UM 204: QUIZ 6
March 15, 2024

Duration. 15 minutes

Maximum score. 10 points

Problem. Show that the Cauchy product of two absolutely convergent series is absolutely convergent. Clearly state any theorems that you are using.

Let $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} b_n$ be two absolutely convergent series. Since $\sum_{n\in\mathbb{N}} |a_n|$ and $\sum_{n\in\mathbb{N}} |b_n|$ are convergent series of nonnegative terms, their sops are bounded above. Thus, there exist $M, N > 0$ such that

$$\sum_{n=0}^{m} |a_n| < M \quad \text{and} \quad \sum_{j=0}^{m} b_n < N \quad \forall m \in \mathbb{N}.$$

Let

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Then, for $m \in \mathbb{N}$,

$$\begin{aligned}
\sum_{n=0}^{m} |c_n| &= \sum_{n=0}^{m} \left| \sum_{k=0}^{n} a_k b_{n-k} \right| \\
&\leq \sum_{n=0}^{m} \sum_{k=0}^{n} |a_k||b_{n-k}| \\
&= |a_0||b_0| + (|a_0||b_1| + |a_1||b_0|) + \cdots + (|a_0||b_m| + \cdots + |a_m||b_0|) \\
&= |a_0| \sum_{j=0}^{m} |b_j| + |a_1| \sum_{j=0}^{m-1} |b_j| + \cdots + |a_m||b_0| \\
&= \sum_{k=0}^{m} |a_k| \sum_{j=0}^{m-k} |b_j| \\
&\leq N \sum_{k=0}^{m} |a_k| < MN.
\end{aligned}$$

Since the sops of $\sum_{n\in\mathbb{N}} |c_n|$ is bounded above, it coverges.