# Lecture - 4

## Relations :-

Notation :- $A, B$ are sets. Then $A \times B = \{(a,b) \mid a \in A, b \in B\}$ is called cartesian product.

Def'n :- Let $A, B$ be sets. Then a subset $R \subseteq A \times B$ is called a (binary) relation from $A$ to $B$.
If $B = A$, then we say $R$ is a relation on $A$.

Example :- $A$ = Set of $2^{nd}$ year IISc students in the UG BS prog
$B = A$
$R = \{(a,b) \mid a, b \in A$, $a$ & $b$ have the same major$\}$

Properties of relations on $A$ :-

1. Reflexive :- $(a,a) \in R \quad \forall a \in A$
2. Symmetry :- $(a,b) \in R \Rightarrow (b,a) \in R$
3. Antisymmetry :- $(a,b) \in R$ and $(b,a) \in R \Rightarrow b = a$
4. Transitivity :- $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$

If $R$ satisfies 1, 2, 4 ; we call it an equivalence relation.
If $R$ satisfies 1, 3, 4 ; we call it a partial order.

Examples :-

1. $A = \mathbb{N}$, we say $(a,b) \in R$ if $a - b \equiv 0 \pmod 3$
$\boxed{\text{Equivalence relation.}}$
2. $A = [10] := \{1, 2, 3, \ldots, 10\}$. $(a,b) \in R$ if $a = b$
$\boxed{\text{Equivalence, Partial Order}}$
3. $A = [3]$ $(a,b) \in R$ if $a = 1$ or $b = 1$
$\boxed{\text{Symmetric only}}$
4. $A = \mathbb{R}^2$ $(a,b) \in R$ if $a_1^2 + a_2^2 = b_1^2 + b_2^2$
$a \to (a_1, a_2)$, $b \to (b_1, b_2)$

Equivalence

Def'n :- Let $X$ be a set, $\sim_R$ be an equivalence relation on $X$ i.e., we write $a \sim_R b$ whenever $(a,b) \in R$. Then the equivalence class associated to $x \in X$ is

$$[x] = \{ y \in X \mid x \sim_R y \}$$

Def'n :- A (set) partition of a set $X$ is a family $\{ X_\alpha \mid \alpha \in \Lambda \}$ ($\Lambda$ is some indexing set) such that

i, $X_\alpha \cap X_\beta = \phi \quad \forall \ \alpha \neq \beta \in \Lambda$

ii, $X = \bigcup_{\alpha \in \Lambda} X_\alpha$. Standard notation for a set partition, $X = \underbrace{\bigsqcup_{\alpha \in \Lambda}}_{\text{Disjoint union}} X_\alpha$

Property (Fundamental theorem and Equivalence Relation) :-

Let $X$ be a set and $\sim$ be an equivalence relation on $X$. Then the family of equivalence classes $\{ [x] \mid x \in X \}$ forms a partition of $X$. Conversely, every partition of $X$ arises from an equivalence relation.

Proof of property :- Exercise.

Def'n :- Let $X$ be a set and $\sim$ be an equivalence relation on $X$. Then the set $X/\sim \, = \{ [x] \mid x \in X \}$ is called the quotient set of $X$ by the relation $\sim$

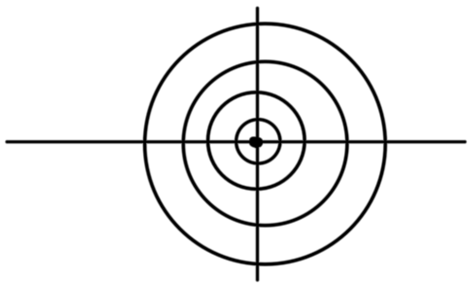
Back to examples :-

1, $\mathbb{N} = \{0, 3, 6, 9, \ldots\} \sqcup \{1, 4, 7, \ldots\} \sqcup \{2, 5, 8, \ldots\}$

$$\mathbb{N}/\sim \, = \{ [0], [1], [2] \} \longleftrightarrow \{0, 1, 2\}$$

2, $\{1, 2, \ldots, 10\}/\sim \, = \{ [1], [2], \ldots, [10] \} \longleftrightarrow \{1, 2, \ldots, 10\}$

4, $\mathbb{R}^2/\sim \, = \mathbb{R}_+ = \{ x \in \mathbb{R} \mid x \geqslant 0 \}$



## Functions :-

Def'n :- Let $A, B$ be sets. A relation $f$ from $A$ to $B$ is a function, if whenever $(a,b), (a,b') \in f$, then $b=b'$. Moreover, for every $a \in A$, $\exists\, b \in B$ st $(a,b) \in f$. Then we write $f: A \to B$ and $f(a) = b$.

A is called the domain and B is called the co-domain/ Range.
For a subset $C \subseteq A$, the image of $C$ under $f$ is $f(C) = \{f(c) \mid c \in C\}$. Then $f(A)$ is called the image of $A$ under $f$. For a subset $D \subseteq B$, the pre-image of $D$ under $f$ is $f^{-1}(D) = \{a \in A \mid f(a) \in D\}$.

Q Why is $f(C)$ a set ? Axiom of Replacement.

Examples:- 1, $A = B = \mathbb{N}$. $f(a) = a{+}{+}$. Then $f(A) = \mathbb{N} \setminus \{0\}$

$$f^{-1}(\{a\}) = \begin{cases} \{a-1\} ; & a > 0 \\ \phi ; & a = 0 \end{cases}$$

2. $A = B = \mathbb{N}$. $f(a) = a{-}{-}$. Not a function.

Def'n:- Two functions $f, g$ with same domain and range $X \to Y$ are equal if $f(x) = g(x)\ \forall\, x \in X$.

Example :- $X = Y = \mathbb{R}_+$, $f(x) = x$, $g(x) = |x|$