UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022
HOMEWORK 6

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 15, 2022**

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1. Let S be a non-empty subset of $\mathbb{R}$ that is bounded above.

1. Show that $\sup S$ belongs to $\overline{S}$.
2. Is $\sup S$ necessarily a limit point of S? Please **justify**.

2. Let X be a metric space and let $E \subseteq X$. Let x_0 be a limit point of E. Show that:

(a) For each $r > 0$, $B(x_0, r)$ contains infinitely many points of E.

(b) There exists a set $S \subseteq E$ such that S is infinite and such that x_0 is the **only** limit point of S.

3. Given a metric space X, we say that a set $E \subseteq X$ is *dense in* X if $\overline{E} = X$. For each $n \in \mathbb{Z}^+$, show that $\mathbb{R}^n$ has a countable dense subset.

Remark. A metric space that contains a countable dense subset is called a *separable metric space.*

4. Let $\mathscr{C}$ be a non-empty at most countable set whose elements are sets. Suppose each $A \in \mathscr{C}$ is at most countable. Prove that

$$B := \bigcup\nolimits_{A \in \mathscr{C}} A$$

is at most countable.

5. Show that a compact metric space is separable.

Given a set S, the *power set* of S is the collection of all subsets of S. That for **any** S this collection is a set is one of the axioms of Set Theory. In the next two problems, $\mathcal{P}(S)$ will denote the power set of S.

6. Let S be a non-empty set. Show that $\mathcal{P}(S)$ has the same cardinality as the set of all functions from S to the set $\{0, 1\}$.

7. Let S be an uncountable set. Show that:

(a) There exists an injective function from S into $\mathcal{P}(S)$.

(b) S does **not** have the same cardinality as $\mathcal{P}(S)$.

Hint. The conclusions of Problem 6 above might be of help, as well as the observation that when

S is countable then, *essentially*, we know the conclusion of part (b).

The following anticipates material to be introduced during the lecture on **February 16.**

8. Recall the construction of the *Cantor middle-thirds set* C, namely

$K_0 := [0,1]$,

$K_n :=$ the union of the 2^n closed intervals obtained by removing from each $I_{n-1}^{(j)}$, $j = 1, \dots 2^{n-1}$, the open interval of length $\mathsf{length}(I_{n-1}^{(j)})/3$ centered at the midpoint of $I_{n-1}^{(j)}$, $\quad n = 1, 2, 3, \dots$,

where $I_{n-1}^{(1)}, \dots I_{n-1}^{(2^{n-1})}$ are the disjoint closed intervals whose union gives K_{n-1}, and

$$C := \bigcap_{n \in \mathbb{N}} K_n.$$

(a) If $I_0^{(j(0))}, I_1^{(j(1))}, I_2^{(j(2))}, \dots$ is a sequence of intervals such that

$$1 \leq j(n) \leq 2^n, \text{ for each } n = 0, 1, 2, \dots, \quad \text{and} \quad I_0^{(j(0))} \supseteq I_1^{(j(1))} \supseteq I_2^{(j(2))} \supseteq \cdots.$$

Show that $\cap_{n \in \mathbb{N}} I_n^{(j(n))}$ is a singleton.

(b) Let $\mathscr{C}$ be the set of **all** nested sequences of intervals $I_0^{(j(0))} \supseteq I_1^{(j(1))} \supseteq I_2^{(j(2))} \supseteq \cdots$ such that

$$1 \leq j(n) \leq 2^n \quad \text{and} \quad I_n^{(j(n))} \subseteq K_n \quad \text{for each } n = 0, 1, 2, \dots.$$

Show that C and $\mathscr{C}$ have the same cardinality.

(c) (A *little* difficult, or *very* cute, depending on your point of view.) Show that $\mathscr{C}$, and therefore C, is uncountable.