UM 204: QUIZ 4
February 9, 2024

Duration. 15 minutes

Maximum score. 10 points

Problem. Given $A, B \subset \mathbb{R}^n$, let

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

Show that if A and B are compact subsets of $\mathbb{R}^n$ (in the standard metric), then so is $A + B$.

APPROACH 1. E is compact if and only if every sequence has a convergent subsequence.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $A+B$. Then, there exist sequences $\{a_n\}_{n\in\mathbb{N}}$ in A and $\{b_n\}_{n\in\mathbb{N}}$ in B such that $x_n = a_n + b_n$ for all $n \in \mathbb{N}$. Since A is compact, by the above characterization, there is a convergent subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} a_{n_k} = a \in A$. Since $\{b_{n_k}\}_{k\in\mathbb{N}}$ is a sequence in the compact set B, there is a convergent subsequence $\{b_{n_{k_\ell}}\}_{\ell\in\mathbb{N}}$ such that

$$\lim_{\ell\to\infty} b_{n_{k_\ell}} = b \in B.$$

Since $\{a_{n_{k_\ell}}\}_{\ell\in\mathbb{N}}$ is a subsequence of the convergent sequence $\{a_{n_k}\}_{k\in\mathbb{N}}$, we have that

$$\lim_{\ell\to\infty} a_{n_{k_\ell}} = a.$$

Combining the two limits above and using algebra of limits, we obtain a convergent subsequence of $\{a_{n_{k_\ell}} + b_{n_{k_\ell}}\}_{\ell\in\mathbb{N}}$ of $\{x_n\}$ whose limit is $a + b \in A + B$. Since $\{x_n\}_{n\in\mathbb{N}}$ was arbitrary seqeunce in $A+B$, $A+B$ is compact.

APPROACH 2. $E \subset \mathbb{R}^n$ is compact if and only E is closed and bounded.

Since A and B are bounded, there exist $p, q \in \mathbb{R}^n$ and $R, S > 0$ such that $A \subset B(p, R)$ and $B \subset B(q, S)$. Then, for any $a \in A$ and $b \in B$,

$$|a + b - p + q| \leq |a - p| + |b - q| < R + S.$$

Thus, $A + B \subset B(p + q, R + S)$. Thus, $A + B$ is bounded.

Now, let $z \in \mathbb{R}^n$ be a limit point of $A + B$. By the sequential characterization of closures, there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $A + B$ such that $x_n \to z$. Now, $x_n = a_n + b_n$ for some $a_n \in A$ and $b_n \in B$. Since A is bounded $\{a_n\}$ has a convergent subsequence, say $\{a_{n_k}\}_{k\in\mathbb{N}}$. Since A is closed, $a = \lim_{k\to\infty} a_{n_k}$ must belong to A. By the algebra of limits and convergent sequences,

$$\lim_{k\to\infty} b_{n_k} = \lim_{k\to\infty} x_{n_k} - a_{n_k} = z - a.$$

Since B is closed, $b = z - a \in B$. Thus, $z = a + b$ for some $a \in A$ and $b \in B$. Since z was an arbitrary limit point of $A + B$, $A + B$ is closed.