UM 204: QUIZ 8
April 05, 2024

Duration. 15 minutes

Maximum score. 10 points

You may cite (without proof) the Riemann integrability of

- monotone functions;
- continuous functions;
- functions with only finitely many discontinuities.

Problem. Let $\{a_n\}_{n\in\mathbb{N}} \subset (0,1)$ be a decreasing sequence such that $\lim_{n\to\infty} a_n = 0$. Let $f : [0,1] \to \mathbb{R}$ be a bounded function such that f is continuous on $[0,1] \setminus \{a_n : n \in \mathbb{N}\}$. Using the ε-characterization of Riemann integrability, show that f is Riemann integrable on $[0,1]$.

Let $\varepsilon > 0$. We must produce a partition $P = P_\varepsilon$ of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$.
Let

$$M = \sup_{x\in[0,1]} |f(x)|.$$

Note that $-M \leq (x) \leq M$ for all $x \in [0,1]$. Since $\lim_{n\to\infty} a_n = 0$, there is an $N \in \mathbb{N}$ such that $a_n \leq a_N < \varepsilon/4M$ for all $n \geq N$. Now consider the interval $[a_N, 1]$. Then, on this interval f is continuous off of $\{a_0, ..., a_{N-1}, a_N\}$. Since it has only finitely many discontinuities in this interval it is Riemann integrable on $[a_N, 1]$. Thus, there is a partition
$P' = \{a_N = y_0 \leq y_1 \leq \cdots \leq y_m = 1\}$ of $[a_N, 1]$ such that

$$U(P', f) - L(P', f) < \varepsilon/2.$$

Now, $P = \{x_0 = 0, x_1 = a_N, x_2 = y_1, ..., x_{m+1} = y_m = 1\}$ is a partition of $[0,1]$. Moreover,

$$\begin{aligned} U(P,f) - L(P,f) &= \left(\sup_{x\in[0,a_N]} f(x) - \inf_{x\in[0,a_N]} f(x)\right)(a_N - 0) + U(P', f) - L(P', f) \\ &\leq 2M\frac{\varepsilon}{4M} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since ε was arbitrary, f is Riemann integrable on $[0,1]$.