Linear Algebra HW#3 - Saaif Ahmed - 661925946

Monday, September 20, 2021 4:09 PM

18) Given $A = \begin{bmatrix} 1 & -2 & 1 & 0 & 3 \\ -3 & 6 & 0 & 2 & -1 \\ -2 & 4 & 3 & 1 & 0 \end{bmatrix}$

Find a basis for $\mathcal{N}(A)$, the null space of A.

Question 18:

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 3 \\ -3 & 6 & 0 & 2 & -1 \\ -2 & 4 & 3 & 1 & 0 \end{bmatrix} R2 = R2 + 3R1 \rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 & 3 \\ 0 & 0 & 3 & 2 & 8 \\ -2 & 4 & 3 & 1 & 0 \end{bmatrix} R3 = R3 + 2R1$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 3 \\ 0 & 0 & 3 & 2 & 8 \\ 0 & 0 & 5 & 1 & 6 \end{bmatrix} \frac{R3}{3}, \frac{R5}{5} \rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{2}{3} & \frac{8}{3} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} R3 = R3 - R2$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{2}{3} & \frac{8}{3} \\ 0 & 0 & 0 & -\frac{7}{15} & -\frac{22}{5} \end{bmatrix} \rightarrow R3\frac{-15}{7} \rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{2}{3} & \frac{8}{3} \\ 0 & 0 & 0 & 1 & \frac{66}{7} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 0 & \frac{17}{7} \\ 0 & 0 & 1 & -\frac{1}{9} & \frac{2}{9} \\ 0 & 0 & 0 & 1 & \frac{22}{7} \end{bmatrix}$$

$$x_1 = 2x_2 - \frac{17}{7}x_5$$

$$x_3 = \frac{1}{9}x_4 - \frac{2}{9}x_5$$

$$x_4 = -\frac{22}{7}x_5\,;$$

$$x_3 = -\frac{4}{7}x_5$$

$$N(A) = span\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{17}{7} \\ 0 \\ -\frac{4}{7} \\ -\frac{22}{7} \\ 1 \end{bmatrix}\right\}$$

21) Given A in problem 18, find bases for the other fundamental subspaces; $\mathcal{C}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A^T)$

Question 21:

$$C(A) = span\left\{\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right\}$$

$$R(A) = span\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$$\begin{bmatrix} 1 & -3 & -2 \\ -2 & 6 & 4 \\ 1 & 0 & 3 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix} \rightarrow Row\ Reduce \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} N(A^T) = \vec{0}$$

23) Consider the block-diagonal matrix $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$. Show that $dim\ \mathcal{N}(A) = dim\ \mathcal{N}(B) + dim\ \mathcal{N}(C)$.

Question 23:

Primary if you row reduce A to UTF then you also do the same for B and C. The pivots of A = pivots B +pivots C

Let A be mxn. Let B m1 x n1 and Let C m2 x n2 where n1+ n2 =n.

Rank(A) = rank(B) + Rank(c)

By rank nullity theorem

dim(N(A)) = n - pivots of A

= n - pivots B - pivots C

=n1 + n2 - rank(B) - rank(C)

=n1 - rank(B) + n2 - rank(C) = dim(N(B)) + dim(N(C))

As desired.

Q.E.D. Samsara Goku

25) Let

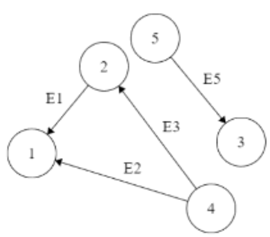
$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

a. Draw a directed graph having A as its incidence matrix

b. Find bases for the four fundamental subspaces; $\mathcal{C}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$, and $\mathcal{N}(A^T)$

Question 25:

A:



B:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C(A) = span\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\} \quad R(A) = span\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}\right\}$$

x_4,x_5 is free

$x_1 = x_4$

$x_2 = x_4$

$x_3 = x_5$

$$N(A) = span\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \rightarrow Row\ Reduce \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_3; x_2 = -x_3\ ; x_4$

$$N(A^T) = span\left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right)$$

26) Let S be the subspace of $\mathbb{R}^4$ given by $S = \{(x_1, x_2, x_3, x_4) : x_1 - 2x_2 + 4x_3 - 3x_4 = 0\}$

a. Find a basis for S and $S^\perp$

b. Find a matrix M such that $S^\perp = \mathcal{N}(M)$

Question 26:

A:

Solve for $x_1 = 2x_2 - 4x_3 + 3x_4$

$S = (2x_2 - 4x_3 + 3x_4, x_2, x_3, x_4)$

Sum as 3 vectors. Linear combination of column vectors for each component

Basis $S = span\left\{\begin{bmatrix}2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-4\\0\\1\\0\end{bmatrix}, \begin{bmatrix}3\\0\\0\\1\end{bmatrix}\right\}$

Let $A = \begin{bmatrix}2 & -4 & 3\\1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ $A^T = \begin{bmatrix}2 & 1 & 0 & 0\\-4 & 0 & 1 & 0\\3 & 0 & 0 & 1\end{bmatrix} \rightarrow Row\ Reduce \rightarrow \begin{bmatrix}1 & 0 & 0 & \frac{1}{3}\\0 & 1 & 0 & -\frac{2}{3}\\0 & 0 & 1 & \frac{4}{3}\end{bmatrix}$

x_4 is free

Basis $S^\perp = span\left\{\begin{bmatrix}-\frac{1}{3}\\\frac{2}{3}\\-\frac{4}{3}\\1\end{bmatrix}\right\}$

B:

Since the a basis of $S^\perp$ is a span, we can make a matrix by combining column vectors where 1 of the is linearly independent from the span of $S^\perp$

$$M = \begin{bmatrix}-\frac{1}{3} & -\frac{2}{3} & \frac{1}{7}\\\frac{2}{3} & \frac{4}{3} & 4\\-\frac{4}{3} & -\frac{8}{3} & \frac{1}{5}\\1 & 1 & 4\end{bmatrix}$$

27) Let A be a 3×4 matrix and B be a 4×5 matrix such that $AB\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^5$. Prove that $rank(A) + rank(B) \leq 4$. You may use the *Rank-Nullity* theorem here.

Question 27:

First we prove that $C(B) \subseteq N(A)$

Let $\vec{x} \in C(B)$ & $\vec{x} \in R^5$

We see that $AB\vec{x} = 0 = A(B\vec{x}) = 0$

Now since $B\vec{x} \in C(B)\ \forall\ \vec{x}$ by the definition of the null space of A we say that $B\vec{x} \in N(A)$ thus $C(B) \in N(A)$

We can now say that $rank(B) \leq \dim N(A)$ Let $A_{m \times n}$ be a matrix

$rank(B) \leq n - rank(A)$

$rank(A) + rank(B) \leq n$

From the problem we see n=4 thus we prove

$rank(A) + rank(B) \leq 4$

As Desired

Q.E.D. Samsara Goku