UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2022

QUIZ 6 **MARCH 21, 2022**

PLEASE NOTE the following:

- **Duration:** 15 minutes
- The quiz is to be written with no access to any books, notes, or study materials.

1. Let $\{a_n\}$ be a real sequence such that $a_1, a_2, a_3, \dots > 0$. Suppose $\limsup_{n\to\infty} a_n = +\infty$. Show that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$$

diverges. Be sure to **justify** your steps.

Note. The above is closely related to Problem 3 of Homework 9.

Solution. Since $(\limsup_{n\to\infty} a_n) \in E[\{a_n\}]$, there exists a subsequence $\{a_{n_j}\}$ such that $\lim_{j\to\infty} a_{n_j} = +\infty$. We compute (noting that $a_n \neq 0$ for every n):

$$\frac{a_n}{1+a_n} = \frac{1}{1+(1/a_n)}. \tag{1}$$

Let $\varepsilon > 0$. Then, as $\lim_{j\to\infty} a_{n_j} = +\infty$, there exists $J \in \mathbb{Z}^+$ such that

$$\begin{aligned} a_{n_j} &> (1/\varepsilon) \;\; \forall j \geq J \\ \Rightarrow 0 < (1/a_{n_j}) &< \varepsilon \;\; \forall j \geq J. \end{aligned}$$

As $\varepsilon > 0$ above was arbitrary, we conclude that

$$\lim_{j\to\infty} (1/a_{n_j}) = 0.$$

From the latter limit and from the theorem on term-wise algebraic combinations of sequences $\lim_{j\to\infty} 1/\left(1 + \frac{1}{a_{n_j}}\right)$ exists and

$$\lim_{j\to\infty} \frac{1}{1+(1/a_{n_j})} = \frac{1}{1+\lim_{j\to\infty}(1/a_{n_j})} = 1.$$

From the above and (1), we get

$$\lim_{j\to\infty} \frac{a_n}{1+a_{n_j}} = 1.$$

In other words $1 \in E[\{a_n/(1+a_n)\}]$. Thus $\{a_n/(1+a_n)\}$ does not converge to 0. Therefore, by the Divergence Test,

$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$$

diverges. $\square$