UM 204 HOMEWORK ASSIGNMENT 2

Posted on January 11, 2024
(NOT FOR SUBMISSION)

- These problems are for self-study. Try these **on your own** before seeking hints.
- Some of these problems will be (partially) discussed at the next tutorial.
- A 15-min. quiz based on this assignment will be conducted at the end of the tutorial section.

Problem 1. Let F and G be ordered fields with the l.u.b. property. In Lecture 04, we defined $h : F \to G$ as

$$h(z) = \sup_G \{w \in \mathbb{Q} : w \leq z\}.$$

Show that h is a field isomorphism, i.e.,

(1) h is a bijection between F and G,
(2) $h(x + y) = h(x) + h(y)$, for all $x, y \in F$,
(3) $h(x \cdot y) = h(x) \cdot h(y)$, for all $x, y, \in F$.

Problem 2. In this problem, you may assume the well-definedness, commutativity and associativty of addition of Dedekind cuts (as defined in Lecture 04). Let $O = \{z \in \mathbb{Q} : z < 0\}$. Verify that O is a Dedekind cut, and $A + O = A$ for all Dedekind cuts A. Let A be a Dedekind cut. Define a Dedekind cut B such that $A + B = O$. You must justify your answer.

Problem 3. Let $a = \{a_n\}_{n \in \mathbb{N}}$ and $b = \{b_n\}_{n \in \mathbb{N}}$ be sequences of rational numbers such that $b_n \neq 0$ for all $n \in \mathbb{N}$. Suppose

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 1.$$

(i) Are a and b equivalent?
(ii) Are a and b equivalent if a is a $\mathbb{Q}$-bounded sequence?

Problem 4. You cannot use the existence (or the l.u.b. property) of the ordered field of real numbers in this problem, so you must work "within" $\mathbb{Q}$.

(1) Show that every $\mathbb{Q}$-bounded monotone sequence of rational numbers is $\mathbb{Q}$-Cauchy.

(2) Consider the following sequence:

$$x_n = \begin{cases} 2, & \text{if } n = 0, \\ x_{n-1} - \dfrac{x_{n-1}^2 - 2}{2x_{n-1}}, & \text{if } n \geq 1. \end{cases}$$

Confirm that $\{x_n\}_{n\in\mathbb{N}}$ is well-defined, i.e., $x_n \neq 0$ for all $n \in \mathbb{N}$. Show that $\{x_n\}_{n\in\mathbb{N}}$ is $\mathbb{Q}$-Cauchy, but not convergent in $\mathbb{Q}$.

Hint. This problem demonstrates that the Monotone Convergence Theorem need not hold within $\mathbb{Q}$*.*

Problem 5. A *digit* is any element of the set $\{0, 1, ..., 9\}$. An *admissible sequence of digits* is a sequence $\{a_n\}_{n\geq 1} \subset S$ satisfying the property: there is no $N \geq 1$ such that $a_n = 9$ for all $n \geq N$. Given $x \in [0, 1)$, we say that an admissible sequence of digits $\{d_n\}_{n\in\mathbb{N}}$ is a *decimal representation* of x if

$$\sup\left\{ D_n = \sum_{j=1}^{n} \frac{d_j}{10^j} : n \geq 1 \right\} = x.$$

Show that every admissible sequence of digits is the decimal representation of a number $x \in [0, 1)$, and conversely, ever $x \in [0, 1)$ admits a unique decimal representation as defined above. *Hint. For the converse, define* d_k *recursively so that* $D_n \leq x < D_n + 10^{-n}$ *for all* $n \geq 1$*.*

Note. In this problem, you may use freely use the standard properties of real numbers.