UM 204 : INTRODUCTION TO BASIC ANALYSIS
SPRING 2022
HOMEWORK 13

Instructor: GAUTAM BHARALI **Assigned: APRIL 12, 2022**

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1. Consider the following simpler (classical?) version of the 2^{nd} Fundamental Theorem of Calculus:

Let $a < b$ be real numbers and let $f : [a,b] \to \mathbb{R}$ be a continuous function. Let F be a primitive of f on $[a,b]$. Then
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Prove the above result by invoking the 1^{st} Fundamental Theorem of Calculus. (The above can be proved by repeating the proof seen in class of the more general version, but that evades the point of the present problem.)

2. Prove the following generalization of the **Weierstrass $\boldsymbol{M}$-test** (see Chapter 7, Theorem 7.10 of "Baby" Rudin for a statement of the classical M-test):

Let X be a metric space and $E \subseteq X$ a non-empty subset. Let $(V, \|\cdot\|_V)$ be a normed vector space over $\mathbb{R}$ or $\mathbb{C}$ that is complete with respect to the metric induced by $\|\cdot\|_V$. Let $\{f_n\}$ be a sequence of V-valued functions defined on E. Suppose that, for each $n \in \mathbb{Z}^+$, there exists a constant $M_n > 0$ such that
$$\|f_n(x)\|_V \leq M_n \quad \forall x \in E, \text{ and } n = 1, 2, 3, \dots,$$
such that the series $\sum_{n=1}^\infty M_n$ is convergent. Then, the series $\sum_{n=1}^\infty f_n$ is uniformly convergent.

3–5. Problems 11, 14 and 15 from "Baby" Rudin, Chapter 7.

6. Let X be a metric space and let $(V, \|\cdot\|_V)$ be a normed vector space over the field $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{B}(X;V)$ be as defined in class. Provide the details supporting the fact (stated in class) that the sup-norm, given by
$$\|f\| := \sup_{x \in X} \|f(x)\|_V\,, \quad f \in \mathcal{B}(X;V),$$
is a norm on $\mathcal{B}(X;V)$.

7. Let $a < b$ be real numbers and let
$$\mathcal{A} := \left\{ [a,b] \ni x \mapsto \int_a^x f(t)\,dt : f \in \mathscr{R}([a,b]) \text{ and } |f(t)| \leq 10 \ \forall t \in [a,b] \right\}.$$
Let $\mathscr{F}$ denote the closure of $\mathcal{A}$ in $\mathcal{B}\big([a,b];\mathbb{R}\big)$ (w.r.t. the sup-metric). **Fix** your favourite polynomial p. Show that there exists a function $g \in \mathscr{F}$ such that
$$p \circ g\left(\frac{a+b}{2}\right) \geq p \circ \varphi\left(\frac{a+b}{2}\right) \quad \forall \varphi \in \mathscr{F}.$$