Mid-Semester Exam

UM 205: Introduction to Algebraic Structures (Winter 2023-24)
Indian Institute of Science

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February 19, 2024
2–4:30pm

Unless otherwise stated, each question is worth 5 marks.

1. Use the standard ordering $\leq$ on the natural numbers to prove trichotomy of the order for natural numbers.

2. Suppose f_1 and f_2 are functions from X to Y and g is an injective function from Y to Z. Prove that if $g \circ f_1 = g \circ f_2$, then $f_1 = f_2$. Give a counterexample to show that the result fails when g is not injective.

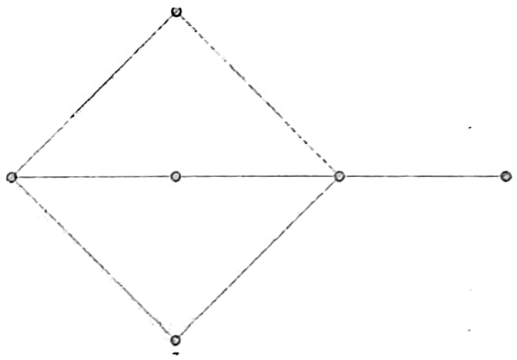
3. Suppose we have defined the integers. Then define addition of rational numbers in the form $a//b$ (as we did in class) and prove that this addition is associative.

4. Evaluate the sum
$$\sum_{k=0}^{n} k^2 \binom{n}{k}.$$

5. **(10 marks)** A permutation π is said to have a *descent* at position i if $\pi_i > \pi_{i+1}$. The set of all descents of π is called the *descent set* of π.
   (a) Find the number of permutations in S_6 with descent set equal to $\{2,4\}$.
   (b) Find the number of permutations in S_6 with descent set being any subset of $\{2,4\}$.

6. Let $S_k(x) = \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} x^n$ be the 'column generating function' of the Stirling numbers of the second kind. Use the recurrence to give an explicit formula for $S_k(x)$.

7. For the graph below, find the number of spanning trees.



8. **(10 marks)** Prove that K_5 minus any edge (the actual edge does not matter) is planar, but K_5 is not planar.