UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022
HOMEWORK 7

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 22, 2022**

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1. Let $k \in \mathbb{N} \setminus \{0, 1\}$. Let $\mathbb{R}^k$ be equipped with the distance introduced in previous assignments (i.e., the Euclidean distance). Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}^k$.

(a) Write $a_n = (a_{n,1}, \dots, a_{n,k})$. Show that $\{a_n\}$ converges if and only if each sequence $\{a_{n,j}\} \subset \mathbb{R}$, $j = 1, \dots, k$, converges.

(b) Let $\{a_n\}$ and $\{b_n\}$ be convergent and let $A = \lim_{n\to\infty} a_n$, $B = \lim_{n\to\infty} b_n$. Show using ($a$) that $\{a_n + b_n\}$ is convergent and that $\lim_{n\to\infty}(a_n + b_n) = A + B$.

(c) Let $\{a_n\}$, $\{b_n\}$, A, and B be as in (b). Show **without** using (a) that $\{a_n + b_n\}$ is convergent and that $\lim_{n\to\infty}(a_n + b_n) = A + B$.

2. Let $\{a_n\}$ be a real sequence that converges to A. Then, show that the sequence of averages

$$\mu_n := \frac{a_1 + \cdots + a_n}{n}, \quad n = 1, 2, 3, \dots,$$

also converges.
Tip. It would help to guess what $\{\mu_n\}$ converges to!

3. Consider the sequence $\{a_n\}$, where $a_1 = 0$ and

$$a_{2n} = a_{2n-1}/2 \quad \text{and} \quad a_{2n+1} = (1/2) + a_{2n}, \quad n = 1, 2, 3, \dots .$$

Determine whether $\{a_n\}$ converges or not. Please give **justifications** for your answer.

4. Let $\mathscr{A} = \{a_n\}$ be a 0–1 sequence (i.e., a sequence whose terms are either 0 or 1) and define $b_n(\mathscr{A}) := a_n/n$, $n = 1, 2, 3, \dots$. Determine, for each $\mathscr{A}$, whether $\{b_n(\mathscr{A})\}$ converges or not.

The following anticipates material to be introduced during the lecture on **February 23.**

5. Let $\{a_n\} \subset \mathbb{R}$. State and prove a result presenting the two properties determining $\liminf_{n\to\infty} a_n$ analogous to those shown in class for $\limsup_{n\to\infty} a_n$.

6. Let $\{a_n\}$ be a real sequence. Define

$$A_k := \inf\{a_k, a_{k+1}, a_{k+2}, \dots\},$$
$$B_k := \sup\{a_k, a_{k+1}, a_{k+2}, \dots\}.$$

Show that

$$\liminf_{n\to\infty} a_n = \lim_{k\to\infty} A_k, \quad \text{and} \quad \limsup_{n\to\infty} a_n = \lim_{k\to\infty} B_k.$$