Lecture - 3

Axioms of Set Theory (ZFC)

↳ (Zermelo-Frenknel Choice)

<u>Def</u>'n :- A set is a well-defined collection of objects, which we call elements.
We write $x \in A$ if x is an element of A.
Note that sets can themselves be objects.

Axioms :-

1, Sets are themselves objects. If A, B are sets, then it makes sense to ask whether A is an element of B or not.
Remark :- There is a branch of set theory called pure set theory in which all objects are sets.
Eg :- $0 \leftrightarrow \{\}$, $1 \leftrightarrow \{\{\}\}$, etc.

2, (Extensionality) 2 sets A, B are equal, written $A = B$, if every element of A is an element of B and vice versa.

3, (Existence) There exists a set $\phi = \{\}$ known as the empty set i.e., $x \notin \phi$ <u>for all objects x</u>.
[We cannot have universal set here]

This doesn't imply that those objects are in a set.

Exercise :- Show that the empty set is unique.
(Use Axiom-2 and Axiom-3)

Lemma (Single Choice) :- Let A be a non-empty set. Then $\exists\, x$ st $x \in A$.
Proof of Lemma :- Suppose not. Then $x \notin A$ for all objects x. But that means A is the empty set.

4, (Singletons & Pairs) If a is an object, $\exists$ a set, $\{a\}$

whose only element is a. Similarly if a and b are objects, $\exists$ a set $\{a, b\}$ whose only elements are a and b.
Example :- ϕ, $\{\phi\}$, $\{\{\phi\}\}$, $\{\phi, \{\phi\}\}$ are all distinct

5. (Unions) Given sets A and B, $\exists$ a set denoted $A \cup B$ which contains all the elements in A, B or both

Lemma :- The union operation is commutative and associative.
Proof of Lemma :- Exercise.
Def'n :- We say that A is a subset of B, denoted $A \subseteq B$ [subset or equals], if every element of A also belongs to B

6. (Axiom of separation or specification) Let A be a set and $P(x)$ be a property for every $x \in A$. Then there exists a set $\{x \in A \mid P(x) \text{ is true}\}$
Exercise :- Prove that such a set is a subset of A
This allows us to define $A \cap B$ and $\underbrace{A \setminus B}$ → A minus B

Recall :- Boolean Algebra of sets.

7. (Replacement) Let A be a set and $P(x, y)$ be a property for every $x \in A$ and objects y such that for every $x \in A$, there exists atmost one object y for which $P(x, y)$ is true.
Then $\exists$ a set $\{y \mid P(x, y) \text{ is true for some } x \in A\}$
Example :- 1, $A = \{7, 9, 22\}$ and $P(x, y)$ is "$y = x{+}{+}$"
This gives $\{8, 10, 23\}$
2, $A = \{7, 9, 22\}$, $P(x, y)$ is "$y = 1$"
This gives $\{1\}$
Example :- $\{1, 2, 3, 1\} = \{3, 1, 2\}$ using Axiom 2
$\Rightarrow$ Don't keep duplicates.

8. (Infinity [Gojo]) There exists a set $\mathbb{N}$ whose objects are natural numbers, an object $0 \in \mathbb{N}$ and the successor $n{+}{+}$ for every $n \in \mathbb{N}$ such that the Peano Axioms hold.

9. → Will be given later as we need relations, functions, &

10. (Regularity / Foundation) If A is a non-empty set, then there exists atleast one $x \in A$ which is either not a set or disjoint from A.

Example:-

$$A = \{\underbrace{\{3,4\}}_{a}, \underbrace{\{3,4,\{3,4\}\}}_{b}\}$$

b fails both conditions
a fails first but satisfies the second.

Reading Assignment:- Russell's Paradox from Wikipedia