UM 204 HOMEWORK ASSIGNMENT 1

Posted on January 04, 2024

(NOT FOR SUBMISSION)

• These problems are for self-study. Try these **on your own** before seeking hints from the instructor/TAs.

• Some of these problems will be discussed at the next tutorial. The TA will not give complete solutions, but will provide hints.

• A 15-min. quiz based on this assignment will be conducted at the end of the tutorial section.

Problem 1. Let $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_Z, \leq_{\mathbb{Z}})$ be as constructed in class. Recall that we identify $n \in \mathbb{N}$ with $[(n,0)] \in \mathbb{Z}$. Show that any non-zero element of $\mathbb{Z}$ is either m or $-m$ for some $m \in \mathbb{N}$.

Problem 2. Recall the construction of $\mathbb{Q}$ as the set of equivalence classes of the relation R on $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ given by $(a,b)R(c,d) \iff ad = bc$. We say that $[(a,b)] \leq [(c,d)]$ if $(bc - ad)(bd) \geq 0$. Using only the arithmetic and order properties of integers, show that the relation $\leq$ is well-defined. Remember you are no allowed to divide yet!

Problem 3. Without assuming the existence of irrational numbers, show that

(a) $(\mathbb{Z}, \leq)$ has the least upper bound property;

(b) $(\mathbb{Q}, \leq)$ does not have the least upper bound property.

You may directly cite any theorem(s) proved in class.

Problem 4. Let F be an ordered field. Recall that $\mathbb{Q} \subset F$. Show that the following two statements are equivalent.

(i) For every $a, b > 0$ in F, there is an $n \in \mathbb{N}$ such that $na > b$.

(ii) For every $a < b$ in F, there is an $r \in \mathbb{Q}$ such that $a < r < b$.

Problem 5. Let F be a field. An absolute value on F is a function $A : F \to \mathbb{R}$ satisfying

(1) $A(x) \geq 0$ for all $x \in F$,

(2) $A(x) = 0$ if and only if $x = 0$,

(3) $A(xy) = A(x)A(y)$ for all $x, y \in F$,

(4) $A(x+y) \leq A(x) + A(y)$ for all $x, y \in F$.

A subset $S \subset F$ is said to be A-bounded if there exists an $M > 0$ such that $A(s) \leq M$ for all $s \in S$. This is a way to define boundedness of sets in the absence of an order relation.

Let $p \in \mathbb{N}$ be a prime number. Define $\nu_p : \mathbb{Z} \to \mathbb{Z} \cup \{\infty\}$ by

$$\nu_p(n) = \begin{cases} \max\{k \in \mathbb{N} : p^k | n\}, & \text{if } n \neq 0, \\ \infty, & \text{if } n = 0. \end{cases}$$

Extend ν_p to $\mathbb{Q}$ by

$$\nu_p(a/b) = \nu_p(a) - \nu_p(b), \quad a, b \in \mathbb{Z}, b \neq 0.$$

Now, define $A_p : \mathbb{Q} \to \mathbb{R}$ by $A_p(x) = e^{-\nu_p(x)}$ if $x \neq 0$, and $A_p(0) = 0$.

(a) Show that A_p is an absolute value on $\mathbb{Q}$.

(b) Show that

$$A_p(x+y) \leq \max\{A_p(x), A_p(y)\}, \quad \forall x, y \in \mathbb{Q}.$$

(c) Show that $\mathbb{Z}$ is A_p-bounded.

You may use basic facts about factorization without proof, but clearly state what you are using.