Lecture - 13

Inclusion- Exclusion Principle

Let $A_1, A_2, \dots, A_n$ be finite sets. Suppose we know the sizes of A_i's and all possible intersections. We want to know $\left| \bigcup_i A_i \right|$

Thm (Principle of Inclusion- Exclusion (PIE)/ Sieve Formula) :-
$A_1, A_2, \dots, A_n$ finite sets, then

$$\#\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{j=1}^{n} (-1)^{j-1} \sum_{1 \le i_1 < i_2 < \dots < i_j \le n} \#(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_j})$$

where the inner sum runs over all j-subsets $(i_1, i_2, \dots i_j)$ of $[n]$.

Proof :- Let $x \in \bigcup A_i$ and $S \subseteq [n]$ st $x \in A_i \ \forall\, i \in S$ and $x \notin A_i \ \forall\, i \notin S$
x contributes to RHS iff the set $\{i_1, i_2, \dots, i_j\}$ is a subset of S.
Also for every subset of S, x contributes
$s = \#S$, then the number of times x appears on the RHS is $(-1)^0 s + (-1)^1\binom{s}{2} + (-1)^2\binom{s}{3} \dots\dots (-1)^{s-1}\binom{s}{s}$ $[j = s]$
$= -(1-1)^s + 1 = 1$

Example (Coat check problem/ Probleme des rencontres) :-
n guests attend a concert and leave their coats at the entrance. At the end each leave with a random coat. Find Prob st no guests get their own coat. In terms of permutations, we want to find the number of permutations in S_n without fixed points. These are called derangements.

Let $A_i = \{\pi \in S_n \mid \pi_i = i\}$ for $1 \le i \le n$
Then $\bigcup_{A_i} = \{\pi \in S_n \mid \pi$ has atleast one fixed point.

$\# A_i = (n-1)!$; $\# A_i \cap A_j = (n-2)!$, etc
By PIE, $\# \bigcup_i A_i = \sum_{j=1}^{n} (-1)^{j-1} (n-j)! \binom{n}{j} = \sum_{j=1}^{n} (-1)^{j-1} \frac{n!}{j!}$

Thus, no., of derangements, $d_n = n! - \# \bigcup_i A_i$

$$= \frac{n!}{0!} - \frac{n!}{1!} + \frac{n!}{2!} + \dots + (-1)^n \frac{n!}{n!}$$

$\sum^{n} (-1)^j \frac{n!}{}$

$$= \sum_{j=0}(-1)^j \frac{1}{j!}$$

Thus, $\mathbb{P}(\text{getting a derangement}) = \dfrac{d_n}{n!} = \displaystyle\sum_{j=1}^{n} \frac{(-1)^j}{j!} \xrightarrow{n\to\infty} \frac{1}{e}$

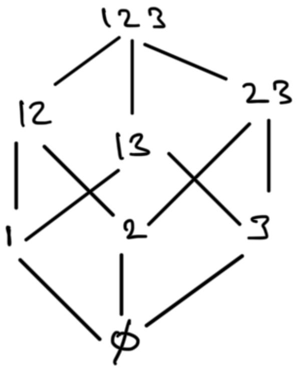
Let P be a poset.

<u>Def'n</u> :- An order ideal/ down set of P is a subset I st if $t \in I$ and $s < t$, then $s \in I$. The principle order ideal generated by $t \in P$ is $V_t = \{s \in P \mid s \leq t\}$

<u>Def'n</u> :- We say that t covers s or s is covered by t if $s<t$ and $\nexists\, u \in P$ st $s<u<t$.

The Hasse diagram of a POSET is the directed graph whose vertex set is P and the edges are cover relations.

<u>Example</u> :- $(2^{[3]}, \subseteq)$



<u>Def'n</u> :- Let P be a poset for which every principle order ideal is finite, then the mobius function $\mu : P\times P \longrightarrow \mathbb{Z}$ is given by $\mu(s,s) = 1 \;\forall\, s \in P$

$$\mu(s,u) = -\sum_{\substack{t \\ s\leq t\leq u}} \mu(t,u) \quad \forall\, s<u \text{ in } P$$

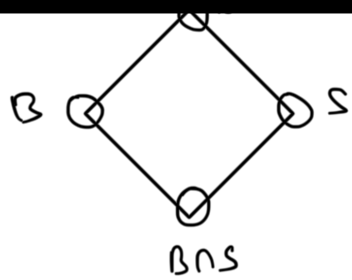
$$\iff \sum_{s\leq t\leq u} \mu(t,u) = \delta_{su}$$

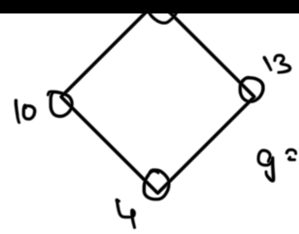
<u>Thm</u> (M<u>obi</u>us <u>Inv</u>ersion For<u>mul</u>a) :- Let P be a poset all of whose principle order ideals are finite. Let $f, g : P \to \mathbb{R}$. Then,

$$g(t) = \sum_{s\leq t} f(s) \quad \forall\, t \in P$$

iff $f(t) = \displaystyle\sum_{s\leq t} \mu(s,t)\, g(s) \;\forall\, t\in P$

<u>Example</u> :- Suppose in a class, 10 play soccer, 13 play basketball and 4 play both. How many people play atleast one game?





g = # players at each level.

Assume the formula for g.

$$\sum_{s\le t}\mu(s,t)\, g(s) = \sum_{s\le t}\mu(s,t)\cdot\sum_{u\le s} f(u)$$

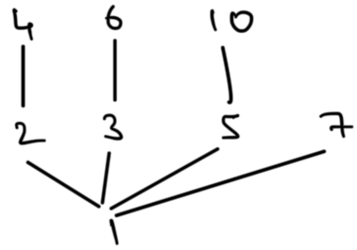
$$= \sum_{u\le t} f(u) \sum_{u\le s\le t}\mu(s,t)$$

$$= \sum_{u\le t} f(u)\,\delta_{u,t} = f(t)$$

Similarly, the other way.

Example :- $(\mathbb{N}>0, \le)$; $a\le b$ if $a|b$

$\mu(1,n)$



Exercise :- $\mu(1,n) = \begin{cases} 0; & \text{if } p^2 | n \\ (-1)^{\text{no. of prime factors}} & \end{cases}$ Prime p

Example :- Set partition of [3] ordered by refinement.