UM 204 (WINTER 2024) - WEEK 7

We were in the midst of showing that:

$\limsup_{n\to\infty} x_n$ is the unique element α in $\overline{\mathbb{R}}$ satisfying

(2) $\alpha \in E$, where E is the set of subsequential limits of $\{X_n\}_{\mathbb{N}}$,
(3) for any $x > \alpha$, there is an $N \in \mathbb{N}$ such that $x_n < x$ for all $n \geq N$.

Suppose (3) does not hold. Then, for every $k \in \mathbb{N}$, there is some $m(k) \geq k$ such that $x_{m(k)} \geq x$. Choose $n_0 = n(0)$ and $n_k = m(n_{k-1})$, $k \geq 1$. Then, $\{x_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence bounded below by x, and any of its subsequential limits will be $\geq x > \sup E$, which is a contradiction because subsequential limits of subsequences are themselves subsequential limits.

For uniqueness, suppose $y < z$ in $\overline{\mathbb{R}}$ such that $y, z \in E$ and both satisfy (3), i.e., if $x > y$, then there is an $N \in \mathbb{N}$ such that $x_n < x$ for all $n \geq N$ (and the analogous statement, if $x > z$). Let

$$x = \begin{cases} z-1, & \text{if } y=-\infty,\ z\in\mathbb{R}, \\ 0, & \text{if } y=-\infty,\ z=+\infty, \\ y+(z-y)/2, & \text{if } y,z\in\mathbb{R}, \\ y+1, & \text{if } y\in\mathbb{R},\ z=+\infty. \end{cases}$$

Note that in all cases, $x < z$. Then, there is some $N \in \mathbb{N}$ such that $x_n < x$ for all $n \geq N$. Thus, all subsequential limits are $< z$, so $z \notin E$, which is a contradiction.

We now give an alternate characterization of lim sup and lim inf.

Theorem 0.1. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a real sequence. Then, the sequences*

$$\begin{aligned} y_n &= \sup\{x_k : k \geq n\}, \\ z_n &= \inf\{x_k : k \geq n\}, \end{aligned}$$

have limits in $\overline{\mathbb{R}}$. Moreover,

$$\begin{aligned} \limsup_{n\to\infty} x_n &= \lim_{n\to\infty} y_n = \inf_{n\geq 0} \sup_{k\geq n}\{x_k\}, \\ \liminf_{n\to\infty} x_n &= \lim_{n\to\infty} z_n = \sup_{n\geq 0} \inf_{k\geq n}\{x_k\}. \end{aligned}$$

where the limits are in $\overline{\mathbb{R}}$.

Proof. Note that y_n and z_n are clearly montonically (decreasing and increasing, respectively) sequences. Monotone sequences always admit limits in $\overline{\mathbb{R}}$.

Let $y = \lim_{n\to\infty} y_n$. We will show that y satisfies (2) and (3) of the above theorem. Let $k > 0$. Then, there is some $N(k) \in \mathbb{N}$ such that

$$y \le y_n = \sup\{x_k : k \ge n\} < y + \frac{1}{k}, \quad \forall n \ge N(k).$$

By the ε-characterization of suprema, for each $n \ge N(k)$, there is some $m(k,n) \ge n$ such that

$$y - \frac{1}{k} \le y_n - \frac{1}{k} < x_{m(k,n)} \le y_n < y + \frac{1}{k}.$$

Choose $n_1 = m(1, N(1))$. Choose $n_2 = m(2, \max\{N(2), n_1\})$. In general, choose $n_k = m(k, \max\{N(k), n_{k-1}\})$. Then, $\{x_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence satisfying

$$|x_{n_k} - y| \le \frac{1}{k}.$$

Thus, y is a subsequential limit.

Now, assume $x > y$. WLOG, assume $x, y \in \mathbb{R}$. Let $\varepsilon = \frac{x-y}{2}$. Then, there is some $N \in \mathbb{N}$ such that

$$y_n \le y + \frac{x-y}{2} < x, \quad \forall n \ge N.$$

But $x_n \le y_n$ for all $n \in \mathbb{N}$. Thus, (3) holds for y. $\square$

Remark. There is a notion of lim sup and lim inf of sequences of sets $A = \{A_j\}_{j\in\mathbb{N}}$.

$$\begin{aligned} \limsup_{n\to\infty} A_n &= \bigcap_{n=0}^{\infty} \bigcup_{k\ge n} A_k, \\ \liminf_{n\to\infty} A_n &= \bigcup_{n=0}^{\infty} \bigcap_{k\ge n} A_k. \end{aligned}$$

The former set is the set of those elements that occur in infinitely many A_j's, while the latter set consists of elements that are in all but finitely many A_j's.

Examples. (1) $\{\sin(n\pi/6)\}_{n\in\mathbb{N}} = 0, 1/2, \sqrt{3}/2, 1, \sqrt{3}/2, 1/2, 0, -1/2, -\sqrt{3}/2, -1, -\sqrt{3}/2, -1/2, 0, \ldots$. Thus,

$$\begin{aligned} \limsup_{n\to\infty} \sin(n\pi/6) &= \lim_{n\to\infty} \sup\{x_k : k \ge n\} = \lim_{n\to\infty} \sqrt{3}/2 = \sqrt{3}/2, \\ \liminf_{n\to\infty} \sin(n\pi/6) &= \lim_{n\to\infty} \inf\{x_k : k \ge n\} = \lim_{n\to\infty} -\sqrt{3}/2 = -\sqrt{3}/2. \end{aligned}$$

END OF LECTURE 18