Lecture - 11

Recall :- $\left\{ {n \atop k} \right\}$ = # of set partitions of $[n]$ into k blocks.

Example :- $\left\{ {n \atop 1} \right\} = \left\{ {n \atop n} \right\} = 1$ (Stirling numbers of 2^{nd} kind.

$n \backslash k$	0	1	2	3	4
0	1	0			
1	0	1			
2	|	1	1		
3	|	1	3	1	
4	|	1	7	6	1

Property :- For $1 \le k \le n$, we have

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$

where $\left\{ {n \atop 0} \right\} = \underset{\downarrow}{\delta_{n,0}}$ and $\left\{ {n \atop k} \right\} = 0$ if $k<0$ or $k>n$ for $n \geq 1$

1 if $i=j$ (δ_{ij})
0 ; otherwise

Proof :- For any set partition S, look at n. If it is a singleton, the rest is a set partition of $(n-1)$ into $(k-1)$ blocks.

If not consider a set partition of $[n-1]$ into k blocks and add n to $\underline{[n-1], k\text{ blocks}}$ into the mix in k ways to obtain a set partition of $[n]$ into k blocks.

Every set partition is obtained in this way and there are no repeats. This gives $k \left\{ {n-1 \atop k} \right\}$ $\blacksquare$

Property :- The # of surjective functions from $[n] \to [k]$ is $k! \left\{ {n \atop k} \right\}$

Proof :- Each block in a set partition of $[n]$ denotes the elements mapping to the same element in $[k]$ and $k!$ is the set of ways of rearranging the blocks.

<u>Corollary</u> :- Let X be a formal indeterminate, then
$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}} = X^n \text{ for all } n \in \mathbb{N}$$

<u>Example</u> :- $1.X^{\underline{1}} + 3.X^{\underline{2}} + 1.X^{\underline{3}}$
$$= X + 3(X)(X-1) + X(X-1)(X-2)$$
$$= \cancel{X} + 3\cancel{X^2} - \cancel{3X} + X^3 - \cancel{3X^2} + \cancel{2X}$$
$$= X^3$$

<u>Proof</u> :- Both sides are polynomials in X of degree n.
It suffices to prove the equality for $n+1$ values of X. We will prove that this holds $\forall x \in \mathbb{N}>0$
RHS counts the number of functions from $[n] \to [X]$.
Consider these functions according to the size of their image. If the image has size k, the function restricted to that subset is surjective.
The # of all possible images is $\binom{X}{k}$ and for each choice, there are $k!\left\{ {n \atop k} \right\}$ surjections by the previous property.
Since this is bijective, the # of functions is
$$\sum_{k=0}^{n} \binom{X}{k} k! \left\{ {n \atop k} \right\} = \frac{X!}{(X-k)!} = X^{\underline{k}}$$

<u>Def'n</u> :- The # of set partitions of $[n]$ is called the n^{th} <u>Bell number</u>, denoted by $B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$

The sequence starts $1, 2, 5, 15, 52, \ldots$

<u>Exercise</u> :- $B_{n+1} = \sum_{i=0}^{n} \binom{n}{i} B_i$; Note that this is not a recurrence.

———x———

Per<u>mutations</u> as <u>cycles</u> :-

$S_n \longrightarrow$ set of permutations of $[n]$
Recall the cycle notation for $x \in S_n$.

$\pi = c_1 \cdot c_2 \cdots \cdots c_k$

where each $c_i = (c_{i,1}, c_{i,2}, \ldots, c_{i,len(c_i)})$

Useful Convention skip singletons.

$\pi = \pi_1 \pi_2 \ldots \pi_k$, where each $\boxed{\pi_i = c_i}$

Lemma:- Let $\pi \in S_n$ and $j \in [n]$. Then $\exists\, i \in [n]$ st $\pi_j^i = j$, where $\pi^i = \underbrace{\pi \circ \pi \circ \pi \circ \cdots \circ \pi}_{i \text{ times}}$

Proof:- Consider $\pi_j, \pi_j^2, \ldots, \pi_j^n$
If any of them equals j, we are done.
If not, by PHP, $\exists\, 1 \le k \le l \le n$ st $\pi_j^k = \pi_j^l$.
Now apply π^{-1} k times $\Rightarrow j = \pi_j^{l-k}$. ▨

Note that the cycle notation is not cannonical.

Example:- $\pi = 6754132$ and $j = 7$.
$\Rightarrow \pi^2 = 3214657 \Rightarrow \pi_7^2 = 7 \Rightarrow l = 2$

Notation for cycles:- (a) Each cycle has its smallest element first.

(b) Arrange cycles in increasing order of the smallest element.

Example:- $\pi^2 = (13)(2)(4)(56)(7)$