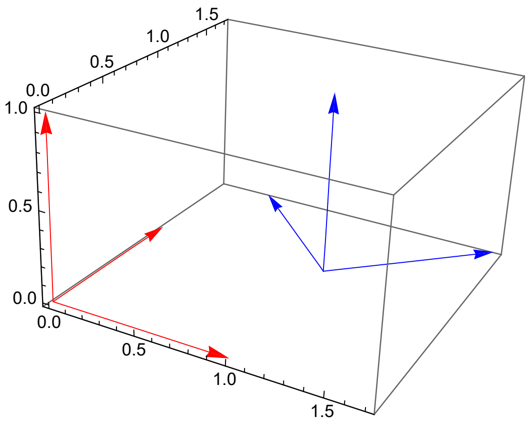
Geometric Transformations



The above figure shows a coordinate system in 'RED' (OX,OY,OZ) indicating Global coordinate system and in 'BLUE' ($O'X'$, $O'Y'$, $O'Z'$) indicating New cordinate system (Local Coordinate system)

Translation

There are two notions of constructing the transformation matrix (setting row or cloumn elements accordingly). This page consists of setting transformation matrix with column elements.

A point in Local coordinate system (x_l, y_l, z_l) need to be transformed to Global coordinate system , let us say it as (x_g, y_g, z_g)

Since we know the Location and orientation of Local coordinate system with respect to Global coordinate system (i.e Origin position and direction cosines of $O'X'$, $O'Y'$, $O'Z'$)

(x_l, y_l, z_l) is wrt O' (New cordinate system)
(x_g, y_g, z_g) is wrt to O (Global Coordinate system)

O' wrt to O is (x_o, y_o, z_0)

Transformation Matrix:

For a 3D spatial coordinates we would construct a 4 x1 Matrix where last element of the column would be 1. If you are familiar with this notation we may recollect that it would be easy when dealing with Scaling feature.

So, (x_o, y_o, z_o) is constructed as $\begin{pmatrix} x_l \\ y_l \\ z_l \\ 1 \end{pmatrix}$

$$\mathbf{T} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & -\boldsymbol{x_o} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & -\boldsymbol{y_o} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & -\boldsymbol{z_o} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

$X = \mathrm{TX}'$

$$\begin{pmatrix} x_g \\ y_g \\ z_g \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -x_o \\ 0 & 1 & 0 & -y_o \\ 0 & 0 & 1 & -z_o \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_l \\ y_l \\ z_l \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_g \\ y_g \\ z_g \\ 1 \end{pmatrix} = \begin{pmatrix} x_l - x_o \\ y_l - y_o \\ z_l - z_o \\ 1 \end{pmatrix}$$

By comparing both sides, above condition is what we learnt in high school

Rotation

By cosidering the Coordinate systems used in Transformation matrix to formulate Rotation matrix. Again, to mention there are many ways to formulate the matrix with right usage of sign convention.

Rotation Matrix:

About X:

$$\mathbb{R}_X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{Cos}[\alpha] & -\mathrm{Sin}[\alpha] & 0 \\ 0 & \mathrm{Sin}[\alpha] & \mathrm{Cos}[\alpha] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

About Y:

$$\mathbb{R}_Y = \begin{pmatrix} \mathrm{Cos}[\beta] & 0 & \mathrm{Sin}[\beta] & 0 \\ 0 & 1 & 0 & 0 \\ -\mathrm{Sin}[\beta] & 0 & \mathrm{Cos}[\beta] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

About Z:

$$\mathbb{R}_Z = \begin{pmatrix} \mathrm{Cos}[\gamma] & -\mathrm{SIn}[\gamma] & 0 & 0 \\ \mathrm{Sin}[\gamma] & \mathrm{Cos}[\gamma] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transforming Coordinates from Local to Global coordinate system

To convert a point in Local coordinate system into Global coordinate system a mere Translation is not sufficient.
As Local coordinate system is aligned at an angle to Global system, Local system need to be rotated correspondingly.

So, we have (OX, OY, OZ) and ($O'X'$, $O'Y'$, $O'Z'$).

We follow as below:

Step-1:
We try to coinside O'Z' with OZ after translation.
So, we try to slam O'Z' into XZ plane by rotating it about X-axis
We rotate $+\alpha$ angle for this purpose

Step-2:
Next we try to rotate it about Y-axis and align O'Z' with OZ.
We rotate $-\beta$ angle for this purpose

Step-3:
Now O'Z' is aligned with OZ , but we are not done. We still need to align O'X' and O'Y'.
This is acheived by rotating about Z-axis
We rotate $+\delta$ angle for this purpose

```
anglecalc[o_, dc_] :=
Module[{p, v1, v2, v3, tempv2, u, α, β, δ, tr, rα, rβ, temp},
p = o; (*New Coordinate system Origin*)
  v1 = dc[[1]]; (*New Coordinate system X-axis dc*)
  tempv2 = dc[[2]];
  v2 = Normalize[((v1) × (tempv2)) × (v1)];
  (*New Coordinate system Y-axis dc*)
  v3 = Normalize[v1 × v2]; (*New Coordinate system Z-axis dc*)
  (*end of Data of New Coordinate Sysytem*)

  (*direction of vector and calculation of α and β*)
  u = Normalize[v3];
If[u[[2]] == 0 && u[[3]] == 0, α = 0,
   α = ArcCos[u[[3]] / (Sqrt[u[[2]]^2 + u[[3]]^2])]];
β = ArcSin[u[[1]]];
  (*end of calculation of α and β*)

tr = {{1, 0, 0, -p[[1]]}, {0, 1, 0, -p[[2]]}, {0, 0, 1, -p[[3]]}, {0, 0, 0, 1}};
rα = {{1, 0, 0, 0}, {0, Cos[α], -Sin[α], 0}, {0, Sin[α], Cos[α], 0}, {0, 0, 0, 1}};
rβ = {{Cos[-β], 0, Sin[-β], 0},
    {0, 1, 0, 0}, {-Sin[-β], 0, Cos[-β], 0}, {0, 0, 0, 1}};

temp = rβ.rα.tr.{{v1[[1]]}, {v1[[2]]}, {v1[[3]]}, {1}};
δ = If[temp[[1]][[1]] == 0 && temp[[2]][[1]] == 0,
    0, ArcTan[temp[[1]], temp[[2]]][[1]]];

{α, β, δ}
]
```

The above ***anglecalc*** Module takes Origin coordinates and direction cosines of O’X’ , O’Y’ in List format and returns {α, β, δ}
Where α,β,δ are Angles that need to be rotated about X,Y,Z axis respectively to align Local Coordinate system with Global Coordinate system

Example

```
neworigin = {0, 0, 1};
newDCs = {{1/√2, 1/√2, 0}, {-1/√2, 1/√2, 0}};
anglecalc[neworigin, newDCs]
```

$\{0, 0, \frac{\pi}{4}\}$

We can observe two things from above exaple, angles α,β,δ are independent of neworigin & The new DC’s are nothing but O’X’ and O’Y’ aligned at an angle 45^0 to Global coordinate system.