UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2025
HOMEWORK 11

Instructor: GAUTAM BHARALI **Assigned: MARCH 29, 2025**

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1. Find a rational number that approximates $(7.9)^{1/3}$ and estimate the error of approximation by:

- defining $f(x) := x^{1/3}$ for all $x \in [0, +\infty)$ — where $x^{1/3}$, for $x \geq 0$, denotes the unique non-negative cube-root of x — and letting your approximation be the Taylor polynomial $T_1 f(7.9; \alpha)$ for an appropriate $\alpha > 0$; and
- selecting an appropriate interval $[a, b] \subsetneq [0, +\infty)$ and applying Taylor's Theorem to $f|_{[a,b]}$.

With your choices, what is the best (i.e., smallest) upper bound for

$$|(7.9)^{1/3} - T_1 f(7.9; \alpha)|$$

predicted by Taylor's Theorem?

Note. You may assume — **no explanations** needed — that the function $(0, +\infty) \ni x \longmapsto x^{-p}$ is a decreasing function for any $p > 0$.

2. Let $a < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be a bounded function. Show that for any two partitions $\mathbb{P}_1$ and $\mathbb{P}_2$ on the interval $[a, b]$,

$$L(\mathbb{P}_1, f) \leq U(\mathbb{P}_2, f).$$

3. Define the function $f : \mathbb{R} \to \{0, 1\}$ as follows:

$$f(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Fix two real numbers $a < b$. Give an expression for each of the Riemann sums $L(\mathbb{P}, f)$ and $U(\mathbb{P}, f)$. Is $f|_{[a,b]} \in \mathscr{R}([a, b])$?

4. Let $a < b$ be real numbers and suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

(i) Let $\alpha, \beta \in \mathbb{R}$ be such that $a \leq \alpha < \beta \leq b$. Show that $f|_{[\alpha,\beta]} \in \mathscr{R}([\alpha, \beta])$.

(ii) Let $c \in (a, b)$. By *(i)*, we know that $f|_{[a,c]} \in \mathscr{R}([a, c])$ and $f|_{[c,b]} \in \mathscr{R}([c, b])$. Show that

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

5. Let $a < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$. Suppose $\mathsf{range}(f) \subseteq [\alpha, \beta]$ and suppose $\phi : [\alpha, \beta] \to \mathbb{R}$ is a continuous fuction. Show that $\phi \circ f$ is Riemann integrable on $[a, b]$.

6. Let $a < b$ be real numbers and let $f, g \in \mathscr{R}([a,b])$. Let p and q be positive real numbers such that $p^{-1} + q^{-1} = 1$. Prove **Hölder's inequality:**

$$\left| \int_a^b fg(x)\, dx \right| \leq \left[\int_a^b |f(x)|^p dx \right]^{1/p} \left[\int_a^b |g(x)|^q dx \right]^{1/q},$$

by completing the outline provided by parts (a)–(c) of Problem 10 in "Baby" Rudin, Chapter 6, taking $\alpha = \mathsf{id}_{[a,b]}$.

7. Review/Self-study. Please review by April 2 the statement and the proof of the result studied in UMA101 called the "First Fundamental Theorem of Calculus," which is presented as Theorem 6.20 in Rudin's book.

8–9. Problems 7 and 15 from "Baby" Rudin, Chapter 6.