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HW3 580

1. Let $k > 1$ be a fixed constant. Suppose that $X \sim \text{Ber}(1/k)$, so X is a Bernoulli random variable with parameter $1/k$. Let $Y = kX$.

   (a) Find $E(Y)$.

   (b) Find $P(Y \geq k)$; compare this to Markov's inequality.

a) Let $k > 1$ be a fixed constant. Suppose $X \sim \text{Ber}\left(\frac{1}{k}\right)$. Let $Y = kX$.

$E(Y) = E(kX) = k\,E(X)$ by Theorem 1 on expectations

$$= k\left(\sum_{x_i : f_X(x_i) > 0} x_i f_X(x_i)\right)$$

Recall: For Bernoulli with parameter $p = \frac{1}{k}$:

$$f_X(x) = \begin{cases} p^x (1-p)^{(1-x)} & \text{for } x \in \{0,1\} \\ 0 & \text{Otherwise} \end{cases}$$

$$= \begin{cases} \left(\frac{1}{k}\right)^x \left(1-\frac{1}{k}\right)^{(1-x)} & \text{for } x \in \{0,1\} \\ 0 & \text{Otherwise} \end{cases}$$

So, $E(Y) = k\left(0 \times \left(1-\frac{1}{k}\right) + \left(\frac{1}{k} \times 1\right)\right)$

$= 1$

b) We want to find $P(Y \geq k)$.

$$
\begin{aligned}
P(Y \geq k) &= 1 - P(Y < k) \\
&= 1 - P(kX < k) \\
&= 1 - P(X < 1) \\
&= 1 - P(X = 0) \\
&= 1 - f(0) \\
&= 1 - \left(1 - \frac{1}{k}\right) \\
&= 1 - \left(\frac{k-1}{k}\right) \\
&= \frac{k - k + 1}{k} = \frac{1}{k}.
\end{aligned}
$$

Markov's inequality:

If $P(Y<0) = 0$, then for $k > 0$: $P(Y \geq k) \leq \frac{E(Y)}{k}$

In this case, $P(Y<0) = P(kX<0) = P(X<0) = 0$ because support for X is the set $\{0, 1\}$.

Then, Markov's inequality gives us that $P(Y \leq k) \leq \frac{E(Y)}{k} = \frac{1}{k}$.

Our result is consistent with Markov's inequality and in this case equality is achieved.

2. Suppose that X has the exponential distribution with parameter $\lambda > 0$.

(a) Find $E(X)$.

(b) Find the mode of the distribution of X.

Hint: look at a plot of the pdf.

(c) Find the CDF $F_X(x)$, and hence the median for X.

(d) Show that

$$E(X) = \int_0^\infty (1 - F_X(x))dx.$$

a) From Lecture notes,

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then, $$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_0^\infty x \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty x e^{-\lambda x}\, dx$$

Let us focus on $I = \int_0^\infty x e^{-\lambda x}\, dx.$

Integrating by parts,

$$\int_0^\infty x e^{-\lambda x}\, dx = -\frac{x}{\lambda} e^{-\lambda x}\Big|_0^\infty - \int_0^\infty -\frac{1}{\lambda} e^{-\lambda x}\, dx$$

$$= -\frac{x}{\lambda} e^{-\lambda x}\Big|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda x}\, dx$$

$$= -\frac{x}{\lambda} e^{-\lambda x}\Big|_0^\infty + \frac{1}{\lambda}\left(-\frac{e^{-\lambda x}}{\lambda}\right)\Big|_0^\infty$$

$$= -\frac{x}{\lambda} e^{-\lambda x}\Big|_0^\infty - \frac{1}{\lambda^2}\left(e^{-\lambda x}\right)_0^\infty$$

Since $\lambda > 0$, $e^{-\lambda x} \to 0$ as $x \to \infty$.

So, $$\int_0^\infty x e^{-\lambda x}\, dx = 0 - \frac{1}{\lambda^2}\left(0 - e^0\right)$$

$$= \frac{1}{\lambda^2}$$

Thus, $E(x) = \lambda \cdot \frac{1}{\lambda^2} = \frac{1}{\lambda}$.

b) The mode is the point at which the the density is the highest. We can see from the pdf of X that density $= 0$ for $x < 0$. For $x \geq 0$, $f_X(x) = \lambda e^{-\lambda x} \Rightarrow f_X'(x) = -\lambda^2 e^{-\lambda x} < 0$. Therefore, the density is decreasing in x and so the highest density will occur at the left endpoint of the support, that is, at $x = 0$. The density is then given $f_X(x) = \lambda$.

c) $$F_X(x) = \int_{-\infty}^{x} f(t)\, dt$$

If $x < 0$, then $f(x) = 0 \Rightarrow F_X(x) = 0$ for $x < 0$. Since, $P(x = 0) = 0$ then $P(X \leq x) = 0$ for $x = 0$. Hence, $F_X(x) = 0$ for $x = 0 \Rightarrow F_X(x) = 0$ for $x \leq 0$.

So, suppose $x > 0$. Then,

$$F_X(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt$$

$$= 0 + \int_{0}^{x} \lambda e^{-\lambda t}\,dt$$

$$= \lambda \int_{0}^{x} e^{-\lambda t}\,dt = \lambda \left[-\frac{1}{\lambda} e^{-\lambda t} \Big|_{0}^{x} \right]$$

$$= -e^{-\lambda x} + 1$$

$$= 1 - e^{-\lambda x}$$

Thus, $F_X(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 - e^{-\lambda x} & \text{for } x > 0 \end{cases}$

Median of X : m such that $P(X \le m) = \frac{1}{2}$

So, $F_X(m) = \frac{1}{2} \Rightarrow m > 0$. Then :

$$1 - e^{-\lambda m} = \frac{1}{2} \Rightarrow e^{-\lambda m} = \frac{1}{2}$$

$$\Rightarrow \quad -\lambda m = \ln\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad m = -\frac{1}{\lambda} \ln\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad m = -\frac{1}{\lambda}\left(\ln(1) - \ln(2)\right)$$

$$\Rightarrow \quad m = \frac{\ln(2)}{\lambda}.$$

Therefore, the median for X is $m = \frac{\ln(2)}{\lambda}$.

d) We will show that $E(X) = \int_0^\infty (1 - F_X(x))\,dx$

From part a), $E(X) = \frac{1}{\lambda}$ (L.H.S of 1)

R.H.S of 1):

$$\int_0^\infty (1 - F_X(x))\,dx = \int_0^\infty 1 - 1 + e^{-\lambda x}\,dx$$

$$= \int_0^\infty e^{-\lambda x}\,dx$$

$$= \left(-\frac{1}{\lambda} e^{-\lambda x}\right)\Big|_0^\infty$$

Since $\lambda > 0$, $e^{-\lambda x} \to 0$ as $x \to \infty$. Thus,

$$\int_0^\infty (1 - F_X(x))\,dx = 0 - \left(-\frac{1}{\lambda}\right) = \frac{1}{\lambda} = E(X)$$

Thus, L.H.S = R.H.S and so $E(X) = \int_0^\infty (1 - F_X(x))\,dx$.

3. Suppose that X is a continuous random variable, which is always positive, so that $P(X > 0) = 1$. Show that in this case the relationship given in Qu. 2(d) still holds (in other words, without assuming X has an exponential distribution).

Hint: Replace F_X in the expression above with the integral of the pdf (remember not to use x as the name of the dummy variable 'dblah' in your integral). This gives you a double integral. Now reverse the order in which you do the two integrals. It will help to draw a picture of the region you are integrating over.

3. We want to show that $E(X) = \int_0^\infty (1 - F_X(x))\,dx \quad (1)$

Notice, we can rewrite $1 = \int_0^\infty f_X(t)\,dt$ since $P(0<X<\infty)=1$ is given.

Also, $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$ by definition. Thus:

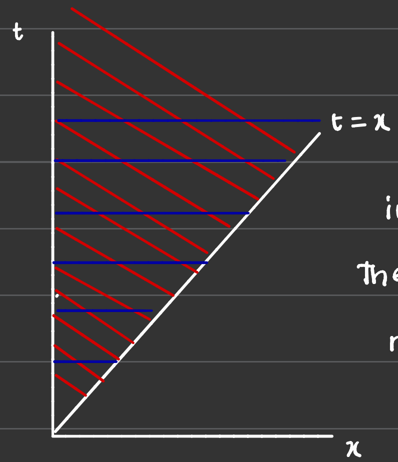
$$1 - F_X(x) = \int_0^\infty f_X(t)\,dt - \int_{-\infty}^{x} f_X(t)\,dt$$

$$= \int_0^\infty f_X(t)\,dt - \int_0^{x} f_X(t)\,dt \quad (\text{since } P(X>0)=1)$$

$$= \int_x^\infty f_X(t)\,dt$$

Substituting this back into $\int_0^\infty (1 - F_X(x))\,dx = \int_0^\infty \int_x^\infty f_X(t)\,dt\,dx$

Now, we want to switch the order of integration. We will first draw the region we are integrating over:



We have that x goes from 0 to infinity and t goes from x to infinity. The region of integration is shaded in red.

Now, if we reverse the order of integration, that is, integrate in x first we will have that x ranges from 0 to t as is indicated by the blue lines. We also have that t ranges from 0 to ∞.

Therefore,

$$\int_0^\infty (1 - F_X(x))\,dx = \int_0^\infty \int_x^\infty f_X(t)\,dt\,dx = \int_0^\infty \int_0^t f_X(t)\,dx\,dt$$

$$= \int_0^\infty \left(x f_X(t) \right)\Big|_0^t \,dt$$

$$= \int_0^\infty t f_X(t)\,dt$$

t is just a dummy variable so we can write it as the following:

$$\int_0^{\infty} (1 - F_X(x))\, dx = \int_0^{\infty} x\, f_X(x)\, dx$$

Now, let us focus on $E(X)$. By definition:

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_{-\infty}^{0} x\, f_X(x)\, dx + \int_0^{\infty} x\, f_X(x)\, dx$$

However, $P(X > 0) = 1 \Rightarrow P(X \leq 0) = 0 \Rightarrow$ for $x \leq 0$, $f_X(x) = 0$

So, $E(X) = \int_{-\infty}^{0} x\, f_X(x)\, dx + \int_0^{\infty} x\, f_X(x)\, dx$

$$= 0 + \int_0^{\infty} x\, f_X(x)\, dx$$

$$= \int_0^{\infty} x\, f_X(x)\, dx = \int_0^{\infty} (1 - F_X(x))\, dx \text{ from above.}$$

Thus, $E(X) = \int_0^{\infty} (1 - F_X(x))\, dx$.

4. This question aims to make concrete the moment generating function via simulation in R: (*Note: this is a computational exercise; almost no theoretical work is required.*)

(a) Simulate 1000 values from $X \sim \mathcal{N}(5,1)$ distribution, that is, the Normal Distribution with mean 5, variance 1.

(b) Let $V = e^{0.01X}$. Using your simulations from (a), obtain 1000 simulations from V. Using your simulations find approximately $E(V) = E(e^{0.01X})$. *Hint: Both of these steps are very easy in* R.

(c) Let $W = e^{-0.01X}$. Using your simulations from (a), obtain 1000 simulations from W. Using your simulations find approximately $E(W) = E(e^{-0.01X})$.

(d) Using your answers to (b) and (c), find $\frac{\partial}{\partial t}E(e^{tX})\big|_{t=0}$ approximately. Is this what you expect? Explain.

Hint:

$$\frac{\partial}{\partial t}E(e^{tX})\Big|_{t=0} \approx \frac{E(e^{\delta X}) - E(e^{-\delta X})}{2\delta},$$

for small δ.

b) $E(V) = E(e^{0.01x}) = 1.0510504$

c) $E(W) = 0.95112$

d) Using the hint, $\frac{\partial}{\partial t} E(e^{tx})\Big|_{t=0} \simeq \frac{E(e^{\partial x}) - E(e^{-\partial x})}{2\partial}$ for small ∂

(letting $\partial = 0.01$) $= \dfrac{\text{Answer from (b)} - \text{Answer from (c)}}{2 \times 0.01}$

$= 5.019175 \simeq 5$

This is what I expected. From class, we have that $E(X) = \frac{\partial}{\partial t} M_X(t)\Big|_{t=0}$

Using the definition of $M_X(t)$: $E(X) = \frac{\partial}{\partial t} E(e^{tX}) \Big|_{t=0}$

Since, X is a normal random variable with mean $= 5$ we have that

$E(X) = 5$. Hence, $\frac{\partial}{\partial t} E(e^{tX}) \Big|_{t=0} = 5$ and that is what we approximated.

R Code for parts a, b, c and d:

```r
set.seed(100)

n <- 1000 # number of simulated values

delta <- 0.01

X <- rnorm(n,5, 1) # generating simulated values

V <- exp(delta*X)  # simulated values of V

E_V<- mean(V)

W <- exp(-delta*X) # simulated values of W

E_W <- mean(W)

d <- (E_V-E_W)/(2*delta)  # derivative of E(e^tX) evaluated at t = 0

```

5. A random variable X with support on $\mathbb{R}$ has pdf

$$f_X(x) = e^{-c|x|}$$

where $c > 0$ is a fixed constant.

(a) Find c.

(b) Find the mgf $M_X(t) = E[e^{tX}]$ for X. *Hint: It may be helpful to split the integral into two pieces.*

(c) State the range of values of t for which $M_X(t)$ is defined. In other words, the range of values of t for which $E[e^{tX}] < \infty$.

(d) Using your answer to (b), or otherwise, find $E[X]$, $E[X^2]$ and $V(X)$.

a) $f_X(x) = e^{-c|x|}$

By Theorem 1 (ii) of pdfs:

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-c|x|}\,dx = 1 \quad (\text{since the support is } \mathbb{R})$$

$$\Rightarrow \int_{-\infty}^{0} e^{cx}\,dx + \int_{0}^{\infty} e^{-cx}\,dx = 1$$

$$\Rightarrow \frac{1}{c} e^{cx} \Big|_{-\infty}^{0} + \left(-\frac{1}{c} e^{-cx}\right)\Big|_{0}^{\infty} = 1$$

As $x \to -\infty$, $e^{cx} \to 0$ given that $c > 0$. Similarly, $e^{-cx} \to 0$

as $x \to \infty$.

Thus, we have that:

$$\frac{1}{c} - 0 + 0 - \left(-\frac{1}{c}\right) = 1$$

$$\Rightarrow \frac{2}{c} = 1$$

$$\Rightarrow c = 2.$$

b) $M_X(t) = E\left[e^{tX}\right]$

$$= \int_{-\infty}^{\infty} e^{tx} e^{-2|x|} \, dx$$

Splitting the integral into two parts:

$$M_X(t) = \int_{-\infty}^{0} e^{tx} \cdot e^{2x} \, dx + \int_{0}^{\infty} e^{tx} \cdot e^{-2x} \, dx$$

$$= \underbrace{\int_{-\infty}^{0} e^{(t+2)x} \, dx}_{\substack{\text{blows up if} \\ t+2 \leq 0 \Rightarrow t \leq -2.}} + \underbrace{\int_{0}^{\infty} e^{(t-2)x} \, dx}_{\substack{\text{blows up if} \\ t-2 \geq 0 \Rightarrow t \geq 2}}$$

$$= \frac{1}{t+2}\left[e^{(t+\frac{1}{2})x}\right]\Bigg|_{-\infty}^{0} + \frac{1}{t-2}\left[e^{(t-\frac{1}{2})x}\right]\Bigg|_{0}^{\infty}$$

Since, we have that $t > -2$ and $t < 2$ we have that $M_X(t)$ will be defined for $-2 < t < 2$. In that case, $0 < t+2$ and so as $x \to -\infty$ we have $e^{(t+2)x} \to 0$. Similarly, $t-2 < 0$ and so as $x \to \infty$, we have that $e^{(t-2)x} \to 0$.

Therefore, $M_X(t) = \frac{1}{t+2}\left[e^{(t+2)x}\right]\Big|_{-\infty}^{0} + \frac{1}{t-2}\left[e^{(t-2)x}\right]\Big|_{0}^{\infty}$

$$= \frac{1}{t+2}(1-0) + \frac{1}{t-2}(0-1)$$

$$= \frac{1}{t+2} - \frac{1}{t-2} = \frac{\cancel{t}-2-\cancel{t}-2}{(t+2)(t-2)}$$

$$= \frac{-4}{t^2-4}.$$

So, $M_X(t) = \begin{cases} \frac{-4}{t^2-4} & -2 < t < 2 \\ \infty & \text{otherwise} \end{cases}$

c) The range of values of t for which $M_X(t)$ is defined, that is, the range of values of t for which $E[e^{tX}] < \infty$ is given by $-2 < t < 2$ as discussed above. Hence, $M_X(t)$ is defined for $|t| < 2$.

d) $E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0}$

$$\frac{d}{dt} M_X(t) = \frac{-(-4)(2t)}{(t^2-4)^2} = \frac{8t}{(t^2-4)^2}$$

$$\frac{d}{dt} M_X(t) \Big|_{t=0} = \frac{0}{16} = 0.$$

So, $E(X) = 0$.

$$E(X^2) = \frac{d^2}{dt^2} M_X(t) \Big|_{t=0}$$

$$\frac{d^2}{dt^2} M_X(t) = \frac{8(t^2-4)^2 - 8t(t^2-4).2.2t}{(t^2-4)^4} = \frac{8(t^2-4) - 32t^2}{(t^2-4)^3}$$

$$= \frac{-24t^2 - 32}{(t^2-4)^3}$$

$$\frac{d^2}{dt^2} M_X(t) \Big|_{t=0} = \frac{(-32)}{(-4)^3} = \frac{-32}{-64} = \frac{1}{2}.$$

So, $E(X^2) = \frac{1}{2}$.

then, $V(X) = E(X^2) - (E(X))^2$ by T2 on expectations

$= \frac{1}{2}$.

6. Suppose that X and Y are random variables, and $Y = a + bX$. Find an expression for the mgf for Y, $M_Y(t)$ in terms of the mgf for X, $M_X(t)$.

By definition :

$$M_X(t) = E(e^{tX})$$

$$M_Y(t) = E(e^{tY})$$

Using $Y = a + bX$:

$$M_Y(t) = E(e^{t(a+bX)})$$

$$= E(e^{at} \cdot e^{bXt})$$

$$= e^{at} E(e^{btX})$$ (for a given t, e^{at} is constant and we can use T1 on expectations)

Let $k = bt$. Then, $M_Y(t) = e^{at} E(e^{kX})$

$$= e^{at} M_X(k)$$ by definition

$$= e^{at} M_X(bt).$$

7. Let X be a random variable with mgf $M_X(t)$. For $t > 0$ define the random variables $Z_t = e^{tX}$.

(Here we are defining a set of random variables, one for each value of t.)

(a) Find a relation between $P(X \geq a)$ and $P(Z_t \geq e^{ta})$.

(b) By using the result from (a) and applying Markov's inequality find an upper bound on $P(X \geq a)$ in terms of $M_X(t)$.

Now suppose that X is a Poisson distribution with parameter λ.

(c) Use Markov's inequality, as given in the Lecture 4, Slide 4 to obtain an upper bound on $P(X \geq k\lambda)$ where $k > 0$.

Since X is a Poisson random variable $M_X(t) = \exp\left[\lambda(e^t - 1)\right]$.

(d) Use the result in (b) to find an upper bound on $P(X \geq k\lambda)$, for $k \geq 1$, in terms of t, k and λ.

(e) By differentiating your answer to (d) with respect to t, find the value of t that gives the tightest upper bound on $P(X \geq k\lambda)$, and state this upper bound.

(f) Let $\lambda = 2$ and $k = 3$, compute the upper bound from (c), the upper bound from (e) and by using `ppois` in `R`, find the exact probability $P(X \geq k\lambda)$.

Hint: when computing the exact probability be careful to include the probability that $P(X = 6)$.

a) $P(Z_t \geq e^{ta}) = P(e^{tX} \geq e^{ta})$

Notice: $e^{tX} \geq e^{ta} \Rightarrow tX \geq ta \Rightarrow X \geq a$. (since $t > 0$ so the inequalities do not get reversed)

So, $P(Z_t \geq e^{ta}) = P(e^{tX} \geq e^{ta}) = P(X \geq a)$.

b) Notice: $P(Z_t < 0) = P(e^{tX} < 0) = 0$. Also, notice that $e^{ta} \geq 0$.

Thus, we can apply Markov's inequality:

$$P(X \geq a) = P(Z_t \geq e^{ta}) \leq \frac{E(Z_t)}{e^{ta}} = \frac{E(e^{tX})}{e^{ta}} = \frac{M_X(t)}{e^{ta}}$$

Hence, $P(X \geq a) \leq \dfrac{M_X(t)}{e^{ta}}$.

c) Suppose that X is a Poisson distribution with parameter λ.

Then,

$$f(x) = \begin{cases} \dfrac{\lambda^x e^{-\lambda}}{x!} & \text{for } x \in \{0, 1, \cdots\} \\ 0 & \text{otherwise} \end{cases}$$

Since, the support for X is non-negative real numbers we have that $P(Y<0) = 0$. Also, $k > 0$ and $\lambda > 0$ and so $\lambda k > 0$. Therefore, we can apply Markov's inequality to $P(X \geq k\lambda) \leq \frac{E(X)}{k\lambda}$. Since, X is a Poisson random variable we have that $E(X) = \lambda$. Thus:

$$P(X \geq k\lambda) \leq \frac{\lambda}{kX} = \frac{1}{k}.$$

$$\Rightarrow P(X \geq k\lambda) \leq \frac{1}{k}.$$

d) From part b), $P(X \geq a) \leq \frac{M_X(t)}{e^{ta}}$.

Then, $P(X \geq k\lambda) \leq \frac{M_X(t)}{e^{tk\lambda}} = \frac{e^{(\lambda(e^t - 1))}}{e^{tk\lambda}}$

e) We will differentiate $\frac{e^{(\lambda(e^t - 1))}}{e^{tk\lambda}}$ with respect to t.

$$\frac{d}{dt}\left[\frac{e^{(\lambda(e^t-1))}}{e^{tk\lambda}}\right]$$

$$= \frac{e^{tk\lambda}\left(e^{\lambda(e^t-1)}\cdot \lambda e^t\right) - e^{(\lambda(e^t-1))}\cdot k\lambda e^{tk\lambda}}{(e^{tk\lambda})^2}$$

$$= \frac{e^{t\cancel{k}\lambda}\left[\lambda e^{\lambda e^t - 1 + t} - k\lambda e^{(\lambda(e^t-1))}\right]}{(e^{tk\lambda})^2}$$

$$= \frac{\lambda e^{\lambda e^t - \lambda + t} - k\lambda e^{(\lambda e^t - \lambda)}}{e^{tk\lambda}}$$

Setting the derivative equal to 0 :

$$\frac{\lambda e^{\lambda e^t - \lambda + t} - k\lambda e^{(\lambda e^t - \lambda)}}{e^{tk\lambda}} = 0$$

$$\Rightarrow \lambda e^{\lambda e^t - \lambda}\cdot e^t - k\lambda e^{(\lambda e^t - \lambda)} = 0$$

$$\Rightarrow \lambda e^{\lambda e^t - \lambda}\left(e^t - k\right) = 0$$

Notice : $\lambda e^{\lambda e^t - \lambda} \neq 0$ because then either $\lambda = 0$ or $e^{\lambda e^t - \lambda} = 0$. However, $\lambda > 0 \Rightarrow \lambda \neq 0$ and $e^{\lambda e^t - \lambda} \neq 0$.

Therefore , $\lambda e^{\lambda e^t - \lambda}\left(e^t - k\right) = 0 \Rightarrow e^t = k$

$$\Rightarrow t = \ln(k)$$

When $t = \ln(k)$ we will get the tightest upper bound, that is,

$$\frac{e^{(\lambda(e^t-1))}}{e^{tk\lambda}}$$ will be lowest.

Plugging in $t = \ln(k)$:

$$\frac{e^{(\lambda(e^{\ln(k)}-1))}}{e^{k\ln(k)\lambda}} = \frac{e^{(\lambda k-\lambda)}}{e^{\ln(k^{k\lambda})}} = \frac{e^{\lambda k-\lambda}}{k^{k\lambda}}.$$

So, $P(X \geq k\lambda) \leq \dfrac{e^{\lambda k-\lambda}}{k^{k\lambda}}$ and this is the tighest upper bound.

f) Let $\lambda = 2$ and $k = 3$.

From c) : $P(X \geq k\lambda) = P(X \geq 6) \leq \frac{1}{k} = \frac{1}{3} = 0.333$

$$\Rightarrow P(X \geq 6) \leq \frac{1}{3}.$$

From e) : $P(X \geq 6) \leq \dfrac{e^{\lambda k-\lambda}}{k^{k\lambda}} = \dfrac{e^4}{3^6} = 0.07489 \approx 0.075$

The exact probability that $P(X \geq k\lambda) = 0.01656361$. Here is the R code used to calculate this:

```
k = 3
lambda = 2

# probability of X being less than equal to 5

q <- ppois(k*lambda-1,lambda)

# probability of X being greater than 5 = probability of X being greater than
 equal to 6

p <- 1-q
```

8. Let X be a standard normal random variable.

(a) Find the mgf for X;
As always, remember to specify the values of t for which it is defined.

(b) Use (a) to find $E(X)$, $E(X^2)$, $V(X)$ and $E(X^3)$.

a) Let X be a standard normal random variable. Then :

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

By definition : $M_X(t) = E(e^{tX})$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \cdot e^{-\frac{x^2}{2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - \frac{x^2}{2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2tx + t^2) + \frac{1}{2}t^2}\,dx \quad \text{(completing the square)}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} \cdot e^{\frac{1}{2}t^2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} dx$$

$$= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx$$

Let $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2}$. $f(x)$ looks like the pdf of the standard normal distribution and in fact it is the pdf of the normal distribution with the mean shifted as $x \to x-t$. The variance is still 1 and so $f(x)$ is the pdf of a normal distribution with mean $= t$ and variance $= 1$.

By Theorem 1 (ii) of density functions: $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx = 1$$

Therefore, $M_X(t) = e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx = e^{\frac{t^2}{2}}$ and it is defined for all t.

b) $E(X) = \left. \frac{d}{dt} M_X(t) \right|_{t=0}$

$$\frac{d}{dt} M_X(t) = t e^{\frac{t^2}{2}}$$

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = 0 \Rightarrow E(X) = 0.$$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0}$$

$$\frac{d^2}{dt^2} M_X(t) = e^{\frac{t^2}{2}} + t.te^{\frac{t^2}{2}}$$

$$= e^{\frac{t^2}{2}} + t^2 e^{\frac{t^2}{2}}$$

Then, $\frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = 1+0 = 1$

$\Rightarrow E(X^2) = 1.$

$V(X) = E(X^2) - (E(X))^2$ by T2 of expectations

$= 1.$

$$E(X^3) = \frac{d^3}{dt^3} M_X(t)\Big|_{t=0}$$

$$\frac{d^3}{dt^3} M_X(t) = te^{\frac{t^2}{2}} + 2te^{\frac{t^2}{2}} + t^2 e^{\frac{t^2}{2}}.t$$

$$= (3t+t^3)e^{\frac{t^2}{2}}$$

$$\frac{d^3}{dt^3} M_X(t) \Big|_{t=0} = 0$$

So, $E(X^3) = 0$.