UM 204: QUIZ 1

Jan. 12, 2024

Duration. 15 minutes

Maximum score. 10 points

You are allowed to compute limits of real sequences without proof.

Problem. Let $A : \mathbb{Q} \to [0, \infty)$ be an absolute value function on $\mathbb{Q}$, i.e.,

(1) $A(x) = 0$ if and only if $x = 0$,

(2) $A(xy) = A(x)A(y)$ for all $x, y \in \mathbb{Q}$,

(3) $A(x + y) \leq A(x) + A(y)$ for all $x, y \in \mathbb{Q}$.

Suppose there is a $C > 0$ such that $A(n) \leq C$ for all $n \in \mathbb{N}$. Show that

$$A(x + y) \leq \max\{A(x), A(y)\}, \quad \forall x, y \in \mathbb{Q}.$$

Hint. Estimate $A\left((x + y)^m\right)$ from above, take the m^{th} root, and take limits as $m \to \infty$.

For any $x, y \in \mathbb{Q}$, we have by the multiplicativity and sub-additivity of A that

$$\begin{aligned} A(x+y)^m = A((x+y)^m) = A\left(\sum_{j=0^m} \binom{m}{j} \cdot x^j \cdot y^{m-j}\right) &\leq \sum_{j=0}^{m} A\left(\binom{m}{j}\right) A(x)^j A(y)^{m-j} \\ &\leq (m+1)C \max\{A(x), A(y)\}^m. \end{aligned}$$

Taking m^{th} roots on both sides, and taking limits as $m \to \infty$, we have that

$$A(x + y) \leq \max\{A(x), A(y)\} \lim_{m\to\infty} (m + 1)^{1/m} C^{1/m} = \max\{A(x), A(y)\}.$$