UM 204 (WINTER 2024) - WEEK 4

Examples. Closed intervals are closed sets. $E = [0,1)$ is neither open nor closed. $\overline{E} = [0,1]$, $E^\circ = (0,1)$ and $\partial E = \{0,1\}$. Finite sets are always closed. If d is the discrete metric on a set X, then every set is both open and closed.

Proposition 0.1. *A set E is closed if and only if its complement, $E^c = X \setminus E$, is open.*

Proof. Suppose E is closed. Let $z \in X \setminus E$. Since z is not an accumulation point of E, there is an $\varepsilon > 0$ such that $B(z;\varepsilon) \cap E \supseteq \{z\}$. However, $z \notin E$, so $B(z;\varepsilon) \subseteq X \setminus E$.

Suppose E is open. Let $z \in E$. There is an $\varepsilon > 0$ such that $B(z;\varepsilon) \subseteq E$, i.e., $B(z;\varepsilon)$ contains no point of $X \setminus E$. Thus, z is not an accumulation point of $X \setminus E$.

Suppose E contains an accumulation point z of $X \setminus E$. T □

Corollary 0.2. *The intersection of an arbitrary collection of closed sets is closed. The union of a finite collection of closed sets is closed.*

Proposition 0.3. *(i) The closure of a set is closed.*
(ii) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.
(iii) $\overline{E}$ is the smallest closed set containing E, $\overline{E} = \cap_{E \subset F : F \text{ is closed}} F$.

Proof. (i) Let $z \in X$ be an accumulation point of $\overline{E}$. Then, for every $\varepsilon > 0$, there is a $w_\varepsilon \in \overline{E} \cap B(z;\varepsilon)$ such that $w_\varepsilon \neq z$. Either $w_\varepsilon \in E$, or $w_\varepsilon \in \overline{E} \setminus E$. In the former case, let $u_\varepsilon = w_\varepsilon$. In the latter case, let $\delta = \varepsilon - d(z, w_\varepsilon)$. Then, there is some $u_\varepsilon \in E \cap B(w_\varepsilon;\delta)$. But $B(w_\varepsilon;\delta) \subseteq B(z;\varepsilon)$. Thus, for every $\varepsilon > 0$, there is a $u_\varepsilon \in E \cap B(z;\varepsilon)$. Thus, $z \in \overline{E}$.
(ii) We will show the contrapositive. Let $z \in X \setminus \overline{B}$. Then, there is an $\varepsilon > 0$ such that $B(z;\varepsilon) \subseteq B^c \subseteq A^c$. Thus, $z \in \left(\overline{A}\right)^c$.
(iii) Let's call the intersection E'. By (i) and (ii), we already have that $\overline{E} \subseteq E'$. Conversely, $\overline{E}$ is a closed set containing E, so $E' \subseteq \overline{E}$. □

Consider the following two questions:

(1) Is $(0,1)$ an open set?
(2) Is $\mathbb{Q}$ a closed set?

These questions are ambiguous if you do not state the underlying metric space. For instance, $(0,1)$ is open in the standard metric on $\mathbb{R}$, but not open in the standard metric on $\mathbb{R}^2$.

Definition 0.4. Given a metric space (X,d) and a subsets $E \subset Y$, we say that E is open (clsoed) relative to Y if E is an open (closed) subset of $(Y, d|_Y)$.

Proposition 0.5. *(Relative topology) Let (X,d) be a metric space and $E \subset Y \subset X$. Then E is open (closed) relative to Y if and only if there is an open (closed) set $F \subset X$ such that $F \cap Y = E$.*

Proof. Suppose E is open in (Y,d). For every $z \in E$, there is an $\varepsilon_Z > 0$ such that

$$B_{d_Y}(z;\varepsilon_Z) = \{y \in Y : d(z,y) < \varepsilon_z\} = B_d(z;\varepsilon_z) \cap Y$$

is contained in E. Let

$$F = \cup_{z \in E} B_d(z;\varepsilon_Z).$$

Then, F is open in X, and $E = F \cap Y$. If E is closed, then $Y \setminus E$ is open, and there is some open $G \subset X$ such that $G \cap Y = Y \setminus E$. Now, $F = X \setminus G$ is closed and $F \cap Y = (X \setminus G) \cap Y = Y \setminus (Y \setminus E) = E$.

Conversely, suppose $E = F \cap Y$ for some closed set $F \subset X$. Let $z \in Y$ be an accumulation point of E. Then, clearly z is an acc. point of F, and thus, $z \in F \cap Y = E$. The case when F is open now follows easily (why?). $\square$

END OF LECTURE 9

0.1. **Compactness.** A prototypical example of a closed set is the closed interval $[0,1]$. However, the interval $[0,\infty)$ is also closed, and we know that continuous functions don't behave as well on such intervals as they do on intervals of the form $[a,b]$.

Definition 0.6. A subset $E \subset (X,d)$ is said to be bounded if there is a $p \in X$ and $M > 0$ such that $E \subseteq B(p;M)$.

We are hoping to extract the essential features of the intervals $[a,b]$, but in general metric spaces, even closed and boundedness doesn't quite cut it.

Definition 0.7. Let $E \subset (X,d)$. An open cover of E in X is a collection $\mathscr{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ of open sets in X such that $E \subset \cup_{\alpha \in \Lambda} U_\alpha$. The set E is said to be compact if every open cover $\{U_\alpha\}_{\alpha_\Lambda}$ admits a finite subcover, i.e., there exist $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that $E \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$.

Examples. 1. Any finite set $E = \{p_1, \ldots, p_k\}$ is compact (in any metric space). Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of E. For each p_j, there is some $\alpha_j \in \Lambda$ such that $p_j \in U_{\alpha_j}$. Thus, $\{U_{\alpha_j}\}_{j=1}^k$ is a finite subcover.
2. $(0,1)$ is not compact space (in the standard metric). Let

$$U_n = \left(\frac{1}{n+1}, \frac{1}{n}\right), \quad n \in \mathbb{N}_+.$$

Then, $(0,1) \subset \cup_{n=1}^\infty U_n$ (why?) but $\{U_n\}_{n=1}^\infty$ admits no finite subcover (why?).

3. $[0,1]$ is compact in the standard metric. In fact, any rectangle $[a_1,b_1]\times\cdots\times[a_n,b_n]$ is compact in $\mathbb{R}^n$ but it takes a bit of effort to prove. The effort is worth it because one then obtains, as a consequence, a part of the following theorem.

Theorem 0.8. *Let $E\subset\mathbb{R}^n$. T.F.A.E.*

(1) *E is compact.*
(2) *E is closed and bounded.*
(3) *Every infinite subset of E admits a limit point in E.*

Remark The equivalence between (1) and (2) is referred to as the Heine Borel theorem.
Proof. The proof relies on many results. First, we show that (1)$\Rightarrow$ (2) in any metric space.

Theorem 0.9. *Let (X,d) be a metric space and $E\subset X$ be a compact set. Then, E is closed and bounded.*

Proof. We will show that E^c is open. Let $p\in E^c$. For any $x\in E$, let $U_x=B_d\left(x;\frac{1}{2}d(x,p)\right)$. Note that $U_x^c\supset B_d\left(p;\frac{1}{2}d(x,p)\right)$. Now, $E\subset\cup_{x\in E}U_x$. By the compactness of E, there exist $x_1,\dots,x_n\in E$ such that $E\subset U_{x_1}\cup\cdots\cup U_{x_n}$. Thus, $E^c\supseteq U_{x_1}^c\cup\cdots\cup U_{x_n}^c\supseteq B_d\left(p;\delta\right)$, where $\delta=\min\{d(x_j,p):j=1,\dots,n\}$.

For boundedness of E, fix $p\in X$, and observe that $E\subset\cup_{r>0}B_d(p;r)$. The compactness of E yields an $R>0$ such that $E\subset B_d(p;R)$. $\square$

Next, we establish that (2)$\Rightarrow$ (1) in $\mathbb{R}^n$. Our strategy is based on the observation that any bounded set E is contained in a rectangle of the form $[0,R]^n$ for some $R>0$ (why?). Then, we prove that

Theorem 0.10. *All rectangles of the form $[0,R]^n$, $R>0$, are compact.*

Theorem 0.11. *A closed subset of a compact set is compact.*

Proof of Theorem 0.10. Fix $R>0$ and let $I_0=[0,R]^n$. Note that

$$\operatorname{diam}(I_0)=\max\{\|x-y\|:x,y\in I_0\}\le\sqrt{n}R.$$

Suppose $\mathscr{U}=\{U_\alpha\}$ is an open cover of I_0 that does not contain a finite subcover of I_0. Now consider a sequence of rectangles $I_0\supset I_1\supset I_2\supset\cdots$ constructed (inductively) as follows. Conider the 2^n rectangles of the form $J_1\times\cdots\times J_n$, where each J_ℓ is either $[0,R/2]$ or $[R/2,R]$. These rectangles partition I_0, and at least one of them is not covered by a finite subcover of $\mathscr{U}$. Call such a rectangle I_1. Subdivding I_1 further and repeating this process, we obtain rectangles $I_0\supset I_1\supset I_2\supset\cdots$ such that

(1) no I_j is covered be a finite subcover of $\mathscr{U}$,
(2) $\operatorname{diam}(I_j)\le\frac{R\sqrt{n}}{2^j}=2^{-j}C$, $j\in\mathbb{N}$, which implies that $I_j\subset B(z;2^{-j}C)$ for all $z\in I_j$.

END OF LECTURE 10

We now use the following fact:

Lemma 0.12. *If $I_0 \supset I_1 \supset \cdots$ is a sequence of closed rectangles, then $\cap_{j=1}^{\infty} I_j$ is non-empty.*

Assuming this fact, we let $a \in \cap_{j=0}^{\infty} I_j$. Then, there is some $\alpha \in \Lambda$ such that $a \in U_\alpha$. So, there is some $\varepsilon > 0$ such that $B(a;\varepsilon) \subseteq U_\alpha$. Now, by the Archimedean property of $\mathbb{R}$, there is some $j \in \mathbb{N}$ such that $2^{-j}C < \varepsilon$. Thus, $I_j \subseteq B(a;2^{-j}C) \subseteq B(a;\varepsilon) \subset U_\alpha$. But I_j was not covered by a finite subcover of $\mathscr{U}$! Thus, $mathscrU$ admits a finite subcover of I_0. ☐

Lemma 0.12 is the higher-dimensional version of the nested interval property discussed in Lecture 16 of UM 101. The proof generalizes easily from the case of $\mathbb{R}$ and is left as an exercise.

Proof of Theorem 0.11. Let $E \subseteq F \subseteq X$, where F is compact in X and E is closed in X. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of E. Let $V = E^c$. Then, $\mathscr{U} = \{V\} \cup \{U_\alpha\}$ is an open cover of F. By the compactness of F, $\mathscr{U}$ contains a finite subcover $\mathscr{V}$ of F. Either the cover contains V, in which case we throw it out and the rest cover E, or the subcover only consists of finitely many elements from $\{U_\alpha\}$, in which case, once again, we get a finite subcover of E. ☐

We will show that (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

For (1) $\Rightarrow$ (3), let $F \subset E$ be an infinite set. Suppose no point of E is a limit point of F. Then, for every $z \in E$, there is a ball $B(z;\varepsilon_z)$ that contains no point from F, other than possibly z. This gives an open cover of E, and each element of F is in exactly one element of this cover. This cover cannot contain a finite subcover.

For (3) $\Rightarrow$ (2), first assume that E is not bounded. Then, we can inductively construct a sequence $\{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ such that $\|x_j\| > \|x_{j-1}\| + 1$. Such a sequence cannot have a limit point (why?).

Next, assume E is not closed. Let $z \in \overline{E} \setminus E$. There is a subsequence of distinct points $\{x_j\}_{j\in\mathbb{N}} \subset E$ such that $\|z - x_j\| < 1/j$. Thus, $S = \{x_j : j \in \mathbb{N}\}$ is an infinite subset of E. For any $y \in \mathbb{R}^n$ such that $y \neq z$, we have that

$$\|x_j - y\| \geq \|y - x_0\| - \|x_j - x_0\| \geq \|y - x_0\| - \frac{1}{j} > \frac{1}{2}\|y - x_0\|$$

for sufficiently large j. Thus, y is not a limit point of S and the only limit point of S is not in E. ☐

Corollary 0.13. *Every bounded infinite subset of $\mathbb{R}^n$ has a limit point in $\mathbb{R}^n$.*

Theorem 0.14. *If $\mathscr{K} = \{K_\alpha\}_{\alpha\in\Lambda}$ is a collection of compact subsets of (X,d) such that every finite subcollection of $\mathscr{K}$ has nonempty intersection, then $\cap_{\alpha\in\Lambda} K_\alpha$ is nonempty.*

Proof. Suppose the claim does not hold. Then, there is some $K = K_\alpha \in \mathscr{K}$ such that no element in K is in every other $K_\beta \in \mathscr{K}$. I.e.,

$$K \subset \cup_{\alpha\in\Lambda} K_\alpha^c.$$

Thus, $\{K_\alpha^c\}$ is an open cover of K. By compactness, there exist $\alpha_1, ..., \alpha_k \in \Lambda$ such that

$$K \subseteq K_{\alpha_1}^c \cup \cdots \cup K_{\alpha_k}^c = \left(K_{\alpha_1} \cup \cdots \cup K_{\alpha_k}\right)^c.$$

But, the finite intersection $K \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_k}$ is empty, which contradicts the hypothesis on $\mathscr{K}$. $\square$

Corollary 0.15. *Suppose $K_0 \supseteq K_1 \supset K_2 \supseteq \cdots$ is a collection of nonempty compact sets in a metric space. Then, $\cap_{j=0}^{\infty} K_j =\neq \emptyset$.*

Theorem 0.16. *Let $E \subset Y \subset (X, d)$. Then, E is compact relative to Y if and only if E is compact in X.*

Proof. We will only sketch the idea: every open cover of E in Y extends to an open cover of E in X, and every open cover of E in X restricts to an open cover of E in Y (by the characterization of relatively open sets in Y). $\square$

END OF LECTURE 11