Linear Algebra HW#7 - Saaif Ahmed - 661925946

Friday, October 29, 2021 12:18 PM

63) Let A be a 2×2 matrix over $\mathbb{R}$ such that $A^6 = I$. Find all possible minimal polynomials of A and characterize the set of all real, 6th order 2×2 matrices. (See the end of the notes on Diagonalization and Similar Matrices)

Question 63:

$x^6 - 1$ is the minimal polynomial. Find factos as

$(x^2 - 1)(x^4 + x + 1)$

Complete the square

$(x + 1)(x - 1)(x^2 + x + 1)(x^2 - x + 1)$ are the minimal polynomials

For $x - 1$ then $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

For $x + 1 = -1(x - 1) = -1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$(x + 1)(x - 1) \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$x^2 + x + 1 \rightarrow \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$

$x^2 - x + 1 \rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

66) Let A and B be similar such that $A = M^{-1}BM$. Prove the following:

i. If λ_i is an eigenvalue of A, then it is also an eigenvalue of B

ii. If $\mathbf{v}_i$ is an eigenvector of A, then $M\mathbf{v}_i$ is an eigenvector of B

iii. A and B have the same characteristic polynomial

iv. A and B have the same minimal polynomial

Hint: For iv. use the fact that $(M^{-1}BM)^n = M^{-1}B^nM$

Question 66:

i: for $\overrightarrow{v_i} \in$ A where $\overrightarrow{v_i}$ is an eigen vector. We have that $A\overrightarrow{v_i} = \lambda_i\overrightarrow{v_i}$

$M^{-1}BM\overrightarrow{v_i} = \lambda_i\overrightarrow{v_i}$

$MM^{-1}BM\overrightarrow{v_i} = M\lambda_i\overrightarrow{v_i}$

$BM\overrightarrow{v_i} = M\lambda_i\overrightarrow{v_i}$

$BM\overrightarrow{v_i} = \lambda_i M\overrightarrow{v_i} \rightarrow B = \lambda_i$

Thus λ_i is an eigen value of B

ii: We know that λ_i is present in both A and B. Which by nature means the eigenvector corrresponding to λ_i, $\overrightarrow{v_i}$ is present in both matrices. We can follow the proof above to see that for an eigenvector $\overrightarrow{v_i} \in$ A that $M\overrightarrow{v_i} \in$ B

iii: $A = M^{-1}BM \rightarrow B = MAM^{-1}$

$A \rightarrow \det(\lambda I - A)$; $B \rightarrow \det(\lambda I - B)$

From above $\det(\lambda I - MAM^{-1}) \rightarrow \det(\lambda MIM^{-1} - MAM^{-1}) \rightarrow \det(M(\lambda IM^{-1} - AM^{-1}))$

$\det(M) * \det\left((\lambda I - A)M^{-1}\right) \rightarrow$

$\det(M) * \det\left((\lambda I - A)\right) * \det(M^{-1})$

$= \frac{1}{\det(M)} * \det(M) * \det\left((\lambda I - A)\right) = \det(\lambda I - A)$

Thus we show that $\det(\lambda I - B) = \det(\lambda I - A)$ thus the characteristic polynomials are the same as desired

iv: Let $f(x)$ be some polynomial and let $f(A) = \vec{0} = a_nA^n + \cdots + a_0I$ representing the miminal polynomial for A.

Now we say that $f(B) = F(MAM^{-1})$

We have that $f(B) = a_n(MAM^{-1})^n + \cdots + a_0I$

We can pull out the $M\&M^{-1}$ from each term and end up with $f(B) = M(a_nA^n + \cdots + a_0 I)M^{-1}$

Simplifies to $f(B) = M\vec{0}M^{-1} = \vec{0}$ from our claim prior.

Thus A and B have the same minimal polynomial as desired.

71) Find the (complex) diagonalization of the matrix $A = \begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix}$

Question 71:

$$\begin{bmatrix} 4-\lambda & -1 \\ 5 & 2-\lambda \end{bmatrix} \rightarrow \lambda^2 - 6\lambda + 13$$

$$\frac{6 \pm \sqrt{36+4(13)}}{2} = 3 \pm 2i$$

Eigen vectors are $\begin{bmatrix} \frac{1}{5} \pm \frac{2i}{5} \\ 1 \end{bmatrix}$

$diag(A) = S\Lambda S^{-1}$

$$S^{-1} = \begin{bmatrix} \frac{5}{4}i & \frac{1}{2} - \frac{1}{4}i \\ -\frac{5}{4}i & \frac{1}{2} + \frac{1}{4}i \end{bmatrix}$$

$$S\Lambda = \begin{bmatrix} -0.2 - \frac{8}{5}i & -0.2 + \frac{8}{5}i \\ 3-2i & 3+2i \end{bmatrix}$$

$$S\Lambda S^{-1} = \begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix}$$

72) Use linear algebra to solve the system of differential equations

$$\begin{aligned} x_1' &= -x_1 + 4x_2 \\ x_2' &= 4x_1 + 5x_2 \end{aligned}$$

with inital values $x_1(0) = -2$, $x_2(0) = 6$

Question 72:

$\vec{x}(0) = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ $A = \begin{bmatrix} -1 & 4 \\ 4 & 5 \end{bmatrix}$

$(-1-\lambda)(5-\lambda) - 16 = \lambda^2 - 4\lambda - 21 = (\lambda - 7)(\lambda + 3)$

For $\lambda = -3 \rightarrow \overrightarrow{v_1} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

For $\lambda = 7 \rightarrow \overrightarrow{v_2} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Guess that $\vec{x} = c_1 e^{-3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Solve for $\vec{c}$ using $\vec{x}$

$$\begin{bmatrix} -2 & \frac{1}{2} \\ 1 & 2 \end{bmatrix}^{-1} = \frac{2}{9} \begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \frac{2}{9} \begin{bmatrix} -7 \\ -10 \end{bmatrix}$$

$$\vec{x} = -\frac{14}{9} e^{-3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \left(-\frac{20}{9}\right) e^{7t} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

76) Suppose that (s_n) is the linear-recursive sequence defined by

$$s_{n+1} = 3s_n - 2s_{n-1}, \quad s_0 = 0, \quad s_1 = 1$$

using the method outlined in the lecture, define an appropriate difference equation and use it to find a closed form for s_n. Then find the limit of the sequence (t_n) where $t_n = s_{n+1}/s_n$ for $n \geq 1$.

Question 76:

$\vec{s_n} = \begin{bmatrix} s_{n+1} \\ s_n \end{bmatrix} A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix} \rightarrow \lambda^2 - 3\lambda + 3 \rightarrow \lambda = 2,1$

$\gamma = 2\,, \psi = 1$

$$A^n s_1 = \begin{bmatrix} \gamma & \psi \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \gamma^n & 0 \\ 0 & \psi^n \end{bmatrix} \begin{bmatrix} \gamma & \psi \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\gamma - \psi} \begin{bmatrix} \gamma & \psi \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \gamma^n(1-\psi) \\ \psi^n(\gamma - 1) \end{bmatrix}$$

$$= \frac{1}{\gamma - \psi} \begin{bmatrix} \gamma^{n+1}(1-\psi) - \psi^{n+1}(1-\gamma) \\ \gamma^n(1-\psi) - \psi^{n+1}(1-\gamma) \end{bmatrix}$$

$$= \frac{1}{\gamma - \psi} \begin{bmatrix} \gamma^{n+2} - \psi^{n+2} \\ \gamma^{n+1} - \psi^{n+1} \end{bmatrix}$$

$$s_n = \frac{2^n - 1^n}{2-1}$$

gamma is bigger so $\gamma^{n-1} = 2^{n-1}$

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \gamma = 2$$