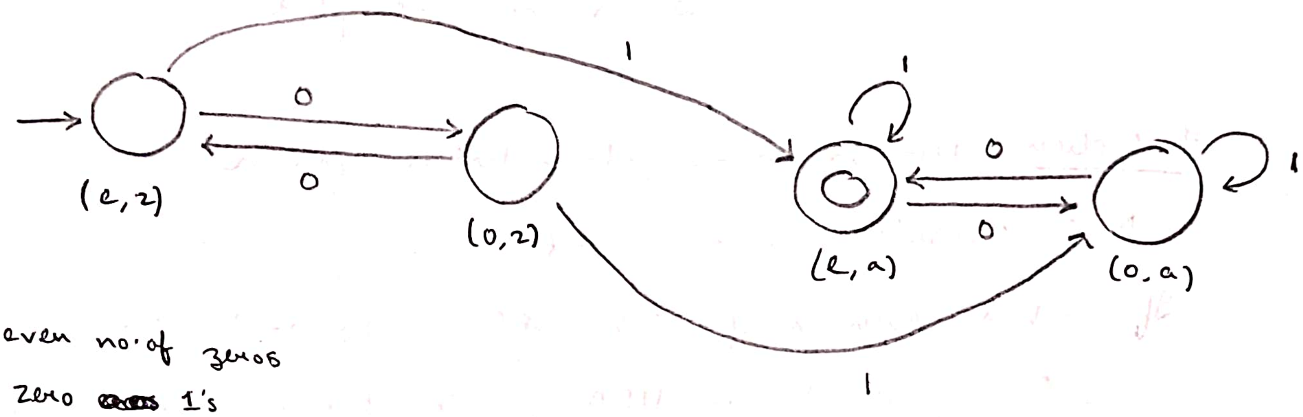
AUTOMATA THEORY (UMC 205)

NAME - SWARNAVA CHAKRABORTY

Assignment - 1

SR.No. - 23746

1) The following is the DFA for even no. of 0's and atleast one 1.



e ::= even no. of zeros
z ::= zero 1's
o ::= odd no. of zeroes
a ::= atleast one 1

2) Base 3 representations of odd numbers has odd number of 1's, as else the number becomes is even.

Proof: Let the base 3 rep. be $x_1 x_2 x_3 \ldots x_n$ for $n \in \mathbb{N} \setminus \{0\}$

$\therefore$ The number $= x_1 \times 3^{n-1} + x_2 \times 3^{n-2} + \cdots + x_n \times 3^0$

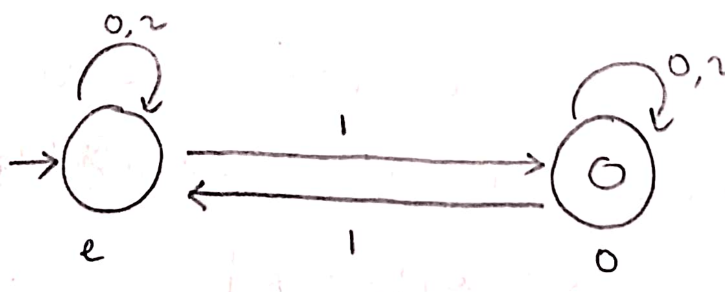
Each power of 3 is odd.

If the number itself is odd, then there needs to be an odd no. of odd values for the $x_1, \ldots, x_n$

But, $x_1, \ldots x_n \in \{0,1,2\}$

So, only odd number is 1

Now, we construct a DFA :- (over $\{0,2,1\}$)



DFA A

e ::= even 1's , o ::= odd 1's

Now, given any $w \in \{0,1,2\}^*$, consider the followi[ng]

CLAIM: w is accepted by A iff w has odd no.

We use induction on length of w. Let length$(w) = n$

Base Case: $n = 0$, thus w has zero 1's, i.e., even no.

And, $\delta(e, \epsilon) = e$, which is not a final sta[te]

$\therefore$ w is not accepted. ☒

Induction: Let us assume claim holds for n.

Now, consider $w.a$, $a \in \{0,1,2\}$

If w had even no. of 1's, DFA was on state e.

Adding $a = 1$, DFA goes to state o, $w.a$ now has odd no. of 1's, and gets accepted.

Else, it remains on same state e, $w.a$ having odd 1's

If w had odd no. of 1's, DFA was on state o.

Adding $a = 1$, DFA goes to state e, $w.a$ now has ~~odd~~ even no. of 1's, and is not accepted

Else, it remains on same state o, $w.a$ having odd no. of a's, getting accepted. ☒

So, the DFA A accepts all base-3 representations of odd numbers.

3) The states of the corresponding DFA,

a) given $Q = \{p, q, r, s\}$

will be 2^Q, i.e., power set of Q

The start state of the DFA will be $\{p\}$

The final states will be

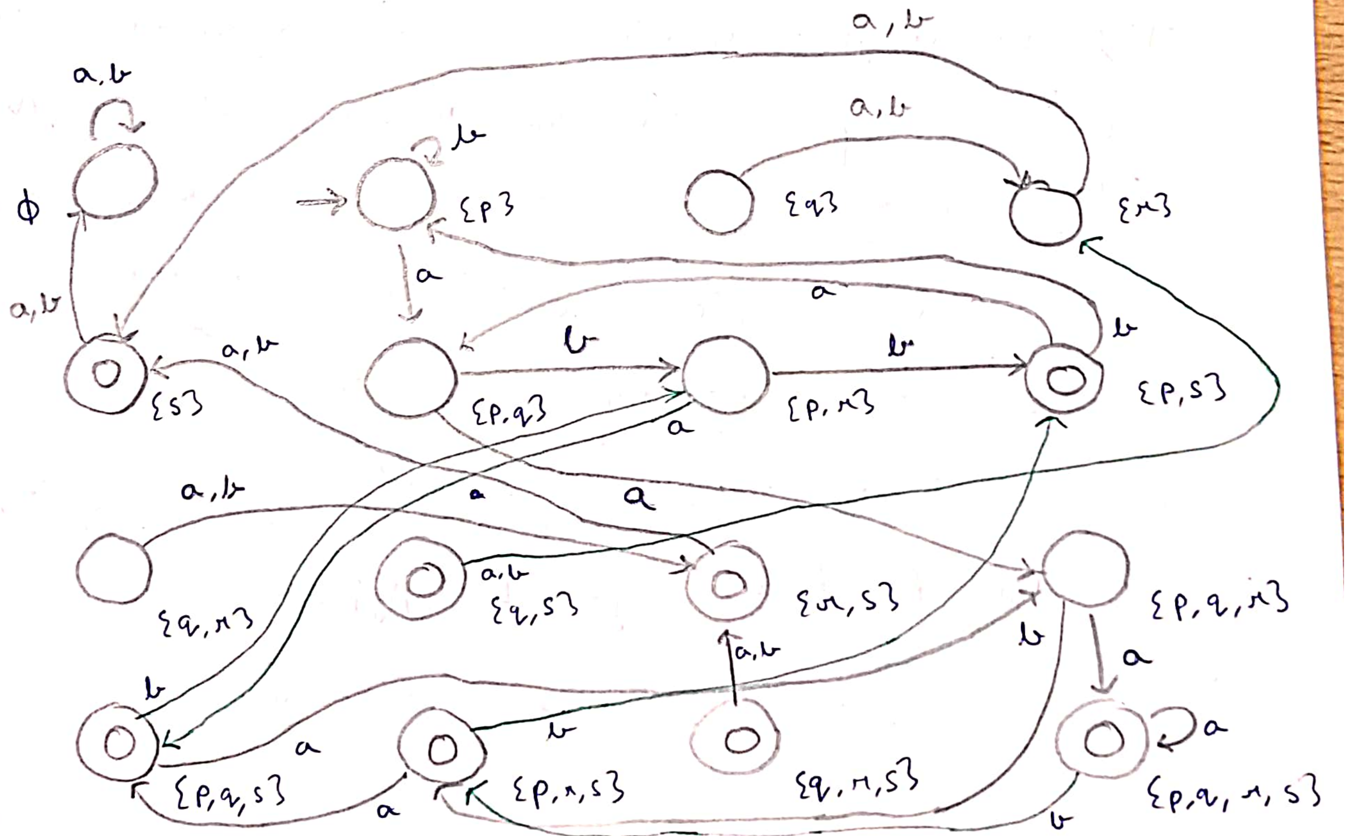
$\{x \in Q : x \cap \{s\} \neq \phi\}$

state | x

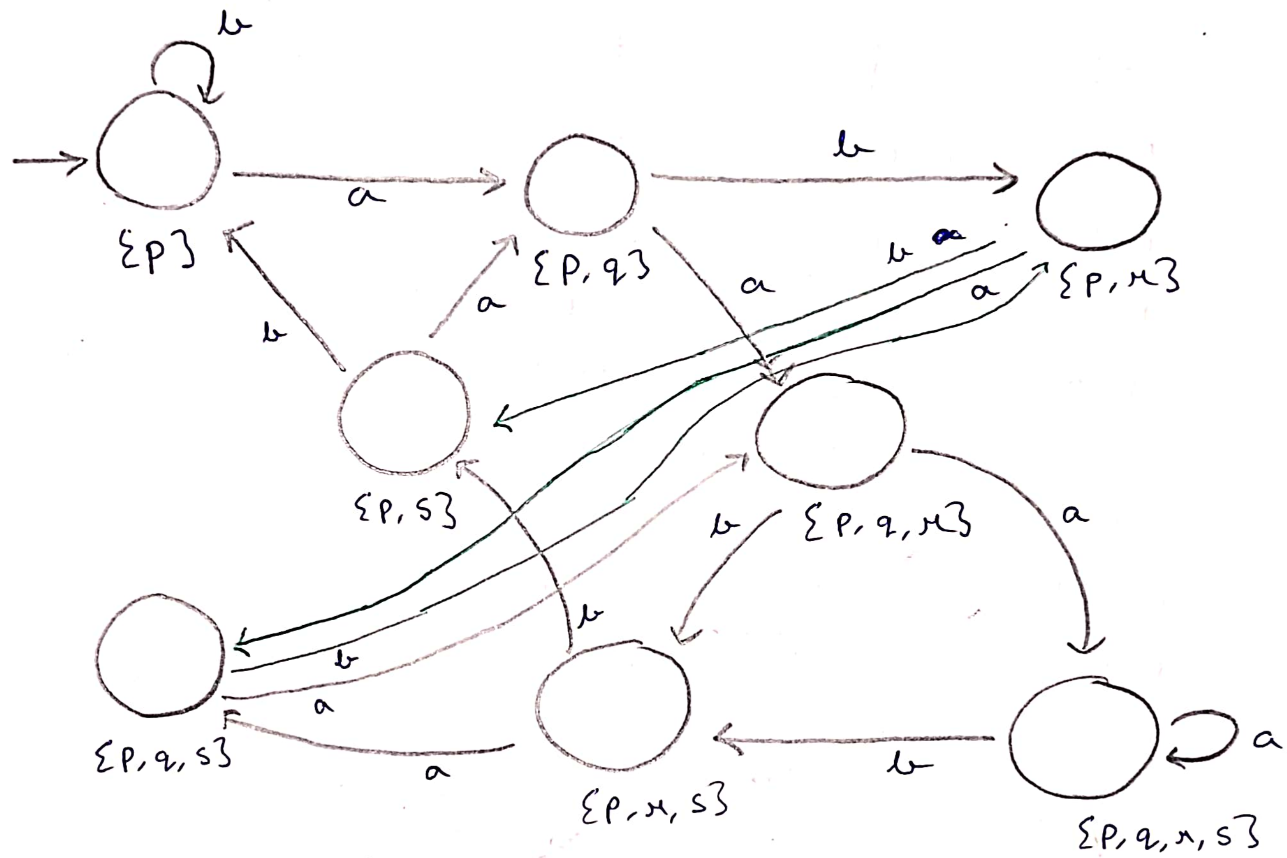
ϕ

, we will make the table of states and transitions

State \ x	a	b
ϕ	ϕ	ϕ
$\rightarrow \{p\}$	$\{p, q\}$	$\{p\}$
$\{q\}$	$\{r\}$	$\{r\}$
$\{r\}$	$\{s\}$	$\{s\}$
$\{s\}$	ϕ	ϕ
$\{p, q\}$	$\{p, q, r\}$	$\{p, r\}$
$\{p, r\}$	$\{p, q, s\}$	$\{p, s\}$
$\{p, s\}$	$\{p, q\}$	$\{p\}$
$\{q, r\}$	$\{r, s\}$	$\{r, s\}$
$\{q, s\}$	$\{r\}$	$\{r\}$
$\{r, s\}$	$\{s\}$	$\{s\}$
$\{p, q, r\}$	$\{p, q, r, s\}$	$\{p, r, s\}$
$\{p, q, s\}$	$\{p, q, r\}$	$\{p, r\}$
$\{p, r, s\}$	$\{p, q, s\}$	$\{p, s\}$
$\{q, r, s\}$	$\{r, s\}$	$\{r, s\}$
$\{p, q, r, s\}$	$\{p, q, r, s\}$	$\{p, r, s\}$



b) As can be seen in the DFA drawn, "
states {q}, {r}, {q,s}, {q,r}, {q,r,s}, {r,
are not reachable from the start state.
So, we can redraw the DFA with only t
8 reachable states :



4) The language described is not regular. Let it be L.
We will prove this by showing that me, the person playing the demon game, will always have a winning strategy against the demon.

Let the demon give me a $k \geq 0$.
I will choose $x = cc$, $y = (/*)^k$, $z = cc(*/)^k cc$
By def^n of L, $xyz \in L$, $|y| \geq k$
Now, demon gives me u, v, w s.t. $uvw = y$

$v = 1$ or $v = *$, we choose $i = 0$

Doing these ensures we start k-1 comments but end k comments in xuv^0wy

$\therefore xuv^0wy \notin A$

Else, we choose $i = 2$

Doing this ensures we start more comments than we end in xuv^2wy

$\therefore xuv^0wy \notin A$

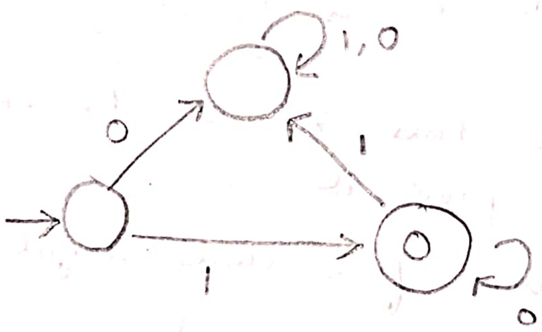
So, no matter what k and decomposition u, v, w the demon gives, I have a winning strategy, which shows that L is not a regular language. ☒

5) a) Let us take $X = \{2^n \mid n \geq 0\}$

So, $binary(X) = \{1, 10, 100, \ldots\}$

$= \{10^k \mid k \geq 0\}$

We make DFA :



It can be seen that this DFA accepts 10^k, $k \geq 0$ and nothing else.

$\therefore$ This DFA accepts $binary(X)$

$\Rightarrow binary(X)$ is a regular language.

Now, $unary(X) = \{1^{2^n} \mid n \geq 0\}$

So, $\text{lengths}(\text{unary}(X)) = \{2^n \mid n \geq 0\}$

Clearly, this is not an ultimately periodic set

Proof → Assume this is ultimately periodic.

$\therefore \exists\, m, p$ s.t. $m \in \text{lengths}(\text{unary}(X))$

$\Rightarrow m+p \in \text{lengths}($

Say $m \in \text{lengths}(\text{unary}(X))$

$\therefore\ m = 2^k$ for some $k \geq 0$

~~If $p \neq 2^k$, $m+p = 2^k + p$~~

If $p \neq 2^k(2^{m'}-1)$ for some $m' \geq 0$, $m+p \neq 2^{k+m'}$

$\Rightarrow m+p \notin \text{lengths}(\text{unary}($

If $p = 2^k(2^{m'}-1)$ for some $m' \geq 0$, $m+p = 2^{k+m'} \in \text{lengths}(\text{unary}$

Now, $m+p+p = 2^k(2^{m'}+2^{m'}-1)$

$= 2^{k+2m'} - 2^k \neq 2^l$ for $l \geq 0$

$\Rightarrow m+p+p \notin \text{lengths}(\text{unary}(X))$

CONTRADICTION.

So, $\text{unary}(X)$ is not regular

Thus, statement (a) is false.

b) As (a) is false, this has to be true.

Let us try to prove it.

We will use the fact that regular languages satisfy the ultimately periodic property.

Now,

$X = \{\text{len}(a) \mid a \in \text{unary}(X)\}$ (by defn)

Now, given $\text{unary}(X)$ is regular, so, the lengths of this set will be ultimately periodic

i.e., X is ultimately periodic.

is the union of a finite set and a finite no. of arithmetic progressions with same difference.

binary (X) is constructed from, X,

binary(X) is also a union of a finite set and arithmetic progressions with the same period.

Now, we will construct a DFA for binary (X).

Let the period of the arithmetic progressions be p

Let us take one AP → $\{a_1, a_1+p, a_1+2p, \ldots\}$

Let us write the DFA, $A = (Q, s, \delta, F)$

$$Q = \{\epsilon, 0, 1, 2, \ldots, p-1\}$$

$$s = \{\epsilon\}$$

$$F = \{a_1 \bmod p\}$$

Let us define the transition,

$$\delta(\epsilon, a) = a \qquad a \in \{0, 1\}$$

$$\delta(q, a) = (2q + a) \bmod p \qquad q \neq \epsilon$$

This is deterministic, hence a DFA.

We can see this DFA accepts the above language given by the AP.

So, the above language is regular.

Similarly, we have finite no. of other A.P.s

$\{a_2, a_2+p, \ldots\}$, $\{a_3, a_3+p, \ldots\}$, $\{a_k, a_k+p \ldots\}$

By closure of regular languages, the union of all these APs are regular.

Also, every finite language is regular.

So, the whole thing, i.e., union of finite set and the A.P.s is regular (CLOSURE)

$\Rightarrow$ binary(X) is regular

find a subset L of $\{a, b\}^*$ such that neither nor its complement has an infinite regular subset, we need to ensure that both L and its complement intersect every infinite regular language.

This means that for every infinite regular language R,

$$R \cap L \neq \phi \quad \text{and} \quad R \cap L^c \neq \phi$$

Now, as the set of regular languages is countable, let us enumerate them

$R_1, R_2, \ldots$ i.e., R_n where $n \in \mathbb{N} \setminus \{0\}$

Now, we select a word $w_i \in R_i$ and add it to L and select another word $w_i' \in R_i$ s.t. it is not in L

In other words, add $w_i' \in R_i$ to L^c s.t. $w_i' \neq w_i$

(As R_i is infinite, existence of w_i and w_i' is guaranteed)

We do this for all $i \in \mathbb{N} \setminus \{0\}$

CLAIM : $R_i \cap L \neq \phi$, $R_i \cap L^c \neq \phi \quad \forall i \in \mathbb{N} \setminus \{0\}$

Proof → Assume not. So, $\exists i \in \mathbb{N} \setminus \{0\}$ s.t.

$$R_i \cap L = \phi$$

But $\exists w_i \in R_i$ s.t. $w_i \in L$

So, CONTRADICTION

Similarly, assume $\exists j \in \mathbb{N} \setminus \{0\}$ s.t.

$$R_j \cap L^c = \phi$$

But $\exists w_j' \in R_j$ s.t. $w_j' \in L^c$

So, CONTRADICTION

$\therefore R_i \cap L \neq \phi, R_i \cap L^c \neq \phi \quad \forall i \in \mathbb{N} \setminus \{0\}$

7) For this q^n, we will use a property (lemma) to prove it
proof:

LEMMA → For languages over a singleton alphabet, i.e., $|\Sigma|$
we have the following:

if $L \in \Sigma^*$, L is regular $\Leftrightarrow$ lengths (L) is ultimately periodic

Now, chosen $L \subseteq \{a\}^*$. For some $S \subseteq \mathbb{N}$, this is nothing but $L = \{a^n \mid n \in S\}$

So, clearly, as $L^* = \{\epsilon\} \cup L \cup L^2 \cup L^3 \cup \ldots$

$$\therefore \text{lengths}(L^*) = \{0\} \cup \bigcup_{k=1}^{\infty} \left(\sum_{i=1}^{k} S\right)$$

(A, B sets, $A + B = \{x+y \mid x \in A, y \in B\}$)

$\therefore$ For L^*, we need a DFA that accepts all strings a^n where $n \in$ lengths (L^*)

Now, we construct the DFA, (Q, s, δ, F):

$Q = \{q_0, q_1, \ldots, q_{p-1}\}$ for some period $p \in \mathbb{N} \setminus \{0\}$

$s =$ ~~[illegible]~~ q_0

$\delta(q_i, a) = q_{(i+1) \bmod p}$

$F = \{q_{n \bmod p} \mid n \in \text{lengths}(L^*)\}$

Now, if L is finite, there are finitely many elements in lengths (L). Thus, lengths (L^*) will eventually repeat with a period equal to the greatest common divisor of the differences between these lengths.

If L is infinite, ~~[illegible]~~ we use the pigeonhole principle and say that after some point $n_0 \in$ lengths (L^*), DFA must enter a cycle since it has finitely many states.

lengths (L^*) is ultimately periodic.

$\therefore$ L^* is a regular language (By Lemma)

$|\Sigma|$

~~Let this be~~

8) Let this DFA be A.

$\therefore$ $L(A) = L_{su}^{\{s,t,u\}}$ (By defn)

Now,

$$L_{su}^{\{s,t,u\}} = L_{su}^{\{s,u\}} + L_{st}^{\{s,u\}} \cdot \left(L_{tt}^{\{s,u\}}\right)^* \cdot L_{tu}^{\{s,u\}} \quad -(1)$$

Now,

$$L_{su}^{\{s,u\}} = L_{su}^{\{s\}} + L_{su}^{\{s\}} \cdot (L_{uu}^{\{s\}})^* \cdot L_{uu}^{\{s\}}$$

and

$$L_{st}^{\{s,u\}} = L_{st}^{\{s\}} + L_{su}^{\{s\}} \cdot (L_{uu}^{\{s\}})^* L_{ut}^{\{s\}}$$

$$L_{tt}^{\{s,u\}} = L_{tt}^{\{s\}} + L_{tu}^{\{s\}} \cdot (L_{uu}^{\{s\}})^* L_{ut}^{\{s\}}$$

$$L_{tu}^{\{s,u\}} = L_{tu}^{\{s\}} + L_{tu}^{\{s\}} \cdot (L_{uu}^{\{s\}})^* L_{uu}^{\{s\}}$$

Now, $L_{su}^{\{s\}} = \{1\} \equiv 1$ (Regular expressions)

$L_{uu}^{\{s\}} = \{\epsilon, 01\} \equiv 01 + \epsilon$

$L_{st}^{\{s\}} = \{0\} \equiv 0$

$L_{ut}^{\{s\}} = \{1, 00\} \equiv 1 + 00$

$L_{tt}^{\{s\}} = \{\epsilon, 0, 10\} = 0 + 10 + \epsilon$

$L_{tu}^{\{s\}} = \{11\} = 11$

$\therefore$ $L_{su}^{\{s,u\}} = 1 + 1\cdot(01)^* \cdot (01 + \epsilon)$ $\quad [(01+\epsilon)^* = (01)^*]$

$L_{st}^{\{s,u\}} = 0 + 1\cdot(01)^* \cdot (1+00)$

7)

$$L_{tt}^{\{s,u\}} = (0 + 10 + \epsilon) + 11.(01)^*(1+00)$$

$$L_{tu}^{\{s,u\}} = 11 + 11(01)^*(01+\epsilon)$$

FOR SIMPLIFI... WE WILL USE SOME RESULTS FROM KOZEN

Simplifying a bit,

$$L_{su}^{\{s,u\}} = 1.(01)^*(01+\epsilon)$$
$$= (10)^* 1 (01+\epsilon) \quad (\text{By } (\alpha\beta)^*\alpha = \alpha(\beta\alpha)^*)$$
$$= (10)^*(101 + 1) \quad (\text{By distributivity})$$
$$= (10)^*(10+\epsilon)1$$
$$= (10)^* 1 \quad (\alpha^*\alpha + \alpha^* = \alpha^*)$$

$$L_{st}^{\{s,u\}} = 0 + 1.(01)^*(1+00)$$
$$= 0 + (10)^* 1 (1+00)$$
$$= (10)^* 11 + (10)^* 100 + 0$$
$$= (10)^* 11 + (10)^* 0$$
$$= (10)^*(11+0)$$

$$L_{tt}^{\{s,u\}} = \epsilon + 0 + 10 + 1(10)^* 1 (1+00)$$
$$= \epsilon + 0 + 10 + 1(10)^* 11 + 1(10)^* 100$$
$$= \epsilon + 0 + 1(10)^* 11 + 1(\epsilon + (10)^* 10) 0$$
$$= \epsilon + 0 + 1(10)^* 11 + 1(10)^* 0$$

$$L_{tu}^{\{s,u\}} = 11 + 11(01)^*(01+\epsilon)$$
$$= 11 + 11(01)^*$$
$$= 11(01)^* = 1(10)^* 1$$

Now,

$$\cancel{L_{su}^{\{s,t,u\}} = (10)^* 1 + (10)^*(11+0)(\epsilon + 0 + 1(10)^* 11 + 1(10)^* 0)^* 1(10)^* 1}$$

$$\ldots t,u\} = (10)^*1 + (10)^*(11+0)\left(\epsilon + 0 + 1(10)^*11 + 1(10)^*0\right)^* 1(10)^*1$$

$$= (10)^*1 + (10)^*(11+0)\left(0 + 1(10)^*(11+0)\right)^* 1(10)^*1$$

$$= (10)^*1 + (10)^*(11+0)\,0^*\left(1(10)^*(11+0)\cdot 0^*\right)^* 1(10)^*1$$

$$= \left(\epsilon + (10)^*(11+0)\,0^*\left(1(10)^*(11+0)\,0^*\right)^*1\right)(10)^*1$$

Let $\alpha = (10)^*(11+0)\,0^*$, $\beta = 1$

$\therefore \quad \alpha(\beta\alpha^*\beta) = \alpha\beta\alpha^*\beta = (\alpha\beta)^*\alpha\beta$

$\therefore \quad \epsilon + \alpha(\beta\alpha^*\beta) = \epsilon + (\alpha\beta)^*\alpha\beta = (\alpha\beta)^*$

$$\therefore \; L_{su}^{\{s,t,u\}} = \left((10)^*(11+0)\,0^*1\right)^*(10)^*1$$