Brahe, Kepler, Mars, Ellipses & Gravity

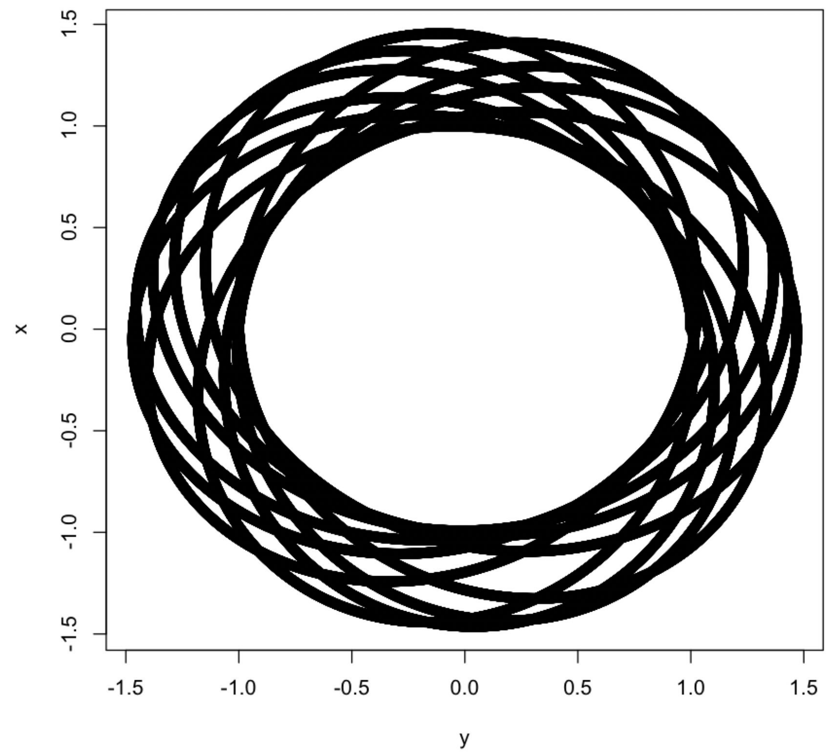
Perhaps the oldest, (very) non-trivial data analytics example

strand
PRECISION MEDICINE SOLUTIONS
FOR BETTER HEALTHCARE DECISIONS

Questions

- What trajectory does a planet move in?
- Can these trajectories be obtained from “simple” observations?
- What physical law explains these trajectories?
- What has data analysis got to do with all this?
- How was this analysis done ~400 years ago (1580-1610)?

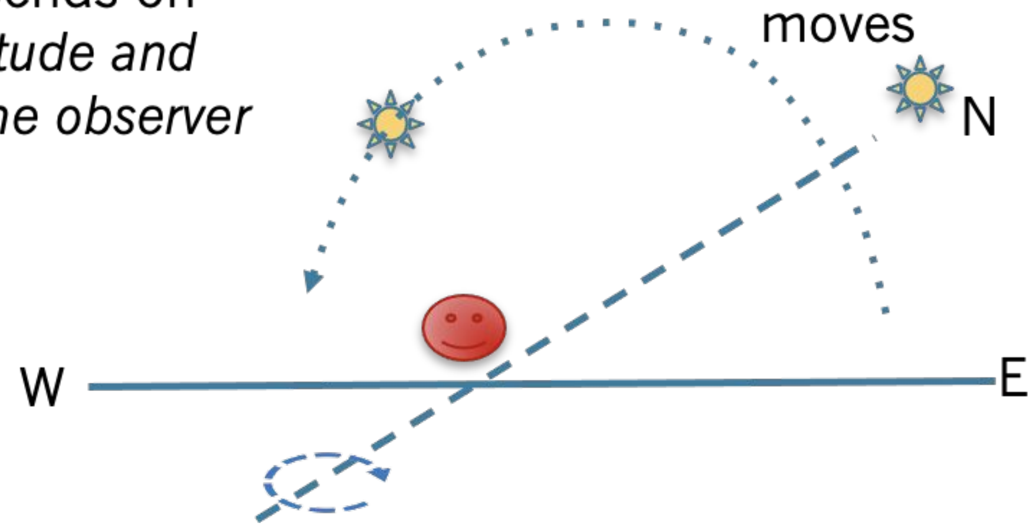
A Strange Trajectory



Could planets move in such a trajectory? How would one know?

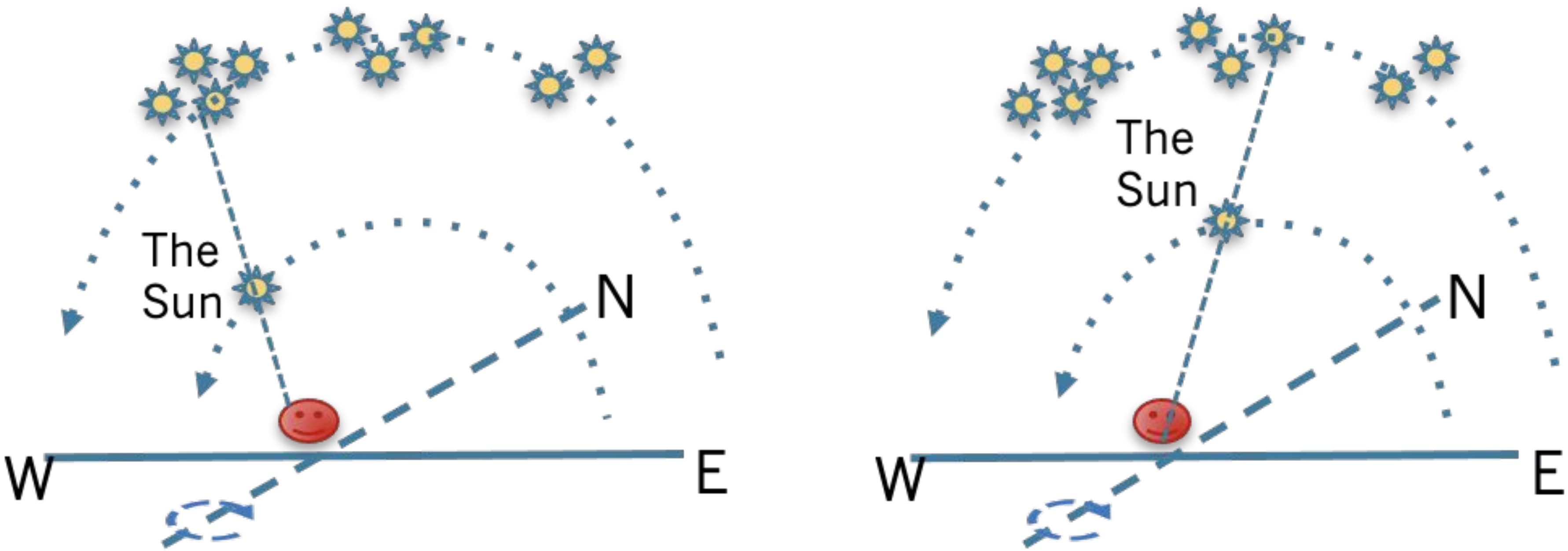
Observing Objects in the Sky

The sun, the stars and planets trace a daily circle of radius that depends on *their latitude and that of the observer*



How do we remove the confounding effects of the Earth's rotation?

Observing Objects in the Sky

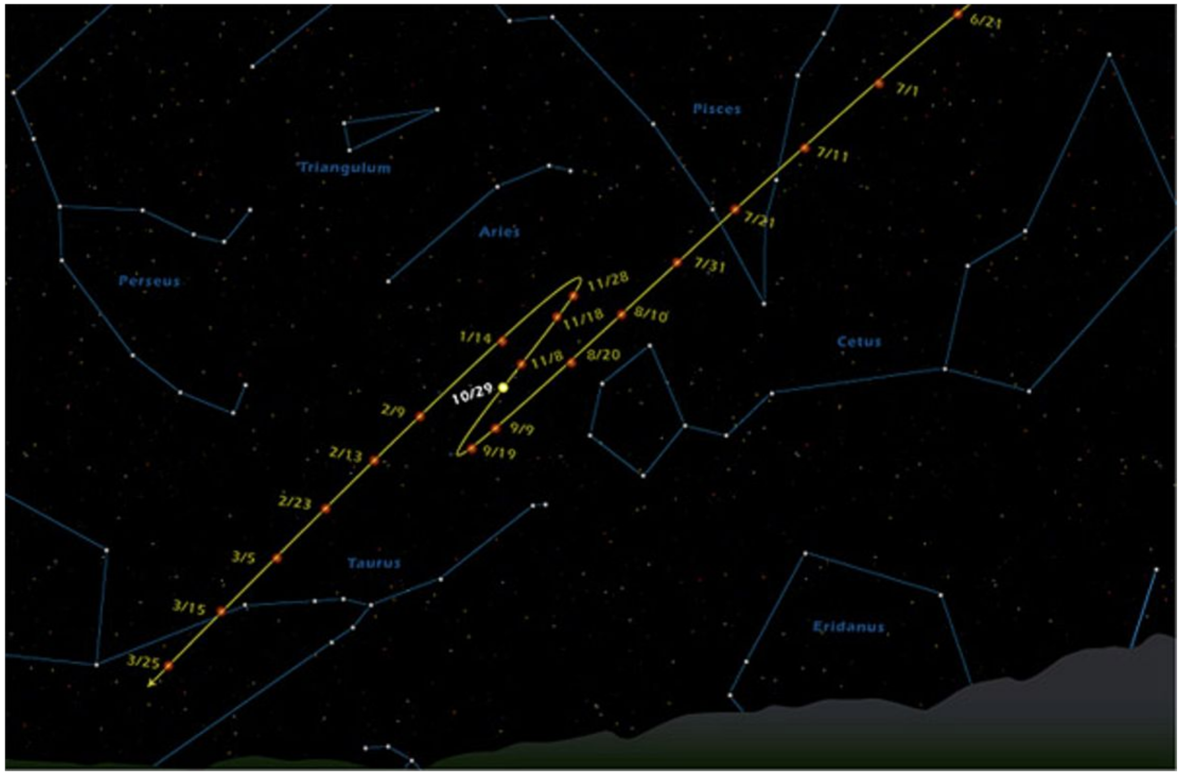


Which star lies behind? And how does that change as the months progress? The simple periodicity of ~365 days is easy to see

Questions

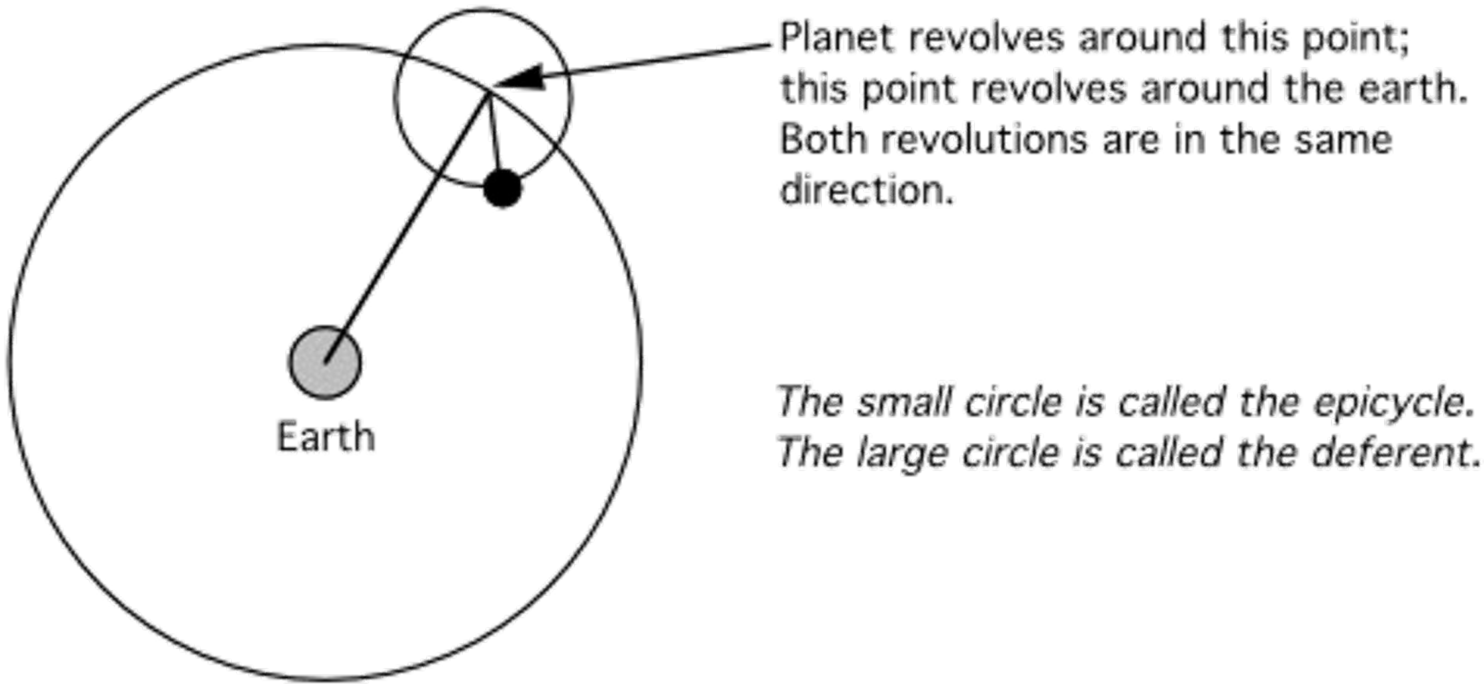
- The Sun is bright; how does one observe the star behind it?
- Use the rising/setting trick
- But requires making observations at the horizon

Mars' Eerie Zig-Zag



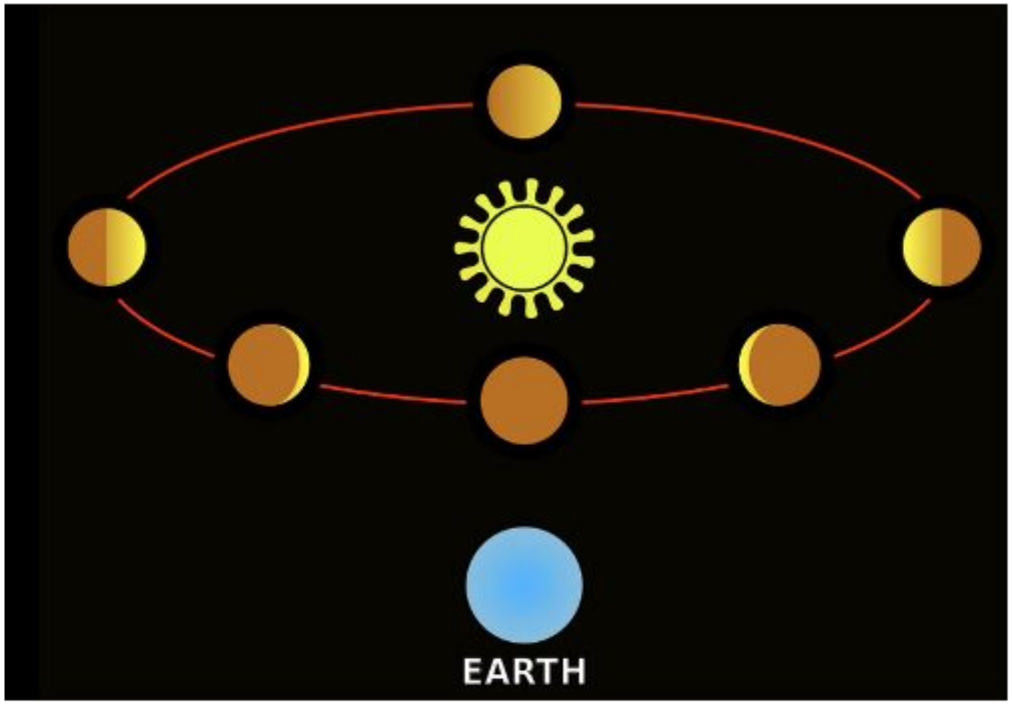
Mars moves in a strange zig-zag? Why?

A Complex Explanation



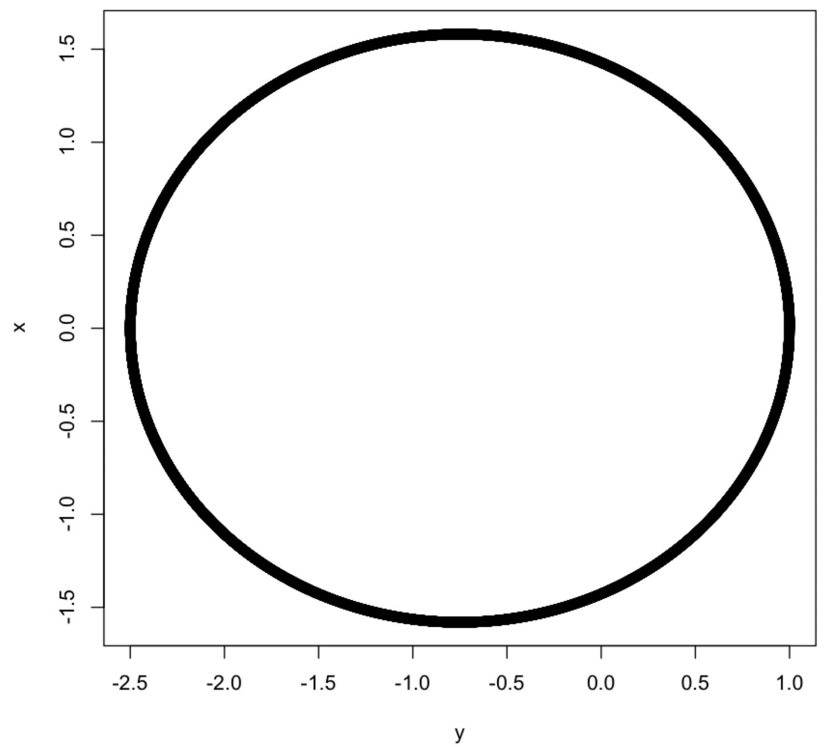
Could this epicyclic model be the right model for Mars?

The Phases of Venus



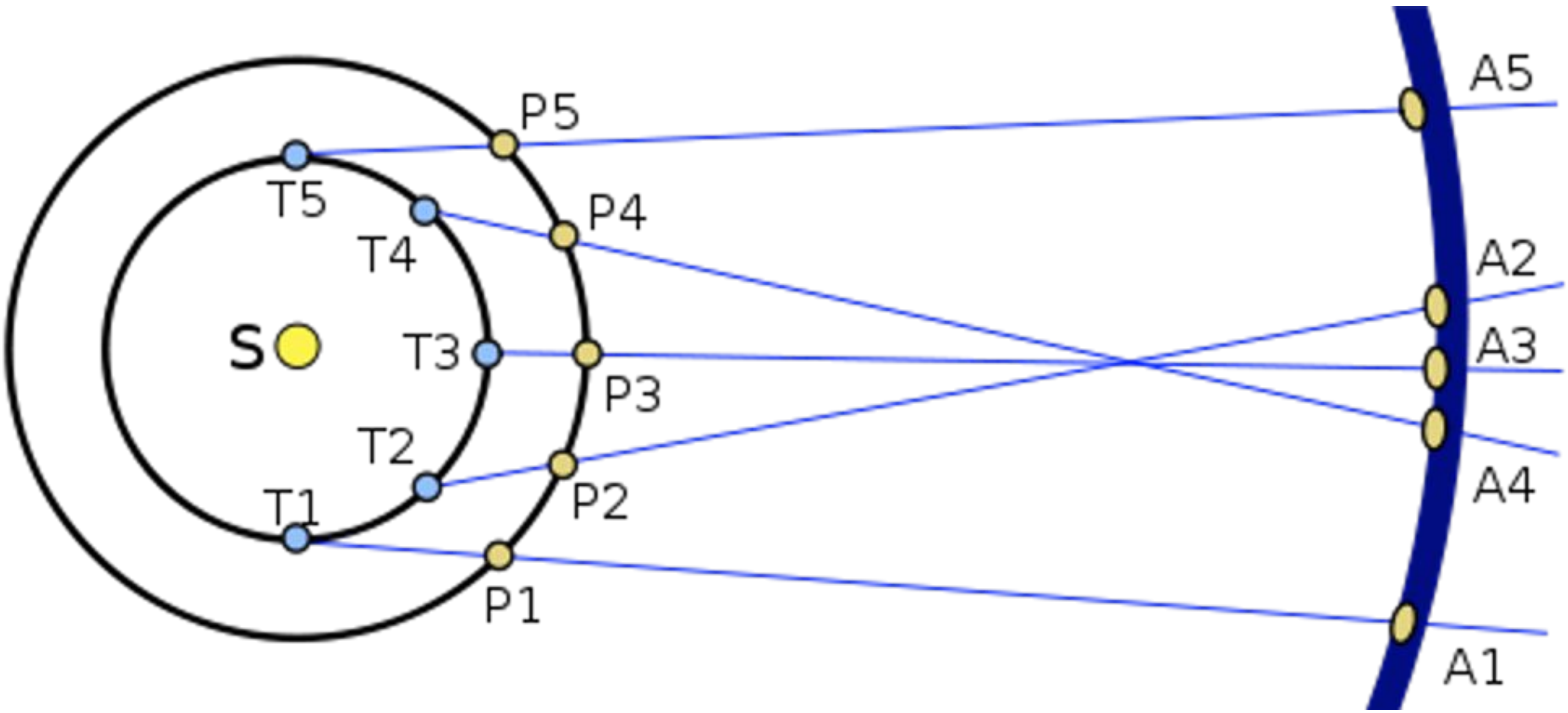
Full Venus is smaller than its crescent version! Why?

A More Likely Trajectory



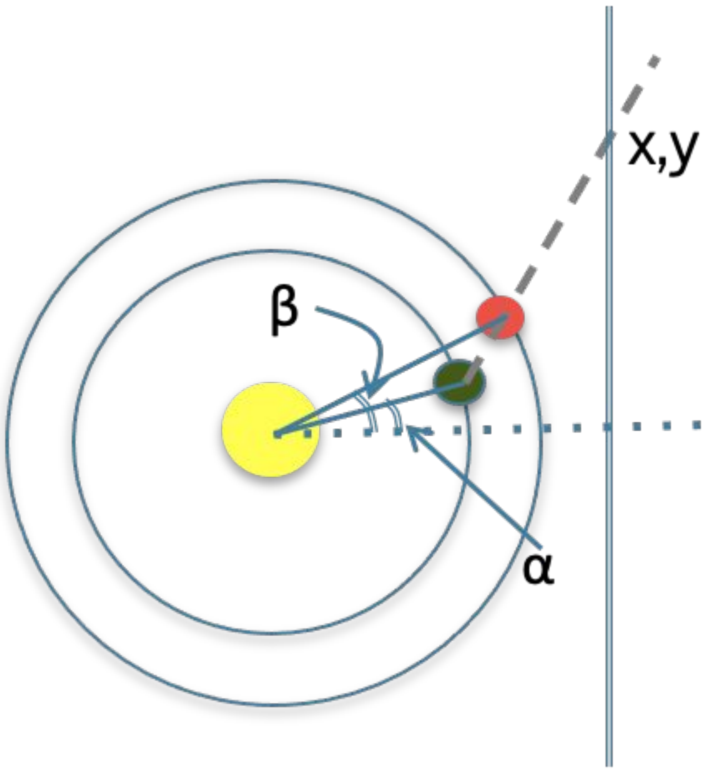
A simple closed curve, perhaps a circle, around the Sun?

Why the Zig-Zag?

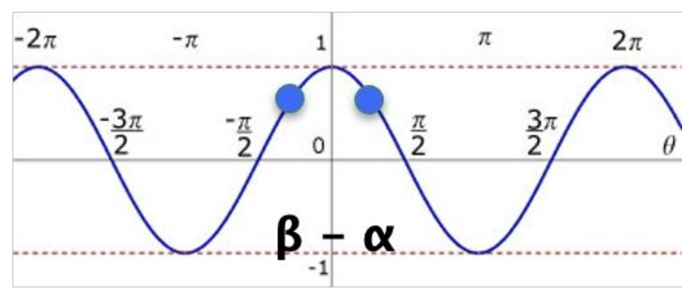


Heliocentric models explain Mars' zig zag

Explaining the Zig Zag

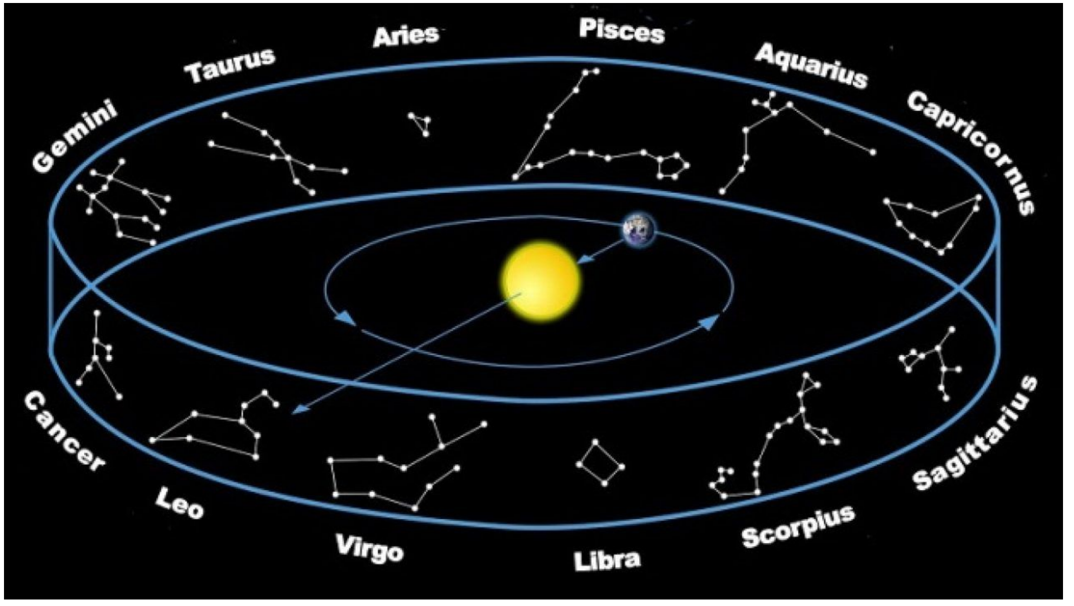


- $y = \frac{M\sin\beta - E\sin\alpha}{M\cos\beta - E\cos\alpha}(x - E\cos\alpha) + E\sin\alpha$
- For large and constant x, $y \sim \frac{M\sin\beta - E\sin\alpha}{M\cos\beta - E\cos\alpha}x$
- $\dot{y} = x\frac{M^2\dot{\beta} + E^2\dot{\alpha} - ME[\dot{\beta}+\dot{\alpha}]\cos(\beta-\alpha)}{(M\cos\beta - E\cos\alpha)^2}$
- $\dot{y} = 0$ if $\cos(\beta - \alpha) = \frac{M^2\dot{\beta} + E^2\dot{\alpha}}{ME[\dot{\beta}+\dot{\alpha}]}$



The derivative is 0 when β-α is close to 0; hence the zig-zag happens close to an Opposition (all three in a straight line)

The Zodiac

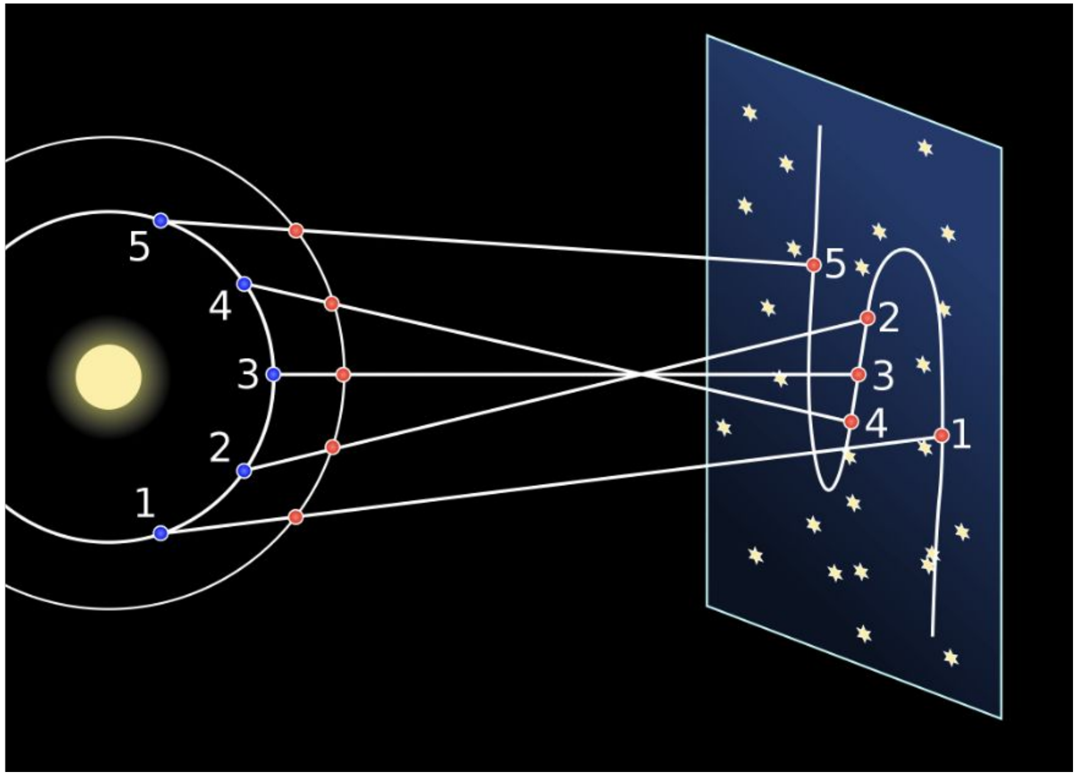


The stars that lie behind the Sun form the Zodiac; the plane on which the Earth and Sun lie is the Ecliptic

The Zodiac

Most planets stay very close to the Ecliptic. Mars' orbit is only slightly inclined relative to the Ecliptic plane.

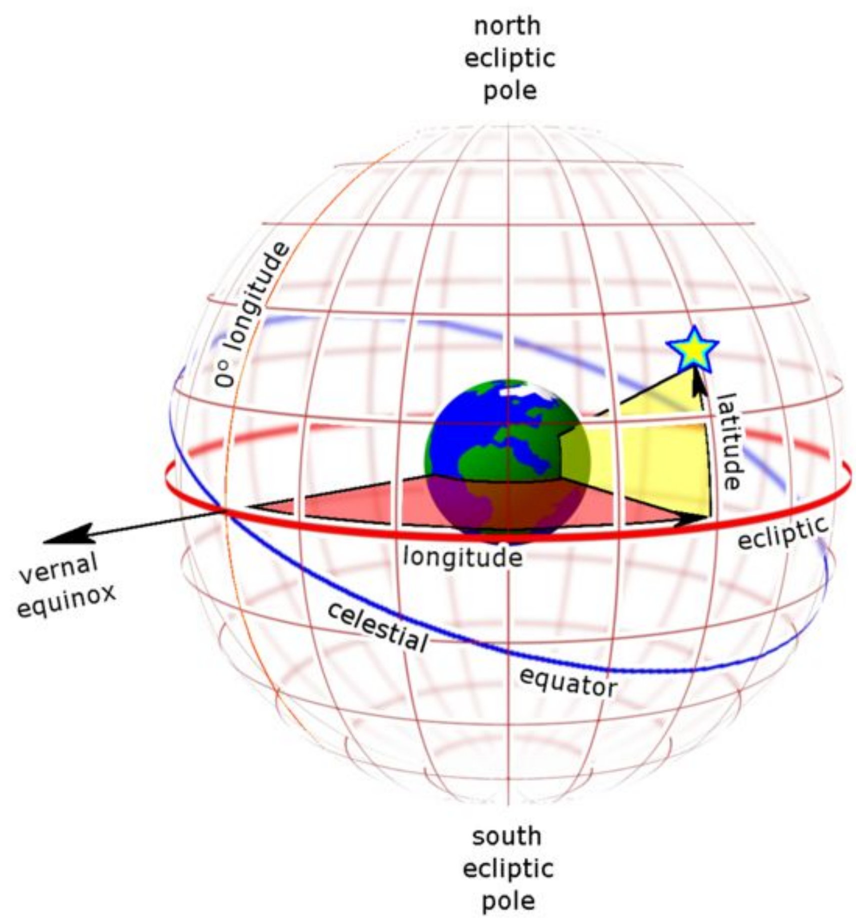
Slight Non-Coplanarity



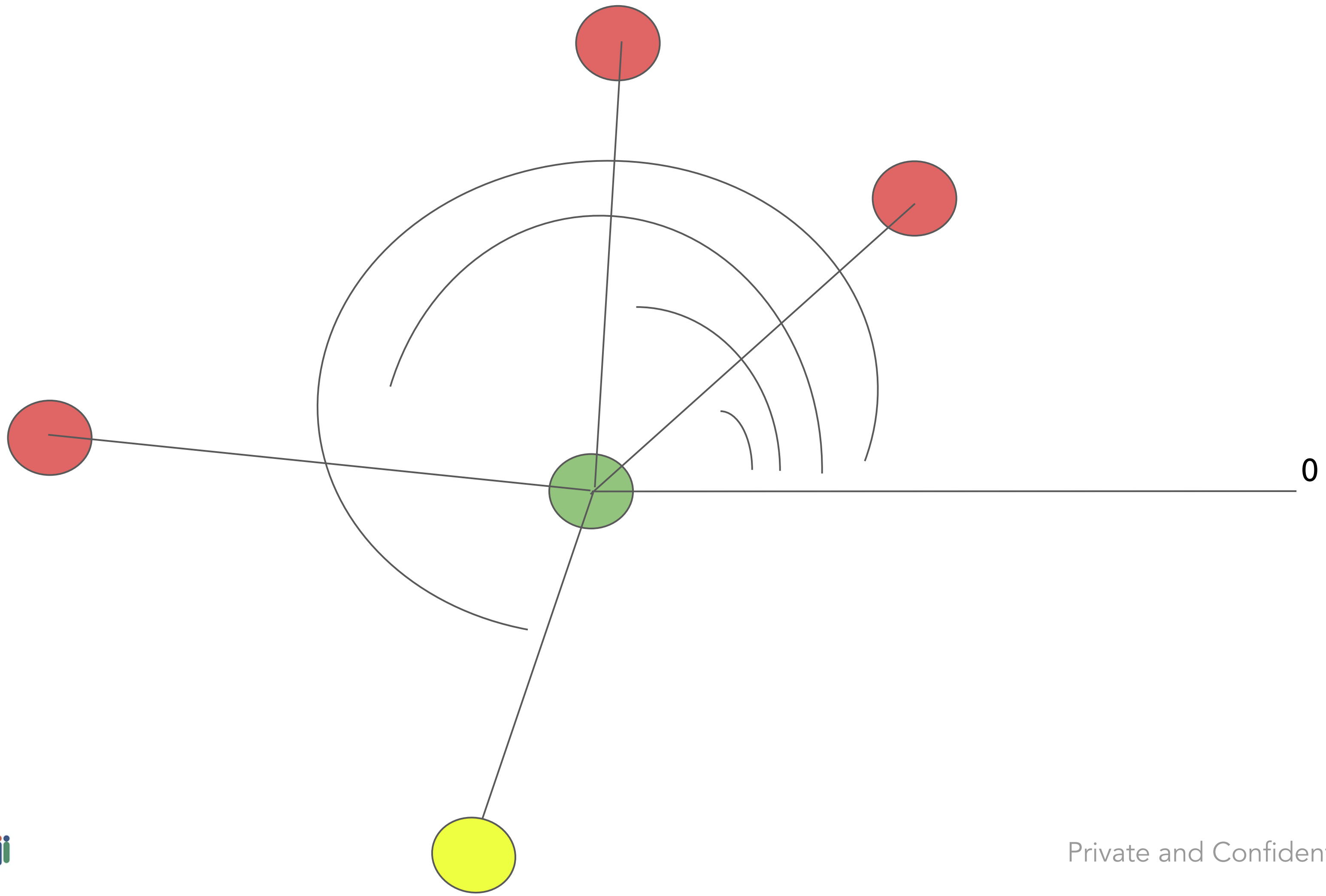
Mars' orbit is not coplanar with the ecliptic plane, though only slightly so; that explains the shift in the zig-zag

Calibrating Movement

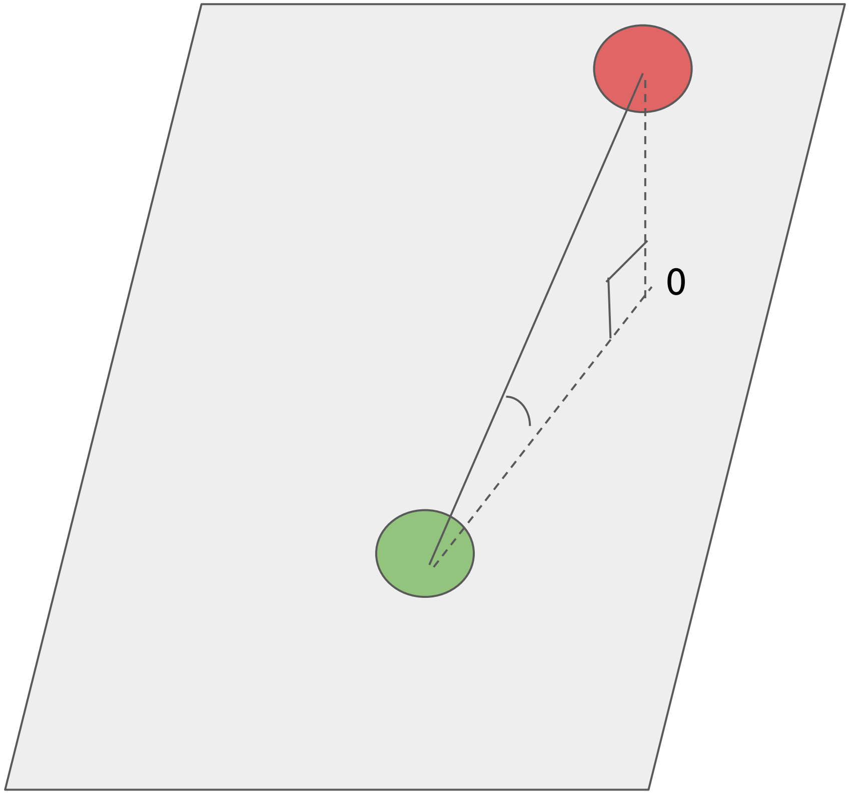
- Longitude: Angle along the Ecliptic relative to a zero point (for now arbitrarily assigned, though the spring equinox point was often the conventional zero)
- Latitude: drop a perpendicular to the ecliptic and measure the angle between the planet and the ecliptic



Longitude

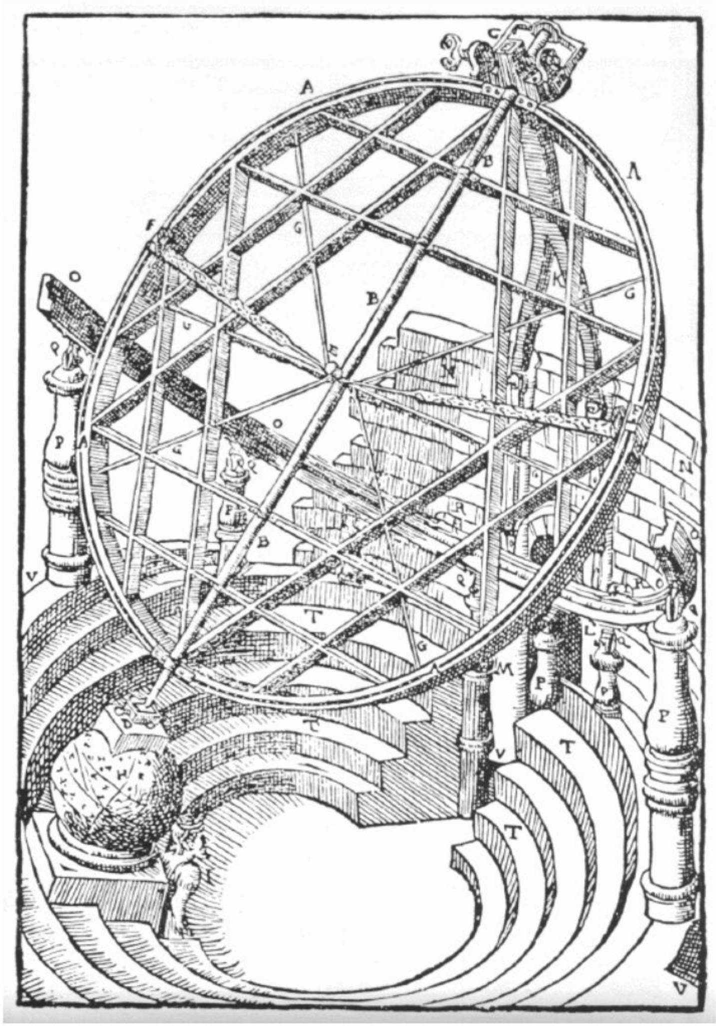


Latitude



Tycho Brahe

- Master observer, created a series of instruments to make latitude and longitude measurements precise
- Error: mostly <2′ of an arc, occasionally 4′
- Observed Mars religiously from 1580-1601 when he died
- Kepler obtained the data post his death and spent the next several years in its analysis



1585 Great Equatorial Armillary

Accuracy of Tycho Brahe's Measurements (from his Wikipedia article)

He aspired to a level of accuracy in his estimated positions of celestial bodies of being consistently within an arcminute of their real celestial locations, and also claimed to have achieved this level. But, in fact, many of the stellar positions in his star catalogues were less accurate than that. The median errors for the stellar positions in his final published catalog were about 1.5', indicating that only half of the entries were more accurate than that, with an overall mean error in each coordinate of around 2'.[76] Although the stellar observations as recorded in his observational logs were more accurate, varying from 32.3" to 48.8" for different instruments,[77] systematic errors of as much as 3' were introduced into some of the stellar positions Tycho published in his star catalog — due, for instance, to his application of an erroneous ancient value of parallax and his neglect of polestar refraction.[78] Incorrect transcription in the final published star catalogue, by scribes in Tycho's employ, was the source of even larger errors, sometimes by many degrees.[c]

Celestial objects observed near the horizon and above appear with a greater altitude than the real one, due to atmospheric refraction, and one of Tycho's most important innovations was that he worked out and published the very first tables for the systematic correction of this possible source of error. But, as advanced as they were, they attributed no refraction whatever above 45° altitude for solar refraction, and none for starlight above 20° altitude.[82]

The Data

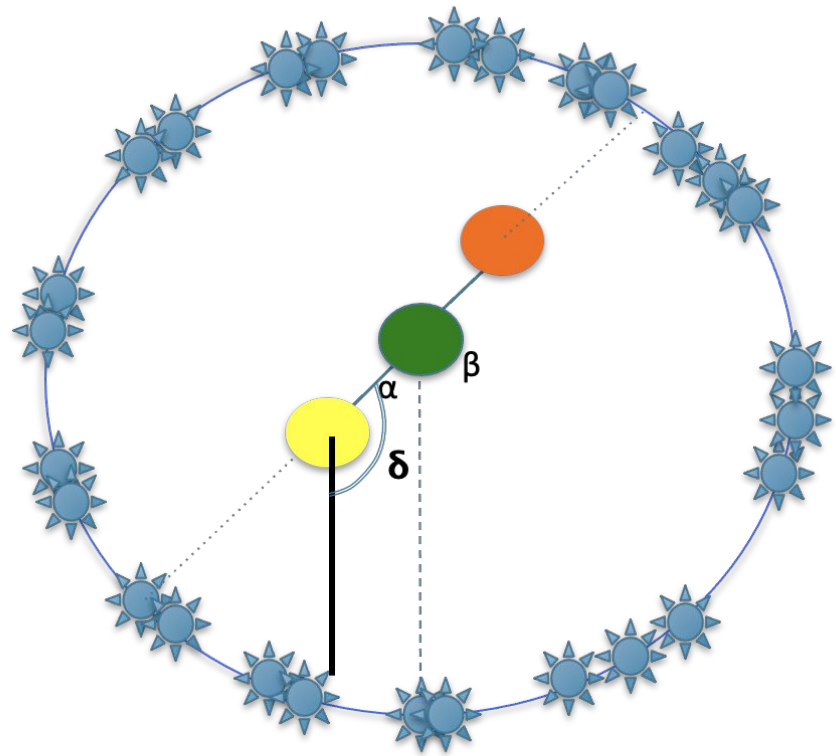
Over 20 years of observations, with the following for each observation

- Time
- Geocentric Longitudes
- Geocentric Latitudes
- With an accuracy largely of ~2′ with the occasional observation being off by ~4′
- After corrections for optics etc
    -

Only some parts of this data are easily accessible though

Oppositions

- If Mars goes around the Sun
- Then it is useful to deal with Heliocentric longitudes (δ) than with Geocentric longitudes (β)
- However, one can only measure Geocentric longitudes
- But at oppositions, both longitudes are identical



Time, latitude and longitude for 12 oppositions is accessible

The Oppositions Data

Time [Year, Mon, Day, Hour, Min]

```
[[1580,11,18,1,31],[1582,12,28,3,58],[1585,1,30,19,14],[1587,3,6,7,23],[1589
,4,14,6,23],[1591,6,8,7,43],[1593,8,25,17,27],[1595,10,31,0,39],[1597,12,13,
15,44],[1600,1,18,14,2],[1602,2,20,14,13],[1604,3,28,16,23]]
intervals = [1108947, 1101076, 1100889, 1108740, 1130480, 1165544, 1146672, 1115465, 1102938, 1100171,
1104610] #in minutes, between successive records
```

Latitudes [Deg, Min, N or s]

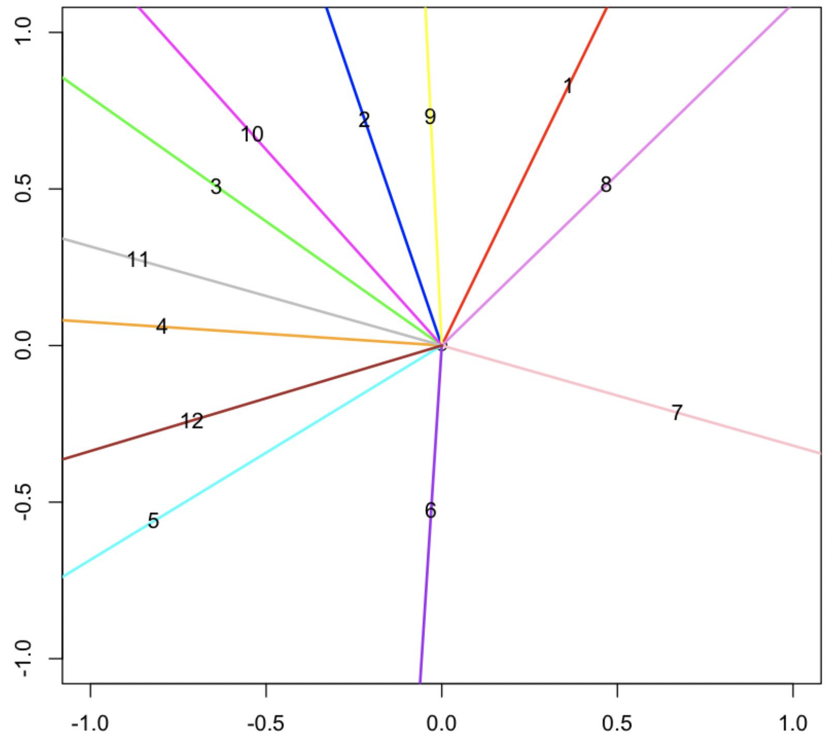
```
[[1,40,'N'],[4,6,'N'],[4,32,'N'],[3,41,'N'],[1,12,'N'],[4,0,'S'],[6,2,'S'],[0,8,'
N'],[3,33,'N'],[4,30, 'N'],[4,10, 'N'],[2,26,'N']]
```

Longitudes [Deg, Min, Sec, Zodiac 30° Band]

```
[[6,28,35,2], [16,55,30,3], [21,36,10,4], [25,43,0,5], [4,23,0,7],  [26,43,0,8],
[12,16,0,11], [17,31,40,1], [2,28,0,3], [8,38,0,4], [12,27,0,5], [18,37,10,6]]
```

Time, heliocentric longitude and geocentric latitude for 12 oppositions

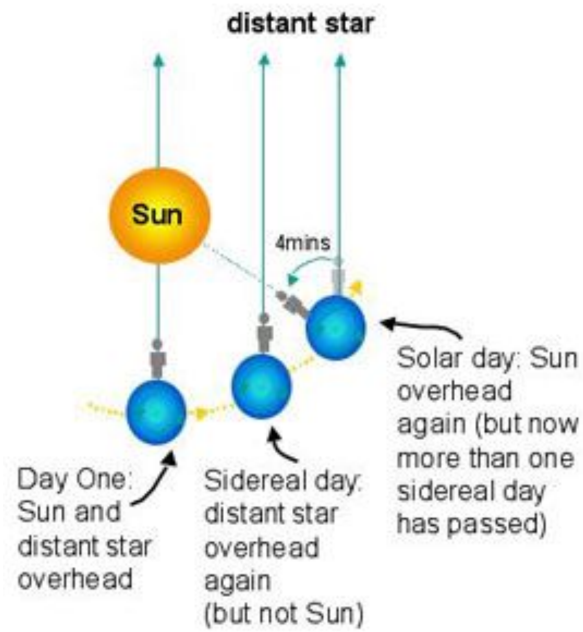
Longitudes from the Oppositions Data



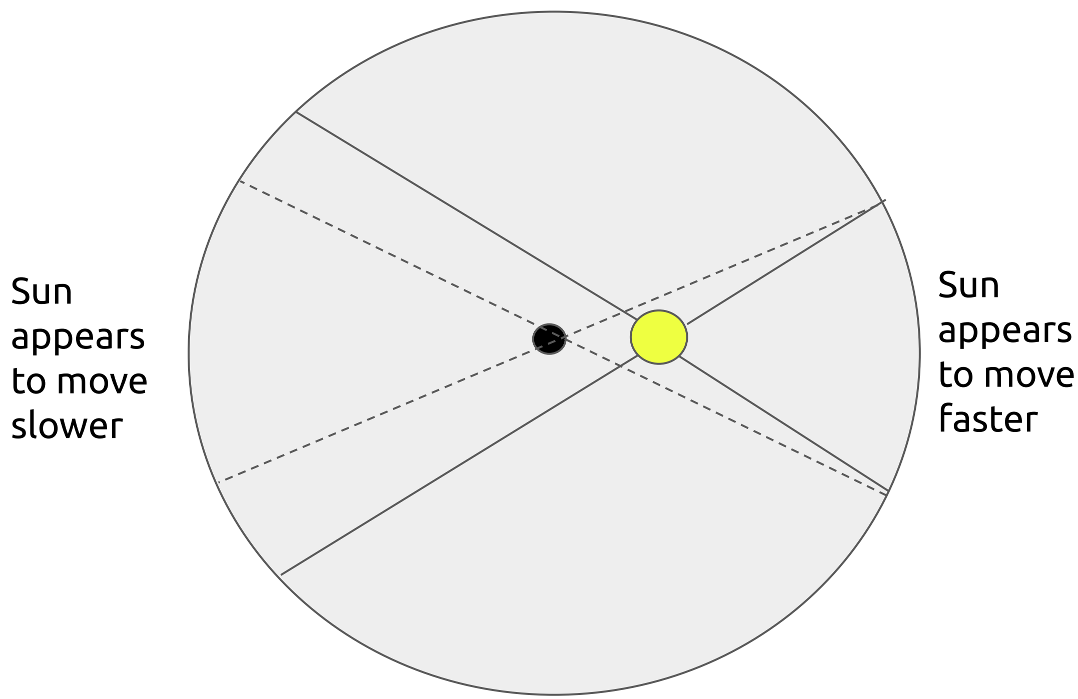
The Sun is taken as the fixed center here; the +ve x axis is the zero longitude. Where on these lines is Mars?

Cadence

- Uniform speed? Or not?
- Earth is faster in Dec-Jan, slower in Jun-Jul!
- The Analemma (plot the Sun in the sky at noon everyday)
- How about Mars?



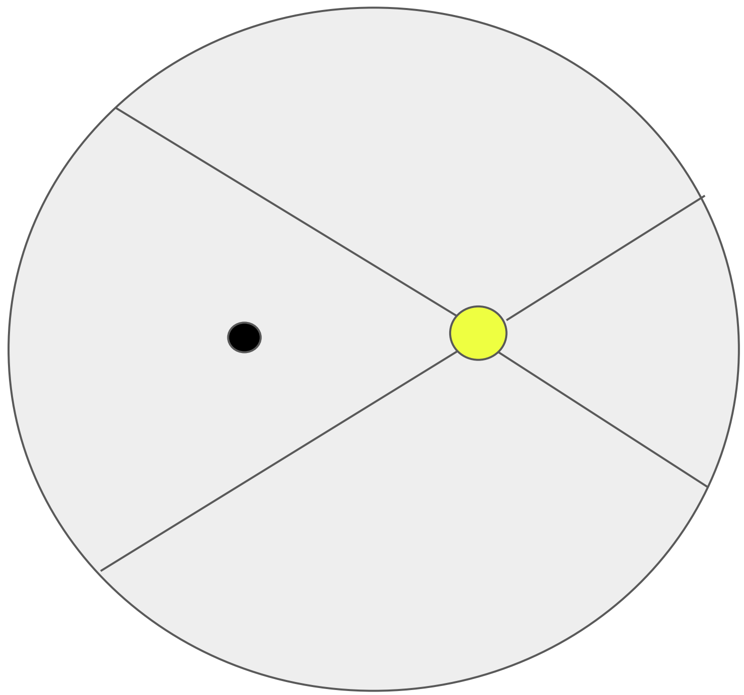
Perhaps an Off Center Sun



Uniform angular speed about the center implies varying angular speed about the Sun.

The Imaginary Equant Hypothesis

Sun appears to move slower



Sun appears to move faster

Uniform angular speed not around the center of the circle but around some other unknown point

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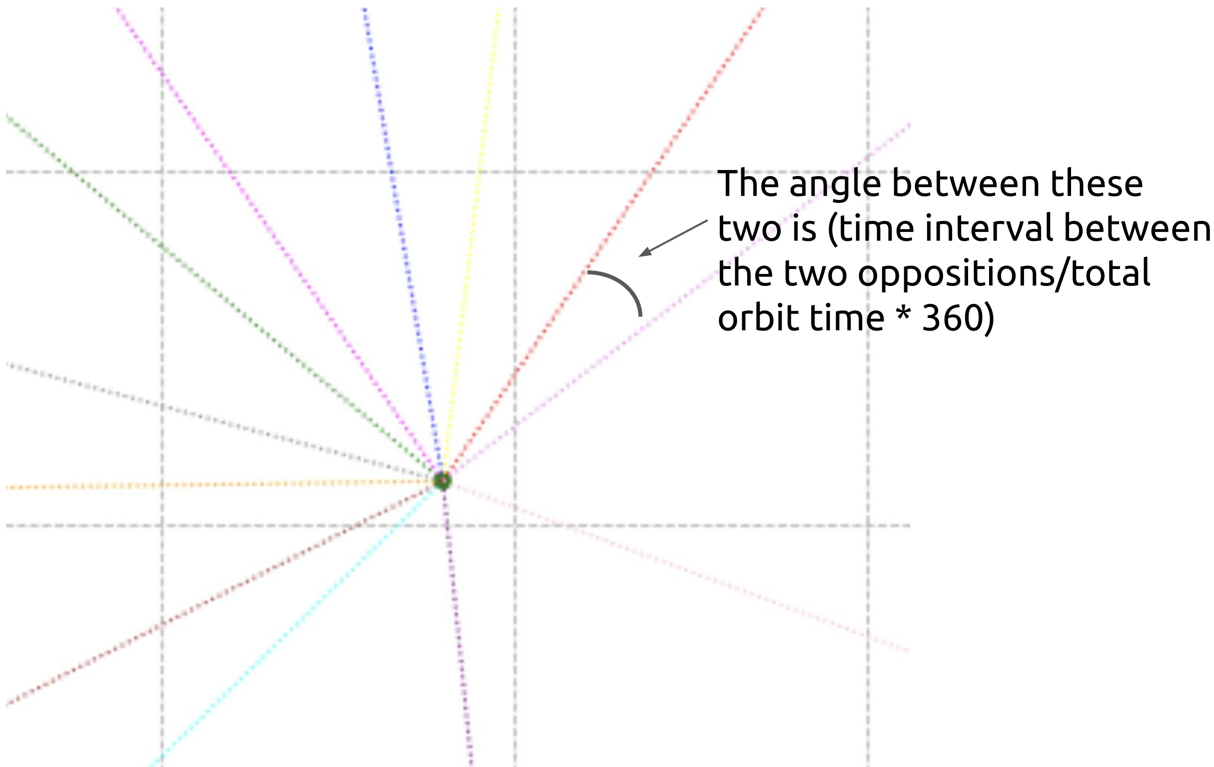
Orbit Time Periods

- How do we determine the time taken by Mars to go around its orbit once?
- Geocentric observations confound periodicity
- Use latitudes
  - Latitude 0 is easily observable from the Earth
  - Happens twice in each orbit
- The period as calculated from Brahe's observations turns out to be ~687 days
- With enough observations, this can be pinned down more precisely as well, which is what Kepler did
- For our analysis, since we are using only a subset of the data, we will treat it as an unknown parameter to be estimated, since we need precision
- We will assume periodicity though (i.e, each round around the orbit takes the same amount of time)

Using the Equant

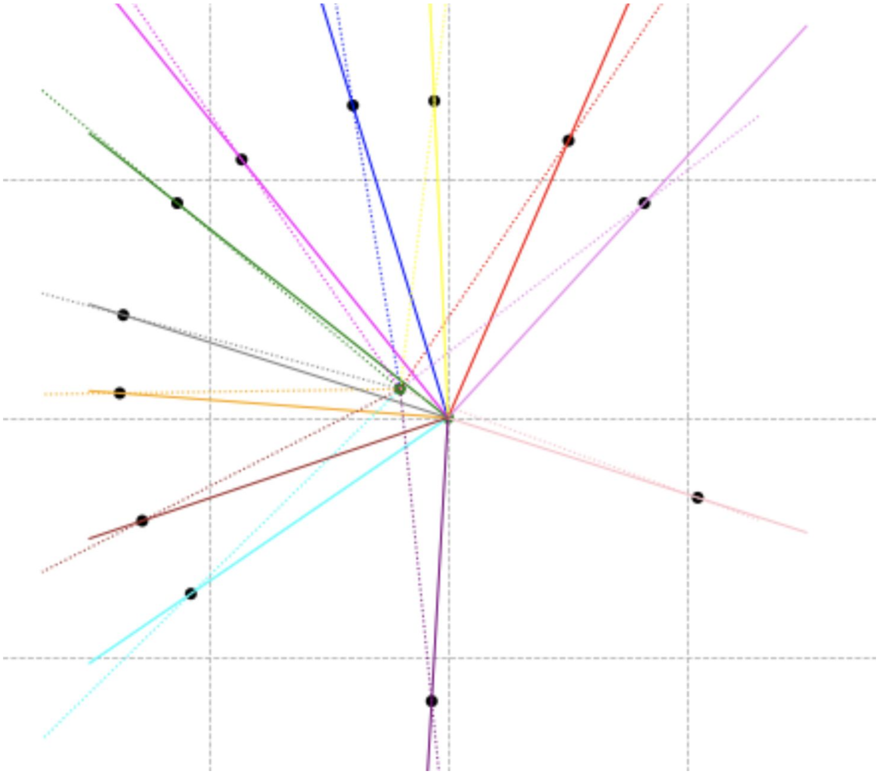


Each dotted line points towards Mars' position at the time of the resp opposition. Time intervals between oppositions provide the angles between the dotted lines, centered at the Equant.

Equant Longitudes

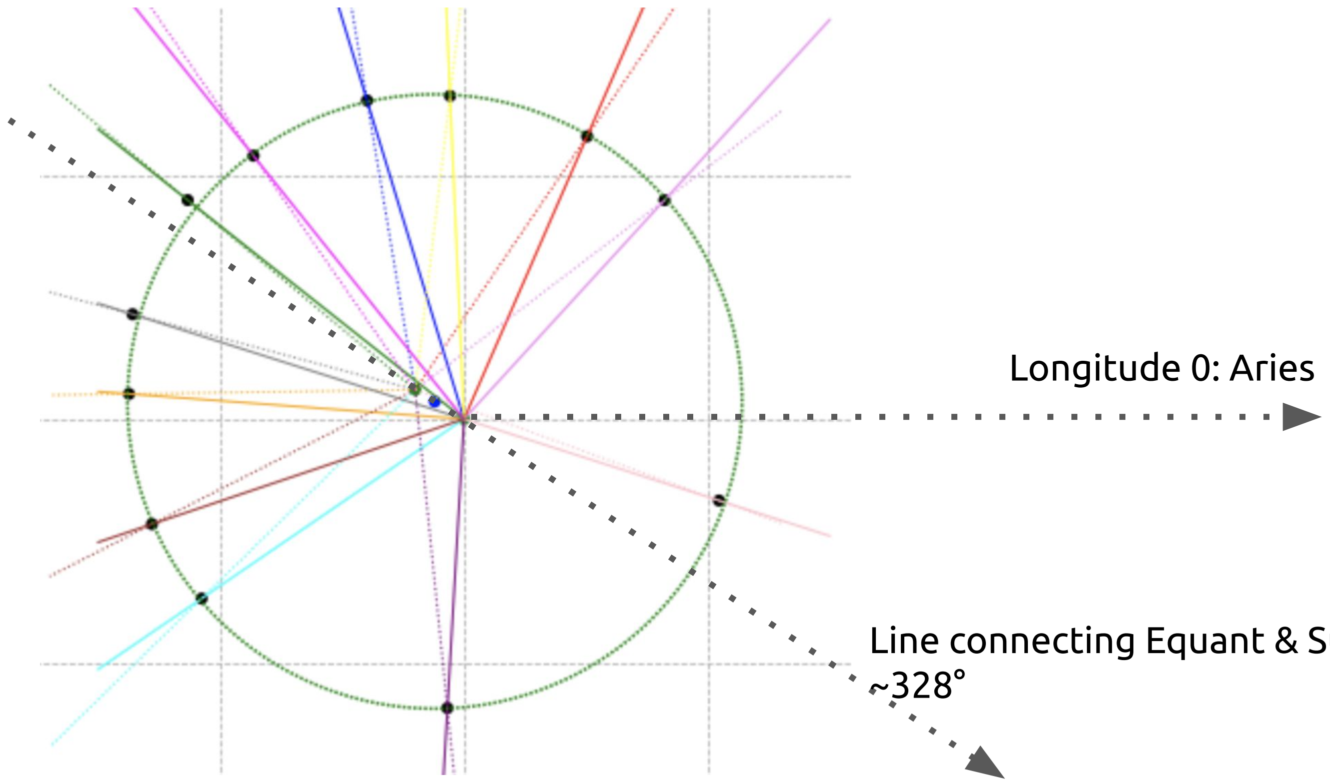
- The position of the Equant and the rotation of the the system of dotted lines around the Equant are unknowns. Once fixed, we get a set of dotted line directions that are called Equant longitudes

- The intersections of the resp dotted and solid lines yield the positions of Mars (black dots)



Optimize over all possible Equant positions and rotations of the dotted line system so the black dots lie as close to a circle as possible

Fitting an Orbit



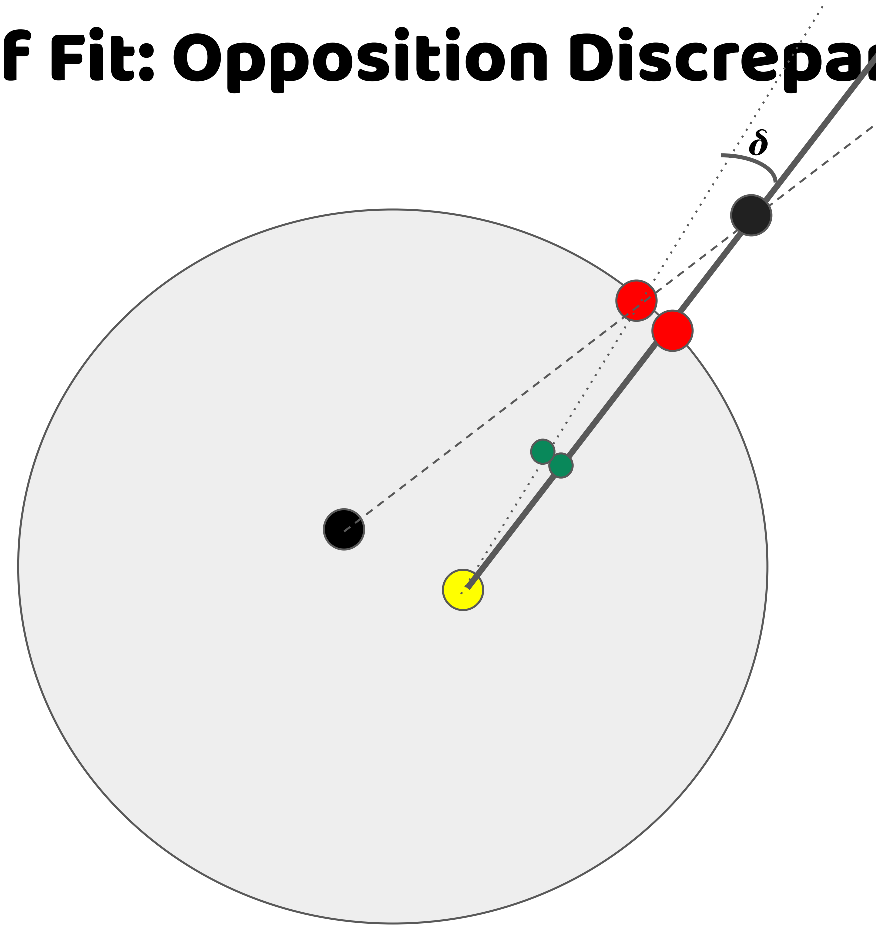
The Circle fit to the black dots makes for a close but not a perfect fit. The center (blue dot) is in between the Sun and the Equant. What is the right goodness of fit measure?

Defining the Goodness of Fit

- Given an orbit model
    - Say a circle
    - With a specified center, radius and position for the sun
    - And a specified Equant
    - And a specified direction for the Equant 0 longitude
    - And a specified orbit time period
- What is the discrepancy between the model and the observation?
- The orbit model predicts a position on the orbit for each opposition
- The heliocentric longitude observations indicate another position on the orbit for each opposition
- If these two positions coincide for all observations, then our orbit model is perfect
- If not, then we have some discrepancy

This discrepancy should be less than the error in each observation for the orbit model to be acceptable.

Goodness of Fit: Opposition Discrepancy



If δ<4′ then there exists an position of the Earth so the longitudes measured from the Earth for the Sun and for Mars are within 4′ of the actual resp observations, which is within the range of observational error

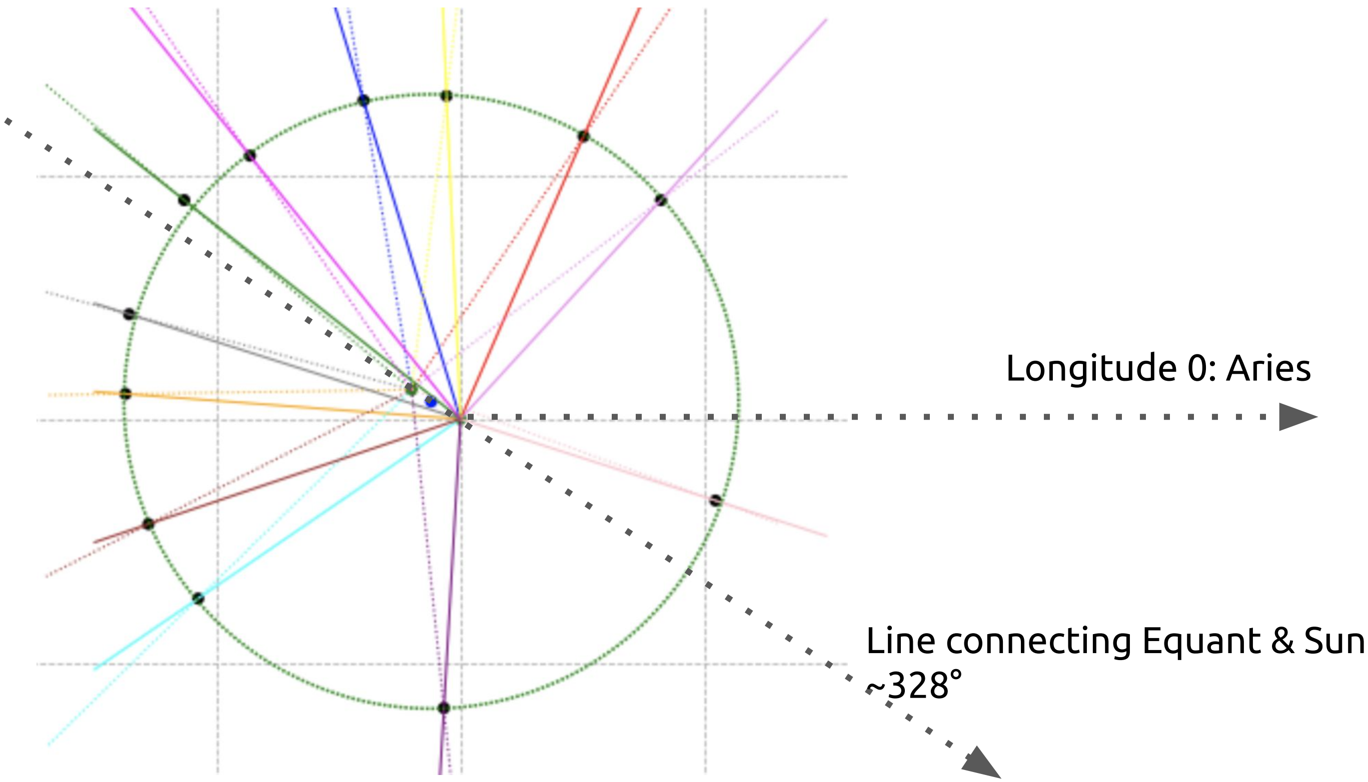
The Problem, then..

- Circular Orbit for Mars, with arbitrary center (c) and radius (r)
- Uniform angular speed (s) for Mars around an arbitrary Equant (e)
- The zero of the Equant longitudes (z)
- Determine c, e, z, s, r so as the minimize the max Oppositions Discrepancy
- What is the value of this discrepancy?

How do you optimize?

- Determine c, e, z, s, r so as the minimize the max Oppositions Discrepancy
- c: just one parameter (the angle)
- e: two parameters (angle and distance from the Sun)
- z: one parameter (angle)
- Do a discretized exhaustive search over these 4 parameters
- That leaves s and r on the inner loop
- For s, do a discretized search in the neighborhood of 687 days
- For r, do a discretized search in the neighborhood of the average distance of the black dots from the center
- A binary search might also work for the two parameters above, given likely single local minimum
- Inside the innermost loop, compute the discrepancy for each opposition

Fitting an Orbit

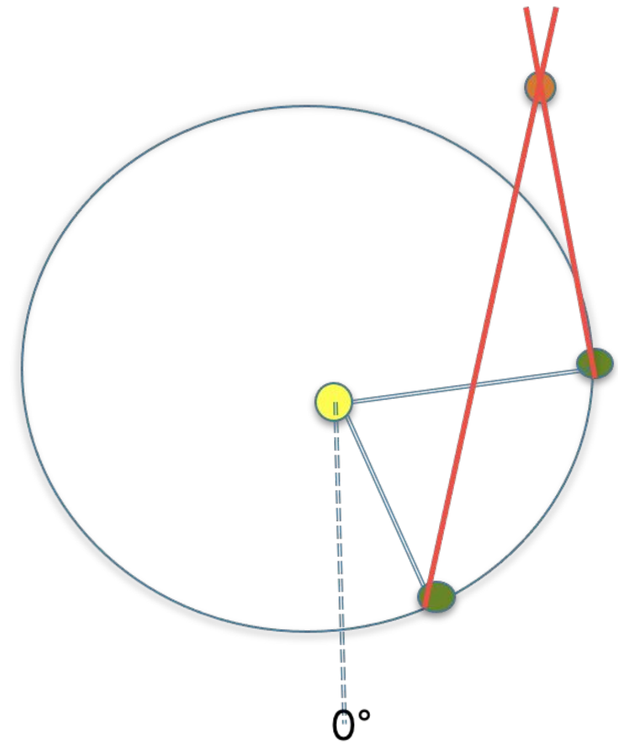


With a circular orbit with center between the Sun and the Equant, the discrepancies are

```
[2.2365144524494895, -3.2120110368510866, -3.5669725653222106, -1.06042215541062, -3.6399623095565166,
-2.1601790100240748, 2.980086019452943, 3.6599934592104133, 0.8100027563293606, 0.22279241557410429,
3.2673586389416465, 1.8050469915310428]
```

What is the Radius of this Circle?

- We can only measure angles; then how do we get distances?
- The triangulation trick
- The time period is T=~687 Earth days (Kepler would have used a more precise value)
- Then take pairs of observations, the two in a pair being T apart in time.
- The Earth varies between the two observations in a pair, but Mars stays the same
- As earlier, you have geocentric longitudes for the Sun and for Mars



The distance between the Earth and the Sun is assumed to be 1 AU (of course, this assumes Earth's orbit is a circle with the Sun at the center, otherwise 1AU will have to be redefined appropriately)

The Triangulations Data

[Heliocentric Longitude of Earth, Geocentric Longitude of Mars each in Deg, Min]

[159,23,135,12] and [115,21,182,8]

[5, 47, 284, 18] and [323, 26, 346, 56]

[85, 53, 3, 4] and [41, 42, 49, 42]

[196, 50, 168, 12] and [153, 42, 218, 48]

[179, 41, 131, 48] and [136, 6, 184, 42]

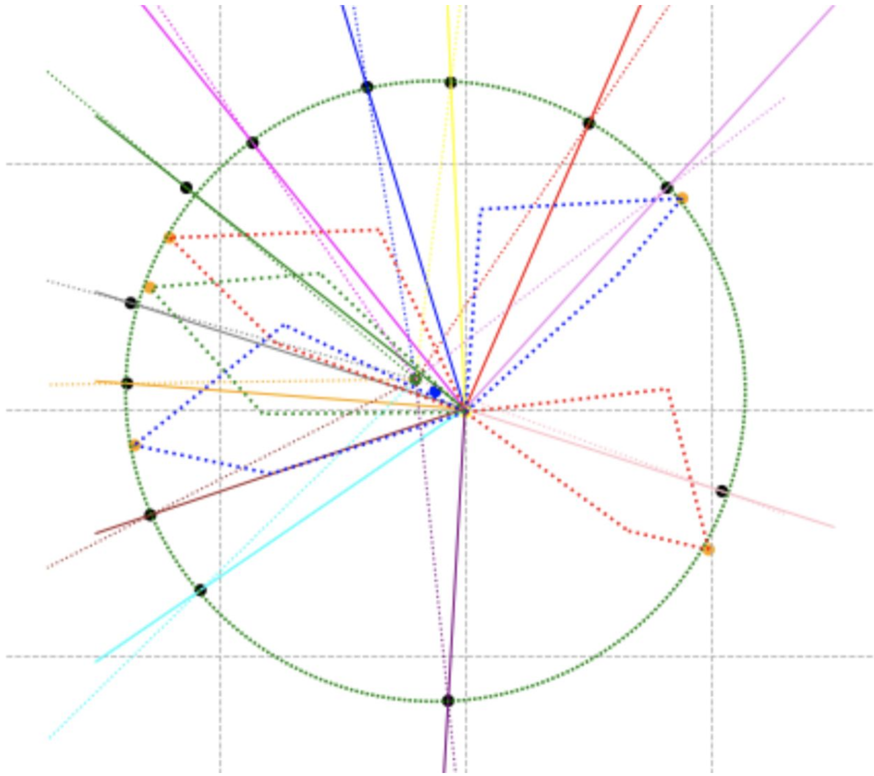
http://faculty.a.edu/saustin/3110/mars.pdf

TABLE I: Tycho's observations of Mars from Kepler's Astronomia nova.

Date	θ (Heliocentric Long. of Earth)	ϕ (Geocentric Long. of Mars)
1585 Feb. 17	159° 23′	135° 12′
1587 Jan. 5	115° 21′	182° 08′
1591 Sep. 19	5° 47′	284° 18′
1583 Aug. 6	323° 26′	346° 56′
1593 Dec. 7	85° 53′	3° 04′
1595 Oct. 25	41° 42′	49° 42′
1587 Mar. 28	196° 50′	168° 12′
1589 Feb. 12	153° 42′	218° 48′
1585 Mar. 10	179° 41′	131° 48′
1587 Jan. 26	136° 06′	184° 42′

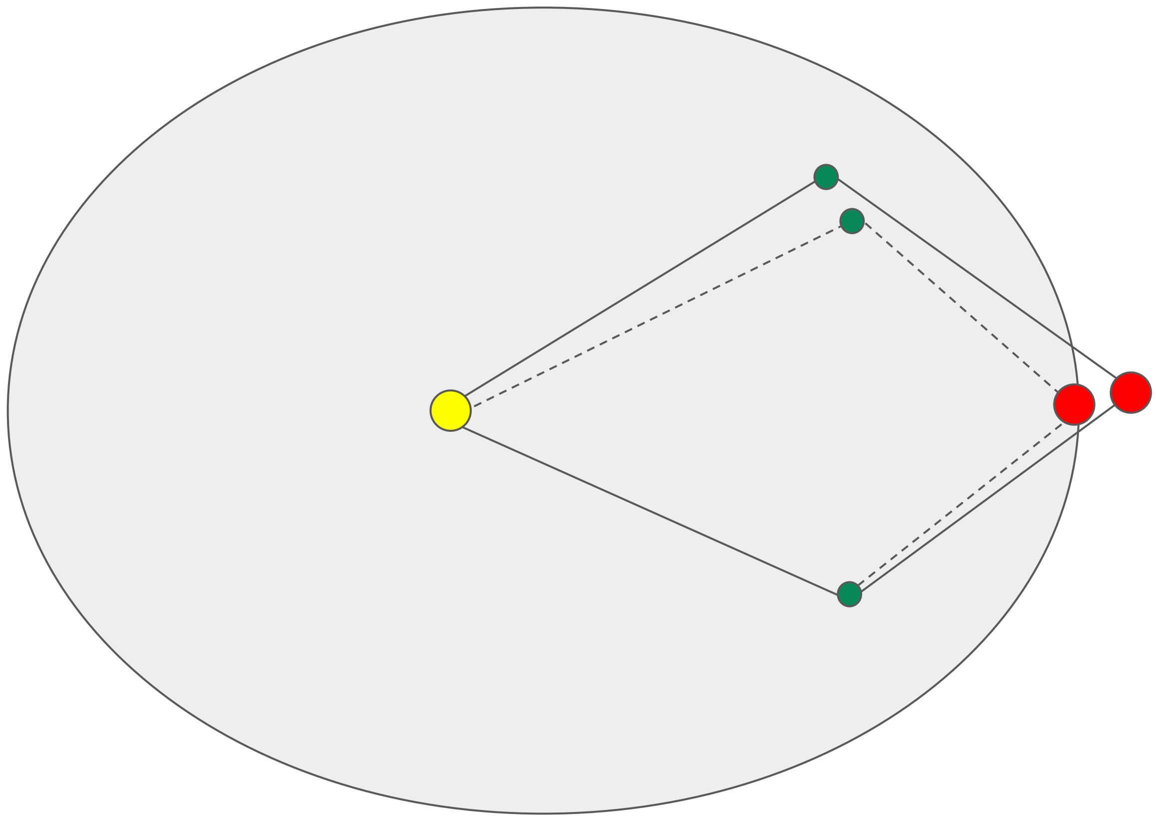
While we have only dates in this picture, Tycho Brahe's observations are likely to have had the hour and min as well

Superimposing Triangulations



Assuming the Earth has a circular orbit centered at the Sun, scale the radius of this orbit appropriately so the triangulation positions of Mars come close to the circular orbit for Mars. The radius of Mars' circle is ~1.5x of Earth's circle.

Triangulations Discrepancy

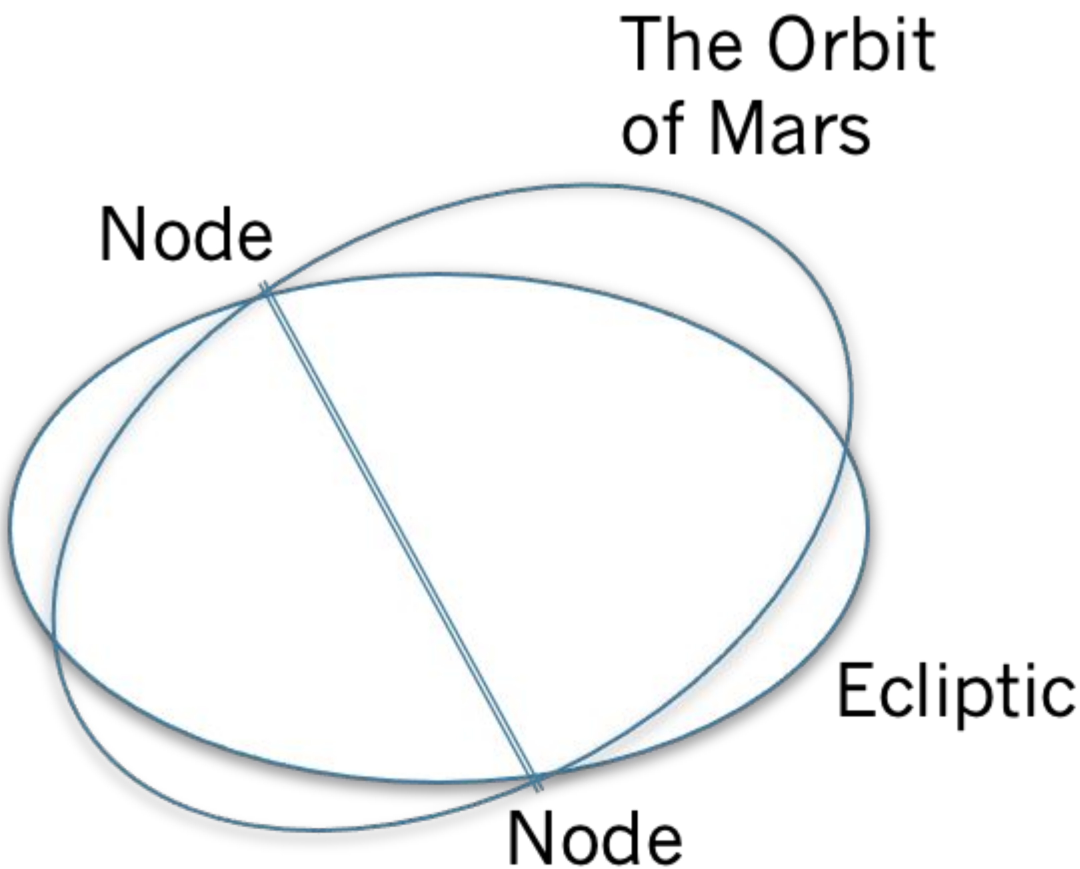


How much do we need to tweak each of the 4 angles here so Mars lies on the proposed orbit? If that is within 4′ each then the proposed orbit is consistent with the observations, up to the observational error of 4′ per angle

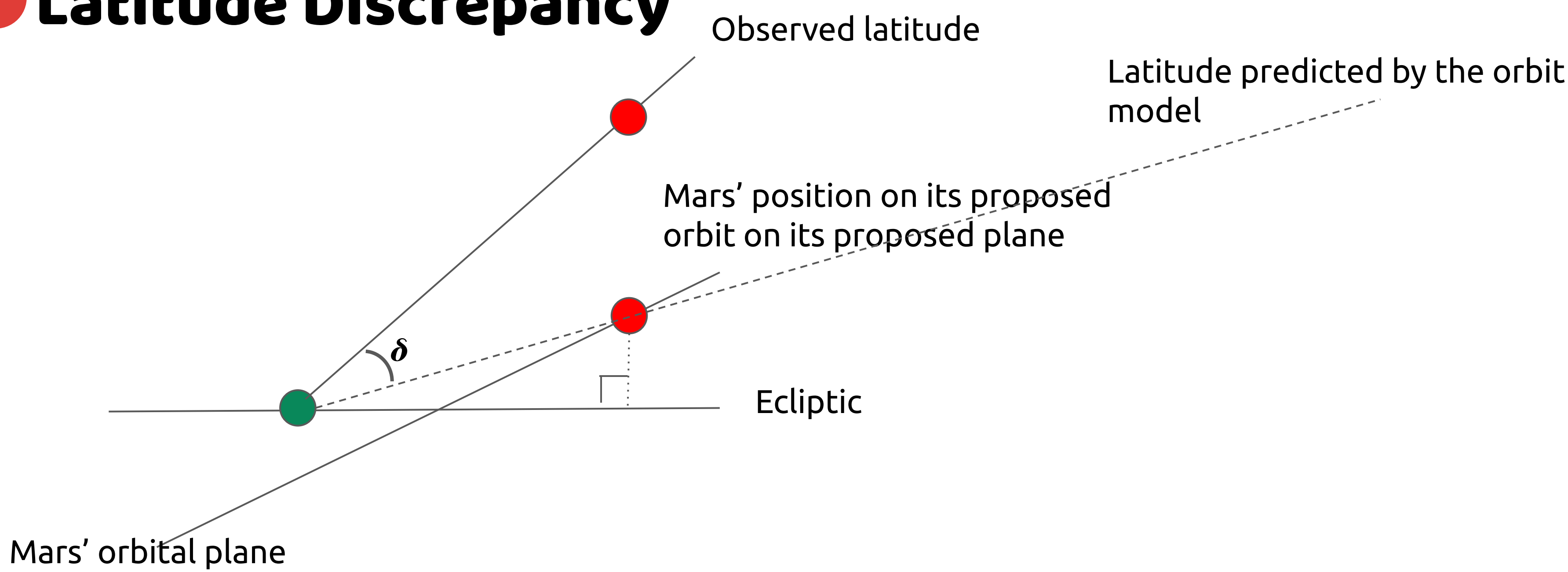
The Problem, then..

- Circular Orbit for Mars, with arbitrary center (c) and radius (r)
- Circular Orbit for Earth, centered at the Sun with radius 1AU
- Uniform angular speed (s) for Mars around an arbitrary Equant (e)
- The zero of the Equant longitudes (z)
- Determine c, e, z, s, r so as the minimize the max Oppositions Discrepancy and Triangulations DIscrepancy
- What is the value of this discrepancy?
- Just a bit too large given the accuracy of Tycho Brahe's observations (but not way way off either, so at least some of the parameters are in the neighborhood)
- So one or more assumptions above is not true

Non-coplanarity



Latitude Discrepancy

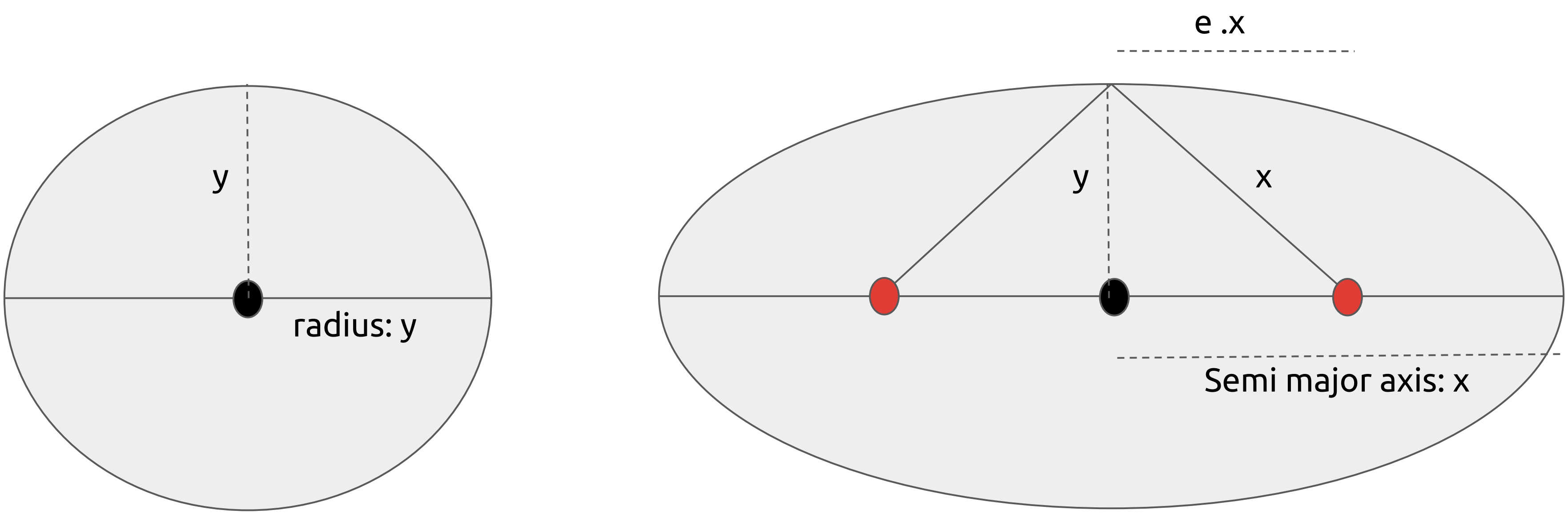


The angle between the observed and predicted latitudes is the discrepancy

So the Challenge

- Non-circular orbit for Mars?
- What other curves? Center/radius?
- Non-circular orbit for Earth? Center? Same as Mars, or different?
- Cadence model: If not Equant, then what?
- Same for both Earth and Mars or different?
- What intersection line between the ecliptic and Mars' orbit plane?
- What inclination between the two planes?
- Find values for all the above parameters so as to minimize the max Oppositions Long. Discrepancy, Triangulations Long. Discrepancy and Oppositions Lat. Discrepancy

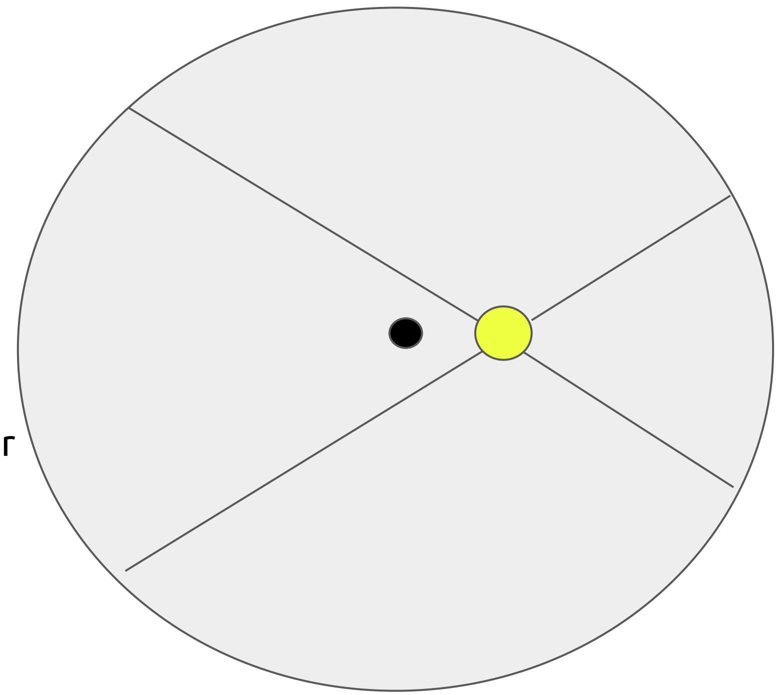
Kepler Leap 1: Generalizing Circles by Ellipses



Just differential stretching on the two orthogonal axes. The center is stretched apart into two so called foci, with e being the eccentricity

Kepler Leap 2: Eliminating the Equant

The Sun appears to move slower, takes more time and covers a lesser area



The Sun appears to move faster, takes lesser time and covers a lesser area

Instead of uniform angular speed around the Equant, equal areas in equal time around the Sun or the focus of the ellipse, or another point in the ellipse

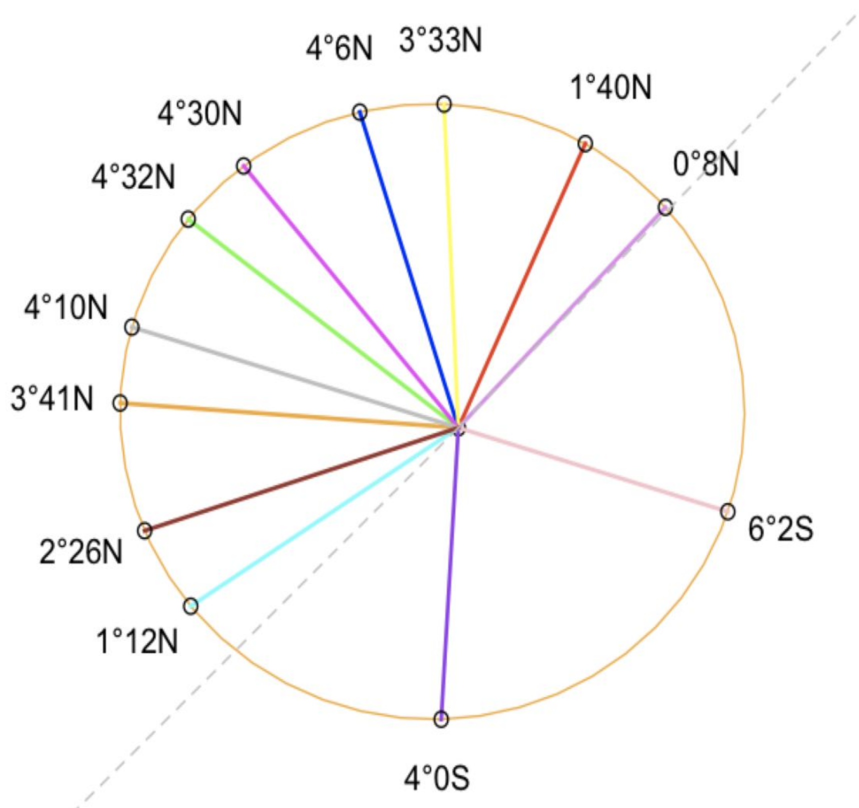
The Full Problem: A Candidate Project

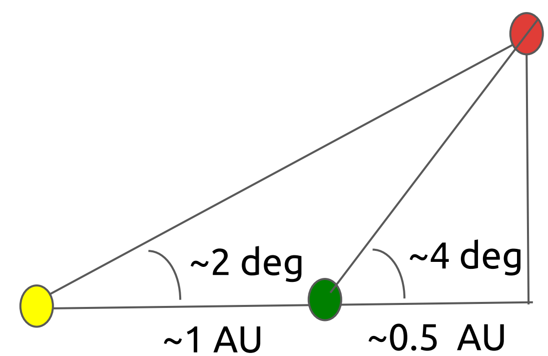
- Major axis for Earth is 1AU
- Elliptical Orbit for Earth, with arbitrary focus (f), major axis direction (d), eccentricity e
- Line of intersection of Mars' plane with the ecliptic (l)
- Inclination of Mars' plane relative to the ecliptic (i)
- Elliptical orbit for Mars on its plane, with arbitrary focus (f'), major axis direction (d'), eccentricity e'
- Major Axis for Mars (m)
- The areal speed for Earth about its focus (s)
- The areal speed for Mars about its focus (s')
- Find values for f, d, e, l, i, f', d', e', m, s, s' so as to minimize the max Oppositions Long. Discrepancy, Triangulations Long. Discrepancy and Oppositions Lat. Discrepancy
- Remember all work on longitudes needs to be done by projecting Mars' orbit on the ecliptic place

Narrowing Ranges

- Earth's eccentricity e is likely small (the analemma width) ~0.02
- Earth's focus (f) likely close to the Sun (from the previous circular fit) ~0.02
- Earth's major axis direction d unconstrained (0-365 deg)
- Line of intersection of Mars' plane with the ecliptic l: see next slide
- Inclination of Mars' plane relative to the ecliptic i: see next slide
- Mars' focus f: could constrain it to be the same as Earth's
- Major axis direction d' roughly 328 degrees, from the previous circular fit
- Major Axis for Mars m roughly 1.5AU from our previous circular fit
- Eccentricity for Mars e': roughly distance between center and sun in the circular fit divided by 1.5 AU
- The areal speed for Earth about its focus s (~π/365)
- The areal speed for Mars about its focus s' (~π 2.25/687)

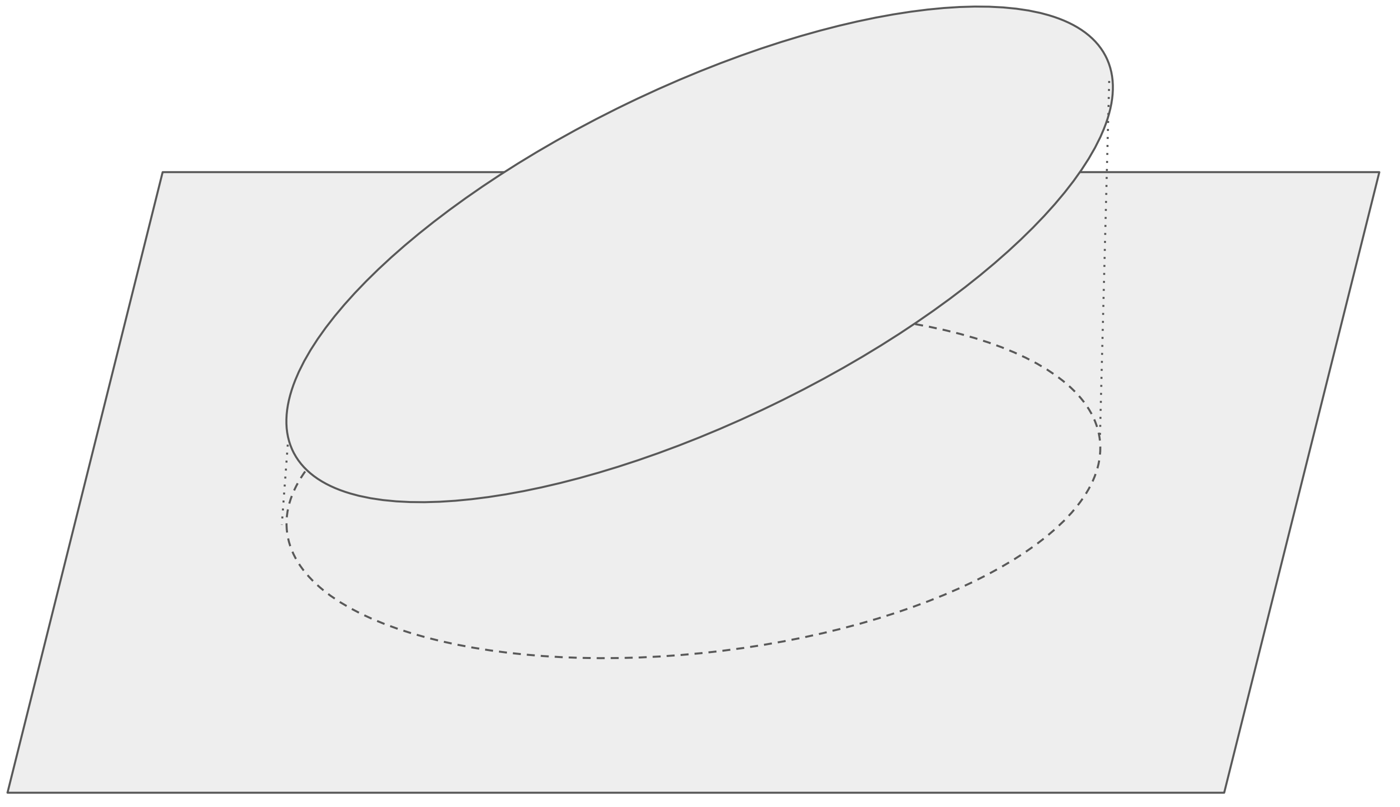
Non Coplanarity: Opposition Latitudes





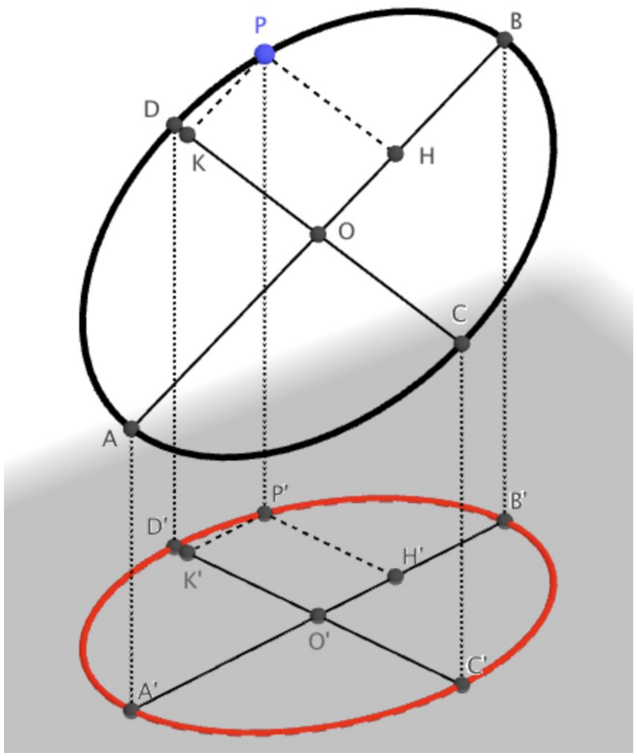
Opposition latitudes are indicative of l and i

Ellipses Stay Ellipses on Projection



This will be convenient to switch between the ecliptic and Mars' plane.
How does one prove this?

Ellipses Stay Ellipses on Projection



$$\frac{OH^2}{OB^2} + \frac{OK^2}{OD^2} = 1.$$

$$\frac{O'H'^2}{O'B'^2} + \frac{O'K'^2}{O'D'^2} = 1.$$

https://math.stackexchange.com/questions/3534227/the-projection-of-an-ellipse-is-still-an-ellipse

Proportions on a line are preserved on projection. There is a flaw though in this proof.

A Linear ALgebraic Proof

- Consider an ellipse on the x-y plane with center at the origin that you want to project to another plane P
- The ellipse has the form $\mathbf{x^T A^T A x = 1}$
- $\mathbf{x}$ and $\mathbf{A}$ are 2D but think of them as 3D with the z coordinate 0
- If $\mathbf{A}$ is a diagonal matrix, this gives the conventional equation for the ellipse
- A linear transformation $\mathbf{Pz}$ projects any given 3D point $\mathbf{z}$ to plane P
- $\mathbf{P}$ not invertible in general (given $\mathbf{Pz}$, $\mathbf{z}$ is not unique)
- However $\mathbf{z}$ is unique on the x-y plane, so a pseudo-inverse $\mathbf{P^\dagger}$ exists relative to this plane
- Then $\mathbf{P^\dagger P x = x}$ for $\mathbf{x}$ on the x-y plane
- $\mathbf{(x^T P^T P^{\dagger T})\ A^T A\ (P^\dagger P x) = 1}$
- $\mathbf{(Px)^T\ (AP^\dagger)^T\ (AP^\dagger)\ (Px) = 1}$
- Hence the projected points $\mathbf{Px}$ form an ellipse
- Note: above assumes plane P passes through the center, convince yourself that is ok

Optimization

- Given a set of parameters
- With range constraints on each parameter
- And an objective function
- Find values of all the parameters that minimize the objective function
- So far, we have outlined a brute force grid search, with some intelligent narrowing down of parameter ranges
- Are there better methods?

Greedy Descent

- Start with an arbitrary valid assignment to the parameters
- Repeatedly find a neighboring assignment that does better on the objective function and move to it
- Where neighbors are defined as per the discretization of the range for each parameter
- Typically tweak a single parameter at a time, perhaps go round robin over parameters
- Could get stuck in local minima
- Do we have any in our case? You can explore this in the project.

Simulated Annealing

- How to jump out of local minima
- Occasionally, allow movement to a neighboring assignment that is worse on the objective function
- So if the neighbor is better, move to it
- If the neighbor is worse, move to it with a probability that is proportional to exp((currentValue - neighbourValue) /Temperature)
- Of course, you will need to normalize the above
- Initially, start with high temperature so the probability of accepting the move to a worse neighbor is high
- As you get closer to the objective function becomes 4', drop the temperature, so you start heading towards the current local minima
- You will need to experiment with setting the initial temperature and the cooling schedule

Other Options: Gradient Descent

- If you can write the objective function in a closed form and obtain the partial derivative for each parameter, the vector of partial derivative values is the gradient
- Repeatedly move in that gradient direction by a certain step size
- Small step sizes take time, bigger step sizes overshoot the local minimum
- Often use convex approximations in the local neighborhood to get to the local minimum of this approximated function in one shot
- Is gradient descent applicable here? Explore in the project

Newton and Gravitation

- Several decades later, Newton tried to explain the physics of ellipses and of equal areas in equal time
- He had to come up with a few tools for this
- Orthogonal decomposition: write a vector as a sum of two orthogonal vectors
- Define Force as vector that changes the velocity (and not the position directly)
- Calculus to handle continuously varying quantities

Equal Areas in Equal Time

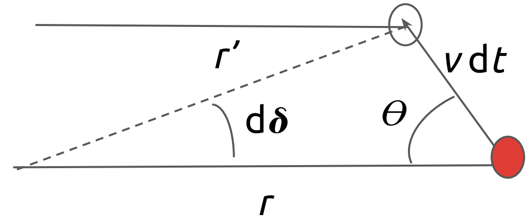
- Area traversed in this infinitesimal time step is

  $0.5\ r\ v\ \sin\theta\ dt$

- Area traversed in the next step is

  $0.5\ \ \mathrm{sqrt}[\ (r - v\cos\theta\ dt)^2 + (v\sin\theta\ dt)^2\ ]$

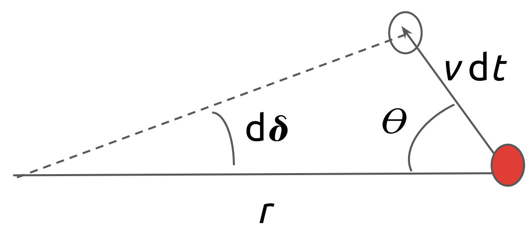
  $[\ (v\cos\theta + a\ dt\ )\sin d\delta\ +\ v\sin\theta\cos d\delta\ ]\ dt$



It follows that the rate of change of areas is a constant; also known as the conservation of angular momentum; holds regardless of a gravitational law

Speed and Energy

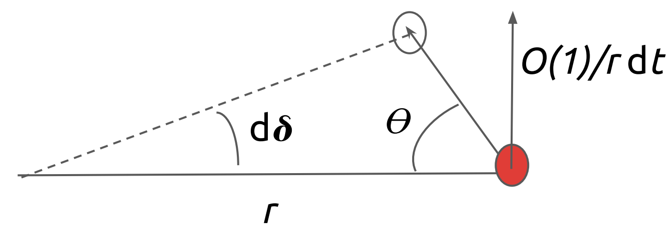
- Speed square v^2
- Speed square after the infinitesimal time step $(v\ cos\Theta + a\ dt)^2 + (v\ sin\Theta)^2$
- Difference $2\ a\ vcos\Theta\ \ dt = \ 2\ a\ dr$
- Integrating $v^2 - v_0^{\ 2} = 2 \int a\ dr$
- Where $v_0^{\ 2}$ is the initial speed squared
- And the integration is from the initial r_0 to the current r
- If $a = O(1)/r^2$ then $v^2 - v_0^{\ 2} = 2\ (O(1)/r - O(1)/r_0)$



This is the familiar law of conservation of energy

Leading up to the Ellipse..

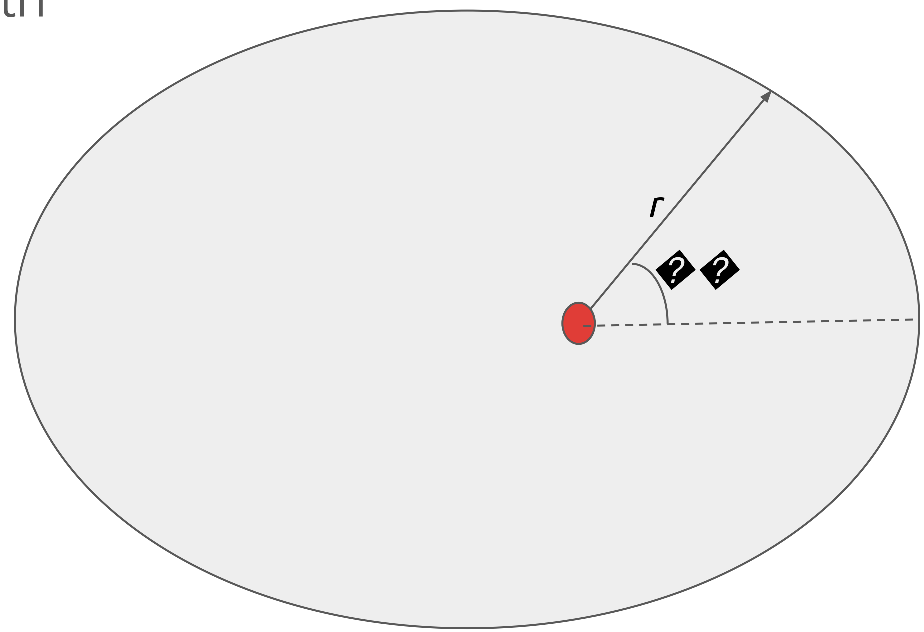
- $v^2 = v_0^2 + 2\,(1/r - 1/r_0) = O(1) + O(1)/r$
- Vertical component of velocity is ***O(1)/r*** on account of equal areas in equal time
- Horizontal component squared is therefore $O(1)+O(1)/r-O(1)/r^2$
- $d\boldsymbol{\delta} = O(1)/r\, dt\, /r$
- $dr = \mathrm{sqrt}\,[O(1)+O(1)/r-O(1)/r^2]\, dt$
- $d\boldsymbol{\delta} = O(1)\, dr\, /\, r^2\, \mathrm{sqrt}\,[O(1)+O(1)/r-O(1)/r^2]$



So we have r in terms of $\boldsymbol{\delta}$; what curve does the integration suggest?

And the Ellipse..

- The polar form of an ellipse with resp to the focus
- *cos* $\boldsymbol{\delta}$ *= 1/e (a/r -1)*
- *r = a/ (1 + e cos* $\boldsymbol{\delta}$*)*
- The numerator a is the latus rectum
- e is the eccentricity



This happens only when the radial acceleration $a=O(1)/r^2$