# Linear Algebra HW#9 - Saaif Ahmed - 661925946

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92) Use linear algebra to find the minimum value of the paraboloid, $f(x_1, x_2) = 2x_1^2 + 4x_1x_2 + 3x_2^2 - 4x_1 + 5x_2 + 8$. Be sure to justify why the value you obtain is indeed a minimum.

Question 92:

$A = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

$\nabla Q = \vec{0} \to A\vec{x} - \vec{b} = \vec{0}$

$\frac{1}{2}\begin{bmatrix} 2 & 2 & 4 \\ 2 & 3 & -5 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & -9 \end{bmatrix} \to \vec{x} = \frac{1}{2}\begin{bmatrix} 11 \\ -9 \end{bmatrix}$

$f\left(\frac{11}{2}, -\frac{9}{2}\right) = \frac{121}{2} - \frac{99}{1} + \frac{243}{4} - 22 - \frac{45}{2} + 8 = -19$

Eigen of A

$\lambda^2 - 5\lambda + 2$

$\lambda = \frac{5 \pm \sqrt{17}}{2}$ Thus since all $\lambda > 0$ A is positve definite the $Q(\overrightarrow{x_c}) = -19$ is a minimum.

94) Let $A$ be symmetric, positive definite. Prove that the diagonal entries of $A$ are all strictly positive.

Question 94:

Let A by $n \times n$ such that $\vec{x}^T A\vec{x} > \vec{0}\ \forall\ \vec{x} \in R^n$

Now let us assume that a given diagonal entry of this matrix is not positive.

So let $a_{ii}$ be any diagonal entry of A such that $a_{ii} \leq 0$

Consider the standard basis vector $\overrightarrow{e_i} = \begin{bmatrix} 0 \\ \cdots \\ 1 \\ 0 \end{bmatrix} \in R^n$ with only a 1 at the $i^{th}$ position.

Now we consider

$\overrightarrow{e_i}^T A\overrightarrow{e_i} = 0 * a_{11} * 0 + \cdots + 1 * a_{ii} * 1 + \cdots + 0 * a_{nn} * 0$

If $a_{ii} \leq 0 \to \overrightarrow{e_i}^T A\overrightarrow{e_i} \leq 0$

But this cannot be true since A is positive definite. Therefore a contradiction is found thus all the diagonal entries of A are all strictly positive.

As Desired.

95) Consider the Rayleigh quotient, $R(x_1, x_2) = \dfrac{5x_1^2 + 6x_1x_2 + 13x_2^2}{x_1^2 + x_2^2}$ where $\mathbf{x} \neq \mathbf{0}$. Find a matrix $A$ such that $R = \dfrac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$ and then find the minimum and maximum value of $R$. (Hint: Use the spectral decomposition of $A$)

Question 95:

Let $\vec{x}$ be an arbitrary vector $\in R^2$

$Spec\ decomp(A) = \sum \lambda_i \overrightarrow{q_i}\overrightarrow{q_i}^T$

(the Eigen vectors form a basis for whatever space they span)

$\vec{x} = c_1q_1 + c_2q_2$

$A = \begin{bmatrix} 5 & 3 \\ 3 & 13 \end{bmatrix}$

$(5-\lambda)(13-\lambda) - 9 = 0\,;\ \ \lambda = 14,4$

$\vec{x}^T A\vec{x} = 14c_1^2\overrightarrow{q_1}^T\overrightarrow{q_1} + c_1c_2 4\overrightarrow{q_1}^T\overrightarrow{q_2} + c_1c_2 14\overrightarrow{q_2}^T\overrightarrow{q_1} + 4c_2^2\overrightarrow{q_2}^T\overrightarrow{q_2}$

Eigen vectors are orthonormal and orthogonal

$= 14c_1^2 + 4c_2^2$

$\vec{x}^T\vec{x} = c_1^2 + c_2^2$

$$R = \frac{14c_1^2 + 4c_2^2}{c_1^2 + c_2^2}$$

$$4 = \frac{4(c_1^2 + c_2^2)}{c_1^2 + c_2^2} < \frac{14c_1^2 + 4c_2^2}{c_1^2 + c_2^2} < \frac{14(c_1^2 + c_2^2)}{c_1^2 + c_2^2} = 14$$

Thus the min of R is 4 and the max of R is 14.

97) Find the spectral decomposition of the Hermitian matrix

$$A = \begin{bmatrix} 2 & 3+3i \\ 3-3i & 5 \end{bmatrix}$$

Question 97:

$(2-\lambda)(5-\lambda) - (18) = \lambda^2 - 7\lambda - 8 \to \lambda = 8, -1$

$\begin{bmatrix} 3 & 3+3i \\ 3-3i & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1+i \\ 3-3i & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1+i \\ 0 & 0 \end{bmatrix}$ thus for $\lambda = -1\ \overrightarrow{v_1} = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$

$\begin{bmatrix} -6 & 3+3i \\ 3-3i & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} - \frac{1}{2}i \\ 3-3i & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} - \frac{1}{2}i \\ 0 & 0 \end{bmatrix}$ thus for $\lambda = 8\ \overrightarrow{v_2} = \begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix}$

$$Spec\ Decomp(A) = \begin{bmatrix} \dfrac{-1-i}{\sqrt{2i+1}} & \dfrac{1+i}{2\sqrt{\dfrac{i}{2}+1}} \\ \dfrac{1}{\sqrt{2i+1}} & \dfrac{1}{\sqrt{\dfrac{i}{2}+1}} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} \dfrac{-1-i}{\sqrt{2i+1}} & \dfrac{1}{\sqrt{2i+1}} \\ \dfrac{1+i}{2\sqrt{\dfrac{i}{2}+1}} & \dfrac{1}{\sqrt{\dfrac{i}{2}+1}} \end{bmatrix}$$

98) Let $A^* = A$ and show that $\mathbf{x}^* A\mathbf{x} \in \mathbb{R}$ for all $\mathbf{x} \in \mathbb{C}^n$.

Question 98:

Consider both

$< \vec{x}, A\vec{x} > = \vec{x}^* A\vec{x} \in C$ where $\vec{x} \in C$

and

$< A\vec{x}, \vec{x} > = (A\vec{x})^* \vec{x} \rightarrow \vec{x}^* A^* \vec{x} = \vec{x}^* A\vec{x}$

As such we can also say

$< Ax, x > = \overline{< x, Ax >}$

and

$< x, Ax > = \overline{< x, Ax >}$

Thus we've showed that the conjugate of a complex number is that same complex number

And this is only possible if that number is real.

Thus $\vec{x}^* A\vec{x} \in \mathrm{R}\ \forall\ \vec{\mathrm{x}} \in \mathrm{C}$