UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2022

HOMEWORK 4

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 1, 2022**

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1. Provide details to the outline below to establish the following:

> **Result 1.** *Given any rational $a > 0$ and any $n \in \mathbb{N} \setminus \{0, 1\}$, there exists a real number $r > 0$ such that $r^n = a$.*

(*a*) Fix $a \in \mathbb{Q}^+$, $n \in \mathbb{N} \setminus \{0, 1\}$ and write

$$r := \{x \in \mathbb{Q}^+ : x^n < a\} \cup 0^* \cup \{0\}.$$

Show that r is a cut and that $r > 0^*$.

(*b*) Prove by mathematical induction that for any $k \in \mathbb{N} \setminus \{0\}$

$$\{z \in \mathbb{Q} : z \leq x_1 \dots x_k,\ x_1, \dots, x_k \in \mathbb{Q}^+ \text{ and } x_j^n < a \text{ for } j = 1, \dots, k\} = r^k.$$

(*c*) Now prove that $r^n = a^*$.

(*d*) Finally, explain in a sentence or two how part (*c*) can be interpreted as the conclusion of Result 1.

You can **freely** use — i.e., without proof and without citing any specific result from the section *Fields* in "Baby" Rudin — any corollary of $\mathbb{Q}$ being an ordered field.

2. Result 1 above can be extended to cover all positive **real** numbers, namely:

> **Result 2.** *Given any real $a > 0$ and any $n \in \mathbb{N} \setminus \{0, 1\}$, there exists a real number $r > 0$ such that $r^n = a$.*

Since the hypothesis of Result 2 is more general than that of Result 1, the steps in the proof of Result 2 will not be as "self-evident" as in the case of Result 1. Its proof will also require the least upper bound property of $\mathbb{R}$. **Read** the proof of Theorem 1.21 from "Baby" Rudin.

Remark. There is a *small* error in the statement of Theorem 1.21 (although the proof correctly establishes our Result 2). What is the error?

3. Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x, y) := |x - y|$. Show, by appealing to the properties of an ordered field (please cite the relevant result(s) from the section *Fields* in "Baby" Rudin that you use) that d is a metric.

4. A *graph* $G := G(V, E)$ is a pair of sets (V, E), where V is a non-empty finite set, and $E \subset T(V)$, where

$$T(V) := \{\{x, y\} : x, y \in V,\ x \neq y\}.$$

The set V is called the set of *vertices of G* and E is called the set of *edges of G*. Consider the following definitions:

- Given $x \neq y \in V$, a *path joining x to y* is a finite collection of edges $\{\{x_j, y_j\} \in E : j = 0, \dots, N\}$ such that $x_0 = x$, $y_{j-1} = x_j$, $j = 1 \dots N$, and $y_N = y$. The *length* of a path is the number of edges contained in it.

- The graph $G(V, E)$ is said to be *connected* if, for each $x \neq y \in V$, there is at least one path joining x to y.
- If $G(V, E)$ is a connected graph, define the function $d : V \times V \to [0, \infty)$ by

$$d(x, y) \;=\; \begin{cases} 0, & \text{if } x = y, \\ \min\{\text{length}(P) : P \textit{ is a path joining } x \textit{ to } y\}, & \text{if } x \neq y. \end{cases}$$

Given any connected graph $G = G(V, E)$, is (V, d) a metric space ? If yes, then justify, else give a counterexample.

5. Is it possible in a metric space X for some subset $A \subset X$, $A \neq \varnothing$ and $A \neq X$, to be **both** open and closed? If you think so, then give an example of (X, d) for which this happens, else give a proof that this is not possible.

6. Given a metric space X and a set $S \subseteq X$, we define the *interior of* S, denoted by S°, as the set of all interior points of S.

(a) Argue that S° is an open set.

(b) Show that S° is the largest open set contained in S (i.e., that if $G \subseteq S$ and G is open, then $G \subseteq S^\circ$). Conversely, show that if $\Omega \subseteq S$ is an open set with the property that for any $G \subseteq S$ that is open, $G \subseteq \Omega$, then $\Omega = S^\circ$.

7. Let G be a non-empty open set of $\mathbb{R}$. Show that every point in G is a limit point of G; please **justify** this fully.

The following anticipates material to be introduced in the lecture on **February 2.**

8. Let X be a metric space and $\{S_\alpha : \alpha \in A\}$ an arbitrary non-empty set of subsets of X. State whether the correct relation **in general** should be $B \supseteq C$ or $B \subseteq C$ or $B = C$, where

$$B = \bigcup_{\alpha \in A} \overline{S}_\alpha \quad \text{and} \quad C = \overline{\bigcup_{\alpha \in A} S_\alpha}.$$

If $B \neq C$ in general, then provide an example showing that the relevant inclusion could be a strict inclusion.