UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022

QUIZ 3 **FEBRUARY 14, 2022**

PLEASE NOTE the following:

- This quiz must be completed **and scanned** within **15 minutes** of the start-time!
- Your scanned **PDF** file must reach your TA within 3 minutes beyond the above-mentioned duration.

1. Let X be a metric space and let A and B be two non-empty, disjoint **closed** subsets of X. Show that there exist open sets U and V with $A \subset U$ and $B \subset V$ such that $U \cap V = \varnothing$.

Note. The above is related to **some** extent to Problem 6 of Homework 5.

Before studying the solution, we should note that the solution milks a technique introduced in the proof that a compact set is closed. When one had to show that a point in $X \setminus K$, K compact, is an interior point, the compactness of K was crucial. Under the present set-up, when points in $X \setminus A$ and $X \setminus B$ are interior points of the respective sets, compactness is not needed to make that technique work.

Solution. Since $A \cap B = \varnothing$ and B is closed, each $a \in A$ belongs to $X \setminus B$ and there exists a number $r(a) > 0$ such that

$$B(a, r(a)) \cap B = \varnothing. \tag{1}$$

By the same reasoning, for each $b \in B$, there exists a number $r(b) > 0$ such that

$$B(a, r(b)) \cap A = \varnothing. \tag{2}$$

Let us define

$$U := \bigcup\nolimits_{a \in A} B(a, r(a)/2) \quad \text{and} \quad V := \bigcup\nolimits_{b \in B} B(b, r(b)/2).$$

By construction, $A \subset U$ and $B \subset V$. As U and V are unions of open balls, which are open sets, U and V are open sets.

We must show that $U \cap V = \varnothing$. Suppose not. Then there exists a point $x_0 \in U \cap V$. Thus, by definition, there exist points $a_0 \in A$ and $b_0 \in B$ such that

$$x_0 \in B(a_0, r(a_0)/2) \cap B(b_0, r(b_0)/2).$$

By the triangle inequality and the above statement,

$$d(a_0, b_0) \leq d(a_0, x_0) + d(x_0, b_0) < \frac{r(a_0)}{2} + \frac{r(b_0)}{2} \leq \max\{r(a_0), r(b_0)\}.$$

If $r(a_0) \geq r(b_0)$, then the above inequality tells us that $b_0 \in B(a_0, r(a_0))$, which contradicts (1), while if $r(b_0) \geq r(a_0)$, then the above inequality tells us that $a_0 \in B(b_0, r(b_0))$, which contradicts (2). Thus, the assumption that $U \cap V \neq \varnothing$ must be false. $\square$.