

Lecture - 5

$$f: A \rightarrow B, \quad C \subseteq A, \quad f(C)$$

1. $f(C)$ can be shown to be a set also by the axiom of specification
2. One can show that the cartesian product $A \times B$ is a set by Axiom 4 (Singletons & Pairs)
Identity (a, b) by $\{\{a\}, \{a, b\}\}$. Then show that $(a, b) = (a', b')$ iff $a = a'$ and $b = b'$
3. Axiom of replacement \Rightarrow Axiom of specification

Def'n:- Let $f: X \rightarrow Y, g: Y \rightarrow Z$. Then the composition $g \circ f: X \rightarrow Z$ is given by $g \circ f(x) = g(f(x))$

Lemma:- Function composition is associative i.e., $f: Z \rightarrow W, g: Y \rightarrow Z, h: X \rightarrow Y$
 $\Rightarrow f \circ (g \circ h) = (f \circ g) \circ h: X \rightarrow W$

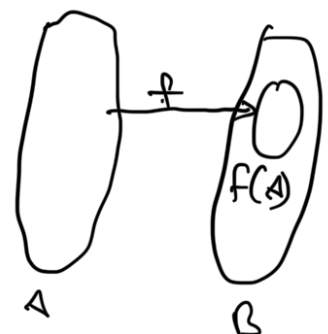
Proof of Lemma:- Exercise

Types of functions:-

1. $f: A \rightarrow B$ is injective or one to one if $f(a) = f(b) \Rightarrow a = b$
2. $f: A \rightarrow B$ is surjective or onto if $f(A) = B$.
3. $f: A \rightarrow B$ is bijective if it is both 1, and 2,
4. $f: A \rightarrow A$ is an involution if $f(f(a)) = a \quad \forall a \in A$

If f is a bijection, then for every $b \in B, \exists$ a unique $a \in A$ st $f(a) = b$. Then a is the inverse of f at b denoted $a = f^{-1}(b)$. Thus $f^{-1}: B \rightarrow A$ is also a valid function.

If f is injective, then $f^{-1}: \underbrace{f(A)}_{\text{Not } B} \rightarrow A$ is also well-defined.



ZFC axioms (continued) :-

11, (Axiom of power sets)

Let X, Y be sets, there exists a set denoted Y^X consisting of all functions from X to Y .

Ques:- If $|X| = m, |Y| = n, |Y^X| = ? \rightarrow n^m$

Exercise:-

Let X be a set. The collection $\{Y \mid Y \text{ is a subset of } X\}$ is a set, called the power set of X denoted 2^X .

12, (Axiom of Unions)

Let A be a set whose elements are also sets. Then \exists a set $\bigcup A$, whose elements are the elements of A .

Thus $x \in \bigcup A$ if $x \in S$ for some $S \in A$

Remarks :-

1, Axiom 12, with Axiom 4, (pairs) \Rightarrow Axiom 5, (finite union)

2, Let I be a set (indexing) such that A_α is a set for every $\alpha \in I$. Then $\{A_\alpha \mid \alpha \in I\}$ is a set (in replacement axiom)

$\Rightarrow \bigcup_{\alpha \in I} A_\alpha$ is a set (by Axiom 12)

Axioms 1-12 give the Zermelo-Fraenkel axioms of set theory.

Def'n:- Two sets X, Y have equal cardinality/size if there exists a bijective function $f: X \rightarrow Y$. Let $n \in \mathbb{N}$. A set A has cardinality of n if it has the same cardinality as $[n] := \{1, 2, \dots, n\}$

Exercise:- Cardinality is an equivalence relation on sets.

Example:- $X = \mathbb{N}, Y = 2\mathbb{N}$. Then $f: X \rightarrow Y$ is a bijection given by $f(x) = 2x$ is a bijection $\Rightarrow |X| = |Y|$ but $Y \subsetneq X$

\downarrow
Notation for cardinality of X .

Def'n:- A set is finite if it has cardinality n

Exercise :- \mathbb{N} is infinite.

Def'n:- A set is countable / countably infinite if it has the same cardinality as \mathbb{N} . A set is at most countable if it is either finite or countable, and is uncountable if it is infinite but not countable.

Exercise :-

1. Let $m < n \in \mathbb{N}$. Then,
 - a, there is no surjective $f: [m] \rightarrow [n]$
 - b, there is no injective $f: [n] \rightarrow [m]$
2. Let $m, n \in \mathbb{N}$. There exists a bijective $f: [m] \rightarrow [n]$ iff $m = n$

$$X, Y \text{ countable} \Rightarrow X \cup Y \text{ countable}$$

2. $\{(n, m) \in \mathbb{N} \times \mathbb{N} \mid 0 \leq m \leq n\}$ is countable

$\Rightarrow \mathbb{Q}$ is countable.

00		
10	11	
20	21	22
30	31	32
40	41	42
50	51	52
60	61	62
70	71	72
80	81	82
90	91	92

Cantor's theorem:- Let X be an arbitrary set. Then \overline{X} and 2^X cannot have the same cardinality.

Proof:- Suppose they do. Then \exists a bijection $f: X \rightarrow 2^X$
Consider $A = \{x \in X \mid x \notin f(x)\} \Rightarrow A \subseteq X$
 $\Rightarrow A \subseteq 2^X$ [By def'n]

Thus $\exists x$ st $f(x) = A$

Case-I:- $x \in A \Rightarrow x \notin f(x) \Rightarrow x \notin A$ by def'n of A

Case-II:- $x \notin A \Rightarrow x \notin f(x) \Rightarrow \{x \in A \text{ by def'n of } A\}$
 $\uparrow \quad \quad \quad \nearrow$
 Contradiction.

Exercise:- Show that 2^X has larger cardinality than X

men X.