UM 204: QUIZ 5
March 1, 2024

Problem. Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be bounded real sequences such that

$$\lim_{n\to\infty} b_n = b > 0.$$

Prove that

$$\limsup_{n\to\infty}(a_n b_n) = (\limsup_{n\to\infty} a_n)b.$$

Let $a = \limsup_{n\to\infty} a_n$ and $c = \limsup_{n\to\infty} a_n b_n$, which are in $\mathbb{R}$ since the given sequences are bounded, and the product of two bounded sequences is bounded. We showed in class that a is a subsequential limit of $\{a_n\}_{n\in\mathbb{N}}$. Thus, there is a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} a_{n_k} = a$. Since every subsequence of a convergent sequence converges to the limit of the sequence, we also have that $\lim_{k\to\infty} b_{n_k} = b$. By algebra of convergent sequences,

$$\lim_{k\to\infty} a_{n_k} b_{n_k} = ab.$$

Since the limit superior is the supremum of subsequenctial limits, we have that

$$c = \limsup_{n\to\infty} a_n b_n \geq ab.$$

Suppose $c > ab$. Then, there is a subsequence of $\{a_n b_n\}$, say $\{a_{m_k} b_{m_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} a_{m_k} b_{m_k} = c.$$

Since $\lim_{n\to\infty} b_n = b > 0$, we know that there is an $N \in \mathbb{N}$ such that $b_n > 0$ for all $n \geq N$. Thus, dropping the first $N-1$ terms, we have that

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}.$$

In particular,

$$\lim_{n\to\infty} a_{m_k} b_{m_k} \frac{1}{b_{m_k}} = \frac{c}{b} > \frac{ab}{b},$$

since $b > 0$. Thus, the subsequence $\{a_{m_k}\}_{k\in\mathbb{N}}$ converges to a limit greater than the limit superior of $\{a_n\}_{n\in\mathbb{N}}$. This is not possible. Thus,

$$c = ab.$$