HW-2

3) $n\backslash 0 = 0\backslash n$, $n \in \mathbb{N}$

To prove: $n = 0$

Proof:- $a\backslash b$ denotes the equivalence class of (a,b) under $\sim_Z$

Let $\sim_Z$ be equivalence relation on $\mathbb{N}\times\mathbb{N}$ given by

$$(a,b) \sim_Z (c,d) \iff a+d = c+b$$

Given $n\backslash 0 = 0\backslash n$.

i.e the equivalence class $n\backslash 0$ is equal to the equivalence class $0\backslash n$.

We know that, equivalence class $a\backslash b$.

$$a\backslash b = [(a,b)] = \{(c,d) \in \mathbb{N}\times\mathbb{N} : a+d = c+b\}.$$

$n\backslash 0 = 0\backslash n$

$\Rightarrow [(n,0)] = [(0,n)]$

By equality of sets, both sets consists of same elements or, all elements in A belong to B and all elements in B belong to A.

$A = [(n,0)]$

$B = [(0,n)]$

$(n,0)$ is an element of A

Hence $(n,0) \in B$ (By def'n of equality of sets)

$\Rightarrow (n,0) \in \{(c,d) \in \mathbb{N}\times\mathbb{N} : 0+d = c+n\} = [(0,n)]$

$c = n$, $d = 0$

$0+0 = n+n$.

$n+n = 0+0$

$n+n = 0$ [by Peano add'n]

$n+0 = n$

$n+n = 0$.

We know that, if $m,n \in \mathbb{N}$ and $m+n = 0$ then $m = n = 0$ (By Lemma)

$n \in \mathbb{N}$ $n+n = 0 \Rightarrow n = n = 0$

Hence $n = 0$ //

(5) (a). $f : \mathbb{N} \to \mathbb{Z}$

$f(n) := n\backslash 0$ for each $n \in \mathbb{N}$

To prove: f is injective.

Proof:- ~~f is in.~~

$g : A \to B$ g is injective $\Longleftrightarrow$

if $g(x_1) = g(x_2)$ then $x_1 = x_2$.

If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ then f is injective.

$f(x_1) = f(x_2)$

$\Rightarrow x_1\backslash 0 = x_2\backslash 0$

$\Rightarrow x_1 + 0 = x_2 + 0$ (By equating the equivalence class

i.e. $(x_1, 0) \in \{(c,d) \in \mathbb{N}\times\mathbb{N} : x_2 + d = c + 0\} = [(x_2, 0)]$

$\Rightarrow x_2 + 0 = x_1 + 0$)

$x_1 = x_2$ [By Peano add'n $n + 0 := n \ \forall n \in \mathbb{N}$

Here $x_1, x_2 \in \mathbb{N}$ as $(x_1, 0), (x_2, 0) \in \mathbb{N}\times\mathbb{N}$]

Hence $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\Rightarrow$ f is injective. ▨

4) $(m',n') \sim_{\mathbb{Z}} (m,n)$ and $(a',b') \sim_{\mathbb{Z}} (a,b) \Leftrightarrow ((m +_{\mathbb{N}} a),(n +_{\mathbb{N}} b)) \sim_{\mathbb{Z}} ((m' +_{\mathbb{N}} a'),(n' +_{\mathbb{N}} b'))$

$(a,b) \sim_{\mathbb{Z}} (c,d) \Leftrightarrow a +_{\mathbb{N}} d = c +_{\mathbb{N}} b$

$(m',n') \sim_{\mathbb{Z}} (m,n) \Rightarrow (m' +_{\mathbb{N}} n) = m +_{\mathbb{N}} n' \longrightarrow ①$

$(a',b') \sim_{\mathbb{Z}} (a,b) \Rightarrow a' +_{\mathbb{N}} b = a +_{\mathbb{N}} b' \longrightarrow ②$

To prove:-

$((m +_{\mathbb{N}} a),(n +_{\mathbb{N}} b)) \sim_{\mathbb{Z}} ((m' +_{\mathbb{N}} a'),(n' +_{\mathbb{N}} b'))$

$(m +_{\mathbb{N}} a) +_{\mathbb{N}} (n' + b') = (m' +_{\mathbb{N}} a') + (n +_{\mathbb{N}} b)$

$LHS = (m +_{\mathbb{N}} a) +_{\mathbb{N}} (n' +_{\mathbb{N}} b')$

$= m +_{\mathbb{N}} (a +_{\mathbb{N}} n') +_{\mathbb{N}} b'$ (By ~~comm~~ associativity in Peano add'n)

$= m +_{\mathbb{N}} (n' +_{\mathbb{N}} a) +_{\mathbb{N}} b'$ (By commutativity in Peano add'n)

$= (m +_{\mathbb{N}} n') +_{\mathbb{N}} (a +_{\mathbb{N}} b')$ (By associativity in Peano add'n)

$= (m' +_{\mathbb{N}} n) +_{\mathbb{N}} (a' +_{\mathbb{N}} b)$ [By ① & ②]

$= m' +_{\mathbb{N}} (n +_{\mathbb{N}} a') +_{\mathbb{N}} b$ (By associativity in Peano add'n)

$= m' +_{\mathbb{N}} (a' +_{\mathbb{N}} n) +_{\mathbb{N}} b$ (By commutativity in Peano add'n)

$= (m' +_{\mathbb{N}} a') +_{\mathbb{N}} (n +_{\mathbb{N}} b)$ (By associativity in Peano add'n)

$= RHS$

Hence $LHS = RHS$

i.e $((m +_{\mathbb{N}} a),(n +_{\mathbb{N}} b)) \sim_{\mathbb{Z}} ((m' +_{\mathbb{N}} a'),(n' +_{\mathbb{N}} b'))$

Hence the given statement is well defined. $\boxtimes$

(b) To prove:- (i) $f(m +_{\mathbb{N}} n) = f(m) + f(n)$

(ii) $f(m \times_{\mathbb{N}} n) = f(m) \times f(n) \ \forall m, n \in \mathbb{N}$

Proof:-

(i) $f(m +_{\mathbb{N}} n) = (m +_{\mathbb{N}} n) \setminus 0$ (By $f(n) = n \setminus 0 \ \forall n \in \mathbb{N}$) $\to$ (1)

$f(m) + f(n) = (m \setminus 0) + n \setminus 0$

$= (m +_{\mathbb{N}} n) \setminus (0 +_{\mathbb{N}} 0)$ [By def'n of $+$

$(a \setminus b) + c \setminus d := (a +_{\mathbb{N}} c) \setminus (b +_{\mathbb{N}} d)$]

$= (m +_{\mathbb{N}} n) \setminus 0$ [By Peano add'n $n + 0 := n \ \forall n \in \mathbb{N}$]

$= f(m +_{\mathbb{N}} n)$ (By (1))

Hence $f(m +_{\mathbb{N}} n) = f(m) + f(n)$

(ii) $f(m \times_{\mathbb{N}} n) = ((m \times_{\mathbb{N}} n) \setminus 0)$ (By def'n $f(n) := n \setminus 0 \ \forall n \in \mathbb{N}$ Here $m \times_{\mathbb{N}} n \in \mathbb{N}$) $\to$ (2)

$f(m) \times f(n) = (m \setminus 0) \times (n \setminus 0)$

$= ((m \times_{\mathbb{N}} n) +_{\mathbb{N}} (0 \times_{\mathbb{N}} 0)) \setminus ((m \times_{\mathbb{N}} 0) +_{\mathbb{N}} (0 \times_{\mathbb{N}} n))$

$= ((m \times_{\mathbb{N}} n) + 0) \setminus (0 +_{\mathbb{N}} 0)$ [By Peano multip'n) $0 \times m = m \times 0 = 0$

$= (m \times_{\mathbb{N}} n) \setminus 0$ [By Peano add'n] $n + 0 := n \ \forall n \in \mathbb{N}$

$= f(m \times_{\mathbb{N}} n)$ (By (2)).

(2) By def'n of function. $f,g : X \to Y$

$f \subset X \times Y$ (Cartesian product)

(i) for each $x \in X$, $\exists y \in Y$ such that $(x,y) \in f$

(ii) (x,y) & $(x,y') \in f \Rightarrow y = y'$

$\Rightarrow (x, f(x)) \in f$

$\Rightarrow f = \{(x, f(x)) \mid x \in X\}$

Since, $f, g : X \to Y$

$f = \{(x, f(x)) \mid x \in X\}$, $g = \{(x, g(x)) \mid x \in X\}$

$f = g$

$\{(x, f(x)) \mid x \in X\} = \{(x, g(x)) \mid x \in X\}$

By def'n of equality / Axiom of equality,

if all the elements in A are in set B & all elements in B are in set A then A=B.

$f = g$

$\{(x, f(x)) \mid x \in X\} = \{(y, g(y)) \mid y \in X\}$

~~Take~~

Let a be an arbitrary element ~~such~~ from X

i.e $a \in X$ be an arbitrary element.

$(a, f(a)) \in f$ [by def'n of f]

$(a, f(a)) \in g$ (by axiom of equality i.e all elements in A are in set B)

$(a, f(a)) = (y, g(y))$

By def'n of ordered pair,

$a = y$ & $f(a) = g(y)$

$f(a) = g(a)$ for some $a \in X$ arbitrary

Since $f(a) = g(a)$ for some arbitrary $a \in X$

$f(a) = g(a)$ $\forall a \in X$ i.e $f(x) = g(x)$, $\forall x \in X$ $\blacksquare$

$= (a' \times_{\mathbb{Z}} (b \times_{\mathbb{Z}} y') \times_{\mathbb{Z}} y) +_{\mathbb{Z}} ((b' \times_{\mathbb{Z}} x') \times_{\mathbb{Z}} b \times_{\mathbb{Z}} y)$

[By associativity of $\mathbb{Z}$
$a \times (b \times c) \times d = (a \times b) \times (c \times d)$]

[By commutativity of $\mathbb{Z}$
$c \times d = d \times c$
$\cancel{a \times b = b \times a}$
here $c = b$, $d = b' \times x'$
$b \times (b' \times x') = (b' \times x') \times b$]

$= \cancel{(a' \times_{\mathbb{Z}} b)} \times_{\mathbb{Z}}$

$= (a' \times_{\mathbb{Z}} (y' \times_{\mathbb{Z}} b) \times_{\mathbb{Z}} y) +_{\mathbb{Z}} ((b' \times_{\mathbb{Z}} x') \times_{\mathbb{Z}} (b \times_{\mathbb{Z}} y))$

[By commutativity of $\mathbb{Z}$]

$= ((a' \times_{\mathbb{Z}} y') \times_{\mathbb{Z}} (b \times_{\mathbb{Z}} y)) +_{\mathbb{Z}} (b' \times_{\mathbb{Z}} x') \times_{\mathbb{Z}} (b \times_{\mathbb{Z}} y)$

[By associativity of $\mathbb{Z}$]

$= ((a' \times_{\mathbb{Z}} y') +_{\mathbb{Z}} (b' \times_{\mathbb{Z}} x')) \times_{\mathbb{Z}} (b \times_{\mathbb{Z}} y)$

[By distributivity of $\mathbb{Z}$
$(c+d)e = ce + de$]

Hence

$((a \times_{\mathbb{Z}} y) +_{\mathbb{Z}} (b \times_{\mathbb{Z}} x)) \times_{\mathbb{Z}} (b' \times_{\mathbb{Z}} y') =$

$((a' \times_{\mathbb{Z}} y') +_{\mathbb{Z}} (b' \times_{\mathbb{Z}} x')) \times_{\mathbb{Z}} (b \times_{\mathbb{Z}} y)$

By def'n $\sim_{\mathbb{Q}}$

$((a \times_{\mathbb{Z}} y) +_{\mathbb{Z}} (b \times_{\mathbb{Z}} x), (b \times_{\mathbb{Z}} y)) \sim_{\mathbb{Q}} ((a' \times_{\mathbb{Z}} y') +_{\mathbb{Z}} (b' \times_{\mathbb{Z}} x'), (b'$

Hence addition is well defined.

$\blacksquare$

(1) To show add'n is welldefined

$$(a|b) + (x|y) := ((a \times_{\mathbb{Z}} y) +_{\mathbb{Z}} (b \times_{\mathbb{Z}} x)) \,/\, (b \times_{\mathbb{Z}} y)$$

We have to show;

a/f $(a,b) \sim_{\mathbb{Q}} (a',b')$ & $(x,y) \sim_{\mathbb{Q}} (x',y')$

then $((a \times_{\mathbb{Z}} y) +_{\mathbb{Z}} (b \times_{\mathbb{Z}} x), (b \times_{\mathbb{Z}} y)) \sim_{\mathbb{Q}} ((a' \times_{\mathbb{Z}} y') +_{\mathbb{Z}} (b' \times_{\mathbb{Z}} x'), (b' \times_{\mathbb{Z}} y'))$

where

$(w,x) \sim_{\mathbb{Q}} (y,z) \Longleftrightarrow (w \times_{\mathbb{Z}} z) = (y \times_{\mathbb{Z}} x)$

implies

$(a,b) \sim_{\mathbb{Q}} (a',b') \Rightarrow (a \times_{\mathbb{Z}} b') = (a' \times_{\mathbb{Z}} b) \longrightarrow (1)$

$(x,y) \sim_{\mathbb{Q}} (x',y') \Rightarrow (x \times_{\mathbb{Z}} y') = (x' \times_{\mathbb{Z}} y). \longrightarrow (2)$

To prove:-

$$((a \times_{\mathbb{Z}} y) +_{\mathbb{Z}} (b \times_{\mathbb{Z}} x), (b \times_{\mathbb{Z}} y)) \sim_{\mathbb{Q}} ((a' \times_{\mathbb{Z}} y' +_{\mathbb{Z}} (b' \times_{\mathbb{Z}} x'), (b' \times_{\mathbb{Z}} y'))$$

Proof:- $((a \times_{\mathbb{Z}} y) +_{\mathbb{Z}} (b \times_{\mathbb{Z}} x)) \times_{\mathbb{Z}} (b' \times_{\mathbb{Z}} y')$

$= ((a \times_{\mathbb{Z}} y) \times_{\mathbb{Z}} (b' \times_{\mathbb{Z}} y')) +_{\mathbb{Z}} ((b \times_{\mathbb{Z}} x) \times_{\mathbb{Z}} (b' \times_{\mathbb{Z}} y'))$

[By distributivity of $\mathbb{Z}$]

$= ((a \times_{\mathbb{Z}} (y \times_{\mathbb{Z}} b') \times_{\mathbb{Z}} y') +_{\mathbb{Z}} ((b \times_{\mathbb{Z}} (x \times_{\mathbb{Z}} b') \times_{\mathbb{Z}} y')$

(By associativity of $\mathbb{Z}$]

$= (a \times_{\mathbb{Z}} (b' \times_{\mathbb{Z}} y) \times_{\mathbb{Z}} y') +_{\mathbb{Z}} (b \times_{\mathbb{Z}} (b' \times_{\mathbb{Z}} x) \times_{\mathbb{Z}} y')$

[By commutativity of $\mathbb{Z}$ over $\times$]

$= ((a \times_{\mathbb{Z}} b') \times_{\mathbb{Z}} (y \times_{\mathbb{Z}} y')) +_{\mathbb{Z}} ((b \times_{\mathbb{Z}} b') \times_{\mathbb{Z}} (x \times_{\mathbb{Z}} y'))$

[By associativity of $\mathbb{Z}$ over $\times$]

$= ((a' \times_{\mathbb{Z}} b) \times_{\mathbb{Z}} (y \times_{\mathbb{Z}} y')) +_{\mathbb{Z}} ((b \times_{\mathbb{Z}} b') \times_{\mathbb{Z}} (x' \times_{\mathbb{Z}} y)).$

[By (1) & (2)]

$= ((a' \times_{\mathbb{Z}} b) \times_{\mathbb{Z}} (y' \times_{\mathbb{Z}} y)) +_{\mathbb{Z}} (b \times_{\mathbb{Z}} (b' \times_{\mathbb{Z}} x') \times_{\mathbb{Z}} y)$

[By commutativity of $\mathbb{Z}$

$y \times_{\mathbb{Z}} y' = y' \times_{\mathbb{Z}} y$

By associativity of $\mathbb{Z}$

$(a \times b) \times (c \times d) = a \times (b \times c) \times d$.

① Let $P(n)$ be a statement such that $\Sigma(S(n))$ is true ①

For $n=0$,

$P(0) \Rightarrow \Sigma(S(0)) \Rightarrow \Sigma(1)$ is true (given)

$P(0)$ is true

Assume $P(n)$ is true.

i.e assume $\Sigma(S(n))$ is true

if $P(n)$ is true $\Rightarrow \Sigma(S(n))$ is true

$\Rightarrow \Sigma(S(S(n)))$ is true

(If $\Sigma(m)$ is true then $\Sigma(S(m))$ is true)

Here $m = S(n)$

$S(m) = S(S(n))$

Hence $\Sigma(S(S(n)))$ is true

i.e $P(S(n))$ [By ①]

Hence $P(0)$ is true & $P(S(n))$ is true whenever $P(n)$ is true

$\therefore$ $P(n)$ is true $\forall n \in \mathbb{N}$ (By Peano Axiom 5 induction)

Hence $P(n)$ is true $\forall n \in \mathbb{N}$

i.e $\Sigma(S(n))$ is true $\forall n \in \mathbb{N}$ (By ①)

if n is natural no. then $S(n)$ is also natural no. (Peano Axiom 2)

By axiom 3, 0 is not successor of any Natural number.

$\Sigma(S(n))$ is true $\forall n \in \mathbb{N}$

Hence $\Sigma(m)$ is true $\forall m \in \mathbb{N} - \{0\}$

i.e $\Sigma(n)$ is true for every natural number $n \neq 0$.

$\boxtimes$