UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2022

HOMEWORK 2

Instructor: GAUTAM BHARALI **Assigned: JANUARY 18, 2022**

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1. Let S be a non-empty set, and let $\sim$ be an equivalence relation on S. Recall that, for any $s \in S$, the *equivalence class of* s — denoted by $[s]$ — is defined as

$$[s] := \{x \in S : x \sim s\}.$$

You may assume without proof that the collection $[s]$ **is** a set. Assuming furthermore (if required) that any auxiliary collections that you need to construct are sets (which **can be shown** rigorously by the axioms of Set Theory), show that $\sim$ partitions S into disjoint equivalence classes.

2. Consider the following subsets of $\mathbb{N} \times \mathbb{N}$:

$$\begin{aligned} \mathsf{diag} &:= \{(m, m) : m \in \mathbb{N}\}, \\ \mathcal{P} &:= \{(m, n) \in \mathbb{N} \times \mathbb{N} : m \times m \leq n\}, \end{aligned}$$

where "$\leq$" denotes the usual order, and "$\times$" denotes Peano multiplication, on $\mathbb{N}$. Define:

$$\text{for } m, n \in \mathbb{N}, \ \ m \preceq n \iff (m, n) \in (\mathsf{diag} \cup \mathcal{P}).$$

Is $\preceq$ and order on $\mathbb{N}$? Give justifications.

3. Review: Recall (and study) the definition, in algebra, of a **field.**

The following anticipates material to be introduced in the lecture on **January 19.**

4. Consider the formalisation of the relation $\leq$ on $\mathbb{N}$:

$$\text{for } m, n \in \mathbb{N}, \ \ m \leq n \iff \exists a \in \mathbb{N} : n = m + a.$$

Show that $\leq$ is an order on $\mathbb{N}$.

5. Let $(S, \leq)$ be an ordered set having the least upper bound property. Let $A \subseteq S$ be a non-empty bounded set. Show that A has a unique least upper bound.