UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022

HINTS TO/SKETCHES OF SOLUTIONS TO MID-SEMESTER PROBLEMS

Instructor: GAUTAM BHARALI **March 1, 10:00 to 11:30 a.m.**

Instructions: PLEASE READ CAREFULLY

a) Please note that **Problems 1–4 are compulsory,** and you are required to solve **any one out of Problems 5 & 6.**

b) You may freely use **without proof:**
- any result, related to the topics **in the syllabus** for this exam, of which a precise statement — whether proven or not — was given in the lectures.
- any standard property of an ordered field (which you can use tacitly: i.e., **without naming** said property).
- any result stated as a homework problem **except, of course,** if a problem below itself was previously given in an assignment!

PLEASE NOTE: In no case, except Problem 3, are complete solutions provided below! What follows are **hints** for solving a problem (or, at best, **sketches** of the solutions meant to help you through the difficult parts). The hints/sketches are meant to encourage you to **think.**

1. Let S be a non-empty subset of $\mathbb{Z}$ that is bounded above.

(a) (1 mark: **easy!**) Why does $\sup S$ exist in $\mathbb{R}$?

(b) (3 marks) Prove, with **justifications,** that $\sup S \in \mathbb{Z}$.

Remark. It might help to rely on a result stated in homework, although that is not the only approach to a solution.

Preliminaries. Several solutions to part (b) attempted in the exam used the assertion that, given $n \in \mathbb{Z}$, there exists no integer q satisfying $n < q < (n+1)$. Strictly speaking, this requires a proof because, while $\leq$ on $\mathbb{N}$ is intuitive and a precise statement of its relationship with Peano arithmetic was stated in the lectures, the same was **not** the case with $\mathbb{Z}$. (Similar remarks apply to invoking the Well-Ordering Principle.) It is for these reasons that the **Remark** above was made. This will inform the hints below.

Hints to the solution of part (b)*:* First establish the following:

A set of integers has no limit point in $\mathbb{R}$.

By Problem 1 in Assignment 6, $\sup S \in \overline{S}$. As S is a set of integers, by the above fact $\sup S$ cannot be a limit point of S. Thus, $\sup S \in S \subset \mathbb{Z}$.

2. (5 marks) Let X be a metric space and let d denote the metric on it. Let Y be a non-empty proper subset of X. Recall that we can view Y itself as a new metric space with the metric $d_Y := d|_{Y \times Y}$. Let $A \subseteq Y$. Let $\overline{A}^Y$ denote the closure of A relative to Y: i.e., the closure of A viewing it as a subset of the metric space (Y, d_Y). Prove that $\overline{A}^Y = Y \cap \overline{A}$ (where $\overline{A}$ denotes the closure of A in the original metric space).

Hints to the solution: By definition, $\overline{A}^Y = A \cup (A')^Y$, where $(A')^Y$ is the set of all limit points of A with respect to the metric d_Y. Using the fact that $B_Y(a,r) = B(a,r) \cap Y$ for $a \in Y$ and $r > 0$, show that

$$A \cup (A')^Y \subseteq Y \cap \overline{A}. \tag{1}$$

Next, suppose $x \in Y \cap \overline{A}$. If $x \notin A$, then for each $r > 0$, there exists $a_r \in A$ such that $a_r \in A \cap B(x,r)$ and $a_r \neq x$. Since $A \subseteq Y$,

$$a_r \in A \cap B(x,r) = A \cap (Y \cap B(x,r)) = A \cap B_Y(x,r),$$

and $a_r \neq x$. Thus, $x \in (A')^Y$. Since this is true for any $x \in Y \cap \overline{A}$, $x \notin A$, show that this gives

$$A \cup (A')^Y \supseteq Y \cap \overline{A}. \tag{2}$$

From (1) and (2), the result follows.

3. (5 marks) Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$. Suppose $a_n \leq b_n$ for $n = 1, 2, 3, \dots$. Prove that

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n.$$

Remark. If you wish to use the conclusions of a problem in Assignment 7, then recall that you can do so without proof, but please give a **clear** statement of what you are using.

Solution: Let us write $A := \liminf_{n\to\infty} a_n$ and $B := \liminf_{n\to\infty} b_n$. If $B = +\infty$ or $A = -\infty$, then there is nothing to prove. Thus, it suffices to assume that $B < +\infty$ and $A > -\infty$.

We need the analogue of part (a) Theorem 3.17 in "Baby" Rudin describing the lower limit of $\{b_n\}$, which is that B itself is a subsequential limit. Thus, there exists a subsequence $\{b_{n_j}\} \subset \{b_n\}$ such that $\lim_{j\to\infty} b_{n_j} = B$.

We do not know, in general, whether $\{a_{n_j}\}$ converges! But $E[\{a_{n_j}\}] \neq \varnothing$. Thus, pick a convergent subsequence $\{a_{n_{j_i}}\}$ (**note:** this is the key trick in this solution). By our "it suffices" assumption, $\lim_{i\to\infty} a_{n_{j_i}} \in \mathbb{R}$. Thus, by the theorem on termwise algebraic combinations of two real sequences, we have

$$0 \leq \lim_{i\to\infty} \left(b_{n_{j_i}} - a_{n_{j_i}}\right) = \lim_{i\to\infty} b_{n_{j_i}} - \lim_{i\to\infty} a_{n_{j_i}} = B - \lim_{i\to\infty} a_{n_{j_i}}.$$

Thus, $\lim_{i\to\infty} a_{n_{j_i}} \leq B$. By definition of B, we have $B \leq A$.

Remark. It is also possible to appeal to Problem 6 in Assignment 7 to solve this problem, but it results in a more wordy solution.

4. Let $X =$ the set of all sequences in $\mathbb{R}$ and write

$$d\left(\{x_n\}, \{y_n\}\right) := \sup\left\{\min\{1, |x_n - y_n|\} : n \in \mathbb{Z}^+\right\}.$$

- (a) (3 marks) It turns out that d is a metric on X. Prove the triangle inequality for d.
- (b) (3 marks) Define $E := \left\{\{x_n\} : x_n \in [-1,1] \text{ for each } n \in \mathbb{Z}^+\right\}$. Determine whether or not E is a compact subset of X. Give **justifications** for your answer.

Hints to the solution: The **cleanest** way to solve part (a) is to first show

$$\rho(x,y) \;:=\; \min\{1, |x-y|\},\ x,y\in\mathbb{R}, \quad \textit{is a metric on } \mathbb{R}. \tag{3}$$

Now, consider three sequences $A=\{a_n\},\ B:=\{b_n\},\ C:=\{c_n\}\in X$. Fix some $m\in\mathbb{Z}^+$. Then, there exists an $n_0\equiv n_0(m)\in\mathbb{Z}^+$ such that

$$d(A,C)-(1/m) \;=\; \sup\left\{\rho(a_n,c_n) : n\in\mathbb{Z}^+\right\}-(1/m) \;\le\; \rho(a_{n_0}c_{n_0}).$$

By (3), the triangle inequality for ρ gives us $\rho(a_{n_0},c_{n_0})\le\rho(a_{n_0},b_{n_0})+\rho(b_{n_0},c_{n_0})$. By definition, $\rho(a_{n_0},b_{n_0})\le d(A,B)$ and $\rho(b_{n_0},c_{n_0})\le d(B,C)$. From these three inequalities:

$$d(A,C)-(1/m) \;\le\; d(A,B)+d(B,C),$$

and this holds true for **any** arbitrary $m\in\mathbb{Z}^+$. Thus, the triangle inequality follows.

Preliminary comment on part (b). It helps to guess what the answer must be. Also, be aware that it gets **extremely** messy to show that E is not compact from first principles; this in **not** the approach to take!

To respond to part (b), we must show that E is **not** compact. It is easiest to rely on the fact that if E were compact, then any infinite set $S\subseteq E$ would have a limit point in E. Take $S=\big\{\{a_{m,n}\}: n=1,2,3,\dots\big\}\subseteq E$, where

$$a_{m,n} \;:=\; \begin{cases} 1, & \text{if } n=m,\\ 0, & \text{otherwise,}\end{cases}$$

for each $m=1,2,3,\dots$, and show that E has no limit points in E.

SOLVE ANY *ONE* OUT OF THE NEXT TWO PROBLEMS.

THE NEXT TWO PROBLEMS WILL BE ASSIGNED FOR HOMEWORK (WITH SUITABLE HINTS).

5. (5 marks) Let $\{a_n\}$ be a real sequence, and define

$$\begin{aligned}\Delta_n \;&:=\; a_{n+1}-a_n,\\ \mu_n \;&:=\; \frac{a_1+\cdots+a_n}{n},\quad n=1,2,3,\dots\,.\end{aligned}$$

Assume that the sequences $\{\mu_n\}$ and $\{n\Delta_n\}$ are convergent. Is $\{a_n\}$ convergent? Give a proof if this is true, else provide a $\{a_n\}$ with the stated properties that is not convergent.

6. (5 marks) Let G be a non-empty bounded open subset of $\mathbb{R}$. Prove that G is the union of an at most countable collection of **disjoint** non-empty open intervals.

Tip + remark. You may use freely **without proof** the fact that $\mathbb{R}$ (equipped with the usual metric) is a separable metric space. The above result is true without the assumption of boundedness of G; the latter assumption just eliminates certain cases to be considered and shortens the proof.