Lecture - 12

Recall :- Notation for cycles, $\pi = c_1 c_2 \cdots c_k$

$$c_i = (c_{i,1}, c_{i,2}, c_{i,3}, \dots, c_{i,k})$$

a, Each element has its smallest element first

b, Cycles are arranged in increasing order of their smallest elements.

Example :- $3214657 \in S_7$

$$(13)(2)(4)(56)(7)$$

Def'n :- The cycle type of permutation σ, denoted type(σ), is the partition formed by arranging the lengths of its cycles in weakly decreasing order.

Example :- $\sigma = 6754132 = (1635)(27)(4)$

$\sigma^2 = 3214657 = (13)(2)(4)(56)(7)$

$\text{type}(\sigma) = (4,2,1) = \langle 1^1, 2^1, 3^0, 4^1 \rangle$

$\text{type}(\sigma^2) = (2,2,1,1,1) = \langle 1^3, 2^2 \rangle$

Frequency notation for partitions :-

Write $\lambda = \langle 1^{a_1}, 2^{a_2}, \dots, n^{a_n} \rangle$ for $\lambda \vdash n$

where $a_i = \#\ i\text{'s in } \lambda$

$$\sum i a_i = n$$

Thm :- The # of permutations in S_n with cycle type $\lambda = \langle 1^{a_1}, 2^{a_2}, \dots, n^{a_n} \rangle$ is

$$\frac{n!}{(a_1!\, a_2! \cdots a_n!)\, 1^{a_1} 2^{a_2} \cdots n^{a_n}}$$

Proof :- For every permutation $\sigma \in S_n$ in 1 line notation. Insert parenthesis. So that we first have a_1, 1 cycle., then a_2, 2 cycles., and so on i.e.,

$$\underbrace{(\sigma_1)(\sigma_2)\cdots(\sigma_{a_1})}_{a_1 \text{ 1cycle}} \cdot \underbrace{(\sigma_{a_1+1} \cdots \cdots}_{a_2 \text{ 2cycles}}$$

Thus we have a map from S_n to S_n

This is not surjective since all elements in the range have cycle type λ.

Question :- How many times does a permutation $\pi \in S_n$ with cycle type λ appear?

Answer :- Fix a length j :- $(c_1^{(1)}, c_2^{(1)}, \dots, c_j^{(1)}) \cdot (c_1^{(2)}, c_2^{(2)}, \dots, c_j^{(2)}) \cdots (c_1^{(a_j)}, c_2^{(a_j)}, \dots, c_j^{(a_j)})$

the same cycle occurs in j ways by cyclic notation.
Since we have a_j cycles, this gives us a factor j^{a_j} among all the $a_j - j$ cycles, we get π if these cycles are permuted and this gives a factor $a_j!$
Note that this is independent of π.
Repeat these calculations for $j=1$ to n and note that the rearrangements are independent.
Thus, the # of ways π appears is
$$\prod_{j=1}^{n} j^{a_j} \cdot a_j! \text{ which gives us the result.}$$

<u>Example</u> :- $n=3$, $\lambda = (2,1) = \langle 1^1, 2^1 \rangle$
$$\# \frac{6}{2} = 3$$
<u>Example</u> :- $\lambda = (n) = \langle n^1 \rangle \Rightarrow \# \frac{n!}{(n)(1)^n!} = (n-1)!$

which is no. of cyclic permutations of length n.

<u>Def'n</u> :- The # of permutations in S_n with k-cycles is denoted $\left[{n \atop k}\right]$ read 'n cycle k' and is called the (unsigned) stirling number of first kind.

<u>Remark</u> :- Using the prev thm to calculate $\left[{n \atop k}\right]$ is very inefficient.
<u>Property</u> :- For $1 \le k \le n$, we have $\left[{n \atop k}\right] = \left[{n-1 \atop k-1}\right] + (n-1)\left[{n-1 \atop k}\right]$ with $\left[{0 \atop k}\right] = \delta_{k,0}$
$\left[{n \atop k}\right] = 0$ if $k < 0$ or $k > n$

<u>Proof</u> :- Split permutations in S_n with k cycles according to whether n is a singleton cycle or not. If it is, then other elements form a permutation in S_{n-1} with $k-1$ cycles. If not, it is a part of cycle and is uniquely obtained by inserting before an element of S_{n-1} with k cycles.
Check that this inserting procedure is bijective.

n\k	0	1	2	3	4
0	1	0	0	..	...
1	0	1	0	.	.
2	0	1	1	0	...
3	⋮	2	3	1	0...
4	⋮	6	11	6	1

Property :- Let $n \in \mathbb{N}$, then $\sum_{k=0}^{n} \left[{n \atop k}\right] x^k = x^{\overline{n}} = x(x+1)\cdots\cdots(x+n-1)$

Proof :- Induct on n.

For $n=1$, this holds.

Let $G_n(x) = \sum_{k=0}^{n} \left[{n \atop k}\right] x^k$. Assume this holds for $n-1$.

$$(x+n-1)\,G_n(x) = (x+n-1)\left[\sum_{k=0}^{n-1} \left[{n-1 \atop k}\right] x^k\right]$$

$$= \sum_{k=0}^{n-1} \left[{n-1 \atop k}\right] x^{k+1} + (n-1)\sum_{k=0}^{n-1} \left[{n-1 \atop k}\right] x^k$$

$$= \sum_{k=1}^{n} \left(\left[{n-1 \atop k-1}\right] + (n-1)\left[{n-1 \atop k}\right]\right) x^k$$

$$= \sum_{k=1}^{n} \left[{n \atop k}\right] x^k + \left[{n \atop 0}\right] x^0 = \sum_{k=0}^{n} \left[{n \atop k}\right] x^k$$

$$= G_n(x)$$

Corollary :- $\sum_{k=0}^{n} (-1)^k \left[{n \atop k}\right] x^k = x^{\underline{n}} = x(x-1)\cdots(x-n+1)$

Proof :- Exercise (Replace x with $(-x)$ in prev proof)

Recall :- $V = \mathbb{Q}[x]$ is a vector space (of polynomials with rational co-efficients).

A natural basis is $B_1 = \{1, x, x^2, \ldots\}$

We also have $B_2 = \{1, x, x(x-1), \ldots\}$ → Falling Factorial.

Let S be the $\mathbb{N} \times \mathbb{N}$ matrix whose $(n,k)^{th}$ entry is $\left\{{n \atop k}\right\}$, then $\sum_k \left\{{n \atop k}\right\} x^{\underline{k}} = x^k$. Show that S is the transition matrix from B_2 to B_1. Similarly, Let s be the $\mathbb{N} \times \mathbb{N}$ matrix whose $(n,k)^{th}$ entry is $(-1)^{n-k}\left[{n \atop k}\right]$, the above corollary shows that s is the transition matrix from B_1 to B_2

Thm :- $s \cdot S = S \cdot s = \mathbb{1}$

Many Identities relate the two :-

1. $\sum_k \binom{n}{k} \left\{{k \atop m}\right\} = \left\{{n+1 \atop m+1}\right\}$
2. $\sum_k \left[{n \atop k}\right] \binom{k}{m} = \left[{n+1 \atop m+1}\right]$
3. $\sum_k (-1)^{m+1} \left\{{n+1 \atop m+1}\right\} \left[{k \atop m}\right] = \binom{n}{m}$