UM 204 : INTRODUCTION TO BASIC ANALYSIS

SPRING 2025

HOMEWORK 8

Instructor: GAUTAM BHARALI **Assigned: MARCH 8, 2025**

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1. Let p be a real number. Use Cauchy's Condensation Test to give a necessary and sufficient condition on p such that

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges.

Note. The result that you have been asked above to establish is sometimes called the *p-series test.*

2. Problem 24 from Chapter 3 of "Baby" Rudin.

Note. We had *very briefly* discussed in class that, given any incomplete metric space (X, d), we can construct a metric space (X^*, Δ) that **is** complete and that there is a distance-preserving map $\phi : X \to X^*$ such that $\phi(X)$ is dense in (X^*, Δ). The above problem establishes the latter over several steps.

3. Fix $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let X be a metric space, $E \subseteq X$, p a limit point of E, and $g : E \to \mathbb{F}$ a function such that $a \in \mathbb{F}$ is the limit of g at p. Suppose $a \neq 0$. Then, show that there exists an open neighbourhood N_p of p such that $g(x) \neq 0$ for all $x \in E \cap N_p$.

4. Let $n \in \mathbb{N} \setminus \{0, 1\}$. Prove from the definition and first principles (i.e., without using any results on sums/products of continuous functions), that $f(x) = x^n$, $x \in \mathbb{R}$, is continuous on $\mathbb{R}$.

5. Let X and Y be metric spaces, and let $f, g : X \to Y$ be two continuous functions. Let $E \subsetneq X$ be a proper dense subset. Suppose $f(x) = g(x)$ for each $x \in E$. Show that $f = g$.