UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022
HOMEWORK 10

Instructor: GAUTAM BHARALI **Assigned: MARCH 22, 2022**

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1. Any rational number x can be written uniquely as $x = m/n$, where $m \in \mathbb{Z}$, $n \in \mathbb{Z}_+$, and such that there is no $d \in \mathbb{N} \setminus \{0, 1\}$ dividing both m and n — with the understanding that we take $n = 1$ when $x = 0$. (You may use this fact **without proof.** You have learnt in UM205 what "d divides m (or n)" means.) Define $f : \mathbb{R} \to \mathbb{Q}$ as follows:

$$f(x) := \begin{cases} 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 1/n, & \text{if } x \in \mathbb{Q}, \end{cases}$$

where n is uniquely associated to $x \in \mathbb{Q}$ as explained above. Show that f is continuous at each irrational point and discontinuous at each rational point.

2. Let X and Y be metric spaces, $E \subseteq X$, and let $f : E \to Y$ be a function. Show that f is continuous at each isolated point of E.

Note. If $p \in E$ is an isolated point, then the class of sequences in $E \setminus \{p\}$ that converge to p is vacuous. So, informally, one expects the above fact owing to the truth of vacuous implications. However, we **cannot** appeal to the sequential definition for the limit of f at p since that definition is valid only at limit points of E! Thus, a **formal** proof would require a different approach.

3. Let $n \geq 2$, $n \in \mathbb{Z}_+$. Prove from the definition and first principles (i.e., without using any results on sums/products of continuous functions), that $f(x) = x^n$, $x \in \mathbb{R}$, is continuous on $\mathbb{R}$.

4. Let X and Y be metric spaces, and let $f, g : X \to Y$ be two continuous functions. Let $E \subsetneq X$ be a proper dense subset. Suppose $f(x) = g(x)$ for each $x \in E$. Show that $f = g$.

5. Let $p \in \mathbb{Q}^+$ be fixed. Show that $f(x) = x^p$, $x \in [0, +\infty)$, is continuous on $[0, +\infty)$.

Hints. (i) The assumptions stated in the preamble of Problem 6 of Assignment 8 about x^p hold true for all $x \geq 0$ (i.e., not just for $x > 1$). (ii) It might not be pleasant to solve the above problem **entirely** from first principles!

6. This problem is on set theory and isn't really about analysis. However, we have made use of some of the statements below. (E.g., the third statement below was used in proving that the continuous image of a compact set is compact.)

Let S_1 and S_2 be non-empty sets and let $f : S_1 \to S_2$. Let $\mathscr{A}$ be a non-empty subset of $\mathcal{P}(S_1)$ and let $\mathscr{B}$ be a non-empty subset of $\mathcal{P}(S_2)$. Prove the following:

$$\begin{aligned} f\big(\cup_{A \in \mathscr{A}} A\big) &= \cup_{A \in \mathscr{A}} f(A), \\ f\big(\cap_{A \in \mathscr{A}} A\big) &\subseteq \cap_{A \in \mathscr{A}} f(A), \\ f^{-1}\big(\cup_{B \in \mathscr{B}} B\big) &= \cup_{B \in \mathscr{B}} f^{-1}(B), \\ f^{-1}\big(\cap_{B \in \mathscr{B}} B\big) &= \cap_{B \in \mathscr{B}} f^{-1}(B). \end{aligned}$$

7. The following problem was given in the mid-semester exam. It is being presented again, this time with hints.
Let G be a non-empty bounded open subset of $\mathbb{R}$. Complete the outline below to show that G is the union of an at most countable collection of **disjoint** non-empty open intervals.

(a) Consider the following relation on G:

$$a \sim b \iff [\min(a, b), \max(a, b)] \subset G,$$

where, for any real numbers $x \leq y$, $[x, y]$ is the set $\{t \in \mathbb{R} : x \leq t \leq y\}$. Show that $\sim$ is an equivalence relation on G.

(b) Prove that each equivalence class of $\sim$ is an open interval.

(c) Using the fact that $\mathbb{R}$ (equipped with the usual metric) is a separable metric space, show that the set of equivalence classes of $\sim$ is at most countable.

Remark. The above result is true without the assumption of boundedness of G. This assumption just eliminates certain annoyances and certain cases that would otherwise have to be considered.

8. Review/Self-study. The following topics were studied in UM101 (and, barring the Chain Rule, with rigorous proofs): differentiability, differentiability of algebraic combinations of differentiable functions, the Chain Rule, points of local maximum/minimum, Rolle's Theorem, Lagrange's Mean Value Theorem and applications. Please **review** this material from pages 104–108 from Rudin's book, excluding Theorem 5.9, by March 29.