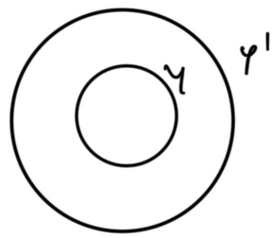
Lecture - 7

Recall Zorn's lemma $(X, \leq)$ a poset such that every $Y \subseteq X$ has an upper bound $\Rightarrow$ X has a maximal element

Zorn's lemma $\Rightarrow$ Axiom of Choice :- Let $X \neq \phi$ be non-empty. Consider the set P of pairs (Y, f), $Y \subseteq X$ and f is a choice function on Y. Introduce a partial order $\leq$ on P by setting $(Y, f) \leq (Y', f')$ if $Y \subseteq Y'$ and $f = f'|_Y$
$(P, \leq)$ is non-empty since $\forall\, x \in X$, we have an obvious choice function on $\{x\}$



Let C be a chain in P. Then let $\bar{Y} = \bigcup\limits_{(Y,f) \in C} Y$ and

define $\bar{f}$ by setting $\bar{f}(S) = f(S)$ for any S st f is defined on S (Note that $\bar{f}$ is well-defined). Then $(\bar{Y}, \bar{f})$ is an upper bound for C.
By Zorn's lemma, $\exists$ a maximal element (Z, g) of P.
If $x \in X \setminus Z$, we can extend f from Z to $Z \cup \{x\}$ by setting $g(S) = x$ for every S containing x. But now, we have a new maximal element. [Contradiction]
$\Rightarrow$ g is a choice function for X.

Done with set theory.

Integers

Def'n :- An integer is an expression of the form $a \backslash b$ where $a, b \in \mathbb{N}$. Two integers are equal, $a \backslash b = c \backslash d$ if $a + d = b + c$. Let $\mathbb{Z}$ denote the set of integers.

Exercise :- Show that the notion of equality is an equivalence relation. (Done in UM204)

Def'n:- The sum (resp. product) of $a\backslash b$ and $c\backslash d$ is defined by

$$(a\backslash b) + (c\backslash d) = (a+c)\backslash(b+d)$$

(resp) $$(a\backslash b) * (c\backslash d) = (ac+bd)\backslash(ad+bc)$$

Property:- Both are well-defined.

Since $n\backslash 0$ behaves like n $\forall\, n \in \mathbb{N}$, we identify $\mathbb{N} \equiv \{n\backslash 0 \mid n \in \mathbb{N}\}$

Def'n:- If $a\backslash b \in \mathbb{Z}$, then its negation is $-(a\backslash b) = (b\backslash a)$
In particular if $n \in \mathbb{N}$, we identify $-n = 0\backslash n$

Laws of Algebra:- Let $x, y, z \in \mathbb{Z}$.

(1) $x+y = y+x$
(2) $x+(y+z) = (x+y)+z$
(3) $x+0 = 0+x = x$
(4) $x+(-x) = (-x)+x = 0$
(5) $xy = yx$
(6) $(xy)z = x(yz)$
(7) $x\cdot 1 = 1\cdot x = x$
(8) $x(y+z) = xy + xz$
(9) $(y+z)x = yx + zx$

These properties make $\mathbb{Z}$ a commutative ring.

Omitting (5) will give a ring.

Subtraction:-
For $x, y \in \mathbb{Z}$, we write $x-y = x+(-y)$

Property:-
(i) $a, b \in \mathbb{Z}$ st $ab = 0 \Rightarrow a = 0$ or $b = 0$
(No Zero Divisors)
(ii) $a, b, c \in \mathbb{Z}$ st $ac = bc$ and $c \neq 0 \Rightarrow a = b$
(Cancellation)

Order properties are exactly as for $\mathbb{N}$.
Lemma:- Trichotomy holds.

Rationals:-
Def'n:- A rational number is of the form $a/\!/b$, where

$a, b \in \mathbb{Z}$ and $b \neq 0$. $a//0$ = never a rational number

Two rational numbers are equal, $a//b = c//d$ if $ad = bc$

Let $\mathbb{Q}$ denote the set of rationals.

Example :- $3//4 = -6//-8 \neq 4//3$

Def'n :- The sum (resp. product) of rational numbers is given by

$$(a//b) + (c//d) = (ad+bc // bd)$$

(resp.) $$(a//b) * (c//d) = (ac // bd)$$

Property :- These are well-defined.

Note that $\left(a//1\right) + \left(b//1\right) = (a+b)//1$ and $\left(a//1\right) * \left(b//1\right) = (ab)//1$

$\Rightarrow$ We can identify $\mathbb{Z}$ with $\{a//1,\ a \in \mathbb{Z}\}$

$\Rightarrow$ $0 \equiv 0//1$

Def'n :- The reciprocal of a non-zero rational $a//b$ is $(a//b)^{-1} = b//a$ (Check that this is well-defined).

Note that numerator / denominator of $a \in \mathbb{Q}$ is not well-defined.

Laws of Algebra for $\mathbb{Q}$:-

Assume as those for $\mathbb{Z}$ (1-9)

10, If $x \neq 0$, $x \cdot x^{-1} = x^{-1} \cdot x = 1$

$\longrightarrow$ 1-10 define a field.

Omitting 5, gives a division ring.

Weddesburn's Little theorem :-

Every finite division ring is a field.

Quotient :- For $x, y \in \mathbb{Q}$, $y \neq 0$, the quotient of x, y is $x/y = x * (y^{-1})$

Check that $\boxed{a/b = a//b}$

Check that [illegible]

<u>Def'n</u> :- We say that $x \in \mathbb{Q}$ is positive (resp. negative) if $x = a/b$ for $a, b \in \mathbb{N}$ (resp $x = -y$ for some positive y).

Usual ordering properties $\Rightarrow$ $\mathbb{Q}$ is an ordered field.

Read Section 4.3 (Abs values and exponents)

<u>Property</u> :- Let $x \in \mathbb{Q}$. Then $\exists$ a unique $n \in \mathbb{Z}$ st $n \leq x \leq n+1$. This is denoted $\lfloor x \rfloor \to$ floor of x.

<u>Property</u> :- If $x, y \in \mathbb{Q}$ st $x < y$, $\exists$ $z \in \mathbb{Q}$ st $x < z < y$.

<u>Thm</u> :- There is no $x \in \mathbb{Q}$ st $x^2 = 2$