Lecture - 15

$\boxed{\text{Mid} \longrightarrow 17^{th} \text{ Feb } 2:30 \text{ PM}}$

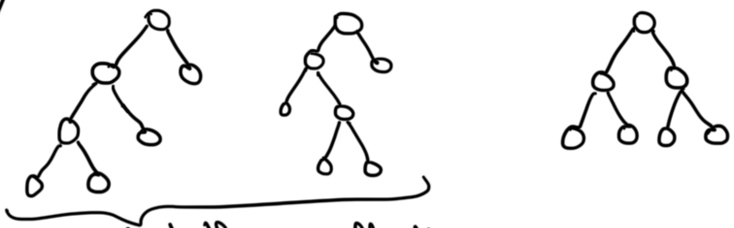
Recall :- Catalan Numbers :- $C_n = \frac{1}{n+1}\binom{2n}{n}$

Counts # of valid words in 2n pairs of Paranthesis

Examples :- 1. Ballot Problem

2 candidates A, B. 2n voters (either to A or to B).
At Each state A has atleast equal votes wrt B.
At the end, $\#_{votes}(A) = \#_{votes}(B)$

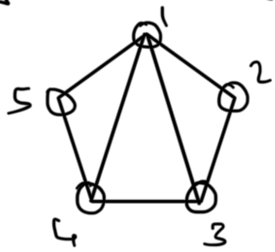
2. Complete Binary Tree with n+1 leaves.



And their reflections

Here every node has either 0 or 2 children

3. Triangulation of (n+2) - gon



$\times 5$ rotations

Enumerative Combinatorics by R. Stanley has more than 200 exercises related to Catalan numbers.

OGF :-

$(a_n) \longrightarrow \sum a_n x^n$ (ogf)

Def'n :- Given a sequence $(a_n)_{n\geq 0}$, the formal power series $A(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n$ is called the exponential generating function (egf).

Example :- $a_n \equiv 1 \Rightarrow A(x) = \sum_{n\geq 0} \frac{1}{n!} x^n = e^x$

Exercise :- Suppose $a_{n+1} = (n+1)(a_n - n+1)$; $n \geq 0$ and $a_0 = 1$
Show that egf is $A(x) = \frac{1}{1-x} + xe^x$
and conclude that $a_n = n! + n$

Property (Product Formula) :- Let $A(x)$ and $B(x)$ be egf's for (a_n) and (b_n) respectively. Then the sequence whose egf is $A(x)\cdot B(x)$ is (c_n), with $(c_n) = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$ [Binomial convolution].

Exercise :- Suppose (a_n) has egf $A(x)$. Then (a_{n+1}) has $\boxed{A'(x)}$

Derivative.

Example :- Recall the bell numbers B_n which counted the # of set partitions of $[n]$.

They satisfy $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \cdot 1$

Let $F(x)$ be the egf of $(B_n)_{n\geq 0}$. From the product formula and the above exercise, we get

$$F'(x) = F(x)e^x \quad \text{with } F(0)=1$$

$$\int \frac{dy}{y} = \int e^x dx$$

$$= \log y = e^x - 1 \quad ; \quad \boxed{F(x) = e^{(e^x - 1)}}$$

which is a valid FPS.

Reference :- generating functionology by H. Wilf

Thm (Exponential Formula) :- Let a_n be the number of ways to build some structure on an n-set with $a_0 = 0$ and h_n be the # of ways to partition $[n]$ and build the same structure on the blocks with $h_0 = 1$. If $H(x)$, $A(x)$ are the egfs of (h_n) and (a_n) resp., $H(x) = e^{A(x)}$

Proof :- The # of ways of partitioning n into k blocks and building the same structure on each block is $\frac{A(x)^k}{k!}$

using the product formula and since the order doesn't matter, Then $H(x) = 1 + \sum_{k\geq 1} \frac{A(x)^k}{k!}$ which is well-defined as an FPS

$$= e^{A(x)} \quad //$$

Example :- Let h_n = # of involutions in S_n ($\pi \in S_n$ st $\pi^2 = id$)

If π is an involution, then π has each cycle of length 1 or 2.

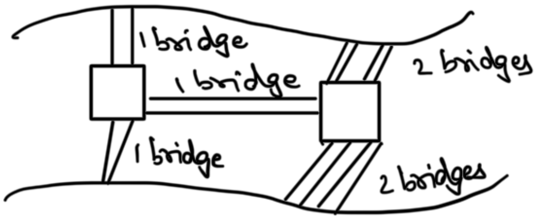
In terms of exp formula,

$$a_n = \begin{cases} 1; & n = 1, 2 \\ 0; & \text{otherwise} \end{cases} \implies \boxed{A(x) = x + \frac{x^2}{2}}$$

$\implies$ From the exp formula, $H(x) = e^{\left(x + \frac{x^2}{2}\right)}$

Graph Theory :-

Konigsberg Bridge Problem :-



Folklore Question :- Can one traverse all the bridges one by one without doubling back or repeating a bridge?

Euler (1736) solved this problem in the -ve (proved it is not possible)

Key Insight :- Only Connectivity matters.