UM 204 - MIDTERM EXAMINATION Solutions

Problem 1. (*24 points*) For each of the statements below, determine whether it is (necessarily) true or (sometimes) false. If you circle TRUE, you must provide a proof. If you circle FALSE, you must provide a counterexample and justify it.

(*a*) In any metric space, a bounded sequence always admits a convergent subsequence.

TRUE $\boxed{\text{FALSE}}$

Let d be the discrete metric on $\mathbb{N}$. Then, $(\mathbb{N}, d)$ is complete. To see this, let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in this metric. Then, letting $\varepsilon = 1/2$, we obtain an $N \in \mathbb{N}$ such that $d(x_n, x_m) < 1/2$ for all $m, n \geq N$. By the definition of the discrete metric, this is only possible if $x_n = x_m$ for all $m, n \geq N$. Thus, $\{x_n\}_{n\in\mathbb{N}}$ is eventually constant, and therefore, convergent.

Now, let $x_n = n$ for each $n \in \mathbb{N}$. Then, each $x_n \in B(0; 2)$. Thus, $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence. However, for any subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$, $d(x_{n_k}, x_{n_\ell}) = d(n_k, n_\ell) = 1$ whenever $k \neq \ell$. Thus, the sequence has no convergent subsequences.

(*b*) The function $d : \mathbb{R}^k \times \mathbb{R}^k \to [0, \infty)$ given by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{2y}\|, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbf{k}},$$

is a metric on $\mathbb{R}^k$.

TRUE $\boxed{\text{FALSE}}$

Let $\mathbf{x} = (\mathbf{2}, ..., \mathbf{2})$ and $\mathbf{y} = (\mathbf{1}, ..., \mathbf{1})$. Then, $\mathbf{x} \neq \mathbf{y}$, but $d(\mathbf{x}, \mathbf{y}) = \mathbf{0}$. Thus, d is not a metric.

(c) In any metric space, the closure of a connected set is connected.

$\boxed{\text{TRUE}}$ FALSE

Let E be a connected set in a metric space (X, d). Suppose $\overline{E}$ is not connected. Then, there exist nonempty separated sets A, B such that $\overline{E} = A \cup B$. We claim that $A' = A \cap E$ and $B' = B \cap E$ are nonempty and separate E.

(i) Suppose A' is empty. Then, $A \subset \overline{E} \setminus E$. Since $\overline{E} = A \cup B$, it must be that $E \subseteq B$. Thus, $A \subseteq \overline{E} \subseteq \overline{B}$, which contradicts the fact that A and B are separated. So, A', and by the same logic, B' are nonempty.

(ii) Note that $E = (A \cup B) \cap E = (A \cap E) \cup (B \cap E) = A' \cup B'$.

(iii)Note that $A' \cap \overline{B'} \subseteq A \cap \overline{B} = \emptyset$. Similarly, $\overline{A'} \cap B' = \emptyset$.

(d) Given subsets $E_1, E_2, \ldots$ of a metric space (X, d),

$$\bigcap_{j=1}^{\infty} E_j^{\circ} = \left(\bigcap_{j=1}^{\infty} E_j \right)^{\circ}.$$

TRUE $\boxed{\text{FALSE}}$

Let $E_j = (-1/j, 1/j)$, $j \in \mathbb{N}$. Then, the LHS is $\{0\}$, but the RHS is the empty set.

Problem 2. (*10 points*) Recall the construction of $\mathbb{Q}$ as the set of equivalence classes of the relation R on $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ given by $(a,b)R(c,d) \iff ad = bc$. We say that $[(a,b)] \leq [(c,d)]$ if $(bc - ad)(bd) \geq 0$. Using only the arithmetic and order properties of integers, show that the relation $\leq$ is well-defined.

Suppose $(a,b)R(a',b')$ and $(c,d)R(c',d')$, i.e.,

$$\begin{aligned} ab' &= a'b \\ cd' &= c'd. \end{aligned}$$

We claim that $(bc - ad)bd \geq 0 \iff (b'c' - a'd')b'd' \geq 0$. For this, observe that

$$\begin{aligned} (a'c')^2(bc - ad)(bd) = (a'bc'c - aa'c'd)(a'b)(c'd) &= (ab'c'c - aa'cd')(ab')(cd') \\ &= (ac)^2(b'c' - a'd')(b'd'). \end{aligned}$$

Thus,

$$(a'c')^2(bc - ad)(bd) \geq 0 \iff (ac)^2(b'c' - a'd')(b'd') \geq 0. \tag{1}$$

Now, we will repeatedly use the fact that for any nonzero integer p and integer q,

$$p^2 q \geq 0 \iff q \geq 0. \tag{2}$$

Case 1. $a = 0$. Since b and b' are nonzero, this case occurs if and only if $a' = 0$. Then, $(bc - ad)bd = b^2(cd)$ and $(b'c' - a'd')b'd' = (b')^2(c'd')$. Now, since $b, b', d, d' \neq 0$,

$$b^2(cd) \geq 0 \iff cd \geq 0 \iff (d')^2 cd \geq 0 \iff c'd'(d)^2 \geq 0 \iff c'd' \geq 0 \iff (b')^2 c'd' \geq 0.$$

Thus, we are done in this case.

Case 2. $c = 0$. This is analogous to the case above.

Case 3. $(ac)^2 > 0$. In this case, neither $a = 0$, nor $c = 0$. Thus, neither $a' = 0$, nor $c' = 0$. So this case occurs precisely when $(a'c')^2 > 0$. Our main claim follows from (1) and (2).

Problem 3. (*12 points*) Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be the sequence given by

$$\begin{aligned} x_0 &= 0; \quad x_1 = 1; \\ x_n &= \frac{1}{2}(x_{n-1} + x_{n-2}), \quad n \geq 2. \end{aligned}$$

Show that $\{x_n\}_{n\in\mathbb{N}}$ is a convergent sequence. (You may cite limits of standard sequences and series without proof.)

Note that, for $n \geq 2$,

$$|x_n - x_{n-1}| = \frac{1}{2}|x_{n-1} - x_{n-2}| = \frac{1}{2^{n-1}}|x_1 - x_0| = \frac{1}{2^{n-1}}.$$

Thus, for any $n > m \geq 2$, we have that

$$\begin{aligned} |x_n - x_m| \leq |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m| \leq \frac{1}{2^{n-1}} + \cdots + \frac{1}{2^m} &= \frac{1}{2^m}(1 + \cdots + \frac{1}{2^{n-m-1}}) \\ &= \frac{1}{2^m}\frac{1 - 1/2^{n-m}}{1 - 1/2} \leq \frac{1}{2^{m+1}}. \end{aligned}$$

Now, let $\varepsilon > 0$. By the A. P. of reals, there is an $N \in \mathbb{N}$ be such that $2^{N+1} > 1/\varepsilon$. Then, for all $m, n \geq N$, we have that $|x_n - x_m| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of $\mathbb{R}$, $\{x_n\}_{n\in\mathbb{N}}$ is a convergent sequence.

Problem 4. (*12 points*) Let $\mathscr{R}$ denote the set of equivalence classes[1] of Cauchy sequences of rational numbers. The equivalence class of $\{a_n\}_{n\in\mathbb{N}}$ is denoted by $[\{a_n\}_{n\in\mathbb{N}}]$. We say that $\alpha = [\{a_n\}_{n\in\mathbb{N}}]$ is positive if there is some positive rational $c > 0$ and some $N \in \mathbb{N}$ such that

$$a_n > c, \quad \forall n \geq N.$$

We say that α is negative if there is some negative rational $c < 0$ and some $N \in \mathbb{N}$ such that

$$a_n < c, \quad \forall n \geq N.$$

Assume that these are well-defined notions. Show that, for any $\alpha \in \mathscr{R}$, **one and only one** of the following holds:

(a) α is the equivalence class of the constant 0 sequence;

(b) α is positive;

(c) α is negative.

You may not use the fact that $\mathscr{R}$ is the set of real numbers. You may only use the properties of the ordered field $(\mathbb{Q}, +, \cdot, \leq)$.

Case 1. Suppose $\{a_n\}_{n\in\mathbb{N}}$ is equivalent to the constant 0 sequence. Then, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n| < \varepsilon$ for all $n \geq N$. Thus, there is no rational $c > 0$ such that $|a_n| > c$ for large enough n. Thus, (b) and (c) cannot hold for this sequence, and we are done.

Case 2. Suppose $\{a_n\}_{n\in\mathbb{N}}$ is not equivalent to the constant 0 sequence. Then, there exists a rational $\varepsilon_0 > 0$ such that for every $M \in \mathbb{N}$, there is some $n(M) \geq M$ such that

$$|a_{n(M)}| > \varepsilon_0.$$

Since $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, for $\varepsilon = \varepsilon_0/2$, there is an $N \in \mathbb{N}$ such that, for all $m, n \geq N$,

$$|a_n - a_m| < \varepsilon.$$

We may choose $m = n(N) \geq N$. Then, for all $n \geq N$,

$$|a_n - a_{n(N)}| < \varepsilon = \varepsilon_0/2 \quad \text{OR} \quad a_{n(N)} - \varepsilon_0/2 < a_n < a_{n(N)} + \varepsilon_0/2.$$

Case a. $a_{n(N)} > 0$. Then, $a_n > \varepsilon_0 - \varepsilon_0/2$ for all $n \geq N$. By definition, α is positive.
Case b. $a_{n(N)} < 0$. Then, $a_n < -\varepsilon_0 + \varepsilon_0/2 = -\varepsilon_0/2$ for all $n \geq N$. By definition, α is negative.

It is clear that (b) and (c) cannot occur simultaneously since this would be mean that there is an $N \in \mathbb{N}$ such that $a_n > 0$ and $a_n < 0$ for all $n \geq N$, which is absurd.

[1] Recall that two rational sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are said to be equivalent if, for every rational $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n - b_n| < \varepsilon$ for all $n \geq N$.

Problem 5. (*12 points*) Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is said to be *upper semicontinuous at* $p \in X$ if, for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converging to p,

$$\limsup_{n\to\infty} f(x_n) \leq f(p).$$

A function is said to be upper semicontinuous on X if it is upper semicontinuous at each point of X. Prove that if f is upper semicontinuous on X, then

$$U_a = \{x \in X : f(x) < a\}$$

is open for any $a \in \mathbb{R}$.

We will show that that $F_a = X \setminus U_a = \{x \in X : f(x) \geq a\}$ is closed for all $a \in \mathbb{R}$. Suppose not, i.e., there is some $a \in \mathbb{R}$ such that F_a is not closed. Thus, there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset F_a$ that converges to some $p \notin F_a$. I.e., $f(x_n) \geq a$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} x_n = p$, but $f(p) < a$. By the upper semicontinuity of f, we have that $\limsup_{n\to\infty} f(x_n) \leq f(p)$. Now, if a sequence is a bounded below by a, then so is its limit superior. Thus, we obtain that

$$a \leq \limsup_{n\to\infty} f(x_n) \leq f(p) < a,$$

which is a contradiction.