UM 204 (WINTER 2024) - WEEK 11

Recall that

Examples. (1) Continuous functions need not pull back compact sets to compact sets. For example, consider $f(x) : \mathbb{R} \to [-1,1]$ given by $f(x) = \sin(x)$.
(2) Discontinuous maps may map compact sets to compact sets. For example, consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \operatorname{sgn}(x)$.

Definition 0.1. A function $f : X \to Y$ between metric spaces is said to be bounded if $f(X)$ is a bounded set in Y.

Proof of Theorem **??**. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of $f(X)$. Since f is continuous, $\{f^{-1}(U_\lambda)\}_{\lambda\in\Lambda}$ is an open cover of X. By compactness of X, there exist $\lambda_1, ..., \lambda_n \in \Lambda$ such that

$$X = f^{-1}(U_{\lambda_1}) \cup \cdots \cup f^{-1}(U_{\lambda_n}).$$

Now, using that $f(A_1 \cup \cdots \cup A_n) = f(A_1) \cup \cdots \cup f(A_n)$ and that $f(f^{-1}(A)) \subseteq A$, we have that

$$f(X) = f\left(f^{-1}(U_{\lambda_1}) \cup \cdots \cup f^{-1}(U_{\lambda_n})\right) f(f^{-1}(U_{\lambda_1})) \cup \cdots \cup f(f^{-1}(U_{\lambda_n})) \subseteq U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}.$$

Thus, we have found a finite subcover of $f(X)$ from the given arbitrary cover. □

Corollary 0.2. *Any $\mathbb{R}$-valued continuous function on a compact metric space attains its maximum and minimum.*

Proof. Let $f : X \to \mathbb{R}$ be a continuous function on a compact metric space X. By the above theorem $f(X)$ is compact in $\mathbb{R}$, and thus $\sup f(X)$ and $\inf f(X)$ exist, and are contained in $f(X)$. □

Corollary 0.3. *If $f : X \to Y$ is a bijective continuous map and X is compact, then f^{-1} is also continuous. (Such an f is called a homeomorphism.)*

Example. Consider $f : (0,1) \cup (1,2] \to (-1,1)$ given by

$$f(x) = \begin{cases} x, & x \in (0,1), \\ x-2, & x \in (1,2]. \end{cases}$$

Then, f is continuous and bijective, but $f^{-1}([-\varepsilon,\varepsilon]) = (0,\varepsilon] \cup [2-\varepsilon, 2]$ is not compact, and thus f^{-1} is not continuous.

Proof. We need to show that f^{-1} pulls back open sets in X to open sets in Y, i.e., $f(V)$ is open in Y whenever V is open in X. Since X is compact,

$$C = X \setminus V$$

is compact. Thus, $f(C)$ is compact. But for a bijective map $f : X \to Y$,

$$f(X \setminus A) = Y \setminus f(A).$$

Thus, $f(V) = f(X \setminus C) = Y \setminus f(C)$ is open. $\square$

Some questions. (1) Continuous functions map convergent sequences to convergent sequences. What kind of functions map Cauchy sequences to Cauchy sequences? Consider the example of $f(x) = 1/x$ on $(0, 1)$.

(2) Suppose $S \subset X$ and you have a continuous function on S. When can you extend f to a continuous function on X? Consider again $f(x) = x$ on $(0, 1)$, but also consider $f(x) = \sin(1/x)$ on $(0, 1)$.

(3) In many applications, the underlying metric space is itself a space of continuous functions. Say, for example,

$$\mathscr{C}([0, 1]) = \{f : [0, 1] \to \mathbb{R} : f \text{ is continuous}\}$$

with

$$d(f, g) = \sup_X |f - g|.$$

When is a set of functions in this metric space compact? You will encounter a theorem about this in more advanced courses.

Definition 0.4. Let $f : X \to Y$ be a function between metric spaces. We say that f is uniformly continuous on X if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $d_X(x, y) < \delta$ for $x, y \in X$, then $d_Y(f(x), f(y)) < \varepsilon$, or equivalently

$$f(B_X(x, \delta)) \subset B_Y(f(x), \varepsilon) \quad \forall x \in X.$$

Theorem 0.5 (Sequential characterization)**.** *Assignment 07!*

Remark. The key feature in the above definition is the independence of δ from $x \in X$. A function that is continuous, but not uniformly continuous is $f(x) = 1/x$ on $(0, 1)$. Write a rigorous proof.

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Theorem 0.6. *A continuous function on a compact set is uniformly continuous.*

Proof. Let $\varepsilon > 0$. Then, for each $x \in X$, there is a $\delta_x > 0$ such that

$$f(B_X(x;\delta_x)) \subset B(f(x);\varepsilon/2). \tag{0.1}$$

Since X is compact, the open cover $\{B_X(x,\delta_x/2)\}_{x\in X}$ admits a finite subcover, say

$$B_X(x_1;\delta_{x_1}/2),\dots,B_X(x_n;\delta_{x_n}/2).$$

Let $\delta = 1/2\min\{\delta_{x_1},\dots,\delta_{x_n}\}$. Let $p,q \in X$ such that $d(p,q) < \delta/2$. Since $p \in X$, there is some $j \in \{1,\dots,n\}$ such that $p \in B_X(x_j;\delta_{x_j}/2)$. Thus,

$$d(q,x_j) \le d(p,q) + d(q,x_j) < \delta.$$

By (0.1),

$$d(f(p),f(q)) \le d(f(p),f(x_j)) + d(f(x_j),f(q)) < 2\frac{\varepsilon}{2} = \varepsilon.$$

□

As a consequence, we obtain the extreme value theorem.

Corollary 0.7. *Let $f : X \to \mathbb{R}$ be a continuous function on a compact metric space X. Then, f attains its maximum and minimum on X.*

Proof. Since $Y = f(X) \subseteq \mathbb{R}$ is a compact space, $\sup f(X)$ and $\inf f(X)$ exist (since Y is nonempty and bounded) and are contained in Y (since Y is closed). □

Corollary 0.8. *Let $f : X \to Y$ be a bijective, continuous function. Assume X is compact. Then, f^{-1} is continuous.*

Proof. Let $g = f^{-1}$. We wish to show that if $V \subset X$ is an open set, then $g^{-1}(V)$ is open in Y. But, $g^{-1} = f$. Thus, we need to show that $f(V)$ is open in Y whenever V is open in X.

Since X is compact, $X \setminus V$ is compact. Thus, $f(X \setminus V)$ is compact, and therefore closed in Y. But, since f is bijective,

$$f(X \setminus V) = Y \setminus f(V).$$

Thus, $f(V)$ is open in Y.

□

Exercise. Give an example of a continuous and bijective function, whose inverse is not continuous.

Theorem 0.9. *Let $f : X \to Y$ be a continuous function between metric spaces. Then, $f(X)$ is connected whenever X is connected.*

Proof. Suppose $f(X)$ is not connected. Then, there exist nonempty sets $A, B \subset f(X)$ such that

$$\begin{aligned} f(X) &= A \cup B \\ A \cap \overline{B} &= \overline{A} \cap B = \emptyset. \end{aligned}$$

Let $A_* = f^{-1}(A)$ and $B_* = f^{-1}(B)$. Then, since $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, we have that

$$X = f^{-1}(f(X)) \subseteq A_* \cup B_*.$$

Since $A_* \subseteq f^{-1}(\overline{A})$ and the RHS is closed, $\overline{A_*} \subseteq f^{-1}(\overline{A})$. Thus, $f(\overline{A_*}) \subseteq f(f^{-1}(\overline{A})) \subseteq \overline{A}$. Also, $f(B_*) \subseteq B$. Thus,

$$f(\overline{A_*} \cap B_*) \subseteq f(\overline{A_*}) \cap f(B_*) \subseteq \overline{A} \cap B = \emptyset \Rightarrow \overline{A_*} \cap B_* = \emptyset.$$

□

0.1. **Discontinuities of functions on $\mathbb{R}$.** Consider the following examples:

(1) The Heaviside step function:

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

(2) The sign function:

$$f(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

(3)

$$f(x) = \begin{cases} \frac{3x - x^2}{x^2 - x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

(4)

$$f(x) = \begin{cases} 1/x, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

(5)

$$f(x) = \begin{cases} \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

(6) The Dirichlet function:

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

The first five functions are continuous everywhere except at $x = 0$. The sixth function is continuous nowhere.

Definition 0.10. Given a function $f : (a, b)$ and $c \in (a, b)$ such that f is discontinuous at C, we say that

(1) f has a discontinuity of the first kind at c if

$$\lim_{x \to c^-} f(x) \quad \text{and} \quad \lim_{x \to c^+} f(x) \quad \text{exist;}$$

(2) f has a discontinuity of the second kind at c if

$$\lim_{x \to c^-} f(x) \quad \text{or} \quad \lim_{x \to c^+} f(x) \quad \text{does not exist.}$$

Functions in examples (1)-(3) have discontinuities of the first kind, while 0 is a discontinuity of the second kind in examples (4)-(5). All the discontinuities are of the second kind in the final example.

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In general, the set of discontinuities can be quite bad, but in the case of monotone functions, we can say something more.

Theorem 0.11. *Let $f : (a, b) \to \mathbb{R}$ be a monotonically increasing function, i.e., $f(x) \leq f(y)$ whenever $(x, y) \subseteq (a, b)$. Then, f only has discontinuities of the first type. In particular,*

$$\lim_{x \to c^-} f(x) \leq f(c) \leq \lim_{x \to c^+} f(x),$$

for all $c \in (a, b)$.

Proof. Let $c \in (a, b)$. Consider the set

$$A = \{f(t) : t \in (a, c)\}.$$

Clearly A is nonempty. Moreover, $f(t) \leq f(c)$ for all $t < c$. Thus, A is bounded above. We claim that

$$\lim_{x \to c^-} f(x) = \sup A \leq f(c).$$

Let $\varepsilon > 0$ and $s = \sup A$. Then, there is a $y \in (a, c)$ such that $s - \varepsilon < f(y) \leq s$, since $s - \varepsilon$ is not an upper bound of A. Let $\delta = c - y$. Then, for all $z \in (c - \delta, c) = (y, c)$, we have that

$$f(c) \geq f(z) \geq f(y) > f(c) - \varepsilon.$$

Since ε was arbitrary,

$$\lim_{x \to c^-} f(x) = s.$$

A similar argument gives the existence of the right-hand limit (and the inequality) as well.

$\square$

Corollary 0.12. *The set of discontinuities of a monotone function on any interval is at most countable.*

Proof. Let f be a monotonically increasing function. Let D denote its set of discontinuities. We construct an injective map from D into $\mathbb{Q}$.

Given any $c \in D$, we know that

$$\lim_{x \to c^-} f(x) < \lim_{x \to c^+} f(x).$$

By the density of $\mathbb{Q}$ in $\mathbb{R}$, there is some $r_c \in \mathbb{Q}$ such that

$$\lim_{x \to c^-} f(x) < r_c < \lim_{x \to c^+} f(x).$$

Now, let $c \neq d \in D$. WLOG, assume $c < d$. Then,

$$r_c < \lim_{x \to c^+} f(x) < \lim_{x \to d^-} f(x) < r_d.$$

Thus, $c \mapsto r_c$ is an injective map from D into $\mathbb{Q}$. $\square$

Example. The set of discontinuities of a monotone function need not be discrete! Let $D \subset (a, b)$ be a dense countable set. Enumerate $D = \{x_1, x_2, \ldots\}$. Define

$$f(x) = \sum_{n : x_n < x} \frac{1}{n^2}.$$

The function is well defined because of the absolute convergence of the $\sum_{n \in \mathbb{N}_+} 1/n^2$.

(1) f is increasing: if $x < y$, then there is some $x_j \in D \cap (x, y)$. Thus, $f(y) - f(x) > \frac{1}{j^2} > 0$.
(2) f is discontinuous at every point of D: let $x_j \in D$. For any $x > x_j$, $f(x) - f(x_j) \geq \frac{1}{j^2}$. Thus, $\lim_{x \to x_j^+} f(x) > f(x_j)$.
(3) f is continuous everywhere else: Exercise.

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