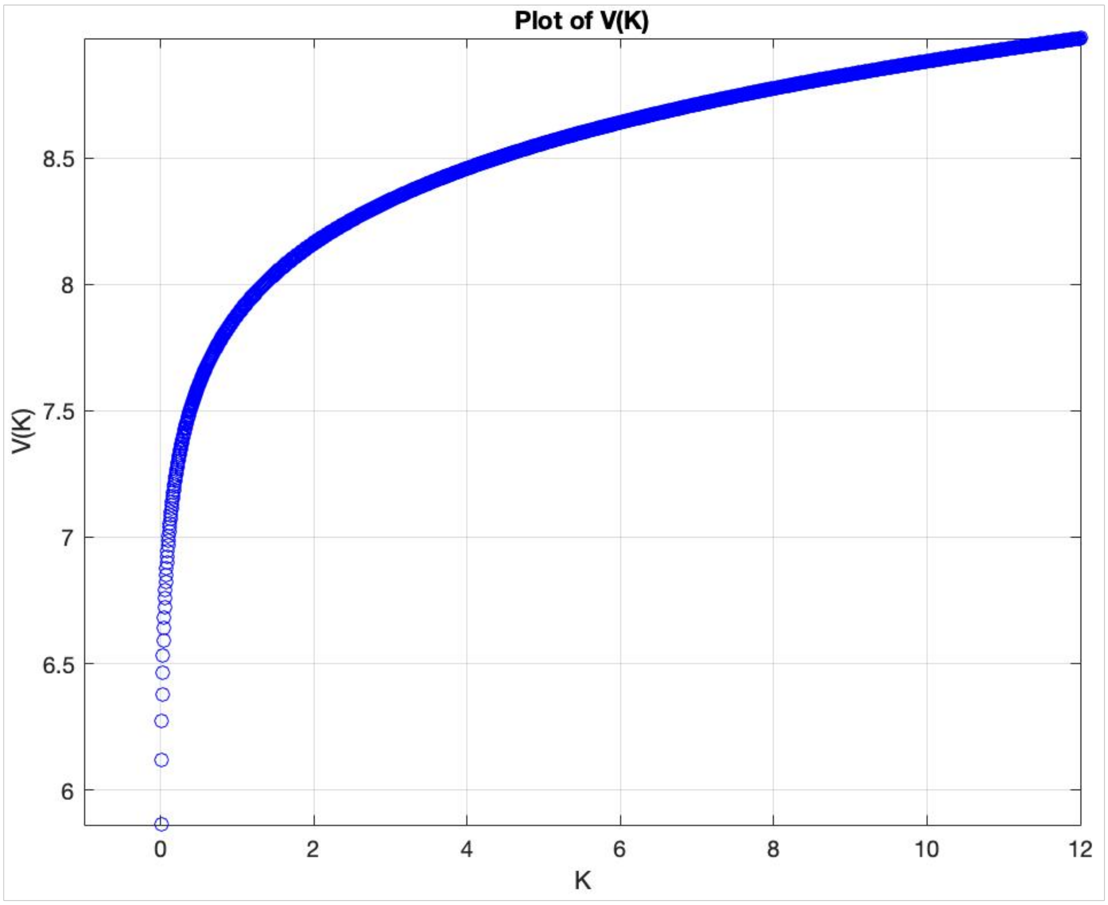


Figure 1: Plot of $V(K)$ over the range $K \in [0, 12]$

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(b) Assume $u(C_t) = \ln(C_t)$ and $F(K_t) = K_t^\alpha$ with $\delta = 1$.

i. Use value function iterations to approximate the value function for your choice of $(\alpha, \beta)$.

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**Solution:**

For this problem we let $\alpha = 0.3$ and $\beta = 0.6$ as Problem 1 (a). The approximation of $V(K)$ is plotted below over the range $K \in [0, 12]$.

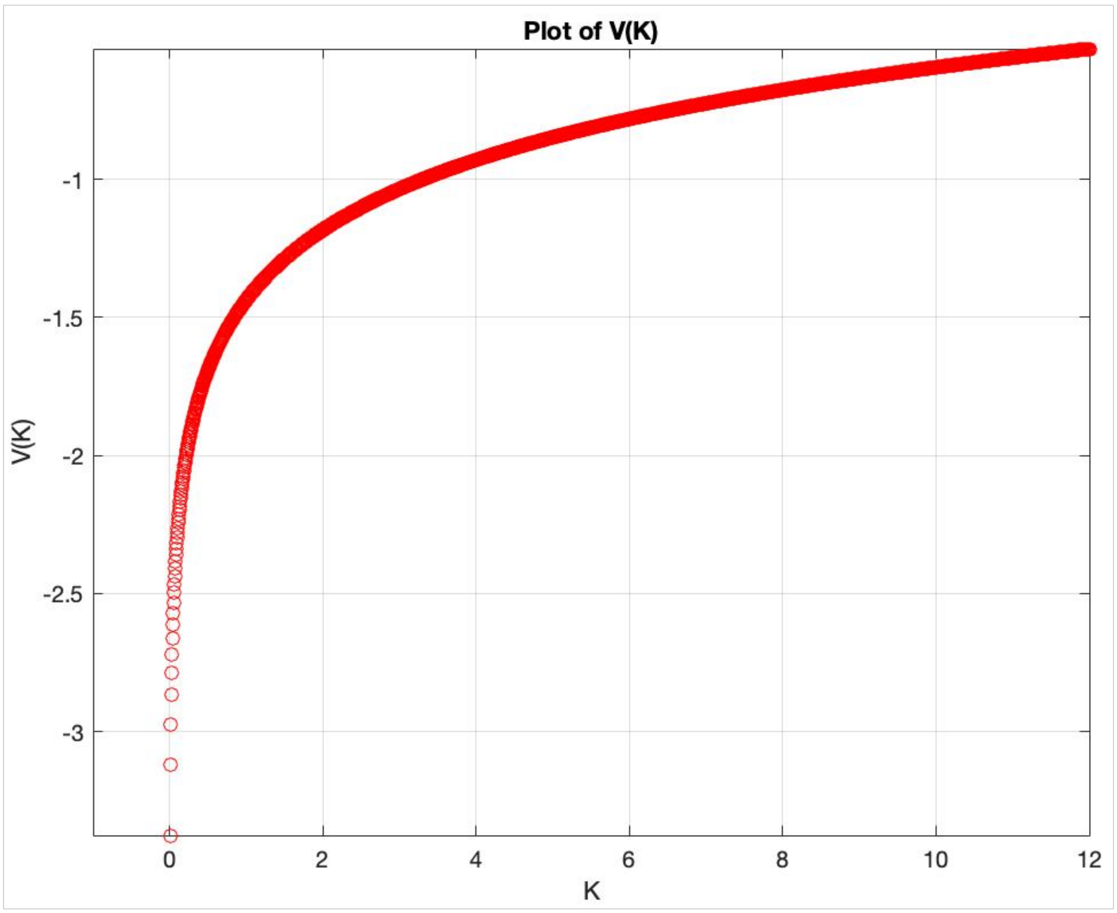


Figure 1: Plot of $V(K)$ over the range $K \in [0, 12]$ for $\alpha = 0.3$ and $\beta = 0.6$

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ii. Use guess and verify to obtain the value function analytically.

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**Solution:**

Using $u(C_t) = \ln(C_t)$, $F(K_t) = K_t^\alpha$ and $\delta = 1$, we can rewrite the Bellman equation as the following:

$$
\begin{aligned}
V(K_t) &= \max_{K_t} \ln(K_t^\alpha - K_{t+1}) + \beta V(K_{t+1}) \\
&= \max_{K_t} F(K_t, K_{t+1}) + \beta V(K_{t+1})
\end{aligned}
$$

We will guess that the solution takes the form $V(K) = A\ln(K) + B$. We will solve for the coefficients $A$ and $B$, thereby showing that $V(K)$ solves the Bellman equation. We will start by using the first order conditions of the Bellman equation:

$$\begin{aligned}
\frac{\partial F(K_t, K_{t+1})}{K_{t+1}} &= -\frac{1}{K_t^\alpha - K_{t+1}} + \beta V'(K_{t+1}) = 0 \\
\implies \beta V'(K_{t+1}) &= \frac{1}{K_t^\alpha - K_{t+1}}
\end{aligned}$$

Using our guess of $V(K) = A\ln(K) + B \implies V'(K) = \frac{A}{K}$:

$$\begin{aligned}
\beta \frac{A}{K_{t+1}} &= \frac{1}{K_t^\alpha - K_{t+1}} \\
\implies \beta A \times \frac{1}{\frac{K_{t+1}}{K_t^\alpha}} &= \frac{1}{1 - \frac{K_{t+1}}{K_t^\alpha}} \qquad (1)
\end{aligned}$$

We can also use the Envelope Theorem to get the following:

$$V'(K_t) = \frac{\partial F(K_t, K_{t+1})}{K_t} = \frac{\alpha K_t^{\alpha-1}}{K_t^\alpha - K_{t+1}}$$

Using our guess of $V(K) = A\ln(K) + B$ we have that $V'(K) = \frac{A}{K}$. Thus:

$$\begin{aligned}
\frac{A}{K_t} &= \frac{\alpha K_t^{\alpha-1}}{K_t^\alpha - K_{t+1}} \\
\implies \alpha K_t^\alpha &= A\left(K_t^\alpha - K_{t+1}\right) \\
\implies AK_{t+1} &= (A - \alpha) K_t^\alpha \\
\implies \frac{K_{t+1}}{K_t^\alpha} &= \frac{A - \alpha}{A}
\end{aligned}$$

Plugging this into (1):

$$\begin{aligned}
\beta A \times \frac{1}{\frac{A-\alpha}{A}} &= \frac{1}{1 - \frac{A-\alpha}{A}} \\
\implies \beta A \frac{A}{A - \alpha} &= \frac{A}{\alpha} \\
\implies \beta A \alpha &= A - \alpha \\
\implies A - \beta A \alpha &= \alpha \\
\implies & \boxed{A = \frac{\alpha}{1 - \alpha\beta}}
\end{aligned}$$

Plugging this value of $A$ into the FOC:

$$\begin{aligned}
&\beta \frac{A}{K_{t+1}} = \frac{1}{K_t^\alpha - K_{t+1}} \\
\implies &\frac{\alpha\beta}{(1-\alpha\beta)K_{t+1}} = \frac{1}{K_t^\alpha - K_{t+1}} \\
\implies &(1-\alpha\beta)K_{t+1} = \alpha\beta(K_t^\alpha - K_{t+1}) \\
\implies &K_{t+1} = \alpha\beta K_t^\alpha
\end{aligned}$$

Now, let us recall the Bellman equation but evaluated at the optimal $K_t$:

$$\begin{aligned}
&V(K_t) = \ln(K_t^\alpha - K_{t+1}) + \beta V(K_{t+1}) \\
\implies &V(K_t) - \beta V(K_{t+1}) = \ln(K_t^\alpha - K_{t+1})
\end{aligned}$$

Plugging in our guess of $V(K) = A\ln(K) + B$ and expressions for $K_{t+1}$ and $A$:

$$\begin{aligned}
&A\ln(K_t) + B - \beta\left(A\ln(K_{t+1}) + B\right) = \ln(K_t^\alpha - K_{t+1}) \\
\implies &\frac{\alpha}{1-\alpha\beta}\ln(K_t) + B - \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta K_t^\alpha) - \beta B = \ln(K_t^\alpha - \alpha\beta K_t^\alpha) \\
\implies &B\,(1-\beta) = \ln(K_t^\alpha - \alpha\beta K_t^\alpha) - \frac{\alpha}{1-\alpha\beta}\ln(K_t) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta K_t^\alpha) \\
\implies &B\,(1-\beta) = \ln((1-\alpha\beta)K_t^\alpha) - \frac{1}{1-\alpha\beta}\ln(K_t^\alpha) + \frac{\alpha\beta}{1-\alpha\beta}\left(\ln(\alpha\beta) + \ln(K_t^\alpha)\right) \\
\implies &B\,(1-\beta) = \ln(1-\alpha\beta) + \ln(K_t^\alpha) - \frac{1}{1-\alpha\beta}\ln(K_t^\alpha) + \frac{\alpha\beta}{1-\alpha\beta}\left(\ln(\alpha\beta) + \ln(K_t^\alpha)\right) \\
\implies &B\,(1-\beta) = \frac{(1-\alpha\beta)\ln(K_t^\alpha) - \ln(K_t^\alpha) + \alpha\beta\ln(K_t^\alpha)}{1-\alpha\beta} + \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) \\
\implies &B(1-\beta) = \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) \\
\implies &\boxed{B = \frac{1}{1-\beta}\left(\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)\right)}
\end{aligned}$$

Therefore, the solution is:

$$\boxed{V(K_t) = \frac{\alpha}{1-\alpha\beta}\ln(K_t) + \frac{1}{1-\beta}\left(\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)\right)}$$

For our choice of $(\alpha, \beta) = (0.3, 0.6)$:

$$\boxed{V(K_t) = 0.3659\ln(K_t) - 1.4372}$$

iii. Compare (plot) the value function obtained in (i) and (ii).

**Solution:**

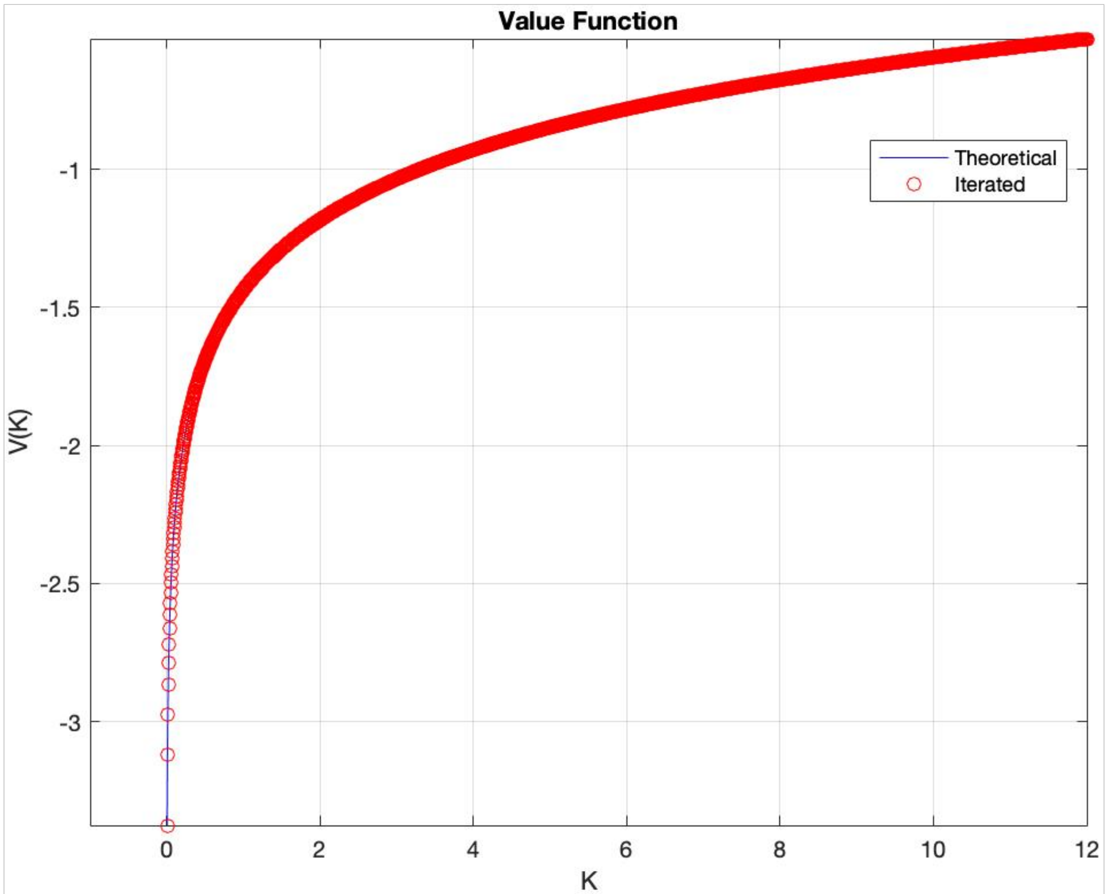


Figure 1: Plot of analytical and iterated $V(K)$ over the range $K \in [0, 12]$ for $\alpha = 0.3$ and $\beta = 0.6$

The iterated value function matches the analytical value function almost exactly. The iterated value is off for smaller values of $K$ at the $4^{\text{th}}$ or $5^{\text{th}}$ significant digit. Therefore, the value function iteration does exceedingly well in approximating the analytical value function.

## 2 Problem 2:

Consider a Rust-type model. The data generating process is as follows:

- $x_t$ can take on 11 values, i.e. $x_t \in \{0, 1, 2, \ldots, 10\}$
- At any time period,

$$x_{t+1} = \begin{cases} \min\{x_{t+1}, 10\}, & \text{with probability } \lambda \\ x_t & \text{with probability } 1 - \lambda \end{cases}$$

  Note that $\lambda$ does not depend on the replacement decision $i_t$.

- The agent's replacement decision is made at the beginning of each period and is effective immediately (i.e. $i_t = 1$, the agent uses machine with mileage 0 and the next period state $x_{t+1}$ is 1 with probability $\lambda$ and 0 with probability $1 - \lambda$).
- The per-period maintenance cost for a bus with mileage $x$ is $C(x, \theta) = \theta_1 x + \theta_2 x^2$
- The cost of replacement is $\theta_3$
- That is, the per-period utility function is given by

$$u(x_t, i_t, \epsilon_{0t}, \epsilon_{1t}; \boldsymbol{\theta}) = \begin{cases} -\theta_1 x_t - \theta_2 x_t^2 + \epsilon_{0t}, & \text{if } i_t = 0, \\ -\theta_3 + \epsilon_{1t}, & \text{if } i_t = 1 \end{cases}$$

  where $(\epsilon_{0t}, \epsilon_{1t})$ follows the iid Type 1 Extreme Value distribution.

- The decision maker discounts the future by $\beta = 0.95$.

(a) Write down the ML estimator for $\lambda$ and solve for it (note that this does not require any nonlinear optimization).

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**Solution:** We will start by noting that when $x_t = 10$, $x_{t+1} = 0$ with probability 1. Otherwise, the $x_t$ switching values at a given time period can be thought of as a Bernoulli random variable. So if we let $X_t = 1$ if $x_{t-1}$ switches value next period and 0 otherwise, we get $X_i \sim \text{Bernoulli}(\lambda)$. The likelihood is then given by:

$$L(\lambda) = \prod_{i=1}^{T} \lambda^{x_i} (1 - \lambda)^{1 - x_i}$$

Taking the log:

$$\log L(\lambda) = l(\lambda) = \log(\lambda) \sum_{i=1}^{T} x_i + \log(1 - \lambda) \sum_{i=1}^{T} (1 - x_i)$$

Since $X_i = 1$ when $x_{t-1}$ switches value, we can see that if we have $k$ switches in the data then $\sum x_i = k$. Therefore,

$$L(\lambda) = \log(\lambda)k + \log(1-\lambda)(T-k)$$

The first order condition is:

$$\begin{aligned} &\frac{k}{\lambda} - \frac{T-k}{1-\lambda} = 0 \\ \implies &\lambda T = k \\ \implies &\widehat{\lambda} = \frac{k}{T} \end{aligned}$$

So, the MLE estimate of $\lambda$ is simply going to be the number of times state variable switched over total number of data points. However, since $x_t = 10$ means that it never switches unless $i_t$ switches we will not consider some of these data points.Essentially, we do not want to deal with a string of $x_t = 10$ in the data because that is deterministic. We want to keep the data points with $x_t = 10$ such that $i_t = 1$ so we can capture the effect of $\lambda$ in moving to $x_{t+1} = 0$ or 1. Let $T_{10}$ be the number of data points such that $(x_t = 10, i_t = 0)$. Therefore, we can write $\widehat{\lambda}$ in our case as the following:

$$\boxed{\widehat{\lambda} = \frac{1}{T - T_{10}} \sum_{i=1}^{T-T_{10}} i_{t-1}x_t + (1 - i_{t-1})(x_t - x_{t-1})}$$

This is because in the case of $i_{t-1} = 1$ switching means $x_t = 1$ and in the case of $i_t = 0$ switching means that we move ahead by 1.

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(b) Set $\theta_1 = 0.3$, $\theta_2 = 0.0$, $\theta_3 = 4.0$ and solve the DP problem using $\lambda = 0.8$. Report the probability of replacement for every state. To do so, try both (1) the choice-specific value function formulation and (2) the integrated value function formulation. Compare the results. Note: assume the Euler constant is 0.5772.

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**Solution:** We will assume conditional independence for this problem to simplify the problem. We use the following formula to calculate the probability of replacement for each state:

$$\Pr(i_t = 1|x_t, \theta) = \frac{\exp\left(\bar{V}^1(x_t, \theta)\right)}{\exp\left(\bar{V}^1(x_t, \theta)\right) + \exp\left(\bar{V}^0(x_t, \theta)\right)}$$

Using (1) the choice-specific value function formulation, the probability of replacement along with the choice-specific value functions are:

| $x_t$ | $V_0(x_t)$ | $V_1(x_t)$ | $\Pr(i_t = 1\|x_t, \theta)$ |
|---|---|---|---|
| 0 | -5.6359 | -9.6359 | 0.0180 |
| 1 | -6.7892 | -9.6359 | 0.0548 |
| 2 | -7.7132 | -9.6359 | 0.1276 |
| 3 | -8.4543 | -9.6359 | 0.2348 |
| 4 | -9.0613 | -9.6359 | 0.3602 |
| 5 | -9.5745 | -9.6359 | 0.4847 |
| 6 | -10.0237 | -9.6359 | 0.5958 |
| 7 | -10.4291 | -9.6359 | 0.6885 |
| 8 | -10.8039 | -9.6359 | 0.7628 |
| 9 | -11.1549 | -9.6359 | 0.8204 |
| 10 | -11.4642 | -9.6359 | 0.8616 |

Finally, using (2) the integrated value function formulation, the probability of replacement along with the value functions are:

| $x_t$ | $V_0(x_t)$ | $V_1(x_t)$ | $\Pr(i_t = 1\|x_t, \theta)$ |
|---|---|---|---|
| 0 | -5.6361 | -9.6361 | 0.0180 |
| 1 | -6.7894 | -9.6361 | 0.0548 |
| 2 | -7.7133 | -9.6361 | 0.1276 |
| 3 | -8.4545 | -9.6361 | 0.2348 |
| 4 | -9.0615 | -9.6361 | 0.3602 |
| 5 | -9.5747 | -9.6361 | 0.4847 |
| 6 | -10.0239 | -9.6361 | 0.5958 |
| 7 | -10.4293 | -9.6361 | 0.6885 |
| 8 | -10.8041 | -9.6361 | 0.7628 |
| 9 | -11.1551 | -9.6361 | 0.8204 |
| 10 | -11.4644 | -9.6361 | 0.8616 |

The probability of replacement for every state is exactly the same from (1) and (2). However, the number of iterations required to reach desired tolerance was much lower using the integrated value function formulation than it was for the choice-specific value function formulation.

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(c) Simulate the model over $T = 5000$ time periods, using random draws (download "draw.out") from the course website) and the policy function obtained in part(b). Set $x_1 = 0$.

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**Solution:**

The solution is provided in the code. Using the simulated data and part (a), we find that $\widehat{\lambda} = 0.7960$.

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(d) Estimate parameters of the model using MLE with a nested fixed point algorithm:

i. Guess initial parameter values (you can use $\theta_1 = 0.3, \theta_2 = 0.0, \theta_3 = 4.0$ as your initial guess);

ii. Solve DP problem ;

iii. Calculate probability of replacement at each state

iv. Use model predictions (the probability calculated in the previous step) and data to derive a likelihood;

v. Search over parameter values by repeating steps [i]-[iv].

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**Solution:** Using a grid search over the values for $\theta_1, \theta_2$ and $\theta_3$ we find that the following parameter values maximize the likelihood:

$$\boxed{\widehat{\theta}_1 = 0.3,\ \widehat{\theta}_2 = 0,\ \widehat{\theta}_3 = 4.}$$

We searched over $[0,1]$ for $\theta_1$, $[0,0.1]$ for $\theta_2$ and $[0,5]$ for $\theta_3$.

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(e) Using the estimated parameters, conduct the following policy experiments:

i. Calculate the expected long-run replacement probability for bus engines by forward simulation;

**Solution:**

The expected long-run replacement probability for bus engines by forward simulation is $\boxed{0.2042}$.

ii. Calculate the expected long-run replacement probability for bus engines by simulating the steady-state distribution and compare the result with the previous one;

**Solution:**

The expected long-run replacement probability for bus engines by simulating the steady-state distribution is $\boxed{0.2082}$.

iii. Suppose that the government reduces replacement cost $\theta_3$ by 10 percent with an investment subsidy. Predict the effect of this subsidy on the long-run replacement probability.

**Solution:**

The expected long-run replacement probability due to this subsidy is $\boxed{0.2382}$. The long-run replacement probability is predicted to increase and this makes sense as the cost of replacement is now reduced by the subsidy.