Lecture - 9

Quiz - 2 Tomorrow (Set Theory)

Strong Induction :- Let $P(n)$ be a property $\forall\, n \in \mathbb{N}$ so that

i. $P(m)$ is true for some $m \in \mathbb{N}$

ii. If $P(n)$ is true for $m \leq n \leq n_0$, then $P(n_0+1)$ is also true.

Then $P(n)$ is true $\forall\, n \geq m$.

Example :- Let $f(0) = 1$, $f(1) = 2$ and $f(n+1) = f(n-1) + 2f(n)$ for $n \geq 1$. Then show that $f(n) \leq 3^n$.

$f(0) = 3^0 = 1$; $f(1) = 2 < 3^1 \Rightarrow$ Result holds for $n = 0, 1$

Assume result holds for integers $0, 1, 2, \ldots, n$

Then
$$f(n+1) = f(n-1) + 2f(n)$$
$$\leq 3^{n-1} + 2(3^n)$$
$$= 7\,3^{n-1} \leq 9 \cdot 3^{n-1} = 3^{n+1}$$
$$\Rightarrow \boxed{f(n+1) \leq 3^{n+1}}$$

Generalisation :- Prove statement about all integers which leave remainder 2 when divided by 5. Here, first prove for $n = 2$ and then if the result holds for n, prove for $n+5$.

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Permutations

Def'n :- The arrangement of atmost countably many objects in a linear order such that each object occurs exactly once is called permutation.

Property :- The no. of permutations of n objects is $n!$
$= n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1$ read "n factorial"

By convention $0! = 1$

Example :- $n = 3$, $\{1, 2, 3\}$

123, 132, 213, 231, 312, 321, $3! = 3 \times 2 \times 1 = 6$

Several ways of thinking of permutations :-

1. Bijection functions from $[n]$ to $[n]$
2. Two-line notation Eg:- $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 1 & 4 & 3 & 5 & 8 & 6 & 9 & 2 \end{pmatrix}$ → Reg order → Irreg "
3. One-line notation. Some Eg:- 7 1 4 3 5 8 6 9 2
4. Cycle Notation :- $(1\,7\,6\,8\,9\,2)(3\,4)(5)$
   Choose one element and use 2 line notation to traverse

Thm (Stirling's formula) :- $\boxed{n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$, where

$a_n \sim b_n$ means $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

Property :- The # of permutations of length of n objects is $\frac{n!}{(n-k)!} = n(n-1)(n-2) \cdots (n-k+1)$

Notation :- The pochhammer symbol or rising factorial $a^{\overline{n}} = (a)_n = a(a+1)\cdots\cdots(a+n-1)$. Similarly, falling factorial is $a^{\underline{n}} = a(a-1) \cdots (a-n+1)$.
Thus $\frac{n!}{(n-k)!} = n^{\underline{k}} = (n-k+1)^{\overline{k}}$ $\boxed{\text{No } {}^nP_k}$

In general, suppose we have a_1 elements of type 1, ... and a_j elements of type j st $a_1 + a_2 + \cdots + a_j = n$

Then a linear ordering of these elements is called a multi-set permutation.

Property :- The # of multiset permutations of $1^{a_1}, 2^{a_2}, \ldots, j^{a_j}$ is the multinomial co-efficient,

1 occurs a_1 times

$$\binom{n}{a_1 \; a_2 \; \ldots \; a_j} = \frac{n!}{a_1!\,a_2!\cdots\cdots a_j!}$$

When $j=2$, we write $\binom{n}{a} \equiv \binom{n}{a_1 \;\; a_2}$ and this is called the binomial co-efficient.

Question :- What if we have objects $1, \dots, j$ and we want to count multi-set with arbitrary repeats each of size n objects ?

Example :- $j = 3$, $n = 2 \Rightarrow 11, 12, 13, 22, 23, 33$

Property :- The # of multisets of size n among j objects is $\binom{n+j-1}{n} = \left(\!\binom{j}{n}\!\right)$ called "j multichoose n".

Property :- The # of j element subsets from an n-set is $\binom{n}{j} = \dfrac{n^{\underline{j}}}{j!}$

Conventions about binomial co-efficients :-

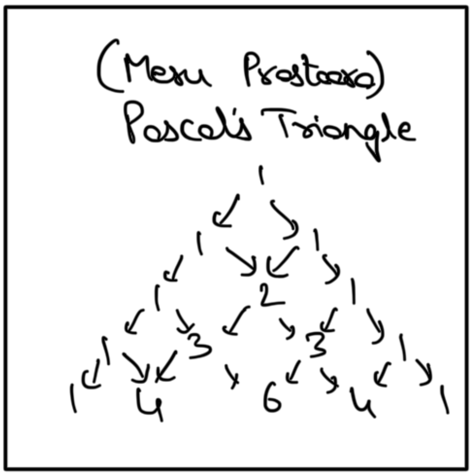
1. $\binom{n}{j} = 0$ if $j > n$ or $j < 0$

2. If $n > 0$, then $\binom{-n}{j} = (-1)^j \binom{n+j-1}{j}$

3. The binomial theorem states $\sum_{k \geq 0} \binom{r}{k} x^k = (1+x)^r$

   If $r \in \mathbb{N}$, this is automatically a finite sum.
   If not, this is an infinite sum and we need $|x| < 1$ for the result to hold.
   In that case, $\binom{r}{k} := \dfrac{r^{\underline{k}}}{k!} = \dfrac{r(r-1)\cdots(r-k+1)}{k!}$

4. Not obvious from the formula that $\binom{n}{k} \in \mathbb{N}$
   So, use induction to prove.



Property :- $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

Binomial coefficients satisfy many identities

1. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

2. $\sum_{k=0}^{n} k \binom{n}{k} = n \cdot 2^{n-1}$

3. $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

Thm (Chu - Vandermonde identity) :- For $m, n, k \in \mathbb{N}$

$$\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{m+n}{k}$$