UM 204 HOMEWORK ASSIGNMENT 8

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(NOT FOR SUBMISSION)

The next quiz is NOT based on this assignment. See the Teams page for the syllabus of the quiz.

Problem 1. Suppose $f : (0, \infty)$ is twice differentiable and

$$\begin{aligned} M_0 &= \sup_{z \in (0,\infty)} |f(z)|, \\ M_1 &= \sup_{z \in (0,\infty)} |f'(z)|, \\ M_2 &= \sup_{z \in (0,\infty)} |f''(z)|, \end{aligned}$$

are all finite. Show that

$$M_1^2 \leq 4 M_0 M_2.$$

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $\lim_{x\to\infty} f(x)$ exists. *See Chapter 4 of Rudin's book for the definition of limits at infinity.*

(1) Suppose $\lim\limits_{x\to\infty} f'(x)$ exists. Show that

$$\lim_{x\to\infty} f'(x) = 0.$$

(2) Suppose f is twice differentiable and f'' is a bounded function. Show that $\lim\limits_{x\to\infty} f'(x)$ exists.

Problem 3. Let $\mathbf{f} : (0, 1) \to \mathbb{R}^n$ be a differentiable vector-valued function such that $\|\mathbf{f}\|$ is a constant function on $(0, 1)$. Show that $\langle \mathbf{f}(x), \mathbf{f}'(x)\rangle = 0$ for all $x \in (0, 1)$.

Problem 4. Recall the following function from the previous assignment: $f : [0, 1] \to [0, 1]$ given by

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q},\ \text{with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N}_{>0},\ \gcd(p, q) = 1, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Use the ε-characterization of Riemann integrability to show that f is Riemann integrable.
Hint. Given $\varepsilon > 0$, let $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \varepsilon$. Show that $f(x) < \frac{1}{n+1}$ for all $x \in [0, 1] \setminus S_n$, where $S_n = \{\frac{p}{q} : p, q \in \mathbb{N}$ and $1 \leq p \leq q \leq n\}$. On the other hand, for $x \in S_n$, $f(x) \leq 1$. Now choose an appropriate partition P of $[0, 1]$ so that $U(P, f) < \varepsilon$. Why does this prove the claim?

Problem 5. Problems 11 & 12 from Chapter 6 in Rudin's book. Take $\alpha(x) = x$.

Problem 6. Let $f : [0, 1] \to \mathbb{R}$ be a continuous monotone function. Show that the graph of f is a rectifiable curve with length at most 2.

Bonus question. Is there a monotone continuous function on $[0, 1]$ whose graph has length 2?