Lecture - 10

Digression :- $\binom{n}{k}$ is the binomial co-efficient.

q - analogues :- q - indeterminate $\hookrightarrow$ Variable

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+\cdots+q^{n-1} \text{ so that } [n]_1 = n$$

q - factorial :- $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$

q - binomial co-efficient :- $\binom{n}{k}_q = \dfrac{[n]_q!}{[k]_q!\,[n-k]_q!}$ [Gaussien Polynomial]

Thm (Multinomial thm) :- Fix $n, k \in \mathbb{N}$ and $x_1, x_2, \ldots, x_k$. Then

$$\sum_{\substack{0 \le a_1, \ldots, a_k \le n \\ \sum_{i=1}^{k} a_i = n}} \binom{n}{a_1, \ldots, a_k} x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} = (x_1 + x_2 + \cdots + x_k)^n$$

Idea of proof :- Bijection b/w multiset-permutations and terms on LHS.

$\boxed{\text{Ch-5}}$

Def'n :- A weak composition of $n \in \mathbb{N}$ is a sequence $(a_1, a_2, \ldots, a_k) \in \mathbb{N}^k$ where $a_i \geq 0$ and $a_1 + a_2 + \cdots + a_k = n$. If each $a_i > 0$, this is called a (strict) composition.
(Indistinguishable balls in distinguishable boxes)

Example :- $n = 2, k = 3$
$(2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)$

Property :- The # of weak compositions of n into k parts is $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$

Proof :- $\underbrace{00\cdots0}_{n}\underbrace{11\cdots1}_{k-1 \text{ dividers}}$. All possible arrangements.

Exercise :- Construct an explicit bijection b/w these and multisets of $[k]$ of length n.

Corollary :- The # of (strict) compositions of n into k parts is $\binom{n-1}{k-1}$

Note that this is 0 if $\boxed{k > n}$

Corollary :- The total # of compositions of n is 2^{n-1}.

Proof :- Binomial Thm.

Example :- $n = 3$

$$\left.\begin{array}{l}(3) \leftarrow 1 \\ \left.\begin{array}{l}(1,2) \\ (2,1)\end{array}\right\} \leftarrow 2 \\ (1,1,1) \leftarrow 1\end{array}\right.$$

$$\overline{4} = 2^{3-1} = 2^2$$

Exercise :- Find a bijection b/w these and subsets of $[n-1]$

Def'n :- An (integer) partition of n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of weakly decreasing +ve integers which sum to n.

If λ is a partition of n, we write $\lambda \vdash n$ and $|\lambda| = n$

Each λ_i is called a part. The # of parts of λ is called the length of λ, $(l)\lambda$

Example :- $n = 5$; $(4,1) \vdash 5$

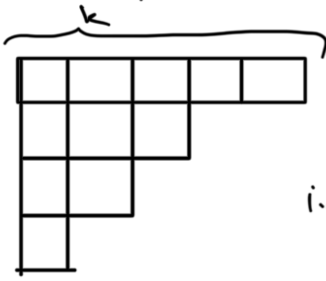
5
4 1
3 2
3 1 1
2 2 1
2 1 1 1
1 1 1 1 1

Property (Crazy) :- The # of partitions of n into exactly k parts is equal to the # of partitions of n with largest part.

Proof :- Transpose the young diagram

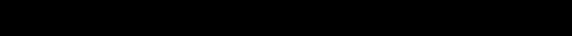
Def'n :- The Young / Ferrers diagram of a partition λ is a left justified array of boxes so that the i^{th} row contains λ_i boxes.

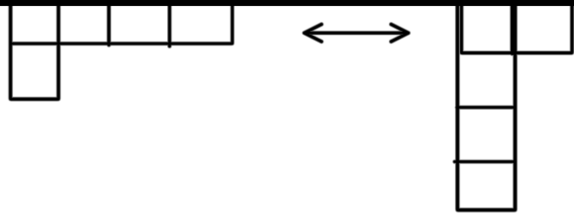
Example :- $\lambda = (5,3,2,1) \vdash 11$



(English Convention i.e, Matrix notation)

$(4,1) \vdash 5$





Def'n :- The conjugate of a partition λ, denoted λ', is the partition whose young diagram is the transpose of that of λ. λ is said to be self-conjugate if $\lambda = \lambda'$

Exercise :- The # of self-conjugate partitions of n is equal to the # of partitions of n with distinct odd parts

Example :- $n = 8$

4 2 1 1	7 1
3 3 2	5 3

Exercise (Euler) :- # of partitions of n into odd parts is equal to the # of partitions of n into distinct parts.

Example :- $n = 5$

5	5
3 1 1	4 1
1 1 1 1 1	3 2

Let $p(n)$ = # of partitions of n. $\boxed{\text{No simple formula}}$

Thm (The Hardy - Ramanujan Formula) :-

$$p(n) \sim \frac{1}{4\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

Set Partitions :-

Def'n :- A set partition of $[n]$ is a collection of non-empty called blocks subsets whose union is $[n]$ and so that no element is in more than one subset. (Distinguishable balls in indistinguishable boxes.

Example :- $n = 3$, $\{\{1,2,3\}\}$, $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$, $\{\{1\},\{2,3\}\}$, $\{\{1\},\{2\},\{3\}\}$

Def'n :- The # of set partitions of $[n]$ into k blocks is called the stirling number of the second kind, denoted by $\left\{ n \atop k \right\}$, read 'n set k'.

Example :- $\left\{ n \atop 1 \right\} = 1 = \left\{ n \atop n \right\}$, $\left\{ 4 \atop 2 \right\} = 7$; 123|4, etc

12|34, etc

[redacted]