UM 204 (WINTER 2024) - WEEK 13

Theorem 0.1. *For a bounded function*

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx.$$

Proof. Suppose P is a partition of $[a,b]$ and P^* is a refinement of P, i.e., $P \subset P^*$. Then, we claim that

$$L(P,f) \le L(P^*,f), \quad U(P^*,f) \le U(P,f).$$

This can be done by induction on $\#P^* \setminus \#P$. When P^* has one more point than P, say $P = \{x_0, x_1, \ldots, x_n\}$ and P^* additionally contains $y \in [x_{j-1}, x_j]$ for some $J \in \{1, \ldots, n\}$, then

$$L(P^*,f) - L(P,f) = \left(\inf_{[x_{j-1},y]} f\right)(y - x_{j-1}) + \left(\inf_{[y,x_j]} f\right)(x_j - y) - \left(\inf_{[x_{j-1},x_j]} f\right)(x_j - x_{j-1}) \ge 0.$$

Now, if P_1 and P_2 are two arbitrary partitions of $[a,b]$, then $P^* = P_1 \cup P_2$ is a refinement of both. Thus,

$$L(P_1,f) \le L(P^*,f) \le U(P^*,f) \le U(P_2,f).$$

Taking supremum over all partitions on the right, and then infimum over all partitions on the left yield the claim. □

Theorem 0.2. *A bounded function $f : [0,1] \to \mathbb{R}$ is Riemann integrable if and only if, for every $\varepsilon > 0$, there is a partition P_ε of $[a,b]$ such that*

$$U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon \tag{0.1}$$

Remarks. (1) If (0.1) holds for some P, then it holds for every refinement of P.

(2) If (0.1) holds for some partition $P = \{x_0 \le x_1 \le \cdots \le x_n\}$, then, for arbitrary points $s_j, t_j \in [x_{j-1}, x_j]$, we have that

$$\sum_{j=1}^{n} |f(s_j) - f(t_j)| \triangle x_j < \varepsilon.$$

(3) If (0.1) holds for some partition $P = \{x_0 \le x_1 \le \cdots \le x_n\}$ and R.I. $f : [a,b] \to \mathbb{R}$, then for arbitrary points $t_j \in [x_{j-1}, x_j]$,

$$\left| \sum_{j=1}^{n} f(t_i) \triangle x_j - \int_a^b f(x)\,dx \right| < \varepsilon.$$

Exercise. Rediscover, on your own, the proof of all the theorems on Riemann integration that we covered in UM 101.

Theorem 0.3. *Every monotone function on* $[a,b]$ *is Riemann integrable.*

Theorem 0.4. *Every continuous function on* $[a,b]$ *is Riemann integrable.*

Theorem 0.5. *Any function on* $[a,b]$ *that has at most finitely many discontinuities is Riemann integrable.*

Theorem 0.6 (Properties of the integral)**.**

Theorem 0.7 (FTOC I)**.**

Theorem 0.8 (FTOC II)**.**

Theorem 0.9 (Integration by substitution)**.**

Theorem 0.10 (Integration by parts)**.**

Theorem 0.11 (Composition)**.** *Suppose* $f:[a,b]\to\mathbb{R}$ *is R.I.. Assume* $f([a,b])\subset[m,m]$. *Suppose* $\phi:[m,M]\to\mathbb{R}$ *is continuous. Then,* $h=\phi\circ f$ *is R.I.*

Proof. Let $\varepsilon>0$. Since ϕ is unif. cont. on $[m,M]$, there is a $\delta\in(0,\varepsilon)$ such that for all $x,y\in[m,M]$, whenever $|x-y|<\delta$, we have that $|\phi(x)-\phi(y)|<\varepsilon$. Since f is R.I. on $[a,b]$, there is a partition $P=\{x_0\le x_1\le\cdots\le x_n\}$ of $[a,b]$ such that

$$U(P,f)-L(P,f)<\delta^2.$$

Let

$$\begin{aligned}
I_j &= [x_{j-1},x_j]\\
M_j &= \sup_{I_j} f\\
m_j &= \inf_{I_j} f\\
M_j^* &= \sup_{I_j} \phi\circ f\\
m_j^* &= \inf_{I_j} \phi\circ f.
\end{aligned}$$

Now, let $A\subset\{1,\dots,n\}$ be the set of those $j\in\{1,\dots,n\}$ such that $M_j-m_j<\delta$. Then, for all $j\in A$,

$$M_j^*-m_j^*\le\varepsilon.$$

Let $B=\{1,\dots,n\}\setminus B$. Then, $M_j-m_j\ge\delta$ for all $j\in B$. But,

$$\delta\sum_{j\in B}|I_j|\le\sum_{j\in B}(M_j-m_j)|I_j|\le\sum_{j=1}^{n}(M_j-m_j)|I_j|=U(P,f)-L(p,f)<\delta^2.$$

Thus,

$$\sum_{j\in B} |I_j| < \delta < \varepsilon.$$

Finally, we have that

$$U(P,\phi\circ f) - L(P,\phi\circ f) = \sum_{j\in A}(M_j^* - m_j^*)|I_j| + \sum_{j\in B}(M_j^* - m_j^*)|I_j| \le \varepsilon(b-a) + 2\sup_{[m,M]}|\phi|\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\phi\circ f$ is R.I. on $[a,b]$. □

END OF LECTURE 31

0.1. **Integration of vector-valued functions.** A bounded vector-valued function $f = (f_1,\dots,f_n) : [a,b] \to \mathbb{R}^n$ is Riemann integrable, if each of its components are.

Theorem 0.12. *If $f = (f_1,\dots,f_n) : [a,b] \to \infty$ is R.I., then so is $\|f\|$, and*

$$\left\|\int_a^b f(x)dx\right\| \le \int_a^b \|f(x)\|dx.$$

Proof. nSince x^2 is continuous on $\mathbb{R}$ and $\sqrt{x}$ is continuous on $[0,\infty)$, $\|f\| = \sqrt{f_1^2+\cdots+f_n^2}$ is R.I. on $[a,b]$. Now,

$$\int_a^b < f(x), g(x) > dx \le \int_a^b \|f(x)\|\|g(x)\|dx.$$

Let $g_j = \int_a^b f_j(x)dx$. Then, the LHS is

$$\left\|\int_a^b f(x)dx\right\|^2$$

and the RHS is $\left\|\int_a^b f(x)\right\| dx \cdot \int_a^b \|f(x)\|dx$. □

0.2. **Rectifiable curves.** A continuous function from $[a,b] \to \mathbb{R}^n$ is called a curve. If γ is injective, it is called an arc. If $\gamma(a) = \gamma(b)$, it is called a closed curve. If γ is a closed curve such that γ is injective on $[a,b)$, then it is called a simple closed curve.

Definition 0.13. Let γ be a curve on $[a,b]$. Let $P + \{x_0 \le x_1 \le \cdots \le x_n\}$ be a partition of $[a,b]$. Let

$$\begin{aligned}\Lambda(P,\gamma) &= \sum_{j=1}^n \|\gamma(x_j) - \gamma(x_{j-1})\|,\\ \Lambda(\gamma) &= \sup\{\Lambda(P,\gamma) : P \text{ is a partition of } [a,b]\}.\end{aligned}$$

We say that γ is rectifiable if $\Lambda(\gamma) < \infty$, in which case $\Lambda(\gamma)$ is called the length of γ.

In general, the length of a rectifiable curve need not be a Riemann integral. However, the following theorem gives us a large class of rectifiable curves for which this is the case.

Theorem 0.14. *Let $\gamma : [a,b] \to \mathbb{R}^n$ be a continuously differentiable curve. Then, γ is rectifiable and*

$$\Lambda(\gamma) = \int_a^b \|\gamma'(t)\|\,dt.$$

Proof. Let $P = \{x_0 = a \le x_1 \le x_2 \cdots \le x_n = b\}$ be an arbitrary partition of $[a,b]$. Then, by the fundamental theorem of calculus and the triangle inequality for integrals,

$$\Lambda(P,\gamma) = \sum_{j=1}^n \|\gamma(x_j) - \gamma(x_{j-1})\| = \sum_{j=1}^n \left\| \int_{x_{j-1}}^{x_j} \gamma'(t)\,dt \right\| \le \sum_{j=1}^n \int_{x_{j-1}}^{x_j} \left\|\gamma'(t)\,dt\right\| \le \int_a^b \|\gamma'(t)\|\,dt.$$

Thus, $\Lambda(\gamma)$ is finite and at most $\int_a^b \|\gamma'(t)\|\,dt$.

Conversely, let $\varepsilon > 0$. Since γ' is uniformly continuous on $[a,b]$, there is a $\delta > 0$ such that

$$\|\gamma'(x) - \gamma'(y)\| < \varepsilon \text{ whenever } \|x - y\| < \delta.$$

Let $P = \{x_0 = a \le x_1 \le x_2 \cdots \le x_n = b\}$ be a partition such that $\Delta x_j < \delta$. Then, for any fixed $j \in \{1, \ldots, n\}$,

$$\begin{aligned}
\int_{x_{j-1}}^{x_j} \|\gamma'(t)\|\,dt \le \int_{x_{j-1}}^{x_j} \|\gamma'(x_{j-1})\|\,dt + \varepsilon\Delta x_j &= \left\| \int_{x_{j-1}}^{x_j} \gamma'(x_{j-1})\,dt \right\| + \varepsilon\Delta x_j \\
&= \left\| \int_{x_{j-1}}^{x_j} \left(\gamma'(x_{j-1}) - \gamma'(t) + \gamma'(t)\right) dt \right\| + \varepsilon\Delta x_j \\
&\le \left\| \int_{x_{j-1}}^{x_j} \gamma'(x_{j-1}) - \gamma'(t)\,dt \right\| + \left\| \int_{x_{j-1}}^{x_j} \gamma'(t)\,dt \right\| + \varepsilon\Delta x_j \\
&\le \int_{x_{j-1}}^{x_j} \left\|\gamma'(x_{j-1}) - \gamma'(t)\right\| dt + \left\|\gamma(x_j) - \gamma(x_{j-1})\right\| + \varepsilon\Delta x_j \\
&\le 2\varepsilon\Delta x_j + \left\|\gamma(x_j) - \gamma(x_{j-1})\right\|.
\end{aligned}$$

Summing boths sides from $j = 1$ to n, we obtain that

$$\int_a^b \|\gamma'(t)\|\,dt \le 2\varepsilon(b-a) + \Lambda(P,\gamma) \le 2\varepsilon(b-a) + \Lambda(\gamma).$$

Taking $\varepsilon \to 0$, we obtain the reverse inequality. □

END OF LECTURE 32

1. Sequences and series of functions

In this chapter, we will consider sequences of the form

$$\{f_0, f_1, \ldots\},$$

and series of the form

$$\sum_{j=0}^{\infty} f_j \leftrightarrow \left\{ s_n = \sum_{j=0}^{n} f_j \right\}_{n\in\mathbb{N}},$$

where each f_j is a $\mathbb{C}$-valued function on some subset of $\mathbb{R}$ (or between metric spaces in the former case).

1.1. **Pointwise convergence.** Suppose, for each $x \in [a,b]$, the sequence $\{f_j(x)\}_{x\in\mathbb{N}}$ converges. Let

$$f_\infty(x) = \lim_{n\to\infty} f_j(x).$$

We say that the sequence $\{f_j\}_{n\in\mathbb{N}}$ converges pointwise to f_∞. Similarly, the sum $\sum_{j=0}^{\infty} f_j$ converges pointwise to the function s if

$$s(x) = \sum_{j=0}^{\infty} f_j(x)$$

for all $x \in [a,b]$.

Examples. (1) Let $f_n(x) = x^n$ on $[0,1]$. Note that

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{if } x \in [0,1), \\ 1, & \text{if } x = 1. \end{cases}$$

Thus, the pointwise limit of a sequence of continuous functions need not be continuous.

(2) It is clear from the previous example that the p.w. limit of differentiable functions need not be differentiable. Even when the p.w. limit is differentiable, we may not have that

$$\lim_{n\to\infty} f_n'(x) = \left(\lim_{n\to\infty} f_n\right)'(x).$$

Take, for example $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$ on $[0,1]$. Then, $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$. However, the poitnwise limit of $\left\{f_n'(x) = \sqrt{n}\cos(nx)\right\}$ does not exist.

(3) Let $f_n(x) = \lim_{m\to\infty} (\cos(m!\pi x))^{2n}$ on $[0,1]$. Note that

$$f_n(x) = \begin{cases} 1, & \text{if } m!x \in \mathbb{N}, \\ 0. & \text{otherwise.} \end{cases}$$

Since the set of discontinuities of each f_n is finite, f_n is R.I. on $[0,1]$. However,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1, & \text{if } m!x \in \mathbb{N} \text{ for some } m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

This is the Dirichlet function, we have shown to be non R.I. on $[0,1]$.

(4) Let $f_n(x) = n^2 x(1-x^2)^n$ on $[0,1]$. Note that

$$\lim_{n\to\infty} f_n(x) = 0$$

for all $x \in [0,1]$. However,

$$0 = \int_0^1 \lim_{n\to\infty} f_n(x) \neq \lim_{n\to\infty} \int_0^1 f_n(x) = \lim_{n\to\infty} \frac{n^2}{2n+2} = \infty.$$

1.2. **Uniform convergence.** Note that in the first example, if we let — for a fixed $\varepsilon > 0$,

$$N(x) = \min\{N \in \mathbb{N} : |f_n(x) - f_\infty(x)| < \varepsilon\},$$

then we obtain the function

$$N(x) = \begin{cases} \frac{\ln(\varepsilon)}{\ln(x)}, & x \in (0,1) \\ 0, & x = 0,1. \end{cases}$$

Clearly, the "rate of convergence" varies from point to point, and no common N can be found that will work for all x.

Definition 1.1. A sequence of functions $\{f_j\}_{j\in\mathbb{N}}$ on $E \subset \mathbb{R}$ is said to be uniformly convergent to a function f on E if, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$|f_j(x) - f(x)| < \varepsilon, \quad \forall n \geq N, x \in E.$$

The series $\sum_{j=0}^n f_j$ converges uniformly to a function s on E if the s.o.p.s. $\{s_j\}_{j\in\mathbb{N}}$ converges uniformly to s.

As an exercise, state and prove the Cauchy characterization of uniform convergence. Also, prove the following

Theorem 1.2 (Weierstrass M-test)**.** *Let $\{f_n\}_{\mathbb{N}}$ be a sequence of $\mathbb{C}$-valued functions on a set E. Suppose, there is a sequence $\{M_n\}_{\mathbb{N}} \subset \mathbb{R}$ such that*

$$|f_n(x)| \leq M_n$$

for all $x \in E$ and $n \in \mathbb{N}$, and $\sum_{\mathbb{N}} M_n < \infty$. Then, the series $\sum_{\mathbb{N}} f_n$ converges absolitely and uniformly on E.

END OF LECTURE 33