Lecture - 19

Recall :- Matrix Tree theorem (MTT)

G is a graph, L is laplacian, L_0 is its reduced laplacian, then the # of Spanning trees of G = $\det L_0$

Cayley's thm :- # of spanning trees is n^{n-2} [labelled $[n]$ vertices]

Proof :- For K_n, $L = \begin{bmatrix} n-1 & & -1 \\ & \ddots & \\ -1 & & n-1 \end{bmatrix}_{n \times n}$

$$\Rightarrow L_0 = \begin{bmatrix} n-1 & & -1 \\ & \ddots & \\ -1 & & n-1 \end{bmatrix}_{(n-1)\times(n-1)}$$

$R_1 \to R_1 + R_2 + \dots + R_{n-1}$

$$L_0 \to \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -1 & n-1 & & -1 \\ & & \ddots & \\ & -1 & & n-1 \end{bmatrix} = L_0'$$

Add the first row to every other row.

$$L_0' \to \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & n & & 0 \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & 0 & \\ 0 & 0 & & n \end{bmatrix} = L_0''$$

L_0'' is upper ~~triangular~~ [illegible].

$\Rightarrow \det L_0 = \det L_0'' = 1 \cdot n^{n-2} = n^{n-2}$. $\square$

To prove Cauchy Binet, we need

Lemma :- Let A be a $n\times m$ matrix, B be $m\times n$ matrix. Then
$$\lambda^m \det(\lambda I_n + AB) = \lambda^n \det(\lambda I_m + BA)$$

Note that this implies $\det(I_n + AB) = \det(I_m + BA)$

Proof :- Use the idea of block matrices.
CHECK that

(1)
$$\begin{array}{cc} \left[\begin{array}{c|c} \lambda I_n & A \\ \hline B & \lambda I_m \end{array}\right] & \begin{array}{c} n \\ m \end{array} \\ \begin{array}{cc} n & m \end{array} & (n+m)\times(n+m) \end{array} = \underbrace{\left[\begin{array}{c|c} I_n & 0 \\ \hline B & I_m \end{array}\right]}_{\det 1} \left[\begin{array}{c|c} \lambda I_n & 0 \\ \hline 0 & \lambda I_m - BA \end{array}\right] \cdot \underbrace{\left[\begin{array}{c|c} I_n & A \\ \hline 0 & I_m \end{array}\right]}_{\det 1}$$

(2)
$$\begin{array}{cc} \left[\begin{array}{c|c} \lambda I_n & A \\ \hline B & \lambda I_m \end{array}\right] & \begin{array}{c} n \\ m \end{array} \\ \begin{array}{cc} n & m \end{array} & (n+m)\times(n+m) \end{array} = \underbrace{\left[\begin{array}{c|c} I_n & A \\ \hline 0 & I_m \end{array}\right]}_{\det 1} \left[\begin{array}{c|c} \lambda I_n - AB & 0 \\ \hline 0 & I_m \end{array}\right] \underbrace{\left[\begin{array}{c|c} I_n & 0 \\ \hline B & I_n \end{array}\right]}_{\det 1}$$

Note that each matrix on both RHSs are upper or lower Δlar block matrices and det of such matrices is the product of det of diagonal. Replace A by $-A$ (B resp)

Recall Cauchy Binet Formula :- $A_{n\times m}$, $B_{m\times n}$
$$\det(AB) = \sum_{S\in\binom{[m]}{n}} \det(A_{[n],S}) \cdot \det(B_{S,[n]})$$

Proof :- Use the previous lemma and compare the co-efficient of λ^{m-n} in
$$\lambda^{m-n}\det(\lambda I_n + AB) = \det(\lambda I_m + BA)$$
LHS gives the constant term in $\det(\lambda I_n + AB)$ which is simply $\det(AB)$

RHS gives the sum of all $n\times n$ principal minors of BA
$\hookrightarrow$ Same rows and columns.

which is $\displaystyle\sum_{S\in\binom{[m]}{n}} \det((BA)_{S,S})$
$$= \sum_{S\in\binom{[m]}{n}} \det(B_{S,[n]}) \cdot \det(A_{[n],S}) \qquad \square$$

———X———

Planar Graphs :-

Def'n :- A graph that can be drawn in the plane without edges intersecting at non-vertices is called a planar graph. A planar graph together with its planar embedding is called a plane graphs.

Example :-

These are isomorphic as graphs but not as plane graphs.

Note that edges need not be straight lines in a planar graph.

Fact (Jordan-Curve Thm) :- A plane graph partitions the plane (i.e., $\mathbb{R}^2$) into disjoint regions called faces (including the unbounded face (F_4)).

The above graphs are not isomorphic planar, because of adjacency of faces.

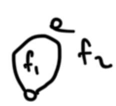
Euler's Thm :- Let G be a connected plane graph (possibly with loops and ||el edges) with v vertices, e edges and f faces. Then $v - e + f = 2$

Proof :- Induction on e.

If $e = 1$, then

$v = 2, e = 1, f = 1$ [unbounded face only]

or



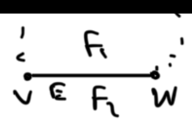
$v = 1, e = 1, f = 2$

Suppose the result holds for all graphs on $e-1$ edges. Let G be a connected graph with e edges. Suppose, we can find an edge E in G st $\underbrace{G/E}_{G'}$ is connected

This means E is a part of cycle in G.

That means there are 2 faces

that means there are 2 faces
on each side of E.

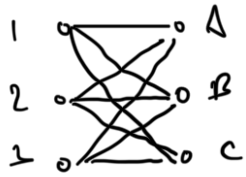


Removing E merges these two faces.
Thus G' has v vertices, e-1 edges and f-1 faces.
$\Rightarrow v-(e-1)+(f-1)=2$
$\Rightarrow v-e+f=2$
On the other hand, suppose no such E exists. Then G must be a tree.
For a tree, $e = v-1$ and $f=1$
$\Rightarrow v-e+f = v-v+1+1 = 2$

Remark :- All planar graphs can be embedded on a sphere
$\Rightarrow$ The same result and proof holds for polyhedra.

Def'n :- A bipartite graph $G=(V,E)$ is one where we can partition $V = V_1 \sqcup V_2$ st no edge joins vertices within V_1 or V_2.
The complete bipartite graph $K_{m,n}$ is the bipartite graph where $|V_1| = m$, $|V_2| = n$ and $\{v_1, v_2\} \in E$ $\forall$ $v_1 \in V_1$ and $v_2 \in V_2$

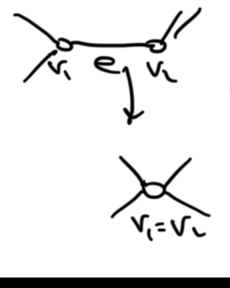
Corollary :- $K_{3,3}$ is not a planar.
Proof :- $K_{3,3}$ has $v=6$, $e=9$. If it were planar, $6-9+f=2 \Rightarrow f=5$. But all its faces are 4 sided. That means we should have 20 edges, except that each edge is shared by 2 faces i.e., we need 10 edges. But we have 9 only.



(Contrd)

Exercise :- Prove K_5 is also not planar using a similar argument.

FYI :- [For Your Information]
A minor of a graph is obtained by deleting vertices or edges of G, or contracting edges in G.



Contraction

Kuratowski's thm :- A graph is planar iff it doesnot contain K_5 or $K_{3,3}$ as a minor.