UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2022

HOMEWORK 12

Instructor: GAUTAM BHARALI **Assigned: APRIL 4, 2022**

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1. Find a rational number that approximates $\sqrt{10}$ by

- choosing an appropriate interval $[a, b] \subsetneq \mathbb{R}$, taking $f : [a, b] \to \mathbb{R}$ to be $f(t) := \sqrt{t}$, $t \in [a, b]$, and
- letting your approximation be the Taylor polynomial $T_{3,f}(t; x_0)$ — with $x_0 = 9 \in [a, b]$ — evaluated at an appropriate point $x \in [a, b]$,

where your choice of a and b is such that the worst error

$$|\sqrt{10} - T_{3,f}(x; x_0 = 9)|$$

predicted by Taylor's Theorem is the least possible. Now, with your choices of a and b, give an **explicit** upper bound on the above error.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f^{(n)}(x)$ exists at each $x \in \mathbb{R}$ and for each $n \in \mathbb{Z}^+$. Let $a \in \mathbb{R}$ be a point at which $f^{(n)}(a) = n$ for each $n \in \mathbb{N}$ and such that

$$f^{(n)}\Big|_{[0,+\infty)} \text{ is a decreasing function, and } f^{(n)}\Big|_{[0,+\infty)} \geq 0 \text{ for } n = 1, 2, 3, \dots .$$

Find the least $n \in \mathbb{Z}^+$ for which we can be certain that

$$|f(a + 1) - T_{n,f}(a + 1; a)| < 1/500.$$

3. Define the function $f : \mathbb{R} \to \{0, 1\}$ as follows:

$$f(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that $f|_{[a,b]} \notin \mathscr{R}([a, b])$ for any $a < b$.

4. Let $a < b$ be real numbers and suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

(*i*) Let $\alpha, \beta \in \mathbb{R}$ be such that $a \leq \alpha < \beta \leq b$. Show that $f|_{[\alpha,\beta]} \in \mathscr{R}([\alpha, \beta])$.

(*ii*) Let $c \in (a, b)$. By (*i*), we know that $f|_{[a,c]} \in \mathscr{R}([a, c])$ and $f|_{[c,b]} \in \mathscr{R}([c, b])$. Show that

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

5. Let $a < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be a continuous, non-negative function. Suppose $\int_a^b f(x)\,dx = 0$. Prove that $f \equiv 0$.

6. Let $a < b$ be real numbers and let $f, g \in \mathscr{R}([a, b])$. Let p and q be positive real numbers such that $p^{-1} + q^{-1} = 1$. Prove **Hölder's inequality:**

$$\left| \int_a^b fg(x)\,dx \right| \leq \left[\int_a^b |f(x)|^p dx \right]^{1/p} \left[\int_a^b |g(x)|^q dx \right]^{1/q},$$

by completing the outline provided by parts (a)–(c) of Problem 10 in "Baby" Rudin, Chapter 6, taking $\alpha = \mathsf{id}_{[a,b]}$.

7–8. Problems 7 and 15 from "Baby" Rudin, Chapter 6.

The following problems look ahead to the lecture of April 8, so that you get an opportunity to have some discussion on the concept of *uniform convergence* during your last tutorial.

9. Let X and Y be metric spaces and let $\{f_n\}$ be a sequence of Y-valued continuous functions on X. Assume that there is a function $f : X \to Y$ such that $f_n \longrightarrow f$ uniformly. Let p be a limit point of X. Show that for any sequence $\{x_n\} \subset X \setminus \{p\}$ that converges to p, $f_n(x_n) \to f(p)$ as $n \to \infty$.

10. Let X be a metric space and let $\{f_n\}$ and $\{g_n\}$ be sequences of $\mathbb{R}$-valued functions on X.

- (i) If $\{f_n\}$ and $\{g_n\}$ are uniformly convergent, then show that $\{f_n + g_n\}$ is uniformly convergent.
- (ii) By taking X to be a suitable subset of $\mathbb{R}$, construct examples of $\{f_n\}$ and $\{g_n\}$ that are uniformly convergent but $\{f_n g_n\}$ is **not** uniformly convergent.