# UM 204 (WINTER 2024) - WEEK 3

Recall that $\mathbb{R} := \{$equivalence classes of equivalent $\mathbb{Q}$-Cauchy sequences$\}$

**Addition & multiplication.** Exercise!

(i) Let $a = \{a_n\}$ and $b = \{b_n\}$ be Cauchy sequences. Then, $a + b = \{a_n + b_n\}$ and $a \cdot b = \{a_n \cdot b_n\}$ are Cauchy sequences.

(ii) Say $aRa'$ and $bRb'$, then $(a+b)R(a'+b')$ and $(a\cdot b)R(a'\cdot b')$.

(iii) Associativity and commutativity of $+$ and $\cdot$.

(iv) $0_{\mathbb{R}} = [\{a_n\}]$, where $a_n \equiv 1$ and $1_{\mathbb{R}} = [\{b_n\}]$, where $b_n \equiv 1$.

(v) $-[a] = [-a]$ for all $[a] \in \mathbb{R}$. Here, we must first show that if $a$ is a Cauchy sequence, then so is $-a$.

(vi) If $[a] \neq 0_{\mathbb{R}}$, then there is some Cauchy sequence $\widetilde{a} = \{\widetilde{a}_n\}_{\mathbb{N}}$ such that and $[\widetilde{a}] = [a]$ and $\widetilde{a}$ nonvanishing, i.e., $\widetilde{a}_n \neq 0$ for all $n \in \mathbb{N}$. We claim that
- $1/\widetilde{a} := \{1/\widetilde{a}_n\}_{\mathbb{N}}$ is a Cauchy sequence,
- $[a \cdot 1/\widetilde{a}] = 1_{\mathbb{R}}$,
- If $\widetilde{b}$ is any other non-vanishing Cauchy sequence such that $[\widetilde{b}] = [a]$, then $[1/\widetilde{b}] = [1/\widetilde{a}]$.

Thus, we may define $1/[a]$ as $[1/\widetilde{a}]$.

(vii) $\cdot$ distributes over $+$.

**Order.** Exercise!

(i) For each $[a] \in \mathbb{R}$, exactly one of the following holds: $[a] = 0_{\mathbb{R}}$, $[a] > 0$ or $-[a] > 0$.

(ii) Transitivity of order holds.

(iii) Addition preserves order.

(iv) Multiplication preserves positivity.

**Theorem 0.1.** *The ordered set* $(\mathbb{R}, \leq)$ *constructed above satisfies the Archimedean property.*

*Proof.* Let $[a], [b] > 0$. For any $m \in \mathbb{N}$, let $[m]$ denote the equivalence class of the constant sequence $\{m, m, \ldots\}$. Since $\{b_n\} > 0$ is $\mathbb{Q}$-Cauchy, it is $\mathbb{Q}$-bounded, say by a rational number $M > 0$. Thus,

$$b_n \leq M, \quad \forall n \in \mathbb{N}. \tag{0.1}$$

Since $[a] > 0$, there is a rational $c > 0$ and $N \in \mathbb{N}$ such that $0 < c < a_n$ for all $n \geq N$. By the Archimedean property of $\mathbb{Q}$, there is some $m \in \mathbb{N}$ such that $mc > M + c$. Thus, we have that for $n \geq N$,

$$0 < c < mc - M < ma_n - b_n.$$

Thus, by the definition of '$> 0$' on $\mathbb{R}$, we have that $[m][a] - [b] > 0$. $\square$

**Theorem 0.2.** *The ordered field $\mathbb{R}$ satisfies the least upper bound property.*

*Proof.* Let $A \subset \mathbb{R}$ be a nonempty set, say containing $x_0$, that is bounded above, say by $M$. We wish to construct a $\mathbb{Q}$-Cauchy sequences of rational numbers whose equivalence class represents the supremum of $A$. We do this by creating a "window" of width $\frac{1}{n}$ near the supposed supremum, and finding a rational number there (for each $n \in \mathbb{N}$).

Fix $n \in \mathbb{N}$. Let

$$U_n = \left\{ m \in \mathbb{Z} : \frac{m}{n} = \left[ \left\{ \frac{m}{n}, \frac{m}{n}, \ldots \right\} \right] \text{ is an upper bound of } A \right\}.$$

By the Archimedean property of $\mathbb{R}$, $U_n$ is non-empty and bounded below (in $\mathbb{Z}$) (why?). Thus, it admits a least element (why?), say $m_n$. To summarize

$$\frac{m_n}{n} \text{ is an upper bound of } A \text{ but } \frac{m_n - 1}{n} \text{ is not.}$$

We show that $a = \{a_n = m_n / n\}_{n \in \mathbb{N}}$ is a $\mathbb{Q}$-Cauchy sequence such that $[a] = \sup A$. Let $\varepsilon > 0$ be a rational number. By the Archimedean property of $\mathbb{Q}$, there is an $N \in \mathbb{N}$ such that $1 < N\varepsilon$. Let $n, n' \geq N$. Then, $\frac{m_n}{n} > \frac{m_{n'}-1}{n'}$ because the former is an u.b. of $A$, but the latter isn't. Similarly, $\frac{m_{n'}}{n'} > \frac{m_n - 1}{n}$. Combining these inequalities yields that

$$-\varepsilon < -\frac{1}{N} \leq -\frac{1}{n'} < \frac{m_n}{n} - \frac{m_{n'}}{n'} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus, $\{a_n\}$ is a $\mathbb{Q}$-Cauchy sequence.

By the definition of the sequence $\{a_n\}$, $x \leq a_k = [\{a_k, a_k, \ldots,\}]$ for each $x = [\{x_n\}] \in A$ and $k \geq 1$. We claim that $x \leq [\{a_n\}]$ for all $x \in A$. Suppose not. Then, there is an $x = [\{x_n\}] \in A$ such that $[\{a_n\}] < [\{x_n\}]$, i.e., there is some rational $c > 0$ and $N_1 \in \mathbb{N}$ such that

$$0 < c < x_n - a_n, \quad \forall n \geq N_1.$$

Let $\varepsilon = c/2$. Since $\{x_n\}$ is $\mathbb{Q}$-Cauchy, there is an $N_2 \in \mathbb{N}$ such that

$$-\frac{c}{2} < x_n - x_m < \frac{c}{2}, \quad \forall n, m \geq N_2.$$

Choose $N_0 = \max\{N_1, N_2\}$. Then, for any $m \geq N_0$

$$a_{N_0} + c < x_{N_0} < x_m + \frac{c}{2} \Rightarrow \frac{c}{2} < x_m - a_{N_0}.$$

This implies that $[\{x_n\}] > a_{N_0}$, which is a contradiction.

For the final step of the proof, let $y = [\{y_n\}]$ be an upper bound of $A$. A similar argument as above shows that $y \geq [\{(m_n - 1)/n\}] = [\{m_n / n\}] = [a]$. □

**Theorem 0.3 (Existence of $n^{\text{th}}$ roots).** *For every real $x > 0$ and every $n \in \mathbb{N}^*$, there is a unique $y > 0$ such that $y^n = x$.*

*Proof.* See Assignment 03. $\square$

END OF LECTURE 6

0.1. **Complex numbers.** We briefly discuss complex numbers, although we will not develop differential calculus on the complex plane in this course. Formally, the a complex number is an ordered 2-tuple $(a,b)$ of real numbers. The set of complex numbers is denoted by $\mathbb{C}$ (instead of $\mathbb{R}\times\mathbb{R}$), and the following operations make $\mathbb{C}$ a *normed field.*

(1) $(a,b)+(c,d)=(a+c,b+d)$.
   $(0,0)$ is the additive identity and $(-a,-b)=-(a,b)$ for any $(a,b)$.
(2) $(a,b)\cdot(c,d)=(ac-bd,ad+bc)$.
   $(1,0)$ is the multiplicative identity and $\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)=(a,b)^{-1}$ for any $(a,b)\neq(0,0)$.
(3) $|(a,b)|=\sqrt{a^2+b^2}$.

The above operations, when restricted to complex numbers of the form $(a,0)$, Note that $(0,1)^2=(-1,0)$, i.e., $(0,1)$ is a square-root of $-1$ in this field. It is denoted by $i$, and one always writes $(a,b)$ as $a+ib$ to indicate that it is a complex number. Another important operation on $\mathbb{C}$ is that of conjugation: $a+ib\mapsto\overline{a+ib}=a-ib$.

**Theorem 0.4** (The Cauchy–Schwarz Inequality)**.** *Let $x_1,\dots,x_n,y_1,\dots,y_n\in\mathbb{C}$. Then*

$$\left|\sum_{j=1}^n a_j\overline{b_j}\right|^2\leq\sum_{j=1}^n|a_j|^2\sum_{j=1}^n|b_j|^2,$$

*with equality if and only if there is a $\lambda\in\mathbb{C}$ such that $b_j=\lambda a_j$ for all $j$.*

*Proof.* Let $\sum_{j=1}^n a_j\overline{b_j}=A$. Assume $B=\sum_{j=1}^n|b_j|^2\neq0$, else there is nothing to prove. We know that

$$\left|\sum_{j=1}^n a_j\overline{a_j}\right|^2=\sum_{j=1}^n|a_j|^2\geq0.$$

We introduce a complex parameter $\lambda=u+iv$, to obtain that

$$\begin{aligned}0\leq\sum_{j=1}^n(a_j+\lambda b_j)\overline{(a_j+\lambda b_j)} &= \sum_{j=1}^n|a_j|^2+\overline{\lambda}a_j\overline{b_j}+\lambda\overline{a_j}b_j+|\lambda|^2|b_j|^2\\ &= \sum_{j=1}^n|a_j|^2+2u\Re(A)+2v\Im(A)+(u^2+v^2)\sum_{j=1}^n|b_j|^2:=F(u,v).\end{aligned}$$

Applying some multi-variable calculus to $F(u,v)$, we have that $F$ attains a global minimum when $\dfrac{\partial F}{\partial u}(u,v) = \dfrac{\partial F}{\partial v}(u,v) = 0$, which happens at $(u_0, v_0) = \left(-\frac{\Re(A)}{\sum_{j=1}^n |b_j|^2}, -\frac{\Im(A)}{\sum_{j=1}^n |b_j|^2}\right)$. But

$$0 \le F(u_0, v_0) = \sum_{j=1}^n |a_j|^2 - \frac{|A|^2}{\sum_{j=1}^n |b_j|^2}.$$

For equality, it must be that the minimum attained is 0. I.e., $\sum_{j=1}^n |a_j + \lambda_0 b_j|^2 = 0$. This yields the equality case. □

## 1. Metric spaces

1.1. **Definition and examples.** To discuss convergence of real sequences, we were not really relying on the algebraic properties of $\mathbb{R}$, but rather the notion of close-ness (or distance) between two real numbers. This notion makes sense in a much more general context.

**Definition 1.1.** A metric space is pair $(X, d)$ consisting of a set $X$ and a “distance" $d : X \times X \to [0,\infty)$ such that for all $x, y, z \in X$,

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) (symmetry) $d(x, y) = d(y, x)$;
(3) (triangle inequality) $d(x,z) \le d(x, y) + d(y,z)$.

**Examples** 1. Let $X = \mathbb{R}$. Then, $d(x, y) = |x - y|$ is a distance function.
2. (Real Euclidean space). The vector space $\mathbb{R}^n$ can be endowed with an inner product

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n.$$

The length of a ‘vector’ $x \in \mathbb{R}^n$ is then given by

$$\|x\| = \langle x, x\rangle^{1/2}.$$

We claim that $d(x, y) = \|x - y\|$ is a distance function on $\mathbb{R}^n$. It is known as the Euclidean metric on $\mathbb{R}^n$. Properties (1) and (2) are clear, and (3) follows from:

**Theorem 1.2.** *For $a, b, c \in \mathbb{R}^n$, we have that*

$$\Big|\|a\| - \|b\|\Big| \le \|a + b\| \le \|a\| + \|b\|.$$

*Proof.* The C-S inequality says that

$$|\langle a, b\rangle| \le \|a\|\|b\|.$$

Using this, we have

$$\|a + b\|^2 = \langle a + b, a + b\rangle = \langle a, a\rangle + 2\langle a, b\rangle + \langle b, b\rangle \le \|a\|^2 + 2\|a\|\|b\| + \|b\|^2 = (\|a\| + \|b\|)^2.$$

Now, given $a', b' \in \mathbb{R}^n$, let $a, b \in \mathbb{R}^n$ such that $a' = a + b$ and $b' = -b$. Then, by the inequality just proved, we have that

$$\|a'\| - \|b'\| \leq \|a' + b'\|.$$

Exchanging the roles of $a'$ and $b'$, we also have that $\|b'\| - \|a'\| \leq \|a' + b'\|$. □

3. (Discrete metric) Let $X$ be any set. Define, for $x, y \in X$

$$d(x, y) = \begin{cases} 1, & x \neq y, \\ 0 & x = y. \end{cases}$$

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4. Let $p \geq 1$. Define, for $x, y \in \mathbb{R}^n$

$$d_p(x, y) = \left( \sum_{j=1}^{n} |x_j - y_j|^p \right)^{1/p}.$$

Then, $d_p$ is a metric. The triangle inequality is non-trivial to prove (Assignment 03).

5. The subset of any metric space is also a metric space via the restriction of the metric.

1.2. **Metric topology.** The notion of a metric allows us to define analogues of open and closed intervals in metric spaces. The idea of an open set is one where there is wiggle-room at every point (think $(0,1)$).

**Definition 1.3.** Let $(X, d)$ be a metric space.

(1) Given $p \in X$ and $\varepsilon > 0$, the open ball centered $p$ with radius $\varepsilon$ is the set

$$B_d(p; \varepsilon) = B(p; \varepsilon) = \{x \in X : d(x, p) < \varepsilon\}.$$

This set is also referred to as the $\varepsilon$-neighborhood of $p$. The closed ball centered at $p$ with radius $\varepsilon$ is the set $\{x \in X : d(x, p) \leq \varepsilon\}$.

In the standard metric on $\mathbb{R}$, $B(p; \varepsilon) = (p - \varepsilon, p + \varepsilon)$.

(2) Given a subset $E \subset X$,a $p \in X$ is an interior point of $E$ if there is some $\varepsilon > 0$ such that

$$B_d(p; \varepsilon) \subseteq E.$$

The collection of all interior points of $E$ is called the interior of $E$, denoted by $E^\circ$.

**Remark.** By definition $E^\circ \subseteq E$.

(3) A set $E \subset X$ is open if $E = E^\circ$, i.e., for every $p \in E$, there is an $\varepsilon_p > 0$ such that $B_d(p; \varepsilon_p) \subseteq E$.

**Remark.** The empty set is open.

(4) The collection of open sets of $(X, d)$ is called the $d$-metric topology on $X$.

**Proposition 1.4.** *Every open ball is an open set.*

*Proof.* Let $q \in B(p;\varepsilon)$. Let $\delta = \varepsilon - d(p,q)$. Then, for any $z \in B(q;\delta)$,

$$d(z,p) \le d(z,q) + d(q,p) < \varepsilon.$$

□

**Proposition 1.5.** *The union of an arbitrary collection of open sets is open. The intersection of a finite collection of open sets is open.*

*Proof.* Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets, and $U = \cup_\Lambda G_\alpha$. Let $z \in U$. Then, $z \in G_\alpha$ for some $\alpha \in \Lambda$. Thus, there is some $\varepsilon > 0$ such that $B_d(z;\varepsilon) \subseteq G_\alpha \subseteq U$.

Let $\{G_1, \dots, G_n\}$ be a collection of open sets, and $U = \cap_{j=1}^n G_j$. Let $z \in U$. Then, $z \in G_j$ for all $j = 1, \dots, n$. Thus, there is an $\varepsilon_j > 0$ such that $B(z;\varepsilon_j) \subseteq G_j$. Let $\varepsilon = \min_{1\le j\le n}\{\varepsilon_j\}$. Then, $B(z;\varepsilon) \subseteq U$. □

**Example.** Let $d$ be the discrete metric on a (non-empty) set $X$. Given $z \in X$, note that $B(z;\varepsilon) = \{z\}$ if $\varepsilon \le 1$, and $B(z;\varepsilon) = X$ if $\varepsilon > 1$. Thus, every set is open in this topology.

The idea of a closed set is one where you cannot approach points outside the set.

**Definition 1.6.** Let $(X,d)$ be a metric space, and $E \subset X$.

(1) A $p \in X$ is a limit point of $E$ if every neighborhood $B(p;\varepsilon)$ contains a point $q \ne p$ from $E$.
**Remark.** If $p$ is a limit point of $E$, then every neighborhood of $p$ contains infinitely many points from $E$.

(2) A $p \in E$ is said to be an isolated point of $E$ if $p$ is not an accumulation point of $E$, i.e., there is some $\varepsilon > 0$ such that $B(p;\varepsilon) \cap E = \{p\}$.

(3) $E$ is said to be closed if $E$ contains all its accumulation points.
**Remark.** The empty set has no limits points, and is therefore, closed.

(4) The closure of $E$ is the set $\overline{E} = E \cup \{z \in X : z \text{ is an accumulation point of } E\}$.

(5) The boundary of $E$ is the set $\partial E = \overline{E} \setminus E^\circ$.

END OF LECTURE 8