Saaif Ahmed - Assignment #3

Sunday, September 29, 2019 4:03 PM

Problem: 5.10

(j) 3 divides $n^3 + 5n + 6$

Proof By Induction:

Base: P(0) : $0^3 + 5(0) + 6 = 6 = 3(2)$
P(0) is true

$P(n) \rightarrow P(n+1)$
Direct Proof:
Assume P(n) is True

n=3k where k ∈ **Z**

$(3k+1)^3 + 5(3k+1) + 6$
$27k^3 + 27k^2 + 9k + 1 + 15k + 5 + 6$
$3(9k^3) + 3(9k^2) + 3(3k) + 3(4)$ <--divisible by 3

P(n+1) is True

$P(n) \rightarrow P(n-1)$
Direct Proof:
Assume P(n) is True

n=3k where k ∈ **Z**

$(3k-1)^3 + 5(3k-1) + 6$
$27k^3 - 27k^2 + 9k - 1 + 15k - 5 + 6$
$3(9k^3) - 3(9k^2) + 3(3k)$ <-- divisible by 3

P(n-1) is True

P(n) is True $\forall n \in \boldsymbol{Z}$

Assignment #3

Sunday, September 29, 2019 4:28 PM

Problem 5.12

(i) $n! \geq n^n e^{-n}$ for all $n \geq 1$

Proof by Induction:

Base Case: P(1) : $1! \geq 1^1 e^{-1}$

$$1 \geq \frac{1}{e}$$

P(1) is True

$P(n) \rightarrow P(n+1)$

Direct Proof:

$$(n+1)! \geq (n+1)^{n+1} e^{-(n+1)}$$

$$(n+1)(n!) \geq (n+1)^{n+1} * e^{-n} * e^{-1}$$

Knowing that: $\frac{1}{\left(1+\frac{1}{n}\right)^n} \leq e$

$$(n+1)(n!) \geq (n+1)^{n+1} * e^{-n} * \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$(n+1)(n!) \geq (n+1)^{n+1} * e^{-n} * \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$(n+1)(n!) \geq (n+1)^{n+1} * e^{-n} * \left(\frac{n}{n+1}\right)^n$$

$$(n!) \geq (n+1)^n * \left(\frac{n}{n+1}\right)^n * e^{-n}$$

$$n! \geq \frac{(n+1)^n\,(n^n)}{(n+1)^n} * e^{-n}$$

$$n! \geq n^n * e^{-n}$$

$P(n+1)$ is True

Assignment #3

Sunday, September 29, 2019 4:45 PM

Problem 5.18

(a) $H_1 + H_2 + \cdots + H_n = (n+1)H_n - n$

Proof by Induction:

Base Case: P(1) : $H_1 = (1+1)H_1 - 1$

$1 = 2(1) - 1 = 1 = 1$

$P(1)$ is True

$P(n) \to P(n+1)$

Direct Proof:

Assume P(n) is True

$H_1 + H_2 + \cdots + H_n + H_{n+1} = (n+2)H_{n+1} - (n+1)$

$(n+1)H_n - n + H_{n+1}$

$(n+1)\left(H_{n+1} - \frac{1}{n+1}\right) - n + H_{n+1}$

$(n+1)(H_{n+1}) - 1 - n + H_{n+1}$

$n(H_{n+1}) + H_{n+1} - (1+n) + H_{n+1}$

$n(H_{n+1}) + 2(H_{n+1}) - (n+1)$

$H_{n+1}(n+2) - (n+1)$

$P(n+1)$ is True

Assignment #3

Sunday, September 29, 2019 5:12 PM

Problem 5.60

(a) Answer: Perimeter is 42

(b) Prove by induction that the perimeter is even for all n ≥ 1.

Proof by Induction:

Base Case: P(1): $\square$ ← has 4 sides. 4 = 2(2)

P(1) is True

$P(n) \rightarrow P(n+1)$

Direct Proof:

Assume $P(n)$ is True

Given that the established perimeter is some n where n is even ($n = 2k$ where $k \in \boldsymbol{Z}$) , there are 5 scenarios that can occur when adding some new square (s) into the grid. Square s has 4 sides.

s is bordered by 0 edges:
$n + 4 = 2k + 2(2)$ <-- is even

s is bordered by 1 edges:
$(n-1) + (s-1)$
$(n-1) + 4 - 1$
$(n-1) + 3$
$n + 2 = 2k + 2(1)$ <-- is even

s is bordered by 2 edges:
$(n-2) + (s-2)$
$(n-2) + 4 - 2$
$(n-2) + 2$
$n = 2k$ <-- is even

s is bordered by 3 edges:
$(n-3) + (s-3)$
$(n-3) + 4 - 3$
$(n-3) + 1$
$n - 2 = 2k + 2(-1)$ <-- is even

s is bordered by 4 edges:
$(n-4) + (s-4)$
$(n-4) + 0$
$n - 4 = 2k + 2(-2)$ <-- is even

For $P(n+1)$ of adding a new square s the resulting perimeter is always even so $P(n+1)$ is True.

Assignment #3

Sunday, September 29, 2019 5:29 PM

Problem 6.6

(a) $\frac{H_1}{1} + \frac{H_2}{2} \ldots + \frac{H_n}{n} \leq \frac{1}{2}\,(H_n)^2 + 1$. What goes wrong?

Proof by Induction:

Base: P(1) : $1 \leq \frac{1}{2}(1)^2+1$

$1 < 1 + \frac{1}{2}$

P(1) is True

$P(n) \rightarrow P(n+1)$

Direct Proof:

Assume $P(n)$ is True

$\frac{H_1}{1} + \frac{H_2}{2} \ldots + \frac{H_n}{n} + \frac{H_{n+1}}{n+1} \leq \frac{1}{2}\,(H_{n+1})^2 + 1$

Knowing that : $\frac{H_1}{1} + \frac{H_2}{2} \ldots + \frac{H_n}{n} \leq \frac{1}{2}\,(H_n)^2 + 1$

$\frac{1}{2}\,(H_n)^2 + 1 + \frac{H_{n+1}}{n+1}$

$\frac{1}{2}\left(H_{n+1} - \frac{1}{n+1}\right)^2 + 1 + \frac{H_{n+1}}{n+1}$

$\frac{1}{2}\left((H_{n+1})^2 + \frac{1}{(n+1)^2} - 2(\frac{H_{n+1}}{n+1})\right)^1 + 1 + \frac{H_{n+1}}{n+1}$

$\frac{1}{2}(H_{n+1})^2 + \frac{1}{2(n+1)^2} - \frac{H_{n+1}}{n+1} + 1 + \frac{H_{n+1}}{n+1}$

$\frac{1}{2}(H_{n+1})^2 + \frac{1}{2(n+1)^2} + 1 \leq \frac{1}{2}\,(H_n)^2 + 1$

This where this proof fails. There is an extra $\frac{1}{2(n+1)^2}$ term in the proof that we cannot get rid of. In the current state the proof, the Direct Proof is failing to prove the implication.

Assignment #3

Sunday, September 29, 2019 5:50 PM

Problem 6.6 (continued)

(b) $\frac{H_1}{1} + \frac{H_2}{2} \ldots + \frac{H_n}{n} \leq \frac{1}{2}\ (H_n)^2 + \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2})$. Why is this claim stronger?

Proof by Induction:

Base: P(1) : $1 \leq \frac{1}{2}(1)^2 + \frac{1}{2}(1)$

$1 \leq \frac{1}{2} + \frac{1}{2}$

P(1) is True

$P(n) \rightarrow P(n+1)$

Direct Proof:

Assume $P(n)$ is True

$$\frac{H_1}{1} + \frac{H_2}{2} \ldots + \frac{H_n}{n} + \frac{H_{n+1}}{n+1} \leq \frac{1}{2}\ (H_{n+1})^2 + \frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2}\right)$$

Knowing that : $\frac{H_1}{1} + \frac{H_2}{2} \ldots + \frac{H_n}{n} \leq \frac{1}{2}\ (H_n)^2 + \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2})$

$$\frac{1}{2}\ (H_n)^2 + \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}) + \frac{H_{n+1}}{n+1} \leq \frac{1}{2}\ (H_{n+1})^2 + \frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2}\right)$$

$$\frac{1}{2}\left(H_{n+1} - \frac{1}{n+1}\right)^2 + \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}) + \frac{H_{n+1}}{n+1}$$

$$\frac{1}{2}\left((H_{n+1})^2 + \frac{1}{(n+1)^2} - 2(\frac{H_{n+1}}{n+1})\right)^1 + \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}) + \frac{H_{n+1}}{n+1}$$

$$\frac{1}{2}(H_{n+1})^2 + \frac{1}{2(n+1)^2} - \frac{H_{n+1}}{n+1} + \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}) + \frac{H_{n+1}}{n+1}$$

$$\frac{1}{2}(H_{n+1})^2 + \frac{1}{2(n+1)^2} + \frac{1}{2}(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2})$$

$$\frac{1}{2}(H_{n+1})^2 + \frac{1}{2}\left(\frac{1}{(n+1)^2} + \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right) \leq \frac{1}{2}\ (H_{n+1})^2 + \frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2}\right)$$

$P(n+1)$ is True

This claim is stronger because it holds true for all $n \geq 1$. It is also stronger since there is no contradiction unlike the other proof. Furthermore it can stated that $\frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2}\right)$ is actually just a another way to write $\frac{1}{2}\ (H_{n+1})^2$. Which means the RHS of the claim is just $\leq (H_{n+1})^2$. This is a tighter bound for this infinite sum, which leads to greater accuracy.

Assignment #3

Sunday, September 29, 2019 6:07 PM

Problem 6.45

(a) There is a one-way flight between every pair of cities. Prove that there is at least one special city that can be reached from every other city either directly or via one stop.

Proof by Induction:
Base case: $A \rightarrow B \leftarrow C$
B is the special city. So P(1) is True

$P(n) \rightarrow P(n+1)$
Direct Proof:
Assume $P(n)$ is True where n is some graph of nodes with a special city c

There are three cases of how the new node (q) can connect to n

If q connects directly to c then c remains as the special city.

If q connects directly to at least one of the nodes connected to c then c still serves as the special city because q can reach c through one of the "outer" nodes.

If q does not connect to n at all, then every node in n connects to q. Because n contains c every node in n can reach q through c. Therefore q becomes the new special city.