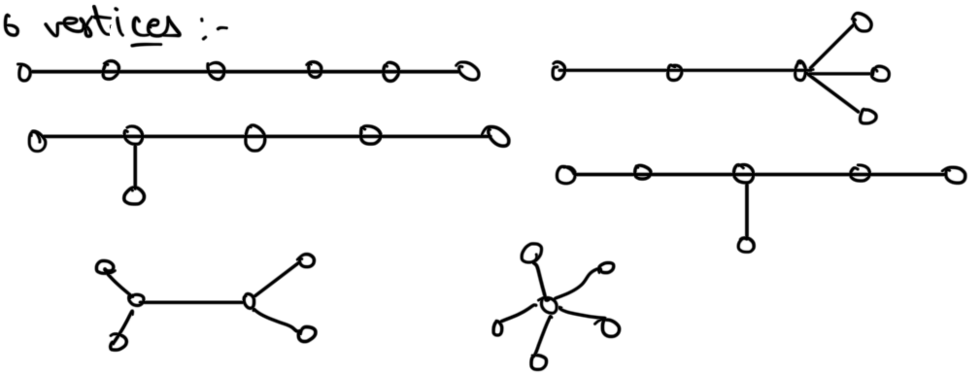
Lecture - 17

Recall:- Trees

We showed
1. deleting any edge in a tree disconnects it.
2. adding a new edge to a tree creates a cycle.
3. There is a unique path b/w any 2 vertices.

Def'n:- A vertex in a graph with degree 1 is called a leaf/pendant vertex.

Trees on 6 vertices:-



Lemma:- Every tree on $n \geq 2$ vertices has atleast 2 leaves.

Proof:- Let $(v_1, \dots v_k)$ be a maximal (i.e., can't be extended) path in the tree. Then both v_1, v_k must be leaves, since otherwise the path could have been made longer. Note that we couldnot have edges from v_1 and v_k to any other v_i because it is a tree.

Thm:- All trees on n vertices have $n-1$ edges. Conversely a connected graph with n vertices and $n-1$ edges must be a tree.

Proof:- $\Rightarrow$ Induction on n. True for $n=1$ (trivially). Assume the result holds for all trees on n-vertices. Let T be a tree on $n+1$ vertices. By the previous lemma, there must exist a leaf in T, say l. Delete l and its incident edge to get a tree T' (note that T' is connected). Since T' has n vertices, it must have $n-1$ edges. Now add back l to get n edges for T'.

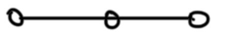
Exercise:- Prove $\Leftarrow$

Def'n:- A graph without cycles is called a forest.

Exercise:- A forest on n vertices with k trees (i.e., components) has $n-k$ edges.

Counting labelled trees with labels $[n]$:-

Examples :-

$n=3$  P_3 $\quad \# = \frac{3!}{2!} = 3$

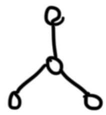
1 2 3 $\longleftrightarrow$ 3 2 1

1 3 2 $\longleftrightarrow$ 2 3 1

2 1 3 $\longleftrightarrow$ 3 1 2

[3! ways and pairs are identical]

$n=4$  P_4 $\quad \# = \frac{4!}{2!} = 12$

 $\quad \# = \frac{4!}{3!} = 4$

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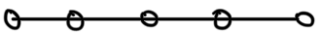
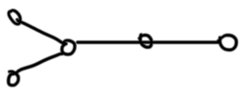
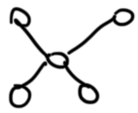
$n=5$  P_5 $\quad \# = \frac{5!}{2!} = 60$

 $\quad \# = \frac{5!}{2!} = 60$

 $\quad \# = \frac{5!}{4!} = 5$

$= 125$

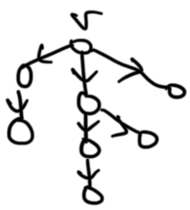
$$\boxed{\Rightarrow P_n = n^{n-2}}$$

Thm (Cayley's Formula) :- The # of trees in vertices labelled $[n]$ is n^{n-2}

Def'n :- A rooted tree $T(v)$ is a tree T together with a specified vertex called the root.

A branching of a rooted tree $T(v)$ is an orientation of T in which every edge is directed away from v.

Example :-

T: 

Arborescence.

Note that there is exactly one incoming edge to every vertex except the root in a branching.

Proof of Thm :- We will show that the # of branchings on n vertices is n^{n-1}.

Build a branching on n vertices by starting with the edgeless graph on n vertices one edge at a time.

$$\begin{array}{ccccc} \circ & \circ & \circ & \cdots & \circ \\ 1 & 2 & 3 & \cdots & n \end{array}$$

Initially, there are n components.

At the k^{th} stage, we have $n-k$ components, each of which is a branching.

At the k^{th} stage, we can choose the starting vertex in n ways. but we can choose the ending vertex in $n-k-1$ ways.

(Only the roots of other branchings can be chosen).

The process ends at the $(n-1)^{th}$ stage.

The # of ways is $\underbrace{n(n-1)}_{\text{at } 1^{st} \text{ stage}} \cdot \underbrace{n(n-2)}_{\text{at } 2^{nd} \text{ stage}} \cdots\cdots \underbrace{n(1)}_{\text{at } (n-1)^{th} \text{ stage}}$

$$= n^{n-1} \cdot (n-1)!$$

Note that each branching can be converted in $(n-1)!$ ways (independent of which branching we choose) since each branching has $n-1$ edges and each permutation of the edges gives the same branching.

$\Rightarrow$ Total # of branchings $= \dfrac{n^{n-1} \cdot (n-1)!}{(n-1)!} = n^{n-1}$

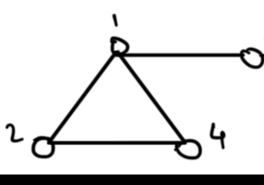
Since there were n choices for the root in any tree, # labelled trees $= [n]$.

Remark :- See the textbook for another proof.

Def'n :- Let $G = (V, E)$ and $|V| = n$. The adjecency matrix of G is the $n \times n$ matrix indexed by V whose entries are

$$A(v, w) = \begin{cases} 1 ; & \text{if } \{v, w\} \in E \\ 0 ; & \text{otherwise.} \end{cases}$$

Example :-



$\Rightarrow A = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Note that A is symmetric.

Exercise :- Let $k \in \mathbb{N}$. The $(u,v)^{th}$ entry of A^k counts the # of <u>walks</u> from u to v of length k.