Introduction to Turing Machines

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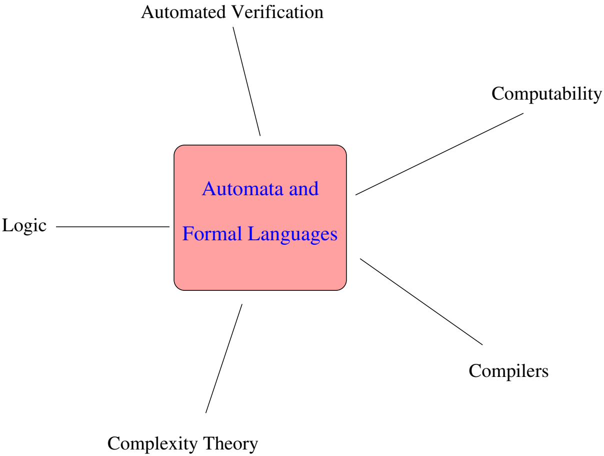
Outline

Role of Automata Theory in other subjects



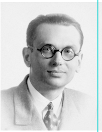
Brief history of logic and computability

David Hilbert 1928: Entscheidungsproblem (deciding validity of FO logic)

Alonzo Church

1936: Undecidability of Entsche
using Lambda–calculus

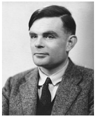
Kurt Godel



1929: Completeness of FO logic
1931: Incompleteness of FO arithmetic
1931: Primitive Recursive Functions

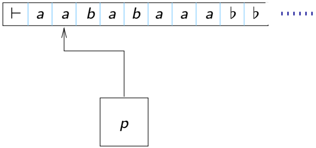
Alan Turing Kleene, Rosser, Scot, Rabin, ...

1936: Undecidability of Entscheidungsproblem using TMs



1930 1935 1940

How a Turing machine works



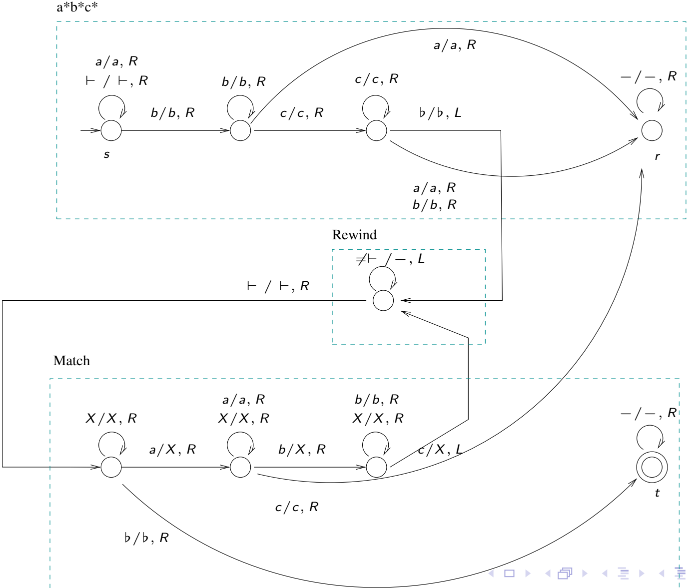
- Finite control
- Tape infinite to the right
- Each step: In current state p, read current symbol under the tape head, say a: Change state to q, replace current symbol by b, and move head left or right.

$$(p, a) \to (q, b, L/R).$$

How a Turing machine works

- Special designated accept state t and reject state r. These states are assumed to be "sink" states.
- TM accepts its input by entering state t.
- TM rejects its input by entering state r.
- TM never falls off the left end of the tape (i.e. it always moves right on seeing '$\vdash$').

Example TM for $a^n b^n c^n$



Exercise: TM for adding numbers in unary

Design a TM that accepts $\{1^m\#1^n\#1^{n+m} \mid m, n \geq 0\}$.

Turning machines more formally

A Turing machine is a structure of the form

$$M = (Q, A, \Gamma, s, \delta, \vdash, \flat, t, r)$$

where

- Q is a finite set of states,
- A is the finite input alphabet,
- Γ is the finite tape alphabet which contains A,
- $s \in Q$ is the start state,
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the (deterministic) transition relation,
- $\vdash \in \Gamma$ is the left-end marker.
- $\flat \in \Gamma$ is the blank tape symbol.
- $t \in Q$ is the accept state.
- $r \in Q$ is the reject state.

Configurations, runs, etc. of a Turing machine

- A configuration of M is of the form $(p, y\flat^\omega, n) \in Q \times \Gamma^\omega \times \mathbb{N}$, which says "$M$ is in state p, with "non-blank" tape contents y, and read head positioned at the n-th cell of the tape.
- Initial configuration of M on input w is $(s, \vdash w\flat^\omega, 0)$.
- 1-step transition of M: If $(p, a) \to (q, b, L)$ is a transition in δ, and $z(n) = a$: then

$$(p, z, n) \overset{1}{\Rightarrow} (q, s_b^n(z), n-1).$$

- Similarly, if $(p, a) \to (q, b, R)$ is a transition in δ, and $z(n) = a$: then

$$(p, z, n) \overset{1}{\Rightarrow} (q, s_b^n(z), n+1).$$

- M accepts w if $(s, \vdash w\flat^\omega, 0) \overset{*}{\Rightarrow} (t, z, i)$, for some z and i.
- M rejects w if $(s, \vdash w\flat^\omega, 0) \overset{*}{\Rightarrow} (r, z, i)$, for some z and i.

Language accepted by a Turing machine

- The Turing machine M is said to halt on an input if it eventually gets into state t or r on the input.
- Note that M may not get into either state t or r on a particular input w. In that case we say M loops on w.
- The language accepted by M is denoted $L(M)$ and is the set of strings accepted by M.
- A language $L \subseteq A^*$ is called recursively enumerable if it is accepted by some Turing machine M.
- A language $L \subseteq A^*$ is called recursive if it is accepted by some Turing machine M which halts on all inputs.

Computability and languages

- Notion of a function $f : \mathbb{N} \to \mathbb{N}$ being "computable" (informally if we can give a "finite recipe" or "algorithm" to compute $f(n)$ for a given n.)
- We say f is computable if we have a TM M that given $\vdash 0^n$ as input, outputs $0^{f(n)}$ on its tape, and halts.

- View f as a language

$$L_f = \{0^n \# 0^{f(n)} \mid n \in \mathbb{N}\}.$$

- Then f is computable iff L_f is recursive.