UM 204 (WINTER 2024) - WEEK 12

1. Differentiation

Throughout this section, we will only consider real or vector-valued functions on open intervals of the real line.

1.1. Introduction.

Definition 1.1. Let $f:(a,b)\to\mathbb{R}$ and $c\in(a,b)$. We say that f is differentiable at c with derivative $f'(c)$ if

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=f'(c),$$

or equivalently,

$$\lim_{x\to c}\frac{f(x)-f(c)-f'(c)(x-c)}{x-c}=0.$$

The function is differentiable in (a,b) if f is differentiable at each $c\in(a,b)$. Moreover, the function $c\mapsto f'(c)$ is the derivative function of f.

As an exercise, revisit the proofs of the following results:

Theorem 1.2. *If $f:(a,b)\to\mathbb{R}$ is differentiable at $c\in(a,b)$, then f is continuous at c.*

Theorem 1.3. *If $f,g:(a,b)\to\mathbb{R}$ are differentiable at c, then*

(1) *$f+g$ is differentiable at c and $(f+g)'(c)=f'(c)+g'(c)$,*
(2) *fg is differentiable at c, and $(fg)'(c)=f(c)g'(c)+f'(c)g(c)$,*
(3) *assuming $g'(c)\neq 0$, f/g is differentiable at c, and $(f/g)'(c)=\frac{g(c)f'(c)-f(c)g'(c)}{g(c)^2}$.*

Theorem 1.4. *Suppose $f:(a,b)\to\mathbb{R}$ is differentiable at $c\in(a,b)$. Suppose $f((a,b))\subset(p,q)$. and $g:(p,q)\to\mathbb{R}$ is differentiable at $f(c)$. Then, $g\circ f:(a,b)\to\mathbb{R}$ is differentiable at c, and*

$$(g\circ f)'(c)=g'(f(c))\cdot f'(c).$$

1.2. **Mean value theorems and applications.** As we have already seen before, derivatives are useful in detecting point of extrema.

Theorem 1.5. *Let $f : (a,b) \to \mathbb{R}$ be a function. Suppose f has a local extremum at $c \in (a,b)$ and f is differentiable at c, then $f'(c) = 0$.*

Proof. WLOG, assume that c is a point of local maximum (otherwise, consider $-f$). Then, there is some $\delta > 0$ such that

$$f(x) \le f(c) \quad \forall x \in B(c;\delta) \subset (a,b).$$

Thus, if $x \in (c-\delta, \delta)$, then

$$\frac{f(x) - f(X)}{x-c} \ge 0,$$

and if $x \in (c, c+\delta)$,

$$\frac{f(x) - f(c)}{x-c} \le 0.$$

Taking limits as $x \to c$, we obtain that $0 \le f'(c) \le 0$, which yields the claim. $\square$

Remark. Similar reasoning also allows us to deduce the sign of the derivative of a monotone function (though the converse requires something more).

Example. Consider the function

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

When $x \neq 0$, the differentiability of the function follows from the result about sums, products and quotients of differentiable functions. Furthermore, we obtain that

$$f'(x) = 2x \sin(1/x) - \cos(1/x), \quad x \neq 0.$$

At $x = 0$, we directly compute

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = h \sin(1/h) = 0.$$

Thus, $f'(x)$ exists for all $x \in \mathbb{R}$, by it is not continuous at $x = 0$.

Theorem 1.6 (Intermediate Value Property)**.** *Let $f : (p,q) \to \mathbb{R}$ is differentiable and $[a,b] \subset (p,q)$. Suppose $f'(a) < \lambda < f'(b)$. Then, there is some $c \in (a,b)$ such that $f'(c) = \lambda$.*

Proof. Let $g(x) = f(x) - \lambda x$. Then, $g'(a) < 0$ and $g'(b) > 0$. Since $g'(a)$, there is a $\delta > 0$ such that for all $x \in (a, a+\delta)$,

$$\frac{g(x) - g(a)}{x-a} < g'(a) - \frac{g'(a)}{2} < 0.$$

Thus, $g(x) < g(a)$ for all $x \in (a, a+\delta)$. Similarly, since $g'(b) > 0$, we obtain that $g(x) < g(b)$ for all $x \in (b - \delta', b)$ for some $\delta' > 0$. This implies that the minimum of g on $[a,b]$ is attained at some $c \in (a,b)$. $\square$

Exercise. Show that if a function has a discontinutiy of the first kind, then there is an interval on which it does not satisfy the intermediate value property.

Corollary 1.7. *A derivative function on an interval of* $\mathbb{R}$ *cannot have discontinuities of the first kind.*

Theorem 1.8 (Cauchy's)**.** *Let* $f, g : [a, b] \to \mathbb{R}$ *be continuous functions that are differentiable on* (a, b)*. Then, there is some* $c \in (a, b)$ *such that*

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

Remarks. 1. The special case where $g(x) = x$ and $f(a) = f(b)$ is called Rolle's theorem.
2. The special case where $g(x) = x$ is call the mean value theorem.
3. One interpretation is: the ratio of the global averages of two functions is attained at some point by the ration of their derivatives.
4. If $x \mapsto (f(x), g(x))$ parametrizes a path in $\mathbb{R}^2$, then the line joining the endpoints is parallel to the tangent line at some point in the path.

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Proof. Let

$$h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x), \quad x \in [a, b].$$

Then, by the algebraic laws of continuous and differentiable functions, h is continuous on $[a, b]$ and differentiable on (a, b). Further, note that

$$h(a) = h(b).$$

Then, since h is continuous, it attains its extrema on $[a, b]$. Either h is constant on $[a, b]$, in which case $h'(x) = 0$ everywhere, and c can be chosen to be any point (a, b).

Otherwise, there is some $t \in (a, b)$ where either $h(t) > h(a)$ or $h(t) < h(a)$. In this case, an extrema is attained at some $c \in (a, b)$, and by the previous theorem $h'(c) = 0$. □

Remark. The mean value theorem allows us to deduce the monotonicity of a function based on the sign of its derivative.

Theorem 1.9 (Taylor's theorem)**.** *Let* $n \in \mathbb{N}$*. Suppose* $f : [a, b] \times \mathbb{R}$ *is such that* $f, f', \ldots, f^{(n)}$ *exist on* (a, b)*, and extend continuously to* $[a, b]$*. Then, there is some* $c \in (a, b)$ *such that*

$$f(b) = f(a) + f'(a)(b - a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b - a)^{n-1} + \frac{f^{(n)}(c)}{n!}(b - a)^n.$$

Proof. Suppose we knew how to prove this result for the special case $f(a) = \cdots = f^{(n-1)}(a) = 0$. That is, for a function satisfying these conditions, there is a $c \in (a,b)$ such that

$$f(b) = \frac{f^{(n)}(c)}{n!}(b-a)^n.$$

Then we try to modify the given function to satisfy this condition. Let

$$F(x) = f(x) - \sum_{k=0}^{\ell} c_k (b-a)^k.$$

In order that $F(a) = \cdots = F^{(n-1)}(a) = 0$, we must have that $\ell = n-1$, and

$$c_j = \frac{f^{(j)}(a)}{j!}.$$

Applying the special (yet unproved) case to observe that there is a $c \in (a,b)$ such that

$$f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k = F(b) = \frac{F^{(n)}(c)}{n!} = \frac{f^{(n)}(c)}{n!}.$$

Now, consider the special case. Let '

$$g(x) = f(x) - f(b)\frac{(x-a)^n}{(b-a)^n}.$$

Then, $g(a) = \cdots g^{(n-1)}(a) = 0$ and $g(b) = 0$. By Rolle's theorem, there is a $c_1 \in (a,b)$ such that

$$g'(c_1) = 0.$$

Now, applying Rolle's theorem to g' on $[a, c_1]$, we obtain a $c_1 \in (a, c_1)$ such that

$$g''(c_2) = 0.$$

Continuing this way, we obtain $a < c_n < c_{n-1} < \cdots < c_1 < b$ such that

$$g^{(n)}(c_n) = f^{(n)}(c_n) - \frac{n!}{(b-a)^n} f(b) = 0.$$

$\square$

Theorem 1.10 (L'Hospital's Rule)**.** *Suppose* $-\infty \le a < b \le +\infty$*. Let* $f, g : (a,b) \to \mathbb{R}$ *be differentiable functions such that* $g(x) \neq 0$ *and* $g'(x) \neq 0$ *for all* $x \in (a,b)$*. Suppose* $A \in [-\infty, \infty]$ *such that*

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = A.$$

Then, in each of the following situations,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = A.$$

(1) *if* $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0.$,
(2) *if* $\lim_{x\to a^+} g(x) = +\infty$,
(3) *if* $\lim_{x\to a^+} g(x) = -\infty$.

Proof. We will only tackle the case when $a, A \in \mathbb{R}$.

Given $\varepsilon > 0$, we obtain a $\delta > 0$ such that for all $x \in (a, a+\delta)$,

$$A - \varepsilon < \frac{f'(x)}{g'(x)} < A + \varepsilon.$$

Let $y \in (a, a+\delta)$.

Assume (i) holds. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $(a, a+\delta)$ such that

(1) $z_n < y$ for all $n \in \mathbb{N}$,
(2) $\lim_{n\to\infty} z_n = a$,
(3) $g(z_n) \neq g(y)$.

The last feature is possible since $g'(x) \neq 0$ on $(a, a+\delta)$. Thus, if $z_1 > \cdots > z_{n-1}$ has been constructed, there is always a $z_n \in (1/n, z_{n-1})$ such that $f(z_n) \neq f(y)$. Now, by the GMVT, there exists a $w_n \in (z_n, y) \subset (a, a+\delta)$ such that

$$A - \varepsilon < \frac{f(y) - f(y_n)}{g(y) - g(y_n)} = \frac{f'(w_n)}{g'(w_n)} < A + \varepsilon$$

Taking limits as $n \to \infty$, we obtain that

$$A - \varepsilon < \frac{f(z)}{g(z)} < A + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have that

$$\lim_{x\to a^+} \frac{f(z)}{g(z)} = A.$$

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Assume that (ii) holds. Then, since $\lim_{x\to a^+} g(x) = +\infty$, there is some $0 < \delta' < \delta$ such that for all $z \in (a, a+\delta')$, $g(z) > \max\{g(y)+1, 0\}$. Then,

$$\frac{g(z) - g(y)}{g(z)} > 0.$$

Once again, for each $y \in (a, a+\delta')$, there is some $\theta \in (y, z)$ such that

$$A - \varepsilon < \frac{f(z) - f(y)}{g(z) - g(y)} = \frac{f'(\theta)}{g'(\theta)} < A + \varepsilon.$$

Thus,

$$(A-\varepsilon)\frac{g(y)-g(z)}{g(z)} < \frac{f(z)-f(y)}{g(z)-g(y)}\frac{g(z)-g(y)}{g(z)} < (A+\varepsilon)\frac{g(z)-g(y)}{g(z)}$$

or

$$(A-\varepsilon)\left(1-\frac{g(y)}{g(z)}\right)+\frac{f(y)}{g(z)} < \frac{f(z)}{g(z)} < (A+\varepsilon)\left(1-\frac{g(y)}{g(z)}\right)+\frac{f(y)}{g(z)}.$$

Keeping y fixed, let $z \to a^+$. Then,

$$A-\varepsilon < \frac{f(z)}{g(z)} < A+\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we are done. The third case is similar to the second one. □

Bad uses. In both cases, in can be directly shown (using the squeeze theorem, basic bounds, etc.) that the limits exist, and are 1 and 0, respectively.

However, an imprecise use of L'Hospital's rule will yield:

$$\lim_{x\to\infty}\frac{2x}{2x+\sin(x)} \text{ " = " } \lim_{x\to\infty}\frac{2}{2+\cos(x)} \text{ " = " } \lim_{x\to\infty}\frac{0}{-\sin(x)} = 0.$$

$$\lim_{x\to 0}\frac{x^2\sin(1/x^2)}{x} \text{ " = " } \lim_{x\to\infty} 2x\sin(1/x^2)-4x^{-2}\cos(1/x^2) = DNE.$$

1.3. **Vector-valued functions.**

Definition 1.11. Given $f:(a,b)\to\mathbb{R}^n$, we say that f is differentiable at $c\in(a,b)$ with $f'(c)\in\mathbb{R}^k$ as derivative if

$$\lim_{x\to c}\left\|\frac{f(x)-f(c)}{x-c}-f'(c)\right\| = 0.$$

Remarks. 1) If $f=(f_1,\dots,f_n)$, then the differentiability of f at c is equivalent to the all f_j's at being differentiable at c, with

$$f'(c) = (f_1'(c),\dots,f_n'(c)).$$

2) The sum and inner product of differentiable vector-valued functions is also differentiable and the derivatives behave in the expected fashion.

Warning! Several theorems fail in this context.

(1) Let $f(x)=e^{ix}=\cos(x)+i\sin(x)$ (or think of it as $f(x)=(\cos(x),\sin(x))$. Then, $f'(x)=ie^{ix}$, which is never $(0,0)$ for $x\in[0,2\pi]$, but $f(2\pi)=f(0)=0$.

(2) Let $f(x) = x$ and $g(x) = x + x^2 e^{i/x^2}$ on $(0,1)$. It's clear that $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$. Moreover, $\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 0$. For this, note that

$$g'(x) = 1 + \left(2x - \frac{2i}{x}\right) e^{i/x^2}.$$

Thus,

$$|g'(x)| \geq \left|2x - \frac{2i}{x}\right| - 1 \geq \frac{2}{x} - 1 \to +\infty,$$

as $x \to 0$, whereas $f'(x) = 1$. However,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = 1.$$

Thus, L'Hosptial's rule doesn't hold.

Theorem 1.12 (Mean Value Inequality)**.** *Suppose $f : [a,b] \to \mathbb{R}^n$ is continuous on $[a,b]$ and differentiable on (a,b). Then, there is a $c \in (a,b)$ such that*

$$\|f(b) - f(a)\| \leq \|f'(c)\|(b-a).$$

Proof. Consider $F : x \mapsto < f(b) - f(a), f(x) >$. Then, there is some $c \in (a,b)$ such that

$$\|f(b) - f(a)\|^2 = F(b) - F(a) = F'(c)(b-a) < f(b) - f(a), f(c) > (b-a) \leq \|f(b) - f(a)\| \|f'(c)\|(b-a).$$

by the Cauchy-Schwartz inequality. □

2. Riemann Integration

Once again, we will focus on $\mathbb{R}$-valued functions on the real line (or its subsets). Riemann integration was the first rigorous theory of integration, and is quite useful for practical purposes.

Definition 2.1. Let $f : [a,b] \to$ be a bounded function. Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a,b]$, i.e.,

$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b.$$

Let

$$\begin{aligned} \triangle x_i &= x_i - x_{i-1}, \\ M_i &= \sup_{x\in[x_{i-1},x_i]} f(x) \\ m_i &= \inf_{x\in[x_{i-1},x_i]} f(x), \quad i \in \{1, ..., n\}. \end{aligned}$$

Let

$$\begin{aligned} U(P,f) &= \sum_{i=1}^{n} M_i x_i \\ L(P,f) &= \sum_{i=1}^{n} m_i x_i. \end{aligned}$$

Then, the upper integral and lower integral of f are defined as

$$\begin{aligned} \overline{\int_a^b} f\,dx &= \inf\{U(P,f) : P \text{ is a partition of } [a,b]\}, \\ \underline{\int_a^b} f\,dx &= \sup\{L(P,f) : P \text{ is a partition of } [a,b]\}. \end{aligned}$$

The function f is said to be Riemann integrable if

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

in which case this quantity is denoted by $\int_a^b f\,dx$.

Note that for every partition P,

$$\inf_{[a,b]} f(b-a) \le L(P.f) \le U(P,f) \le \sup_{[a,b]} f(b-a).$$

So, the definition makes sense for every bounded function.

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