Linear Algebra HW#5 - Saaif Ahmed - 661925946

Friday, October 15, 2021 5:47 PM

42) Let $S = Span(\mathbf{v}_1, \mathbf{v}_2)$ where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \\ -1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$. Find the projection matrix onto S and use it to find the vector in S that is closest to $(4, 1, 2, -2)$.

Question 42:

$$A(A^TA)^{-1}A^T\vec{x} \text{ where } A = \begin{bmatrix} 1 & 3 \\ 3 & 0 \\ 4 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 27 & -2 \\ -2 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 27 & -2 \\ -2 & 11 \end{bmatrix}^{-1} = \frac{1}{293}\begin{bmatrix} 11 & 2 \\ 2 & 27 \end{bmatrix}$$

Now we finish the equation.

$$\frac{1}{293}\begin{bmatrix} 1 & 3 \\ 3 & 0 \\ 4 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 11 & 2 \\ 2 & 27 \end{bmatrix}\begin{bmatrix} 1 & 3 & 4 & -1 \\ 3 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 2 \\ 2 & 27 \end{bmatrix} * \begin{bmatrix} 1 & 3 & 4 & -1 \\ 3 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 33 & 42 & -9 \\ 83 & 6 & -19 & 25 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 3 & 0 \\ 4 & -1 \\ -1 & 1 \end{bmatrix} * \begin{bmatrix} 17 & 33 & 42 & -9 \\ 83 & 6 & -19 & 25 \end{bmatrix} = \begin{bmatrix} 266 & 51 & -15 & 66 \\ 51 & 99 & 126 & -27 \\ -15 & 126 & 187 & -61 \\ 66 & -27 & -61 & 34 \end{bmatrix}$$

Answer: $\frac{1}{293}\begin{bmatrix} 266 & 51 & -15 & 66 \\ 51 & 99 & 126 & -27 \\ -15 & 126 & 187 & -61 \\ 66 & -27 & -61 & 34 \end{bmatrix}\vec{x}$

$$\frac{1}{293}\begin{bmatrix} 266 & 51 & -15 & 66 \\ 51 & 99 & 126 & -27 \\ -15 & 126 & 187 & -61 \\ 66 & -27 & -61 & 34 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{293}\begin{bmatrix} 953 \\ 609 \\ 562 \\ 47 \end{bmatrix}$$

Answer: $\frac{1}{293}\begin{bmatrix} 953 \\ 609 \\ 562 \\ 47 \end{bmatrix}$

43) Use linear algebra to find the line of best fit given the set of points $\{(-1, 11), (0, 4), (1, -2), (2, 1), (3, -6)\}$

Question 43:

$$A^TA = \begin{bmatrix} 5 & 5 \\ 5 & 15 \end{bmatrix} A^T\vec{y} = \begin{bmatrix} 8 \\ -29 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 5 \\ 5 & 15 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 29 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 15 & -5 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} 8 \\ -29 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{21}{10} \end{bmatrix}$$

Answer: $y = \frac{21}{10}x - \frac{1}{2}$

44) Find the QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix}$$

Question 44:

Let $u_1 = a_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} < a_1, e_1 > = \vec{a}^T \vec{e_1} = \sqrt{2}$ $e_1 = \frac{u_1}{||u_1||} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$

$u_2 = a_2 - < a_2, e_1 > e_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $e_2 = \frac{u_2}{||u_2||} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$u_3 = a_3 - < a_3, e_1 > e_1 - < a_3, e_2 > e_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ $e_3 = \frac{u_3}{||u_3||} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Answer: $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \end{bmatrix} = Q \quad R = \begin{bmatrix} \sqrt{2} & \sqrt{2} & -2\sqrt{2} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & -\sqrt{\frac{2}{3}} \end{bmatrix}$

45) The following problem deals with an arbitrary projection matrix P. Recall that P is a projection if and only if $P^2 = P$.

a. Prove that $I - P$ is also a projection

b. Show that if $\mathbf{x} \in \mathcal{C}(P)$, then $P\mathbf{x} = \mathbf{x}$. Deduce that P is the indentity operator on $\mathcal{C}(P)$ and hence P is a projection onto its range.

c. Show that $\mathcal{N}(I - P) = \mathcal{C}(P)$ and vise-versa. Deduce that $I - P$ a projection onto the kernel of P.

Question 45:

A:

Since $P^2 = P$ we calculate $(I - P)^2 = I^2 - 2IP - P^2$
By definition $I^2 = I$ & $P^2 = P$ we have that $I - 2P + P = I - P$
Since $(I - P)^2$ is a projection then $I - P$ is a projection
As Desired

B:

Let $Py = x : y \in C(P)$
Now try $P(Py) = P^2 y$. Since P is a projection onto it's range then $\mathrm{P}^2 = \mathrm{P}$
As such $P^2 y = Py = Px$
Thus it is clear to see that $y = x$ thus $Px = x : x \in C(P)$
As Desired

C:

Let $x \in N(I - P)$ thus $(I - P)x = \vec{0}$ thus we have that $x - Px = \vec{0}$
From $Px = x$ we know that $x \in C(P)$
We have that $x \in C(P)$ and $x \in N(I - P)$. Thus we see that these are subspaces of each other. As these are subspaces of each other we have that $N(I - P) \subseteq C(P)$ & $C(P) \subseteq N(I - P)$ thus they are equal
$C(P) = N(I - P)$ and vice versa follows from reordering the subset equation. As Desired.

49) Let let $\mathbf{u} \in \mathbb{R}^n$ be such that $||\mathbf{u}|| = 1$ and define the householder matrix, $H = I - 2\mathbf{u}\mathbf{u}^T$. Prove the following:

a. H is symmetric, $H^T = H$
b. H is self-invertible, $H = H^{-1}$
c. H is orthogonal
d. H acts as the identity on the hyperplane, $W = \{\mathbf{v} : \mathbf{u}^T\mathbf{v} = 0\}$

Question 49:

A:

$(I - 2\vec{u}\vec{u}^T)^T = I^T - 2(\vec{u}\vec{u}^T)^T = I - 2(\vec{u}^T)^T\vec{u}^T$
$I - 2\vec{u}\vec{u}^T$
Thus $H^T = H$ thus it as symmetric as desired.

B:

$H * H = (I - 2\vec{u}\vec{u}^T)(I - 2\vec{u}\vec{u}^T) = I - 2(2\vec{u}\vec{u}^T) + 4(\vec{u}\vec{u}^T)(\vec{u}\vec{u}^T)$
$= I - 4(\vec{u}\vec{u}^T) + 4\vec{u}(\vec{u}^T\vec{u})^T\vec{u}^T = I - 4(\vec{u}\vec{u}^T) + 4(\vec{u}\vec{u}^T)$
$= I$
Since $H * H$ is equal to I we show that it is self invertible as this is a propety of inverse matrices.

C:

Unit vectors are defined as $\vec{u}^T\vec{u} = 1 = ||u||$
We show that $H * H^T = 1$
Primarily we know that H is symmetric so we can say $H * H^T = 1$
Secondly we know that H is self-invertible so we can say that $H * H = 1$
Thus we can safely say that H is orthogonal as desired.

D:

Let $W = \{\vec{v} : \vec{u}^T\vec{v} = \vec{0}\}$
$H(v) = (I - 2\vec{u}\vec{u}^T)\vec{v} = I\vec{v} - 2(\vec{u})(\vec{u}^T)\vec{v} = \vec{v} - 2(\vec{0}) = \vec{v} \; \forall \; \vec{v} \in W$
Because the action of H on any vector v in the subspace W results in the same vector v it is an identity on the hyper plane.