UM 204 HW-9

2) Given, p is an isolated pt.

So, $\exists\, r > 0$ s.t. $B(p, r) \cap E = \{p\}$

Choose any $\varepsilon > 0$.

Take $\delta = r$

$\therefore$ Take x : $d_X(x, p) < \delta$ and $x \in E$

$\Rightarrow x \in B(p, r) \cap E$

$\Rightarrow x = p$

$\Rightarrow d_Y(f(x), f(p)) = 0$

$\Rightarrow d_Y(f(x), f(p)) < \varepsilon$

By defn, f is continuous at p

3) Given, p is a limit pt. of E

So, $\exists\, \{x_n\} \subset E$ s.t. $\{x_n\} \to p$

Let $\{x_n\}$ be such an arbitrary seq.

"$\Rightarrow$" f is continuous at p.

So, choose any $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$d_Y(f(x), f(p)) < \varepsilon$ whenever

$d_X(x, p) < \delta$ and $x \in E$ — (1)

Now, $\exists\, N \in \mathbb{P}$ s.t.

$d_X(x_n, p) < \delta \quad \forall\, n \geq N$

as $x_n \in E$; using (1).

$d_Y(f(x_n), f(p)) < \varepsilon \quad \forall\, n \geq N$

$\Rightarrow \lim_{n \to \infty} f(x_n) = f(p)$

As $\{x_n\}$ is arbitrary, $\lim_{x \to p} f(x) = f(p)$

"$\Leftarrow$" $\lim_{x \to p} f(x) = f(p)$

So, as $\{x_n\} \to p \Rightarrow \{f(x_n)\} \to f(p)$

Now, assume the contradiction of continuity, i.e.,
assume $\exists \, \varepsilon_0 > 0$ s.t. $\forall \delta > 0$, $\exists \, x_\delta \in E$ s.t.

$$d_Y(f(x_\delta), f(p)) \geq \varepsilon_0 \quad \text{whenever} \quad d_X(x_\delta, p) < \delta$$

If we choose $\delta = 1/n \; \forall n \in \mathbb{P}$ and $x_n := x_\delta \; \forall n \in \mathbb{P}$

we have $\{x_n\}$ s.t. $d_X(x_n, p) < 1/n$

$$\Rightarrow \{x_n\} \to p$$

But $d_Y(f(x_n), f(p)) \geq \varepsilon_0 \quad \forall n \in \mathbb{P}$

$\Rightarrow \{f(x_n)\}$ does not cvg. to $f(p)$

CONTRADICTION.

So, f is continuous at p.

4) "$\Rightarrow$" E is compact

Take any open cover $\mathscr{C}$ ~~[illegible]~~

$\therefore \exists \, i_1, \ldots i_k \in \mathbb{P}$ s.t. $G_{i_j} \in \mathscr{C}$ and

$$E \subseteq \bigcup_{j=1}^{k} G_{i_j} \quad - (1)$$

As G_is are open sets in X,

$G_i \cap E$ are open relative to E

Take $\mathscr{C}' = \{G_{i_j} \cap E \mid j \in \mathbb{P}\}$ is an open cover rel. to E

From (1), $E \subseteq \bigcup_{j=1}^{k} (G_{i_j} \cap E)$

~~As~~ We get a finite cover of E w.r.t. E

As $\mathscr{C}$ was arbitrary, E is a compact metric space.

"$\Leftarrow$" E is a compact metric space w.r.t. $d|_{E\times E}$

Let $\mathcal{C}$ be an open cover w.r.t. E

~~s.t. ...~~

So, $\exists$ $i_1, \dots i_k \in \mathbb{P}$ s.t. $A_{i_j} \in \mathcal{C}$

and $E \subseteq \bigcup_{j=1}^{k} A_{i_j}$

As A_{i_j}'s are open rel. to E,

$\exists$ $G_{i_j} \subseteq_{open} X$ s.t. $A_{i_j} = G_{i_j} \cap E$

$\therefore$ $E \subseteq \bigcup_{j=1}^{k} G_{i_j}$

Let $\mathcal{C}' = \{G \mid G \cap E \in \mathcal{C}\}$
$G \subseteq_{open} X$

This is an open cover in X

$\therefore$ we have found a sub-cover

$\Rightarrow$ E is compact in X

5) (i) $f\left(\bigcup_{A\in\mathcal{A}} A\right) = \bigcup_{A\in\mathcal{A}} f(A)$

Let $y \in f\left(\bigcup_{A\in\mathcal{A}} A\right)$ $\therefore$ $\exists$ $x \in \bigcup_{A\in\mathcal{A}} A$ s.t. $y = f(x)$

$\Rightarrow \exists x \in A$ for some $A \in \mathcal{A}$

$\Rightarrow f(x) \in f(A)$,

$\Rightarrow y \in f(A)$,

$\Rightarrow y \in \bigcup_{A\in\mathcal{A}} f(A)$

Let $z \in \bigcup_{A\in\mathcal{A}} f(A) \Rightarrow z \in f(A)$ for some $A \in \mathcal{A}$

$\Rightarrow \therefore \exists x' \in A$ s.t. $z = f(x')$

$\Rightarrow x' \in \bigcup_{A\in\mathcal{A}} A$ s.t. $z = f(x')$

$\Rightarrow z \in f\left(\bigcup_{A\in\mathcal{A}} A\right)$.

$\therefore$ (i) $\boxtimes$

(ii) $f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A)$

Let $y \in f\left(\bigcap_{A\in\mathcal{A}} A\right)$, $\exists\, x \in \bigcap_{A\in\mathcal{A}} A$ s.t. $y = f(x)$

$\Rightarrow x \in A \quad \forall A \in \mathcal{A}$

$\Rightarrow f(x) \in f(A) \quad \forall A \in \mathcal{A}$

$\Rightarrow y \in f(A) \quad \forall A \in \mathcal{A}$

$\Rightarrow y \in \bigcap_{A\in\mathcal{A}} f(A) \quad \forall A \in \mathcal{A}$

$\boxtimes$

(iii) $f^{-1}\left(\bigcup_{B\in\mathcal{B}} B\right) = \bigcup_{B\in\mathcal{B}} f^{-1}(B)$

Let $x \in f^{-1}\left(\bigcup_{B\in\mathcal{B}} B\right) \Rightarrow f(x) \in \bigcup_{B\in\mathcal{B}} B$

$\Rightarrow f(x) \in B$ for some $B \in \mathcal{B}$

$\Rightarrow x \in f^{-1}(B)$

$\Rightarrow x \in \bigcup_{B\in\mathcal{B}} f^{-1}(B)$

Let $x' \in \bigcup_{B\in\mathcal{B}} f^{-1}(B) \Rightarrow x' \in f^{-1}(B)$ for some $B \in \mathcal{B}$

$\Rightarrow f(x') \in B$

$\Rightarrow f(x') \in \bigcup_{B\in\mathcal{B}} B$

$\Rightarrow x' \in f^{-1}\left(\bigcup_{B\in\mathcal{B}} B\right)$ $\boxtimes$

(iv) SAME ARGUMENTS.

6) (i) We can look at seq. $\{x_n\}$ which does not satisfy this property, i.e.,

$$\{x_n\} \subset X \text{ s.t. } \{x_n : n \in \mathbb{P}\} \cap X \setminus E \text{ is finite.}$$

i.e., $\exists N \in \mathbb{P}$ s.t. $x_1, \dots x_N \in X \setminus E$ and

$$x_n \in E \quad \forall n > N$$

Now, we know from 101 that, as

$$\{x_n\} \to p \quad , \quad \{x_{N+n}\} \to p$$

Let $y_n := x_{N+n} \quad \forall n \in \mathbb{P}$

So, $\{y_n\} \subset E$ and $\{y_n\} \to p$

By defn, $\tilde{f}(p) = \lim_{n \to \infty} f(y_n)$

[ALREADY DONE IN CLASS]

So, any seq. which does not satisfy prop. 1. is by defn continuous at p, so no use checking these seq.

(ii) ~~Assume~~

Let $\{x_n\}$ be a seq. with prop. 1

Define $\{y_n\}$ as follows,

if $x_n \in E$, $y_n := x_n$

else, $x_n \in E' \setminus E$, $\therefore \exists q \in B(x_n, 1/n), q \in E$

So, $\{y_n\} \subset E$ $\qquad y_n := q$

Now, $d_Y(\tilde{f}(x_n), \tilde{f}(x_m)) \leq d_Y(\tilde{f}(x_n), \tilde{f}(y_{n^*}))$
$+ d_Y(\tilde{f}(y_{n^*}), \tilde{f}(y_{m^*}))$
$+ d_Y(\tilde{f}(y_{m^*}), \tilde{f}(x_m))$

$\forall m, n, m^*, n^* \in \mathbb{P}$ — (1)

~~As $\tilde{f}|_E = f$, $\tilde{f}(y_n) = f(y_n)$ $\forall n \in \mathbb{P}$~~

~~As~~

~~If $x_n \in E$, $d_Y(\tilde{f}(x_n), \tilde{f}(y_n)) = 0$~~

~~By constr^n~~

~~If $x_n \notin E \Rightarrow x_n \in E' \setminus E$~~

~~$\therefore \exists \{x_k^{n*}\}_{k \in \mathbb{P}} \subset E$ s.t. $\{x_k^{n*}\} \to x_n$~~

~~s.t. $d_X(x_k^{n*}, x_n) < 1/k$ $\forall k \in \mathbb{P}$~~

~~Let us take a $\{x_k^{n*}\}$ s.t.~~

~~$x_n^{n*} = y_n$~~

As $\tilde{f}|_E = f$. $\tilde{f}(y_k) = f(y_k)$ $\forall k \in \mathbb{P}$

If $x_n \in E$, $d_Y(\tilde{f}(x_n), \tilde{f}(y_n)) = 0$

[By construct^n]

If $x_n \notin E \Rightarrow x_n \in E' \setminus E$

$\therefore \exists \{x_k^{n*}\}_{k \in \mathbb{P}} \subset E$ s.t. $\{x_k^{n*}\} \to x_n$

and $d_X(x_k^{n*}, x_n) < \frac{1}{n}$

Choose $\varepsilon > 0$. as $\{f(x_k^{n*})\} \to \{f(x_n)\}$ as $k \to \infty$

$\exists K \in \mathbb{P}$, $d_Y(\tilde{f}(x_k^{n*}), \tilde{f}(x_n)) < \varepsilon/3$ $\forall k \geq K$

[illegible]

1) • Continuous at irrational pts.

Let $p \in \mathbb{R} \setminus \mathbb{Q}$.

Choose $\varepsilon > 0$, $\exists N \in \mathbb{P}$ s.t. $\frac{1}{N} < \varepsilon$

Now, we define $S = \{ r \in \mathbb{Q} \mid r \in (p-1, p+1)$ and $r = \frac{m}{n}$ s.t. $1 \le n \le N \}$

For each n, $n(p-1) < m < n(p+1)$

So, S is finite.

Define $\delta = \min_{r \in S} |r - p| < 1$

Now, let $|x - p| < \delta$

(i) if $x \in \mathbb{R} \setminus \mathbb{Q}$, $|f(x) - f(p)| = 0 < \varepsilon$

(ii) if $x \in \mathbb{Q}$, $x = \frac{m}{n}$

As $|x - p| < \delta$, $x \notin S$

and as $\delta < 1$, $x \in (p-1, p+1)$

$\therefore$ $n > N$ (ELSE, $x \in S$)

$\Rightarrow \frac{1}{n} < \frac{1}{N} < \varepsilon$

$\Rightarrow f(x) < \varepsilon \Rightarrow |f(x)| < \varepsilon$

$\therefore$ f is continuous at p

As p was arbitrary, $\boxtimes$

• Discontinuous at rational pt.

Let $p \in \mathbb{Q}$, So, $p = \frac{m_p}{n_p}$

Let $\varepsilon_0 = \frac{1}{2n_p}$

$\forall \delta > 0$, $\exists x \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $|x - p| < \delta$

and $|f(x) - f(p)|$

$= \frac{1}{n_p} > \varepsilon_0$

So, not continuous at p. As p was arbitrary $\boxtimes$