Linear Algebra HW#2 - Saaif Ahmed - 661925946

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13) Determine whether or not the following three vectors in $\mathbb{R}^3$ linearly independent

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix},\ v_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Question 13:

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 5 \\ 1 & 1 & 3 \end{bmatrix} R2 = R2 - 3R1 \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} R3 = R3 - R1 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the reduced matrix is not full rank it is singular and hence its columns and rows are linearly dependent.

14) Consider the system

$$\begin{aligned} 2x + 3y - z &= 3 \\ 4x - 2y + z &= d \\ cx + y &= 7 \end{aligned}$$

a. Find the value of c that makes the system singular

b. Given the value of c in a., find the value of d such that the resulting system has an infinity of solutions

Question 14:

A:

$$\begin{bmatrix} 2 & 3 & -1 & 3 \\ 4 & -2 & 1 & d \\ c & 1 & 0 & 7 \end{bmatrix} R2 \longleftrightarrow R3 \begin{bmatrix} 2 & 3 & -1 & 3 \\ c & 1 & 0 & 7 \\ 4 & -2 & 1 & d \end{bmatrix} R1 = \frac{R1}{2} \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \\ c & 1 & 0 & 7 \\ 4 & -2 & 1 & d \end{bmatrix}$$

$$R2 = R2 - cR1 \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 - \frac{3}{2}c & \frac{c}{2} & 7 - \frac{3}{2}c \\ 4 & -2 & 1 & d \end{bmatrix} R3 = R3 - 4R1 \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 - \frac{3}{2}c & \frac{c}{2} & 7 - \frac{3}{2}c \\ 0 & -8 & 3 & d - 6 \end{bmatrix}$$

$$\frac{R2}{\left(1 - \frac{3}{2}c\right)} \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & \frac{c}{2 - 3c} & \frac{14 - 3c}{2 - 3c} \\ 0 & -8 & 3 & d - 6 \end{bmatrix} R3 = R3 + 8R2 \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & \frac{c}{2 - 3c} & \frac{14 - 3c}{2 - 3c} \\ 0 & 0 & 3 + \frac{8c}{2 - 3c} & \frac{2d - 3dc - 6c + 100}{2 - 3c} \end{bmatrix}$$

$$3 + \frac{8c}{2 - 3c} = 0 \rightarrow \frac{8c}{2 - 3c} = -3 \rightarrow 8c = -3(2 - 3c) \rightarrow 8c = -6 + 9c \rightarrow -c = -6 \rightarrow c = 6$$

Answer: $c = 6$ makes the system singular

B:

$$\frac{2d - 3d(6) - 6(6) + 100}{2 - 3(6)} = 0 \rightarrow 2d - 3d(6) - 6(6) = -100 \rightarrow d = 4$$

Answer: $d = 4$ makes the system have infinity solutions

15) Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 4 & 0 \\ 4 & -7 & 9 \end{bmatrix}$$

find the LU factorization for A and use it to solve the system

$$\begin{aligned} 2x + y + z &= 6 \\ 2x + 4y &= 0 \\ 4x - 7y + 9z &= 42 \end{aligned}$$

Question 15:

$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 4 & 0 \\ 4 & -7 & 9 \end{bmatrix} R2 = R2 - R1 \to \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -1 \\ 4 & -7 & 9 \end{bmatrix} R3 = R3 - 2R1 \to \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -1 \\ 0 & -9 & 7 \end{bmatrix} R3 = R3 + 3R2$

$U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$ $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$

Solve $L\vec{c} = \vec{b} \to \begin{bmatrix} 1 & 0 & 0 & 6 \\ 1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 42 \end{bmatrix}$ $c_1 = 6, c_2 = -6,\ 12 + 18 + c_3 = 42 \to c_3 = 12$

Solve $U\vec{x} = \vec{c} \to \begin{bmatrix} 2 & 1 & 1 & 6 \\ 0 & 3 & -1 & -6 \\ 0 & 0 & 4 & 12 \end{bmatrix}$ $z = 3,$

$3y - 3 = -6, y = -1\ \ 2x - 1 + 3 = 6\ x = 2$

Answer: $x = 2, y = -1, z = 3$

17) Let V be a vector space such that $dimV = n$. Show that any set of n (non-zero) vectors, $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, satisfying $\mathbf{v}_i^T\mathbf{v}_j = 0$ for $i \neq j$ is a basis for V.

Question 17:

Let B be defined arbitrarily as a set of non-zero vectors where $\{v_1, \ldots, v_n : v_i^T v_j = 0 : i \neq j\}$ where $v_i \in V\ \forall\ i$

Because B has n vectors in it then it is clear that $B \subseteq V$

The condition that $v_i^T v_j = 0\ \forall\ i$ and $\forall\ j$ where $j \neq i$ means that each possible combination of vectors are orthogonal, and given that B is a set of non-zero vectors then B is linearly independent.

Proof of above

Linear independence is $\sum c_i v_i = 0$ if and only if $c_i = 0\ \forall\ i$

Consider $v_j^T(\sum c_i v_i)\ \forall\ i\ \&\ j$

Inner product is linear so $c_i\left(\sum v_j^T v_i\right)$

Based on the definition of the set: $\sum v_j^T v_i = 0$ where $i \neq j$ but for when $j = i$ we have that $\sum v_j^T v_i = \left|\left|v_j\right|\right|^2$ so $c_i \left|\left|v_j\right|\right|^2 = 0$. Meaning that c_i has to be $0\ \forall\ i$. Thus it is linearly indpendent.

Now since that B is made of $\{v_1, \ldots, v_n : v \in V\}$, and that $\dim(V) = n$, it clear that $V = span(B)$. This is because $|B| = n = \dim(V)$

For an a finite dimensional vector space V where $\dim(V) = n$ then B being a set of n linearly independent vectors is a basis.

Thus B is a basis of V as desired.

20) Find the general solution to $A\mathbf{x} = \mathbf{b}$ given that

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & 1 & 1 & 3 \\ 3 & 1 & 3 & 7 \end{bmatrix} \quad \text{and } b = \begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$$

Question 20:

$$\begin{bmatrix} 1 & 0 & 2 & 4 & 2 \\ 2 & 1 & 1 & 3 & 6 \\ 3 & 1 & 3 & 7 & 8 \end{bmatrix} R2 = R2 - 2R1 \rightarrow \begin{bmatrix} 1 & 0 & 2 & 4 & 2 \\ 0 & 1 & -3 & -5 & 2 \\ 3 & 1 & 3 & 7 & 8 \end{bmatrix} R3 = R3 - 3R1$$

$$\begin{bmatrix} 1 & 0 & 2 & 4 & 2 \\ 0 & 1 & -3 & -5 & 2 \\ 0 & 1 & -3 & -5 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 4 & 2 \\ 0 & 1 & -3 & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivots = x_1, x_2 Free = x_3, x_4

Set $x_3 \& x_4 = 0$

$x_1 = 2, x_2 = 2$

$$\overrightarrow{x_p} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$R\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & -3 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -2x_3 - 4x_4$

$x_2 = 3x_3 + 5x_4$

Answer: The general solution is $\vec{x} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 5 \\ 0 \\ 1 \end{bmatrix}$