Lecture - 14

Generative Functions :-

Def'n :- A Formal Power Series (FPS) $R([x])$ consists of formal sums of the form $\sum_{n=0}^{\infty} a_n x^n$ where $a_n \in \mathbb{R} \ \forall n$

Remarks :- We donot worry about the convergence since x is a formal indeterminate.

Example :- 1. $\sum_{n=0}^{\infty} n! x^n$ makes sense.

2. We can replace $\mathbb{R}$ by any field or even a commutative ring.

3. This FPS is a "completion" of $R[x]$

Example :- $a_n = 1 \ \forall n \Rightarrow A(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{1}{1-x}$

Def'n :- Given a sequence $(a_n)_{n \geq 0}$ the FPS, $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is called the ordinary generating function (ogf) of (a_n)

Example :- Suppose $(a_n)_{n \geq 1}$ satisfies $a_n = 4a_{n-1} - 100$ and $a_0 = 50$. Want an explicit formula for a_n.

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= 50x^0 + \sum_{n=1}^{\infty} (4a_{n-1} - 100) x^n$$

$$= 50 + 4\sum_{n=1}^{\infty} a_{n-1} x^n - 100 \sum_{n=1}^{\infty} x^n = 50 + 4x A(x) - \frac{100x}{1-x}$$

$$(1-4x) A(x) = 50 - \frac{100x}{1-x}$$

$$\boxed{A(x) = \frac{50}{1-4x} - \frac{100x}{(1-4x)(1-x)}}$$

$$= \frac{100}{3(1-x)} + \frac{50}{3(1-4x)}$$

$$= \frac{100}{3} \sum_{n \geq 1} x^n + \frac{50}{3} \sum_{n \geq 0} 4^n x^n$$

$$\boxed{a_n = \frac{100}{3} + 4^n \left(\frac{50}{3}\right)}$$

Property :- Let $A(x)$ and $B(x)$ be ogf's for (a_n) and (b_n) respectively. Then the sequence for which ogf is $A(x)\cdot B(x)$ is given by $c_n = \sum_{k=0} a_k b_{n-k}$ [Discrete Convolution].

Example :- $e^x = \sum_{n\geq 0} \frac{1}{n!} x^n$; $e^{x+1} \neq e^x . e$ [At the level of FPS]

$$e^{x+1} = \sum_{n\geq 0} \frac{1}{n!}(x+1)^n \longleftarrow \text{Any co-eff is an infinite sum.}$$

Recall :- Partitions $P(n) = \#$ of partitions of n.

Thm (Euler):- The ogf of $(P_n)_{n\geq 0}$ is $\sum_{n\geq 0} P_n q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i}$

Proof :- Expand the RHS $(1+q+q^2+\dots)(1+q^2+q^4+\dots)\dots\dots$
Look at all the terms that contribute to the co-efficient of q^n. CHECK that this is a bijection b/w set of terms that contribute and $\{\lambda \vdash n\}$ ▨

Exercise :- If (a_n) has ogf $A(x)$, then $(a_0 + a_1 + \dots + a_n)$ is $\frac{A(x)}{1-x}$. Two modifications of Euler's thm.

1, Restrict the set of parts to $S \subseteq \mathbb{N}$
Then RHS $= \prod_{i\in S} \frac{1}{1-q^i}$

2, Restrict the no. of times each part appears.
If each part is allowed to appear atleast thrice, then RHS $= \prod_{i\geq 1}(1+q^i+q^{2i}+q^{3i})$

Recall Euler's odd distinct theorem.

\# of partitions of n into odd parts $=$ \# of partitions of n into distinct parts.

Proof :-
$$\sum_{n\geq 0} P_{odd}(n)\, q^n = \prod_{i\geq 1} \frac{1}{1-q^{2i-1}} \cdot \frac{1-q^{2i}}{1-q^{2i}}$$
$$= \prod_{i\geq 1} \frac{1-q^{2i}}{1-q^i} = \prod_{i\geq 1}(1+q^i)$$
$$= \sum_{n\geq 0} P_{distinct}(n)\cdot q^n \quad ▨$$

Example :- Let c_n be the no. of valid words in n pairs of parenthesis

Example :- $n=3$:- $((()))$, $(())()$, $()()()$, $()(())$, $(()())$

Let $C(x) = \sum c_n x^n$

$n \geq 0$

Claim :- Every valid word ω can be expressed in the form $\omega = (\omega_1)(\omega_2)$

First time a valid subword is formed.

where ω_1, ω_2 are valid words with k, $n-1-k$ pairs of parenthesis respectively,

then $c_n = \sum_{k=0}^{n-1} c_k c_{n-1-k}$ for $n \geq 1$ and $c_0 = 1$

By product formula $C(x) - 1 = C(x) \cdot xC(x)$

$$[C(x)]^2 x - C(x) + 1 = 0$$

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

Expand $c_n = \frac{1}{n+1}\binom{2n}{n}$ [Catalan numbers]