Pumping Lemma and Ultimate Periodicity

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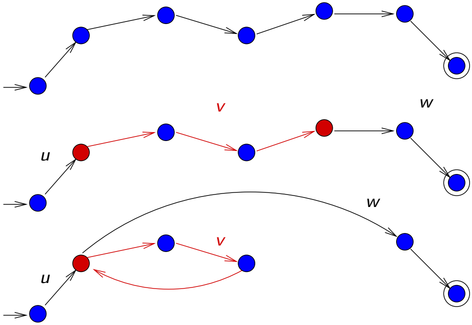
Outline

Two necessary conditions for regularity

- Pumping Lemma: Any "long enough" word in a regular language must have a "pump."
- Lengths of words in a regular language are "ultimately periodic."

Pumping lemma for regular languages

Based on a simple observation:
In a given a DFA $\mathcal{A}$, if a path p in it is longer than the number of states in $\mathcal{A}$ then p must have a loop in it.



So if uvw is accepted along this path, then so is uw, uv^2w, ...

Pumping lemma statement

Pumping Lemma

For any regular language L there exists a constant k, such that for any word $t \in L$ of the form xyz with $|y| \geq k$, there exist strings u, v, w such that:

1. $y = uvw$, $v \neq \epsilon$, and
2. $xuv^iwz \in L$, for each $i \geq 0$.

Game induced by a language L

A play in G_L:

Demon	You
Provides a k.	
	Choose $t \in L$, with decomposition x, y, z, and $\lvert y \rvert \geq k$.
Provides decomposition of y into uvw, with $v \neq \epsilon$.	
	Choose $i \geq 0$.

Demon wins the play if $xuv^iwz \in L$, otherwise You win.

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- If L is regular then Demon has a winning strategy in G_L.
- Equivalently: If You have a winning strategy in G_L, then L is not regular.

Pumping Lemma is *not* a sufficient condition for regularity

- There exist non-regular languages L for which the Demon has a winning strategy in G_L.

Example applications of Pumping Lemma

Describe Your strategy to beat the Demon in the games for:

- $\{a^n b^n \mid n \geq 0\}$.
- $\{w \in \{a, b\}^* \mid \#_a(w) = \#_b(w)\}$.
- $\{a^{2^n} \mid n \geq 0\}$.

Exercise

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Consider the language

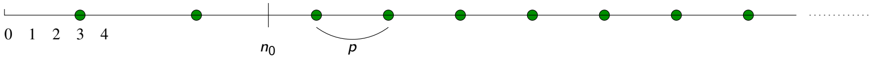
$$L = \{w \cdot w \mid w \in \{0,1\}^*\}$$

Is this language regular? Justify your answer.

Two problems to think about

1. If $L \subseteq \{a\}^*$, show that L^* is regular.
2. Show that there exists a language $L \subseteq A^*$ such that neither L nor its complement $A^* - L$ contains an infinite regular set.

Ultimately periodic sets



A subset X of $\mathbb{N}$ is ultimately periodic if

- There exist $n_0 \geq 0$, $p \geq 1$ in $\mathbb{N}$, such that for all $m \geq n_0$,

$$m \in X \text{ iff } m + p \in X.$$

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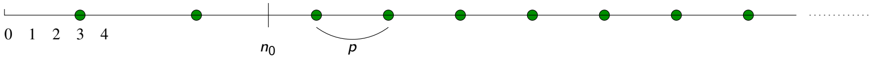

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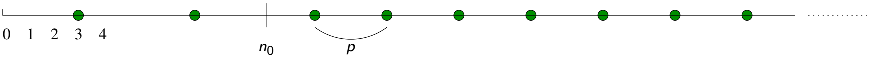

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- Or equivalently: $X = F \cup A_1 \cup \cdots \cup A_k$, for some finite set F and arithmetic progressions A_i of same period p.

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- $\{10, 12, 14, 16, \ldots\} \cup \{5, 10, 15, \ldots\}$ is u.p.
- $\{0, 2, 4, 8, 16, 32, \ldots\}$ is *not* u.p.

Ultimate Periodicity of Regular Languages

For $L \subseteq A^*$ define $lengths(L) = \{|w| \mid w \in L\}$.

Claim

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Proof:

- Argue for language over single-letter alphabet.
- Infer for general language.

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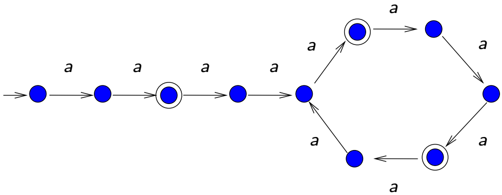

$$lengths(L(\mathcal{A})) = \{2\} \cup \{5, 11, 17, \ldots\} \cup \{8, 14, 20, \ldots\}.$$