UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022
HOMEWORK 9

Instructor: GAUTAM BHARALI **Assigned: MARCH 15, 2022**

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1. Let $\sum_{n=1}^{\infty} a_n$ be a non-negative series that converges. Let $\varepsilon > 0$. Determine whether or not the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{(1/2)+\varepsilon}}$$

converges. Please justify your answer.

Tip. Consider an inequality that you learned in UM102.

2. Determine **all** the $p \in \mathbb{R}^+$ for which the series

$$\sum_{n=1}^{\infty} \left(\sqrt{1+n^{2p}} - n^p\right)$$

converges.

3. (Problem 11(a) from Chapter 3 of "Baby" Rudin) Let $a_1, a_2, a_3, \dots > 0$ and assume that the series $\sum_{n=1}^{\infty} a_n$ diverges. Show that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$$

also diverges.

4. Determine whether or not the series

$$\sum_{n=1}^{\infty} \frac{\left(2+(-1)^n\right)^n}{3^n}$$

converges. Please justify your answer.

5. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n},$$

and let S_n denote its n-th partial sums, $n = 1, 2, 3, \dots$

- (a) Show that the sequences $\{S_1, S_3, S_5, \dots\}$ and $\{S_2, S_4, S_6, \dots\}$ are monotone and bounded.
- (b) Examine the conclusions of (a) to deduce that the above series converges.

6. (A *little* difficult, or *very* cute, depending on your point of view.) Complete the following outline for a proof that the interval $[0, 1)$ is uncountable. Given a number $x \in [0, 1)$, let $\mathcal{I}_0(x) := [0, 1)$ and define the intervals

$$\mathcal{I}_{n+1}(x) := \begin{cases} \big[\inf \mathcal{I}_n(x), \mu_n(x)\big), & \text{if } x < \mu_n(x), \\ \big[\mu_n(x), \sup \mathcal{I}_n(x)\big), & \text{if } x \geq \mu_n(x), \end{cases}$$

for $n = 0, 1, 2, \dots$, where $\mu_n(x) := \big(\inf \mathcal{I}_n(x) + \sup \mathcal{I}_n(x)\big)/2$: i.e., the midpoint of $\mathcal{I}_n(x)$. Let $\mathfrak{S}$ denote the set of all sequences in $\{0, 1\}$. We now define a function $F : [0, 1) \to \mathfrak{S}$ as follows: write $F(x) = \{s_n(x)\}$ where

$$s_n(x) := \begin{cases} 0 & \text{if } x < \mu_{n-1}(x), \\ 1, & \text{if } x \geq \mu_{n-1}(x), \end{cases}$$

for $n = 1, 2, 3, \dots$.

(*a*) Show that the series

$$\sum_{n=1}^{\infty} \frac{s_n(x)}{2^n}$$

converges, and that its sum is x. (**Remark.** This problem shows that the "binary representation" of x — i.e., the expression "$0.s_1(x)\, s_2(x)\, s_3(x) \dots$", which is analogous to the common decimal expressions for real numbers — exists.)

(*b*) Show that F is **not** surjective (use the conclusion of (*a*) above).

(*c*) Show that $\mathfrak{S} \setminus \mathsf{range}(F)$ is countable.

(*d*) Use the conclusions of (*a*)–(*c*) to show that $[0, 1)$ is uncountable.