Linear Algebra HW#10 - Saaif Ahmed - 661925946

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102) Let A be any matrix over $\mathbb{R}$ and consider the symmetric products, AA^T and A^TA.

a) Prove that AA^T and A^TA are both postive semi-definite. I.e. $\mathbf{x}^T(AA^T)\mathbf{x}$ and $\mathbf{x}^T(A^TA)\mathbf{x}$ are both non-negative.

b) Prove that AA^T and A^TA have the same non-zero eigenvalues.

Question 102:

A:

$(Ax)^T Ax$

$||(Ax)||^2$ *this is always positive or* 0

$x^T AA^T x$

$(A^T x)^T A^T x$

$||A^T x||^2 \geq 0$

As desired

B:

$AA^T \vec{v} = \lambda \vec{v}$: lambda is a non zero eigen value of AAT

$A^T AA^T \vec{v} = \lambda A^T \vec{v}$

$A^T A(A^T \vec{v}) = \lambda A^T \vec{v}$

Thus $A^T \vec{v}$ is an eigen vector of $A^T A$ with the same eigen value λ

105) Given $A = \begin{bmatrix} 3 & 0 & 4 & -1 & 2 \\ 1 & -1 & 3 & 0 & 1 \end{bmatrix}$, find its singular values and the **dimensions** of its four fundamental subspaces.

Question 105:

$AA^T = \begin{bmatrix} 30 & 17 \\ 17 & 12 \end{bmatrix} \rightarrow \lambda^2 - 42\lambda + 71 = 0\,; \sqrt{\lambda_1} = \sqrt{21 - \sqrt{370}}, \sqrt{\lambda_2} = \sqrt{\sqrt{370} + 21}$

Singular values: $\sqrt{21 - \sqrt{370}}, \sqrt{\sqrt{370} + 21}$

$rank(A) = 2$

$\dim(C(A)) = 2$

$\dim(R(A)) = 2$

$\dim(N(A)) = 3$

$\dim(N(A^T)) = 0$

108) Find and sketch the image of the unit circle under the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ where

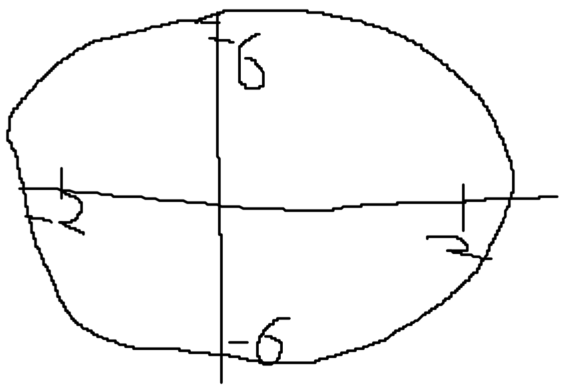
$$\begin{aligned} T\mathbf{e}_1 &= 2\mathbf{e}_1 + \mathbf{e}_2 \\ T\mathbf{e}_2 &= -3\mathbf{e}_1 + 6\mathbf{e}_2 \end{aligned}$$

Question 108:

$$T\begin{bmatrix}e_1\\e_2\end{bmatrix} = \begin{bmatrix}2 & 1\\-3 & 6\end{bmatrix}\begin{bmatrix}e_1\\e_2\end{bmatrix}$$

For a unit circle we want $x_1 = \cos(t)\ ; x_2 = \sin(t)$

$$\begin{bmatrix}2\cos(t) + \sin(t)\\-3\cos(t) + 6\sin(t)\end{bmatrix}$$



110) Find the psuedoinverse of the matrix

$$A = \begin{bmatrix}1 & -1 & 1\\1 & 1 & 2\end{bmatrix}$$

and compute the projections A^+A and AA^+

Question 110:

$$A^TA = \begin{bmatrix}2 & 0 & 3\\0 & 2 & 1\\3 & 1 & 5\end{bmatrix};\ AA^T = \begin{bmatrix}3 & 2\\2 & 6\end{bmatrix}$$

$-\lambda(\lambda-7)(\lambda-2) = 0$

$\lambda = 7$

$$\begin{bmatrix}-5 & 0 & 3\\0 & -5 & 1\\3 & 1 & -2\end{bmatrix} \text{ thus eigen vector is } \begin{bmatrix}\frac{3}{5}\\\frac{1}{5}\\1\end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix}0 & 0 & 3\\0 & 0 & 1\\3 & 1 & 3\end{bmatrix} \text{ thus the eigen vector is } \begin{bmatrix}-\frac{1}{3}\\1\\0\end{bmatrix}$$

$\lambda = 7$

$$\begin{bmatrix}-4 & 2\\2 & -1\end{bmatrix} \text{ thus the eigen vector is } \begin{bmatrix}-\frac{1}{2}\\1\end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} \text{ thus the eigen vector is } \begin{bmatrix}-2\\1\end{bmatrix}$$

$$\frac{1}{\sqrt{7}}\begin{bmatrix}\frac{3}{5}\\\frac{1}{5}\\1\end{bmatrix} * \begin{bmatrix}-\frac{1}{2} & 1\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}-\frac{1}{3}\\1\\0\end{bmatrix} * \begin{bmatrix}-2 & 1\end{bmatrix} = A^+$$

$$A^+ = \frac{1}{\sqrt{7}}\begin{bmatrix}\frac{-9+10\sqrt{14}}{30} & \frac{9\sqrt{2}-5\sqrt{7}}{15\sqrt{2}}\\\frac{-1-10\sqrt{14}}{10} & \frac{\sqrt{2}+5\sqrt{7}}{5\sqrt{2}}\\\frac{-1}{2} & 1\end{bmatrix}$$

$$A^+A = \begin{bmatrix}\frac{9\sqrt{14}+70}{210\sqrt{2}} & \frac{9\sqrt{14}-70}{70\sqrt{2}} & \frac{9}{10\sqrt{7}}\\\frac{\sqrt{14}-70}{70\sqrt{2}} & \frac{3\sqrt{14}+210}{70\sqrt{2}} & \frac{3}{10\sqrt{7}}\\\frac{1}{2\sqrt{7}} & \frac{3}{2\sqrt{7}} & \frac{3}{2\sqrt{7}}\end{bmatrix}$$

$$AA^+ = \begin{bmatrix}\frac{40\sqrt{14}-21}{30\sqrt{7}} & \frac{3\sqrt{14}-20}{15\sqrt{2}}\\-\frac{21+10\sqrt{14}}{15\sqrt{7}} & \frac{5\sqrt{2}+6\sqrt{7}}{15}\end{bmatrix}$$

111) Let M be the Markov matrix

$$M = \begin{bmatrix} 0.4 & 0.9 \\ 0.6 & 0.1 \end{bmatrix}.$$

with steady-state $M^\infty = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$. Prove that M^n converges to M^∞ in the matrix norm by showing that

$$\lim_{n\to\infty} ||M^n - M^\infty|| = 0.$$

Recall that $||A||$ equals the largest singular value of A

Question 111:

Compute M^n at a given n

$$M^1 = \begin{bmatrix} 0.4 & 0.9 \\ 0.6 & 0.1 \end{bmatrix}$$

at some n

$$\lambda = 1, -\frac{1}{2}$$

$$\overrightarrow{v_1} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}; \overrightarrow{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$M^n = \begin{bmatrix} \frac{3}{2} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -0.5^n \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\lim_{n\to\infty} \left\| \begin{bmatrix} \frac{3}{2} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -0.5^n \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -1 \\ 1 & 1 \end{bmatrix}^{-1} - \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \right\|$$

$$\lim_{n\to\infty} \left\| \begin{bmatrix} \frac{3}{2} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -0.5^\infty \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -1 \\ 1 & 1 \end{bmatrix}^{-1} - \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \right\|$$

$$\lim_{n\to\infty} \left\| \begin{bmatrix} \frac{3}{2} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix} - \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \right\|$$

$$\lim_{n\to\infty} \left\| \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} - \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \right\|$$

$$\lim_{n\to\infty} \left\| \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\| = 0$$

As desired.