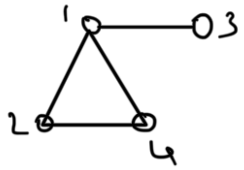
Lecture - 18

Recall, A = adjecency matrix of a graph G

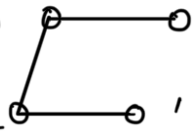
<u>Example</u> :- G_1 =



$$\Rightarrow A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

<u>Exercise</u> :- $(A^k)_{u,v}$ = # walks from u to v of length k.

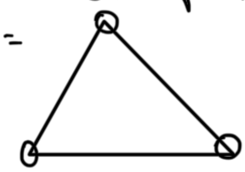
<u>Def'n</u> :- A spanning tree of a graph $G(V,E)$ is a subgraph T with $V(T) = V$ st T is a tree

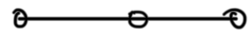
<u>Example</u> :- Spanning tree of $G \in \{$  , ... $\}$

The graph on n vertices with all possible edges is called a complete graph, denoted K_n.

We have shown that # of spanning trees of K_n is n^{n-2}

K_3 =





1 2 3

1 3 2

2 1 3

<u>Def'n</u> :- The laplacian L of a graph G is the matrix given by $L = D - A$, where $D = \text{diag}(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$

A = Adjacency matrix.

All the off-diagonal entries are negative.

Note that rows and columns sums of L is 0 [Trivial]

$$L\,[\text{for above graph } (G_1)] = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

Exercise :- For any graph, $\det L = 0$

<u>Def'n</u> :- A reduced Laplacian L_0 of a graph is obtained

by deleting the same row and column from L.

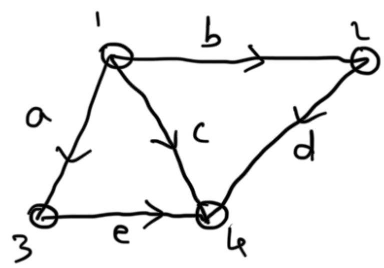
Thm (Kirchoff's matrix Thm) :- Let G be a graph and L_0 be a reduced laplacian L_0 of G. Then the # of spanning trees of G is det L_0.

Note that this doesn't depend on row and column being deleted.

Def'n :- Let $G = (V, E)$ with $|V| = n$, $|E| = m$. Orient the edges of G in the same way. Then the incidence matrix I is the $n \times m$ matrix given by

$$I_{v,e} = \begin{cases} -1; & v \in e \text{ st } v \text{ is the tail of } e \\ +1; & v \in e \text{ st } v \text{ is the head of } e \\ 0; & \text{otherwise.} \end{cases}$$

Example :-



$$I = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} & \\ \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

$n = 4$
$m = 5$

Let I_0 be the reduced incidence matrix with some row deleted.

Exercise :- $L = I \cdot I^T$
$L_0 = I_0 I_0^T$ for the corresponding row

$\Big\}$ Independent of orientation.

Thm (The Cauchy-Binet Formula) :-

Let A be an $n \times m$ matrix and B be an $m \times n$ matrix with $n \leq m$.

Recall that $\binom{[m]}{n}$ is the collection of n-subsets of $[m]$.
If $S \in \binom{[m]}{n}$, let $A_{[n],S}$ (resp $B_{S,[n]}$) be the $n \times n$ matrix formed by taking all rows (resp columns) and columns (resp rows) indexed by S.

Then $\boxed{\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_{[n],S}) \cdot \det(B_{S,[n]})}$

Proof of Kirchoff's matrix tree theorem :-

We will show that the # of spanning trees of G is $\det(I_0 I_0^T)$ using Cauchy-Binet theorem.

Note that L_0 is an $(n-1)\times(n-1)$ matrix

$\Rightarrow$ we need to take $S \in \binom{[n]}{n-1}$

Suppose $S \subset E$, $|S| = n-1$

This corresponds to the term $\det\big(\underbrace{(I_0)_{[n-1],S}}_{B}\big) \cdot \det\big(\underbrace{(I_0^T)_{S,[n-1]}}_{B^T}\big)$

[WLOG, we deleted the last row and column]

$$= \det(B^2) = (\det B)^2$$

We now claim that $\det B^2 = 1$ iff S corresponds to a spanning tree and is 0 otherwise.

Proof of Claim :- By induction. Check the base case.

If there exists a vertex i of degree 1 in S. Then the i^{th} row of B has a single non zero entry, either ± 1. Expand $\det B$ along the i^{th} row, and use the induction hypothesis on $S \setminus \{\text{that edge}\}$, which is a spanning tree of $G \setminus \{i\}$ if S is a spanning tree of G.

If no vertex in S has degree 1, then S is disconnected and must have a cycle.

Since it has $n-1$ edges, S can't be a spanning tree. Look at the rows corresponding to vertices that form a cycle.

Now, we use the fact that these rows or columns in B must be linearly dependent.

$\Rightarrow \det B = 0$

This proves the result. $\blacksquare$