Quiz 3

UM 205: Introduction to Algebraic Structures (Winter 2023-24)
Indian Institute of Science

Instructor: Arvind Ayyer

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9

Name: Aditya Gupta Id. No.: 22205

1. Prove a simple compact formula for $\left\{ n \atop 2 \right\}$.

2. The *complement* of a graph G is the graph $\bar{G}$ with the same vertex set and whose edges are pairs of non-adjacent vertices in G. Show that if G is not connected, then $\bar{G}$ is connected.

1) We use the recurrence relation:

$$\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k \cdot \left\{ n-1 \atop k \right\}$$

$$k=2 \Rightarrow \left\{ n \atop 2 \right\} = \left\{ n-1 \atop 1 \right\} + 2 \cdot \left\{ n-1 \atop 2 \right\} = 1 + 2 \cdot \left\{ n-1 \atop 2 \right\}$$

Let $a_n = \left\{ n \atop 2 \right\} \Rightarrow a_n = 1 + 2a_{n-1}$, $n \geq 3$, $a_2 = 1$

Let $a_n = 2^{n-1} - 1$ We show this holds by induction

$a_2 = 2^{2-1} - 1 = 1$ Holds

$a_{n+1} = 1 + 2a_n = 1 + 2(2^{n-1} - 1) = 2^n - 1 = 2^{(n+1)-1} - 1$ Holds

$\therefore a_n = 2^{n-1} - 1$

$\therefore \left\{ n \atop 2 \right\} = 2^{n-1} - 1$ ✓ 5

2) We have that G is not connected

Let $= (V, E)$. Then $\exists\, v' \in V$ st there is no edge from v' to any other vertex Why? Consider:

Now consider $\bar{G}_1 = (V, \bar{E})$. Since it has all non adjacent vertices of G as edges, $\forall\, v \in V \setminus \{v'\}$, $\exists$ an edge (v', v) in $\bar{E}$.

Thus the vertex v' has an edge to every other vertex. Now if we want a path from v_1 to v_2, we have two cases:

i) v_1 or v_2 is v' $\Rightarrow$ edge already in $\bar{E}$

ii) v_1 & v_2 not v' $\Rightarrow$ consider the path made by edges (v_1, v') & (v', v_2). ~~Thus~~ These edges are in $\bar{E}$ and form a path from v_1 to v_2.

Hence there is a path from every vertex v_1 to every other vertex v_2. Thus $\bar{G}$ is connected.

4