Linear Algebra HW#1 - Saaif Ahmed - 661925946

Tuesday, September 7, 2021 10:15 PM

3) Prove the parallelogram law: $||\mathbf{x}+\mathbf{y}||^2 + ||\mathbf{x}-\mathbf{y}||^2 = 2\left(||\mathbf{x}||^2 + ||\mathbf{y}||^2\right)$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Question 3:

Direct Proof

Let $x, y \in R^n$ chosen arbitrarily

$||x+y||^2 = < x+y, x+y >$ Aka the inner product

$=< x,x >+< x,y >+< y,x >+< y,y >$

By nature of inner products

$= ||x||^2 + 2 < x,y > +||y|^2$

$||x-y||^2 = < x-y, x-y >$

$=< x,x >$ - $< x,y >$ - $< y,x >+< y,y >$

By nature of iner products

$= ||x||^2 - 2 < x,y > +||y||^2$

Thus

$||x||^2 + 2 < x,y > +||y||^2 + ||x||^2 - 2 < x,y > +||y||^2$

$= 2(||x||^2 + ||y||^2)$

As desired.

5) Show that the following subsets of $\mathbb{R}^n$ are subspaces:

a. $W_1 = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \sum c_i \mathbf{v}_i\}$ where $\mathbf{v}_i \in \mathbb{R}^n$ for each i. (This is the set of linear combinations of the vectors $\mathbf{v}_i$)

b. $W_2 = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{y} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n\}$ where A is an $n \times n$ matrix.

Question 5b:

Proof:

Since $\vec{0} \in \mathrm{R}^n$ and for any $A_{n\times n}$ matrix -> $A\vec{0} = \vec{0}$ thus $\vec{0} \in \mathrm{W}_2$

Let $\overrightarrow{x_1}, \overrightarrow{x_2} \in W_2$

Then $\overrightarrow{x_1} = A\overrightarrow{y_1}$; $\overrightarrow{x_2} = A\overrightarrow{y_2}$

Now since $c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} = c_1 A\overrightarrow{y_1} + c_2 A\overrightarrow{y_2} : c_1, c_2 \in \boldsymbol{R}$

$= A(c_1\overrightarrow{y_1} + c_2\overrightarrow{y_2})$ where $(c_1\overrightarrow{y_1} + c_2\overrightarrow{y_2}) = z \in R^n$ because R^n is by definition closed under linear combinations

Therefore $c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} = Az$ where $z \in R^n$ as shown above

Thus W_2 is closed under linear combinations

As desired.

6) Show that the following subsets of $\mathbb{R}^n$ are not subspaces:

a. $S = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$ where $\mathbf{b} \neq \mathbf{0} \in \mathbb{R}^n$.

b. Any non-empty, finite subset of $\mathbb{R}^n$. (Clarification: by non-empty I mean containing at least one non-zero vector)

Question 6b:

Proof:

The scalar field is infinite $c \in R$ where R is the real numbers.
Let $\overrightarrow{x_1}, \overrightarrow{x_2} \in W$ where W is the subspace.
Then by definition it has to follow that $c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} \in W$ where $c_1, c_2 \in R$ (R being the infinite field of scalars).

But because W is defined as finite and the scalars are infinite. It is easy to show that you can choose $c_1 \& c_2$ such that $c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} \notin$ W

Therefore W is not closed under linear combinations and is not a subspace.
As desired.

9) Solve the following systems using any method

a.
$$\begin{aligned} x + 3y &= 5 \\ 2x - y &= 3 \end{aligned}$$

b.
$$\begin{aligned} x + 3y + z &= 9 \\ 4x + 7y - 2z &= 27 \\ -2y + 4z &= -10 \end{aligned}$$

Question 9b:

$$\begin{bmatrix} 1 & 3 & 1 & 9 \\ 4 & 7 & -2 & 27 \\ 0 & -2 & 4 & -10 \end{bmatrix} R2 = R2 - 4R1 \rightarrow \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & -5 & -6 & -9 \\ 0 & -2 & 4 & -10 \end{bmatrix} R2 = -\frac{1}{5}R2 \rightarrow \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & 1 & \frac{6}{5} & \frac{9}{5} \\ 0 & -2 & 4 & -10 \end{bmatrix}$$

$$R3 = 2R2 + R3 \rightarrow \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & 1 & \frac{6}{5} & \frac{9}{5} \\ 0 & 0 & \frac{32}{5} & -\frac{32}{5} \end{bmatrix} R3 = \frac{5}{32}R3 \rightarrow \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & 1 & -\frac{6}{5} & \frac{9}{5} \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$R2 = \frac{6}{5}R3 + R2 \rightarrow \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} R1 = R1 - R3 \rightarrow R1 = R1 - 3R2 \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Answer: $x = 1, y = 3, z = -1$

10) Let A and B be invertible. Prove that

a. $(AB)^{-1} = B^{-1}A^{-1}$
b. $(A^{-1})^T = (A^T)^{-1}$

Question 10b:

Direct Proof
Let A be an arbitrary invertible matrix
Let $D = A^{-1}$
Thus $(D)^T$. We know that for any matrix C that $CI = C$. And for any matrix C with inverse C^{-1} that $CC^{-1} = I$ where I is the identity matrix.

From our definition $AD = I$.
$(AD)^T = (I)^T \rightarrow (A^T)(D^T) = I$ (Transpose properties)
Solving for D^T: $D^T = (A^T)^{-1}I \rightarrow D^T = (A^T)^{-1}$
Replace D^T with our substitution and thus we show since $D^T = A^{-1}$ that $(A^{-1})^T = (A^T)^{-1}$
As desired.

11) Given $A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & 2 \\ 2 & 2 & -1 \end{bmatrix}$, compute the following

a. A^2
b. A^T
c. A^{-1}

Question 11:

A:

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & 2 \\ 2 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 4 \\ 2 & 0 & -10 \\ -4 & 2 & 13 \end{bmatrix} \textbf{Answer: } \begin{bmatrix} 5 & 10 & 4 \\ 2 & 0 & -10 \\ -4 & 2 & 13 \end{bmatrix}$$

(1*1)+(2*-2)+(4*2)=5	(1*2)+(2*0)+(4*2) = 10	(1*4)+(2*2)+(4*-1)=4
(-2*1)+(0*-2)+(2*2)=2	(-2*2)+(0*0)+(2*2)=0	(-2*4)+(0*2)+(2*-1)=-10
(2*1)+(2*-2)+(-1*2)=-4	(2*2)+(2*0)+(-1*2)=2	(2*4)+(2*2)+(-1*-1)=13

B:

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & 2 \\ 2 & 2 & -1 \end{bmatrix}^T = A \rightarrow [a]_{ij}\ ; A^T = [a]_{ji} \text{ Square Matrix. Symmetric flip across diagonal}$$

$$\textbf{Answer: } \begin{bmatrix} 1 & -2 & 2 \\ 2 & 0 & 2 \\ 4 & 2 & -1 \end{bmatrix} = A^T$$

C:

$$\begin{bmatrix} 1 & 2 & 4 & 1 & 0 & 0 \\ -2 & 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & -1 & 0 & 0 & 1 \end{bmatrix} R2 = 2R1 + R2 \rightarrow \begin{bmatrix} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 4 & 10 & 2 & 1 & 0 \\ 2 & 2 & -1 & 0 & 0 & 1 \end{bmatrix} R3 = R3 - 2R1$$

$$\begin{bmatrix} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 4 & 10 & 2 & 1 & 0 \\ 0 & -2 & -9 & -2 & 0 & 1 \end{bmatrix} R2 = \frac{R2}{4} \rightarrow \begin{bmatrix} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -2 & -9 & -2 & 0 & 1 \end{bmatrix} R1 = R1 - 2R2$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -2 & -9 & -2 & 0 & 1 \end{bmatrix} R3 = R3 + 2R2 \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & -4 & -1 & \frac{1}{2} & 1 \end{bmatrix} R3 = \frac{-R3}{4}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{-1}{8} & -\frac{1}{4} \end{bmatrix} R1 = R1 + R3 \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} & -\frac{5}{8} & -\frac{1}{4} \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{-1}{8} & -\frac{1}{4} \end{bmatrix} R2 = R2 - \frac{5}{2}R2$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} & -\frac{5}{8} & -\frac{1}{4} \\ 0 & 1 & 0 & \frac{-1}{8} & \frac{-9}{16} & \frac{5}{8} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{-1}{8} & -\frac{1}{4} \end{bmatrix} \textbf{Answer: } \begin{bmatrix} \frac{1}{4} & -\frac{5}{8} & -\frac{1}{4} \\ \frac{-1}{8} & \frac{-9}{16} & \frac{5}{8} \\ \frac{1}{4} & \frac{-1}{8} & -\frac{1}{4} \end{bmatrix}$$