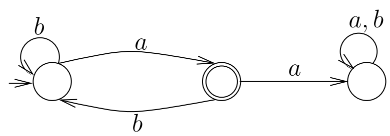
Solutions to ATC Quiz 1

Problem 1. Consider the DFA over the alphabet $\Sigma = \{a, b\}$ given below.



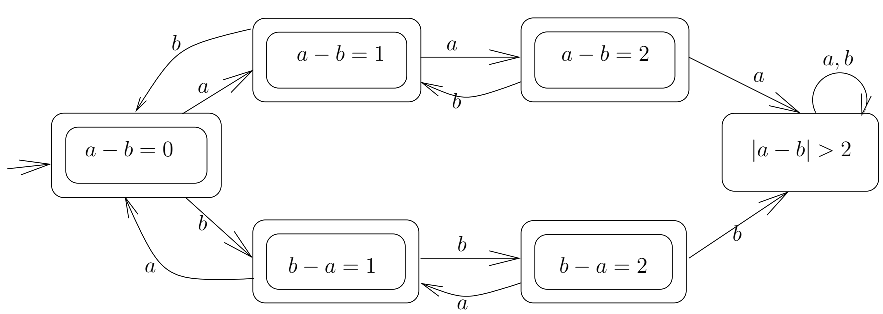
Describe the language accepted by the automaton.

Solution.

$$L(\mathcal{A}) = \{w \in \Sigma^* \mid \; w \text{ ends in } a \text{ and } w \text{ has no substring } aa\}.$$

Problem 2. Consider the language of all strings over the alphabet $\{a, b\}$ which satisfy the property that in *every prefix* the difference between the number of a's and b's is at most 2. Thus, $aabab$ is in the language, while $abaaab$ is not. Give the state diagram of DFA for this language. Label your states meaningfully.

Solution. Below is the automaton $\mathcal{A}$ that accepts the language of all strings whose every prefix has $|a - b| \le 2$, i.e., the number of a's and b's differ by at most 2.

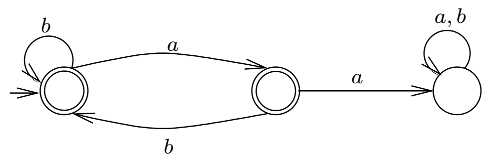


Problem 3. For a language $L \subseteq A^*$, the language of prefixes of L, denoted $\mathit{pref}(L)$, is defined to be the set

$$\{u \in A^* \mid \; \exists v \in A^* \text{such that } u \cdot v \in L\}$$

(a) Give a DFA that accepts the language $pref(L(\mathcal{A}))$ for the DFA $\mathcal{A}$ of Q1.

Solution. Below is the automaton $\mathcal{A}$ that accepts the language $perf(L(\mathcal{A}))$ of Q1.



(b) Prove that the class of regular language is closed under the prefix operation.

Solution. Let L be any regular language and $\mathcal{A} = (Q, \Sigma, \delta, s, F)$ be an automaton such that $L = \mathcal{L}(\mathcal{A})$. We need to show that there exists an automaton accepting the language $pref(L)$. This will prove that the class of regular sets is closed under the *pref* operation.

Let $\mathcal{B} = (Q, \Sigma, \delta, s, F_{\mathcal{B}})$ be the automaton constructed froom $\mathcal{A}$, where

$$F_{\mathcal{B}} = \{q \in Q \mid \exists x \in \Sigma^* \text{ s.t. } \hat{\delta}(q, x) \in F\}.$$

Claim 1. $\mathcal{L}(\mathcal{B}) = pref(L)$

We will use the following lemma to prove Claim 1.

Lemma 1. For any strings $u, v \in \Sigma^*$, $\widehat{\delta}(s, u \cdot v) = \hat{\delta}(\widehat{\delta}(s, u), v)$.

Proof: By induction on length of v. Let $P(n)$ be the statement:

For all $u.v \in \Sigma^*$ with $|v| = n$, we have $\widehat{\delta}(s, u \cdot v) = \widehat{\delta}(\widehat{\delta}(s, u), v)$.

Base case: For $n = 0$, we have $v = \epsilon$, and $\widehat{\delta}(s, u \cdot v) = \widehat{\delta}(s, u) = \widehat{\delta}(\widehat{\delta}(s, u), \epsilon) = \widehat{\delta}(\widehat{\delta}(s, u), v)$.

Induction step: Suppose $P(n)$ is true for some $n \in \mathbb{N}$. Consider $u, v \in \Sigma^*$ such that $|v| = n + 1$. Let $v = w \cdot a$. Then

$$\begin{aligned}
\widehat{\delta}(s, u \cdot v) &= \widehat{\delta}(s, u \cdot wa) \\
&= \delta(\widehat{\delta}(s, u \cdot w), a) \\
&= \delta(\widehat{\delta}(\widehat{\delta}(s, u), w), a) \text{ (by induction hypothesis)} \\
&= \widehat{\delta}(\widehat{\delta}(s, u), wa) \\
&= \widehat{\delta}(\widehat{\delta}(s, u), v).
\end{aligned}$$

This completes the proof of the lemma.

Returning to the proof of Claim 1, we first show $pref(L) \subseteq \mathcal{L}(\mathcal{B})$.

Let u be any string in $\mathit{pref}(L)$. Let v be such that $u \cdot v \in L$. Then

$$\begin{aligned}
& \widehat{\delta}(s, u \cdot v) \in F \\
\Longrightarrow\ & \widehat{\delta}(\widehat{\delta}(s, u), v) \in F \text{ (by Lemma 1)} \\
\Longrightarrow\ & \widehat{\delta}(q, v) \in F, \text{ where } q = \widehat{\delta}(s, u) \\
\Longrightarrow\ & q \in F_{\mathcal{B}} \text{ (by construction of } \mathcal{B}) \\
\Longrightarrow\ & u \in \mathcal{L}(\mathcal{B}).
\end{aligned}$$

Conversely we argue that $\mathcal{L}(\mathcal{B}) \subseteq \mathit{pref}(L)$. Let $u \in \mathcal{L}(\mathcal{B})$. Then $\widehat{\delta}(s, u) \in F_{\mathcal{B}}$. Let $\widehat{\delta}(s, u) = q$. Since $q \in F_{\mathcal{B}}$, there exists $v \in \Sigma^*$ such that $\widehat{\delta}(q, v) \in F$. But by Lemma 1 this means that $\widehat{\delta}(s, u \cdot v) \in F$. Hence $u \cdot v \in L$ and hence $u \in \mathit{pref}(L)$.

This completes the proof.