NFAs and Closure

Deepak D'Souza

Department of Computer Science and Automation
Indian Institute of Science, Bangalore.

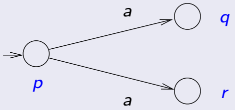
18 January 2024

Outline

Nondeterministic Finite-state Automata (NFAs)

- Allows multiple start states.
- Allows more than one transition from a state on a given letter.

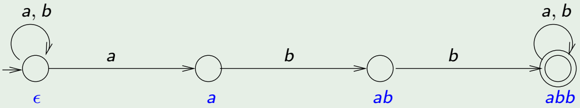
Non-deterministic transitions



- A word is accepted if there is some path on it from a start to a final state.

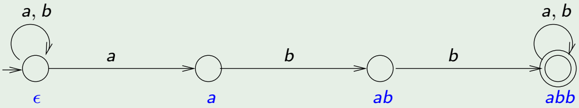
Example NFAs

NFA for "contains *abb* as a subword"

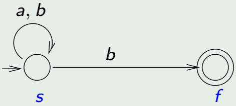


Example NFAs

NFA for "contains *abb* as a subword"



NFA for "last letter is a *b*"



Exercise

Exercise

Give an NFA for the language of strings over $\{a, b\}$ in which the 3rd-last letter is a b.

NFA definition

We can mathematically define an NFA over an alphabet A to be a structure of the form:

$$\mathcal{A} = (Q, S, \Delta, F), \text{ where}$$

- Q is a finite set of states.
- $S \subseteq Q$ is the set of start states,
- $\Delta : (Q \times A) \rightarrow 2^Q$, and
- $F \subseteq Q$ is the set of final states.

Define relation $p \xrightarrow{w} q$ which says there is a path from state p to state q labelled w.

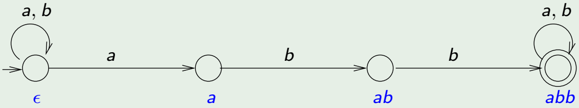
- $p \xrightarrow{\epsilon} p$
- $p \xrightarrow{ua} q$ iff there exists $r \in Q$ such that $p \xrightarrow{u} r$ and $q \in \Delta(r, a)$.

Define $L(\mathcal{A}) = \{w \in A^* \mid \exists s \in S, f \in F : \ s \xrightarrow{w} f\}$.

NFA to DFA conversion: Subset Construction

Example: How do we determinize this NFA?

NFA for "contains *abb* as a subword"



NFA to DFA conversion: Subset Construction

Formal construction:

Let $\mathcal{B} = (Q, S, \Delta, F)$ be an NFA over alphabet A.

Construct a DFA

$$\mathcal{A} = (2^Q, S, \delta, G),$$

where

- for any set of states $X \subseteq Q$, δ is given by

$$\delta(X, a) = \{q \in Q \mid \exists p \in X \text{ with } q \in \Delta(p, a)\}.$$

- $G = \{X \subseteq Q \mid X \cap F \neq \emptyset\}$.

NFA to DFA conversion: Subset Construction

Correctness: To prove that $L(\mathcal{B}) = L(\mathcal{A})$.

Argue that the transition function δ of the subset automaton satisfies the property:

$$\widehat{\delta}(X, w) = \{q \mid \exists p \in X : \; p \xrightarrow{w}{}^{*} q\}.$$

Closure under concatenation and Kleene iteration

- Concatenation of languages:

$$L \cdot M = \{u \cdot v \mid u \in L, \ v \in M\}.$$

- Kleene iteration of a language:

$$L^* = \{\epsilon\} \cup L \cup L^2 \cup L^3 \cup \cdots ,$$

where

$$\begin{aligned} L^n &= L \cdot L \cdots L \ (n \text{ times}). \\ &= \{w_1 \cdots w_n \mid \text{each } w_i \in L\}. \end{aligned}$$