UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022

QUIZ 2 **FEBRUARY 7, 2022**

PLEASE NOTE the following:

- This quiz must be completed **and scanned** within **15 minutes** of the start-time!
- Your scanned **PDF** file must reach your TA within 3 minutes beyond the above-mentioned duration.

1. Consider $\mathbb{R}^n$, where $n \in \mathbb{N} \setminus \{0\}$, and endow it with the metric

$$d(x, y) := \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \quad \forall x, y \in \mathbb{R}^n,$$

where we write $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. You can assume **without proof** that d as above is a metric on $\mathbb{R}^n$. Let G be a non-empty open set of $\mathbb{R}^n$ endowed with the metric d. Show that every point in G is a limit point of G; please **justify** this fully.

Note. The above is closely related to Problem 7 of Homework 4.

Solution. Consider an arbitrary point $a = (a_1, \dots, a_n)$. As a is an interior point, there exists $r_a > 0$ such that $B(a, r_a) \subseteq G$. Now, consider any $R > 0$, and write $r = \min\{r_a, R\}$. From the conclusion of a problem in Homework 3

$$\exists q \in \mathbb{Q} \text{ such that } a_1 < q < a_1 + r. \tag{1}$$

Write $x_R := (q, a_2, \dots, a_n)$. Then, by (1):

$$d(a, x_R) = (q - a_1) \in (0, r).$$

Thus, $a \neq x_R$ and

$$x_R \in B(a, r) \subseteq B(a, R),$$
$$x_R \in B(a, r) \subseteq B(a, r_a) \subseteq G.$$

Since an $x_R \neq a$ satisfying the last two relations can be found for each $R > 0$, a is by definition a limit point of G. □

Remark. There are many other choices (clearly) for $x_R \neq a$ such that $x_R \in B(a, R) \cap G$, but the one presented above is the one where the arithmetic for showing that $x_R \in B(a, R) \cap G$ is the **simplest.** Furthermore, given the requirement for a full justification, (1) and the words preceding it are, at the very least, expected for full marks.