UM 204 (WINTER 2024) - WEEK 10

In general, convergence of $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} b_n$ does not suffice to grant the convergence of the Cauchy product.

Example. Let $a_n = b_n = (-1)^n/n$, $n \in \mathbb{N}_+$. Then, the product is

$$\sum_{n\in\mathbb{N}_+} \sum_{k=1}^{n} \frac{(-1)^{n+1}}{\sqrt{(n+1-k)(k)}}.$$

Now, note that

$$\left|\sum_{k=1}^{n} \frac{(-1)^{n+1}}{\sqrt{(n+1-k)(k)}}\right| \geq \frac{n}{\sqrt{(n+1)(n)}} \to 1 \neq 0.$$

Thus, by the divergence test the product series does not converge.

Definition 0.1. A series $\sum_{n\in\mathbb{N}} a_n$ of complex numbers is said to be absolutely convergent if $\sum_{n\in\mathbb{N}} |a_n|$ is convergent. The series is conditionally convergent if it is convergent but not absolutely convergent.

We recall, from UM 101, that if a series converges absolutely, then it converges.

Theorem 0.2 (Mertens)**.** *Suppose*

(1) $\sum_{n\in\mathbb{N}} a_n$ *is absolutely convergent,*
(2) $\sum_{n\in\mathbb{N}} a_n = A$,
(3) $\sum_{n\in\mathbb{N}} b_n = B$.

Then the product series

$$\sum_{n\in\mathbb{N}} \left(\sum_{j=0}^{n} a_j b_{n-j}\right)$$

converges to AB.

Proof. Let $\{A_n\}_{n\in\mathbb{N}}$, $\{B_n\}_{n\in\mathbb{N}}$ and $\{C_n\}_{n\in\mathbb{N}}$ be the sops of the three series, respectively. Note that

$$\begin{aligned} C_n &= a_0b_0 + (a_0b_1 + a_1b_0) + \cdots + (a_0b_n + \cdots + a_nb_0) \\ &= a_0B_n + a_1B_{n-1} + \cdots + a_nB_0 \\ &= a_0(B + B_n - B) + \cdots + a_n(B + B_0 - B) \\ &= A_nB + a_0\beta_n + \cdots + a_n\beta_0, \end{aligned}$$

where $\beta_n = B_n - B$ for all $n \in \mathbb{N}$. Since, $\lim_{n\to\infty} A_n B = AB$, it suffcies to show that $\lim_{n\to\infty}(a_0\beta_n + \cdots + a_n\beta_0) = 0$. This is similar to a problem we have faced before, where the number of summands is increasing as n increases. We use a similar trick of truncating the sum as before.

Let $\alpha = \sum_{n\in\mathbb{N}} |a_n|$. Since $\lim_{n\to\infty} \beta_n = 0$, given $\varepsilon > 0$, there is an $m \in \mathbb{N}$ such that $|\beta_n| < \varepsilon$ for all $n \geq m$. Now, for $n \geq m$,

$$|a_0\beta_n + \cdots + a_n\beta_0| \leq |\beta_0 a_n + \beta_1 a_{n-1} + \cdots + \beta_m a_{n-m}| + \varepsilon\alpha.$$

Now, for a fixed m, let $n \to \infty$. Then, $a_n, ..., a_{n-m} \to 0$, so we obtain that for $n \geq m$,

$$|a_0\beta_n + \cdots + a_n\beta_0| \leq \varepsilon\alpha$$

Since $\varepsilon > 0$ was arbitrary, we are done. $\square$

Theorem 0.3 (Abel)**.** *If $\sum_{n\in\mathbb{N}} a_n$, $\sum_{n\in\mathbb{N}} b_n$ and the product series converge to A, B and C, respectively, then $C = AB$.*

Proof. Later! $\square$

0.1. **Rearrangements.** Conditionally convergent series have a surprising feature. The series can be "rearranged" to obtain any sum!

Example. Consider

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

Let s denotes its sum. It is clear that

$$s < 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

Now, consider the rearrangement

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots.$$

We claim that it converges. Exercise. Say the limit is ℓ. Let $\{S_n\}_{n\in\mathbb{N}}$ denote its sops. Then,

$$S_{3k} = \frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} > 0.$$

Thus, $\{S_{3k}\}_{k\in\mathbb{N}_+}$ is strictly increasing sequence. Thus, $\ell = \lim_{k\to\infty} S_{3k} > S_3 = \frac{5}{6} = s$.

This cannot happen with the following series

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

Definition 0.4. Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijective function. Let $b_n = a_{f(n)}$ for every $n \in \mathbb{N}$. Then, $\sum_{n\in\mathbb{N}} b_n$ is a rearrangement of $\sum_{n\in\mathbb{N}} a_n$.

Theorem 0.5. *Let $\sum_{n\in\mathbb{N}} a_n$ be an absolutely convergent series with limit L. Then, every rearrangement converges to L.*

Proof. Let $\{A_n\}_{n\in\mathbb{N}}$ denote the sops of the given series. Fix a bijective function $f : \mathbb{N} \to \mathbb{N}$. Let $\{B_n\}$ denote the sops of the rearrangement $\sum_{n\in\mathbb{N}} a_{f(n)}$.

Let $\varepsilon > 0$. By absolute convergence, there is an N_1 such that

$$\sum_{j=p}^{q} |a_j| < \varepsilon/2 \quad \forall p, q \geq N_1. \tag{0.1}$$

Now choose $N_2 = \max\{f^{-1}(0), ..., f^{-1}(N_1)\}$, so that $0, 1, ..., N_1$ are all contained in $\{f(0), ..., f(N_2)\}$. Now, for any $p \geq N_2$, the terms $a_0, ..., a_{N_1}$ will cancel from the difference $B_p - A_p$, and the difference will be a sum of $a_j' s$, where each j is some integer greater than N_1. Thus, by 0.1 and the triangle inequality,

$$|B_p - A_p| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the two sops must converge to the same sum.

□

We will state, but not prove, the remarkable theorem of Riemann alluded to at the beginning of this section.

Theorem 0.6. *Let $\sum_{n\in\mathbb{N}} a_n$ be a conditionally convergent series. Given $\alpha \leq \beta$ in $\overline{\mathbb{R}}$, there is a rearrangement $\sum_{n\in\mathbb{N}} b_n$ of the given series such that*

$$\liminf_{n\to\infty} B_n = \alpha \qquad \limsup_{n\to\infty} B_n = \beta.$$

where $\{B_n\}_{n\in\mathbb{N}}$ denotes the sops of the rearrangement.

END OF LECTURE 22

1. Functional limits and continuity

1.1. **Definitions.**

We define what it means for a function have a limit as it approaches a limit point of its domain.

Definition 1.1. Let X and Y be metric space. Let $E \subset X$, $f : E \to Y$, and p be a limit point of E. We say that

$$\lim_{x\to p} f(x) = q$$

if there is some $q \in Y$ satisfying: for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$d_Y(f(x), q) < \varepsilon$$

whenever

$$0 < d_X(x, p) < \delta.$$

Remarks. (1) f need not be defined at p.
(2) Even if f is defined at p, q need not coincide with $f(p)$.

It is often useful to work with the following characterization.

Theorem 1.2. *Let X, Y, E, p and f be as above. Then,*

$$\lim_{x \to p} f(x) = q$$

if and only if, for every sequence $\{p_n\}_{n \in \mathbb{N}} \subset E$ such that $\lim_{n\to\infty} p_n = p$ and $p_n \neq p$ for all $n \in \mathbb{N}$, we have that

$$\lim_{n \to \infty} f(p_n) = q.$$

Proof. Suppose $\lim_{x\to p} f(x) = q$. Let $\{p_n\}_{\mathbb{N}} \subset E \setminus \{p\}$ be a sequence such that $\lim_{n\to\infty} p_n = p$. We will show that

$$\lim_{n \to \infty} f(p_n) = q.$$

Let $\varepsilon > 0$. Then, by definition, there is a $\delta > 0$ such that

$$\text{if } x \in B_X(p;\delta) \cap (E \setminus \{p\}), \text{ then } f(x) \in B_Y(q;\varepsilon). \tag{1.1}$$

By the definition of sequential limits, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $p_n \in B_x(p;\delta)$. Moreover, by assumption, $p_n \subset E \setminus \{p\}$ for all $n \in \mathbb{N}$. Thus, by (1.1), for all $n \geq N$, $f(x_n) \in B_y(q;\varepsilon)$. Since $eps > 0$ was arbitrary, the claim holds.

Conversely, suppose for every sequence $\{p_n\}_{n\in\mathbb{N}} \subset E$ such that $\lim_{n\to\infty} p_n = p$ and $p_n \neq p$ for all $n \in \mathbb{N}$, we have that

$$\lim_{n \to \infty} f(p_n) = q.$$

Suppose the limit of f as x approaches p is not q. Then, there is an $\varepsilon > 0$ such that for every $\delta > 0$, there is an $q_\delta \in E \setminus \{p\}$ such that $f(q_\delta) \notin B_Y(q;\varepsilon)$. Let $p_n = q_{1/n}$ for all $n \in \mathbb{N}_+$. Then,

$$0 < d_X(p_n, p) < 1/n \quad \text{but} \quad d_Y(f(p_n), q) \geq \varepsilon.$$

This contradicts the assumptions. ☐

The sequential characterization now grants uniqueness of functional limits and all the algebraic laws of limits: see Definition 4.3 and Theorem 4.4 in Rudin's book (Exercise).

Definition 1.3. Let X and Y be metric spaces. Let $E \subset X$, $f : E \to Y$, and $p \in E$. We say that f is continuous at p if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\text{if } x \in B_X(p;\delta) \cap E, \text{ then } f(x) \in B_Y(f(p);\varepsilon),$$

or equivalently,

$$f\left(B_X(p;\delta) \cap E\right) \subseteq B_Y(f(p);\varepsilon).$$

We say that f is continuous on E if f continuous at every $p \in E$.

Theorem 1.4. *Let X and Y be metric spaces. Let $E \subset X$, $f : E \to Y$, and $p \in E$. Then, f is continuous at p if and only if, for every sequence $\{p_n\} \subset E$ such that $\lim_{n\to\infty} p_n = p$, we have that $\lim_{n\to\infty} f(p_n) = f(p)$.*

Remark. Note that a function is always continuous at isolated points of its domain.

(Exercise) Theorems 4.7, 4.9 and 4.10, and the examples in 4.11 are now simple consequences.

END OF LECTURE 23

1.2. **Continuity and topology.** To define the notion of continuity, one does not really need a notion of metric, but rather an appropriate notion of open (or closed) sets.

Recall that for a function $f : A \to B$ and $E \subset B$, the pre-image of E under f is

$$f^{-1}(E) = \{x \in A : f(x) \in E\}.$$

Theorem 1.5. *Let $f : X \to Y$ be a function between two metric spaces. The following are equivalent.*

(1) *f is continuous on X.*
(2) *For every open set $U \subseteq Y$, $f^{-1}(U)$ is open in X,*
(3) *For every closed set $C \subseteq Y$, $f^{-1}(C)$ is closed in X.*

Proof. We first show that (2) and (3) are equivalent. Suppose (2) holds. Let $C \subseteq Y$ be a closed subset. Let $U = Y \setminus C$. Since U is open, $f^{-1}(U)$ is open in X. But,

$$f^{-1}(C) = f^{-1}(Y \setminus U) = X \setminus f^{-1}(U).$$

Thus, $f^{-1}(C)$ is closed in X. The converse holds by a similar argument.

We now show the equivalence of (1) and (2). Suppose f is continuous on X. Let $U \subseteq Y$ be an open set. we show that $f^{-}(U)$ is open in X. Let $p \in f^{-1}(U)$. Since $f(p) \in U$, there is an $\varepsilon > 0$ such that

$$B_Y(f(p);\varepsilon) \subseteq U.$$

By continuity of f at p, there is a $\delta > 0$ such that

$$f\left(B_X(p;\delta)\right) \subseteq B_Y(f(p);\varepsilon)$$

Thus, using that $A \subseteq f^{-1}(f(A))$ and $f^{-1}(A) \subseteq f^{-1}(B)$ whenever $A \subseteq B$,

$$B_X(p;\delta) \subseteq f^{-1}\left(f\left(B_X(p;\delta)\right)\right) \subseteq f^{-1}\left(B_Y(f(p);\varepsilon)\right) \subseteq f^{-1}(U).$$

Since $p \in f^{-1}(U)$ was arbitrary, $f^{-}(U)$ is open.

Conversely, let $p \in X$. Let $\varepsilon > 0$. Consider the open set $U = B_Y(f(p),\varepsilon)$ in Y. Thus, $f^{-1}\left(B_Y(f(p),\varepsilon)\right)$ is open in X. But $p \in f^{-1}\left(B_Y(f(p),\varepsilon)\right)$. Thus, there is a $\delta > 0$ such that

$$B_X(p;\delta) \subseteq f^{-1}\left(B_Y(f(p),\varepsilon)\right).$$

Thus, using the fact that $f(f^{-1}(A)) \subseteq A$,

$$f\left(B_X(p;\delta)\right) \subseteq f\left(f^{-1}\left(B_Y(f(p),\varepsilon)\right)\right) \subseteq B_Y(f(p);\varepsilon).$$

Since $p \in X$ was arbitrary, f is continuous on X. $\square$

Examples. (1) Continuous functions need not map open sets to open sets. Even if we assume that the function is surjective. For example, consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x(x-1)(x-2)$. Then, $f((0,1)) = (0, 2/2\sqrt{3}]$.

(2) Continuous functions need not map closed sets to closed sets. For example, consider $f : \mathbb{R} \to (-\infty, 1/e]$ given by $f(x) = xe^{-x}$. Then, $f([1,\infty)) = (0, 1/e]$.

Theorem 1.6. *Let f be a continuous function on a compact metric space X. Then, $f(X)$ is compact.*

Examples. (1) Continuous functions need not pull back compact sets to compact sets.

(2) Discontinuous maps may map compact sets to compact sets.

END OF LECTURE 24