Linear Algebra HW#8 - Saaif Ahmed - 661925946

Friday, November 12, 2021 12:57 AM

84) Find the spectral decomposition of the symmetric matrix

$$A = \begin{bmatrix} -3 & 4 & 0 \\ 4 & 12 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Question 84:

$(-\lambda^3 + 11\lambda^2 + 34\lambda - 104) = -(\lambda + 4)(\lambda^2 - 15\lambda + 26)$

$= -(\lambda + 4)(\lambda - 2)(\lambda - 13) = 0$

$\lambda = -4, 2, 13$

$\begin{bmatrix} 1 & 4 & \square \\ 4 & 16 & \square \\ \square & \square & 6 \end{bmatrix} \overrightarrow{v_1} = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -5 & 4 & \square \\ 4 & 10 & \square \\ \square & \square & 0 \end{bmatrix}$ linearly independent we have that $\overrightarrow{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -16 & 4 & \square \\ 4 & -1 & \square \\ \square & \square & -11 \end{bmatrix} \overrightarrow{v_3} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$

Unit vectors are

$\begin{bmatrix} -\frac{4}{17} \\ \frac{1}{17} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{17} \\ \frac{4}{17} \\ 0 \end{bmatrix}$

Answer: $A = \begin{bmatrix} -\frac{4}{17} & 0 & \frac{1}{17} \\ \frac{1}{17} & 0 & \frac{4}{17} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -4 & \square & \square \\ \square & 2 & \square \\ \square & \square & 13 \end{bmatrix} \begin{bmatrix} -\frac{4}{17} & \frac{1}{17} & 0 \\ 0 & 0 & 1 \\ \frac{1}{17} & \frac{4}{17} & 0 \end{bmatrix}$

88) Prove that $A^T = A$ is negative definite, i.e that $\mathbf{x}^T A \mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$, if and only if all the eigenvalues of A are strictly negative.

Question 88:

Assume that A is negative definite then $\vec{x}^T A \vec{x} < 0 \ \forall \ \vec{x}$. Now let λ be any eigen value for A . Then let $\vec{v} \neq \vec{0}$ be such that $A\vec{v} = \lambda \vec{v}$. Now $\vec{v}^T A \vec{v} = \lambda \vec{v}^T \vec{v} \rightarrow \lambda = \frac{\vec{v}^T A \vec{v}}{\vec{v}^T \vec{v}}$

The top of this fraction is <0 because A is negative definite and the bottom is the 2-norm of the vector v which is always positive. Thus negative divided by positive is still negative thus all eigen values $\lambda < 0$ as desired

Now the other way

Assume that the eigen values are negative. Then by the spectral theorem $A = \sum \lambda_i q_i q_i^T$ such that $\lambda_i < 0$ and that $A\lambda_i = \lambda_i q_i$ Now let $\vec{x}$ be chosen arbitrarily such that $\sum c_i \overrightarrow{q_i}$ and consider

$\vec{x}^T A \vec{x} = < \vec{x}, A\vec{x} > = < \vec{x}, \sum c_i A \overrightarrow{q_i} > = < \vec{x}, \sum c_i \lambda \overrightarrow{q_i} > = < \sum c_j \overrightarrow{q_j}, \sum c_i \lambda \overrightarrow{q_i} >$

$= \sum c_j^2 \lambda_j < 0$

Thus A is negative definite as desired.

Thus proved both ways.

90) Let A be a symmetric, positive definite matrix. Prove that A is invertible and that A^{-1} is also symmetric, positive definite. What are the eigenvalues of A^{-1}?

Question 90:

A is positive definite and symmetric. Therefore all eigen values are >0 and A is invertible so the det(A) >0.

Consider

$(AA^{-1})^T = I = {A^{-1}}^T A^T = {A^{-1}}^T A = I$

Consider from this

${A^{-1}}^T A A^{-1} = A^{-1} \rightarrow {A^{-1}}^T = A^{-1}$

Now let λ be an eigen value of A we have that

$A\vec{v} = \lambda\vec{v} \rightarrow A^{-1}A\vec{v} = \lambda A^{-1}\vec{v}$

$\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$

Thus the eigen values are the reciprocal of the eigen values of A

91) Show that the following matrix is positive definite and find the Cholesky factorization

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 3 & -3 \\ 2 & -3 & 5 \end{bmatrix}$$

Question 91:

It is real and symmetric thus it is positive definite

$I_{11} = \sqrt{a_{11}} = 1$

$I_{21} = \frac{a_{21}}{I_{11}} = -1$

$I_{22} = \sqrt{a_{22} - I_{21}^2} = \sqrt{3 - -1^2} = \sqrt{2}$

$I_{31} = \frac{a_{31}}{I_{11}} = \frac{2}{1} = 2$

$I_{32} = \frac{a_{32} - (I_{31}I_{21})}{I_{22}} = (-3) - (-2)/\sqrt{2} = \frac{\sqrt{2}}{-2}$

$I_{33} = \sqrt{a_{33} - I_{31}^2 - I_{32}^2} = \sqrt{5 - 4 - \frac{-\sqrt{2}}{2}^2} = \frac{\sqrt{2}}{2}$

$A = L\,L^T$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & \sqrt{2} & 0 \\ 2 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 2 \\ 0 & \sqrt{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$