Lecture - 16

Recall :- Konigsberg Problem [Basis for graph theory]

Def'n :- A simple graph G is an ordered pair $G=(V,E)$, where V is a set of vertices and E is a collection of 2-element subsets of V called edges.

We will focus on the case where $|V|$ and $|E|$ are finite.
In general, we could have loops $e=(v,v)$ or parallel edges.

We say that $v \in V$ is incident to $e \in E$ if $v \in e$
We say that v_1 is adjacent to v_2 if $\{v_1, v_2\} \in E$.

A walk is a sequence of vertices $(v_1, v_2, \dots, v_n)$ where $\{v_i, v_{i+1}\} \in E \ \forall \ i \in [n-1]$
A walk in which all vertices are distinct is called path. A walk in which all edges are distinct is called a trail.
A walk/path/trail is closed if $v_n = v_1$
A trail is Eulerian if all edges in the graph are used.

Def'n :- A graph $G=(V,E)$ for which there is a path from v_1 to v_2 $\forall \ v_1, v_2 \in V$ is said to be connected.

The degree of a vertex is the # of edges to which it is incident.

Thm (Euler) :- A connected graph G, possibly with multiple edges has a closed Eulerian trial iff all vertices of G have even degree.

Proof :- $\Longrightarrow$ Suppose G has closed Eulerian trial. Look at each time a vertex V is visited. Everytime, we enter V, we exit it as well. And all edges are distinct Moreover the starting and ending is the same.
$\Longrightarrow$ Each vertex is incident to an even # of edges.

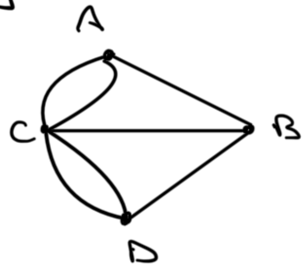
$\Longleftarrow$ All vertices have an even degree and G is

connected. Pick a vertex v and pick an edge e_1 st $v \in e_1$. Let v_1 be the other vertex of e_1. Pick another edge $e_2 \ni v_1$ to another vertex v_2. Continue this way. Since G is finite, we claim that we will return to v forming a closed trial.

If $C_1 = G$, we are done. If not; choose $w_1 \in C_1$ st w_1 is incident to an edge not in C_1 (since G is connected.) Continue the same way as before to form another closed trial C_2 and we are done if $G = C_1 \cup C_2$. If not continue this way to form $G = C_1 \cup C_2 \cup C_3 \cup \ldots \cup C_k$ where each C_i is a Eulerian trial.

Remark :- The proof is allowed for $\|$el edges.

Back to Konigsberg :- The corresponding graph



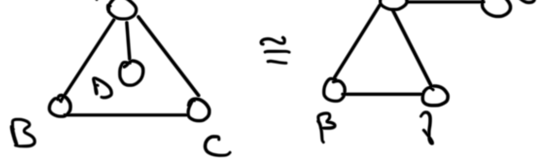
$\because \deg(A) = \deg(B) = \deg(D) = 3$
This is not a Eulerian trial.

Digression :- A closed path (aka cycle) is said to be Hamiltonian if it visits every vertex exactly once.

Dirac's Theorem :- If $|V| = n$ and every vertex has degree atleast $n/2$, then the graph admits a Hamiltonian cycle.

Def'n :- Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if $\exists$ a bijection $f: V_1 \to V_2$ st $\{v_1, v_2\} \in E_1$ iff $\{f(v_1), f(v_2)\} \in E_2$

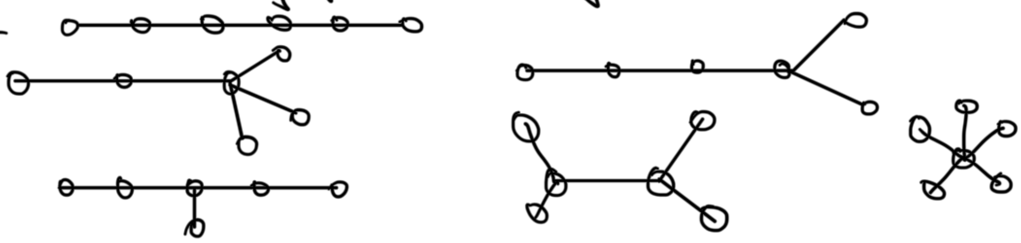
Example :-



Recall that a closed path is called a cycle.

Def'n :- A connected graph without cycles is called a tree.

Example :-



Property :- Let T be a tree. Then,

1, deleting any edge in T disconnects it.

2, adding a new edge to T creates a cycle.

3, For any 2 vertices $v_1, v_2 \in V(T)$, there is a unique path from v_1 to v_2

Proof :- 3, If there were 2 paths from v_1 to v_2, then we get a cycle.

2, We already have a unique path b/w a, b. If we add edge $\{a, b\}$, we will get a cycle.