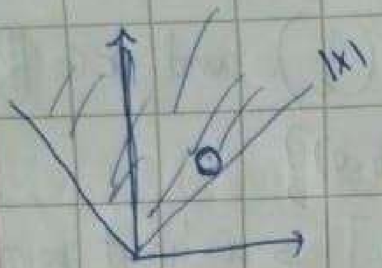
(a)



for any $\underset{\shortparallel}{p}\ (x_1, x_2) \in S$,

take $r = \dfrac{x_2 - |x_1|}{2}$

Claim:

$B(p, r) \subseteq S$.

Take $p \neq (y_1, y_2) \in B(p, r) \Rightarrow |x_1 - y_1| < r$

$\Rightarrow |x_2 - y_2| < r$

$\Rightarrow 0 < x_1^2 + y_1^2 - 2x_1y_1 + x_2^2 + y_2^2 - 2x_2y_2 < x_2^2 + x_1^2 - 2x_2|x_1|$

$\Rightarrow 2x_1y_1 + 2x_2y_2 > y_1^2 + y_2^2 + 2x_2|x_1| > 0$

$\Rightarrow y_2^2 - y_1^2 = |x_1 - y_1|^2 + |y_2 - x_2|^2 + (x_2^2 - x_1^2) + 2y_2x_2 + 2x_1y_1$

> 0

$\Rightarrow y_2 > |y_1|$

(b) For any $x \in S^c$,

$$r_0 = \frac{d(x, (a_1, a_2)) - r}{2} \text{ works}$$

any $y \in B(x, r_0)$,

$$d(\cancel{y}, (a_1, \cancel{a_2})) \leq d(y, x) + d(\cancel{x}, (a_1, a_2))$$

$$\leq r_0 + d(x, a_1, a_2)$$

$$\cancel{d(y,x)} \leq d(y, (a_1, a_2)) + \cancel{d(x, (a_1, a_2))}\ d(y, x)$$

$d(x, a)$

$$\leq d(y, (a_1, a_2)) + r_0$$

$$\cancel{r < d(x,a) - 2r_0} \Rightarrow d(y, (a_1, a_2)) > r$$

(c) neither closed nor open

$$\left\{b_1 - \frac{1}{m}\right\}_{m \geq 0} \xrightarrow{m \to \infty} b_1 \qquad m \text{ chosen s.t. } b_1 - \frac{1}{m} \in [a_1, b_1)$$

for any $r > 0$

$B((a_1, \quad a_n), r)$ contains

$$(a_1 - \frac{r}{2}, \quad , a_n - \frac{r}{n}) \notin S$$

(e) <u>closed</u>

$$S^c = B(0,1) \cup \left(\overline{B(0,1)}^c\right) \text{ is open.}$$

topologist's sine curve.

(d) $\cdot$ $S = \{(x,y) \mid y = \sin(\frac{1}{x}) \text{ and } x \in \mathbb{R}^+\}$

$$S' = S \cup \{(0,0)\}$$

Claim: every $p \in \{0\} \times [-1,1]$ is a limit point of S'

first consider, $\sin^{-1} : [-1,1] \longrightarrow [0, 2\pi]$

For each $p = (0, p_y) \in \{0\} \times [-1,1]$,

$\forall n \geq 1, x_n^{(p)} = \dfrac{1}{2\pi n + \sin^{-1}(p_y)} \leq \dfrac{1}{n}$

$y_n^{(p)} = \sin\left(\frac{1}{x_n^{(p)}}\right) = p_y$

$\Rightarrow \underbrace{(x_n^{(p)}, y_n^{(p)})}^{\in S'} \xrightarrow[n \to \infty]{} p$, but $p \notin S'$

S' is not closed !!

S' is not open in $\mathbb{R}^2$, take $x_0 = \frac{2}{\pi}$

$\sin(\frac{1}{x_0}) = 1$

any $B(x_0, r) \not\subseteq S'$

(o.t.w, $(x_0, 1 + r/2) \in S'$ but $\sin(\frac{1}{x_0}) = 1$
$\Rightarrow \Leftarrow$)

(Q2) $A = [n] \subseteq \mathbb{R}$

$B(k,1) \cap [n] = \phi$, $k \in [n]$

So no point is a limit point of the set.

$$A = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right\} \sqcup \left\{\frac{1}{2}+1, \frac{1}{4}+1, \dots\right\}$$

$$\dots \sqcup \left\{\frac{1}{2}+(n-1), \frac{1}{4}+(n-1), \dots\right\}$$

$$\sqcup \{0, 1, 2, \dots (n-1)\}$$

$\frac{1}{2^i} + k \xrightarrow[k\to\infty]{} k$, so $\{0, 1, \dots (n-1)\}$ is are its limit points, we'll show these are the only limit points.

take any $\frac{1}{2^n} + k$, $n \geq 1$, $B\left(\frac{1}{2^n}+k, \frac{1}{2^{n+2}}\right) \cap A = \phi$

is not a limit point.

(Q3)

$A_n = \left[0, 1-\frac{1}{n}\right)$, $n \geq 2$

$\bar{A}_n = \left[0, 1-\frac{1}{n}\right]$

$$\overline{\bigcup_{n\geq 2} A_n} = [0,1]$$

$$\bigcup_{n\geq 2} \overline{A_n} = \bigcup_{n\geq 2} \left[0, 1-\frac{1}{n}\right] = [0,1)$$

$$\bigcup_{n\geq 2} \overline{A_n} \subseteq \overline{\bigcup_{n\geq 2} A_n}$$

Sps. $p \in \bigcup \overline{A_n}$ $\exists n$ s.t. $p \in \overline{A_n}$

if $p \in A_n$, ~~we~~ $\Rightarrow p \in \overline{\bigcup A_n}$

So we'll assume $p \in \overline{A_n} \setminus A_n$,

for any $r > 0$, $B(p, r) \cap A_n \neq \phi$

$$A_n \subseteq \bigcup A_n \subseteq \overline{\bigcup A_n}$$

$$\cancel{\Rightarrow B(p,r) \cap \left(\overline{\bigcup A_n}\right) \neq \phi}$$

$$\Rightarrow A_n \setminus \{p\} \subseteq \overline{\bigcup A_n} \setminus \{p\}$$

$$\phi \neq B(p,r) \cap (A_n \setminus \{p\}) \subseteq B(p,r) \cap \left(\overline{\bigcup A_n} \setminus \{p\}\right)$$

We have shown,

if $p \in \bigcup \overline{A_n} \Rightarrow p \in \overline{\bigcup A_n}$ ∎

(Q4) ~~choose~~ Sps. $\sup A \notin A$.

$$B\left(\sup A, \frac{1}{n}\right) = \left(\sup A - \frac{1}{n}, \sup A + \frac{1}{n}\right)$$

contains at least ~~oth~~ one element of A other than $\sup A$ itself

(or else $\sup A - \frac{1}{n^2}$ would work and contradict the choice of $\sup A$)

Say $\overset{\in A}{a_n} \in B\left(\sup A, \frac{1}{n}\right)$ and this works $\forall n \geq 1$

(infact, $a_n \xrightarrow{n \to \infty} \sup A$)

Now given any $B(\sup A, r)$ we can find n s.t. $\frac{1}{n} < r$, $n \geq 1$ (archimidean property)

$$B\left(\sup A, \frac{1}{n}\right) \subseteq B(\sup A, r)$$

and w.k.t. $B\left(\sup A, \frac{1}{n}\right) \cap \left(A \setminus \{\sup A\}\right) \neq \phi$

If $\sup A \in A \Rightarrow \sup A \in \overline{A}$.

$((\Leftarrow))$ $F = Y \cap A$, A is closed in (X,d)

$$Y \setminus F = Y \setminus (Y \cap A)$$
$$= [X \setminus (Y \cap A)] \cap Y \quad \text{since } Y \subseteq X$$
$$= (X \setminus A) \cap Y$$

$(X \setminus A)$ is open in (X,d)

So $\forall p \in Y \setminus F$, $p \in (X \setminus A)$

$\exists r_p > 0$ s.t. $B(p, r_p) \subseteq X \setminus A$

$$B(p, r_p) \cap Y \subseteq (X \setminus A) \cap Y$$
$$\| $$
$$B_Y(p, r_p)$$

$\Rightarrow Y \setminus F$ is open in $(Y, d_Y) \Leftrightarrow F$ is closed in (Y, d_Y)

(S6) $A=\{p\}$

$\forall r>0\ B(p,r)\cap A\setminus\{p\}=\phi$, so p is not a limit point, in fact, A has no limit points.

So A is closed vacuosly.

(S7)

$$\mathcal{U}^{int}=\bigcup_{\substack{\mathcal{U}\subseteq A\\ \mathcal{U}\text{ open}}}\mathcal{U}=A^{\circ}$$

$A^{\circ}\subseteq\mathcal{U}$ since $\forall x\in A^{\circ}$, by def^n, x is an interior point of A, $\exists B(x,r)\subseteq A$

$$\Rightarrow x\in B(x,r)\subseteq\bigcup_{\substack{\mathcal{U}\subseteq A\\ \mathcal{U}\text{ open}}}\mathcal{U}$$

$\mathcal{U}^{int}\subseteq A^{\circ}$

Sps $x\in\mathcal{U}^{int}$, w.k.t $\exists\,\mathcal{U}$ s.t.

$x\in\underset{\text{open}}{\mathcal{U}}$ and $\mathcal{U}\subseteq A$

But $\mathcal{U}$ open is an open nbh of every point so $\exists B(x,r)\subseteq\mathcal{U}\Rightarrow x$ is an interior point of A

$\Rightarrow x\in A^{\circ}$.

Now $\exists q$ s.t. $q \in (x, x') \subseteq (x - \frac{1}{n}, x + \frac{1}{n})$

We are done since for any $B(x, r)$ we can find N s.t.

$$B(x, \tfrac{1}{N}) \subseteq B(x, r)$$

and

$$q \in B(x, \tfrac{1}{N}) \setminus \{x\}.$$

$F^c = Y \setminus F$

(B) $((\Rightarrow))$ F^c is open w.r.t. (Y, d_Y)

since all points of F^c is an interior point,
$\exists\, r_x$ s.t. $B_Y(x, r_x) \subseteq Y$ for each $x \in Y$

$$F^c = \bigcup_{x \in Y} B_Y(x, r_x)$$

$$= \bigcup_{x \in Y} \left(B_X(x, r_x) \cap Y\right)$$

$$= \underbrace{\bigcup_{x \in Y} B_X(x, r_x)}_{V} \cap Y = V \cap Y$$

V is open in (X, d)

$$F = Y \setminus F^c = (X \setminus F^c) \cap Y$$

$$= \left[(X \setminus V) \cup (X \setminus Y)\right] \cap Y$$

$$= (X \setminus V) \cap Y$$

$X \setminus V$ is closed in (X, d) $\Longleftrightarrow$ V is open in (X, d)

(Q7) Sps. $\overline{\mathbb{Q}} = \mathbb{R}$.

~~a~~ for each x in $\mathbb{R}$ and each $n \geq 1$,

$$B(x, \tfrac{1}{n}) \setminus \{x\} \cap \mathbb{Q} \neq \phi$$

(because x is a limit point of $\mathbb{Q}$)

Now sps. we are given $a, b \in \mathbb{R}$, $a \neq b$, a.

We want to recover the density of $\mathbb{Q}$ in $\mathbb{R}$ i.e. show $\exists q \in \mathbb{Q}$ s.t.

$$a < q < b$$

This follows from,

$$B(b, \tfrac{1}{N}) \setminus \{b\} \cap \mathbb{Q} \neq \phi$$

where $\frac{1}{N} \leq \frac{(b-a)}{2}$ ← archimedian prop tells you such an N exsists

Conversely assume the density of $\mathbb{Q}$ in $\mathbb{R}$.

For all $n \geq 1$ and each x in $\mathbb{R}$,

$(x - \frac{1}{n}, x + \frac{1}{n})$ contains x' s.t. $x' \neq x$

($x' = x + \frac{1}{n^2}$ works for instance)