HW-11:

1. Find a rational number that approximates $(7.9)^{1/3}$ and estimate the error of approximation by:

- defining $f(x) := x^{1/3}$ for all $x \in [0, +\infty)$ — where $x^{1/3}$, for $x \geq 0$, denotes the unique non-negative cube-root of x — and letting your approximation be the Taylor polynomial $T_1 f(7.9; \alpha)$ for an appropriate $\alpha > 0$; and
- selecting an appropriate interval $[a, b] \subsetneq [0, +\infty)$ and applying Taylor's Theorem to $f|_{[a,b]}$.

With your choices, what is the best (i.e., smallest) upper bound for

$$|(7.9)^{1/3} - T_1 f(7.9; \alpha)|$$

predicted by Taylor's Theorem?

Note. You may assume — **no explanations** needed — that the function $(0, +\infty) \ni x \longmapsto x^{-p}$ is a decreasing function for any $p > 0$.

Let $\alpha = 8 \to f(\alpha) = 2 \left(= 8^{1/3}\right)$ | $T_1 f(7.9; \alpha) = f(8) + f'(8)(7.9 - 8)$

Also; $f'(t) = \frac{1}{3} t^{-2/3}$ Hence :→ $= 2 + \frac{1}{3} \cdot \frac{1}{4}(-0.1) = 2 - \frac{1}{120} = \boxed{\frac{239}{120}}$ ← Req. Appxn!

$f^{(2)}(t) = \frac{1}{3} \cdot \left(\frac{-2}{3}\right) t^{-5/3} = -\frac{2}{9} t^{-5/3}$

Then; $(7.9)^{1/3} = T_1 f(7.9; \alpha) + \frac{f^{(2)}(c)}{2!}(7.9 - \alpha)^2 = \frac{239}{120} + \frac{f^{(2)}(c)}{2}(0.01)$

$\longrightarrow \left|(7.9)^{1/3} - T_1 f(7.9; \alpha)\right| = \frac{1}{2}|f^{(2)}(c)|(0.01) \leq \frac{1}{2}|f^{(2)}(7.9)|(0.01) = \frac{1}{2} \cdot \left(\frac{2}{9}\right)(7.9)^{-5/3}(0.01)$ ← Upper bd.

2. Let $a < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be a bounded function. Show that for any two partitions $\mathbb{P}_1$ and $\mathbb{P}_2$ on the interval $[a, b]$,

$$L(\mathbb{P}_1, f) \leq U(\mathbb{P}_2, f).$$

Let $\mathbb{P}^*$ be common refinement of $\mathbb{P}_1$ & $\mathbb{P}_2$.

$\Rightarrow \underbrace{L(\mathbb{P}_1, f) \leq L(\mathbb{P}^*, f)}_{\text{Thm on L-35}} \underbrace{\leq U(\mathbb{P}^*, f)}_{(\because m_j \leq M_j)} \underbrace{\leq U(\mathbb{P}_2, f)}_{\text{Same-Thm}} \longrightarrow$ Done!

3. Define the function $f : \mathbb{R} \to \{0, 1\}$ as follows:

$$f(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Fix two real numbers $a < b$. Give an expression for each of the Riemann sums $L(\mathbb{P}, f)$ and $U(\mathbb{P}, f)$. Is $f|_{[a,b]} \in \mathscr{R}([a, b])$?

Let $\mathbb{P}$ be a part'n on $[a,b]$. As $[x_{j-1}, x_j]$ contains a rat'l no., an irrat'l no. $\forall j = 1, \ldots, n$. we have →

$L(\mathbb{P}, f) = \sum_{j=1}^{n} m_j \Delta x_j = \sum_{1}^{n} 0 \cdot \Delta x_j = 0$

$U(\mathbb{P}, f) = \sum_{j=1}^{n} M_j \Delta x_j = \sum_{1}^{n} 1 \cdot \Delta x_j = b - a$

$\boxed{\text{NO!}}$, $f|_{[a,b]} \notin \mathscr{R}([a,b])$ b/c $\overline{\int_a^b} f(x)\,dx = \inf \underset{b-a}{U(\mathbb{P}, f)} \neq \sup \underset{0}{L(\mathbb{P}, f)} = \underline{\int_a^b} f(x)\,dx$

4. Let $a < b$ be real numbers and suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

(i) Let $\alpha, \beta \in \mathbb{R}$ be such that $a \le \alpha < \beta \le b$. Show that $f|_{[\alpha,\beta]} \in \mathscr{R}([\alpha, \beta])$.

(ii) Let $c \in (a, b)$. By (i), we know that $f|_{[a,c]} \in \mathscr{R}([a, c])$ and $f|_{[c,b]} \in \mathscr{R}([c, b])$. Show that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

(i) As $f \in \mathcal{R}([a,b])$; for $\varepsilon > 0$ $\exists$ $\mathbb{P}$ of $[a,b]$ with $U(\mathbb{P},f) - L(\mathbb{P},f) < \varepsilon$

Now consider refinement $\mathbb{P}^*$: $x_0 \le \dots \le x_K = \alpha < \dots < x_m = \beta < \dots < x_n$

and Q partitions $[\alpha,\beta]$ s.t. Q: $x_K \le \dots \le x_m$

$$\Rightarrow U(Q, f|_{[\alpha,\beta]}) - L(Q, f|_{[\alpha,\beta]}) = \sum_{i=k+1}^{m} M_i \Delta x_i - \sum_{i=k+1}^{m} m_i \Delta x_i = \sum_{i=k+1}^{m} (M_i - m_i)\Delta x_i$$

How?

$$\le U(\mathbb{P}^*, f) - L(\mathbb{P}^*, f)$$

$$\le U(\mathbb{P}, f) - L(\mathbb{P}, f)$$

$(\because$ $U(\mathbb{P}^*, f) \le U(\mathbb{P}, f)$
$L(\mathbb{P}, f) \le L(\mathbb{P}^*, f))$

so $f|_{[\alpha,\beta]} \in \mathcal{R}([\alpha,\beta]) \leftarrow < \varepsilon$

(ii) As $f|_{[a,c]} \in \mathcal{R}([a,c])$ & $f|_{[c,b]} \in \mathcal{R}([c,b]) \Rightarrow \exists$ Part^{n}s Q of $[a,c]$ & R of $[c,b]$ s.t.

$$U(f; Q) - L(f; Q) < \frac{\epsilon}{2}, \qquad U(f; R) - L(f; R) < \frac{\epsilon}{2}.$$

Let $P = Q \cup R$. Then

$$U(f; P) - L(f; P) = U(f; Q) - L(f; Q) + U(f; R) - L(f; R) < \epsilon,$$

which proves that f is integrable on $[a, b]$.

Finally, with the partitions P, Q, R as above, we have

$$\begin{aligned}\int_a^b f &\le U(f; P) = U(f; Q) + U(f; R)\\ &< L(f; Q) + L(f; R) + \epsilon\\ &< \int_a^c f + \int_c^b f + \epsilon.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_a^b f &\ge L(f; P) = L(f; Q) + L(f; R)\\ &> U(f; Q) + U(f; R) - \epsilon\\ &> \int_a^c f + \int_c^b f - \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we see that $\int_a^b f = \int_a^c f + \int_c^b f$.

5. Let $a < b$ be real numbers and let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$. Suppose $\text{range}(f) \subseteq [\alpha, \beta]$ and suppose $\phi : [\alpha, \beta] \to \mathbb{R}$ is a continuous fuction. Show that $\phi \circ f$ is Riemann integrable on $[a,b]$.

↳ ϕ is then unif. cont.

6.11 Theorem *Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$, $m \le f \le M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathscr{R}(\alpha)$ on $[a, b]$.*

Needs Pf??, Used directly in Rudin

A continuous function f from a closed bounded interval [a, b] into $\mathbb{R}$ is uniformly continuous

Proof Choose $\varepsilon > 0$. Since ϕ is uniformly continuous on $[m, M]$, there exists $\delta > 0$ such that $\delta < \varepsilon$ and $|\phi(s) - \phi(t)| < \varepsilon$ if $|s - t| \le \delta$ and $s, t \in [m, M]$.

Since $f \in \mathscr{R}(\alpha)$, there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that

Recall → $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$, $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2. \tag{18}$$

Let M_i, m_i have the same meaning as in Definition 6.1, and let M_i^*, m_i^* be the analogous numbers for h. Divide the numbers $1, \ldots, n$ into two classes: $i \in A$ if $M_i - m_i < \delta$, $i \in B$ if $M_i - m_i \ge \delta$.

For $i \in A$, our choice of δ shows that $M_i^* - m_i^* \le \varepsilon$.

For $i \in B$, $M_i^* - m_i^* \le 2K$, where $K = \sup|\phi(t)|$, $m \le t \le M$. By (18), we have

→ Here let $I_1 = \{i : M_i - m_i < \delta\}$
$I_2 = \{i : M_i - m_i \geq \delta\}$

→ if $i \in I_1 \Rightarrow$ for any $x, y \in [x_{i-1}, x_i]$
$\Rightarrow |f(x) - f(y)| \le M_i - m_i < \delta$

So; $|\phi(f(x)) - \phi(f(y))| < \varepsilon$
$\Rightarrow M_i' - m_i' < \varepsilon$

$$\delta \sum_{i \in B} \Delta\alpha_i \le \sum_{i \in B} (M_i - m_i)\, \Delta\alpha_i < \delta^2 \tag{19}$$

so that $\sum_{i \in B} \Delta\alpha_i < \delta$. It follows that

$$\begin{aligned} U(P, h, \alpha) - L(P, h, \alpha) &= \sum_{i \in A} (M_i^* - m_i^*)\, \Delta\alpha_i + \sum_{i \in B} (M_i^* - m_i^*)\, \Delta\alpha_i \\ &\le \varepsilon[\alpha(b) - \alpha(a)] + 2K\delta < \varepsilon[\alpha(b) - \alpha(a) + 2K]. \end{aligned}$$

Since ε was arbitrary, Theorem 6.6 implies that $h \in \mathscr{R}(\alpha)$.

6. Let $a < b$ be real numbers and let $f, g \in \mathscr{R}([a,b])$. Let p and q be positive real numbers such that $p^{-1} + q^{-1} = 1$. Prove **Hölder's inequality:**

$$\left| \int_a^b fg(x)\, dx \right| \le \left[\int_a^b |f(x)|^p dx \right]^{1/p} \left[\int_a^b |g(x)|^q dx \right]^{1/q},$$

by completing the outline provided by parts (a)–(c) of Problem 10 in "Baby" Rudin, Chapter 6, taking $\alpha = \text{id}_{[a,b]}$.

Prove the following statements.

(*a*) If $u \ge 0$ and $v \ge 0$, then

$$uv \le \frac{u^p}{p} + \frac{v^q}{q}. \quad \text{—} (\alpha)$$

Equality holds if and only if $u^p = v^q$.

(*b*) If $f \in \mathscr{R}(\alpha)$, $g \in \mathscr{R}(\alpha)$, $f \ge 0$, $g \ge 0$, and

$$\int_a^b f^p\, d\alpha = 1 = \int_a^b g^q\, d\alpha,$$

then

$$\int_a^b fg\, d\alpha \le 1.$$

(*c*) If f and g are complex functions in $\mathscr{R}(\alpha)$, then

$$\left| \int_a^b fg\, d\alpha \right| \le \left\{ \int_a^b |f|^p\, d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q\, d\alpha \right\}^{1/q}.$$

This is *Hölder's inequality*. When $p = q = 2$ it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

(a) if $v^q = 0 \Rightarrow$ ineq. reduces to $0 \le \frac{1}{p} u^p \to$ True for $u \geq 0$ & equality holds if & only if $u^p = 0$

if $v^q > 0 \to$ let $t = \frac{u^p}{v^q} \geq 0 \Rightarrow$ so ineq. becomes $t^{1/p} \le \frac{1}{p} \cdot t + \frac{1}{q}$ (divide (α) by v^q b.s. & see $uv^{1-q} = t^{1/p}$ as $p = \frac{q}{q-1}$)

$\Rightarrow$ let $f(t) = t^{1/p} - \frac{t}{p} - \frac{1}{q}$ $\left(\Rightarrow f(1) = 1 - \frac{1}{p} - \frac{1}{q} = 0\right)$

So we've to prove for $t \geq 0$; $f(t) \le 0$ & $f(t) = 0$ iff $t = 1$

Use LMVT $\to$ fix $t \geq 0$ $\exists\, \gamma$ b/w 1 and t s.t. $(\gamma \neq 1, t)$

$$f(t) = f(t) - f(1) = f'(\gamma)(t-1) = \left(\frac{1}{p}\gamma^{1/p-1} - \frac{1}{p}\right)(t-1)$$

$$= \frac{1}{p}\left(\gamma^{\frac{1}{p}-1} - 1\right)(t-1) = \frac{1}{p}\left(\gamma^{-1/q} - 1\right)(t-1) \le 0 \quad (\text{for } t \geq 0)$$

and equality holds iff $t = 1$. Hence done!

(b) as $f, g \in \mathcal{R}(\cdot) \Rightarrow fg \in \mathcal{R}$ (thm 6.13, done incls)

also, f^p & $g^q \in \mathcal{R}$ (thm 6.11/probs & $f \geq 0, g \geq 0$)

Now, use ineq. of (a) $\Rightarrow$ $fg \le \frac{f^p}{p} + \frac{g^q}{q} \xrightarrow[\text{“}\int\text{”}]{\text{integrate}} \int_a^b fg(x)\,dx \le \frac{1}{p}\int f^p + \frac{1}{q}\int g^q$

$= \frac{1}{p} + \frac{1}{q} = 1$ ☑

(c) $\left|\int_a^b fg(x)\,dx\right| \le \int_a^b |fg(x)|\,dx = \int_a^b |f(x)||g(x)|\,dx$

if $I = \int_a^b |f|^p \neq 0$ & $J = \int_a^b |g|^q \neq 0 \to$ Use (b) on $\frac{|f|}{c}$ & $\frac{|g|}{d}$ where $c^p = I$, $d^q = J$

$$\Rightarrow \left|\int_a^b fg\right| \le 1 = cd = I^{1/p} J^{1/q} = \left(\left|\int_a^b |f|^p\right|\right)^{1/p} \left(\left|\int_a^b |g|^q\right|\right)^{1/q}$$

On the other hand if one of these intgrals vanishes, say the first since we can always reverse the roles of p and q, then

$$\int_a^b |f|(c|g|)d\alpha \le c^q \frac{1}{q} \int_a^b |g|^q d\alpha$$

for any $c > 0$ and sending $c \to 0$ shows that $\int_a^b |f||g| d\alpha = 0$ so the inequality still holds. □

[or] (c) Put $f = f_1 + if_2$ and $g = g_1 + ig_2$, with f_1, f_2, g_1, g_2 being real functions. By the hypotheses, these real functions are all in $\mathscr{R}(\alpha)$, so that the product

$$fg = (f_1g_1 - f_2g_2) + i(f_1g_2 + f_2g_1) \in \mathscr{R}(\alpha),$$

by Theorem 6.13. Since $|f| = (f_1^2 + f_2^2)^{1/2}$, $|g| = (g_1^2 + g_2^2)^{1/2}$, by Theorem 6.11, we know that $|f|, |g| \in \mathscr{R}(\alpha)$, and so that $|f|\,|g|, |f|^p, |g|^q \in \mathscr{R}(\alpha)$. By Theorem 6.25, we know that

$$\left|\int_a^b fg\,d\alpha\right| \le \int_a^b |fg|\,d\alpha = \int_a^b |f|\,|g|\,d\alpha.$$

Complex functions are indeed vector-valued functions.

Put

$$A = \int_a^b |f|^p\,d\alpha, \qquad B = \int_a^b |f|^p\,d\alpha.$$

If $A \neq 0$ and $B \neq 0$, then

$$\int_a^b \left(\frac{|f|}{A^{1/p}}\right)^p d\alpha = \int_a^b \frac{1}{A}|f|^p\,d\alpha = \frac{1}{A}\int_a^b |f|^p\,d\alpha = 1.$$

Similarly, we have

$$\int_a^b \left(\frac{|g|}{B^{1/q}}\right)^q d\alpha = 1.$$

By the result of part (a), we have

$$\int_a^b \frac{|f|}{A^{1/p}} \cdot \frac{|g|}{B^{1/q}} d\alpha \le 1,$$

~~7~~. 8. **Suppose f is a real function on $(0, 1]$ and $f \in \mathscr{R}$ on $[c, 1]$ for every $c > 0$. Define**

$$\int_0^1 f(x)\,dx = \lim_{c \to 0} \int_c^1 f(x)\,dx$$

if this limit exists (and is finite).

(a) If $f \in \mathscr{R}$ on $[0, 1]$, show that this definition of the integral agrees with the old one.

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

(a) if $f \in \mathscr{R}([0,1]) \Rightarrow f \in \mathscr{R}([0,c])$ & $f \in \mathscr{R}([c,1])$

↳ also that f is bdd. on $[0,1] \Rightarrow |f(x)| \le M \ \forall x \in [0,1]$

Using Thm 6.12 $\rightarrow \left| \int_0^1 f(x)dx - \int_c^1 f(x)dx \right| = \left|\int_0^c f(x)dx\right| \le \int_0^c |f(x)|\,dx \le cM$

$\Rightarrow \lim_{c \to 0} \int_c^1 f(x)dx = \int_0^1 f(x)\,dx$

[OR]

As $f \in \mathscr{R}([0,1]) \rightarrow$ Fix $\varepsilon > 0$; let $M := \sup\{|f(x)| : 0 \le x \le 1\}$.

let $c \in \left(0, \frac{\varepsilon}{4M}\right]$ be fixed.

Consider any partⁿ $\mathbb{P}$ of $[0,1]$ containing c for which $U(\mathbb{P},f) - L(\mathbb{P},f) < \varepsilon/4$. then partⁿ of $[c,1]$ (call $\mathbb{P}'$) formed by points of this partⁿ that lie in this interval certainly satisfy that $U(\mathbb{P}',f) - L(\mathbb{P}',f) < \varepsilon/4$. Moreover, the terms of original upper & lower Riemann sums NOT found in the sums for the smaller interval amount to less than $\varepsilon/4$. So for $c < \frac{\varepsilon}{4M}$ & a suitable partⁿ containing c,

$$\sum M_j \Delta x_j - \varepsilon/4 < \int_0^1 f(x)\,dx < \sum m_j \Delta x_j + \varepsilon/4 \text{ AND}$$

$$\sum M_j' \Delta x_j - \varepsilon/4 < \int_c^1 f(x)\,dx < \sum m_j' \Delta x_j + \varepsilon/4$$

Also that;

$$\left|\sum M_j \Delta x_j - \sum M_j' \Delta x_j\right| < \varepsilon/4 \;\&\; \left|\sum m_j \Delta x_j - \sum m_j' \Delta x_j\right| < \frac{\varepsilon}{4}$$

Combining these ineq. gives $\Rightarrow$ $\left|\int_0^1 f(x)\,dx - \int_c^1 f(x)\,dx\right| < \varepsilon$ ▨

(b) let $f(x) = (-1)^n (n+1)$ for $\frac{1}{n+1} < x \le \frac{1}{n}$; $n = 1,2,\ldots$

then if $\frac{1}{N+1} \le c \le \frac{1}{N}$ we have $\int_c^1 f(x)\,dx = (-1)^N (N+1)\left(\frac{1}{N} - c\right) + \sum_{k=1}^{N-1} \frac{(-1)^k}{k}$

Since $0 \le \frac{1}{N} - c \le \frac{1}{N} - \frac{1}{N+1} = \frac{1}{N(N+1)}$, the 1st term on RHS $\to 0$ as $c \downarrow 0$

So, Integral approaches a limit. $\quad -\underbrace{\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right)}$ → $\ln(1+x)$ with $x=1$ i.e. $\underline{\underline{\ln 2}}$.

However, $\int_c^1 |f(x)|\,dx = (N+1)\left(\frac{1}{N} - c\right) + \sum_{k=1}^{N-1} \frac{1}{k}$,

& in this case 1st term on RHS $\to 0$ as $c \downarrow 0$; while sum tends to ∞.

~~15.~~ 9. Suppose f is a real, continuously differentiable function on $[a, b]$, $f(a)=f(b)=0$, and

$$\int_a^b f^2(x)\,dx = 1.$$

let $u = x \rightarrow du = dx$

$dv = f(x)f'(x)dx \rightarrow V = \int dv = \frac{f^2(x)}{2}$

Prove that

$$\int_a^b xf(x)f'(x)\,dx = -\tfrac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2\,dx \cdot \int_a^b x^2 f^2(x)\,dx > \tfrac{1}{4}.$$

$$\int_a^b \underbrace{x}_{u} \underbrace{f(x)f'(x)dx}_{dv} = x\frac{f^2(x)}{2}\Big|_a^b - \int_a^b 1\cdot\frac{f^2(x)}{2} = x\frac{f^2(x)}{2}\Big|_a^b - \frac{1}{2}$$

$$= \cancel{\frac{b^2 f^2(b)}{2}}^{\,0} - \cancel{\frac{a f^2(a)}{2}}^{\,0} - \frac{1}{2} = -1/2 \,\|$$

By Holder's inequality, we have

$$\left|\int_a^b \underbrace{xf(x)}_{f}\underbrace{f'(x)}_{g}dx\right|^2 \le \int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x)dx,$$

which gives

$$\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x)dx \ge \frac{1}{4}.$$

Since the equality cannot hold in this case, we have the desired result. (Note that if the equality hold, then we have that $\frac{f'^2(x)}{\int_a^b f'^2(x)dx} = \frac{(xf(x))^2}{\int_a^b (xf(x))^2 dx}$. Equivalently, we have $|f'(x)| = M|xf(x)|$, where $M = \sqrt{\frac{\int_a^b f'^2(x)dx}{\int_a^b (xf(x))^2 dx}}$. Since $\int_a^b xf(x)f'(x)dx = -\frac{1}{2}$, we have $f'(x) = -Mxf(x) = Cxf(x)$ ($C = -M$), namely, $\frac{df(x)}{dx} = Cxf(x)$, i.e., $\frac{df(x)}{f(x)} = Cxdx$. Solving this equation gives us that $\ln f(x) = \frac{1}{2}Cx^2 + K'$, namely, $f(x) = Ke^{(1/2)Cx^2}$, where $K = e^{K'} > 0$. But since $f(a) = f(b) = 0$, we have $K = 0$, a contradiction.)