UM 204 (WINTER 2024) - WEEK 5

1. Metric spaces

1.1. **Connected Sets.** So far, we have been motivated by certain desirable properties of continuous functions on $[a, b]$. A property we are yet to revisit is the intermediate value property.

Definition 1.1. A pair of sets $A, B \in (X, d)$ are said to be separated in X if

$$A \cap \overline{B} = \overline{A} \cap B = \emptyset.$$

A set $E \subset (X, d)$ is said to be disconnected if it can be written as a union of two non-empty separated sets in X. The set E is said to be connected if it is not disconnected, i.e., there does not exist a pair of sets $A, B \subset X$ such that

(1) $A, B \neq \emptyset$;
(2) $E = A \cup B$;
(3) A and B are separated in X.

Examples. (1) $[0, 2)$ and $(2, 3)$ are separated in $\mathbb{R}$, but $[0, 2)$ and $[2, 3)$ are not separated in $\mathbb{R}$.
(2) The empty set in any metric space is connected since it can never be written as the union of two nonempty sets.
(3) $\mathbb{Q}$ is disconnected (in $\mathbb{R}$) since $\mathbb{Q} \cap (-\infty, \sqrt{2})$ and $\mathbb{Q} \cap (\sqrt{2}, \infty)$ separate $\mathbb{Q}$ (in $\mathbb{R}$).
(4) Exercise. Show that if $E \subseteq Y \subseteq (X, d)$, then E is connected relative to Y if and only if E is connected in X. Thus, it makes sense to refer to a *connected metric space*.

Theorem 1.2. *Let $E \subset \mathbb{R}$. Then, E is connected in $\mathbb{R}$ if and only if E is convex, i.e., for every $x < y$ in E, $[x, y] \subseteq E$.*

Proof. Suppose E is connected, but E is not convex. Then, there exist $x < y$ in E and $r \in (x, y)$ such that $r \notin E$. The pair

$$A = E \cap (-\infty, r) \qquad B = E \cap (r, \infty)$$

separates E.

Conversely, suppose E is convex, but E is not connected. Let A, B be a pair of sets that separate E. Let $x \in A$ and $y \in B$ (guaranteed because they are nonempty). The set $A \cap [x, y]$ is nonempty and bounded above, thus we may set

$$r = \sup A \cap [x, y].$$

Lemma 1.3 (Exercise)**.** *Given a nonempty and bounded above set $F \subset \mathbb{R}$,* $\sup F \in \overline{F}$.

Assuming this lemma, we complete the proof. Since $r \in \overline{A \cap [x,y]} \subseteq \overline{A}$, by separatedness of A and B, we have that $r \notin B$. Thus, $x \le r < y$, and $r \in A$ (since $r \in E = A \cup B$). Once again, by separatedness of A and B, $r \notin \overline{B}$. Thus, there is an $\varepsilon > 0$ such that $(r-\varepsilon, r+\varepsilon) \cap B = \emptyset$. On the other hand, $(r, r+\varepsilon) \cap A = \emptyset$ since $r = \sup A \cap [x,y]$ □

1.2. **The Cantor Set.** The closed set $E = [0,1]$ has the special property that every point in E is a limit point of E. Such closed sets are called perfect sets. If we want to construct a sparse set like this, we may try to remove points from $[0,1]$. Removing a finite set of points leaves behind a nonclosed set, while removing a finite set of open intervals leaves behind a finite union of closed intervals. Can we get something very sparse, say nowhere dense in $\mathbb{R}$, i.e., E satisfies $\left(\overline{E}\right)^\circ = \emptyset$, or in other words $X = \overline{X \setminus \overline{E}}$?

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Before we answer this question, we briefly discuss ternary expansions. For every $x \in [0,1]$, there is a sequence $\{d_k\}_{k\ge1} \subset \{0,1,2\}$ s.t.

$$x = \sup\left\{D_k = \sum_{j=1}^{k} \frac{d_j}{3^j} : k \ge 1\right\}.$$

Let $I_0 = [0,1/3]$, $I_1 = [1/3,2/3]$ and $I_2 = [2/3,1]$. Given $x \in [0,1]$, let $d_1 = j$ if $x \in I_j$. Then, split I_j into three equal intervals I_{j0}, I_{j1} and I_{j2} of length $\frac{1}{9}$, and let $d_2 = \ell$ if $x \in I_{j\ell}$. Thus,

$$\frac{\ell}{9} \le x - \frac{d_1}{3} = x - D_1 \le \frac{\ell+1}{9}.$$

Continue in this way to produce a sequence $\{d_k\}_{k\ge1} \subset \{0,1,2\}$ such that

$$D_k \le x \le D_k + \frac{1}{3^k}, \quad \forall k \ge 1.$$

Breaking ties. $\boxed{\text{Scheme A}}$ If x is an endpoint of two intervals at any stage, pick the interval on the right. Then, there is a *unique* expansion such that

$$D_k \le x < D_k + \frac{1}{3^k}, \quad \forall k \ge 1.$$

$\boxed{\text{Scheme B}}$ If x is an endpoint of two intervals at any stage, pick the interval on the left. Then, there is a *unique* expansion such that

$$D_k < x \le D_k + \frac{1}{3^k}, \quad \forall k \ge 1.$$

Example.

	x=1/3	x=2/3	x=1/2	x=4/9
Scheme A	0.1000...	0.2000...	0.111...	0.11000...
Scheme B	0.0222...	0.1222...	0.111...	0.10222...

Observations. (1) Only the endpoints of "middle thirds" are going to create ambiguity.
(2) Suppose $x \in (0,1)$ admits two expansions. Let $\{d_k\}_{k\geq 1}$ be the unique expansion by Scheme A. Let

$$k = \min\{\ell : x = D_\ell\}.$$

Then, $d_k = 1$ or 2, and $x = 0.d_1 d_2 ... d_k 000...$ by Scheme A, and $x = 0.d_1 d_2 ... (d_k - 1)222...$ by Scheme B. Thus, there are two possibilities:

$$\begin{aligned} x &= 0.d_1 d_2 ... d_{k-1} 1000... = 0.d_1 d_2 ... d_{k-1} 0222... \\ x &= 0.d_1 d_2 ... d_{k-1} 2000... = 0.d_1 d_2 ... d_{k-1} 1222... \end{aligned}$$

Theorem 1.4. *There exists a set $E \subset [0,1]$ such that*

(1) *E is compact.*
(2) *Every $x \in E$ is a limit point of E.*
(3) *$E^\circ = \emptyset$ (E contains no intervals of positive length).*
(4) *E is uncountable.*

Proof. The idea is to only consider the set of $x \in [0,1]$ that admit at least one ternary expansion consisting only of 0's and 2's. This approach is useful to show uncountability, but an alternate viewpoint is useful show the other properties. Let

$$\begin{aligned} E_0 &= [0,1], \\ E_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], \\ &\vdots \\ E_m &= E_{m-1} \setminus \bigcup_{0 \leq k \leq 3^{m-1}-1} \underbrace{\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)}_{I_{k,m}}. \end{aligned}$$

This gives a sequence $E_0 \supset E_1 \supset \cdots$ of sets such that each E_m is a disjoint union of 2^m closed intervals, each of length (diameter) $\frac{1}{3^m}$. Let

$$E = \cap_{j=0}^{\infty} E_j = [0,1] \setminus \bigcup_{m=0}^{\infty} \bigcup_{k=0}^{3^{m-1}-1} \underbrace{\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)}_{I_{k,m}}.$$

Note that

(1) E is closed and bounded, and therefore, compact.
(2) $E \neq \emptyset$. All the endpoints of E_m will survive (and there are more).
(3) Fix $m \in \mathbb{N}$. Then, any interval of length $\frac{2}{3^m}$ intersects at least one $I_{k,m}$, $k = 0, \ldots, 3^{m-1}$. If there is some $x \in E$ and $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset E$, then simply choose $m > 0$ such that $2/3^m < \varepsilon$. Thus, E has empty interior.
(4) It can be shown rigorously (but we skip the details) that

$$E = \{x \in [0,1] : x \text{ admits one ternary expansion consisting only of 0's and 2's}\}.$$

For a quick sketch, observe that if x is in the right-hand set, then it skips all the "middle-thirds". Conversely, if $x \in E$, let d_k be the first digit for which a choice has to be made. Then, of the two possible choices, one will always lead to a string of only 0's and $'2$ starting from d_k. The first $k - 1$ digits necessarily have to be 0 or 2 since x is neither in the interior of the middle thirds (all removed), nor the endpoint of any of them (source of ambiguity). Now, uncountability follows by a Cantor's diagonalization argument.

□

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2. Sequences and series

2.1. Convergence and subsequential limits.

Definition 2.1. Let (X, d) be a metric space. A sequence is a function from $\mathbb{N}$ to X, conventionally expressed as $\{f(k)\}_{k \in \mathbb{N}}$. A sequence $\{x_k\}_{k \in \mathbb{N}} \subset X$ converges in (X, d) if there is an $x_0 \in X$ such that, for every $\varepsilon > 0$, there exists an $N = N_\varepsilon \in \mathbb{N}$ so that

$$d(x_n, x_o) < \varepsilon, \quad \forall n \geq N.$$

In this case, we say that x_0 is a limit of the sequence, and we write

$$\lim_{n \to \infty} x_n = x_0, \quad \text{or} \quad x_n \to x_0 \text{ as } n \to \infty.$$

If the sequence $\{x_n\}_{n \in \mathbb{N}}$ does not converge, we say that it diverges.

Examples. (1) For sequences in $\mathbb{R}$, this coincides with the notion of convergence we saw previously since $d(y, x) < \varepsilon \iff y \in (x - \varepsilon, x + \varepsilon)$.
(2) Let $x_n = (1/n, -2/n^2) \in \mathbb{R}^2$, $n \in \mathbb{N}_+$. We claim that $\{x_n\}_{n \in \mathbb{N}_+}$ converges to $(0,0)$, i.e., we want to find an $N > 0$ such that $\frac{1}{n^2} + \frac{4}{n^4} < \varepsilon^2$ for all $n \geq N$. For this, observe that

$$\frac{1}{n^2} + \frac{4}{n^4} < \frac{5}{n^2}.$$

By the Archimedean property, there is an $N \in \mathbb{N}$ such that

$$N \geq \sqrt{5}/\varepsilon.$$

We now obtain the desired inequality for all $n \geq N$.

(3) Consider the sequence $\{x_n = (1/n, (-1)^n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^2$. Visually, it seems to have two limits, which is not possible (we see this shortly). Proving divergence at this stage is slightly more tedious, and we wait for some more language.

Theorem 2.2. *Let* $\{x_n\}_{n\in\mathbb{N}} \subset X$.

(1) $\lim\limits_{n\to\infty} x_n = x$ *if and only if every* ε*-neighborhood of* x *contains all but finitely many terms of* $\{x_n\}$.
(2) *If* $\lim\limits_{n\to\infty} x_n = x$ *and* $\lim\limits_{n\to\infty} x_n = y$, *then* $x = y$.
(3) *If* $\{x_n\}$ *is convergent, then the set* $\{x_n : n \in \mathbb{N}\}$ *is bounded.*
(4) *Let* $E \subset X$. *Then,* $x \in \overline{E}$ *iff there is a sequence* $\{x_n\}_{n\in\mathbb{N}} \subset E$ *such that* $\lim_{n\to\infty} x_n = x$.

Proof. Claims (1), (2) and (3) are straightforward (mimic the proofs in $\mathbb{R}$).

For (4), suppose $x \in E$, then simply choose $x_n = x$ for all $n \in \mathbb{N}$. If x is a limit point of E, then for every $n \in \mathbb{N}_+$, there is an $x_n \in E$ such that $d(x_n, x) < 1/n$. Now, for any $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $1/\varepsilon < N$. Then, $d(x_n, x) < \varepsilon$ for all $n \geq N$. Thus, $\{x_n\}$ is the desired sequence. Conversely, suppose $x \notin \overline{E}$, but there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset E$ such that $\lim_{n\to\infty} x_n = x$. Since $X \setminus \overline{E}$ is open, there is an $\varepsilon > 0$ such that $B(x;\varepsilon) \subseteq X \setminus \overline{E} \subseteq x \setminus E$. In particular, $x_n \notin B(x;\varepsilon)$ for all $n \in \mathbb{N}$. This contradicts the fact that $\lim_{n\to\infty} x_n = x$. □

Definition 2.3. Let $x = \{x_n\}_{n\in\mathbb{N}}$ be a sequence in a metric space. Let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ be a strictly increasing sequence, i.e., $n_0 < n_1 < n_2 < \cdots$. Then, the sequence $\{p_{n_k}\}_{k1\mathbb{N}}$ is called a subsequence of x, and if it converges, its limit is called a subsequential limit of x.

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