UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022

QUIZ 8 **APRIL 11, 2022**

PLEASE NOTE the following:

- **Duration:** 15 minutes
- The quiz is to be written with no access to any books, notes, or study materials.

1. Find a rational number that approximates $7^{2/3}$ by

- choosing an appropriate interval $[a, b] \subsetneq \mathbb{R}$, taking $f : [a, b] \to \mathbb{R}$ to be $f(t) := t^{2/3}$, $t \in [a, b]$, and
- letting your approximation be the Taylor polynomial $T_{2,f}(t; x_0)$ — with $x_0 = 8 \in [a, b]$ — evaluated at an appropriate point $x \in [a, b]$,

where your choice of a and b is such that the worst error

$$|7^{2/3} - T_{2,f}(x; x_0 = 8)|$$

predicted by Taylor's Theorem is the least possible. You can report your answer **as a sum of fractions.** Now, with your choices of a and b, give an **explicit** upper bound on the above error.

Note. The above is very similar to Problem 1 of Homework 12.

Solution. Given that $f(t) := t^{2/3}$, $7^{2/3} = f(7)$ and, so, $x = 7$. The desired Taylor polynomial is

$$T_{2,f}(x; x_0 = 8) \;=\; f(8) + f'(8)(x - x_0) + \frac{f^{(2)}(8)}{2!}(x - x_0)^2.$$

We compute

$$f'(t) = \frac{2}{3}t^{-1/3}, \quad f^{(2)}(t) = -\frac{2}{9}t^{-4/3}, \quad f^{(3)}(t) = \frac{8}{27}t^{-7/3}, \;\; \forall t > 0.$$

Thus, the desired approximation is

$$T_{2,f}(7; 8) \;=\; 4 + \frac{2}{3}\left(\frac{1}{2}\right)(-1) + \left(-\frac{1}{9}\right)\left(\frac{1}{16}\right)(-1)^2 \;=\; \frac{527}{144}.$$

To apply Taylor's Theorem to estimate the error of the above approximation, we must take $a < b$ such that $a > 0$ and $x, x_0 \in [a, b]$. Then, Taylor's Theorem applied to $f|_{[a,b]}$ tells us that

$$7^{2/3} \;=\; T_{2,f}(7; 8) + \frac{f^{(3)}(c)}{3!}(7 - 8)^3,$$

where $c \in (7, 8)$. As Taylor's Theorem provides **no further information** on c, we have

$$|7^{2/3} - T_{2,f}(7; 8)| \;\leq\; \frac{1}{3!} \sup_{\xi \in [a,b]} \left| f^{(3)}(\xi) \right|. \tag{1}$$

The worst possible error — i.e., the upper bound given by (1) — is the least possible if we consider the smallest interval $[a, b]$ for which the above analysis makes sense. Thus, we take $a = 7$ and $b = 8$. Finally, as $[7, 8] \ni \xi \mapsto |f^{(3)}(\xi)|$ is a decreasing function, the desired upper bound is

$$\frac{1}{3!} f^{(3)}(7) \;=\; \frac{4}{81(7^{7/3})}.$$