A graph $G = (V, E)$

$v_1, \dots, v_n \in V$. $\rightarrow$ Vertex set

$(v_i, v_j) \sim E \rightarrow$ Edge set

In a directed graph edges have direction.

$$v_i \longrightarrow v_j$$

(v_i, v_j) is an <u>Arc</u>

<u>Adjacent vertices</u>

Two vertices are said to be adjacent if there is an edge (or arc) connecting them

Neighbor of an undirected graph

$u \in V$ is neighbor of $v \in V$ if $(u, v) \in E$

Neighborhood of an undirected graph

$$N_G(v) = \{ u \mid (u, v) \sim E \}$$

Path

Path is a sequence of vertices with the property that there is an edge(arc) between any two consecutive vertices.

Simple Path

A path that does not repeat vertices are called Simple Path

Cycle:

A simple path that begins and ends at the vertex is called a cycle

Parent and Child

If (v_i, v_j) is an arc

then $\boxed{v_i \text{ is parent of } v_j,}$

and $\boxed{v_j \text{ is parent of } v_i}$

Directed Acyclic graphs (DAGs)

A Directed graph with no cycles
is called DAG.

Graphical Models:

Let the random variable X_i be associated with vertex $v_i \in V$. Then $(X_1, X_2, \ldots, X_d, G(V,E))$ is a graphical model

- Bayesian networks
- Markov networks

Bayesian Networks

Let G be a DAG.

$(X_1, \ldots, X_n, G)$ is a Bayesian network if

$$P(X_1, \ldots, X_N) = \prod_{i=1}^{n} P(X_i \mid Pa(X_i))$$

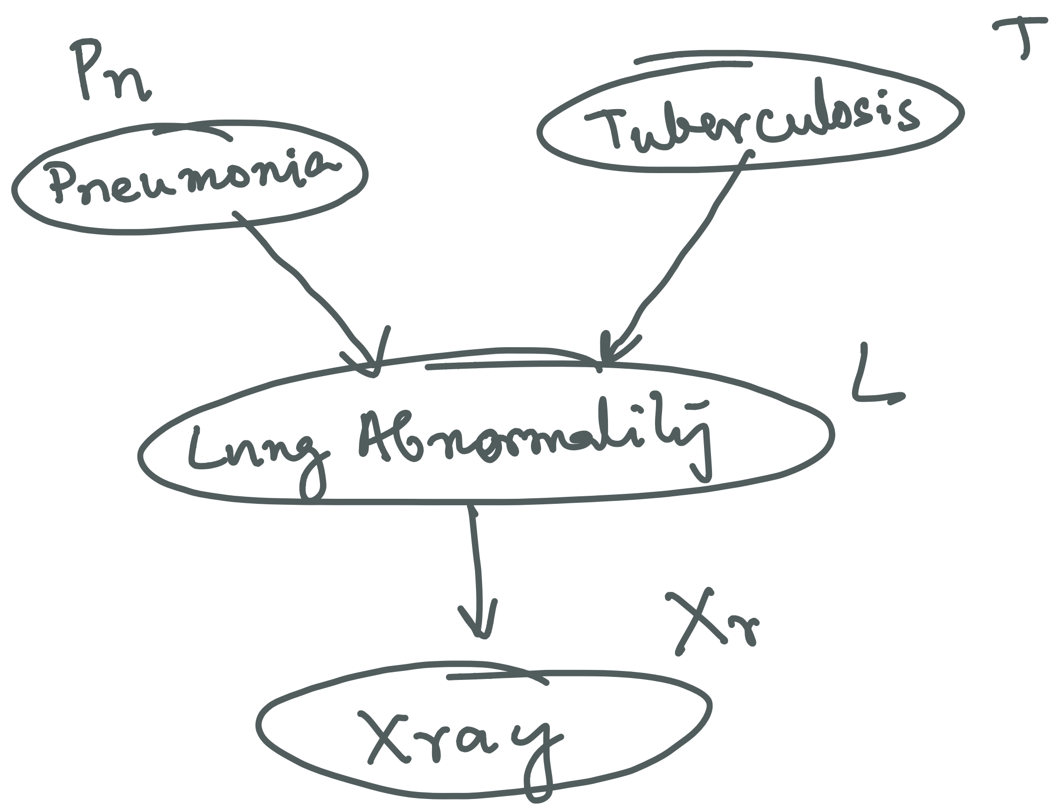
It satisfies the following assumptions

$$X_i \mid X_{-i} = X_i \mid Pa(X_i)$$

$$Pa(X_i) = \left\{ \{X_j \mid j \in \{1, \ldots, i-1\}\} \right\}$$

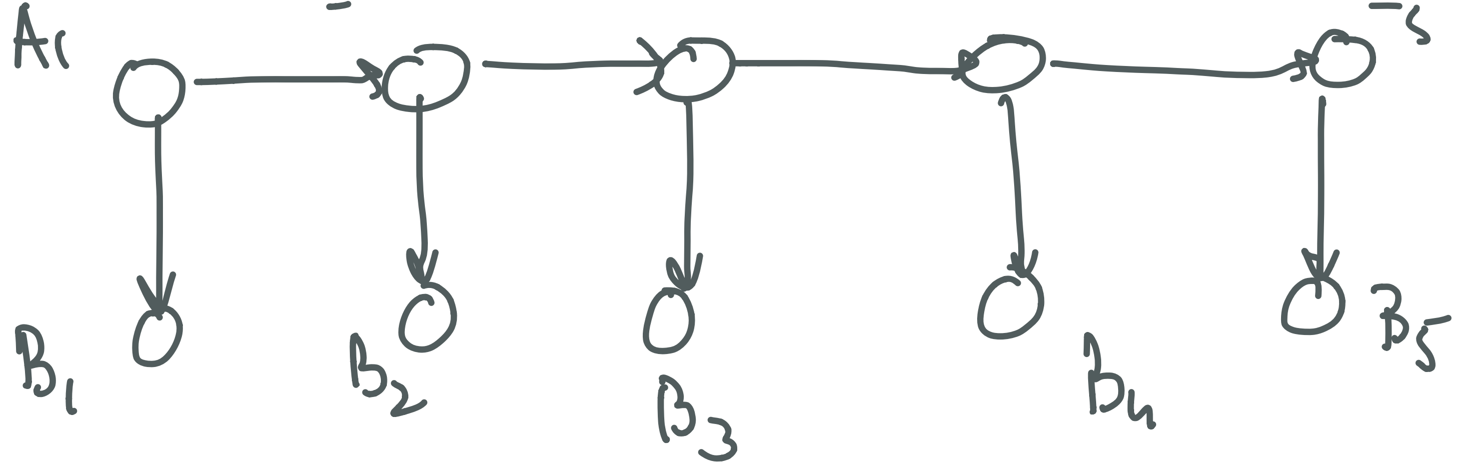
$i \geq 2$

$$X_i \perp\!\!\!\perp \text{Non-desc}(X_i) \mid Pa(X_i)$$



$$P(Pn = p, T = t, L = l, Xr = x)$$

$$= P(Pn = p)\, P(T = t)\, P(L = l \mid Pn = p, T = t)\, P(Xr = x \mid L = l)$$



$$P(A_1, \ldots, A_5, B_1, \ldots, B_5)$$

$$= P(A_1) \prod_{i=2}^{5} P(A_i \mid A_{i-1}) \prod_{i=1}^{5} P(B_i \mid A_i)$$

Conditional Probability Distributions (CPD)

Graphical Models

$(X_1, \ldots, X_n)$ factorizes over G.
if there exist $\psi_c \geq 0$, $c \in C(G)$

$$p(x_1, \ldots, x_d) = \frac{1}{Z} \prod_{c \in C(G)} \psi_c(x_c)$$

Global Markov property

Let $A, B, C \subset V$ be disjoint subsets of V.
Then C separates A, B

$$\Longrightarrow \quad X_A \perp\!\!\!\perp X_B \mid X_C$$

$$P(X_A = x_A, X_B = x_B \mid X_C = x_C) = P(X_A = x_A \mid X_C = x_C)\, P(X_B = x_B \mid X_C = x_C)$$

$A, B, C \subset V$. be disjoint subsets.
C separates A and B if for every path from a vertex $v \in A$ to another vertex $w \in B$ intersects C.

Local Markov Property

$$x_i \mid X_{\sim i} = x_i \mid X_{N(i)}$$

$$X_{\sim i} = \{x_j \mid j \neq i\}$$

$$N(i) = \{ j \mid (i,j) \in E \}$$

Markov Networks

$$P(X_1, \ldots, X_n) = \frac{1}{Z} \prod_{c \in C} \psi_c(x_c)$$

$C \rightarrow$ Maximal Clique

$$P(X_i \mid X_{-i}) = P(X_i \mid X_{N(i)})$$

Hammersly Clifford theorem

If $(X_1, X_2, \ldots, X_d)$ satisfies the local Markov property w.r.t G, then it factorizes over G, provided

$P(X_1 = x_1, \ldots, X_d = x_d) > 0$ for all x