UM 204 - MIDTERM EXAMINATION

Instructor: Purvi Gupta

February 23, 2024 2:00 pm - 5:00 pm (3 hours)

Name of the student : ________________________________

Student number : ________________________________

Instructions

1. This booklet has **16 pages** and **5 questions**. The first question has 4 parts.
2. **Read** every question carefully before attempting to answer it. **Partial credit** is available for partial (but correct) ideas.
3. You may use without proof any results or examples discussed in class or homework assignments. If you do not cite what you are using, you will be penalized.
4. The answer to each question must be written below it. If you run out of space, you may use Pages 14-16, or even attach extra sheets, but these will only be graded if you clearly **tell the reader** to turn to these pages.
5. All the blank spaces can also be used for **scrapwork**.
6. Do not separate any pages from this booklet.

Grading table

Problem #	Points	Score
1	24	
2	10	
3	12	
4	12	
5	12	
Total	70	

Problem 1. (*24 points*) For each of the statements below, determine whether it is (necessarily) true or (sometimes) false. If you circle TRUE, you must provide a proof. If you circle FALSE, you must provide a counterexample and justify it.

(*a*) In any complete metric space, a bounded sequence always admits a convergent subsequence.

TRUE FALSE

(b) The function $d : \mathbb{R}^k \times \mathbb{R}^k \to [0, \infty)$ given by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{2y}\|, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbf{k}},$$

is a metric on $\mathbb{R}^k$.

TRUE FALSE

(c) In any metric space, the closure of a connected set is connected.

TRUE FALSE

(*d*) Given subsets $E_1, E_2, ...$ of a metric space (X, d),

$$\bigcap_{j=1}^{\infty} E_j^{\circ} = \left(\bigcap_{j=1}^{\infty} E_j \right)^{\circ}.$$

(Here, E° denotes the interior of E.)

TRUE FALSE

Problem 2. (*10 points*) Recall the construction of $\mathbb{Q}$ as the set of equivalence classes of the relation R on $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ given by $(a, b)R(c, d) \iff ad = bc$. We say that $[(a, b)] \leq [(c, d)]$ if $(bc - ad)(bd) \geq 0$. Using only the arithmetic and order properties of integers, show that the relation $\leq$ is well-defined.

EXTRA SPACE FOR PROBLEM 2.

Problem 3. (*12 points*) Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be the sequence given by

$$\begin{aligned} x_0 &= 0; \quad x_1 = 1; \\ x_n &= \frac{1}{2}(x_{n-1} + x_{n-2}), \quad n \geq 2. \end{aligned}$$

Show that $\{x_n\}_{n\in\mathbb{N}}$ is a convergent sequence. (You may cite limits of standard sequences and series without proof.)

EXTRA SPACE FOR PROBLEM 3.

Problem 4. (*12 points*) Let $\mathscr{R}$ denote the set of equivalence classes[1] of Cauchy sequences of rational numbers. The equivalence class of $\{a_n\}_{n\in\mathbb{N}}$ is denoted by $[\{a_n\}_{n\in\mathbb{N}}]$. We say that $\alpha = [\{a_n\}_{n\in\mathbb{N}}]$ is positive if there is some positive rational $c > 0$ and some $N \in \mathbb{N}$ such that

$$a_n > c, \quad \forall n \geq N.$$

We say that α is negative if there is some negative rational $c < 0$ and some $N \in \mathbb{N}$ such that

$$a_n < c, \quad \forall n \geq N.$$

Assume that these are well-defined notions. Show that, for any $\alpha \in \mathscr{R}$, **one and only one** of the following holds:

(a) α is the equivalence class of the constant 0 sequence;

(b) α is positive;

(c) α is negative.

You may not use the fact that $\mathscr{R}$ is the set of real numbers. You may use the properties of the ordered field $(\mathbb{Q}, +, \cdot, \leq)$.

[1]Recall that two rational sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are said to be equivalent if, for every rational $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n - b_n| < \varepsilon$ for all $n \geq N$.

EXTRA SPACE FOR PROBLEM 4.

Problem 5. (*12 points*) Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is said to be *upper semicontinuous at* $p \in X$ if, for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converging to p,

$$\limsup_{n\to\infty} f(x_n) \leq f(p).$$

A function is said to be upper semicontinuous on X if it is upper semicontinuous at each point of X. Prove that if f is upper semicontinuous on X, then

$$U_a = \{x \in X : f(x) < a\}$$

is open for any $a \in \mathbb{R}$.

EXTRA SPACE FOR PROBLEM 5.

EXTRA SPACE FOR ANSWERS OR ROUGH WORK

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