Assignment - 4

1. a, $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > |x_1|\}$; $d(x,y) := \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$;
Let $h = x_2 - |x_1| > 0$;
Choose $\delta = \frac{h}{2}$

Any point $(y_1, y_2) \in B((x_1, x_2), \delta)$ satisfies.

$$\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} < \delta$$

$$\Rightarrow |y_1 - x_1| < \delta \text{ and } |y_2 - x_2| < \delta$$

To prove $y_2 > |y_1|$

$$|y_1| \leq |y_1 - x_1| + |x_1|$$

$$\Rightarrow |y_1| < \delta + |x_1|$$

$$-\delta < y_2 - x_2$$
$$y_2 > x_2 - \delta$$

$$y_2 > x_2 - \delta = (x_2 - h) + \frac{h}{2} = |x_1| + \frac{h}{2}$$
$$> |y_1|$$

$$\Rightarrow \boxed{y_2 > |y_1|} \Rightarrow B((x_1, x_2), \delta) \subseteq A \ \forall (x_1, x_2) \in S$$

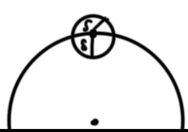
Hence S is open

$$\mathbb{R}^2 \setminus S = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq |x_1|\}$$

Consider $B((x_1, x_1), \delta)$; $(x_1, x_1) \in \mathbb{R}^2 \setminus S \ \forall x_1 \in \mathbb{R}^+$
This ball contains points from S. Hence
$B(x_1, x_2)$ contains the points $(y_1, y_2) \in B((x_1, x_1), \delta)$
where $y_2 \leq |y_1|$ and $y_2 > |y_1|$
$\Rightarrow \mathbb{R}^2 \setminus S$ is not open
$\Rightarrow S$ is not closed.

b, Consider $A = B((a_1 + r, a_2), \delta)$; $\delta > 0$ $(a_1 + r, a_2) \in S$.

$d((a_1 + r, a_2), (y_1, y_2)) < \delta$; $(y_1, y_2) \in A$



All possible vals of distances of (y_1, y_2) from (a_1, a_2) are $(r-\delta, r+\delta)$

$\Rightarrow \exists\ (y_1, y_2)$ st $r < d((y_1, y_2), (a_1, a_2)) < r+\delta \Rightarrow (y_1, y_2) \notin S$

$\Rightarrow A \notin S \Rightarrow S$ is not open.

$\mathbb{R}^2 \setminus S := \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - a_1)^2 + (x_2 - a_2)^2 > r^2\}$

Using similar results (openness for i,) we can say that $\mathbb{R}^2 \setminus S$ is open

$\Rightarrow S$ is closed.

c, $S := [a_1, b_1) \times [a_2, b_2) \times \dots \times [a_n, b_n)$; $a_j, b_j \in \mathbb{R}$, $a_j < b_j$

$\Rightarrow S = \{(x_1, x_2, \dots, x_n) : a_i \le x_i < b_i \ \forall\ i = \{1, 2, \dots, n\}\}$

e, $S = \{(x_1, x_2, \dots, x_n) : x_1^2 + x_2^2 + \dots + x_n^2 = 1\}$; $x' = (x_1, x_2, \dots, x_n)$

Any open ball $A = B(x', \delta)$ $[x' \in S]$ contains points with $d((y_1, y_2, \dots, y_n) \in A, (\underbrace{0, 0, 0, \dots, 0}_{n \text{ times}}))$ can be $>$ or $<$ 1

i.e., range of above d is $(1-\delta, 1+\delta)$ but S contains points whose $d = 1$. Hence S is not open.

$\mathbb{R}^n \setminus S = \{(x_1, x_2, \dots, x_n) : x_1^2 + x_2^2 + \dots + x_n^2 > 1 \text{ or } < 1\}$

Solving similar to i, openness, $\mathbb{R}^n \setminus S$ is open

$\Rightarrow S$ is closed.

3. $B = \bigcup_{A \in \mathscr{C}} \overline{A}$, $C = \overline{\bigcup_{A \in \mathscr{C}} A}$

To prove :- $B \subseteq C$

$$x \in B \Rightarrow x \in \bigcup_{A \in \mathscr{C}} \overline{A}$$

$\Rightarrow x \in \overline{A}$ where $A \in \mathscr{C}$

$x \in \overline{A}$

$\Rightarrow x \in A \cup \{y : y \text{ is a limit point of } A\}$

Case-I :- $x \in A$ or x is a limit point of A.

$x \in A \Rightarrow x \in \bigcup_{A \in \mathscr{C}} A \Rightarrow x \in \overline{\bigcup_{A \in \mathscr{C}} A}$

Case-II $x \in \{y : y \text{ is a limit point of } A\}$

If x is a limit point of A, then by def'n, $\exists$ $\delta > 0$ st $B(x, \delta) \cap A \neq \phi$

$\Rightarrow B(x, \delta) \cap \left(\bigcup_{A \in \mathscr{C}} A \right) \neq \phi \Rightarrow x$ is a limit point of $\bigcup_{A \in \mathscr{C}} A$. $\Rightarrow x \in \overline{\bigcup_{A \in \mathscr{C}} A}$

Thus $B \subseteq C$

Example :- $X = \mathbb{R}$

$\mathscr{C} = \{A_n : n \in \mathbb{P}\}$

where $A_n = (0, 1 - \frac{1}{n}) \; \forall \; n \in \mathbb{P}$

$$\overline{A_n} = \left(0, 1 - \frac{1}{n}\right) \cup \left\{0, 1 - \frac{1}{n}\right\} = \left[0, 1 - \frac{1}{n}\right]$$

$$B = \bigcup_{n \in \mathbb{P}} \overline{A_n} = [0, 1)$$

$C = \overline{\bigcup_{n \in \mathbb{P}} A_n} = \overline{(0, 1)} = [0, 1]$; [$\{0, 1\}$ are limit points of $\cup A_n$]

$B \subsetneq C$ i.e., $B \neq C$

4. $A \subsetneq \mathbb{R}$ is bdd above [A has LUB property]

$d(x, y) := |x - y| \; \forall \; x, y \in \mathbb{R}$

To prove :- $\sup A \in \overline{A}$

$\overline{A} = A \cup \{y : y \text{ is a limit point of } A\}$

Let $x = \sup A$.

$x \geq a \; \forall \; a \in A$

If $b < x$, $\exists$ $a \in A$ st $b < a \leq x$

$\boxed{\text{If } b < x, \ \exists\ a \in A \text{ st } b < a \leq x}$

$x \in \bar{A} \Rightarrow$

Case I :- $x \in A$

$\Rightarrow x \in A \cup \{y : y \text{ is a limit pt of } A\}$

$\Rightarrow x \in \bar{A}$

Case II :- $x \notin A$

To prove :- x is a limit point of A.

For any $\varepsilon > 0$, consider $(x - \varepsilon, x)$

$x - \varepsilon$ is not an upper bound of A, by def'n

$\exists\ a \in A$ st $x - \varepsilon < a \leq x$

$\Rightarrow a \in (x - \varepsilon, x) \cap A$ and $a \neq x$ [$x \notin A$ but $a \in A$]

$\Rightarrow$ If $B(x, \varepsilon)$ is considered, it contains a

$\Rightarrow B(x, \varepsilon) \cap A \neq \phi \Rightarrow x$ is a limit point of A

$\Rightarrow x \in \bar{A}$

Thus $\boxed{\sup A \in \bar{A}}$

5, $A \subseteq X$

A contains all its limit points

To prove :- $X \setminus A$ is open.

$X \setminus A$ is open if $\exists\ \varepsilon > 0$ st $B(x, \varepsilon) \subseteq X \setminus A$ $\forall\ x \in X \setminus A$

$x \in X \setminus A \Rightarrow x \notin A \Rightarrow x$ is not a limit point of A

$\Rightarrow \exists\ \varepsilon > 0$ st $B(x, \varepsilon) \cap A = \phi$

$\Rightarrow B(x, \varepsilon) \subseteq X \setminus A$

Since x is arbitrary, such ε exists $\forall\ x \in X \setminus A$

$\Rightarrow X \setminus A$ is open

$\Rightarrow A$ is closed.

6, To prove :- Each singleton in a metric space is a closed set. i.e., $\{x\}$ $\forall\ x \in X$ is closed.

i.e., $X \setminus \{x\}$ is open.

For any $y \in X \setminus \{x\}$, Fix, $\varepsilon = \dfrac{d(x, y)}{2} > 0$

$y \neq x$

$B(y, \varepsilon) = \{z : d(y, z) < \varepsilon\}$

$$d(x,y) \le d(x,z) + d(z,y)$$
$$\Rightarrow d(x,z) \geq d(x,y) - d(y,z)$$
$$\Rightarrow d(x,z) \geq d(x,y) - \varepsilon$$
$$\Rightarrow d(x,z) \geq \varepsilon$$
$$\Rightarrow x \neq z \Rightarrow z \in X\setminus\{x\} \; \forall \; z \in B(y,\varepsilon)$$
$$\Rightarrow B(y,\varepsilon) \subseteq X\setminus\{x\} \; \forall \; y \in X\setminus\{x\}$$
$\Rightarrow X\setminus\{x\}$ is open

$\Rightarrow \{x\}$ is closed.

8, $Y \subseteq X$, $Y \neq \phi$; $F \subseteq Y$

Proof 1, :- F is closed relative to Y.

$\Rightarrow Y\setminus F$ is open rel to Y.

$\Rightarrow \exists$ open $U \subseteq X$ st

$$Y\setminus F = U \cap Y \quad (1)$$

Let $A := X\setminus U$

$$Y \cap A = Y \cap (X\setminus U)$$
$$= (Y\cap X)\setminus(Y \cap U)$$
$$= Y\setminus(Y\cap U)$$

From (1)

$$Y \cap A = Y\setminus(Y\setminus F)$$
$$= F$$

$\Rightarrow \exists$ a closed $A \subseteq X$ st $F = Y \cap A$.

Proof 2, :- $\exists$ closed $A \subseteq X$ st $F = Y \cap A$

$\Rightarrow X\setminus A$ is open rel to X

Let $U := X\setminus A$

U is open rel to X.

$F = Y \cap A$

$Y\setminus F = Y\setminus(Y\cap A)$

$Y\setminus(Y\cap A)$ contains elements y st $y \in Y$ and $y \notin A$.

$\Rightarrow y \in (Y \cap X)$ $[Y \cap X = Y]$

and $y \notin (Y\cap A)$

$\Rightarrow y \in (Y\cap X)\setminus(Y\cap A)$

$\Rightarrow y \in Y \cap (X\setminus A)$

$\Rightarrow y \in Y \cap U$

$$Y \setminus F = Y \cap U$$

U is open rel to X
$\Rightarrow Y \setminus F$ is open rel to Y
$\Rightarrow$ F is closed rel to X.

9, A = HW3 Property :- For every $x, y \in \mathbb{R}$ with $x<y$, $\exists q \in \mathbb{Q}$ st $x < q < y$.

B = $\mathbb{Q}$ dense in $\mathbb{R}$ with std metric :- For every $x \in \mathbb{R}$ and $\delta > 0$, $\exists q \in \mathbb{Q}$ st $d(x, q) < \varepsilon$ where $d(x,y) = |x-y| \ \forall x, y \in \mathbb{R}$.

$A \Rightarrow B$:-

Let $x, y \in \mathbb{R}$ st $x<y$

By A, $\exists q \in \mathbb{Q}$ st $x<q<y$ $\quad [q - x \in \mathbb{R}^+]$

$\varepsilon := y - x > 0$

Now $\because x < q$

$x - \varepsilon < q$

And $q<y \Rightarrow q < x+\varepsilon$

$\Rightarrow x - \varepsilon < q < x+\varepsilon$

$\Rightarrow x - \varepsilon - x < q - x < x + \varepsilon - x$

$\Rightarrow -\varepsilon < q - x < \varepsilon$

$\Rightarrow |q-x| < \varepsilon$

i.e., $|x-q| < \varepsilon$

$\Rightarrow d(x,q) < \varepsilon$

$\therefore A \Rightarrow B$

$B \Rightarrow A$

$d(x,q) < \varepsilon$ $\boxed{\text{for each } x \in \mathbb{R},\ \varepsilon > 0\ \exists q}$

$\Rightarrow |x-q| < \varepsilon$ $\quad \varepsilon > 0$; Let $y := x + \varepsilon$

i.e., $|q-x| < \varepsilon$

$\Rightarrow -\varepsilon < q - x < \varepsilon$

$-\varepsilon + x < q - x + x < \varepsilon + x$

$\Rightarrow x - \varepsilon < q < x + \varepsilon$ — (1)

$\Rightarrow x - \varepsilon < q < y$

Now we have $x - \varepsilon, y \in \mathbb{R}$ with $x - \varepsilon < y$ and $q \in \mathbb{Q}$ st $x - \varepsilon < q < y$.

2, $i \in \{1, 2, 3, \ldots, n\}$

Let $x_1, x_2, \ldots, x_n$ be n limit points of $A \subseteq \mathbb{R}$, Ensure all limit points are in $[0,1]$

Construction of A :-

$$A_i := \{x_i + \frac{1}{m} \mid m \in \mathbb{N}\}$$

$$A := \bigcup_{i=1}^{n} A_i$$

To prove :-

1, Each x_i is a limit point of A.

Fix $\varepsilon > 0$, $\exists\, m \in \mathbb{P}$ st $\frac{1}{m} < \varepsilon$

$\Rightarrow$ For any $(x_i - \varepsilon, x_i + \varepsilon)$ we find atleast one element of A

$\Rightarrow$ $x_i \; \forall\, i = \{1, 2, \ldots, n\}$ is a limit point of A.

2, $y \neq x_i \in \mathbb{R}$ is not a limit point of A.

Doubt.

7, A° is the largest open set contained in A

$\Rightarrow A^\circ \subseteq A$

To prove :- $A^\circ = \{x \in A : x \text{ is an interior pt of } A\} = B$

i, $A^\circ \subseteq B$

$x \in A^\circ$

$\because A^\circ$ is open $\Rightarrow$ x is an interior pt of A

$\Rightarrow$ $x \in B$

$\Rightarrow A^\circ \subseteq B$

ii, $B \subseteq A^\circ$

$x \in B^\circ$

$\Rightarrow x$ is an interior pt of A

Then $\exists\, r > 0$ st $B(x, r) \subseteq A$

Let $U := B(x, r)$

$\Rightarrow U \subseteq A$

U is a open set [Every open ball is an open set]

$\because$ for each $x \in B$, $\exists$ open set U st $x \in U \subseteq A$

B is an open set in A.

But $A^\circ :=$ the largest open set in A

$\Rightarrow$ $B \subseteq A^\circ$

From 1, 2,

$$\boxed{A^{\circ} = B}$$