UM 204 HOMEWORK ASSIGNMENT 7

Posted on March 14, 2024
(NOT FOR SUBMISSION)

- These problems are for self-study. Try these **on your own** before seeking hints.
- Some of these problems will be (partially) discussed at the next tutorial.
- A 15-min. quiz based on this assignment will be conducted at the end of the tutorial section.

Problem 1. Consider the function $f : \mathbb{R} \to [0, 1]$ given by

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q}, \text{ with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N}_{>0}, \ \gcd(p, q) = 1, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is discontinuous at every rational number, and continuous elsewhere. (Here, we have used the fact that every rational number $x \in \mathbb{Q}$ admits a unique representation of the form p/q, with p and q as described above.)

Problem 2. Let (X, d) be a metric space and $A \subset X$ be a nonempty subset. Define

$$f_A(x) = \inf\{d(x, y) : y \in A\}, \quad x \in X.$$

(a) Show that f_A is uniformly continuous on X.

(b) Prove that $x \in \overline{A}$ if and only if $f_A(x) = 0$.

Problem 3. Show that uniformly continuous functions map Cauchy sequences to Cauchy sequences. Is the converse true?

Problem 4. An F_σ set is a countable union of closed sets. Complete the following steps to prove that the discontinuity set D_f of any function $f : \mathbb{R} \to \mathbb{R}$ is an F_σ set.

(a) Given $f : \mathbb{R} \to \mathbb{R}$ and $\alpha > 0$, f is said to be *α-continuous* at $x \in \mathbb{R}$ if there exists a $\delta > 0$ such that for all $y, z \in B(x; \delta)$, $|f(y) - f(z)| < \alpha$. Show that the set $D^\alpha = \{x \in \mathbb{R} : f \text{ is not } \alpha\text{-continuous at } x\}$ is closed, for each $\alpha > 0$.

(b) Show that $D^\alpha \subset D_f$ for any $\alpha > 0$.

(c) Show that

$$D_f = \bigcup_{n=1}^{\infty} D^{\frac{1}{n}}.$$