UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022

QUIZ 1 **JANUARY 31, 2022**

PLEASE NOTE the following:

- This quiz must be completed **and scanned** within **15 minutes** of the start-time!
- Your scanned **PDF** file must reach your TA within 3 minutes beyond the above-mentioned duration.

1. Recall that if α is a positive cut, then α^{-1} is given by

$$\alpha^{-1} := \{x \in \mathbb{Q} : \exists r \in \mathbb{Q} \text{ such that } r < 1/x \text{ and } r \notin \alpha\} \cup 0^* \cup \{0\}. \tag{1}$$

Now, let β be a **negative** cut.

(a) (6 marks) Give an expression for β^{-1} (please be brief).

Using, **without proof** the fact that the right-hand side of (1) is a cut, and **freely** using (i.e., without proof and without citing any specific result from the section *Fields* in "Baby" Rudin) any corollary of $\mathbb{Q}$ being an ordered field, show — anything **else** will need proof! — that:

(b) (3 marks) $\beta^{-1} \neq \varnothing$.

(c) (1 mark) β^{-1} has the property (C2) of a cut.

Note. The above has some similarities with Problem 2 of Homework 3.

Solution. The principle behind the expression to derive in part (a) is that in any ordered field $(\mathbb{F}, \leq)$, if an element $x > 0$, then $-x < 0$ and, moreover, for any $x \neq 0$, $-(-x)^{-1} = x$.

Thus, to solve part (a), we need to find an expresion for $-(-\beta)^{-1}$. By definition:

$$-(-\beta)^{-1} = \{x \in \mathbb{Q} : \exists r_1 \in \mathbb{Q}^+ \text{ s.t. } -(x + r_1) \notin (-\beta)^{-1}\}, \tag{2}$$

$$(-\beta)^{-1} = \{y \in \mathbb{Q} : \exists r_2 \in \mathbb{Q} \setminus (-\beta) \text{ s.t. } 1/y > r_2\} \cup 0^* \cup \{0\}. \tag{3}$$

For (3), we apply (1) to $(-\beta)$. Since, as β is a negative cut, $(-\beta)$ is presumed positive and we substitute (by the principle stated above) α by $(-\beta)$ in (3). By negation and by de Morgan's law:

$$r_2 \in \mathbb{Q} \setminus (-\beta) \iff -(r_2 + r_3) \in \beta \quad \forall r_3 \in \mathbb{Q}^+, \tag{4}$$

$$-(x + r_1) \notin (-\beta)^{-1} \iff -(x + r_1) > 0 \text{ and } \tfrac{1}{-(x+r_1)} \leq r_2 \quad \forall r_2 \in \mathbb{Q} \setminus (-\beta). \tag{5}$$

By (4), $r_2 > 0$ whenever $r_2 \in \mathbb{Q} \setminus (-\beta)$. Thus, the inequalities in (5) canbe restated as

$$x < -r \text{ and } x \leq -r_1 - \tfrac{1}{r_2} \quad \forall r_2 \in \mathbb{Q} \setminus (-\beta).$$

Combining this with (2)–(4) gives us the desired expression for β^{-1}:

$$\beta^{-1} := \big\{x \in \mathbb{Q} : \text{for some } r_1 \in \mathbb{Q}^+,\, x < -r_1, \text{ and} \\ x \leq -r_1 - \tfrac{1}{r_2} \quad \forall r_2 \in \mathbb{Q}^+ \text{ s.t.} -(r_2 + r_3) \in \beta \ \ \forall r_3 \in \mathbb{Q}^+\big\} \tag{6}$$

As $\beta < 0^*$, by definition $\exists \tau \in 0^*$ such that $\tau \notin \beta$. Now, if $\exists z \in \beta$ such that $\tau \leq z$, then by the property (C2) of β, we would get $\tau \in \beta$, which is a contradiction. We have just shown that

$$z < \tau < 0 \quad \forall z \in \beta. \tag{7}$$

As $\{r_2 \in \mathbb{Q}^+ : -r_2 - r_3 \in \beta \;\; \forall r_3 \in \mathbb{Q}^+\} \subseteq \{r_2 \in \mathbb{Q}^+ : -r_2 + (\tau/2) \in \beta\}$, if we can produce an $x \in \mathbb{Q}$ such that

$(*)$ for some $r_1 \in \mathbb{Q}^+$, $x \leq -r_1 - \frac{1}{r_2} \;\; \forall r_2 \in \mathbb{Q}^+$ s.t. $-r_2 + (\tau/2) \in \beta$,

then x would satisfy all the conditions stated in (6) and would hence be in β^{-1}. But note that for any $r_2 \in \mathbb{Q}^+$ s.t. $-r_2 + (\tau/2) \in \beta$:

$$\begin{aligned} & -r_2 + (\tau/2) \; < \; \tau && \text{[by the inequality (7) above]} \\ \implies & -(1/r_2) \; > \; 2/\tau && \text{[by the properties of an ordered field]} \end{aligned}$$

and this holds for any $r_2 \in \mathbb{Q}^+$ s.t. $-r_2 + (\tau/2) \in \beta$. So, $-1 + (2/\tau)$ satisfies $(*)$ with $r_1 = 1$. By the statement following $(*)$, we have $-1 + (2/\tau) \in \beta^{-1}$, whence the latter is non-empty.

Property (C2) is **immediate** from (6). □