Quiz 5

UM 205: Introduction to Algebraic Structures (Winter 2023-24)
Indian Institute of Science

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1. Find the remainder when 2^{50} is divided by 17.

2. A band of 7 pirates stole a sack of gold coins. When they tried to divide the fortune into equal portions, 3 coins remained. In the ensuing brawl over who should get the extra coins, 3 pirates were killed. The wealth was redistributed, but this time an equal division left 1 coin. Again an argument developed in which 1 pirate was killed. But now the total fortune was evenly distributed among the survivors. Calculate the least number of coins that could have been stolen using the Chinese Remainder Theorem.

1) $2^{0} = 1 \equiv 1 \ (17)$

$2^{1} = 2 \equiv 2 \ (17)$

$2^{2} = 4 \equiv 4 \ (17)$

$2^{3} = 8 \equiv 8 \ (17)$

$2^{4} = 16 \equiv -1 \ (17)$

$\therefore 2^{8} \equiv 1 \ (17) \quad (2^4 \cdot 2^4)$

$\therefore 2^{48} \equiv 1 \ (17) \quad ((2^8)^6)$

$\therefore 2^{50} = 2^{48} \cdot 2^{2} \equiv 4 \ (17)$

Multiply remainders when multiplying numbers
ie if $x \equiv a (m)$, $y \equiv b (m)$
then $xy \equiv ab \ (m)$

$\therefore \boxed{2^{50} \equiv 4 \ (17)}$

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2) Let the no. of coins be x

From question, $x \equiv 3 \ (7)$ $\quad b_1 = 3 \quad m_1 = 7$

$x \equiv 1 \ (4)$ $\quad b_2 = 1 \quad m_2 = 4$

$x \equiv 0 \ (3)$ $\quad b_3 = 0 \quad m_3 = 3$

$m = 7 \cdot 4 \cdot 3 = 84$

$n_1 = \frac{84}{7} = 12$ $\quad n_2 = \frac{84}{4} = 21$ $\quad n_3 = \frac{84}{3} = 28$

$7r_1 + 12s_1 = 1$ $\quad 4r_2 + 21s_2 = 1$ $\quad 3r_3 + 28s_3 = 1$

$r_1 = -5, \ s_1 = 3$ $\quad r_2 = -5, \ s_2 = 1$ $\quad r_3 = -9, \ s_3 = 1$

$\therefore e_1 = 3 \cdot 12 = 36$ $\quad e_2 = 1 \cdot 21 = 21$ $\quad e_3 = 1 \cdot 28 = 28$

$$\therefore x = b_1e_1 + b_2e_2 + b_3e_3$$
$$= 3(36) + 1(21) + 0(28)$$
$$= 129$$

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By Chinese remainder theorem, solutions are
$X = \{129 + m \cdot 84 \mid m \in \mathbb{Z}\}$

Take $m = -1 \Rightarrow x = 129 - 84$
$= 45$

smaller than $m = -1$ would make x negative, which is not practical.

$\therefore$ Least no. of coins $= \boxed{45}$