UM 204 HOMEWORK ASSIGNMENT 3

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(NOT FOR SUBMISSION)

- These problems are for self-study. Try these **on your own** before seeking hints.
- Some of these problems will be (partially) discussed at the next tutorial.
- A 15-min. quiz based on this assignment will be conducted at the end of the tutorial section.

Problem 1. Let $\mathbb{R}$ denote the unique ordered field with the least upper bound property. Let $b > 0$ and $x \in \mathbb{R}$. Complete the following steps to establish the existence of a unique real number b^x. You may assume all the well-known properties of the function $n \mapsto b^n$ for $n \in \mathbb{Z}$.

(A) Case 1. $x = 1/n$, where $n \in \mathbb{Z} \setminus \{0\}$.

(a) Assume $b > 1$ and $n > 0$. Prove that $B = \{t \in \mathbb{R} : t > 0,\ t^n < b\}$ is nonempty and bounded above in $\mathbb{R}$.

(b) Prove that $(\sup B)^n = b$, and if there is a $t > 0$ such that $t^n = b$, then $t = \sup B$.

$$\boxed{\text{Define } \ \mathrm{b}^{1/n} = \begin{cases} \sup B, & \text{if } b > 1, n > 0, \\ \frac{1}{(1/b)^{1/n}}, & \text{if } 0 < b \leq 1, n > 0, \\ (1/b)^{1/-n}, & \text{if } b > 0, n < 0. \end{cases}}$$

(B) Case 2. $x = r \in \mathbb{Q} \setminus \{0\}$.

(a) Prove that if $r = m/n = p/q$, for $m, n, p, q \in \mathbb{Z}$ and $n, q > 0$, then $(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}$.

$$\boxed{\text{Define } b^r = (b^m)^{1/n}.}$$

(b) Prove that $b^{r+s} = b^r b^s$ if $r, s \in \mathbb{Q}$.

(C) Case 3. $x \in \mathbb{R}$.

(a) Assume $b > 1$. Prove that $B(x) = \{b^t : t \in \mathbb{Q} \text{ and } t \leq x\}$ is nonempty and bounded above in $\mathbb{R}$.

(b) Prove that $\sup B(r) = b^r$ if $r \in \mathbb{Q}$.

$$\boxed{\text{Define } \ \mathrm{b}^x = \begin{cases} \sup B, & \text{if } b > 1, \\ (1/b)^{-x}, & \text{if } 0 < b \leq 1. \end{cases}}$$

(c) Prove that $b^{x+y} = b^x b^y$ for all $x, y \in \mathbb{R}$.

Problem 2. Given $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $p > 0$, define

$$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}.$$

(a) Show that if $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{j=1}^{n} |x_j y_j| \le \|x\|_p \|y\|_q, \quad \forall x, y \in \mathbb{R}^n.$$

You may directly use Young's inequality: if $a, b \ge 0$, then $ab \le \dfrac{a^p}{p} + \dfrac{b^q}{q}$.

Hint. Consider $a = \dfrac{|x_j|}{\|x\|_p}$ *and* $b = \dfrac{|y_j|}{\|y\|_q}$.

(b) Let $d_p(x, y) = \|x - y\|_p$, $x, y \in \mathbb{R}^n$. Show that $(\mathbb{R}^n, d_p)$ is a metric space if $p \ge 1$.

Hint. Write $\sum_{j=1}^{n} |x_j + y_j|^p \le \sum_{j=1}^{n} |x_j||x_j + y_j|^{p-1} + |y_j||x_j + y_j|^{p-1}$, *and use* (a).

(c) Show that d_p is not a metric on $\mathbb{R}^n$ if $p \in (0, 1)$.

Problem 3. For $x, y \in \mathbb{R}$, let

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

(a) Show that d is a metric on $\mathbb{R}$.

(b) Show that the d-topology on $\mathbb{R}$ is the same as the topology induced by the standard metric on $\mathbb{R}$. Recall that the topology induced by a metric refers to the collection of all open sets in that metric.

Problem 4. Let p be a prime number. Recall the absolute value A_p on $\mathbb{Q}$ defined in Assignment 02. It follows from Problem 5 in Assignment 2 that

$$d_p(x, y) = A_p(x - y), \quad x, y \in \mathbb{Q},$$

is a metric on $\mathbb{Q}$. Is $\mathbb{Z}$, the set of integers, a closed subset of $(\mathbb{Q}, d_p)$?

Problem 5. Let (X, d) be a metric. For each of the claims below, determine whether it is either true (for all metric spaces) or false (in some metric space), and provide a justification for your answer.

(a) Let $E \subset X$. The set of limit points of E is a closed subset of X.

(b) Let $a \in X$ and $r > 0$. Then. $\overline{B(a; r)} = \{x \in X : d(x, a) \le r\}$.

(c) Every closed and bounded subset $E \subset X$ is compact.

(d) For any subset $E \subset X$, $E^\circ = \left(\overline{E}\right)^\circ$.