UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022

QUIZ 5 **MARCH 14, 2022**

PLEASE NOTE the following:

- **Duration:** 15 minutes
- The quiz is to be written with no access to any books, notes, or study materials.

1. Let $\{a_n\}$ be a real sequence, and define

$$\Delta_n := a_{n+1} - a_n, \quad n = 1, 2, 3, \dots .$$

Suppose $\{\sqrt{n}\Delta_n\}$ is convergent. Then, is $\{a_n\}$ convergent? Give a proof if this is true, else construct a sequence $\{a_n\}$ with the stated property that is not convergent.

Remark / tip. Recall that, in our study of sequences, we aren't repeating what you learnt in UM101. In that spirit, you may use **without** proof any fact of which a precise statement was presented in UM101.

Note. The above is closely related to part (b) of Problem 2 of Homework 8.

Solution. A sequence $\{a_n\}$ with the above property is not, in general, convergent. Consider the following example:

$$a_n := \begin{cases} 0, & \text{if } n = 1, \\ \sum_{k=1}^{n-1} \dfrac{1}{\sqrt{k-1}}, & \text{if } n = 2, 3, 4, \dots . \end{cases}$$

Then

$$\Delta_n = 1/\sqrt{n} \quad \forall n \in \mathbb{Z}^+.$$

Clearly $\{\sqrt{n}\Delta_n\}$ is convergent since it is a constant sequence! However, as a_n is the $(n-1)$-st partial sum, for each $n \geq 2$, of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

the p-series test, with $p = 1/2 < 1$, implies that $\{a_n\}$ does not converge. $\square$

Remarks. (i) Other examples of $\{a_n\}$ that do not converge can, clearly, be constructed. The above is merely a prototypical example. (ii) The "fact... presented in UM101," alluded to in the remark above refers to the p-series test. Recall that, as of March 14, the proof of the p-series test had not been presented in class.