~~6.3~~

5.5

# Quiz 1

UM 205: Introduction to Algebraic Structures (Winter 2023-24)
Indian Institute of Science

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1. Using the Peano axioms, prove the *cancellation law*, namely if $a, b, c \in \mathbb{N}$ so that $a + c = b + c$, then $a = b$.

2. Prove that if $A$ and $B$ are finite sets with cardinalities $m$ and $n$ respectively, and $m < n$, there cannot exist a function $f : A \to B$ which is a bijection.

1) Fix a and b and induct on c.

Let $P(c)$ be the property that $a+c=b+c \Rightarrow a=b$ for $a,b,c \in \mathbb{N}$

Take $c=0$    commutativity of addition needed    -0.5

$\therefore\ a+0 = b+0 \Leftrightarrow a=b$   Thus $P(0)$ is true

Now assume $P(c)$ is true & show $P(c{+}{+})$

$P(c{+}{+}) \rightarrow a+(c{+}{+}) = b+(c{+}{+})$

4.5

$\therefore\ (a+c){+}{+} = (b+c){+}{+}$ (definition of addition)

But $\overset{(a+c)++}{\cancel{a+c}} = \overset{(b+c)++}{\cancel{b+c}} \Rightarrow \overset{a+c=b+c}{\cancel{a=b}}$ ~~since P(c) is true~~ (unique successor)

$\therefore\ \overset{a+c=b+c}{\cancel{(a+c){+}{+} = (b+c){+}{+}}} \Rightarrow \cancel{a{+}{+} = b{+}{+}} \Rightarrow a=b$ (~~unique successor~~) (since $P(c)$ holds)

$\therefore\ P(c{+}{+})$ holds

$\therefore$ By Principle of mathematical induction, $a+c=b+c \Rightarrow a=b\ \forall\ a,b,c \in \mathbb{N}$

2) Bijection holds when function is one-one & onto

$f: A \to B$ where $\# A < \# B$

We show that there can't be onto function.

Suppose func is onto

Then $\forall\ b \in B,\ \exists\ a \in A$ st $f(a) = b$

But ~~f(a) =~~ $\forall\ a \in A,\ \exists!\ b \in B$ st $f(a) = b$

$\therefore$ Every $b \in B$ has a unique preimage $a \in A$

the question has been restated.

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$\therefore\ \# B \leq \# A$ (contradiction)

not well defined (yet)

Hence no bijection possible