UM 204: QUIZ 2

Jan. 19, 2024

Duration. 15 minutes

Maximum score. 10 points

You are not allowed to assume the existence of the ordered field of real numbers.

Problem. Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be two equivalent sequences of rational numbers. Show that $\{a_n\}_{n\in\mathbb{N}}$ is $\mathbb{Q}$-Cauchy if and only if $\{b_n\}_{n\in\mathbb{N}}$ is $\mathbb{Q}$-Cauchy.

Suppose $\{a_n\}$ is a $\mathbb{Q}$-Cauchy sequence. Let $\varepsilon > 0$. Then, there is an $N_1 \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon/3, \quad \forall n, m \geq N_1. \tag{1}$$

By the equivalence of the two sequences, there is an $N_2 \in \mathbb{N}$ such that

$$|a_n - b_n| < \varepsilon/3, \quad \forall n \geq N_2. \tag{2}$$

Let $N = \max\{N_1, N_2\}$, then by (1) and (2), for all $m, n \geq N$,

$$|b_n - b_m| \leq |b_n - a_n| + |a_n - a_m| + |a_m - b_m| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\{b_n\}$ is $\mathbb{Q}$-Cauchy.

Switching the roles of $\{a_n\}$ and $\{b_n\}$, we also obtain the converse.