Lecture - 5

$f : A \to B$, $C \subseteq A$, $f(C)$

1, $f(C)$ can be shown to be a set also by the axiom of specification

2, One can show that the cartesian product $A \times B$ is a set by Axiom 4 (Singletons & Pairs)
Identify (a,b) by $\{\{a\},\{a,b\}\}$. Then show that $(a,b) = (a',b')$ iff $a = a'$ and $b = b'$

3, Axiom of replacement $\Rightarrow$ Axiom of specification

Def'n:- Let $f : X \to Y$, $g : Y \to Z$. Then the composition $g \circ f : X \to Z$ is given by $g\circ f(x) = g(f(x))$

Lemma:- Function composition is associative i.e., $f : Z \to W$, $g : Y \to Z$, $h : X \to Y$
$\Rightarrow f\circ(g\circ h) = (f\circ g)\circ h : X \to W$

Proof of Lemma:- Exercise

Types of functions :-

1, $f : A \to B$ is injective or one to one if $f(a) = f(b) \Leftrightarrow a = b$

2, $f : A \to B$ is surjective or onto if $f(A) = B$.

3, $f : A \to B$ is bijective if it is both 1, and 2,

4, $f : A \to A$ is an involution if $f(f(a)) = a \ \forall a \in A$

If f is a bijection, then for every $b \in B$, $\exists$ a unique $a \in A$ st $f(a) = b$. Then a is the inverse of f at b denoted $a = f^{-1}(b)$. Thus $f^{-1} : B \to A$ is also a valid function.

If f is injective, then $f^{-1} : \underbrace{f(A)}_{\text{Not } B} \to A$ is also well-defined.

ZFC axioms (continued):-

11. (Axiom of power sets)

Let X, Y be sets, there exists a set denoted Y^X consisting of all functions from X to Y.

Ques:- If $|X| = m$, $|Y| = n$, $|Y^X| = ? \to n^m$

Exercise:-

Let X be a set. The collection $\{Y \mid Y \text{ is a subset of } X\}$ is a set, called the power set of X denoted 2^X.

12. (Axiom of Unions)

Let A be a set whose elements are also sets. Then $\exists$ a set $\bigcup A$, whose elements are the elements of A.

Thus $x \in \bigcup A$ if $x \in S$ for some $x \in A$

Remarks:-

1. Axiom 12. with Axiom 4. (pairs) $\Rightarrow$ Axiom 5. (Pairwise union)

2. Let I be a set (indexing) such that A_α is a set for every $\alpha \in I$. Then $\{A_\alpha \mid \alpha \in I\}$ is a set (in replacement axiom)
$\Rightarrow \bigcup_{\alpha \in I} A_\alpha$ is a set (by Axiom 12)

Axioms 1-12 give the Zermelo-Fraenkel axioms of set theory.

Def'n:- Two sets X, Y have equal cardinality/size if there exists a bijective function $f: X \to Y$. Let $n \in \mathbb{N}$. A set A has cardinality of n if it has the same cardinality as $[n] := \{1, 2, \ldots, n\}$

Exercise:- Cardinality is an equivalence relation on sets.

Example:- $X = \mathbb{N}$, $Y = 2\mathbb{N}$. Then $f: X \to Y$ is a bijection given by $f(x) = 2x$ is a bijection $\Rightarrow |X| = |Y|$ but $Y \subsetneq X$

Notation for cardinality of X.

Def'n:- A set is finite if it has cardinality n

for some $n \in \mathbb{N}$. Otherwise it is infinite.

Exercise :- $\mathbb{N}$ is infinite.

Def'n :- A set is countable / countably infinite if it has the same cardinality as $\mathbb{N}$. A set is almost countable if it is either finite or countable, and is uncountable if it is infinite but not countable.

Exercise :- 1. Let $m < n \in \mathbb{N}$. Then,
(a) there is no surjective $f : [m] \to [n]$
(b) there is no injective $f : [n] \to [m]$
2. Let $m, n \in \mathbb{N}$. There exists a bijective $f : [m] \to [n]$ iff $m = n$

Properties of countable sets :-
(1) X, Y countable $\Rightarrow X \cup Y$ countable
Corollary :- $\mathbb{Z}$ is countable
(2) $\{(n,m) \in \mathbb{N} \times \mathbb{N} \mid 0 \leq m \leq n\}$ is countable
$\Rightarrow \mathbb{Q}$ is countable.

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Ques :- Do there exist uncountable sets ?

Cantor's theorem :- Let X be an arbitrary set. Then X and 2^X cannot have the same cardinality.

Proof :- Suppose they do. Then $\exists$ a bijection $f : X \to 2^X$
Consider $A = \{x \in X \mid x \notin f(x)\} \Rightarrow A \subseteq X$
$\Rightarrow A \in 2^X$ [By def'n]

Thus $\exists\ x$ st $f(x) = A$

Two possibilities for whether $x \in A$

Case - I :- $x \in A \Rightarrow x \notin f(x) \Rightarrow x \notin A$ by def'n of A
Case - II :- $x \notin A \Rightarrow x \notin f(x) \Rightarrow x \in A$ by def'n of A
Contradiction.

Exercise :- Show that 2^X has larger cardinality than X

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