UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2022

HOMEWORK 5

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 8, 2022**

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1. Determine whether the following subsets S are open, closed, or neither. For any $n \in \mathbb{Z}^+$, $\mathbb{R}^n$ is endowed with the metric

$$d(x, y) := \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \quad \forall x, y \in \mathbb{R}^n,$$

where we write $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$.

(a) $S := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > |x_1|\}$

(b) $S := \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - a_1)^2 + (x_2 - a_2)^2 \leq r^2\}$ for some (fixed) $(a_1, a_2) \in \mathbb{R}^2$ and $r > 0$

(c) $S := [a_1, b_1) \times [a_2, b_2) \times \dots [a_n, b_n)$, where $a_j, b_j \in \mathbb{R}$ and $a_j < b_j$, $j = 1, \dots, n$

(d) $S := \{(x, \sin(1/x)) : x \in \mathbb{R}^+\} \cup \{(0, 0)\}$

(e) $S := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$

Please give **justifications.**

2. Let X be a metric space and let $Y \subseteq X$, $Y \neq \varnothing$. State and prove a characterisation for a set $F \subseteq Y$ to be closed relative to Y (in terms of Y and open or closed sets in X).

3. Let X be a metric space and let $K_1, \dots, K_n$ be compact sets of X. Prove from the definition that $K_1 \cup \cdots \cup K_n$ is compact.

4. For each set S below, determine using **only** the definition whether or not S is a compact subset of X (i.e., do **not** appeal to any theorems on compactness):

(a) $(X, d) =$ any set containing at least two points, equipped with the 0–1 metric; $S \subset X$ (your answer will depend on the nature of S; please give a **complete** discussion).

(b) The interval $[0, 1)$ in $\mathbb{R}$.

The following anticipates material to be introduced in the lecture on **February 9.**

5. Let X be a metric space and let $K \subseteq X$. Prove that if K is compact then it is bounded.

6. Let X be a metric space and let A and B be two disjoint compact subsets of X. Show that there exist open sets U and V with $A \subset U$ and $B \subset V$ such that $U \cap V = \varnothing$.