UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022
HOMEWORK 3

Instructor: GAUTAM BHARALI **Assigned: JANUARY 25, 2022**

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1. Consider the set
$$A := \{x \in \mathbb{Q}^+ : x^2 < 2\}.$$
In Chapter 1 of Rudin's book, there is an informal discussion of the fact that A has no least upper bound in $\mathbb{Q}$ ("informal" in the sense that it relies on the relationships between the arithmetic operations and the ususal order $\leq$ on $\mathbb{Q}$ that we take for granted). With this discussion as a guide,

- Using the **formal** definition of the order $\leq$ on $\mathbb{Q}$, and
- Quoting the relevant propositions from the section entitled *Fields* in Chapter 1 of Rudin's book in support of your arguments,

give a full and formal proof of the statement: *A has no least upper bound in $\mathbb{Q}$.*

2. If α is a positive cut, then we define
$$\alpha^{-1} := \{x \in \mathbb{Q} : \exists r \in \mathbb{Q} \text{ such that } r < 1/x \text{ and } r \notin \alpha\} \cup 0^* \cup \{0\}.$$

(a) Show that α^{-1} as defined **is** a cut for any $\alpha \neq 0^*$.

(b) Show that α^{-1} as defined is the multiplicative inverse of α.

3. Given any $a \in \mathbb{Q}$, let $a^* := \{x \in \mathbb{Q} : x < a\}$. Prove that
$$a^* b^* = (ab)^* \quad \forall a, b \in \mathbb{Q}^+ \tag{1}$$

Remark. The above statement can be extended to all $a, b \in \mathbb{Q}$, but doing so is cumbersome, and (1) is the key to the latter. It is easy to show that $a^* + b^* = (a + b)^*$ for all $a, b \in \mathbb{Q}$. The last two statements form the analogue of Problem 6 in Homework 1 showing, this time, that the arithmetic operations between **cuts** extend the arithmetic operations on $\mathbb{Q}$.

4. Let $(S, \leq)$ be an ordered set.

(a) Let $A \subseteq S$. Formulate definitions, analogous to those given in class, for "lower bound of A", "A is bounded below", "greatest lower bound of A", and "greatest lower bound property".

(b) Show that if $(S, \leq)$ has the least upper bound property, then it has the greatest lower bound property.

(c) Recall, from Problem 5 in Homework 2, that if A is a non-empty subset of an ordered set $(S, \leq)$ that is bounded above, and $(S, \leq)$ has the least upper bound property, then A has a unique least upper bound. Denote this element as $\sup A$. Thus, by (b) (with the same assumptions on S), if $A \subseteq S$ is non-empty and bounded below, then A has a unique greatest lower bound. Denote this element as $\inf A$. Now, let A be a non-empty subset of $\mathbb{R}$ that is

bounded below and let $-A := \{-x \in \mathbb{R} : x \in A\}$. Argue that $\sup(-A)$ exists, and show that

$$\inf A \;=\; -\sup(-A).$$

The following anticipates material to be introduced in the lecture on **January 28.**

5. Using the Archimedean property of $\mathbb{R}$, prove that if $x, y \in \mathbb{R}$ and $x < y$, then there exists $q \in \mathbb{Q}$ such that $x < q < y$.