UM 204 HOMEWORK ASSIGNMENT 4

Posted on February 01, 2024
(NOT FOR SUBMISSION)

- These problems are for self-study. Try these **on your own** before seeking hints.
- Some of these problems will be (partially) discussed at the next tutorial.
- A 15-min. quiz based on this assignment will be conducted at the end of the tutorial section.

Problem 1. Given $A, B \subset \mathbb{R}^n$, define

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

Endow $\mathbb{R}^n$ with the standard metric.

(1) Is $A + B$ open if both A and B are open?
(2) Is $A + B$ closed if both A and B are closed?

Problem 2. In this problem, (X, d) denotes a metric space and $E \subset X$ denotes a connected subset. For each of the claims below, determine whether it is either true (for all metric spaces) or false (in some metric space), and provide a justification for your answer.

(1) $\overline{E}$ is connected.
(2) E° is connected.
(3) $bE = \overline{E} \setminus E^\circ$ is connected.

Problem 3. Let C denote the Cantor set constructed in Lecture 13. Recall that

$$C = \bigcap_{j=0}^{\infty} E_n,$$

where each E_n is a disjoint union of 2^n closed intervals of length $1/3^n$.

(1) Prove, rigorously, that E contains no interval of the form (a, b), $a < b$.
Hint. Recall the form of the open intervals being removed from $[0, 1]$ to construct C.
(2) Show that every $p \in C$ is a limit point of C.
Hint. Remember that the endpoints of the inervals comprising E_n are in C for all $n \in \mathbb{N}$.
(3) (**Bonus question**[1]) Show that $C + C = [0, 2]$.

[1]Not for the quiz.

Problem 4. Let p be a prime number. Let d_p be the distance on $\mathbb{Q}$ defined in HW03. Show that $\{1/p^n\}_{n\in\mathbb{N}}$ is a divergent sequence, while $\{p^n\}_{n\in\mathbb{N}}$ is a convergent sequence in $(\mathbb{Q}, d_p)$. (**Bonus question**[2]) What about the sequence $\{n\}_{n\in\mathbb{N}}$?

Problem 5. Consider the following metrics on $\mathbb{R}^2$:

$$\begin{aligned} d_1(x,y) &= |x_1 - y_1| + |x_2 - y_2| \\ d_2(x,y) &= \left(|x_1 - y_1|^2 + |x_2 - y_2|^2\right)^{\frac{1}{2}} \\ d_\infty(x,y) &= \max\left\{|x_1 - y_1|, |x_2 - y_2|\right\}, \quad x, y \in \mathbb{R}^2. \end{aligned}$$

Show that if a sequence in $\mathbb{R}^2$ converges in any one of the above metrics, it also converges in the other two.

[2]Not for the quiz.