UM 204 (WINTER 2024) - WEEK 2

Definition 0.1. An ordered field F is said to have the Archimedean property if for every $x, y > 0$, there is an $n \in \mathbb{N}$ such that $nx > y$.

Theorem 0.2. *The field $\mathbb{Q}$ satisfies the Archimedean property.*

Proof. Let $x, y > 0$ be rational numbers. Assume $x \leq y$, else there is nothing to prove. Let $r = x/y \in \mathbb{Q}$. It suffices to show that $\exists n \in \mathbb{N}$ such that $nr > 1$. Write $r = p/q$, $m, n \in \mathbb{N}^*$. Then, $n = q + 1$ works. □

Not all ordered fields have the Archimedean property.

Theorem 0.3. *Let F be an ordered field* **with the l.u.b. property**. *Then, F has the Archimedean property.*

Proof. Let $A = \{nx : n \in \mathbb{N}\}$. Clearly, A is nonempty. Suppose the claim does not hold. Then, y is an upper bound. By the l.u.b. property, $\sup A$ exists. Thus, $\sup A - x < \sup A$ is not an upper bound of A. So, there is an $m \in \mathbb{N}$ such that

$$\sup A - x < mx \implies \sup A < (m+1)x \qquad \text{(due to (\textbf{??}))}.$$

This contradicts the fact that $\sup A$ is an upper bound of A. □

Corollary 0.4. *Let F be as above. Given $x, y \in F$, with $y - x > 1$, there is an $m \in \mathbb{Z}$ such that $x < m < y$.*

Proof. Exercise (and hint for Assignment 01)! □

Theorem 0.5 (Density of $\mathbb{Q}$)**.** *Let F be as above. Given any $x, y \in F$ such that $x < y$, there is a $z \in \mathbb{Q}$ such that $x < z < y$.*

Proof. This follows from the equivalence of the Archimedean property and density of $\mathbb{Q}$ in ordered fields (Assignment 01). □

0.1. **Real numbers.** The lack of square roots in $\mathbb{Q}$ is only the tip of the iceberg. Once again, we can use equivalence relations to “fill the gaps".

Theorem 0.6 (Dedekind)**.** *There exists a unique* ordered field $(\mathbb{R}, +, \cdot, \leq)$ with the least upper bound property, i.e., if G is any other ordered field with the least upper bound property, then there is a an ordered field isomorphism h from $\mathbb{R}$ onto G.*

Proof of uniqueness. Let F and G be ordered fields with the l.u.b. property. Then, both F and G contains $\mathbb{Q}$ as an ordered subfield. Let h be the identity map on $\mathbb{Q}$. Given any $z \in F \setminus \mathbb{Q}$, let

$$A_z = \{w \in \mathbb{Q} : w \leq z\}.$$

Then, by the density of $\mathbb{Q}$ in F, A_z is nonempty and has an upper bound in $\mathbb{Q}$, i.e., there is a $q \in \mathbb{Q}$ such that $w \leq q$ for all $w \in A_z$. Thus, q is an upper bound of A_z in G. By the l.u.b. property of G, we can define

$$h(z) = \text{supremum in } G \text{ of } A_z.$$

We claim that h is the desired ordered field isomorphism. We check that h is order-preserving (and leave the rest as exercise).

Say $z < w$. Clearly, $A_z \subseteq A_w$. By the density of $\mathbb{Q}$ in F, there exist $r, s \in \mathbb{Q}$ such that $z < r < s < w$. Since $a < r$ for all $a \in A_z$, we have that $\sup_G A_z \leq r$. Since $s \in A_w$, we have that $s \leq \sup_G A_w$. Thus,

$$h(z) = \sup_G A_z \leq r < s \leq \sup_G A_w = h(w).$$

Construction I (Dedekind). We will keep this brief.

Definition 0.7. A Dedekind cut is a subset $A \subseteq \mathbb{Q}$ scuh that

(1) (non-trivial) $\emptyset \neq A \neq \mathbb{Q}$,
(2) (closed from left) if $b \in \mathbb{Q}$ is such that $b < a$ for some $a \in A$, then $b \in \mathbb{Q}$,
(3) (no greatest element) if $a \in A$, then there is a $c \in A$ such that $a < c$.

Example. The set $\{x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2\}$ is a Dedekind cut.

Definition 0.8. Let $\mathbb{R} := \{A \subset \mathbb{Q} : A \text{ is a Dedekind cut}\}$. We say that

(1) $A \leq B \iff A \subseteq B$,
(2) $A + B = \{a + b : a \in A \text{ and } b \in B\}$,
(3) For $A, B > 0$, $A \cdot B = \{q \in \mathbb{Q} : q \leq rs \text{ for some positive } r \in A, s \in B\}$. For general $A, B \in \mathbb{R}$, use

$$A \cdot B = -(A \cdot -B) = -(-A \cdot B) = (-A \cdot -B).$$

END OF LECTURE 4

$(\mathbb{R}, \leq)$ has the l.u.b. property assuming $\leq$ is a total order. Let $\alpha \subseteq \mathbb{R}$ be a nonempty subset that is bounded above. Then,

$$\bigcup_{A : A \in \alpha} = \sup \alpha.$$

Proof. First, we show that $C \in \mathbb{R}$, i.e., C is a cut. Since C is a nonempty union of nonempty sets, it is nonempty. Since α is bounded above, there is a $B \subset \mathbb{R}$ such that

$$A \subseteq B \quad \forall A \in \alpha.$$

Thus, $C \subseteq B \subsetneq \mathbb{Q}$.

Let $a \in C$. Then, $c \in A$ for some $A \in \alpha$. Thus, for any rational $b < c$, $b \in A \subset C$. Moreover, since A is a cutm there is a $d \in A \subset C$ such that $b < d$.

Clearly, $A \subseteq C$ for all $A \in \alpha$, i.e., C is an upper bound of α. Suppose D is an upper bound of α such that $D < C$, i.e., $D \subsetneq C$. Then, there is a $c \in C \setminus D$. Since $c \in A$ for some A, $A \not\subset D$, i.e., $A \nleq D$. Since $\leq$ is a total order, $D < A$. $\square$

If you are curious about this construction, you can see the appendix of Chapter 1 in Rudin. In this construction, one is performing an "order completion".

Construction II (Cauchy). This construction goes via "metric completion". There are sequences in $\mathbb{Q}$ that appear to have a limit (e.g., monotone and bounded), but do not converge! We use an equivalence relation on a space of sequences to give a different construction of $\mathbb{R}$. The idea is that if you take the "decimal representation" of an irrational number, say $\sqrt{2} \approx 1.41421356237309504880168872420969 80...$, you may truncate the expansion in different ways to produce different sequences of rational numbers that converge to $\sqrt{2}$. We want to say that any such sequence can be "called" $\sqrt{2}$.

Definition 0.9. A sequence of rational numbers is a function $f : \mathbb{N} \to \mathbb{Q}$. We denote $f(k)$ by a_k and refer to it as the k^{th}-term of the sequence. The function itself is written as $\{a_k\}_{k\in\mathbb{N}}$.

Definition 0.10. A sequence $\{a_k\}_{k\in\mathbb{N}}$ in $\mathbb{Q}$ is said to be

(1) $\mathbb{Q}$-bounded if there is a rational $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$;
(2) $\mathbb{Q}$-Cauchy if for every rational $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon \quad \forall n, m \geq N;$$

(3) convergent in $\mathbb{Q}$ if there is an $\ell \in \mathbb{Q}$ such that, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$|a_n - \ell| < \varepsilon \quad \forall n \geq N.$$

Exercise! Show that convergent in $\mathbb{Q} \Rightarrow \mathbb{Q}$-Cauchy $\Rightarrow \mathbb{Q}$-bounded.

Remark. By a similar proof as seen in UM 101, in Case (3), ℓ is the unique number satisfying the condition, and is called the limit of $\{a_n\}_{\mathbb{N}}$. All the algebraic limits laws hold.

Definition 0.11. Given two sequences of rational numbers, $a = \{a_n\}_{n\in\mathbb{N}}$ and $b = \{b_n\}_{n\in\mathbb{N}}$, a is said to be equivalent to b if, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$|a_n - b_n| < \varepsilon \quad \forall n \geq N.$$

Proposition 0.12. *Let $\mathscr{C}$ denote the space of $\mathbb{Q}$-Cauchy sequences in $\mathbb{Q}$. Then aRb given by 'a is equivalent to b' is an equivalence relation on $\mathscr{C}$.*

Example. Any rational sequence $a = \{a_n\}_{\mathbb{N}}$ converging to $\ell = 0$ is equivalent to the constant 0 sequence (and vice versa). In general, it is quite hard to establish ($\mathbb{Q}$-)Cauchyness!

Definition 0.13. Define $\mathbb{R} := \left\{[a] = [a]_R : a \in \mathscr{C}\right\}$. Given $[a], [b] \in \mathbb{R}$,

(1) $[a] + [b] = [a + b]$,
(2) $[a] \cdot [b] = [a \cdot b]$,
(3) $[a] > 0$ if there is a rational number $c > 0$ and an $N \in \mathbb{N}$ such that $a_n > c$ for all $n \geq N$. We say that $[a] > [b]$, if there is a $[d] > 0$ such that $[a] + [d] = [b]$.

END OF LECTURE 5

Addition & multiplication. Exercise!

(i) Let $a = \{a_n\}$ and $b = \{b_n\}$ be Cauchy sequences. Then, $a + b = \{a_n + b_n\}$ and $a \cdot b = \{a_n \cdot b_n\}$ are Cauchy sequences.

(ii) Say aRa' and bRb', then $(a+b)R(a'+b')$ and $(a \cdot b)R(a' \cdot b')$.

(iii) Associativity and commutativity of $+$ and $\cdot$.

(iv) $0_{\mathbb{R}} = [\{a_n\}]$, where $a_n \equiv 1$ and $1_{\mathbb{R}} = [\{b_n\}]$, where $b_n \equiv 1$.

(v) $-[a] = [-a]$ for all $[a] \in \mathbb{R}$. Here, we must first show that if a is a Cauchy sequence, then so is $-a$.

(vi) If $[a] \neq 0_{\mathbb{R}}$, then there is some Cauchy sequence $\widetilde{a} = \{\widetilde{a}_n\}_{\mathbb{N}}$ such that and $[\widetilde{a}] = [a]$ and $\widetilde{a}$ nonvanishing, i.e., $\widetilde{a}_n \neq 0$ for all $n \in \mathbb{N}$. We claim that
- $1/\widetilde{a} := \{1/\widetilde{a}_n\}_{\mathbb{N}}$ is a Cauchy sequence,
- $[a \cdot 1/\widetilde{a}] = 1_{\mathbb{R}}$,
- If $\widetilde{b}$ is any other non-vanishing Cauchy sequence such that $[\widetilde{b}] = [a]$, then $[1/\widetilde{b}] = [1/\widetilde{a}]$.

Thus, we may define $1/[a]$ as $[1/\widetilde{a}]$.

(vii) $\cdot$ distributes over $+$.

Order. Exercise!

(i) For each $[a] \in \mathbb{R}$, exactly one of the following holds: $[a] = 0_{\mathbb{R}}$, $[a] > 0$ or $-[a] > 0$.

(ii) Transitivity of order holds.

(iii) Addition preserves order.

(iv) Multiplication preserves positivity.

The Archimedean property of $\mathbb{R}$. Let $[a], [b] > 0$. For any $m \in \mathbb{N}$, let $[m]$ denote the equivalence class of the constant sequence $\{m, m, \ldots\}$. Suppose, for all $m \in \mathbb{N}$, we have that $[m][a] \leq [b]$, i.e., $[ma - b] \leq 0$ or $[ma - b] \not> 0$. This means that for every $m \in \mathbb{N}$ and rational $c > 0$, there is a strictly increasing sequence of natural numbers $\{n_k\}_{k \in \mathbb{N}}$ such that $ma_{n_k} - b_{n_k} < c$.

Let $\varepsilon > 0$. Since $\{b_n\}$ is $\mathbb{Q}$-Cauchy, it is $\mathbb{Q}$-bounded, say by a rational number $M > 0$. By the Archimedean property of $\mathbb{Q}$, there is some $m \in \mathbb{N}$ such that $M < m\varepsilon/3$. Let $c = m\varepsilon/3$. Thus, there is sequence $\{n_k\}_{k \in \mathbb{N}}$ of natural numbers such that

$$ma_{n_k} \leq m2\varepsilon/3 \quad \forall k \in \mathbb{N}.$$

Now, since $\{a_n\}$ is $\mathbb{Q}$-Cauchy, there is an $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon/3$ for all $m, n \geq N$. Let $k \in \mathbb{N}$ such that $n_k > N$. Then, for any $n \geq N$,

$$|a_n| \leq |a_n - a_{n_k}| + |a_{n_k}| < \varepsilon.$$

This means that $\lim_{n \to \infty} a_n = 0$, i.e., $[a] = 0$. This is a contradiction.