1 Functions

Definition 1.1. A **function** f from a set A to a set B, denoted as $f : A \to B$, associates to each $a \in A$ an element $f(a) \in B$. The set A is called the **domain** of f and the set B is called the **codomain** of f. We let $\mathrm{Dom}(f)$ denote the domain of f and $\mathrm{Cod}(f)$ denote the codomain of f.

Definition 1.2. The **range** of a real-valued function f, denoted $\mathrm{Rng}(f)$, is the set of values that f outputs; that is,

$$\mathrm{Rng}(f) = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in \mathrm{Dom}(f)\} = \{f(x) \mid x \in \mathrm{Dom}(f)\}$$

Definition 1.3. Let f be a real-valued function. A real-valued function g is called an **inverse** of f if $f(g(x)) = x$ for all $x \in \mathrm{Dom}(g)$ and $g(f(x)) = x$ for all $x \in \mathrm{Dom}(f)$. We denote this by $g = f^{-1}$.

Proposition 1.3.1. If g is an inverse of f, or in other words $g = f^{-1}$, then $\mathrm{Dom}(f) = \mathrm{Rng}(g)$ and $\mathrm{Dom}(g) = \mathrm{Rng}(f)$.

$g(f(x)) = x$ for all $x \in \mathrm{Dom}(f) \implies x \in \mathrm{Rng}(g)$, so $\mathrm{Dom}(f) \subseteq \mathrm{Rng}(g)$.
If $y \in \mathrm{Rng}(g)$, then $y = g(x)$ for some $x \in \mathrm{Dom}(g)$, thus $f(y) = f(g(x)) = x$
Since $y \in \mathrm{Dom}(f)$, $\mathrm{Rng}(g) \subseteq \mathrm{Dom}(f)$
By combining those two inclusions, we get $\mathrm{Dom}(f) = \mathrm{Rng}(g)$.
By symmetry of the definition of inverse functions, $\mathrm{Dom}(g) = \mathrm{Rng}(f)$

Proposition 1.3.2. Show that the inverse of a function is unique.

Let g and h be inverses of f.
By the above propostion, $\mathrm{Dom}(g) = \mathrm{Rng}(f) = \mathrm{Dom}(h)$
Since $x \in \mathrm{Rng}(f)$, there is some $y \in Dom(f)$ such that $f(y) = x$
But then $g(x) = g(f(y)) = y = h(f(y)) = h(x)$
Therefore $g(x) = h(x)$

Definition 1.4. Let f be a function and $A \subseteq \mathrm{Dom}(f)$. The **image** of A under f, denoted $f(A)$, is

$$f(A) = \{y \in \mathrm{Cod}(f) \mid y = f(x) \text{ for some } x \in A\}$$

Let $B \in \mathrm{Cod}(f)$. The **preimage** of B under f, denoetd $f^{-1}(B)$, is

$$f^{-1}(B) = \{x \in \mathrm{Dom}(f) \mid f(x) \in B\}$$

Definition 1.5. The binomial coefficient is denoted as $\binom{n}{k}$