Saaif Ahmed - Assignment #2

Sunday, September 15, 2019 6:15 PM

Problem: 3.53

(a) $\exists x : x^2 = 4$

Let x = 2

$2^2 = 4$

$4 = 4$

Answer: Because 2 ∈ (**N, Q, Z, R**) this claim is true for the sets (**N, Q, R, Z**)

(b) $\exists x : x^2 = 2$

$x^2 = 2$

$x = \pm\sqrt{2}$

Answer: As $\pm\sqrt{2}$ ∈ (**R**) this claim is true only for the set **R**

(c) $\forall x\ (\exists y : x^2 = y)$

Let x = n where n ∈ (**N**)

$x^2 =$ n^2 ∈ (**N**)

Because **N** ⊂ (**Q, R, Z**) y can be in the sets (**N, Q, Z, R**)

Let x = k where k ∈ (**Z**)

$x^2 =$ k^2 ∈ (**N**)

Because **N** ⊂ (**Q, R, Z**) y can be in the sets (**N, Q, Z, R**)

Let x = z where z ∈ (**Q**)

$x^2 =$ z^2 ∈ (**Q**)

Because **Q** ⊂ (**R**) y can be in the sets (**Q, R**)

Let x = r where r ∈ (**R**)

$x^2 =$ z^2 ∈ (**R**)

Because **R** ⊂ (**R**) y can be in the sets (**R**)

Answer: If x ∈ (**N, Z**) then y ∈ (**N, Q, R, Z**). If x ∈ (**Q**) then y ∈ (**Q, R**).
If x ∈ (**R**) then y ∈ (**R**).

(d) $\forall y\ (\exists x : x^2 = y)$

Let y = n where n ∈ (**N**)

Let n = 2

$x^2 = 2$

$x = \pm\sqrt{2}$

Therefore x must be in the set (**R**) if y ∈ (**N**)

Let y = z where z ∈ (**Z**)

Let z = -2

$x^2 = -2$

$x = \pm\sqrt{-2}$

Therefore x does not exist in the sets (**N, Q, R, Z**). Furthermore because z ∈ (**Z**) ⊂ (**Q, R**).
There are no domains other than **R** where this claim is true $\forall y$.

Answer: The claim only holds true when y ∈ (**N**) and x ∈ (**R**).

Assignment #2

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Problem: 4.9

(a) ($n^3 + 5$ is odd) $\rightarrow$ (n is even).

Let the predicate $P(n) = n^3 + 5$ is odd
Let the predicate $Q(n) =$ n is even.
Claim : $P(n) \rightarrow Q(n)$

Direct Proof:

Assume P(n) is true
$n^3 + 5 = 2k + 1$
$n^3 = 2k - 4$
$n^3 = 2(k - 2)$ n^3 is even

If n^3 is even that must mean n is even due to the nature of numbers being multiplied
Therefore Q(n) can never be false. The implication is true.

Proof by Contraposition:

Assume Q(n) is false
$n = 2k + 1$
$n^3 = (2k + 1)^3$
$n^3 = 8k^3 + 12k^2 + 6k + 1$
$n^3 + 5 = 2(4k^3) + 2(6k^2) + 2(3k) + 6$ <-- even

If $n = 2k + 1$ then $n^3 + 5$ will always be even. This means that if Q(n) is false then P(n) is always false and thus the implication holds true.

(b) (3 does not divide n) $\rightarrow$ (3 divides $n^2 + 2$).

Let the predicate $P(n) =$ 3 does not divide n
Let the predicate $Q(n) =$ 3 divides $n^2 + 2$
Claim : $P(n) \rightarrow Q(n)$

Direct Proof:

Assume P(n) is true
n = 3k +1 where k ∈ ($\mathbf{Z}$)
$n^2 + 2 = (3k + 1)^2 + 2$
$n^2 + 2 = (9k^2 + 6k + 1) + 2$
$n^2 + 2 = (9k^2 + 6k + 3)$
$n^2 + 2 = 3(3k^2 + 2k + 1)$ <-- contains a factor of 3

or

n = 3k +2 where k ∈ ($\mathbf{Z}$)
$n^2 + 2 = (3k + 2)^2 + 2$
$n^2 + 2 = (9k^2 + 12k + 4) + 2$
$n^2 + 2 = (9k^2 + 12k + 6)$
$n^2 + 2 = 3(3k^2 + 4k + 2)$ <-- contains a factor of 3

In either case the resultant value for $n^2 + 2$ will contain a factor of 3 and therefore be divisible by 3. Therefore Q(n) can never be false and thus the implication is true.

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Problem: 4.9

(b continued)

Proof by Contraposition:
Assume Q(n) is false
$n^2 + 2 = 3k + 2$ where $k \in$ **(Z)**
$n^2 = 3k$
$\frac{n^2}{3} = k$

In order for k to remain as an integer 3 must be a factor. This due to 3 being a prime number. Any number when divided by a prime number that is not of its prime factors will result in a rational number instead of an integer. If k is to remain as an integer n^2, and further more n must have 3 as one of its factors and so n divides 3 and P(n) is also false.
The implication holds.

or
$n^2 + 2 = 3k + 1$ where $k \in$ **(Z)**
$n^2 + 1 = 3k$

This assumption from Q(n) is invalid because $n^2 + 1$ is never divisible by 3. So the only assumption we can make is $n^2 + 2 = 3k + 2$ where $k \in$ **(Z).**

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Problem: 4.15

(e) $\forall n \in \boldsymbol{Z} : n^2 + 3n + 4$ is even

Let P(n) = n^2+3n+4 is even

Proof for a general n:
$n \in \boldsymbol{Z}$ n is either even or odd

$n = 2k$
$n^2 + 3n + 4 = 2(k^2) + 2(3k) + 4$ <-- This expression results in an even number

or
$n = 2k + 1$
$n^2 + 3n + 4 = 4k^2 + 10k + 8$
$n^2 + 3n + 4 = 2(2k^2) + 2(5k) + 8$ <-- This expression results in an even number

Answer: In either case of n being an even or odd integer, the expression always comes out as even

(w) If a and b are positive real numbers: $ab < 10{,}000 \rightarrow \min(ab) < 100$

Proof by Contraposition:
Let P(n) =ab<10,000
Let Q(n) = min(ab)<100
Claim : P(n) → Q(n)

Assume Q(n) is false
$a = 100 + z$ where $z \in (\boldsymbol{R})$ and $z \geq 0$
$b = 100 + s$ where $s \in (\boldsymbol{R})$ and $s \geq 0$
$a * b = 10{,}000 + 100s + 100z + zs \geq 10{,}000$

Answer: From this we can see that if Q(n) is false then P(n) will always be false and therefore the implication is true.

Assignment #2

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Problem: 4.26

(b) $\neg(P(n) \rightarrow Q(n))$

Answer: To prove this statement you must find a scenario where P(n) is True and Q(n) is false. To disprove the statement you can find any other combination of True and False such as P(n) is False and Q(n) is true.

(d) $\forall x : ((\forall n : P(n)) \rightarrow Q(x))$

Answer: To prove the statement show that Q(x) is always True. Otherwise you can prove that P(n) is always False if Q(x) can either be True or False. To disprove, find an example of P(n) being True and Q(x) being False.

(f) $\exists x : ((\exists n : P(n)) \rightarrow Q(x))$

Answer: To prove show an example where Q(x) is True. If Q(x) is always False then show that P(n) is also always false. To disprove show an example where Q(x)is False and P(n) is True.

Assignment #2

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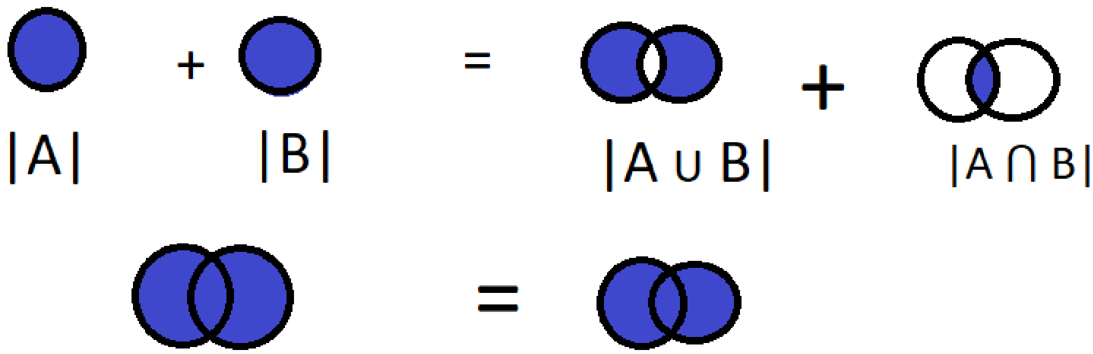
Problem: 4.36

(j) $|A| + |B| = |A \cup B| + |A \cap B|$.



Assignment #2

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Problem: 4.45

(b) $\mathrm{f(n)} = (\mathrm{n}+3)/(\mathrm{n}+1)$

(i) $\mathrm{f(n)} \rightarrow \infty$

Let $\mathrm{n} = \mathrm{C}$

$$\frac{C+3}{C+1} \geq C$$

This statement does not hold for C > 2.

$$\frac{3+3}{3+1} \geq 3$$

$$\frac{6}{4} \geq 3$$

The above statement is False. Because the definition states that $\forall n \geq n_C : (f(n) \geq C)$ the claim here is proven False.

(ii) $\mathrm{f(n)} \rightarrow 1$

Let $\mathrm{n} = \mathcal{E}$

Claim : $\forall n \geq n_{\mathcal{E}} : |f(n) - a| \leq \mathcal{E}$

Proof by Induction:

$\mathrm{P(n)} = |f(n) - a| \leq \mathcal{E}$

$\mathrm{P}(\mathcal{E})$:

$$\frac{\mathcal{E}+3}{\mathcal{E}+1} - 1 \leq \mathcal{E}$$

$$\frac{\mathcal{E}+3-(\mathcal{E})}{\mathcal{E}+1} \leq \mathcal{E}$$

$$\frac{3}{\mathcal{E}+1} \leq \mathcal{E}$$

$\mathrm{P}(\mathcal{E})$ is true.

Implication: $\mathrm{P(n)} \rightarrow \mathrm{P(n+1)}$

Use a Direct Proof:

Assume P(n) is True

$$\frac{(\mathcal{E}+1)+3}{(\mathcal{E}+1)+1} - 1 \leq \mathcal{E} + 1$$

$$\frac{\mathcal{E}+4-(\mathcal{E}+1)}{\mathcal{E}+2} \leq \mathcal{E} + 1$$

$$\frac{3}{\mathcal{E}+2} \leq \mathcal{E} + 1$$

$\mathrm{P}(\mathcal{E}+1)$ is always True and there for P(n+1) is True. We can now confidently say that the claim holds and that $\mathrm{f(n)} \rightarrow 1$ is True.

Assignment #2

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Problem: 4.45

(b) $f(n) = (n+3)/(n+1)$ **(continued)**

(iii) $f(n) \to 2$

Let $n = \mathcal{E}$

Claim : $\forall n \geq n_{\mathcal{E}} : |f(n) - a| \leq \mathcal{E}$

Proof by Induction:

$P(n) = |f(n) - a| \leq \mathcal{E}$

$P(\mathcal{E})$:

$$\frac{\mathcal{E}+3}{\mathcal{E}+1} - 2 \leq \mathcal{E}$$

$$\frac{\mathcal{E}+3-(2\mathcal{E}+2)}{\mathcal{E}+1} \leq \mathcal{E}$$

$$\frac{-\mathcal{E}+1}{\mathcal{E}+1} \leq \mathcal{E}$$

$P(\mathcal{E})$ is true.

Implication: $P(n) \to P(n+1)$

Use a Direct Proof:

Assume $P(n)$ is True

$$\frac{(\mathcal{E}+1)+3}{(\mathcal{E}+1)+1} - 2 \leq \mathcal{E} + 1$$

$$\frac{\mathcal{E}+4-(2\mathcal{E}+4)}{\mathcal{E}+2} \leq \mathcal{E} + 1$$

$$\frac{-\mathcal{E}}{\mathcal{E}+2} \leq \mathcal{E} + 1$$

$P(\mathcal{E}+1)$ is always True and there for $P(n+1)$ is True. We can now confidently say that the claim holds and that $f(n) \to 2$ is True.

Assignment #2

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Problem: 5.7

(f) $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{n}\right)=\frac{1}{n}$

Proof by Induction:

P(n)= $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{n}\right)=\frac{1}{n}$

Base Case: P(2)

$$\left(1-\frac{1}{2}\right)=\frac{1}{2}$$

P(2) is true.

Implication: $P(n) \rightarrow P(n+1)$

Use a Direct Proof:
Assume P(n) is True

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n+1}\right)=\frac{1}{n+1}$$

$$\frac{1}{n}\left(1-\frac{1}{n+1}\right)=\frac{1}{n+1}$$

$$\frac{1}{n}-\left(\frac{1}{(n)(n+1)}\right)=\frac{1}{n+1}$$

$$\frac{(n+1)-1}{n+1(n)}=\frac{1}{n+1}$$

$$\frac{n}{n+1(n)}=\frac{1}{n+1}$$

$$\frac{1}{n+1}=\frac{1}{n+1}$$

P(n+1) is true. The implication is True for all values of $n \geq 2$.