UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022

QUIZ 4 **FEBRUARY 21, 2022**

PLEASE NOTE the following:

- This quiz must be completed **and scanned** within **15 minutes** of the start-time!
- Your scanned **PDF** file must reach your TA within 3 minutes beyond the above-mentioned duration.

1. Let S be a **countable** set.

(a) (8 marks) Show that $\mathcal{P}(S)$ has the same cardinality as the set of all 0–1 sequences (i.e., sequences whose terms are either 0 or 1).

(b) (2 marks) Is also $\mathcal{P}(S)$ countable? Please state a reason for your answer.

Remark. Please **do not** merely so some preliminary work and stop by saying, "This now reduces to Problem 6 of Homework 6." Show the details of your argument. That said, quoting without proof any result shown in class is okay.

Note. As you would have realised, the above is closely related to Problem 6 of Homework 6.

Solution. The key step in the solution of part (a) is to show that $\mathcal{P}(S) \approx \big\{f : S \to \{0,1\}\big\}$. To this end, for $A \in \mathcal{P}(S)$, define the function $\chi_A : S \to \{0,1\}$ as follows:

$$\chi_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Now, define the function $F : \mathcal{P}(S) \to \big\{f : S \to \{0,1\}\big\}$ by $F(A) := \chi_A$. It is surjective because, if we pick a $g \in \big\{f : S \to \{0,1\}\big\}$, by definition $g = \chi_{g^{-1}\{1\}}$ and so $g = F(g^{-1}\{1\})$.

Now consider $A_1, A_2 \in \mathcal{P}(S)$ such that $F(A_1) = F(A_2)$. If $A_1 \neq A_2$ then, we may assume without loss of generality that there exists a point $x_0 \in A_1 \setminus A_2$. By definition, $\chi_{A_1}(x_0) = 1$ but $\chi_{A_2}(x_0) = 0$. This contradicts the fact that $F(A_1) = F(A_2)$. Thus A_1 and A_2 must be equal, which establishes the injectivity of F. This, together with the conclusion of the last paragraph, gives us $\mathcal{P}(S) \approx \{f : S \to \{0,1\}\}$.

Since S is countable, there is a bijective function $\phi : \mathbb{Z}^+ \to S$. For any $g \in \big\{f : S \to \{0,1\}\big\}$, this induces the sequence $g \circ \phi : \mathbb{Z}^+ \to \{0,1\}$. (In an alternative notation, we have the sequence

$$\big\{x_n^g\big\} \subset \{0,1\}, \quad \text{where} \quad x_n^g := g(\phi(n)), \quad n = 1, 2, 3, \dots .)$$

Let $\mathcal{A}$ denote the set of 0–1 sequences and write $G : \big\{f : S \to \{0,1\}\big\} \to \mathcal{A}$ by $G(g) = g \circ \phi$. It is surjective because, if we pick a sequence $\sigma \in \mathcal{A}$, by definition $G(\sigma \circ \phi^{-1}) = \sigma$. Also, if $G(g_1) = G(g_2)$, then

$$g_1 \circ \phi = g_2 \circ \phi \implies g_1 = (g_1 \circ \phi) \circ \phi^{-1} = (g_2 \circ \phi) \circ \phi^{-1} = g_2,$$

which establishes injectivity of G. Thus $\big\{f : S \to \{0,1\}\big\} \approx \mathcal{A}$. Combining this with the last conclusion, we have $\mathcal{P}(S) \approx \mathcal{A}$.

Remark. The above problem could be solved more directly, but it is more transparent — albeit a little more lengthy — to first show that $\mathcal{P}(S) \approx \big\{f : S \to \{0,1\}\big\}$. This step is also **precisely** Problem 6 of Homework 6.

Proof of (b)*:* We have shown in class that $\mathcal{A}$ is **uncountable!** Thus, by part (a), $\mathcal{P}(S)$ cannot be countable. $\square$