Parikh's Theorem for CFLs

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Outline

1. Parikh map
2. Parikh's theorem
3. Proof

Rohit Parikh

Parikh map of a string

- Let $A = \{a_1, \ldots, a_n\}$ be a finite alphabet.
- The Parikh map of a string $w \in A^*$ is a vector in $\mathbb{N}^n$ given by:

$$\psi(w) = (\#_{a_1}(w), \#_{a_2}(w), \ldots, \#_{a_n}(w)).$$

- For example if $A = \{a, b\}$, then $\psi(baabb) = (2, 3)$.
- Parikh map is also called the "letter-count" of a string.
- Extend the map to languages L over A:

$$\psi(L) = \{\psi(w) \mid w \in L\}.$$

- What is $\psi(\{a^n b^n \mid n \geq 0\})$?

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  - $\{(m, n) \mid m \leq n\}$.

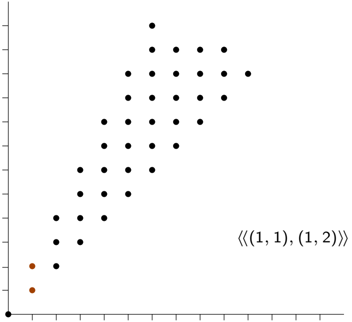
Semi-linear sets of vectors

- The set of vectors generated by a set of vectors $u_1, \ldots, u_k$ in $\mathbb{N}^n$, denoted $\langle\langle u_1, \ldots, u_k \rangle\rangle$, is the set

$$\{d_1 \cdot u_1 + d_2 \cdot u_2 + \cdots + d_k \cdot u_k \mid d_i \in \mathbb{N}\}.$$

- A subset X of $\mathbb{N}^n$ is called linear if there exist vectors $u_0, u_1, \ldots, u_k$ such that

$$X = u_0 + \langle\langle u_1, u_2, \ldots, u_k \rangle\rangle.$$



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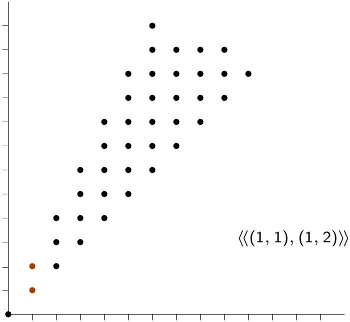

- A set of vectors is called semi-linear if it is a finite union of linear sets.

Parikh's Theorem for CFLs

Theorem (Parikh 1966)

The Parikh map of a CFL is a semi-linear set. That is, if L is a CFL then $\psi(L)$ is semi-linear.

Some corollaries:

- Every CFL is "letter-equivalent" to a regular language.
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Is Parikh's theorem a sufficient condition for context-freeness as well?

- No, since $\psi(\{a^n b^n c^n \mid n \geq 0\}) = \{(n, n, n) \mid n \geq 0\}$ is semi-linear.

Running example and Exercise

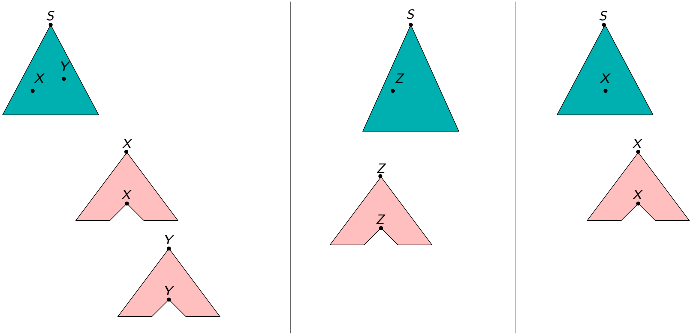
CFG G_1

$$
\begin{array}{rcl}
S & \rightarrow & XC \mid AY \\
X & \rightarrow & aXb \mid ab \\
Y & \rightarrow & bYc \mid bc \\
A & \rightarrow & aA \mid a \\
C & \rightarrow & cC \mid c
\end{array}
$$

- What is the language generated?
- What is the Parikh image of this language?
- Write it as a semi-linear set.

Idea of proof

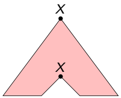
- Group parse trees of G into a finite number of buckets
- Each bucket is represented by a "minimal" parse tree and associated "basic pumps".
- Argue that the set of strings derived in each bucket gives rise to a linear set.



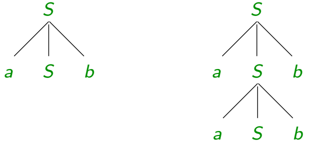
Proof: Pumps

Let us fix a CFG $G = (N, A, S, P)$ in CNF form.

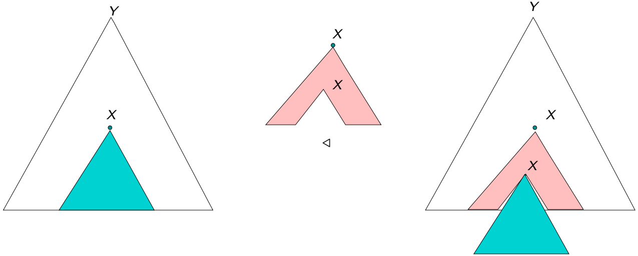
- A pump is a derivation tree s which has at least two nodes, and $yield(s) = u \cdot root(s) \cdot v$, for some terminal strings u, v.



- Example pumps for grammar $S \to aSb \mid SS \mid \epsilon$:



Growing and shrinking with pumps



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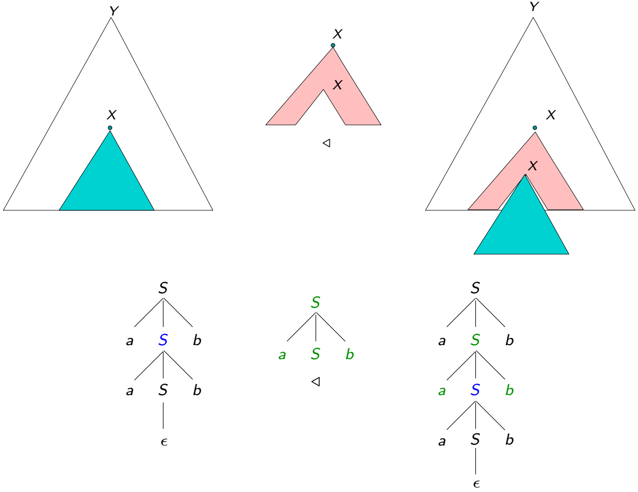
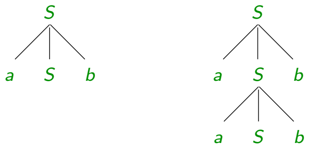


Basic Pumps

A pump is basic if it is $\triangleleft$-minimal. Thus a pump s is a basic pump if it cannot be shrunk by some pump and still remain a pump.



- First pump is basic but second is not.
- How many pumps are there?

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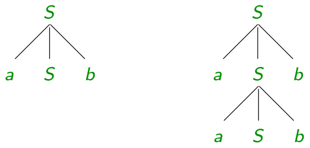

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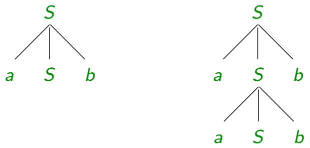
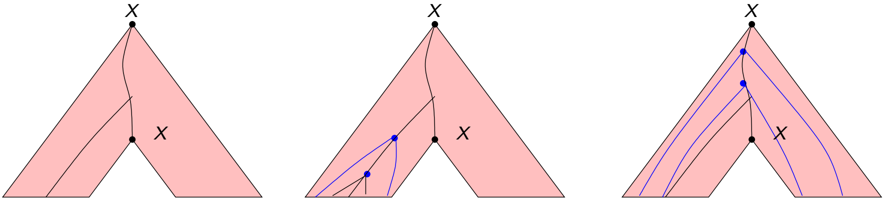


- First pump is basic but second is not.
- How many pumps are there? Infinitely many in general.
- How many basic pumps are there? Finitely many since their height is bounded by $2|N|$ (See argument on next slide).
- Let p be the number of basic pumps.

Basic pumps have height bounded by $2N$

Consider the longest path from root to leaf in a basic pump.
The number of nodes on it is bounded by $2|N| + 1$.



$\leq$ relation on parse trees

- Let s and t be derivation trees of terminal strings starting from start symbol S (i.e. parse trees).
- Then we say $s \leq t$ iff t can be grown from s by basic pumps whose non-terminals are contained in those of s (thus the pumps do not introduce any new non-terminals, and s and t have the same set of non-terminal nodes).
- A parse tree s is thus $\leq$-minimal if it does not contain a basic pump that can be cut out without reducing the set of non-terminals that occur in s.
- $\leq$-minimal trees can be seen to be finite in number: their height is bounded by $(p+1)(|N|+1)$.
- Ex: What are the $\leq$-minimal parse trees for grammar G_1?

Overall strategy of Proof

- Begin with the $\leq$-minimal derivation trees, say $s_1, \ldots, s_k$.
- Associate with each s_i the set of basic pumps whose non-terminals are contained in that of s_i.
- Argue that the set of derivation trees obtained by starting with s_i and growing using the associated basic pumps, (let us call this the "bucket" of parse trees associated with s_i) gives rise to a set of strings whose Parikh map is a linear set.

Exercise

Describe the "buckets" for G_1, and the corresponding linear sets.