# End-Semester Exam

UM 205: Introduction to Algebraic Structures (Winter 2023-24)
Indian Institute of Science

Instructor: Arvind Ayyer

April 22, 2024
9:30am – 12:30pm

**Unless otherwise stated, each question is worth 5 marks.**

1. See separate sheet.
2. See separate sheet.
3. Give a simple formula for the number $f(n)$ of permutations $\sigma$ in $S_n$ which satisfy $|\sigma_i - i| \leq 1$ for all $i \in [n]$. The formula can be explicit, in terms of a recurrence, or as a generating function.
4. Let $G$ be a simple graph. The degrees of all the vertices of $G$ arranged in weakly decreasing order forms a partition $p_G$ called the *ordered degree sequence* of $G$.
   (a) Prove that $p_G$ is a partition of an even integer.
   (b) Show that is is not possible to construct $G$ with 6 vertices such that $p_G = (4, 4, 4, 2, 1, 1)$.
5. Recall that $\nu(n)$ is the number of positive divisors of a positive integer $n$. Prove that $\nu(n)$ is odd if and only if $n$ is a square.
6. Find all inequivalent solutions $x$ to the congruence $15x \equiv 25 \pmod{35}$. Explain why your solutions are inequivalent.
7. Suppose $G$ is a group and $a, b \in G$. Prove that $|ab| = |ba|$.
8. Prove that the dihedral group $D_6$ and the symmetric group $S_3$ are isomorphic, but that $D_{24}$ and $S_4$ are not.
9. For the quotient group $G = \mathbb{Q}/\mathbb{Z}$, find a natural set of representatives of cosets. Then prove that every element in $G$ has finite order, but that there is no upper bound on the order of an element.
10. Suppose $G$ is a group of order $p^n$ for some prime $p$ and $n \geq 1$. Prove that $G$ must have a nontrivial center.

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Additional Booklet No.

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| Course Code | UM-205. |

**UM205 End Sem (Q.1 and Q.2)**

**Q.1** Suppose $A, B$ are sets. First prove that $A \cap B \subseteq A$. Then suppose that $C$ is another set satisfying $C \subseteq A$ and $C \subseteq B$. Now prove that $C \subseteq A \cap B$.

**Q.2** Suppose you select 12 integers (possibly with repetition) $x_1, \ldots, x_{12}$ between 1 and 30 in an arbitrary way. Show that you can always find two integers $i, j$ such that $\gcd(x_i, x_j) > 1$.

**Solution:**