Lecture - 6

Recall :- Cantor's Thm

Def'n :- A has less than or equal cardinality than B if $\exists$ an injection $f : A \to B$.

A has strictly lesser cardinality than B if A has lesser or equal cardinality than B but A doesnot have equal cardinality wrt B.

Corollary :- $2^{\mathbb{N}}$ is uncountable
Corollary :- There is no largest set.

Thm :- $\mathbb{R}$ is uncountable
Proof :- Cantor's diagonal argument.

Recall that cartesian products of 2 sets A and B is a set. Thus, ______ is also the cartesian product of a finite sets.
??

Def'n :- Let I be a (possibly infinite) set, and for $\alpha \in I$, let X_α be a set. The cartesian product,

$$\prod_{\alpha \in I} X_\alpha = \left\{ (x_\alpha)_{\alpha \in I} \in \underbrace{\left(\bigcup_{\beta \in I} X_\beta \right)^I} \;\middle|\; x_\alpha \in X_\alpha \; \forall \; \alpha \in I \right\}$$

Recall :- is the set of functions $I \to \bigcup_{\alpha \in I} X_\alpha$

Exercise :- For any set I and X, $\prod_{\alpha \in I} X = X^I$

Recall the Lemma following the axiom of single choice.

ZFC axioms (cont'd) :-

(13) Let I be a set and $X_\alpha \neq \phi \; \forall \; \alpha \in I$. Then $\prod_{\alpha \in I} X_\alpha$ is non-empty. (Axiom of Choice)

$\Rightarrow$ [illegible]

→ Counter-intuitive because it is non-constructive

→ Useful because it allows one to suppose the existence of functions, on $\mathbb{R}$ (eg).

Def'n:- A choice function on X is a function $f: 2^X \setminus \phi \to X$ st $f(S) \in S$ $\forall$ $S \subseteq X$, $S \neq \phi$.

Exercise:- Existence of a choice function is equivalent to axiom of choice.

Application:-

Lemma:- Let $E \subseteq \mathbb{R}$, $E \neq \phi$ st $\sup(E) < \infty$ i.e., E is bdd above. Then $\exists$ a sequence, $(a_n)_{n \geq 1}$, $a_n \in E$ $\forall$ $n \geq 1$ st $\lim_{n \to \infty} a_n = \sup(E)$

Proof of Lemma:- Let, for $n \in \mathbb{N}$,
$$X_n = \{x \in E \mid \sup(E) - \tfrac{1}{n} \leq x \leq \sup(E)\}$$
Note that $X_n \neq \phi$ as $\sup(E) - \frac{1}{n}$ is not an upper bound for E.
By AOC, we pick $a_n \in X_n$.
By the sandwich thm, we are done.

Recall:- Posets AKA partially ordered sets are of the form $(X, \leq)$

Def'n:- Let $(X, \leq)$ be a poset. A subset $Y \subseteq X$ is called a chain/totally ordered if for any $y, z \in Y$ either $y \leq z$ or $z \leq y$.

Def'n:- Let $(X, \leq)$ be a poset and let $Y \subseteq X$. We say that y is a minimal (resp maximal) element of Y if $y \in Y$ and there is no $z \in X$ st $z < y$ (resp $z > y$)

Example:-

1. $X = 2^{[5]}$, $(X, \subseteq)$ Min ϕ
2. $Y = \{\{1,2\}, \{2\}, \{2,3\}, \{2,3,4\}, \{5\}\}$
   Min :- $\{2\}$, $\{5\}$
   Max :- $\{2,3,4\}$, $\underbrace{\{5\}\{1,2\}}_{??}$
3. $(X = \mathbb{N}, \leq)$ Min :- 0
   Max :- Doesn't exist.

Def'n :- Let $(X, \leq)$ be a poset and $Y \subseteq X$ be a chain. We say that Y is well-ordered if every non-empty subset of Y has a minimal element and $\leq$ is a well-ordering, if it is a total order and X is well ordered.

Example :- $\mathbb{N}$ is well-ordered but $\mathbb{Z}$, $\mathbb{Q}$ are not.

Well-Ordering Principle (or Axiom 13') :- Given any set X, $\exists$ a well-ordering on X

Zorn's Lemma (or Axiom 13'') :- Let $(X, \leq)$ be a non-empty poset st every chain Y of X has an $\underbrace{\text{upper bound}}$. Then X contains atleast one maximal element.
$\to x \in X$ st $y \leq x \; \forall y \in Y$

Axiom 13, 13', 13'' are equivalent.

Proof of $13 \Rightarrow 13''$:- Let P be a poset st every chain has an upper bound. Suppose P has no maximal element.
Using choice function, pick $x_0 \in P$
Since x_0 is not maximal
$\Rightarrow \exists x_1$ st $x_0 < x_1$
Ily x_1 is not maximal $\Rightarrow \exists x_2$ st $x_0 < x_1 < x_2$
--- etc.
Thus, we get a chain $x_0 < x_1 < \cdots$
Let x_ω be the upper bound. But x_ω is not maximal $\Rightarrow \exists x_{\omega+1}$ st $x_\omega < x_{\omega+1}$
Continue this way.

At some point, we will get a chain "larger" than the cardinality of P which is a contradiction.
Thus P has a maximal element.