UM 204: QUIZ 3

February 3, 2024

Duration. 15 minutes

Maximum score. 10 points

You are free to use basic set-theoretic facts.

Problem. Let (X, d) be a metric space, and $E \subseteq X$. Recall that the *boundary* of E is the set

$$bE = \overline{E} \setminus E^\circ.$$

Let $x \in X$. Show that the following are equivalent.

(1) $x \in bE$,

(2) for every $\varepsilon > 0$, there is a $z \in E$ and $w \in E^c$ such that $z, w \in B(x; \varepsilon)$,

(3) $x \in \overline{E} \cap \overline{(X \setminus E)}$.

We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). The key observation is that $x \in \overline{E}$ iff for every $\varepsilon > 0$, there is a $y \in E$ such that $y \in B(x; \varepsilon)$.

• Let $x \in bE$ and $\varepsilon > 0$. Since $x \in \overline{E}$, there is a $z \in E$ such that $z \in B(x; \varepsilon)$. Since $x \notin E^\circ$, $B(x; \varepsilon) \nsubseteq E$. Thus, there is a $w \in E^c$ such that $w \in B(z; \varepsilon)$. Thus, (1) $\Rightarrow$ (2).

• Let $x \in X$ satisfy (2). Then, by the observation made above, x is in both $\overline{E}$ and $\overline{X \setminus E}$. Thus, (2) $\Rightarrow$ (3).

• Let $x \in \overline{E} \cap \overline{X \setminus E}$. Clearly, $x \in \overline{E}$. Now, let $\varepsilon > 0$. Since $x \in \overline{X \setminus E}$, there is a $w \in X \setminus E$ such that $w \in B(x; \varepsilon)$. Thus, $B(x; \varepsilon) \nsubseteq E$. Since $\varepsilon > 0$ was arbitrary, $x \notin E^\circ$. Thus, $x \in \overline{E} \setminus E^\circ$.