UM 204: QUIZ 7

March 22, 2024

Duration. 15 minutes

Maximum score. 10 points

Problem. Let $A \subset \mathbb{R}^n$ be a bounded set. Let $f : A \to \mathbb{R}$ be a uniformly continuous function. Show that $f(A)$ is bounded in $\mathbb{R}$. You may directly cite any result proved in class or stated as a problem in Assignment 07 (posted a week ago).

Suppose $f(A)$ is not bounded. Then, there is a sequence $\{y_n = f(x_n)\}_{n\in\mathbb{N}} \subseteq f(A)$ such that

$$|y_n| > \max\{n, |y_{n-1}| + 1\} \quad \forall n \in \mathbb{N}. \tag{1}$$

Since $\overline{A} \subset \mathbb{R}^n$ is closed and bounded, it is compact. Thus, $\{x_n\}_{n\in\mathbb{N}} \subset \overline{A}$ admits a convergent (and thus, Cauchy) subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$. By Problem 3, $\{y_{n_k} = f(x_{n_k})_{k\in\mathbb{N}}\}$ is a Cauchy sequence in $\mathbb{R}$. Thus, setting $\varepsilon = 1$, there is a $K > 0$ such that for all $k \geq K$, $|y_{n_k} - y_{n_{k+1}}| < 1$. At the same time, by (1),

$$|y_{n_k} - y_{n_{k+1}}| \geq |y_{n_{k+1}}| - |y_{n_k}| \geq 1.$$

This is a contradiction. Thus, $f(A)$ must be bounded.