UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2025

HOMEWORK 7

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 28, 2025**

---

1. Let $\{a_n\}$ be a real sequence and assume that

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Show that $\{a_n\}$ converges to some extended real number.

2. Let $\{a_n\}$ be a real sequence. Define

$$\begin{aligned} A_k &:= \inf\{a_k, a_{k+1}, a_{k+2}, \dots\}, \\ B_k &:= \sup\{a_k, a_{k+1}, a_{k+2}, \dots\}. \end{aligned}$$

Show that

$$\liminf_{n\to\infty} a_n = \lim_{k\to\infty} A_k, \quad \text{and} \quad \limsup_{n\to\infty} a_n = \lim_{k\to\infty} B_k.$$

3. Consider the series

$$\sum\nolimits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n},$$

and let S_n denote its n-th partial sums, $n = 1, 2, 3, \dots$.

(a) Show that the sequences $\{S_1, S_3, S_5, \dots\}$ and $\{S_2, S_4, S_6, \dots\}$ are monotone and bounded.

(b) Examine the conclusions of (a) to deduce that the above series converges.

4. (Problem 11(a) from Chapter 3 of "Baby" Rudin) Let $a_1, a_2, a_3, \dots > 0$ and assume that the series $\sum_{n=1}^{\infty} a_n$ diverges. Show that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{1 + a_n}$$

also diverges.

5. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with terms in $\mathbb{R}$.

(a) Let $\{b_n\} \subset \mathbb{R}$ be a monotone bounded sequence. Show that $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

(b) What can be said about $\sum_{n=1}^{\infty} a_n b_n$ if $\{b_n\} \subset \mathbb{R}$ is assumed merely to be bounded?

6. Determine **all** the $p \in \mathbb{R}^+$ for which the series

$$\sum_{n=1}^{\infty} \left(\sqrt{1+n^{2p}} - n^p\right)$$

converges.

7. Given the power series $\sum_{n=0}^{\infty} c_n z^n$ (where $c_n \in \mathbb{C}$ for each $n \in \mathbb{N}$). Define

$$\alpha := \limsup_{n\to\infty} \sqrt[n]{|c_n|} \quad \text{and} \quad \rho := 1/\alpha$$

(with the understanding the $\rho := 0$ if $\alpha = +\infty$, and $\rho := +\infty$ if $\alpha = 0$).

(a) Show that $\sum_{n=0}^{\infty} c_n z^n$ converges at each $z : |z| < \rho$ and diverges at each $z : |z| > \rho$.

(b) Give an example of $\sum_{n=0}^{\infty} c_n z^n$ showing that it behaves differently for different $z \in \mathbb{C}$ satisfying $|z| = \rho$.