UM 204 HOMEWORK ASSIGNMENT 9

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(NOT FOR SUBMISSION)

Problem 1. Given a double sequence $\{x_{m,n}\}_{m,n\in\mathbb{N}} \subset \mathbb{R}$, a joint limit L of $\{x_{m,n}\}$ is a real number such that for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that if $m, n \geq M$, then $|x_{m,n} - L| < \varepsilon$.

(*a*) Suppose $\{x_{m,n}\}$ admits a joint limit L. Further, suppose $\lim_{n\to\infty} x_{m,n}$ exists for all $m \in \mathbb{N}$, and $\lim_{m\to\infty} x_{m,n}$ exists for all $n \in \mathbb{N}$. Show that

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{m,n} = \lim_{m\to\infty}\lim_{n\to\infty} x_{m,n}.$$

(*b*) Produce an example of a double sequence $\{x_{m,n}\}$ that admits a joint limit, but $\lim_{n\to\infty} x_{m,n}$ does not exist for any $m \in \mathbb{N}$, amd $\lim_{m\to\infty} x_{m,n}$ does not exist for any $n \in \mathbb{N}$.

Problem 2. Let $\{f_n\}$ be a sequence of continuous functions on a metric space X. Suppose $\{f_n\}$ converges uniformly to f on X.

(*a*) Show that for every sequence $\{x_n\} \subset X$ converging to $x \in X$,

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

(*b*) Let $X = [0,1]$. Prove or disprove the following claim.

$$\lim_{n\to\infty}\int_0^{1-\frac{1}{n}} f_n(t)dt = \int_0^1 f(t)dt.$$

Problem 3. Let $f_n(x) = \dfrac{x}{1+(nx)^2}$, $x \in \mathbb{R}, n \in \mathbb{N}$. Let $\{r_n\}$ be an enumeration of the rational numbers. Define

$$h_n(x) = \sum_{k=1}^{n} 2^{-k} f_n(x - a_k).$$

(*a*) Show that f_n converges uniformly to 0.

(*b*) Show that f_n' converges pointwise to a function that is discontinuous at the origin.

(*c*) Show that $\{h_n\}$ converges uniformly to 0.

(*d*) Show that $\{h_n'\}$ converges pointwise to a function whose set of discontinuities is precisely the rational numbers.

Problem 4. Let $X = [0,1]$. Suppose $K \subset \mathcal{C}(X;\mathbb{C})$, where the latter is endowed with the $||\cdot||_\infty$ metric. Show that the following statements are equivalent.

(*a*) K is compact.

(*b*) K is closed, pointwise bounded and equicontinuous.

Problem 5. Let f be a continuous function on the interval $[0, 2\pi]$. Show that

$$\lim_{n\to\infty}\int_0^{2\pi} f(x)\cos(nx)dx = 0.$$

Hint. What if you were allowed to use integration by parts?