UM 204 (WINTER 2024) - WEEK 8

0.1. **Series.** We have already encountered two instances where we wanted to take the sum of infinitely many real numbers:

(1) when discussing decimal and tertiary expansions;
(2) when discussing the "length" of the Cantor set.

We now give a rigorous procedure of doing so.

Definition 0.1. Given a sequence $\{a_n\}_{n\in\mathbb{N}}$ of complex numbers, we say that the infinite series

$$\sum_{n=0}^{\infty} a_n \tag{0.1}$$

converges to a if and only if the sequence of partial sums (sops) $\{s_n\}_{n\in\mathbb{N}}$ given by

$$s_j = a_0 + \cdots + a_j, \quad j \in \mathbb{N},$$

converges to a. In this case, a is called the sum of the series (0.1). If $\{s_n\}_{n\in\mathbb{N}}$ diverges, then we say that the series $\sum_{n=0}^{\infty} a_n$ diverges.

Example. Let $z \in \mathbb{C}$, and $a_n = z^n$, $n \in \mathbb{N}$. Now, note that

$$s_n = z^0 + z^1 + \cdots + z^n.$$

Thus, $(1-z)s_n = 1 - z^{n+1}$, and if $z \neq 1$, then

$$s_n = \frac{1 - z^{n+1}}{1-z}.$$

Now, if $|z| < 1$, then $\{z^{n+1}\}_{n\in\mathbb{N}}$ converges to 0, and thus, $\sum_{n=0}^{\infty} z^n = \lim_{n\to\infty} s_n = \frac{1}{1-z}$.
If $|z| > 1$, then $\{z^{n+1}\}_{n\in\mathbb{N}}$, and therefore, $\{s_n\}_{n\in\mathbb{N}}$ diverges. Thus, the sum $\sum_{n=0}^{\infty} z^n$ diverges.
If $|z| = 1$, then the series also diverges. We consider a couple of special cases, and then give a general argument.

If $z = 1$, then the series is

$$1 + 1 + \cdots +$$

In this case, the sops is $\{n\}_{n\in\mathbb{N}}$, which clearly diverges.

If $z = -1$, the series is

$$1 - 1 + 1 - \cdots$$

In this case, the sops is $\{1 + (-1)^n\}_{n\in\mathbb{N}}$, which also diverges.

Theorem 0.2. *Let* $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$.

(1) $\sum_{n\in\mathbb{N}} a_n$ *converges if and only if for every* $\varepsilon > 0$, *there is an* $N \in \mathbb{N}$ *such that for all* $n \geq m \geq N$,

$$\left|\sum_{j=m}^{n} a_j\right| < \varepsilon.$$

(2) *If* $\sum_{n\in\mathbb{N}} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

(3) *Suppose* $a_n \geq 0$ *for all* $n \in \mathbb{N}$. *Then,* $\sum_{n\in\mathbb{N}} a_n$ *converges if and only of its sops is a bounded sequence.*

Proof. (1) We use the fact that $(\mathbb{C}, |\cdot|)$ is complete. Then, the convergence of $\{s_n\}_{n\in\mathbb{N}}$ is equivalent to its Cauchyness. Thus, $\sum_{n=0}^{\infty} a_n$ converges if and only if $\{s_n\}_{n\in\mathbb{N}}$ is Cauchy. This happens iff for every $\varepsilon > 0$, there is an $\widetilde{N} \in \mathbb{N}$ such that for all $n \geq m \geq \widetilde{N} + 1$,

$$|s_n - s_{m-1}| = \left|\sum_{j=m}^{n} a_j\right| < \varepsilon.$$

(2) Let $n = m$ in the previous part. We obtain: if $\sum_{n\in\mathbb{N}} a_n$ converges, then for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n| < \varepsilon$ for all $n \geq N$.

This is the monotone convergence theorem since $s_n = s_{n-1} + a_n \geq s_{n-1}$ for all $n \geq 1$. □

Now, note that if $|z| = 1$, then $\lim_{n\to\infty} z^n \neq 0$, thus the geometric series cannot converge when $|z| = 1$.

The convergence/divergence of some standard examples such as $\sum_{n\in\mathbb{N}_+} \frac{1}{n}$ and $\sum_{n\in\mathbb{N}} \frac{1}{n^2}$ follows from the Cauchy condensation test.

Theorem 0.3. *Suppose* $a_1 \geq a_2 \geq a_3 \cdots \geq 0$. *Then,* $\sum_{n=1}^{\infty} a_n$ *converges if and only if the series*

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

converges.

Exercise. Read the statements and proofs of the following tests:

(1) Comparison test
(2) Root test
(3) Ratio test

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Proof. Let $\{s_k\}_{k\in\mathbb{N}_+}$ and $\{t_k\}_{k\in\mathbb{N}_+}$ denote the sops of $\sum_{n\in\mathbb{N}_+} a_n$ and $\sum_{n\in\mathbb{N}} 2^k a_{2^k}$, respectively. Note that both the sequences are monotone. Thus, it suffices to show that one is bounded above if and only if the other is bounded above. Note that for a fixed $k \geq 1$ and $n < 2^k$,

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n \\ &\leq a_1 + (a_2 + a_3) + (a_4 + \cdots + a_7) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} = t_k. \end{aligned}$$

So, for $n < 2^k$, $s_n \leq t_k$.

If $n > 2^k$,

$$\begin{aligned} s_n &\geq a_1 + a_2 + (a_3 + a_4) + (a_5 + \cdots + a_8) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\geq \frac{1}{2}a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1} a_{2k} = \frac{1}{2} t_k. \end{aligned}$$

Thus, for $n > 2^k$, $s_n \geq (1/2)t_k$. Thus, the two sops are either both bounded above, or neither are bounded above. $\square$

Corollary 0.4. *Given $p \in \mathbb{R}$, the series $\sum_{n\geq 1} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.*

Proof. If $p \leq 0$, the individual terms do not converge to 0.

If $p > 0$, then the individual terms are decreasing, and we can use the previous theorem. Note that $\sum_{k\in\mathbb{N}} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k}$. Now, use your knowledge of geometric series. $\square$

Recall that we had defined the number e as the sum of the following convergent series in UM 101:

$$\sum_{n=0}^{\infty} \frac{!}{n!}.$$

Recall that the convergence of this sum follows from the fact that $n! \geq 2^{n-1}$, $n \in \mathbb{N}$, and the comparison test. You also established the irrationality of e using this definition. Now we may invoke the following exercise to talk about the natural logarithm, and claim that $\ln(n)$ is increasing in n.

Exercise Work out problem 7, Chapter 1, from Rudin's book. It gives the existence of a unique $x \in \mathbb{R}$ such that $b^x = y$, whenever $b > 1$ and $y > 0$. This will be our definition of $\log_b y$.

Corollary 0.5. *The series*

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

diverges if $p \leq 1$ and converges if $p > 1$.

Proof. Suppose $p \le 0$. Since $\ln(n))^p > 1$ for $n \ge 3$, $\frac{1}{n(\ln n)^p} \ge \frac{1}{n}$ for $n \ge 3$. Thus, by the comparison test, the series diverges.

For $p > 0$, we use that $n(\ln n)^p$ is increasing in n. Thus, by the CCT, we need to examine

$$\sum_{n=2}^{\infty} \frac{2^n}{(2^n)(\ln 2)^p n^p},$$

which diverges if $p \le 1$ and converges if $p > 1$. □

Before moving on, we give an alternate description of e.

Theorem 0.6.

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Proof. We want to show that the following two sequences have the same limit:

$$\begin{aligned}
s_n &= \sum_{j=0}^{n} \frac{1}{n!} \\
t_n &= \left(1+\frac{1}{n}\right)^n = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{n}\right)^j = \sum_{j=0}^{n} \frac{1}{j!} \frac{n(n-1)\cdots(n-j+1)}{n^j} \\
&= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \\
&\le s_n.
\end{aligned}$$

Thus, since we don't yet know that $\lim_{n\to\infty} t_n$ exists, we can say that

$$\limsup_{n\to\infty} t_n \le \lim_{n\to\infty} s_n = e.$$

Since the number of additive terms in each t_n is increasing as n increases, we can't just use the fact that $1/n \to 0$ to get rid of those terms. To keep the number of additive terms fixed, so that we can do this, we use the following trick. Fix $m \in \mathbb{N}$. Now, for $n \ge m \ge 3$,

$$\begin{aligned}
t_n &= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \\
&\ge 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right) \\
&= u_n^m.
\end{aligned}$$

Thus, the sequence $\{u_n^m\}_{n\ge3}$ is a sum of $m+1$ sequences, and taking $n \to \infty 0$ (for m fixed), we get that

$$\liminf_{n\to\infty} t_n \ge \lim_{n\to\infty} u_n^m = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!} = s_m.$$

Now, taking limit $m \to \infty$, we get that

$$\liminf_{n\to\infty} t_n \geq \lim_{m\to\infty} s_m = e.$$

$\square$

An important significance of infinite series is in being able to represent certain functions as power series (especially in complex analysis).

Definition 0.7. Given a sequence $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$, the series (whether convergent or not)

$$\sum_{n=0}^{\infty} a_n z^n$$

is called a power series.

Here, z denotes an arbitrary complex number. The series will converge or diverge, depending on the choice of z. We apply the root test to obtain information about the region of convergence of a power series.

Theorem 0.8. *Given the power series*

$$\sum_{j=0}^{\infty} a_j z^j,$$

let $R = 1/\alpha$, *where*

$$\alpha = \limsup_{n\to\infty} |a_n|^{1/n},$$

where $R = +\infty$ *if* $\alpha = 0$ *and* $R = 0$ *if* $\alpha = +\infty$. *Then the given series converges for* $|z| < R$ *and diverges for* $|z| > R$.

Remark. The behavior for $|z| = R$ can be quite varied (and delicate).

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Examples. (1) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is convergent everywhere.
(2) We already completely described the behavior of $\sum_{n\in\mathbb{N}} z^n$.
(3) The series $\sum_{n=0}^{\infty} z^n/n$ converges for $|z| < 1$ and diverges for $|z| > 1$. It converges for all $|z| = 1$ other than $z = 1$ (to be proved later).

0.2. **Combining series.** I. (Addition) Suppose $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} b_n$ are convergent (to a and b, respectively), then so is $\sum_{n\in\mathbb{N}} a_n + b_n$, and $\sum_{n\in\mathbb{N}} a_n + b_n = a + b$.
II. (Scalar multiplication) Suppose $\sum_{n\in\mathbb{N}} a_n$ converges to a and $c \in \mathbb{C}$. Then, $\sum_{n\in\mathbb{N}} ca_n$ converges to ca.
III. (Termwise product) Suppose $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} b_n$ are convergent (to a and b, respectively), then what can we say about

$$\sum_{n\in\mathbb{N}} a_n b_n?$$

This need not always be true! For example, take $a_n = b_n = \frac{(-1)^n}{n}$.

Theorem 0.9. *Let $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$. Suppose*

(1) *The sops, $\{A_n\}_{n\in\mathbb{N}}$, of $\sum_{n\in\mathbb{N}} a_n$ is bounded,*
(2) $b_0 \geq b_1 \geq \cdots$,
(3) $\lim_{n\to\infty} b_n = 0$.

Then, $\sum_{n\in\mathbb{N}} a_n b_n$ converges.

Remark. Note that their divergent series $\sum a_n$ and $\sum b_n$ that satisfy the hypotheses above!

Proof. We will use the Cauchy criterion. Note that for $q \geq p$,

$$\begin{aligned}\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n - \sum_{n=p}^{q} A_{n-1} b_n &= \sum_{n=p}^{q} A_n b_n - \sum_{m=p-1}^{q-1} A_m b_{m+1} \\ &= \sum_{n=p}^{q-1} A_n (b_n - b_{n-1}) + A_q b_q - A_{p-1} b_p.\end{aligned}$$

Suppose $M > 0$ such that $|A_n| \leq M$ for all $n \in \mathbb{N}$. Then, since $b_n - b_{n+1} \geq 0$ for all $n \in \mathbb{N}$, we have that

$$\left|\sum_{n=p}^{q} a_n b_n\right| \leq 2Mb_p.$$

Since $\{b_n\}_{n\in\mathbb{N}}$ converges to 0, for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $p \geq N$, we have that $2Mb_p < \varepsilon$. This completes the proof. ☐

Corollary 0.10 (The alternating series test)**.** *Suppose $b_0 \geq b_1 \geq \cdots$ and $\lim_{n\to\infty} b_n = 0$. Then,*

$$\sum_{n\in\mathbb{N}} (-1)^n b_n$$

converges.

Proof. Apply the above theorem to $a_n = (-1)^n$ and b_n as given. Note that the sops of $\sum_{n\in\mathbb{N}} a_n$ is $1/2(1 + (-1)^n)$ which is bounded.

☐

Corollary 0.11. *Suppose* $\{b_n\}_{n\in\mathbb{N}}$ *is a decreasing sequence of real numbers converging to* 0, *and such that the radius of convergence of the power series* $\sum_{n\in\mathbb{N}} b_n z^n$ *is* 1. *Then, the power series converges for all* $|z| = 1$, *except possible at* $z = 1$.

Proof. Apply the above theorem to the given $\{b_n\}_{n\in\mathbb{N}}$ and $a_n = z^n$. You must prove that the sops of $\sum_{n\in\mathbb{N}} z^n$ is bounded (Exercise). □

IV. (Product of series) Given $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} b_n$, the product of the two series is defined as the series $\sum_{n\in\mathbb{N}} c_n$ where

$$c_n = \sum_{j=0}^{n} a_j b_{n-j}.$$

This grouping of terms is motivated by looking at the coefficients of z^n in the term-by-term product of

$$(a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots).$$

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