Lecture - 8 : 21st Jan Quiz on Set Theory

Mid Week $\to$ 15th - $\square$

Combinatorics

Reference :- Walk through combinatorics.

Thm (PHP) :- Let $n, k \in \mathbb{N} > 0$ and $n > k$. Suppose, we have to place n balls in k boxes, then there is a box with atleast 2 balls.

Proof :- Suppose not. Then each box contains either 0 or 1 ball. Let m be the no., of boxes containing a ball.

$\Rightarrow$ Total # balls $= m \leq k < n$, Contradication.

Example :- Consider the sequence $(a_n)_{n \geq 1}$ where $a_n = \underbrace{77\ldots\ldots7}_{n}$ in decimal notation.

Then $\exists$ an element of the sequence divisible by 2023.

Proof of Example :- We will prove that one of $a_1, a_2, \ldots, a_{2023}$ is divisible by 2023. Let $r_1, r_2, \ldots, r_{2023}$ be the remainders of these when divided by 2023.

If one of the r_i's $= 0$, we are done.

If not $1 \leq r_i \leq 2022 \ \forall i$

By PHP, $r_i = r_j$ for some $i \neq j$

$\Rightarrow a_j - a_i = \underbrace{7777\ldots77}_{j-i}\underbrace{00\ldots0}_{i} = a_{j-i} \times 10^i$

Now, since $\underbrace{(2023, 10)}_{\text{Co-primes}} = 1$, $2023 \mid a_{j-i}$

Example :- A round robin tournament has n players and all pairs play one game each. Then at any given time, $\exists$ 2 players who played the same no. of games.

Proof of Example :- Let $a_i(t)$ be the # of games played by player i, $i \in \mathbb{N}$

Then $a_i \in \{0, 1, 2, \ldots, n-1\}$

There are n players.

At this point, it is not clear how to use PHP to prove that $a_i = a_j$ for some $i \neq j$

Since # players = # possible games = n.

But we have the fact that if $a_i = 0$, then no a_j can play $n-1$ games and vice versa. Thus only $n-1$ possible games can be played at anytime. Now, PHP can be applied.

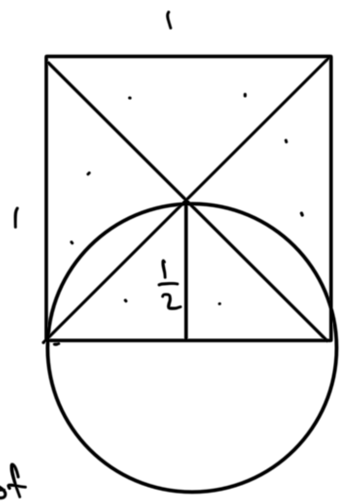
Thm (Generalised PHP) :- Let $r, m, n \in \mathbb{N} > 0$ st $n > m \cdot r$. Suppose, we place n balls in m boxes. Then $\exists$ a box with atleast $r+1$ balls.

Proof :- Exercise.

Example :- Nine points are distributed arbitrarily in a unit square. Show that 3 of them can be covered by a disk of radius $\frac{1}{2}$.

$n = 9$,

Divide the square into 4 regions as shown. By the GPHP, one of these regions has atleast 3 points



Each of these triangular regions has a circumcircle of radius $\frac{1}{2}$.

Many powerful results in math use PHP.

We have stated the principle of mathematical induction as a Peano Axiom(s). Let $P(n)$ be a property st

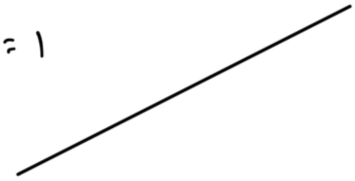
i, $P(m)$ is true for some $m \in \mathbb{N}$

ii, If $P(n)$ is true, then so is $P(n+1)$.

Then $P(n)$ holds $\forall\, n \geq m$

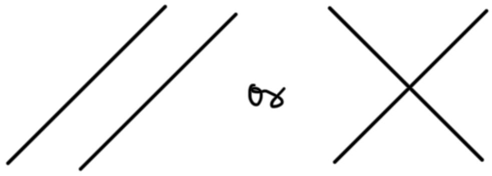
Example :- Let $f(m)$ be the maximal # of regions into which m lines divide the plane for $m \in \mathbb{N} > 0$. Then $f(m) = \binom{m+1}{2} + 1$

$m = 1$



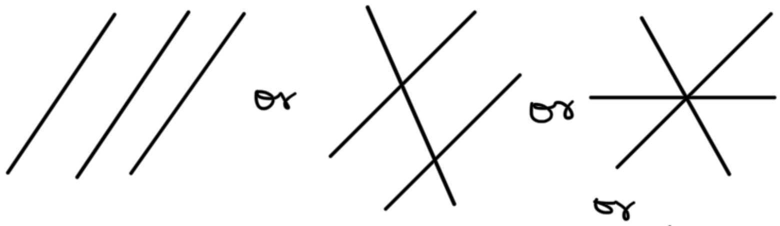
$\Rightarrow f(1) = \binom{2}{2} + 1 = 2$

$m = 2$

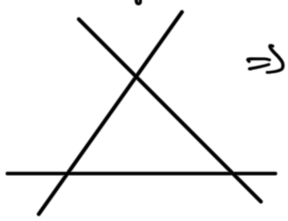


$\Rightarrow f(2) = \binom{3}{2} + 1 = 4$

$m = 3$



or



$\Rightarrow f(3) = \binom{4}{2} + 1 = 7$

Proof of Example :- Works for $m = 1$

Suppose the statement holds for n. Then we have an arrangement of n lines which divide the plane into $f(n)$ regions.

Suppose L is the $(n+1)^{th}$ line added to this arrangement and it intersects k lines.

Then it divides $k+1$ regions into 2.

$\Rightarrow f(n+1) = f(n) + k + 1 \le f(n) + n + 1$

The maximum possible is when L intersects all n lines $\Rightarrow f(n+1) = f(n) + n + 1$

$$= \binom{n+1}{2} + 1 + (n+1)$$

$$= \binom{n+2}{2} + 1$$

Gaps :- 1. L should intersect other lines at distinct points.

2. Such an L should exist.

2. such an E should exist.

3. How do we know this strategy gives the best solution ?

Statement :- All cows have same colour.

Proof by induction:- Equivalently, for any $n \in \mathbb{N} > 0$, any set of n cows have the same colour.

Trivial for $n = 1$

Assume true for n cows.

Suppose we have $n+1$ cows. Arrange them in a line

$$\underbrace{C_1, \overbrace{C_2, \ldots, C_n}^{}}_{}\!, C_{n+1}$$

By hypothesis, $C_1, \ldots, C_n$ have the same colour.

By hypothesis, $C_2, \ldots, C_{n+1}$ have the same colour.

$\Rightarrow C_1, C_2, \ldots, C_{n+1}$ have the same colour. [FAILS for $n = 2$]