UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022

QUIZ 7 **MARCH 28, 2022**

PLEASE NOTE the following:

- **Duration:** 15 minutes
- The quiz is to be written with no access to any books, notes, or study materials.

1. Let $n \geq 2$, $n \in \mathbb{Z}^+$. Show that $f(x) = x^{1/n}$, $x \in [0, +\infty)$, is continuous on $[0, +\infty)$.

Note. The above forms one of the **parts** to solving Problem 5 — if done right — of Homework 10.

Solution. For $y > x \geq 0$

$$\begin{aligned} y > x \;&\Rightarrow\; \{f(y)\}^n - \{f(x)\}^n \;>\; 0 \\ &\Rightarrow\; \{f(y) - f(x)\}\Big(\sum_{j=0}^{n-1}\{f(y)\}^{n-1-j}\{f(x)\}^j\Big) \;>\; 0 \\ &\Rightarrow\; f(y) \;>\; f(x) \qquad\qquad [\,\text{since } f(x) \geq 0 \text{ and } f(y) > 0\,] \end{aligned}$$

Thus, f is an increasing function. Now, fix $a \geq 0$. Let $\varepsilon > 0$. Our analysis comprises two cases:

Case 1. $a = 0$

Since f is strictly increasing, if we take $\delta = \varepsilon^n$, then

$$\begin{aligned} |x - a| < \delta \text{ and } x \geq 0 \;\Rightarrow\; 0 \;\leq\; x \;<\; \delta \;&\Rightarrow\; 0 \;\leq\; f(x) \;<\; \varepsilon \\ &\Rightarrow\; |f(a) - f(x)| \;<\; \varepsilon. \end{aligned} \tag{1}$$

Case 2. $a > 0$

Let $\varepsilon^* := \min(a^{1/n}, \varepsilon)$ and let

$$\delta := \min\big((a^{1/n} + \varepsilon^*)^n - a, a - (a^{1/n} - \varepsilon^*)^n\big).$$

Then, since f is strictly increasing,

$$\begin{aligned} |x - a| \;<\; \delta \;\Rightarrow\; a - \delta \;<\; x \;<\; a + \delta \;&\Rightarrow\; (a^{1/n} - \varepsilon^*)^n \;<\; x \;<\; (a^{1/n} + \varepsilon^*)^n \\ &\Rightarrow\; (a^{1/n} - \varepsilon^*) \;<\; x^{1/n} \;<\; (a^{1/n} + \varepsilon^*) \\ &\Rightarrow\; (a^{1/n} - \varepsilon) \;<\; x^{1/n} \;<\; (a^{1/n} + \varepsilon) \\ &\Rightarrow\; |f(x) - f(a)| \;<\; \varepsilon. \end{aligned} \tag{2}$$

Since, in either case, for any arbitrary $\varepsilon > 0$, we can produce a $\delta > 0$ leading to the conclusions (1) and (2), respectively, f is continuous on $[0, +\infty)$. $\square$