UM 204 HOMEWORK ASSIGNMENT 6

Posted on March 06, 2024
(NOT FOR SUBMISSION)

- These problems are for self-study. Try these **on your own** before seeking hints.
- Some of these problems will be (partially) discussed at the next tutorial.
- A 15-min. quiz based on this assignment will be conducted at the end of the tutorial section.

Problem 1. Determine the precise values of $a, b, c \in \mathbb{R}$ for which

$$\sum_{n=3}^{\infty} \frac{1}{n^a (\ln n)^b (\ln \ln n)^c}$$

converges.

Problem 2. Given series $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} b_n$, recall that the product is defined as the series $\sum_{n\in\mathbb{N}} c_n$, where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$. Assume throughout that $a_0, b_0 > 0$.

(1) Suppose $a_n, b_n \geq 0$ for all $n \in \mathbb{N}_+$. Show that if $\sum_{n\in\mathbb{N}} c_n$ converges, then so must $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} b_n$.

(2) Show that the above is not true in general (i.e., if we don't assume that a_n and b_n are nonnegative for all $n \in \mathbb{N}_+$).

(3) Provide an exmple where $\sum_{n\in\mathbb{N}} a_n$ is absolutely convergent and $\sum_{n\in\mathbb{N}} b_n$ is convergent, but $\sum_{n\in\mathbb{N}} c_n$ is not absolutely convergent. Compare this with Problem 13 in Chapter 3 of Rudin's book.

Problem 3. Let $\{a_j\}_{n\in\mathbb{N}_+}$ be a sequence of integers such that $1 \leq a_n \leq n-1$ for all $n \geq 2$. Give a necessary and sufficient condition on the sequence $\{a_n\}_{n\in\mathbb{N}_+}$ for

$$\sum_{n=1}^{\infty} \frac{a_n}{n!}$$

to be rational.

Problem 4. Let $f : X \to Y$ be a function from one metric space (X, d_X) to another (Y, d_Y). Prove that f is continuous on X if and only if

$$f\left(\overline{A}\right) \subseteq \overline{f(A)} \quad \text{for every } A \subseteq X.$$

Provide an example to show that for a continuous f, $f\left(\overline{A}\right)$ need not coincide with $\overline{f(A)}$ for every subset A of X.

Problem 5. Provide an example of a function f on $\mathbb{R}^2$ such that the restriction of f to every straight line in $\mathbb{R}^2$ is continuous, but f itself is not continuous on $\mathbb{R}^2$. (Hint: f can be constructed to be continuous everywhere except at the origin.)

Problem 6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the functions f_y and f_x given by

$$\begin{aligned} f_y(x) &= f(x, y), \\ f_x(y) &= f(x, y), \end{aligned}$$

are both continuous on $\mathbb{R}$ for all $y \in \mathbb{R}$ and $x \in \mathbb{R}$. Further, assume that the function f_y is monotonically increasing for every $y \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}^2$. Here, $\mathbb{R}^2$ is assumed to have the standard metric, but it might be easier to consider the d_∞-metric (see Assignment 04, Problem 5).