UMA101: Analysis and Linear Algebra

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§1 Basic Set Theory

Definition 1.1. A set is a **well defined** collection of objects. Remarks on the formal definition.

- The meaning of **collection** and **objects** are taken to be self evident.
- A collection is said to be **well defined** if it is described by one rule in a specific list called Axioms of Set Theory (ZF Axioms)

Some of the axioms of set theory are:

- The collection with no elements is a set (Null set)
- Two sets are said to be equal if they consist of the same list of elements
- **Axiom of Union:** Given a non null set C of sets. The collection all of whose elements belong to some set $A \in C$, and such that every element of every $A \in C$ belongs to this collection, is a set. This set is denoted by $\bigcup_{A \in C} A$

Given sets A and B, the set difference $A - B$ is the set of all elements in A that are not in B. (The existence of such a set is guaranteed by the set builder axiom)

Theorem 1.2 (De Morgan's Laws)
Let X be a set. Let C be a non null set whose elements are subsets of X. Then,

1. $X - \bigcap_{A \in C} A = \bigcup_{A \in C}(X - A)$
2. $X - \bigcup_{A \in C} A = \bigcap_{A \in C}(X - A)$

Question: Why do we need to codify the term **well defined**
Answer: It leads to contradictions. For example, consider the following,

$$A = \{x \in S : x \notin x\}$$

where S is the set of all sets. The problem arises when we try to figure out whether $A \in A$ or not. This is called **Russell's Paradox**

§2 Numbers

§2.1 Natural Numbers

Our approach will deviate from Apostol. The natural numbers (denoted as $\mathbb{N}$) is a collection of elements that satisfy a list of rules called the **Peano Axioms**

Peano Axioms

1. There exists a special element in $\mathbb{N}$ called 0
2. For each $n \in \mathbb{N}$, there is an unique element called the successor of n, denoted by $S(n)$
3. 0 is not the successor of any $n \in \mathbb{N}$
4. If m and n are two natural numbers and $S(m) = S(n)$, then $m = n$
5. For each $n \in \mathbb{N}$, let $\Sigma(n)$ denote a statement involving n. If $\Sigma(0)$ is true, and $\Sigma(S(n))$ is true whenever $\Sigma(n)$ is true, then $\Sigma(n)$ is true for all $n \in \mathbb{N}$

Remarks: It can be proven that $\mathbb{N}$ is a set. A natural number is simply any element of $\mathbb{N}$

Arithmetic

- **Peano Addition:** $n + 0 := n, \forall n \in \mathbb{N}$
  $n + S(m) = S(m + n), \forall m, n \in \mathbb{N}$
- **Peano Multiplication:** $n.0 := 0$
  $n.S(m) = n.m + n$

§2.2 Integers and Rational

Integers are denoted by $\mathbb{Z}$. This set was created so as to realise the closure law with respect to subtraction.

Rational are denoted by $\mathbb{Q}$. This set was created to realise closure with respect to division.

Field

A field (denoted by $\mathbb{F}$) is a set equipped with two rules $+$ and $\cdot$ defined for every pair of elements $x, y \in \mathbb{F}$, such that they obey,

- **Axiom 0:** $x + y \in \mathbb{F}$ and $x \cdot y \in \mathbb{F}$
- **Axioms 1-6 given in Apostol**

§2.3 Real Numbers

The real numbers (denoted by $\mathbb{R}$) are collectively the ordered field containing $\mathbb{Q}$ as an ordered sub-field and has the **Lowest Upper Bound** Property.

Definition 2.1. We can equip $\mathbb{Q}$ with a symbol $>$ such that for any pair $a, b \in \mathbb{Q}$, the following properties hold :

- $a = b$ or $a > b$ or $b > a$
- $a, b, c \in \mathbb{Q}$, and if $a > b$ and $b > c$, then $a > c$

We write $a \geq b$ if either $a > b$ or $a = b$

Question: How can we tell $x > y$, $x, y \in \mathbb{Q}^+$
Answer: Step 1: Define $\mathbb{Q}^+ := \left\{ \frac{a}{b} : a, b \in \mathbb{N} - \{0\} \right\}$
In addition to obeying Axioms 0-6, $\mathbb{Q}^+$ also obeys:

- $x, y \in \mathbb{Q}^+ \implies x + y \in \mathbb{Q}^+$
- $0 \notin \mathbb{Q}^+$
- $\forall x \in \mathbb{Q} - \{0\}$, exactly one of x and $-x$ is in $\mathbb{Q}^+$

Step 2: We define for $x, y \in \mathbb{Q}$, $x > y \Leftrightarrow x - y \in \mathbb{Q}^+$
Let $(\mathbb{F}, +, \cdot)$ be a field. We say $\mathbb{F}$ is an ordered field if $\exists$ a set $\mathbb{F}^+ \subset \mathbb{F}$, such that $(\mathbb{F}, \mathbb{F}^+)$ satisfy the above axioms with $\mathbb{Q}$ replaced by $\mathbb{F}$

Using the above rules, we assume that $\mathbb{R}$ is a ordered field. But we run into a few problems.

- We are demanding so many properties from $\mathbb{R}$. Could these co-exist?
- We will assume they co-exist in this course. This is a reasonable assumption because Dedekind constructed the real numbers.

Let $\mathbb{F}$ be an ordered field and let $>$ be given in $\mathbb{F}^+ \subset \mathbb{F}$. Let $S \neq \phi$ be a subset of $\mathbb{F}$

1. $u \in \mathbb{F}$ is called an **upper bound** if $u \geq x$ for all x in S
2. We say S is bounded above if S has some upper bound
3. An element $b \in \mathbb{F}$ is called as a **Least Upper Bound** of S if,
   - b is an upper bound of S
   - If $x \in \mathbb{F}$ such that $x < b$, then x is not an upper bound of S

Problem (Self Thinking)

If $\mathbb{F}$ and S are as above, formulate definition for: **Lower bound of S**, **S is bounded below** and **Greatest lower bound of S**

Definition 2.2. An ordered field has **Least Upper Bounded** property if any $S \neq \phi$, $S \subset \mathbb{F}$ that is bounded above has a **Least Upper Bound** property.

Remark: $\mathbb{Q}$ does not have **Least Upper Bound** (consider the set $S = \{x \in a : 0 < x^2 < 2\}$, this doesn't have a LUB in $\mathbb{Q}$.

Problem 2.3

$\mathbb{N}$ as a subset of either $\mathbb{Q}$ or $\mathbb{R}$ is not bounded above.

Proof. First $\mathbb{N} \subset \mathbb{R}$. Assume $\mathbb{N}$ is bounded above, let $u \in \mathbb{R}$ be the LUB. Thus, $u - 1$ is not an upper bound. Thus, there exists a natural number n such that,

$$n > u - 1$$

But this means that $n + 1 > u$, and $n + 1 \in \mathbb{N}$. Contradiction! $\square$

Theorem 2.4 (Archimedean Property)

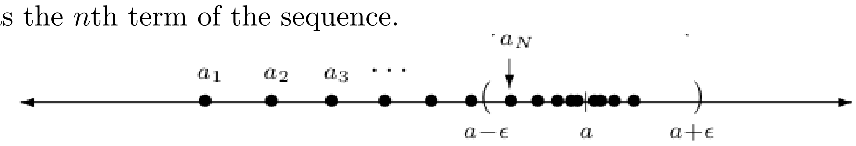
Let $x \in \mathbb{R}$ and $x > 0$. Given $y \in \mathbb{R}$, $\exists n \in \mathbb{P}$ such that $nx > y$

Proof. We have two cases.
Case 1: $y \leq 0$. In this case $n = 1$ works.
Case 2: $y > 0$. Assume that no such n exists. So, $nx \leq y$, $\forall n \in \mathbb{P}$. This implies that $n \leq x^{-1}y$. This contradicts the fact that $\mathbb{N}$ is not bounded above. $\square$

§3 Sequences

Definition 3.1. Let $S \neq \phi$ be a set. A sequence is a function $f : \mathbb{P} \to S$.

Historically, we have denoted sequences as $a_1, a_2, a_3, \cdots$ So, we drop the functional notation.
$f(n) := a_n \in S$, and denote the sequence as $\{a_n\}_{n=1}^{\infty}$, or simply as $\{a_n\}$. a_n is known as the nth term of the sequence.



Definition 3.2. Let $\{a_n\}$ be a sequence in $\mathbb{R}$. We say $\{a_n\}$ converges to L (Alternatively, has limit L), $L \in \mathbb{R}$, if for any $\epsilon > 0$, $\exists N \in \mathbb{P}$ (N depending on ϵ) such that $|a_n - L|$, $\forall n \geq N$

- If $\{a_n\}$ has a limit, we say that it is convergent if $\{a_n\}$ does not converge, we say it is divergent.
- If $\{a_n\}$ has limit L, we denote this as $\lim_{n\to\infty} a_n$

Theorem 3.3

Let $p \in \mathbb{P}$. Then the sequence $\frac{1}{n^p}$ converges to 0.

Proof. Fix $\epsilon > 0$. By the Archimedean property of $\mathbb{R}$, $\exists N \in \mathbb{P}$ such that $N\epsilon > 1 \implies \epsilon > 1/N$. Thus,

$$\begin{aligned}
&\implies \frac{1}{n^p} \leq \frac{1}{n} \leq \frac{1}{N} \leq \epsilon, \forall n \geq N \\
&\implies 0 < \frac{1}{n^p} < \epsilon \forall n \geq N \\
&\implies \left| \frac{1}{n^p} - 0 \right| < \epsilon
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, by definition, $\lim_{n\to\infty} \frac{1}{n^p} = 0$ $\square$

Theorem 3.4

Let $\{a_n\}$ be a convergent sequence in $\mathbb{R}$. Then it has a unique limit.

Proof. Suppose $\{a_n\}$ has limit L and L'. Fix $\epsilon > 0$. By definition, $\exists N \in \mathbb{P}$ such that $|a_n - L| < \epsilon \forall n \geq N$. Also, $\exists N' \in \mathbb{P}$ such that $|a_n - L'| < \epsilon \forall n \geq N'$ Now, we write,

$$|L - L'| \leq |a_n - L'| + |a_n - L| < 2\epsilon \forall n \geq \max N, N'$$

Since ϵ was arbitrary, $L = L'$ □

Theorem 3.5

Let $\{a_n\}$ and $\{b_n\}$ be a sequence in $\mathbb{R}$ with limits L and T respectively.

- $\{ca_n\}$ is convergent and $\lim_{n\to\infty}(ca_n) = cL$
- $\{a_n + b_n\}$ is also convergent and $\lim_{n\to\infty}(a_n + b_n) = L + T$
- $\{a_n b_n\}$ is also convergent and $\lim_{n\to\infty}(a_n b_n) = LT$
- Suppose $b_n \neq 0$ for all n and that $T \neq 0$. Then the sequence $\{\frac{a_n}{b_n}\}$ is convergent and $\lim_{n\to\infty}\{\frac{a_n}{b_n}\}\frac{L}{T}$

Proof. We will only prove part third. As $\{a_n\}$ and $\{b_n\}$ converge, they are bounded. So, $\exists c_1, c_2 > 0$ such that $|a_n| \leq c_1$ and $|b_n| \leq c_2 \forall n$

$$\begin{aligned} a_n b_n - LM &= (a_n b_n - a_n M) + (a_n M - LM) \\ |a_n b_n - LM| &\leq |a_n||b_n - M| + |M||a_n - L| \\ &= c_1|b_n - M| + M'|a_n - L| \end{aligned}$$

Where $M' := \begin{cases} |M| \text{ if } M \neq 0 \\ 1 \text{ if } M = 0 \end{cases}$ By definition,

$$\begin{aligned} &\exists N_1, \text{ such that } |a_n - L'| < \frac{\epsilon}{2M'} \forall n \geq N_1 \\ &\exists N_2, \text{ such that } |b_n - M| < \frac{\epsilon}{2c_1} \forall n \geq N_2 \end{aligned}$$

From these equations, $|a_n b_n - LM| < \epsilon$ for all $n \geq \max N_1, N_2$ □

Problem (Example)

Consider $a_n = \frac{n^2}{n+1} - \frac{n^2+1}{n}$, does $\{a_n\}$ converge?

Proof.

$$\begin{aligned} \{a_n\} &= \frac{n^2}{n+1} - \frac{n^2+1}{n} \\ &= \frac{n^3 - (n+1)(n^2+1)}{n^2+n} \qquad = \frac{-n^2 - n - 1}{n^2 + n} = \frac{-1 - 1/n - 1/n^2}{1 + 1/n} \end{aligned}$$

By theorem of algebraic combination of convergence of sequences, and since limit of denominator $\neq 0$, $\{a_n\}$ converges to the limit 1 □

Definition 3.6. Let $\{a_n\}$ be a sequence in $\mathbb{R}$. We say that $\{a_n\}$ is monotone if

- $a_1 \leq a_2 \leq \cdots$ (monotonic increasing)
- $a_1 \geq a_2 \geq \cdots$ (monotonic decreasing)

Recall that if $S \subset \mathbb{R}$ such that $S \neq \phi$ and bounded above, its **LUB** is unique. This is called the **supremum** of S, denoted by $\sup S$. Similarly, if S is bounded below, it has a unique **GLB** and this is called the **infimum** of S, denoted by $\inf S$

Theorem 3.7 (Monotone Convergence Theorem)
A monotone bounded sequence is convergent.

Proof. We have two cases. We will only show when the sequence is monotone decreasing. The other part is similar and left as an exercise.
Case 1: $\{a_n\}$ is monotone decreasing.
The set $S = \{a_n : n \in \mathbb{P}\}$ is bounded below. Let I be the $\inf S$. Fix $\epsilon > 0$. By definiton, $I + \epsilon$ is **not** a lower bound. So, $\exists N$ such that $a_N < I + \epsilon$

$$\begin{aligned} &\implies I - \epsilon < a_n < I + \epsilon \forall n \geq N \text{ (Because sequence is monotone)} \\ &\implies |a_n - I| < \epsilon \forall n \geq N \end{aligned}$$

Since ϵ was arbitrary, we are done! □

§4 Infinite Series

Definition 4.1. Let $\mathbb{F}$ be a field. An infinite series is an expression of the form $\sum_{n=1}^{\infty} a_n$, where $a_1, a_2, \cdots \in \mathbb{F}$

Associated with a series of the above form, we have a sequence $\{S_n\}$, where $S_n := \sum_{k=1}^{n} a_k$. This is called the nth **partial sum**.

Definition 4.2. Let $\sum_{n=1}^{\infty} a_n$ be a series, $a_n \in \mathbb{R} \forall n$. We say that this series converges if the associated sequence S_n converges. The limit $L := \lim_{n \to \infty} S_n$ is called the sum of the series.

Example: $1 - 1 + 1 - 1 + \cdots$

$$S_n = \begin{cases} 1, \text{if } n \text{ is odd} \\ 0, \text{if } n \text{ is even} \end{cases}$$

In this example, we cannot define the sum because S_n does not converge.

Definition 4.3. We say that a real series $\sum_{n=1}^{\infty} a_n$ is a telescopic series if $\exists$ a real sequence $\{b_n\}$ such that $a_n = b_n - b_{n+1} \forall n \in \mathbb{P}$

- $S_1 = a_1 = b_1 - b_2$
- $S_2 = a_1 + a_2 = b_1 - b_3$
- $S_n = b_1 - b_{n+1}$

Theorem 4.4

Let $\sum_{n=1}^{\infty} a_n$ be a telescoping series. Let $\{b_n\}$ such that $a_n = b_n - b_{n+1} \forall n$. The series converges iff $\{b_n\}$ converges. In this case, $sum_{n=1}^{\infty} a_n = b_1 - \lim_{n\to\infty} b_n$

Proof. We will first prove a lemma.
Lemma: Fix $k \in \mathbb{P}$. Let $\{b_n\}$ be a sequence that converges to L. Let $\alpha_n := b_{n+k} \forall n$, then $\{\alpha_n\}$ converges to L.
Sketch of Proof: Given $\epsilon > 0$, pick N to be the number for $\{b_n\}$. Then the same N works for $\{\alpha_n\}$ because $n + k > n \geq N$, which will deliver the proof.

$$
\begin{aligned}
\sum_{n=1}^{\infty} a_n &\Leftrightarrow \{S_n\} \\
&\Leftrightarrow \{b_1 - b_{n+1}\} \\
&\Leftrightarrow \{-(b_1 - b_{n+1} + b_1)\} \\
&\Leftrightarrow \{b_{n+1}\} \\
&\Leftrightarrow \{b_n\}
\end{aligned}
$$

Where the above denotes that RHS converges $\Leftrightarrow$ LHS converges. Hence, its sum is $(b_1 - \lim_{n\to\infty} b_n)$ □

§4.1 Tests for convergence

Theorem 4.5 (Geometric Series Test)

Series of the form $\sum_{n=1}^{\infty} ar^{n-1}$, where, $a, r \in \mathbb{R}$. Read section 10.8 from **Apostol**.

Theorem 4.6 (Divergence Test)

Let $\sum_{n=1}^{\infty} a_n$ be a real series. If $\sum_{n=1}^{\infty} a_n$ exists then the sequence of nth terms is convergent and $\lim_{n\to\infty} a_n = 0$.If

- $\{a_n\}$ does not converge, or
- $\{a_n\}$ converges but not to 0

We immediately deduce that $\sum_{n=1}^{\infty} a_n$ does not converge. Hence the name **Divergence Test**

Theorem 4.7 (p series test)

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p \in \mathbb{Q}$ converges $\leftrightarrow p > 1$

Theorem 4.8

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be real converging series. Then,

- Let $\alpha \in \mathbb{R}$. Then the series $\sum_{n=1}^{\infty} \alpha a_n$ converges and the sum is $\alpha \sum_{n=1}^{\infty} a_n$
- $\sum_{n=1}^{\infty} a_n + b_n$ converges and the sum is $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

Proof. Let $S_n^{(a)}$ and $S_n^{(b)}$ be partial sums of the two series.

- $\sum_{j=1}^{n} \alpha S_n^{(a)}$. By an earlier theorem, this converges to $\alpha \lim_{n\to\infty} S_n^{(a)}$
- $\sum_{j=1}^{n} a_j + b_j = S_n^{(a)} + S_n^{(b)}$. By an earlier theorem, $\{S_n^{(a)} + S_n^{(b)}\}$ converges to $\lim_{n\to\infty} S_n^{(a)} + \lim_{n\to\infty} S_n^{(b)}$

□

Tests for non-negative series

Theorem 4.9 (Comparision Test)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be non negative series. Suppose $\exists N \in \mathbb{P}$ such that $a_n \leq b_n \forall n \geq N$

- If $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$
- if $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$

Example: Suppose $a_n = \frac{1}{\sqrt{3n^2+7n-1}}$ Does $\sum_{n=1}^{\infty} a_n$ converge?
Hint: Develop an intuition that $\frac{1}{\sqrt{3n^2+7n-1}}$ is **like** $\frac{1}{\sqrt{3}n}$

- By p-series and linearity, looks like $\sum_{n=1}^{\infty} a_n$ diverges
- Need to find **smaller** divergent series

Using the above information, note that

$$3n^2 + 7n - 1 \leq 3n^2 + 7n \leq 4n^2$$

for all $n \geq 7$. Thus, $a_n \geq \frac{1}{2n}$. Since this diverges, we are done by comparision test.
Proof of comparision test second part.

Proof. We will first prove a lemma.
Lemma: Let $\sum_{n=1}^{\infty} a_n$ be a real series. Let $N \in \mathbb{N}\ \{0,1\}$. Then the series $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=N}^{\infty} a_n$ converges.

By our lemma, we can assume WLOG that $a_n \leq b_n \forall n$ (if not, then consider the series $\sum_{n=1}^{\infty} a'_n$ and $\sum_{n=1}^{\infty} b'_n$ where $a'_n = a_{N+n-1}$ and $b'_n = b_{N+n-1}$)

Note that $\{S_n^{(a)}\}$ and $\{S_n^{(b)}\}$ are monotonic increasing sequences. By assumption, $\{S_n^{(a)}\}$ is not bounded above. Now assume that $\{S_n^{(b)}\}$ is bounded. Then $\exists c > 0$ such that $0 \leq S_n^{(b)} \leq c$. But then, $S_n^{(a)} \leq S_n^{(b)} \leq c$ because $a_n \leq b_n$. This is a contradiction. □

Theorem 4.10 (Ratio Test)

Let $\sum_{n=1}^{\infty} a_n$ be a series with only positive terms. Assume $\{\frac{a_{n+1}}{a_n}\}$ converges. Let L denote the limit.

- If $L < 1$, the series converges
- If $L > 1$, the series diverges
- If $L = 1$, we cannot draw any conclusions

Ratio test is effective for a series like $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ where a_n is hard to analyse, but $\frac{a_{n+1}}{a_n}$ is easy to.

Theorem 4.11

Let $a > 1$ be a real number. Let $q \in \mathbb{Q}$ such that $q \geq 0$, then $\lim_{n\to\infty} \frac{n^q}{a^n} = 0$

Sketch. Consider $\sum_{n=1}^{\infty} \frac{n^q}{a^n}$ and show that $L = 1/a < 1$. Thus, $\sum_{n=1}^{\infty} \frac{n^q}{a^n}$ converges, and by an above theorem, $\lim_{n\to\infty} \frac{n^q}{a^n} = 0$

- The sketch can be made more rigorous if $q \in \mathbb{N}$
- The general **Slogan** is that "a^n grows faster than n^q"

□

Proof of Ratio Test. We will proceed case wise.
Case 1: $L < 1$
Fix $\epsilon > 0$ such that $L + \epsilon < 1$ $\exists N \in \mathbb{P}$ such that,

$$L - \epsilon < \frac{a_{n+1}}{a_n} < L + \epsilon \forall n \geq N$$

By a simple induction, we can prove that $a_{N+k} < a_N(L+\epsilon)^k$. Now comparing to the geometric series, and using the fact that $L + \epsilon < 1$, $\sum_{n=1}^{\infty} a_n$ converges.
Case 2: $L > 1$ (Left as an exercise)
Case 3: $L = 1$
Notice that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges □

Theorem 4.12 (Root Test)

Let $\sum_{n=1}^{\infty} a_n$ be a non negative series. Assume that $\lim_{n\to\infty} a_n^{1/n}$ exists. Call this limit R

- If $R < 1$, the series converges
- If $R > 1$, the series diverges
- If $R = 1$, then no conclusions

§4.2 Order for trying tests

If the non negative series $\sum_{n=1}^{\infty} a_n$ is not a telescopic or a geometric series, useful order for trying out tests is:

- Divergence Test
- p-series Test
- Ratio/Root Test
- Comparision Test

Example: $\sum_{n=1}^{\infty} \frac{n(2+(-1)^n)^n}{5^n}$
Hint: Note that the Divergence Test, p-series Test and Ratio/Root test does not work. Hence, use the Comparision Test.

Absolute Convergence

Definition 4.13. Given a real series $\sum_{n=1}^{\infty} a_n$, we say this series **converges absolutely** if the associated series $\sum_{n=1}^{\infty} |a_n|$ converges

Remark: Every test for **non-negative** series is also a test for absolute convergence!
Conditional Convergence: Series which converges but not absolutely.

§5 Limits of Function

- What do we mean by $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- One way to understand is to take a sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = 0$ and observe that $\lim_{n\to\infty} \frac{\sin x_n}{x_n} = 1$, where we must **only** consider sequences $\{x_n\}$ such that $x_n \neq 0 \forall n$
- Also, we must get $\lim_{n\to\infty} \frac{\sin x_n}{x_n}$ to exist, and be independent of any $\{x_n\}$ as above.

Definition 5.1. Let I be an open interval and $p \in I$. Let f be a real valued funciton such that $f(x)$ is defined $\forall x \in I$, except perhaps at $x = p$. Let $A \in \mathbb{R}$, we say that the limit of $f(x)$ as x approaches p is A, denoted by $\lim_{x\to p} f(x) = A$, if for any sequence $\{x_n\} \subset I - \{p\}$ such that $\lim_{n\to\infty} x_n = p$, we have that $\lim_{n\to\infty} f(x_n) = A$

This is an excellent definition in the sense that theorems whose burden can be passed onto sequences and series are easily proven. For an example, consider these theorems.

Theorem 5.2

Let I, p, f be as in the above definition. Let A_1 and A_2 be such that $\lim_{x\to p} f(x) = A_1$ and $\lim_{x\to p} f(x) = A_2$, then $A_1 = A_2$.

Theorem 5.3

Let I be an open interval and $p \in I$. Let f and g be real valued functions such that f and g are defined $\forall x \in I$, except perhaps at $x = p$. Let $c \in \mathbb{R}$. Suppose that $\lim_{x \to p} f(x) = A$ and $\lim_{x \to p} g(x) = B$. Then,

- $cf(x)$ has a limit as x approaches p, and $\lim_{x \to p} cf(x) = cA$
- $f + g$ has a limit as x approaches p, and $\lim_{x \to p} f(x) + g(x) = A + B$
- fg has a limit as x approaches p, and $\lim_{x \to p} f(x)g(x) = AB$

These two theorem are direct corollaries of the corresponding theorems from sequences and series.
The sequencial definition has the problem that given a f and p, we would have to analyse every $\{f(x_n)\}$ for every $\{x_n\}$. This seems **HARD**!
If we fix $\epsilon > 0$ and want to know for which x, $|f(x) - A| < \epsilon$, instead of looking at $x_n \forall n$ sufficiently large, we consider all x sufficiently close to p, and not equal to p. This allows us to get rid of sequences $\{x_n\} \subset I - \{p\}$

Definition 5.4 (ϵ-δ definition)**.** Let I, p, f be as above. Let $A \in \mathbb{R}$. We say that the limit of $f(x)$ as x approaches p is A, if given $\epsilon > 0$, $\exists \delta > 0$ (which depends on ϵ) such that,

$$|f(x) - A| < \epsilon \text{ whenever } x \in I \text{ and } 0 < |x - p| < \delta$$

Two standard limits

Theorem 5.5

$\lim_{x \to p} \frac{\sin x}{x}$ exists and equals 1

Sketch. Won't prove; notice that the proof done in high school can be recasted as demonstration of the sequence definition. The **Missing Ingredient** is the rigorous proof of **Squeeze Theorem** □

Theorem 5.6

Let $k \in \mathbb{P}$ and $p \in \mathbb{R}$. Then $\lim_{x \to p} x^k$ exists and equals p^k

Proof. We proceed by cases.
Case 1: $k = 1$. Nothing to prove **Case 2:** $k \neq 1$. Fix a sequence $\{x_n\} \subset \mathbb{R} - \{p\}$ such that $\lim_{n \to \infty} x_n = p$. Thus, there exists $N_1 \in \mathbb{P}$ such that $|x_n - p| < 1$. Using the triangle inequality, we get that

$$|x_n| < 1 + |p| \forall n \geq N_1$$

Fix a sequence $\{x_n\} \subset \mathbb{R} - \{p\}$ such that $\lim_{n \to \infty} x_n = p$. Fix $\epsilon > 0$. Thus, $\exists N_2 \in \mathbb{P}$ such that,

$$|x_n - p| < \frac{\epsilon}{k(|p| + 1)^{k-1}} \forall n \geq N_2$$

Thus, we get that,

$$
\begin{aligned}
|x^k - p^k| &\le |x_n - p| \sum_{j=0}^{k-1} |x_n|^{k-j-1} |p|^j \\
&\le \frac{\epsilon}{k(|p|+1)^{k-1}} \sum_{j=0}^{k-1} (|p|+1)^{k-j-1} |p|^j \\
&\le k(|p|+1)^{k-1} \forall n \ge \max N_1, N_2
\end{aligned}
$$

Because $\epsilon > 0$ was arbitrary and $\{x_n\} \in \mathbb{R} - \{p\}$ was also arbitrary, we are done. □

Theorem 5.7 (Equivalence of the two definitions)
Let I, p, f be as in our definitions. Suppose that for any $\{x_n\} \subset I - \{p\}$ such that $\lim_{n\to\infty} x_n = p$, we have $\lim_{n\to\infty} f(x_n) = A$, then given $\epsilon > 0$, $\exists \delta > 0$ (depends of ϵ) such that

$$|f(x) - A| < \epsilon \text{ whenever } x \in I \text{ and } 0 < |x - p| < \delta$$

and vice versa.

Proof. We will prove the **vice versa** implication. Fix $\epsilon > 0$ and let δ be as given as above. Let $\{x_n\} \subset I - \{p\}$ such that $\lim_{n\to\infty} x_n = p$. By definition, $\exists N_1 \in \mathbb{P}$ such that,

$$0 < |x - p| < \delta \implies |f(x) - A| < \epsilon$$

Because $\epsilon > 0$ was arbitrary, $\lim_{n\to\infty} f(x_n) = A$. As $\{x_n\}$ was arbitrary, **vice versa** follows □

§6 04/10/23

When we defined limits, we had,

- an **open interval** I and $p \in I$
- an $\mathbb{R}$ valued function f such that $f(x)$ is defined $\forall x \in I$ except perhaps at $x = p$

Reason for the choice of an open interval I are:

1. It is the closest to the definition in Apostol
2. The limit concept is key to **differentiation** , $I \subseteq_{\text{open}} \mathbb{R}$ admits generalization to 2 or more variable.

Let us recall the definition of limit. We say that $\lim_{x\to p} f(x) = A$, iff, given $\epsilon > 0$, $\exists \delta > 0$ (depending on ϵ) such that, $|f(x) - A| < \epsilon$ whenever $\boxed{x \in I}$ AND $0 < |x - p| < \delta$

Notice that the boxed words ensure that we can define a limit even if I is not an open interval.

Now, we will begin our discussion on continuity. Let I be an interval and $f : I \to \mathbb{R}$ be a function.

Definition 6.1. Let f and I be as above. Let $p \in I$. We say that f **is continuous at** p if, $\lim_{x\to p} f(x) = f(p)$. In general, we say that f is continuous if f is continuous $\forall p \in I$

Definition 6.2 ("ϵ- δ" definition of continuity)**.** Let f and I be as above. Let $p \in I$. We say that f is continuous at p, if given any $\epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(p)| < \epsilon$ whenever $x \in I$ AND $|x - p| < \delta$

Note that,

1. Sequential definition of limit (used in proving the theorem on sum and products of limits) is equivalent to the ϵ- δ definition.

2. The above discussion on open interval extends the notion of limits to all intervals.

Theorem 6.3

Let I be an interval, and $p \in I$ and let $f, g : I \to \mathbb{R}$. Let $c \in \mathbb{R}$. Assume f and g are continuous at p. Then,

- $f + g$ is continuous at p
- fg is continuous at p

§7 06/10/23

Last time

1. Introduced the concept of continuity
2. Proved the algebra of continuity theorem.

Today

We will take the theorem on algebra of continuity a bit further. Namely, we will add the following propositions.

Theorem 7.1 (Continuation of the algebra of continuity)

Consider the same hypothesis as in the earlier theorem, we have that,

- cf is continuous at p
- Suppose that $g(p) \neq 0$. Then, $\exists r > 0$, such that, $g(x) \neq 0 \forall x \in (-r+p, r+p) \cap I$, and f/g is continuous at p

§7.1 Examples

First note that the previous theorem allows us to VASTLY expand our stock of continuous functions.

- Any constant function is continuous on $\mathbb{R}$
- The function $f(x) = x$ (which is sometimes denoted by $\mathrm{id}_{\mathbb{R}}$) is continuous on $\mathbb{R}$ (Note that $\epsilon = \delta$ works)
- We can repeatedly apply the previous theorem on the identity function to get that **Any polynomial is continuous on** $\mathbb{R}$

- (Rational Functions) Let f and g be polynomials. Let $I \subseteq \mathbb{R}$ be an interval such that $g(x) \neq 0\ \forall x \in I$. Then f/g (defined on I) is continuous on I.
- Fix $p \in \mathbb{R}$. We will try to show that $\sin x$ is continous at p. For that, we will use the following lemma.

Lemma 7.2

Suppose that f is a function (defined on I) such that $|f(a) - f(b)| \leq c|x - p|$ $\forall a, b \in I$, then this function is continous.

Now we compute,

$$|\sin(\theta + p) - \sin p| = |\sin\theta\cos p + \cos\theta\sin p - \sin p|$$
$$\leq |\sin\theta||\cos p| + |\sin p||\cos\theta| \qquad \leq |\sin\theta| + 2\sin^2\theta/2$$

Now suppose that $\theta \neq 0$, then by the above lemma,

$$\sin(\theta + p) - \sin p \leq |\theta| \left\{ \left| \frac{\sin\theta}{\theta} \right| + \frac{2\sin\theta/2}{|\theta|} \right\}$$

Now use the ϵ- δ definition with $\epsilon = 1$ to get a known c. Finish it as a homework.

§8 09/10/23

Last time

Vastly expanded our stock of continuous functions

Today

Let $S_1, S_2 \subset \mathbb{R}$ be non empty sets and $f : S_1 \to \mathbb{R}$, $g : S_2 \to \mathbb{R}$. We will denote as $f(S_1)$ the set

$$f(S_1) := \{f(x) : x \in S_1\}$$

Suppose $f(S_1)$ is a subset of S_2, then we can define a function which is denoted as $g \circ f$, and it is defined as

$$g \circ f := g(f(x))$$

Theorem 8.1 (Continuity of Composition)

Let $I_1, I_2 \subset \mathbb{R}$ be intervals, let $f : I_1 \to \mathbb{R}$ and $g : I_2 \to \mathbb{R}$. Suppose that $f(I_1) \subseteq I_2$ Let $p \in I_1$. Suppose that f is continuous at p and g is continuous at $f(p)$. Then the function $g \circ f$ is continuous at p

Proof. Fix an $\epsilon > 0$. Write $q := f(p)$. Then $\exists \sigma > 0$ such that $|g(y) - g(q)| < \epsilon$ whenever $y \in I_2$ AND $|y - q| < \sigma$.

By the continuity of f at p, $\exists \delta > 0$, such that $|f(x) - f(p)| < \sigma$ whenever $x \in I_1$ and $|x - p| < \delta$.

By our assumption, whenever $x \in I_1 \implies f(x) \in I_2$. From the above statements, taking $y = f(x)$, we get that $|g(f(x)) - g(f(p))| < \epsilon$ whenever $x \in I_1$ and $|x - p| < \delta$ $\square$

Theorem 8.2 (Bolzano Theorem)

Let $a < b \in \mathbb{R}$. Let $f : a, b \to \mathbb{R}$ be continuous. If $f(a)$ and $f(b)$ have opposite signs, then $\exists c \in (a, b)$ such that $f(c) = 0$

Proof. We will begin by first proving a lemma.

Lemma 8.3 (Sign Preserving Property)

Let I be an interval, $g : I \to \mathbb{R}$ and let $c \in I$. Suppose that g is continuous at c. Suppose that $g(c) \neq 0$. Then $\exists r > 0$ such that $g(x)$ has the same sign as $g(c)$, $\forall x \in (c - r, c + r) \cap I$.

Proof of Lemma. We will only prove for the case $g(c) < 0$. The other case is left as an exercise. By continuity, $\exists r > 0$ such that $|g(x) - g(c)| < |g(c)|/2 \implies g(x) - g(c) < -g(c)/2$ (because $g(c) < 0$). Rearranging, we get that $g(x) < g(c)/2 < 0$. $\square$

Having done this, define,

$$S := \{x \in [\ a, b] \ : f(x) \leq 0\}$$

We immediately notice the following:

- $S \neq \phi$ (because $a \in S$)
- S is bounded

Hence, by the LUB property, $c := \sup S$ exists. $\square$

§9 10/10/2023

Last time

1. Continuity of composition of functions
2. Began proving Bolzano's Theorem

Today

Continuation of Yesterday's proof. Assume that $f(c) \neq 0$. Then $f(c) < 0$ or $f(c) > 0$. We will get a contradiction in either case.
$f(c) > 0 \implies \exists r > 0$ such that,

$$f(x) > 0 \ \forall x \in (c - r, c + r) \cap [a, b]$$

Note that $a \notin (c - r, c + r) \implies a < c - r < c$, i.e, $c - r$ is not even an upper bound of S. Thus, $\exists x' : c - r < x' \leq c$ such that $f(x') \leq 0$. Contradiction!
Do the $f(c) < 0$ case as a homework. $\square$

Theorem 9.1

Let I be an interval, and let $f : I \to \mathbb{R}$ be continuous. Let $x_1 < x_2 \in I$. Suppose that $f(x_1) \neq f(x_2)$. Then, for any $k \in \mathbb{R}$ lying between $f(x_1)$ and $f(x_2)$, $\exists c \in (x_1, x_2)$ such that $f(c) = k$

Proof. Let $g(x) = f(x)|_{[x_1,x_2]} - k : [x_1, x_2] \to \mathbb{R}$. Clearly, this is continuous. Also notice that $\max[f(x_1), f(x_2)] > k$ and $\min[f(x_1), f(x_2)] < k$. So, $g(x_1)$ and $g(x_2)$ have opposite signs. Thus, by using Bolzano's theorem, $\exists c \in (x_1, x_2)$ such that $g(c) = 0$. But, this gives that $f(c) = g(c) + k = k$ $\square$

§9.1 Interlude

A and B are non empty sets. We define the **Cartesian Product** as,

$$A \times B := \{(a, b) : a \in A \text{ and } b \in B\}$$

Notice that the Cartesian Product is a collection, but is this a set?

Answer 1

Take as an axiom that $A \times B$ is a set.

Answer 2

$A \times B$ can be constructed as a set thanks to something called the **Pair Set Axiom**

Definition 9.2. Let A and B be two non empty sets. A relation between two sets, say $\to$, is a subset $S_\to \subseteq A \times B$. We say that $a \to b \Leftrightarrow (a, b) \in S_\to$

§10 11/10/23

Last time

- Bolzano Theorem
- Cartesian Product

Today

Example 10.1

$>$ is a relation between $\mathbb{R}$ and itself. $y > x \Leftrightarrow y - x \in \mathbb{R}^+$. More formally,

$$S_> = \{(y, x) \in \mathbb{R} \times \mathbb{R} : y - x \in \mathbb{R}^+\}$$

Working inductively using the above argument, given non empty sets, $A_1, \cdots, A_n$, we can define

$$A_1 \times A_2 \times \cdots \times A_n := \{(a_1, \cdots, a_n) : a_j \in A_j, j = 1, \cdots, n\}$$

Definition 10.2. Let A and B be non empty sets. A function $f : A \to B$ is a relation $\Gamma_f \subseteq A \times B$ such that,

1. For each $a \in A$, $\exists (a, b) \in \Gamma_f$ (A is the domain of f)
2. For each $a \in A$, $\exists$ a **unique** $b \in B$ such that $(a, b) \in \Gamma_f$ ($b := f(a)$)

Theorem 10.3

Let $f : [-1, 1] \to [-1, 1]$. Suppose f is continuous. Then, $\exists c \in [-1, 1]$ such that, $f(c) = c$

Proof. We proceed by cases.

Case 1: $f(-1) = -1$ or $f(1) = 1$. Trivial. Nothing to prove.

Case 2: $f(-1) \neq -1$ and $f(1) \neq 1$. Define $g(x) = f(x) - x \forall x \in [-1, 1]$. Note that g is continuous. Further,

$$\begin{aligned} g(-1) &= f(-1) - (-1) && > 0 \\ g(1) &= f(1) - 1 && < 0 \end{aligned}$$

Now, by Bolzano's theorem, $\exists c \in (-1, 1)$ such that $g(c) = f(c) - c = 0 \implies f(c) = c$ $\square$

A Short note on the relevance of the above theorem

We proved the $n = 1$ case of the **Brower's Fixed Point Theorem**. To understand the theorem, consider the set,

$$B_n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \sqrt{\sum_{j=1}^{n} x_j^2 \leq 1}\}$$

Theorem 10.4

Any continuous function from B_n to itself has a fixed point i.e $\exists$ a point $(x_1, \cdots, x_n)$ such that, $f(x_1, \cdots, x_n) = (x_1, \cdots, x_n)$

There are many practical applications of this theorem, one of them being **PAGE RANK** of a search engine.

§11 13/10/20

Last time

- More about relations and functions
- Brouwer's Fixed Point Theorem ($\underline{n = 1 \text{ in syllabus}}$)

Today

Let $S \neq \phi$ be a set. Let f be a function $f : S \to \mathbb{R}$ such that

$$\{f(x) \in \mathbb{R} : x \in S\} \text{ (range of } f)$$

is bounded above and below.

- Such an f is called a <u>bounded</u> function
- inf(range) and sup(range) exists.
- Abbreviate terms as inf f and sup f

Theorem 11.1

Let $a < b \in \mathbb{R}$. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is bounded

Proof. Read Theorem 3.11 from Apostol □

Corollary 11.2

Let $[a, b]$ and f be as above. Then,

1. $\inf f$ and $\sup f$ exists
2. $\exists c, d \in [a, b]$ such that, $f(c) = \inf f$ and $f(d) = \sup f$. Thus, f has a point of absolute maximum and a point of absolute minimum.

Proof. For the first part, by the previous theorem, range(f) is bounded and non empty. the first claim follows from the GLB property and 2nd from the LUB property.

For the second part, it suffices to show that $\exists d$ such that, $f(d) = \sup f$. Suppose not, and let $M := \sup f$. Then $M - f(x) > 0 \forall x \in [a, b]$. By our theorem on continuity of algebraic combinations,

$$g(x) := \frac{1}{M - f(x)} \text{ (Continuous on } [a, b])$$

By the last theorem, $\exists c > 0$ such that,

$$\begin{aligned} 0 < \frac{1}{M - f(x)} &\leq C \ \forall x \in [a, b] \\ \implies M - f(x) &\geq 1/c \\ \implies f(x) &\leq M - 1/c \ \forall x \in [a, b] \end{aligned}$$

But $(M - 1/c)$ is not an upper bound of range(f), which contradixts the last line. □

§12 16/10/23

Last time

1. If $f : [a, b] \to_{\text{continuous}} \mathbb{R}$, then f is bounded
2. Corollary: Existance of points of absolute maximum and minimum for above class of functions

Today

Definition 12.1. Let S be a non empty set and $f : S \to \mathbb{R}$. A point $p \in S$ is called a point of **absolute maximum** (respectively, **absolute minimum**) if, $f(p) \geq f(x) \forall x \in S$ (respectively, $f(p) \leq f(x) \forall x \in S$)

§13 Differentiability

Definition 13.1. Let $I \subseteq \mathbb{R}$ be an open interval and $x \in I$. Let $f : I \to \mathbb{R}$. We say that f is differentiable at x if the following limit exists.

$$\lim_{x \to 0} \frac{f(x+h) - f(x)}{h} := f'(x)$$

- If f is differentiable at each $x \in I$, we say that f is **differentiable**
- In the above case, f induces the function $f' : I \to \mathbb{R}$

Theorem 13.2

Let I be an open interval, $a \in I$ and $f : I \to \mathbb{R}$. If f is differentiable at a, then it is continuous at a

Proof. Write

$$f(x) - f(a) = (x-a)\frac{f(x) - f(a)}{x - a} \forall x \in I - \{a\}$$

Since $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists, by our theorem on limit of products, this implies,

$$\lim_{x \to a}(f(x) - f(a)) = 0$$
$$\implies \lim_{x \to a} f(x) = f(a)$$

□

Example 13.3

Show that $f(x) = \cos x$ is differentiable

Solution.

$$\begin{aligned}
\frac{\cos(x+h) - \cos x}{h} &= \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \frac{-\cos x(1 - \cos h)}{h} - \sin x \left(\frac{\sin h}{h}\right) \\
&= -2\cos x \frac{h}{4}\left(\frac{\sin^2(h/2)}{(h/2)^2}\right) - \sin x \left(\frac{\sin h}{h}\right)
\end{aligned}$$

Finish this argument as an homework. □

§14 18/10/23

Last time

Introduction to differentiation.

Today

Theorem 14.1

Let $I \subseteq \mathbb{R}$ be an open interval. Let a be a point in I and let $f, g : I \to \mathbb{R}$. Let $c_1, c_2 \in \mathbb{R}$. Assume that f and g are differentiable at a. Then

- $c_1 f + c_2 g$ is differentiable at a and $(c_1 f + c_2 g)'(a) = c_1 f'(a) + c_2 g'(a)$
- fg is differentiable at a and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$
- Assuming that $g(a) \neq 0$ Then $\exists r > 0$ such that $g(x) \neq 0 \ \forall \ (a - r, a + r)$. Moreover, f/g is differentiable at a and

$$\left(\frac{f}{g}\right)' = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

Proof of part 2. We compute,

$$\begin{aligned}\frac{fg(a+h) - fg(a)}{h} &= \frac{f(a+h)(g(a+h) - g(a)) + g(a)(f(a+h) - f(a))}{h} \\ &= f(a+h)\frac{g(a+h) - g(a)}{h} + g(a)\frac{f(a+h) - f(a)}{h}\end{aligned}$$

Also, we know that, $\lim_{h\to 0} f(a+h) = f(a)$. Now, by using limit of algebraic combinations,

$$\lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} = f(a)g'(a) + f(a)g'(a)$$

□

Theorem 14.2 (Chain Rule)

Let $I_1, I_2 \subseteq \mathbb{R}$ be open intervals. Let $f : I_1 \to \mathbb{R}$ and $g : I_2 \to \mathbb{R}$. Let $f(I_1) \subseteq I_2$ and let $F := g \circ f$. Let $a \in I_1$ and suppose f is differentiable at a and let g is differentiable at $f(a)$. Then, F is differentiable at a and

$$F'(a) = g'(f(a))f'(a)$$

How "Not" to prove the chain rule

Here's a faulty argument.

1. Write:
$$\frac{F(x) - F(a)}{x - a} = \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a}$$

2. We <u>could</u> justify writing $y := f(x)$ and stating
$$\lim_{y\to f(a)} (y - f(a)) = \lim_{x\to a} (f(x) - f(a)) = 0$$
The first limit is self evident and the second one follows by continuity of f

3. But EQUATING:
$$\frac{g(y) - g(f(a))}{y - f(a)} = \frac{g(f(x)) - g(f(a))}{f(x) - f(a)}$$
is a problen because there could, for each $r > 0$, $\exists x_r \in (a - r, a + r) \cap I_1$, $x_r \neq a$ such that $f(x_r) = f(a)$.

Self-tudy

Work on all "applied" problems in 4.12

§15 20/10/23

Last time

1. Differentiation of algebraic combinations
2. Chain rule

Today

Definition 15.1. Let S be a non empty set and let $f : S \to \mathbb{R}$. A point $c \in S$ is called

i) a point of **local** or **relative** maximum if $\exists r > 0$ such that $f(c) \geq f(x) \forall x \in S \cap (c - r, c + r)$

ii) a point of **local** or **relative** maximum if $\exists r > 0$ such that $f(c) \leq f(x) \forall x \in S \cap (c - r, c + r)$

Theorem 15.2

Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$. Let f be differentiable. If $c \in I$ is a point of local maximum or local minimum, then $f'(c) = 0$

Proof. Let c is a point of local extremum. then, $\exists R > 0$ such that ,
$$f(c) \geq 0 \text{ OR}$$
$$f(c) \leq 0$$
Suppose $f'(c) \neq 0$. Then, it suffices to assume $f'(c) > 0$ (WHY?). Define $D_f : I - \{c\} \to \mathbb{R}$, such that,
$$D_f(x) := \frac{f(x) - f(c)}{x - c}$$
By assumption, $\lim_{x \to c} D_f(x)$ exists ($= f'(c)$). Now, by the sign preserving property, $\exists r \in (0, R)$ such that,
$$(c - r, c + r) \subseteq I \text{ AND } D_f(x) > 0 \forall x : 0 < |x - c| < r$$
□

§16 23/10/2023

§16.1 Last time

Point of local maximum or minimum algorithm for determining absolute max and min values

§16.2 Today

Theorem 16.1 (Rolle's Theorem)

Let $a < b \in \mathbb{R}$ and $f : [\,a, b] \ \to \mathbb{R}$ be continuous. Assume f is differentiable on (a, b) and $f(a) = f(b)$. Then $\exists c \in (a, b)$ such that $f'(c) = 0$

Proof. Assume $f'(x) \neq 0 \forall x \in (a, b)$. By assumption, f has a point of absolute maximum and also a point of absolute minimum. By the previous theorem, none of the latter are in (a, b). Thus,

$$\min\{f(a), f(b)\} \leq f(x) \leq \max\{f(a), f(b)\} \forall x \in [\,a, b]$$

This implies that f is a constant, thus, $f'(x) = 0 \forall x \in (a, b)$, a contradiction. □

Theorem 16.2 (Lagrange's Mean Value Theorem)

Let $a < b \in \mathbb{R}$ and $f : [\,a, b] \ \to \mathbb{R}$ and f is continuous. Further assume that f is differentiable on (a, b). Then $\exists c \in c(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Sketch of Proof. Apply Rolle's Theorem to $h : [\,a, b] \ \to \mathbb{R}$ where

$$h(x) := (b - a)f(x) - x(f(b) - f(a))$$

□

Corollary 16.3

Let $a < b \in \mathbb{R}$ and $f : [\,a, b] \ \to \mathbb{R}$ is continuous. Suppose f is differentiable on (a, b), then,

- If $f'(x) > 0 \forall x \in (a, b)$, then f is strictly increasing
- If $f'(x) < 0 \forall x \in (a, b)$, then f is strictly decreasing
- If $f'(x) = 0 \forall x \in (a, b)$, then f is constant

Proof. We will only do part (i). Let $x < y \in [\,a, b]$. Apply LMVT to f (first check all conditions to apply LMVT). Thus, $\exists c \in (x, y)$ such that,

$$f(y) - f(x) = f'(c)(y - x) > 0$$

because $f'(x) > 0$ by hypothesis. Thus, $f(y) > f(x)$. Since, $x < y$ are arbitrary, (i) follows. □

Definition 16.4. Let $S \subseteq \mathbb{R}$ be a non empty set. If $f : S \to \mathbb{R}$ is one one, then $\exists f^{-1} : f(S) \to S$ such that , $f(f^{-1}(y)) = y$, $y \in f(S)$

GENERAL EXAMPLES

- With S as above, if f is either strictly increasing or strictly decreasing, f is one-one

Theorem 16.5

Let $I \subseteq \mathbb{R}$ be an interval and let f be constant. If f is one one, then f is either strictly increasing or strictly decreasing.

§17 25/10/2023

Last time

- 2 mean value theorems (Rolle's and LMVT)
- Invertible functions

Today

We begin with an interesting question. If f is invertible, is ontinuity of f inherited by f^{-1}. The answer is that it depends on the domain of f. Here is the appropriate theorem.

Theorem 17.1

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be an invertible funciton. If f is continuous, then so is f^{-1}

The proof of the above theorem was not done in class, but it's a nice exercise to try out (**Hint:** Try proving one one first) Now, we are ready to talk about derivatives of inverses.

Theorem 17.2

Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be an invertible function. Let $a \in I$. Assume f is continuous and f is differentiable at a. If $f'(a) \neq 0$, then,

- $f(I)$ is an open interval
- f^{-1} is differentiable at $f(a)$ and,

$$(f^{-1})'[\ f(a)]\ \frac{1}{f'(a)}$$

A short sketch/hint. Consider a sequence $\{y_n\} \subset J - \{f(a)\}$. We need to analyse if,

$$\frac{f^{-1}(y_n) - f^{-1}(f(a))}{y_n - f(a)} \text{ has a limit}$$

Now set $x_n := f^{-1}(y_n)$, and try it out as an homework. □

Example 17.3

Find the derivate of $\arcsin x$ (Note that in general $\arcsin x$ is defined only if the sin function is restricted to $[2n - 1/2\pi, 2n + 1/2\pi]$)

solution. Note that if we define,

$$g = f^{-1} \text{ where } f(\theta) = \sin\theta \forall \theta \in [-\pi/2, \pi/2]$$

Let $x \in (-1, 1)$. Any such x is of the form $x = \sin\theta$ for a unique $\theta \in (-\pi/2, \pi/2)$. Thus, we can infer

$$(f^{-1}(\sin\theta) = \frac{1}{\cos\theta})$$

Now complete the argument and prove that

$$(f^{-1}(x) = \frac{1}{\sqrt{1-x^2}})$$

□

§18 27/11/23

Last time

- Continuity and Differentiability of inverse in context of function defined on <u>intervals</u>
- Found the derivative of the $\arcsin x$ function

Today

Definition 18.1. Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$. If f is differentiable on I, write $f^{(1)} := f'$ and $f^{(0)} := f$. We say that f has derivatives of order n, $n \in \mathbb{P}$, if $f^{(n-1)}$ exists and is differentiable where $f^{(n)} := \left[f^{(n-1)}\right]'$ for $1 \leq k \leq n$

Integration

(Discussion) We want to make precise the meaning of $\int_a^b f(x)dx$ (from which $\int f(x)dx$ follows) For $f : [a, b] \to \mathbb{R}$ such that $f \geq 0$, <u>want</u>,

$$\int_a^b f(x)dx = \text{ Area } \left[\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq f(x), x \in [a, b]\}\right]$$

Definition 18.2. Let $a < b \in \mathbb{R}$

- A <u>partition</u> of $[a, b]$ is a finite set of real number $x_0, \cdots, x_n$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$
- A function on $[a, b]$ is called a <u>step function</u> if $\exists$ a partition $a = x_0 < x_1 < \cdots < x_n = b$ and $\exists$ $c_1, c_2, \cdots, c_n \in \mathbb{R}$ such that $s(x) = c_j$ if $x \in (x_{j-1}, x_j)$

Theorem 18.3

Let $a < b \in \mathbb{R}$ and let $s, t : [a, b]$ be step functions.

- cS is a step function for any $c \in \mathbb{R}$
- $s + t$ is a step function for any $c \in \mathbb{R}$

§19 30/10/23

Last time

- Motivation for the integral $\int_a^b f(x)dx$ (Specifically, if $f \geq 0$, integral is the area of the function)
- Defined step functions

Today

Definition 19.1. Let $a < b$ and $s : [a.b] \to \mathbb{R}$ be a step function. Let $a = x_0 < x_1 < \cdots < x_n = b$ and $c_1, \cdots, c_n \in \mathbb{R}$ be the data defining S. Then,

$$\int_a^b s(x)dx = \sum_{j=1}^{n} c_j(x_j - x_{j-1})$$

Remarks: A partition determining S is not unique!

Definition 19.2. Let $a < b \in \mathbb{R}$. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Define

$$S_f^+ := \left\{ \int_a^b s(x)dx : \text{ s is a step function and } s \geq f \right\}$$

$$S_f^- := \left\{ \int_a^b s(x)dx : \text{ s is a step function and } s \leq f \right\}$$

- $S_f^+ \neq \phi$ (As f is bounded, $\exists\ M > 0$ such that $-M \leq f(x) \leq M\ \forall x$, $s^+ := M \geq f$, $s_- := -M \leq f$)
- S_f^+ is bounded below (CHECK: $-M(b-a)$ is a lower bound) and similarly S_f^- is bounded above
- $\bar{I} := \inf S_f^+$ (Reimann upper integral)
- $\underline{I} := \sup S_f^-$ (Reimann lower integral)

Definition 19.3. Let $a < b \in \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be a bounded function. We say that f is **Reimann Integrable** denoted $f \in \mathcal{R}[a, b]$ iff $\bar{I}(f) = \underline{I}(f)$. The common value is called the integral of f, denoted $\displaystyle\int_a^b f(x)dx$

§20 03/11/23

Last time

Defined the Reimann Integral

Today

Theorem 20.1

Let $a < b \in \mathbb{R}$ and $s : [a, b] \in \mathbb{R}$ be a step function. Then $\int_a^b s(x)dx$ is independent of partitions determining S

Sketch. **Step 1:** Special case where $\mathcal{P} \subsetneq \mathcal{P}^*$ such that $\mathcal{P}^* - \mathcal{P} = \{c\}$; $\mathcal{P}^*$ and $\mathcal{P}$ determine partition.

$$\mathcal{P} : a = x_0 < x_1 < \cdots < x_n = b$$

and $\exists$ unique j^* such that $1 \leq j^* \leq n$ such that, $c \in (x_{j^*-1}, x_{j^*})$. If

$$\mathcal{P}^* : a = y_0 < y_1 < \cdots < y_{n+1} = b \implies y_j = \begin{cases} x_j, 0 \leq j \leq j^* - 1 \\ c, j = j^* \\ x_{j-1}, j \geq j^* - 1 \end{cases}$$

Now compare,

$$(\mathcal{P}) \int_a^b s(x)dx \text{ and } (\mathcal{P}^*) \int_a^b s(x)dx$$

Step 2: General case. Given determining partitional, $\mathcal{P}_1 \neq \mathcal{P}_2$, define $\mathcal{P}^* = \mathcal{P}_1 \cup \mathcal{P}_2$. Thus, $\mathcal{P}^* \supsetneq \mathcal{P}_1, \mathcal{P}_2$. Now, use induction + Step 1 to show that,

$$(\mathcal{P}_1) \int_a^b s(x)dx = (\mathcal{P}^*) \int_a^b s(x)dx = (\mathcal{P}_2) \int_a^b s(x)dx$$

$\square$

Theorem 20.2

Let $a < b \in \mathbb{R}$, $s, t : [a, b] \to \mathbb{R}$ be step functions. Let $\alpha \in \mathbb{R}$ and $c \in (a, b)$

- (Homogenity) $\int_a^b \alpha s(x)dx = \alpha \int_a^b s(x)dx$
- (Additivity) $\int_a^b (s + t)(x)dx = \int_a^b s(x)dx + \int_a^b t(x)dx$
- (Comparision Theorem) If $s \geq t$, then $\int_a^b s(x)dx \geq \int_a^b t(x)dx$
- (Additivity wrt interval) $\int_a^b s(x)dx = \int_a^c s(x)dx + \int_b^c s(x)dx$
- (Translation invariance) $\int_a^b s(x)dx = \int_{a+\alpha}^{b+\alpha} s(x - \alpha)dx$

Proof. We will only proof third part. Notice that,

$$\int_a^b s(x)dx - \int_a^b t(x)dx = \int_a^b (s - t)(x)dx \geq 0$$

The inequalities follow by the part one and two of the theorem. $\square$

§21 06/11/23

Last time

Properties of integral of step functions.

Today

First, notice that, the value of the integral of the step function given by the general formula is the same as the reimann integral. This fact is a homework assignment.

Theorem 21.1

Let $a < b \in \mathbb{R}$ and let $f, g : [a, b] \to \mathbb{R}$ such that $f, g \in \mathcal{R}[a, b]$. Let $c \in (a, b)$ and $\alpha \in \mathbb{R}$. Then,

- The first three properties of the theorem on 3/11 hold. (with $f + g$, $\alpha f \in \mathcal{R}[a, b]$)
- (Additivity wrt interval) $f|_{[a,c]} \in \mathcal{R}[a, c]$, $f|_{[c,b]} \in \mathcal{R}[c, b]$ and,
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$
- (Translational invariance) $f(x - \alpha) \in \mathcal{R}[a + \alpha, b + \alpha]$ and
$$\int_a^b f(x)dx = \int_{a+\alpha}^{b+\alpha} f(x - \alpha)dx$$

Proofs are lengthy and omitted. Now, proceeding further, we want to ask a question.

Example 21.2 (Question)

Other than step functions, is there some other class of functions that are integrable?

Answer. This question needs some steps.
Step 1:

Definition 21.3. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. We say that f is uniformly continuous if given $\epsilon > 0$, $\exists\ \delta > 0$ $\boxed{\text{depending ONLY on } \epsilon}$ such that,

$$|f(x) - f(y)| < \epsilon \forall x, y \in I \text{ AND such that } |x - y| < \delta$$

(In general, if f is continuous at $p \in I$, δ depends on both ϵ and p)

Step 2: Examples

- The sine function is uniformly continuous. Recall that,
$$|\sin(\theta + x) - \sin x| \le |\sin\theta| + 2\sin^2\theta/2$$
Note that the LHS doesn't have any term involving x, which is enough for uniform continuity (complete the proof)

- $f(x) = x^n$ is not uniformly continuous on $\mathbb{R}$ if $n \in \mathbb{N} - \{0, 1\}$
- $f(x) = x$ is uniformly continuous (given $\epsilon > 0$, $\delta = \epsilon$ works)

$\square$

§22 7/11/23

Last time

Extended properties of integral of step functions to general Reimann Integrable functions. Intro to uniform continuity.

Today

The general question asked was to find a class of Reimann Integrable functions.

Theorem 22.1
Let $a < b \in \mathbb{R}$. If $f : [a, b] \to \mathbb{R}$ is continuous, then it is uniformly continuous.

This is by itself a deep result. We will not prove this, but it is useful to know that the proof re-uses ideas for "boundedness theorem". The following theorem answers our question.

Theorem 22.2
Let $a < b \in \mathbb{R}$. If $f : [a, b] \to \mathbb{R}$ is continous, then $f \in \mathcal{R}[a, b]$

Homework

- READ statement of "small span theorem" (Theorem 3.13 in Apostol)
- Study proof of the above theorem (Theorem 3.14 in Apostol)

Now, we will introduce some conventions. Let $a < b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be bounded.

- If $f \in \mathcal{R}[a, b]$, set $\int_b^a f(x)dx := -\int_a^b f(x)dx$
- If $c \in [a, b]$, set $\int_c^c f(x)dx = 0$ (This can be justified)

§23 Fundamental Theorem of Calculus

Theorem 23.1 (1st Fundamental Theorem of Calculus)
Let $a < b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$. Let $c \in [a, b]$. Define

$$F(x) := \int_c^x f(t)dt \forall x \in [a, b]$$

Then F is differentiable at each $x \in (a, b)$ at which f is continuous and $F'(x) = f(x)$

Proof. Before proving the theorem, check that if f be defined as above and $c_1, c_2, c_3 \in [a, b]$ (in no particular order). Then,

$$\int_{c_1}^{c_3} f(x)dx = \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx$$

Now, let $x \in (a, b)$ at which f is continous. Let $|h|$ be so small that $x + h \in [a, b]$. Then,

$$\begin{aligned}
\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| &= \left|\frac{1}{h}\left(\int_c^{x+h} f(t)dt - \int_c^c f(t)dt\right) - f(x)\right| \\
&= \left|\frac{1}{h}\int_x^{x+h} f(t)dt - f(x)\right| \\
&= \left|\frac{1}{h}\int_x^{x+h} (f(t)-f(x))dt\right| \\
&\leq \frac{1}{|h|}\int_{\min(x,x+h)}^{\max(x,x+h)} |f(t)-f(x)|dt \text{ (Comparision Theorem)}
\end{aligned}$$

Fix $\epsilon > 0$. By continuity at x, $\exists \delta > 0$ such that, $|f(y) - f(x)| < \delta$ whenever $y \in [a, b]$ and $|x - y| < \delta$. If $t \in [\min(x, x + h), \max(x, x + h)]$ and $|h| < \delta$, then we have $|t - x| < \delta$ and so, $|f(x) - f(y)| < \epsilon$. Now by the above chain of steps,

$$\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| \leq \frac{1}{|h|}\int_{\min(x,x+h)}^{\max(x,x+h)} \epsilon dt$$
$$\forall h : 0 < |h| < \delta$$

Since $\epsilon > 0$ was arbitrary,

$$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x)$$

□

Definition 23.2. Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$. A function $P : I \to \mathbb{R}$ is called a <u>primitive</u> (or anti derivative) if P is differentiable, and $P'(x) = f(x)\forall x \in I$

Theorem 23.3 (2nd Fundamental Theorem of Calculus)
Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ is continuous. Then f admits a primitive. Let P be a primitive and $c \in I$. Then,

$$P(x) = P(c) + \int_c^x f(t)dt\forall x \in I$$

In particular, if $a < b \in I$,

$$\int_a^b f(x)dx = P(b) - P(a)$$

Remarks

- Let f and I be as in the definition of primitve and let f admit a primitive. If P_1 and P_2 are two primitives, then $\exists$ a constant c such that, $P_1 = P_2 + c$ (follows from the corollary to the LMVT)

- Let I be as above and $f : I \to \mathbb{R}$ is continuous. Let $a < b \in I$. If we can guess a primitive for f then we can easily calculate the integral by the above theorem

Corollary 23.4 (Integration by parts)
Let $I \subseteq \mathbb{R}$ be open. Let $f, g : I \to \mathbb{R}$ be differentiable functions such that f' and g' are continuous. Let $c \in I$. Then,

$$f(x)g(x) = f(c)g(c) + \int_c^x f'(t)g(t)dt + \int_c^x f(t)g'(t)dt$$

Let $G : I \to \mathbb{R}$ be continuous. Let P_G be a primitive of G. Let $a < b \in I$

$$\int_a^b f(x)G(x)dx = fP_G\Big|_{x=a}^{b} - \int_a^b f'(x)P_G(x)dx$$

Proof. By hypothesees, $f'g + fg'$ is continuous on I. By the product rule, fg is a primitive of $(f'g + fg')$. The assertion follows from the second FTC. □

§24 10/11/23

Last time

Finished proof of the 1st FTC. Stated 2nd FTC. Integration by parts

Today

Leibnizian notation: Let $I \subseteq \mathbb{R}$ be a non empty interval and $f : I \to \mathbb{R}$ is continuous. The 2nd FTC gives existence of primitive of f, which is non unique.

1. We denote by $\displaystyle\int f(x)dx :=$ a function P, UNDETERMINED up to a constant, such that, $P' = f$ on I
2. If we can guess a specific P as in (a), we write,

$$\int f(x)dx = P(x) + c$$

Logarithms

What is a logarithm?
Answer: Any function $f : (0, \infty) \to (-\infty, \infty)$ such that,

$$f(xy) = f(x) + f(y) \forall x, y \in (0, \infty)$$

Definition 24.1. Let $x \in (0, \infty)$. The natural logarithm (denoted by log) of x is defined as,

$$\log(x) := \int_1^x \frac{1}{x}dt$$

Theorem 24.2 (Properties of logarithm)
Let log be defined as above. Then,

- $\log(xy) = \log x + \log y \ \forall x, y > 0$
- log is strictly increasing
- Range of log $= \mathbb{R}$ (tricky to prove rigorously)

Proof. We will only prove first one. Fix $x, y > 0$. Then,

$$\begin{aligned}
\log(xy) &= \int_1^{xy} \frac{1}{t} dt \\
&= \int_1^{x} \frac{1}{t} dt + \int_x^{xy} \frac{1}{t} dt \\
&= \log(x) + x \int_1^{y} \frac{1}{xt} dt \\
&= \log(x) + \log(y)
\end{aligned}$$

$\square$