UM 204 (WINTER 2024) - WEEK 6

1. Sequences and series

1.1. Convergence and subsequential limits. Contd...

Examples. Let $x = \{x_n = (1/n, (-1)^n)\}_{n\in\mathbb{N}}$ in $\mathbb{R}^2$. Then,

$$y_k = x_{2k} = \left(\frac{1}{2k}, 1\right), z_k = x_{2k+1} = \left(\frac{1}{2k+1}, -1\right),$$

are (convergent) subsequences of x. Thus, their limits, $(0,1)$ and $(0,-1)$ are subsequential limits of x.

Theorem 1.1. *Let $\{x_n\}_{n\in\mathbb{N}} \subset (X,d)$. Then, $\lim_{n\to\infty} x_n = x$ if and only if every subsequence of $\{x_n\}_{n\in\mathbb{N}}$ converges to x.*

Proof. Assume $\lim_{n\to\infty} x_n = x$. Let $\{y_k\}_{k\in\mathbb{N}}$ be a subsequence of $\{x_n\}_{n\in\mathbb{N}}$, i.e.,

$$y_k = x_{n_k}$$

for some choice of $0 \le n_0 < n_1 < n_2 < \cdots$. Let $\varepsilon > 0$. Then, there is an $N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon, \quad \forall n \ge N.$$

But $n_k \ge k$ for all $k \in \mathbb{N}$. Thus, for all $k \ge N$, we have that $n_k \ge N$. Thus

$$d(y_k, x) = d(x_{n_k}, x) < \varepsilon.$$

Conversely, suppose every subsequence of the given sequence converges to x. Since $\{x_n\}_{n\in\mathbb{N}}$ is a subsequence of itself, it converges to x. $\square$

We can now state one more characterization of compactness in metric spaces.

Theorem 1.2. *Let (X,d) be a metric space, and $E \subset X$. T.F.A.E.*

(1) *E is compact.*
(2) *Every infinite subset of E has a limit point in E.*
(3) *Every sequence in E admits a convergent subsequence that converges to a limit in E.*

Remark. We have already shown that (1) $\Rightarrow$ (2) in general, and (2) $\Rightarrow$ (1) in $\mathbb{R}^n$. We will show that, in general, (2) $\iff$ (3). Although, our proof is only complete in the special case of $\mathbb{R}^k$, you are allowed to cite this theorem for a general metric space in this course.

Proof. Suppose E satisfies (2). Let $\{x_n\}_{n\in\mathbb{N}} \subset E$ be a sequence. If the collection $S = \{x_n : n \in \mathbb{N}\}$ is a finite set. Then, there is some $x \in E$ such that $x_n = x$ for infinitely many $n \in \mathbb{N}$. This yields a subsequence that converges to $x \in E$. If S is infinite, let $p \in E$ be a limit point of S. We choose a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ inductively as follows: let $\varepsilon = 1$, then there is some $n_1 \in \mathbb{N}$ such that $d(x_{n_1}, x) < 1$. Now, p is also a limit point of $S \setminus \{x_0, ..., x_{n_1}\}$. Choosing $\varepsilon = 1/2$, we find an $n_2 > n_1$ such that $d(x_{n_2}, x) < 1/2$. Continuing this way, we obtain $n_1 < n_2 < \cdots$ so that

$$d(x_{n_k}, x) < 1/k.$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $1/\varepsilon < N$. Then, for all $k \geq N$, we have that

$$d(x_{n_k}, x) < \varepsilon.$$

Thus, $\lim_{k\to\infty} x_{n_k} = x$.

Suppose E satisfies (3). Let $S \subset E$ be an infinite set that admits no limit points in E. Since S is infinite, we may choose a sequence $\{x_n\}_{n\in\mathbb{N}} \subset E$ such that $x_j \neq x_k$ for an $j \neq k$. Then, by hypothesis, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_{n_k} = x$ for some $x \in E$. Thus, by the sequential characterization of closed sets, $x \in \overline{S}$. You may check that x is a limit point of S in E, which contradicts our assumption. □

Corollary 1.3. *Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^k$ be a bounded sequence. Then, $\{x_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence.*

Proof. Since the sequence is bounded, there is a $p \in \mathbb{R}^k$ and $R > 0$ such that $x_n \in B(p; R)$ for all $n \in \mathbb{N}$. Thus, $\{x_n\}$ is a sequence in the compact (why) set $\overline{B(p;\mathbb{R})}$. The result follows from the above theorem. □

END OF LECTURE 15

1.2. **Cauchy sequences and completeness.** In HW02, we encountered the sequence

$$x_n = \begin{cases} 2, & \text{if } n = 0, \\ x_{n-1} - \dfrac{x_{n-1}^2 - 2}{2x_{n-1}}, & \text{if } n \geq 1. \end{cases}$$

You checked that this sequence is Cauchy, and argued that it does not have a limit within $\mathbb{Q}$. This is an example of a common iteration scheme known as the Newton-Raphson method. Given a 'nice' function f and an initial guess, say x_0, for a root of f, one gets better approximations by considering

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \quad n \geq 1.$$

If f satisfies certain conditions, then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. If, $\{f'(x_n)\}_{n\in\mathbb{N}}$ is bounded,

$$\lim_{n\to\infty} f(x_{n-1}) = \lim_{n\to\infty} f'(x_{n-1})(x_{n-1}-x_n) = 0.$$

If such a sequence converges (to say ℓ), and f is continuous, then we get that $0 = \lim_{n\to\infty} f(x_n) = f(\ell)$.

Definition 1.4. A sequence $\{x_n\}_{n\in\mathbb{N}} \subset (X,d)$ is said to be Cauchy if, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon, \quad \forall n, m \geq N.$$

Example. (Assignment 02) Every bounded above increasing sequence in $\mathbb{Q}$ (or $\mathbb{R}$) is Cauchy. Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}$ such that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, and there is some $M \in \mathbb{Q}$ such that

$$|x_n| < M, \quad \forall n \in \mathbb{N}.$$

Suppose the sequence is not Cauchy. Then, there is an $\varepsilon > 0$ such that for every $k \in \mathbb{N}$, there exist $n > m \geq k$ such that $|x_n - x_m| > \varepsilon$. We build a subsequence iteratively as follows. For $k = 0$, let $m_0 = m$ and $n_0 = n$, as above. Next, let $k = \max\{1, n_0\}$, and choose $m_1 = m$ and $n_1 = n$ as above, and continue so that $n_k > m_k \geq \max\{k, n_0, ..., n_{k-1}\}$. Then, we have that $|x_{n_k} - x_{m_k}| = x_{n_k} - x_{m_k} > \varepsilon$. Thus,

$$x_{n_k} > \varepsilon + x_{m_k} \geq \varepsilon + x_{n_{k-1}} > k\varepsilon + x_{n_0}.$$

Choose $k \in \mathbb{N}$, we can make the RHS greater that M, which is a contradiction.

Theorem 1.5. *Every convergent sequence is Cauchy. Every Cauchy sequence is bounded.*

Definition 1.6. A metric space is said to be complete if every Cauchy sequence is convergent.

Example. $\mathbb{Q}$, $(0,1)$ (both with the standard metric) are not complete.

Theorem 1.7. *Every compact metric space is a complete metric space.*

Proof. Let (X,d) be a complete metric space. Let $\alpha = \{x_n\}_{n\in\mathbb{N}} \subset X$ be a Cauchy sequence. By the sequential characterization of compactness, there is a subsequence $\{x_{n_k}\}$ of α that converges to some $x \in X$. We claim that

$$\lim_{n\to\infty} x_n = x.$$

Let $\varepsilon > 0$. By Cauchyness of α, there is an $N \in \mathbb{N}$ such that

$$d(x_n, x_n) < \varepsilon/2, \quad \forall m, n \geq N.$$

By convergence of $\{x_{n_k}\}$, there is a $K \in \mathbb{N}$ such that

$$d(x_{n_\ell}, x) < \varepsilon/2, \quad \forall \ell \geq K.$$

Now, choose $N_0 = \max\{N, K\}$. Then, for $\ell \geq N_0$, we have that $\ell \geq K$ and $\ell, n_\ell \geq N$. Thus,

$$d(x_\ell, x) \leq d(x_\ell, x_{n_\ell}) + d(x_{n_\ell}, x) < \varepsilon, \quad \forall \ell \geq N_0.$$

□

Theorem 1.8. $(\mathbb{R}^n, |\cdot|)$ *is a complete metric space.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}^n$. Then, it is a bounded sequence, and has a convergent subsequence. Now repeat the above argument. □

END OF LECTURE 16

1.3. **The extended real line and sequences in $\mathbb{R}$.** The extended real number system consists of all real numbers, and two formal symbols, $+\infty$ and $-\infty$. This extended set will be denoted by $\overline{\mathbb{R}}$, and will be endowed with the usual order on $\mathbb{R}$, along with

$$-\infty < x < +\infty, \quad \forall x \in \mathbb{R}.$$

The l.u.b. (g.l.b.) property continues to hold in $(\overline{\mathbb{R}}, <)$, where if a subset $A \subset \mathbb{R}$ is bounded above in $\mathbb{R}$, then it has an l.u.b. in $\mathbb{R}$, and it is not bounded above in $\mathbb{R}$, then

$$\sup A = +\infty.$$

The usual algebraic operations are extend to $\overline{\mathbb{R}}$ as follows (here $x \in \mathbb{R}$):

(1) $x + \infty = \infty$, $x - \infty = -\infty$
(2) If $x > 0$, then $x \cdot (+\infty) = +\infty$ and $x \cdot (-\infty) = -\infty$
(3) If $x < 0$, then $x \cdot (+\infty) = -\infty$ and $x \cdot (-\infty) = +\infty$
(4) $\frac{x}{+\infty} = \frac{x}{-\infty} = 0$

We next consider a concept that gives us a finer understanding of the clustering behavior of sequences beyond 'convergence' and 'divergence'. Before that, we extend the meaning of the symbol $\to$.

Definition 1.9. Given a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, we say that

$$x_n \to +\infty$$

if, for every $M \in \mathbb{R}$, there is some $N \in \mathbb{N}$ such that $x_n \geq M$ for all $n \geq M$. Similarly, we say that

$$x_n \to -\infty$$

if, for every $L \in \mathbb{R}$, there is an $N \in \mathbb{N}$ such that $x_n \leq L$ for all $n \geq N$.

Remark. In the above case, the sequences are NOT considered to be convergent! However, $+\infty$ and $-\infty$ are said to be limits in the extended real line.

Definition 1.10. Given a sequence $\alpha = \{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, let $E \subseteq \overline{\mathbb{R}}$ be the set of subsequential limits of α in the *extended real line*. Then, the limit superior/upper limit and limit inferior/lower limit of α are given by $\sup E$ and $\inf E$, respectively. These are denote by

$$\limsup_{n\to\infty} x_n = \sup E \quad \text{and} \quad \liminf_{n\to\infty} x_n = \inf E.$$

Examples. (1) $x_n = (-1)^n$. You can check that $E = \{-1, 1\}$, so $\limsup x_n = 1$ and $\liminf x_n = -1$.
(2) $x_n = n$. Then $E = \{+\infty\}$. So, $\liminf x_n = \limsup x_n = \infty$.
(3) Let $\{x_n\}$ be an enumeration of all the rational numbers. Then, every (extended) real number occurs as a subsequential limit of this sequence. Thus, $E = \overline{\mathbb{R}}$, $\liminf x_n = -\infty$ and $\limsup x_n = +\infty$.

Theorem 1.11. *(1) Let $\{x_n\}$ and $\{y_n\}$ be sequences such that for some $N \in \mathbb{N}$, $x_n \le y_n$ for all $n \ge N$. Then*

$$\begin{aligned} \limsup x_n &\le \limsup y_n \\ \liminf x_n &\le \liminf y_n. \end{aligned}$$

(2) Let $\{x_n\}_{n\in\mathbb{N}}$ be a real sequence. Then, $x_n \to x \in \overline{\mathbb{R}}$ if and only if $\limsup x_n = \liminf x_n = x$.

Proof. Exercise. □

Theorem 1.12. *Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence and E be it set of subsequential limits (in $\overline{\mathbb{R}}$).*

(1) *E is nonempty and bounded above in $\overline{\mathbb{R}}$.*
(2) $\limsup x_n \in E$.
(3) *If $x > \limsup x_n$, there is an $N \in \mathbb{N}$ such that $x_n < x$ for all $n \ge N$.*
(4) $\limsup x_n$ *is the only (extended) real number satisfying* (2) *and* (3).

Exercise. State and prove an analogous theorem for the existence and properties of the lower limit.

Proof. (1) Note that $+\infty$ is an upper bound in $\overline{\mathbb{R}}$ of any subset of $\mathbb{R}$, so we need to show that E is nonempty.
(a) If $\{x_n\}$ is bounded in $\mathbb{R}$ (say $|x_n| \le M$), then it has a convergent subsequence. In this case, E is nonempty.
(b) If $\{x_n\}$ is not bounded above, we construct a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to +\infty$.
Let $M(1) = 1$. Since $\{x_n\}$ is not bounded above, there is some $n_1 \in \mathbb{N}$ such that

$$x_{n_1} > M(1) = 1.$$

Next, let $M(2) = \max\{2, x_0, ..., x_{n_1}\}$. Since $\{x_n\}$ is not bounded above, there is some $n_2 \in \mathbb{N}$ such that

$$x_{n_2} > M(2) \geq 2.$$

Since $M(2) \geq x_0, x_1, ..., x_{n_1}$, it must be that $n_2 > n_1$.

Suppose $n_1, n_2, ..., n_k$ have been chosen so that $n_1 < n_2 < \cdots < n_k$ and

$$x_{n_k} > k.$$

Let $M(k+1) = \max\{k+1, x_0, ..., x_{n_k}\}$. Then, there must be some $n_{k+1} > n_k$ such that

$$x_{n_{k+1}} > M(k+1) \geq k+1.$$

This yields a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to \infty$.
(c) $\{x_n\}$ is not bounded below. Consider $\{-x_n\}$ and repeat the above argument.

(2) (a) Suppose $\sup E \in \mathbb{R}$. For any $k \in \mathbb{N}$, there is some $e_k \in E$ such that

$$\sup E - \frac{1}{k} < e_k \leq \sup E < \sup E + \frac{1}{k}.$$

Since e_k is a subsequential limit of $\{x_n\}$, there is an increasing sequence

$$n_{k1} < n_{k2} < n_{k3} < \cdots$$

such that

$$|x_{n_{k\ell}} - e_k| < \frac{1}{k}, \quad \forall \ell \geq 1.$$

Let $n_1 = n_{11}$. Since $\lim_{\ell\to\infty} n_{2\ell} = \infty$, there is some $\ell \in \mathbb{N}_+$ such that $n_{2\ell} > n_1$. Let $n_2 = n_{2\ell}$. Continuing this way, we obtain

$$n_1 < n_2 < n_3 < \cdots$$

such that

$$|x_{n_k} - e_k| < \frac{1}{k}.$$

But then,

$$|x_{n_k} - \sup E| < \frac{1}{2k} \to 0.$$

Thus, $\sup E \in E$.
(b) Suppose $\sup E = +\infty$. We claim that $\{x_n\}_{n\in\mathbb{N}}$ is not bounded above, and therefore, by (1)(b), $+\infty$ is a subsequential limit. Suppose not, i.e., $\exists M \in \mathbb{R}$ such that

$$x_n \leq M, \quad \forall n \in \mathbb{N}.$$

Then, any subsequence is bounded above by M, and therefore, any subsequential limit is bounded above by M. Thus, E is bounded above by M, and $\sup E \leq M < \infty$.

(c) Suppose $\sup E = +\infty$. Apply the above argument to $\{-x_n\}$ to show that $\{x_n\}$ is not bounded below, and therefore, $-\infty$ is a subsequential limit.

REST OF THE PROOF TO BE COMPLETED NEXT TIME

$\square$

END OF LECTURE 16