Saaif Ahmed -Assignment #4

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Problem 7.4

(c) $A_0 = 1; A_1 = 2; A_n = 2A_{n-1} - A_{n-2} + 2$ for $n \geq 2$

Tinker:

0	1	2	3	4	5	6
1	2	5	10	17	26	37

Claim: $A_n = n^2 + 1$

Proof by Strong Induction:

Base Case: $P(0) = 1$. Base Case is True

$P(n) \to P(n+1)$

Direct Proof:

Assume $P(1) \wedge P(2) \ldots \wedge P(n-1) \wedge P(n)$ is True

$P(n+1) = 2\big(P(n)\big) - P(n-1) + 2$

$= 2(n^2+1) - (n^2 - 2n + 2) + 2$

$= 2n^2 + 2 \; - n^2 + 2n \; - 2 + 2$

$= n^2 + 2n + 2$

$= (n+1)^2$

$P(n+1)$ is True

The claim is True

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Problem 7.56

(a) What are *M(0, k)*, *M(n, 1)*. What is *M(n, k)* for $n < 2^k$?

Answer: $M(0,k) = 0$; $M(n,1) = n$

$M(n,k)$ for $n < 2^k = \binom{n}{k} + k - 1$

(b) If the first drop is at floor x, how many drops are needed if: (i) The egg breaks? (ii) The egg survives?

Answer: (i) If the egg breaks at floor x then the you need $x - 1$ trials.

(ii) If the egg survives then you need $n - x$ trials;

(c) Give a recursion for M(n, k). Program your recursion to compute M(n, 3) for n = 7, 8, 9, . . .

$$M(n,k) = \{ 0 \quad n = 0$$
$$\binom{n}{k} + k - 1 \quad n \geq 1\}$$

$n = 7$	$n = 8$	$n = 9$
5	5	5

The base case is always testing an egg on floor 1.
This is most optimal method is by testing the three eggs at floors 1, 4, and 7 for each scenario and then using the one that survives to complete the rest. Worst case in this sense refers to when $M(n,k)$ is largest. At these three n the worst case is always at floor 7.

In the $n = 7$ scenario if the egg at floor 7 breaks then you must check floor 4 and 5. This is $3 + 1 + 1 = 5$

In the $n = 8$ scenario if the egg at floor 7 breaks then you must check floor 4 and 5. This is $3 + 1 + 1 = 5$

In the $n = 9$ scenario if the egg at floor 7 breaks then you must check floor 4 and 5. This is $3 + 1 + 1 = 5$.
However if it survives then you must check floor 8 and 9. This is still $3 + 1 + 1 = 5$

In cases of $n = 7,8,9$ the optimal worst case value of $M(n,k)$ always 5.

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Problem 8.6

Give a recursive definition for the set $A = \{1, 2, 2^2, \ldots\}$, the non-negative powers of 2

Recursive Definition for the set A:

(1) $1 \in A$
(2) $x \in A \rightarrow 2(x) \in A$
(3) Nothing else is in A

(a) Proof that every element is a non-negative power of 2:

Proof by Structural Induction:

Base Case: 1

1 is a non-negative power of 2. Base Case is True.

Constructor:

$x = 2^n$ where $n \in \boldsymbol{N}$

$x \in A \rightarrow 2(x) \in A$

Direct Proof:

Assume $P(n)$ is True

$2(x) = 2(2^n)$
$2(x) = 2^{n+1}$

By definition of the set $\boldsymbol{N}$: $n \in \boldsymbol{N} \rightarrow (n+1) \in \boldsymbol{N}$
So 2^{n+1} is also a non-negative power of 2. Constructor is True

Neither the base case nor the constructor allow a negative power of 2 to be in the set A so the set does not contain a negative power of 2.

(b) Proof that every non-negative power of 2 is in your set.

Proof By Induction:

Base Case: $P(0) = 2^0 = 1$
Base Case is True

$P(n) \rightarrow P(n+1)$

Direct Proof:

Assume $P(n)$ is True

$2(x) = 2(2^n)$
$2(x) = 2^{n+1}$

By definition of the set $\boldsymbol{N}$: $n \in \boldsymbol{N} \rightarrow (n+1) \in \boldsymbol{N}$
So 2^{n+1} is in the set A

For every non negative power of 2 to be in A all of $\boldsymbol{N}$ must be present in the exponents of A. Because the constructor starts of with 2^n and it implies 2^{n+1} all values in $\boldsymbol{N}$ will be in A based off the definition of the set $\boldsymbol{N}$ where if $n \in \boldsymbol{N} \rightarrow (n+1) \in \boldsymbol{N}$.

The Base Case provides the first non-negative power of 2 which is 1.

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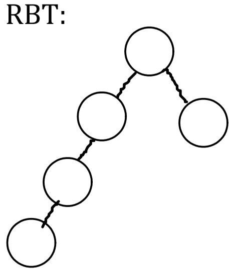
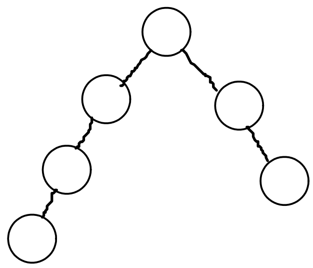
Problem 8.18

(a) Give Examples of RBT and RFBT with 5,6 and 7 vertices.

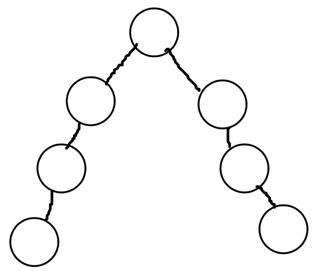
RBT:



5 Vertices

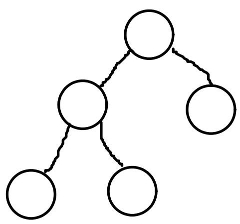


6 Vertices



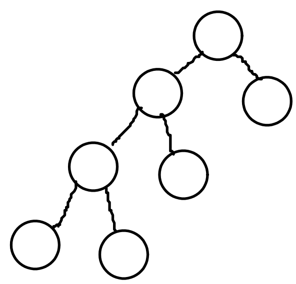
7 Vertices

RFBT:



5 Vertices

6 Vertices (Not possible)



7 Vertices

(b) Prove by structural induction that every RFBT has an odd number of vertices.

Proof by Structural Induction:

Base Case : Single root node vertices $= 1$ <-- odd
Base Case is True

Constructor:
$P(n) = r_1 \wedge r_2$ are an RFBTs and have odd vertices
$P(n+1)$: joining $r_1 \wedge r_2$ to a new root node is an RFBT

$P(n) \rightarrow P(n+1)$
Direct Proof:

Assume $P(n)$ is True.
For vertices in $P(n+1)$
$P(n+1) : (2k+1) + (2k+1) + 1$
$= 4k + 2 + 1$
$= 2(2k+1) + 1$ <--- odd

$P(n+1)$ is True

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Problem 9.3

(b)

$$\sum_{i=1}^{n}\sum_{j=1}^{i} i + j$$

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{i} i + \sum_{j=1}^{i} j\right)$$

$$\sum_{i=1}^{n}\left(i^2 + \frac{1}{2}i(i+1)\right)$$

$$\sum_{i=1}^{n} i^2 + \frac{1}{2}\sum_{i=1}^{n} i^2 + i$$

$$\sum_{i=1}^{n} i^2 + \frac{1}{2}(\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i)$$

$$\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}(\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1))$$

Answer: $\frac{n^3}{2} + n^2 + \frac{n}{2}$

(e)

$$\sum_{i=0}^{n}\sum_{j=0}^{m} 2^{i+j}$$

$$\sum_{i=0}^{n} 2^i \sum_{j=0}^{m} 2^j$$

$$\sum_{i=0}^{n} 2^i \ (2^{m+1} - 1)$$

Answer: $(2^{m+1}-1)(2^{n+1} - 1)$

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Problem 9.37

**Note: I will show the table of values first and then show all no trivial work afterwards.*

	a	b	c	d	e	f
	n^3	2^n	$n!$	$\sum_{i=0}^{n} i^2$	$\sum_{i=1}^{n}\sum_{j=1}^{n} 2^{i+j}$	$\sum_{i=0}^{n} i\sqrt{i}$
i	$g \in O(f)$	$f \in O(g)$	$f \in O(g)$	$g \in O(f)$	$g \in O(f)$	$g \in O(f)$
ii	*both*	$g \in O(f)$	$g \in O(f)$	$g \in O(f)$	*both*	$g \in O(f)$
iii	$f \in O(g)$	$f \in O(g)$	$f \in O(g)$	*both*	$f \in O(g)$	*neither*
iv	*both*	$f \in O(g)$	$g \in O(f)$	$g \in O(f)$	$f \in O(g)$	$g \in O(f)$
v	$g \in O(f)$	$f \in O(g)$	$f \in O(g)$	*both*	$f \in O(g)$	$f \in O(g)$

Section a:

(i) Dominant term is n^2 and therefore n^3 is larger

(iv) Can be written as $2^{\log_2 n^3} * 2^2$ which in Big O Notation drops to just n^3

(v) Polynomials will rise faster than a constant base raised to a logarithm.

Section b:

(iv) Can be written as $2^n * 2^{\log_2 n}$ which reduces to $n2^n$which will always be larger than just 2^n

(v) Can be written as $2^n + 2^{\sqrt{n}}$ in Big O Notation which will always be larger than just 2^n

Section c:

(i)

$n! = n * (n-1) * (n-2) * \cdots * (1)$ <------This product will have n terms

$n^n = n * n * n \ldots * n$ <----- This product will also have n terms

Because the terms in $n^n \geq$ terms in $n!$ and both have the same number of terms. The product of n^n is much larger.

(ii)

$n! = n * (n-1) * (n-2) * \cdots * (1)$ <------This product will have n terms

$n^{\frac{n}{2}} = n * n * n \ldots * n$ <----- This product will have $\frac{n}{2}$ terms

Assuming n is incredibly large, the sheer fact that $n!$ will have twice the amount of terms is enough to claim that it is larger. It does not matter that the individual terms of $n^{\frac{n}{2}}$ are larger. There is more high number multiplication in $n!$ to account for that.

(iii) Can be written as $n * n!$ in Big O Notation. This is always larger than $n!$

(iv) A factorial with a variable base will exceed an exponential term with a constant base whose exponent term is of order 1. So $n!$ is larger.

(v) A factorial with a variable base will not exceed an exponential term with a constant base whose exponent term is order 2 or greater.

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Problem 9.37 (continued)

Section d:

$$\sum_{i=1}^{n} i^2$$

This sum can be expressed as $\frac{1}{6}n(n+1)(2n+1)$ which when expanded will have a highest order term of n^3.

(iv) Can be written $2^{\log_2 n^2}$ which simplifies to n^2 which is less than n^3
(v) Can be written $2^{\log_2 n^3}$ which simplifies to n^3 which is equal to n^3

Section e:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} 2^{i+j}$$
$$\sum_{i=1}^{n} 2^i \sum_{j=1}^{n} 2^j$$
$$\sum_{i=1}^{n} 2^i\,(2^{n+1}-1)$$

$$(2^{n+1}-1)(2^{n+1}-1) = 2^{2n+2} - 2^{n+2} - 1$$

The most significant term from this is 2^{2n} from Big O Notation analysis and this will be used to compare.

(v)

$$\sum_{i=1}^{n}\sum_{j=1}^{i} 2^{i+j}$$
$$\sum_{i=1}^{n} 2^i \sum_{j=1}^{i} 2^j$$
$$\sum_{i=1}^{n} 2^i\,(2^{i+1}-1) = \sum_{i=1}^{n} 2^{(2i+1)} - 2^i = \sum_{i=1}^{n} 2^{2(i+\frac{1}{2})} - \sum_{i=1}^{n} 2^i$$
$$(2)\sum_{i=1}^{n} (2^2)^i - (2^{n+1}-1)$$
$$(2)\left(\frac{1-4^n}{1-4}\right) - (2^{n+1}-1)$$
$$\left(\frac{2-2^{3^n}}{1-4}\right) - (2^{n+1}-1) = \left(\frac{1-2^{3n}}{1-4}\right) - (2^{n+1}-1)$$

Most significant term here is 2^{3n} which is always bigger than 2^{2n}

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Problem 9.37

Section f:

$$\sum_{i=0}^{n} i\sqrt{i}$$

According to the sum of powers rule, this summation will have a highest order term of $\frac{n^{\frac{5}{2}}}{\frac{5}{2}}$ which in Big O Notation is $n^{\frac{5}{2}}$. And this will be used for analysis.

(iii) Because the $g(n)$ can be 0 across multiple values of n there is no real way to compare the two functions

(iv) Can be written $2^{\log_2 n^2}$ which simplifies to n^2 which is less than $n^{2.5}$

(v) Can be written $2^{\log_2 n^3}$ which simplifies to n^3 which is greater than $n^{2.5}$

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Problem 9.44

(a) Give upper and lower bounds and the asymptotic (big-Theta) behavior

$$\sum_{i=1}^{n} \frac{i^2}{i^3+1}$$

Represent as an integral:

$$\int_1^n \frac{i^2}{i^3+1}\, di$$

Let $u = i^3 + 1\,; du = 3i^2 di$

$$\frac{1}{3}\int \frac{du}{u}$$

$$\tfrac{1}{3}(\ln(i^3+1)) \quad \Big|_1^n$$

Lower Bound:

(From 0 to n)

$$\frac{1}{3}\ln(n^3+1) - \left(\frac{1}{3}(\ln 1)\right) = \frac{1}{3}(\ln(\mathrm{n}^3+1))$$

Upper Bound:

(From 1 to $(n+1)^3$)

$$\tfrac{1}{3}\ln\left((n+1)^3+1\right) - \left(\tfrac{1}{3}(\ln 2)\right) = \tfrac{1}{3}\left(\ln\left((\mathrm{n}+1)^3+1\right) - \ln(2)\right)$$