UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2025

HOMEWORK 9

Instructor: GAUTAM BHARALI **Assigned: MARCH 14, 2025**

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1. Any rational number x can be written uniquely as $x = m/n$, where $m \in \mathbb{Z}$, $n \in \mathbb{Z}_+$, and such that there is no $d \in \mathbb{N} \setminus \{0, 1\}$ dividing both m and n — with the understanding that we take $n = 1$ when $x = 0$. (You may use this fact **without proof.** You have learnt in UM205 what "d divides m (or n)" means.) Define $f : \mathbb{R} \to \mathbb{Q}$ as follows:

$$f(x) := \begin{cases} 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 1/n, & \text{if } x \in \mathbb{Q}, \end{cases}$$

where n is uniquely associated to $x \in \mathbb{Q}$ as explained above. Show that f is continuous at each irrational point and discontinuous at each rational point.

2. Let X be a metric space, let $E \subseteq X$, and let $p \in E$ be an isolated point of E. Show that for any metric space Y and any function $f : E \to Y$, f is continuous at p.

3. Let X and Y be metric spaces, let $E \subseteq X$, and let $f : E \to Y$. Let $p \in E$ and assume that p is a limit point of E. Show that f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.

4. Let (X, d) be metric a space and let $E \subseteq X$. Show that E is compact as a subset of X if and only if it is compact as a metric space equipped with the metric $d|_{E \times E}$.

5. This problem is on set theory and isn't really about analysis. However, we have made use, in class, of some of the statements below.
Let S_1 and S_2 be non-empty sets and let $f : S_1 \to S_2$. Let $\mathscr{A}$ be a non-empty subset of $\mathcal{P}(S_1)$ and let $\mathscr{B}$ be a non-empty subset of $\mathcal{P}(S_2)$. Prove the following:

$$\begin{aligned} f\big(\cup_{A \in \mathscr{A}} A\big) &= \cup_{A \in \mathscr{A}} f(A), \\ f\big(\cap_{A \in \mathscr{A}} A\big) &\subseteq \cap_{A \in \mathscr{A}} f(A), \\ f^{-1}\big(\cup_{B \in \mathscr{B}} B\big) &= \cup_{B \in \mathscr{B}} f^{-1}(B), \\ f^{-1}\big(\cap_{B \in \mathscr{B}} B\big) &= \cap_{B \in \mathscr{B}} f^{-1}(B). \end{aligned}$$

The following anticipates material to be introduced in the lecture on **March 17.**

6. Consider the result:

Theorem. *Let X and Y be metric spaces, and let $E \subsetneq X$ be a proper dense subset. Let $f : E \to Y$ be a uniformly continuous function. Suppose Y is complete. Then, there exists a unique continuous function $\widetilde{f} : X \to Y$ such that $\widetilde{f}\big|_E = f$.*

Consider the function $\widetilde{f}$ constructed as a part of the proof — which must be shown to have the properties stated above. Fix $x \in (X \setminus E)$, and let $\{x_n\}$ be a sequence in $X \setminus \{x\}$ that converges to x. Complete the following outline to prove that $\widetilde{f}$ is the unique continuous extension:

(*a*) Explain why it suffices to only consider sequences $\{x_n\}$ such that

$$\{x_n : n \in \mathbb{Z}_+\} \bigcap (X \setminus E) \text{ is an infinite set.} \tag{1}$$

(*b*) Consider a sequence $\{x_n\}$ with the property (1). Construct an auxiliary sequence $\{y_n\} \subset E$ such that for each n for which $x_n \notin E$, y_n is "sufficiently close" to x_n — in an appropriate sense — and converges to x in such a way that you can use its behaviour, plus uniform continuity, to infer that $\{\widetilde{f}(x_n)\}$ is convergent.

(*c*) Deduce that $\{\widetilde{f}(x_n)\}$ converges to $\widetilde{f}(x)$.

(*d*) Now, complete the argument showing that $\widetilde{f}$ is continuous and that it is unique.

7. Review/Self-study. The following topics were studied in UMA101 (and, barring the Chain Rule, with rigorous proofs): differentiability, differentiability of algebraic combinations of differentiable functions, the Chain Rule, points of local maximum/minimum, Rolle's Theorem, Lagrange's Mean Value Theorem and applications. Please **review** this material from pages 104–108 from Rudin's book, excluding Theorem 5.9, by March 21.