Introduction to Context-Free Grammars

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Outline

Example of a context-free grammar: syntax of regular expressions

Syntax of regular expresions over an alphabet $\{a, b\}$:

$$r ::= \emptyset \mid a \mid b \mid r + r \mid r \cdot r \mid r^*$$

Why study Context-Free Grammars?

- Arise naturally in syntax of programming languages, parsing, compiling.
- Characterize languages accepted by Pushdown automata.
- Pushdown automata are important class of system models:
  - They can model programs with procedure calls
  - Can model other infinite-state systems.
- Easier to prove properties of Pushdown languages using CFG's:
  - Pumping lemma
  - Ultimate periodicity
  - PDA = PDA without ϵ-transitions.
- Parsing algo leads to solution to "CFL reachability" problem: Given a finite A-labelled graph, a CFG G, are two given vertices u and v connected by a path whose label is in $L(G)$.

Context-Free Grammars: Example 1

CFG G_1

$$\begin{array}{rcl} S & \rightarrow & aX \\ X & \rightarrow & aX \\ X & \rightarrow & bX \\ X & \rightarrow & b \end{array}$$

Derivation of a string: Begin with S and keep rewriting the current string by replacing a non-terminal by its RHS in a production of the grammar.

Example derivation:

$$S$$

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$$S \quad \Rightarrow aX$$

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Language defined by G, written $L(G)$, is the set of all terminal strings that can be generated by G.

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What is the language defined by G_1 above?

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Example derivation:

$$S \Rightarrow aX \Rightarrow abX \Rightarrow abb.$$

Language defined by G, written $L(G)$, is the set of all terminal strings that can be generated by G.

What is the language defined by G_1 above? $a(a+b)^*b$.

Context-Free Grammars: Example 2

CFG G_2

$$\begin{aligned} S &\rightarrow aSb \\ S &\rightarrow \epsilon. \end{aligned}$$

Example derivation:

$$S$$

Context-Free Grammars: Example 2

CFG G_2

$$
\begin{aligned}
S &\rightarrow aSb \\
S &\rightarrow \epsilon.
\end{aligned}
$$

Example derivation:

$$S \Rightarrow aSb$$

Context-Free Grammars: Example 2

CFG G_2

$$\begin{aligned} S &\rightarrow aSb \\ S &\rightarrow \epsilon. \end{aligned}$$

Example derivation:

$$S \Rightarrow aSb \Rightarrow aaSbb$$

Context-Free Grammars: Example 2

CFG G_2

$$
\begin{aligned}
S &\rightarrow aSb \\
S &\rightarrow \epsilon.
\end{aligned}
$$

Example derivation:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb.$$

Context-Free Grammars: Example 2

CFG G_2

$$\begin{array}{lcl} S & \rightarrow & aSb \\ S & \rightarrow & \epsilon. \end{array}$$

Example derivation:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb.$$

What is the language defined by G_2 above?

Context-Free Grammars: Example 2

CFG G_2

$$\begin{array}{lcl} S & \rightarrow & aSb \\ S & \rightarrow & \epsilon. \end{array}$$

Example derivation:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb.$$

What is the language defined by G_2 above? $\{a^n b^n \mid n \geq 0\}$.

Context-Free Grammars: Example 3

CFG G_3

$$S \rightarrow aSa \mid bSb \mid a \mid b \mid \epsilon.$$

Example derivation:

$$S$$

Context-Free Grammars: Example 3

CFG G_3

$$S \quad \rightarrow \quad aSa \mid bSb \mid a \mid b \mid \epsilon.$$

Example derivation:

$$S \quad \Rightarrow aSa$$

Context-Free Grammars: Example 3

CFG G_3

$$S \quad \rightarrow \quad aSa \mid bSb \mid a \mid b \mid \epsilon.$$

Example derivation:

$$S \quad \Rightarrow aSa \quad \Rightarrow abSba$$

Context-Free Grammars: Example 3

CFG G_3

$$S \quad \rightarrow \quad aSa \mid bSb \mid a \mid b \mid \epsilon.$$

Example derivation:

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What is the language defined by G_3 above?

Context-Free Grammars: Example 3

CFG G_3

$$S \quad \rightarrow \quad aSa \mid bSb \mid a \mid b \mid \epsilon.$$

Example derivation:

$$S \quad \Rightarrow aSa \quad \Rightarrow abSba \quad \Rightarrow abbSbba \quad \Rightarrow abbbba.$$

What is the language defined by G_3 above? Palindromes: $\{w \in \{a, b\}^* \mid w = w^R\}$.

Context-Free Grammars: Example 4

CFG G_4

$$S \quad \rightarrow \quad (S) \mid SS \mid \epsilon.$$

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$$S \quad \rightarrow \quad (S) \mid SS \mid \epsilon.$$

Exercise: Derive "((()())()())".

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S

Context-Free Grammars: Example 4

CFG G_4

$$S \quad \to \quad (S) \mid SS \mid \epsilon.$$

Exercise: Derive "((()())()())".

$$S \quad \Rightarrow (S)$$

Context-Free Grammars: Example 4

CFG G_4

$$S \;\rightarrow\; (S) \mid SS \mid \epsilon.$$

Exercise: Derive "((()())()())".

$$\begin{aligned} S &\Rightarrow (S) \\ &\Rightarrow (SS) \end{aligned}$$

Context-Free Grammars: Example 4

CFG G_4

$$S \quad \rightarrow \quad (S) \mid SS \mid \epsilon.$$

Exercise: Derive "((()())()())".

$$\begin{aligned} S \quad & \Rightarrow (S) \\ & \Rightarrow (SS) \\ & \Rightarrow (SSS) \end{aligned}$$

Context-Free Grammars: Example 4

CFG G_4

$$S \rightarrow (S) \mid SS \mid \epsilon.$$

Exercise: Derive "((()())()())".

$$\begin{aligned} S &\Rightarrow (S) \\ &\Rightarrow (SS) \\ &\Rightarrow (SSS) \\ &\Rightarrow ((S)SS) \end{aligned}$$

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$$S \rightarrow (S) \mid SS \mid \epsilon.$$

Exercise: Derive "((()())()())".

$$\begin{aligned} S &\Rightarrow (S) \\ &\Rightarrow (SS) \\ &\Rightarrow (SSS) \\ &\Rightarrow ((S)SS) \\ &\Rightarrow ((SS)SS) \end{aligned}$$

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$$S \rightarrow (S) \mid SS \mid \epsilon.$$

Exercise: Derive "((()())()())".

$$\begin{aligned} S &\Rightarrow (S) \\ &\Rightarrow (SS) \\ &\Rightarrow (SSS) \\ &\Rightarrow ((S)SS) \\ &\Rightarrow ((SS)SS) \\ &\Rightarrow (((S)S)SS) \end{aligned}$$

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Exercise: Derive "((()())()())".

$$\begin{aligned}
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&\Rightarrow (SS) \\
&\Rightarrow (SSS) \\
&\Rightarrow ((S)SS) \\
&\Rightarrow ((SS)SS) \\
&\Rightarrow (((S)S)SS) \\
&\Rightarrow ((()S)SS) \\
&\Rightarrow ((()(S))SS)
\end{aligned}$$

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CFG G_4

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Exercise: Derive "((()())()())".

$$\begin{aligned}
S &\Rightarrow (S)\\
&\Rightarrow (SS)\\
&\Rightarrow (SSS)\\
&\Rightarrow ((S)SS)\\
&\Rightarrow ((SS)SS)\\
&\Rightarrow (((S)S)SS)\\
&\Rightarrow ((()S)SS)\\
&\Rightarrow ((()(S))SS)\\
&\Rightarrow ((()())SS)\\
&\Rightarrow ((()())(S)S)\\
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\end{aligned}$$

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Exercise: Derive "((()())()())".

$$
\begin{aligned}
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&\Rightarrow ((()(S))SS) \\
&\Rightarrow ((()())SS) \\
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&\Rightarrow ((()())()S) \\
&\Rightarrow ((()())()(S)) \\
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\end{aligned}
$$

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&\Rightarrow ((()S)SS)\\
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&\Rightarrow ((()())SS)\\
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What is the language defined by G_4 above?

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What is the language defined by G_4 above?

Balanced Parenthesis: $w \in \{(,)\}^*$ such that

- $\#_((w) = \#_)(w)$, and
- for each prefix u of w, $\#_((u) \geq \#_)(u)$.

Visualizing balanced parenthesis

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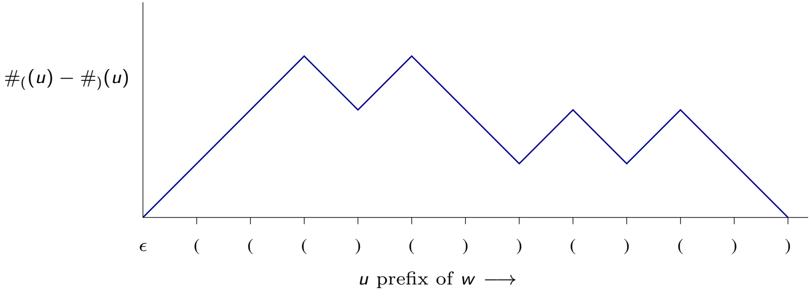

CFGs more formally

A Context-Free Grammar (CFG) is of the form

$$G = (N, A, S, P)$$

where

- N is a finite set of non-terminal symbols
- A is a finite set of terminal symbols.
- $S \in N$ is the start non-terminal symbol.
- P is a finite subset of $N \times (N \cup A)^*$, called the set of productions or rules. A production (X, α) is written as

$$X \to \alpha.$$

Derivations, language etc.

- "α derives β in 0 or more steps, according to G": $\alpha \Rightarrow_G^* \beta$.
- First define $\alpha \stackrel{n}{\Rightarrow} \beta$ inductively:
  - $\alpha \stackrel{0}{\Rightarrow} \alpha$.
  - $\alpha \stackrel{1}{\Rightarrow} \beta$ iff α is of the form $\alpha_1 X \alpha_2$ and $X \to \gamma$ is a production in P, and $\beta = \alpha_1 \gamma \alpha_2$.
  - $\alpha \stackrel{n+1}{\Rightarrow} \beta$ iff there exists γ such that $\alpha \stackrel{n}{\Rightarrow} \gamma$ and $\gamma \stackrel{1}{\Rightarrow} \beta$.
- Define $\alpha \Rightarrow_G^* \beta$ iff there exists $n \in \mathbb{N}$ s.t. $\alpha \stackrel{n}{\Rightarrow} \beta$.
- Sentential form of G: any $\alpha \in (N \cup A)^*$ such that $S \Rightarrow_G^* \alpha$.
- Language defined by G:

$$L(G) = \{w \in A^* \mid S \Rightarrow_G^* w\}.$$

- $L \subseteq A^*$ is called a Context-Free Language (CFL) if there is a CFG G such that $L = L(G)$.

Leftmost derivations

- A leftmost derivation in G is a derivation sequence in which at each step the leftmost non-terminal in the sentential form is re-written.
- Example:

$$\underline{S}$$

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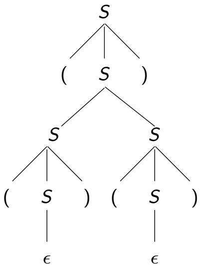
Parse trees

Derivation represented as parse tree:

$$
\begin{aligned}
\underline{S} &\Rightarrow (\underline{S}) \\
&\Rightarrow (\underline{S}S) \\
&\Rightarrow ((\underline{S})S) \\
&\Rightarrow (()\underline{S}) \\
&\Rightarrow (()(\underline{S})) \\
&\Rightarrow (()())
\end{aligned}
$$



- Sentential form can be read off from the leaves of the parse tree in a left-to-right manner.
- Leftmost derivations and parse trees represent eachother.

Proving that a CFG accepts a certain language

CFG G_1

$$
\begin{array}{rcl}
S & \rightarrow & aX \\
X & \rightarrow & aX \\
X & \rightarrow & bX \\
X & \rightarrow & b
\end{array}
$$

Prove that $L(G_1) = a(a+b)^*b$.

Proving that a CFG accepts a certain language

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Prove that $L(G_1) = a(a+b)^*b$.

- Show that $L(G_1) \subseteq L(a(a+b)^*b)$, and $L(a(a+b)^*b) \subseteq L(G_1)$.
- Use induction statement that talks about sentential forms rather than just terminal strings.
- Eg $P(n)$: "If $S \stackrel{n}{\Rightarrow}_{G_1} \alpha$ then α is of the form S, auX, or aub, with $u \in \{a,b\}^*$."
- Follows that all terminal sentential forms are of the form "aub" $\in L(a(a+b)^*b)$.
- For $L(a(a+b)^*b) \subseteq L(G_1)$ use induction statement "If $|u| = n$ then $S \Rightarrow^*_{G_1} auX$."

Proving that a CFG accepts a certain language

CFG G_2

$$\begin{array}{lcl} S & \rightarrow & aSb \\ S & \rightarrow & \epsilon. \end{array}$$

Prove that $L(G_2) = \{a^n b^n \mid n \geq 0\}$.

Proving that a CFG accepts a certain language

CFG G_4

$$S \rightarrow (S) \mid SS \mid \epsilon.$$

Prove that $L(G_4) = \mathrm{BP}$.

Visualizing balanced parenthesis

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- $\#_{(}(w) = \#_{)}(w)$, and
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