UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022
HOMEWORK 8

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Note. Problem 1 has been carried over from Assignment 7. This will allow discussion — if required at this stage — of Problem 1 during the March 14 tutorial.

1. Let $\{a_n\}$ be a real sequence that converges to A. Then, show that the sequence of averages

$$\mu_n := \frac{a_1 + \cdots + a_n}{n}, \quad n = 1, 2, 3, \ldots,$$

also converges.
Tip. It would help to guess what $\{\mu_n\}$ converges to!

2. Let $\{a_n\}$ be a real sequence, and define

$$\begin{aligned} \Delta_n &:= a_{n+1} - a_n, \\ \mu_n &:= \frac{a_1 + \cdots + a_n}{n}, \quad n = 1, 2, 3, \ldots . \end{aligned}$$

(*a*) Assume that the sequences $\{\mu_n\}$ and $\{n\Delta_n\}$ are convergent. Show that $\{a_n\}$ is convergent.

(*b*) Suppose we **only** assume that $\{n\Delta_n\}$ is convergent (i.e., no assumptions on $\{\mu_n\}$ are made). Does the conclusion of part (*a*) still hold true? Give a proof if this is true, else construct a sequence $\{a_n\}$ with the stated property that is not convergent.

Hint. The conclusion of Problem 1 holds true if applied to the sequence $\{n\Delta_n\}$. Examine what the sequence of averages of the first n terms of $\{n\Delta_n\}$ looks like.

3. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with terms in $\mathbb{R}$. Let $\{b_n\} \subset \mathbb{R}$ be a monotone bounded sequence. Show that $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

4. Let V be a vector space over the field $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, equipped with a norm $\|\cdot\|$. Let

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

be convergent series with terms in V whose sums are A and B respectively. Let $c \in \mathbb{F}$. Show that:

(*a*) the series $\sum_{n=1}^{\infty}(a_n + b_n)$ is convergent and its sum is $(A + B)$.

(*b*) the series $\sum_{n=1}^{\infty} c a_n$ is convergent and its sum is cA.

5. Review: Recall (and study) the two parts of the so-called Comparison Test, which was introduced to you in UM101. This is presented as **Theorem 3.25** in Chapter 3 of "Baby" Rudin (where the statement pertaining to **convergent** series is extended to series whose terms are in $\mathbb{C}$).

6. Fix a real number $a > 1$. Recall that if $p \in \mathbb{Q}$ and if $p = m/n$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, then

$$a^p := (a^m)^{1/n}. \tag{1}$$

Now, if now p is an arbitrary real number, then we define

$$a^p := \sup\{a^q : q \in \mathbb{Q} \text{ and } q \leq p\}. \tag{2}$$

Assume, for the moment, that the definition (1) does not depend on the choice of the representative m/n of p and that $a^p > 1$ whenever $p \in \mathbb{Q}^+$.

(a) For $p \in \mathbb{Q}$, show that the two definitions (1) and (2) of a^p agree.

(**Remark.** For $p \in \mathbb{Q}$, the left-hand side (1) is simply shorthand for an operation, given by the right-hand side, in an ordered field that admits n-th roots for any positive field element. Part (a) establishes the fact that (2) is the **extension** of the definition of the p-th power to an arbitrary $p \in \mathbb{R}$.)

(b) For any $x, y \in \mathbb{R}$, show that $a^{x+y} = a^x a^y$.

(c) Let $p \in \mathbb{R} \setminus \mathbb{Q}$. Let $\{q_n\}$ be **any** sequence of rational numbers such that $q_n \to p$. Show that $\{a^{q_n}\}$ is a convergent sequence. Having established this, show that if

$$q_1 \leq q_2 \leq q_3 \leq \cdots,$$

then the limit of $\{a^{q_n}\}$ is a^p as defined by (2)

(**Remark / Tip.** Part (c) provides yet another reason for why the right-hand side of (2) is a natural way to define a^p when $p \in \mathbb{R} \setminus \mathbb{Q}$. For the sequence $\{a^{q_n}\}$ in the first half of part (c), it suffices to show that $\{a^{q_n}\}$ is Cauchy. Also, you may assume **without proof** that $\lim_{j\to\infty} a^{1/j} = 1$.)

(d) How would you define a^p for $p \in \mathbb{R}\setminus\mathbb{Q}$ if $a < 1$? You are not required to establish appropriate analogues of (a) and (c), but to state reasons why (2) must be amended.