Transitive :-

if $a \leq b$ & $b \leq c$ then $a \leq c$

$a \leq b \Rightarrow (a,b) \in (\text{diag} \cup P) \Rightarrow (a,b) \in \text{diag} \Rightarrow (a,b) = (m,m)$

$a = b = m$
since $a = b$ & $b \leq c$
$a \leq c$
we are done

or

$(a,b) \in P$

$a \cdot a \leq b \rightarrow ①$

$b \leq c \Rightarrow (b,c) \in (\text{diag} \cup P) \Rightarrow (b,c) \in \text{diag} \Rightarrow (b,c) = (m,m)$

$b = c = m$
$a \leq b$
$a \leq c$
we are done

or

$(b,c) \in P$

$b \cdot b \leq c \rightarrow ②$

From ① & ②

$a \cdot a \leq b \,,\, b \cdot b \leq c$

$a^2 \leq b \Rightarrow a^2 \cdot b \leq b^2 \; (b \in \mathbb{N})$

$a^4 \leq b^2 \; (\cancel{a^2 b} \;\; a^2 \leq b$
$\quad a^4 \leq b a^2 \,]$

$b^2 \leq c \Rightarrow a^4 \leq c \; [\text{Since } a^4 \geq a^2 \,]$

$a^2 \leq a^4 \leq c$

$\therefore a^2 \leq c$

Hence $a \leq c$

$\therefore$ it is transitive.

$\therefore (\mathbb{N}, \leq)$ is partially ordered.

Let $(2,3)$ $m = 2$ & $n = 3$

Check comparability $2 \leq 3$ or $3 \leq 2$

$2^2 \leq 3 \qquad 3^2 < 2$

false $\qquad$ false.

Hence, $\leq$ not comparable on $\mathbb{N}$

$\therefore$ not totally ordered only partially ordered

HW-3

(1) $diag := \{(m,m) : m \in \mathbb{N}\}$

$P := \{(m,n) \in \mathbb{N} \times \mathbb{N} : m \cdot m \leq n\}$

$m \leq n \Leftrightarrow (m,n) \in (diag \cup P)$.

Reflexivity :- (CHECK)

$m \leq m$

$(m,m) \in$ diagonal

$(m,m) \in (diag \cup P)$

Hence it is reflexive

Antisymmetry :- (CHECK)

if $a \leq b$ & $b \leq a$ then $a = b$

$a \leq b \Rightarrow (a,b) \in (diag \cup P)$

$(a,b) \in diag$ or $(a,b) \in P$.

if $(a,b) \in diag$ then $(a,b) = (m,m)$

$a = m, b = m$

$\Rightarrow a = b = m$.

$a = b$, we are done

if $(a,b) \in P$ then $a \cdot a \leq b \rightarrow (1)$

$b \leq a \Rightarrow (b,a) \in (diag \cup P)$

$(b,a) \in diag$ or $(b,a) \in P$.

if $(b,a) \in diag$ then $(b,a) = (m,m)$

$b = m$ & $a = m$

$\Rightarrow a = b = m$

$a = b$ we are done.

if $(b,a) \in P$, then $b \cdot b \leq a \rightarrow (2)$

From (1) & (2)

$a \cdot a \leq b$, $b \cdot b \leq a$

since $b \in \mathbb{N}$

$a \cdot a \cdot a \cdot a \leq a \cdot a \cdot b \leq b \cdot b$

$a \cdot a \cdot a \cdot a \leq a$

$a \cdot a \cdot a \leq 1$

$a^3 \leq 1$

$a \cdot a \leq b$.

$a \cdot a \cdot b \leq b \cdot b$

$a \cdot a \cdot b \leq a$

$ab \leq 1$ (by cancellation).

$a, b \in \mathbb{N} \Rightarrow a \cdot b \in \mathbb{N}$ (by Peano's arithmetic)

$a \cdot b \in \mathbb{N}$ & $ab \leq 1$

$a = 0$ or $b = 0 \Leftarrow ab = 0$ or $ab = 1 \Rightarrow a = b = 1$ (done)

if $a = 0$

$b \cdot b \leq a$

$b \cdot b \leq 0$

$\Rightarrow b = 0$

if $b = 0$

$a \cdot a \leq b$

$a \cdot a \leq 0$

$\Rightarrow a = 0$

$a = b = 0$

Hence antisymmetric

② $(-1)a = -a \quad \forall a \in \mathbb{Z}$

~~$a = m \backslash n =: \{ (m,n) \in \mathbb{N},\mathbb{N}$ s.t $a + n = m$~~

$a = m \backslash n$ since $a \in \mathbb{Z}$

$-1 = 0 \backslash 1$

$a = a \backslash 0$

$$(-1) \times_{\mathbb{Z}} a = (0 \backslash 1) \times_{\mathbb{Z}} (m \backslash n)$$

$$= \left[(0 \times_{\mathbb{N}} m) +_{\mathbb{N}} (1 \times_{\mathbb{N}} n)\right] \backslash \left[(1 \times_{\mathbb{N}} m) +_{\mathbb{N}} (0 \times_{\mathbb{N}} n)\right]$$

$$= (0 +_{\mathbb{N}} n) \backslash [m +_{\mathbb{N}} 0] \quad \left[\text{By Peano mult'n extended to integer } 0 \times_{\mathbb{N}} n = 0\right]$$

$$= n \backslash m \quad \left[\text{By integer add'n } 0 +_{\mathbb{N}} n = n +_{\mathbb{N}} 0 = n\right]$$

$$= -(m \backslash n) \quad \left[\text{By def'n } -(a \backslash b) = b \backslash a\right]$$

$$= -a$$

$-(m \backslash n) = n \backslash m$

Compute. $n \backslash m + m \backslash n = (n +_{\mathbb{N}} m) \backslash m +_{\mathbb{N}} n$

$$= (m +_{\mathbb{N}} n) \backslash (m +_{\mathbb{N}} n) \quad [m + n = n + m]$$

$$= 0 \quad [a \backslash a = 0]$$

Hence $n \backslash m$ is additive inverse of $m \backslash n$

$-a$ is additive inverse of a.

③ $(-1)a = -a \quad \forall a \in \mathbb{Q}$.

$(-1) = (-1)/1$

$a = m/n$

$$(-1) \times_{\mathbb{Q}} a = ((-1)/1) \times_{\mathbb{Q}} (m/n)$$

$$= (-1 \times_{\mathbb{Z}} m) / 1 \times_{\mathbb{Z}} n$$

$$= (-m)/n \quad \left[\text{By Question 2 \& By multiplication } 1 \times_{\mathbb{Z}} n = n\right] = -(m/n) = -a$$

Compute

$$(-m)/n +_{\mathbb{Q}} m/n = \left((-m) \times_{\mathbb{Z}} n + (m \times_{\mathbb{Z}} n)\right) / n \times_{\mathbb{Z}} n$$

$$= (-m \cdot n + mn) / n \times_{\mathbb{Z}} n$$

$$= 0 / n \times_{\mathbb{Z}} n$$

$$= 0$$

Hence $(-m)/n = -(m/n)$

i.e additive inverse of m/n

(8) Given:- $x, y \in \mathbb{R}$ $x<y$

To prove:- $\exists q$ s.t $x<q<y$

Proof:- $x<y \Rightarrow y-x \in \mathbb{R}^+$

$y-x>0$

$\frac{1}{y-x}>0$

By Archemedian property

$z = \frac{1}{y-x}$

Fix $\varepsilon \geq 0$, $=1$

$\exists n \in \mathbb{P}$ s.t $n\varepsilon > z$

$n \times 1 > \frac{1}{y-x}$

$n > \frac{1}{y-x}$

$y-x>0$

$(y-x)(n) > 1$

$ny - nx > 1$

Since $ny-nx>1$

$\exists m \in \mathbb{Z}$ s.t m lies b/w nx & ny

$y-x>0$

$n(y-x)>0$

$ny > nx$

$ny > m > xn$

$n \in \mathbb{P}$ & $n \neq 0$

$y > \frac{m}{n} > x$

$x < \frac{m}{n} < y$

$\because m \in \mathbb{Z}$ & $n \in \mathbb{P}$

$q = m/n \in \mathbb{Q}$

Hence $\exists q$ s.t $x<q<y$

⑦ Given :- $(\mathbb{F}, \leq)$ be an ordered field having LUB.

$$A \subseteq \mathbb{F}$$

$$-A := \{-x \mid x \in A\}$$

To prove :- Sup $(-A)$ exists &

$$\inf(A) = -\sup(-A)$$

Proof :- Let α be the $\inf(A)$

~~ie $\nexists$ $\alpha' > \alpha$ s.t~~

(i) $\forall x \in A \quad x > \alpha$

(ii) for every $\alpha' > \alpha$, $\exists x \in A$ s.t $x < \alpha'$

$$-A := \{-x \mid x \in A\}$$

(i) $\forall x \in A \quad x > \alpha \Rightarrow \alpha < x$

$$x - \alpha > 0$$

$$\underbrace{x + (-x)}_{0} - \alpha > -x$$

$$-\alpha > -x$$

$\forall x \in A \quad -\alpha > -x$

$\Rightarrow \forall -x \in -A$

i.e $\forall y \in -A \quad -\alpha > y$

(ii) for every $\alpha' > \alpha$, $\exists x \in A$ s t $x < \alpha'$

$\Rightarrow$ for every $-\alpha' < -\alpha$, $\exists -x \in -A$ st $-x > -\alpha'$

for every $\alpha'' < -\alpha$, $\exists y \in -A$ s.t $y > \alpha''$

Hence (i) $\forall y \in -A$, $-\alpha > y$

(ii) for every $\alpha'' < -\alpha$, $\exists y \in -A$ s.t $y > \alpha''$

Hence $-\alpha$ is the $\sup(-A)$

$$\sup(-A) = -\alpha$$

$$-(\sup(-A)) = \alpha$$

$$\inf(A) = -(\sup(-A))$$

(5) For two cuts α, β

$$\alpha \leq \beta \iff \begin{cases} \alpha = \beta \\ \alpha \subsetneq \beta \end{cases}$$

To prove: $\leq$ is a total order

Proof: $\leq$ is reflexive

$\alpha \leq \alpha \iff \alpha = \alpha$ or $\alpha \subsetneq \alpha$.

$\alpha^{=A} = \alpha^{=B}$

$\forall x \in \alpha^{=A}, x \in \alpha^{=B}$

$\forall x \in \alpha^{=B}, x \in \alpha^{=A}$

$\alpha = \alpha$ [By defn of equality

$\leq$ is antisymmetric

$\alpha \leq \beta$ & $\beta \leq \alpha \Rightarrow \alpha = \beta$

$\Downarrow \qquad \Downarrow$

$\alpha = \beta$ or $\alpha \subsetneq \beta \qquad \beta = \alpha$ (or) $\beta \subsetneq \alpha$.

if $\alpha = \beta$ (we are done)

if $\alpha \subsetneq \beta \Rightarrow \exists x \in \beta$ s.t $x \notin \alpha$

~~but~~ $\beta \leq \alpha$.

$\beta = \alpha$ $\forall x \in \beta$ & $x \in \alpha$
contradiction

$\beta \subsetneq \alpha$, $\forall x \in \beta$, $x \in \alpha$
contradiction

Hence $\alpha \subsetneq \beta$ cannot occur.

Hence $\alpha = \beta$

$\leq$ is transitive

$\alpha \leq \beta$ & $\beta \leq \gamma \Rightarrow \alpha \leq \gamma$

$\Downarrow \qquad \Downarrow$

$\alpha = \beta$ or $\alpha \subsetneq \beta \qquad \beta = \gamma$ or $\beta \subsetneq \gamma$

if $\alpha = \beta \Rightarrow \beta \leq \gamma \Rightarrow \alpha \leq \gamma$ (We are done)

if $\alpha \subsetneq \beta \Rightarrow \beta = \gamma$ or $\beta \subsetneq \gamma$

if $\beta = \gamma$

$\Rightarrow \alpha \subsetneq \gamma \Rightarrow \alpha \leq \gamma$ (We are done

if $\beta \subsetneq \gamma$.

$\alpha \subsetneq \beta$ & $\beta \subsetneq \gamma \Rightarrow \alpha \subsetneq \gamma \Rightarrow \alpha \leq \gamma$ (We are done)

Hence is transitivee

reflexive, antisymmetric & transitive.

Comparal

Comparability:- either $\alpha \subset \beta$ or $\beta \subseteq \alpha$

if $\alpha \subseteq \beta \Rightarrow \forall x \in \alpha, x \in \beta$

suppose not

$\cancel{\forall x \in \alpha}$, $\exists x_0 \in \alpha, x_0 \notin \beta$

~~$\Rightarrow$ By (CII) if $x_0 \in \beta$~~

Claim:- $x_0 > y, \forall y \in \beta$

Proof:- suppose not

$\exists y_0 \in \beta$ s.t $y_0 \geq x_0$

$\Rightarrow x_0 \in \beta$ (By CII)

Contradiction

$\forall y \in \beta, x_0 > y \quad x_0 \in \alpha.$

$\Rightarrow \forall y \in \beta, y \in \alpha.$ (By CII)

Hence $\beta \subseteq \alpha$.

$\therefore$ either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$,,

Hence reflexive, anti symmetric, transitive & comparable

$\Rightarrow \leq$ is totally ordered.