UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022
HOMEWORK 11

Instructor: GAUTAM BHARALI **Assigned: MARCH 29, 2022**

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1. Give a proof of Brouwer's Fixed Point Theorem in the $n = 1$ case.

2. Problem 20 from "Baby" Rudin, Chapter 4.

3. Let $a < b$ be real numbers. Fix $n \geq 2$, $n \in \mathbb{Z}_+$. Prove from the **definition** (i.e., without invoking the result on the uniform continuity of a continuous function on a compact domain), that the function $f : [a, b] \to \mathbb{R}$ given by $f(x) = x^n$, $a \leq x \leq b$, is uniformly continuous.

4. Consider the result:

Theorem. *Let X and Y be metric spaces, and let $S \subsetneq X$ be dense subset. Let $f : S \to Y$ be a uniformly continuous function. Suppose Y is complete. Then, there exists a continuous function $\widetilde{f} : X \to Y$ such that $\widetilde{f}\big|_S = f$.*

that was *partially* proved in class. Consider the function $\widetilde{f}$ constructed in that proof — which must be shown to have the properties stated above. Fix $x \in (X \setminus S)$, and let $\{x_n\}$ be a sequence in $X \setminus \{x\}$ that converges to x. Complete the following outline to prove that $\widetilde{f}$ is continuous:

(a) Explain why it suffices to only consider sequences $\{x_n\}$ such that

$$\mathsf{range}(\{x_n\}) \bigcap (X \setminus S) \text{ is an infinite set.} \tag{1}$$

(b) Consider a sequence $\{x_n\}$ with the property (1). Construct an auxiliary sequence $\{y_n\} \subset S$ such that for each n for which $x_n \notin S$, y_n is "sufficiently close" to x_n — in an appropriate sense — and converges to x in such a way that you can use its behaviour, plus uniform continuity, to infer that $\{\widetilde{f}(x_n)\}$ is convergent.

(c) Deduce that $\{\widetilde{f}(x_n)\}$ converges to $\widetilde{f}(x)$.

(d) Now, complete the argument showing that $\widetilde{f}$ is continuous.

The following problems are based on the **review assignment** given in Homework 8.

5–6. Problems 4 and 5 from "Baby" Rudin, Chapter 5.

7. Let $r \in \mathbb{R}$ and let p be a positive real number. Consider the function $f : [-1, 1] \longrightarrow \mathbb{R}$ given by:

$$f(x) := \begin{cases} x^r \sin(1/x^p), & \text{if } 0 < x \leq 1, \\ 0, & \text{if } -1 \leq x \leq 0. \end{cases}$$

Find (i) a necessary & sufficient condition on (r, p) for f to be differentiable at 0; (ii) a necessary & sufficient condition on (r, p) for f to be differentiable at 0 **and** such that f' is continuous at 0.

Note. You may assume **without** proof that the function $\phi_r : x \longmapsto x^r$, $x \in (0, \infty)$, is differentiable on $(0, \infty)$ for any $r \in \mathbb{R}$, and $\phi_r'(x) = rx^{r-1}$ $\forall x \in (0, \infty)$.

8. Given real numbers $a < b$ and a function $f : [a, b] \to \mathbb{R}$, the hypothesis of Lagrange's Mean Value Theorem (i.e., Theorem 5.10 in Rudin's book) imposes **two** conditions on f. Show, by suitable examples, that **each** condition is essential to the conclusion of this theorem. (I.e., give examples of $f : [a, b] \to \mathbb{R}$ that satisfy exactly one of these two conditions and for which the conclusion of Lagrange's Mean Value Theorem is false.)

9. Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be an injective differentiable function.

(*a*) Show that $f(I)$ is an interval.

(*b*) Show that f^{-1} is differentiable at each y belonging to the set

$$\mathscr{R}_f := \{y \in f(I) : f'(f^{-1}(y)) \neq 0\}$$

and that

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))} \quad \text{for each } y \in \mathscr{R}_f.$$

Hint. First deduce that f^{-1} is continuous. If required, you may assume **without** proof that if $f : I \to \mathbb{R}$ is an injective continuous function, then it is either strictly increasing or strictly decreasing.