Linear Algebra HW#4 - Saaif Ahmed - 661925946

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34) Let $L : \mathbb{P}_3 \to \mathbb{P}_3$ be defined by $L[f] = D[(2x+3)f]$ where $D = d/dx$ is the differential operator.

a. Show that L is a linear operator

b. Find $[L]$ with respect to the standard basis $\{1, x, x^2, x^3\}$ and show that L is onto. (Think dimensionality)

c. Use the above result to solve the equation $L[f] = x^2$. You may use a CAS here.

Question 34:

A: $L|f| = \frac{d}{dx}(2xf) + \frac{d}{dx}(3f)$

$= 2f + 2xf' + 3f'$

$= 2f + (2x+3)f'$

Thus L is a linear operator due to the linearity of derivation and multiplication.

B:

$L(1) = [2,0,0,0]$

$L(x) = [3,4,0,0]$

$L(x^2) = 2x^2 + 4x^2 + 6x \to [0,6,6,0]$

$L(x^3) = 2x^3 + 6x^3 + 9x^2 \to [0,0,9,8]$

$$L = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 4 & 6 & 0 \\ 0 & 0 & 6 & 9 \\ 0 & 0 & 0 & 8 \end{bmatrix} = [L]$$

So we see that [L] is by default in upper triangular form. Thus it is full rank. If a matrix is full rank it is onto if the diagonal are not 0 which in this case it is.

Therefore L is onto as desired

QED Samsara Goku

C:

$L[f] = x^2$

$$\begin{bmatrix} 2 & 3 & 0 & 0 & 0 \\ 0 & 4 & 6 & 0 & 0 \\ 0 & 0 & 6 & 9 & 1 \\ 0 & 0 & 0 & 8 & 0 \end{bmatrix} \to use\ a\ CAS$$

Answer: $\begin{bmatrix} \frac{3}{8} \\ -\frac{1}{4} \\ \frac{1}{6} \\ 0 \end{bmatrix}$

36) Given vectors, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, use Gram-Schmidt to fine a corresponsding orthonormal set $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ and show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{q}_1, \mathbf{q}_2\}$.

Question 36:

Let $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ & $q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$u_2 = v_2 - \left(\frac{u_1^T v_2}{u_1^T u_1}\right) u_1$

$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{2} u_1$

$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ thus $q_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$u_3 = v_3 - \frac{u_1^T v_3}{u_1^T u_1} u_1 - \frac{u_2^T v_3}{u_2^T u_2} u_2$

$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ thus $q_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

thus $\{\overrightarrow{q_1}, \overrightarrow{q_2}, \overrightarrow{q_3}\} = \left\{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right\}$

We see that $\frac{1}{\sqrt{2}} v_1 = q_1$ Thus $\overrightarrow{v_1} \in span\ \{\overrightarrow{q_1}\}$

We also see that $q_2 = v_2 - \sqrt{3} q_1$

And since $v_1 \in span\{q_1\}$ therefore $v_2 \in span\ \{q_2\}$

And thus $span\ \{\overrightarrow{v_1}, \overrightarrow{v_2}\} = span\ \{\overrightarrow{q_1}, \overrightarrow{q_2}\}$

37) Assume that $\{\mathbf{u}_1, \dots \mathbf{u}_n\}$ are mutually orthogonal and show that for any $\mathbf{v}$, $\mathbf{v} - \sum_{i=1}^{n} \frac{\mathbf{u}_i^T \mathbf{v}}{\mathbf{u}_i^T \mathbf{u}_i} \mathbf{u}_i$ is orthogonal to each $\mathbf{u}_i$.

Questions 37:

$$< u_j, v - \sum_{i=1}^{N} \frac{u_i^T v}{u_i^T u_i} u_i >$$

By nature of inner products we can say that

$$< u_j, v > - \sum \frac{u_i^T v}{u_i^T u_i} < u_i, u_j >$$

Now $\forall\ i\ , j\ :\ i \neq j$ we have that

$$< u_j, v > - \sum \frac{u_i^T v}{u_i^T u_i} < u_i, u_j > = \vec{0}$$

due to the properties of mutually orthogonal vectors

In the exception case where we have $i = j \rightarrow u_i = u_j$ we have that

$$< u_j, v > - \frac{u_j^T v}{u_j^T u_i} u_j^T u_i = \ u_j^T v \ - \ u_j^T v = \vec{0}$$

Thus we have that for all u_i we have that $< u_j, v - \sum_{i=1}^{N} \frac{u_i^T v}{u_i^T u_i} u_i > = \vec{0}$

Thus it is orthogonal to each u_i