# Chapter 9 Notes: Inner Product Spaces

Sai Nallani

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All vector spaces in this section are finite dimensional over $\mathbb{C}$. If $V$ is finite dimensional over $\mathbb{R}$ or $\mathbb{Q}$, the construction of inner product still works. But it doesn't work for finite dimensional vector spaces over finite fields.

**Definition 0.1.** Let $V$ be a vector space. AN inner product on $V$ is a bilinear map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ satisfying the following axioms:

1. $\langle \mathbf{u}, \mathbf{u} \rangle > 0$ for all nonzero $\mathbf{u} \in V$. (positive definiteness)
2. $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$ for all $\mathbf{u}, \mathbf{v} \in V$. (conjugate symmetry)
3. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ (linearity)
4. $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$. (homogeneity in the first argument)

Define an inner product on $V \cong \mathbb{C}^n$, with vectors $\mathbf{u} = (u_1, \ldots, u_n)$, as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} u_i \overline{v_i}$$

*Proof.*

1. Assume $\mathbf{u} \neq \mathbf{0}$. Then WLOG, $u_1 \neq 0$. $\langle \mathbf{u}, \mathbf{u} \rangle = \sum_{i=1}^{n} (u_i \overline{u_i}) \geq u_1 \overline{u_1} > 0$.
2. Conjugate symmetry
3. Linearity and homogeneity

$\square$

**Example 0.2**

Continuous functions $f : [-\pi, \pi] \to \mathbb{C}$ form an inner product space with inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx$$

## §1 Orthonormal bases

Out of all spanning sets for an inner product space $V$, a basis is the nicest in many ways. Out of all bases, orthonormal bases are the nicest.

**Definition 1.1.** Let $B = \{e_1, \dots, e_n\}$ be a basis for a finite dimensional inner product space $V$. $B$ is called an orthonormal basis if:

$$\langle e_i, e_j \rangle = \delta_{ij}$$

for all $1 \le i, j \le n$.

If $\mathbf{v} = a_1 e_1 + \cdots + a_n e_n$,

$$\langle \mathbf{v}, e_j \rangle = \sum_{i=1}^{n} a_i \langle e_i, e_j \rangle = a_j$$

Fourier's formula: $\mathbf{v} = \langle \mathbf{v}, e_1 \rangle e_1 + \cdots + \langle \mathbf{v}, e_n \rangle e_n$.

From this we have Parseval's identity: $\langle \mathbf{v}, \mathbf{v} \rangle = \sum_{i=1}^{n} |\langle \mathbf{v}, e_i \rangle|^2$ Prove by substituting by Fourier's formula and stuff.

## §2 Exercises

**Problem 2.1.** Prove the following properties for complex numbers $z = a + bi$

1. $\overline{\overline{z}} = \overline{\overline{a + bi}} = \overline{a - bi} = a - (-bi) = a + bi = z$
2. $\overline{z + w} = \overline{a_z + b_z i + a_w + b_w i} = \overline{a_z + a_w + (b_z + b_w)i} = a_z + a_w - (b_z + b_w)i = a_z - b_z i + a_w - b_w i = \overline{z} + \overline{w}$
3. $\overline{z \cdot w} = \overline{(a_z + b_z i)(a_w + b_w i)} = \overline{(a_z a_w - b_z b_w) + (a_z b_w + b_z a_w)i} = (a_z a_w - b_z b_w) - (a_z b_w + b_z a_w)i = (a_z - b_z i)(a_w - b_w i) = \overline{z}\overline{w}$
4. $|z| = |a + bi| = \sqrt{a^2 + b^2} = \sqrt{a^2 + (-b)^2} = |a - bi| = |\overline{z}|$
5. $\overline{z^n} = \overline{r^n e^{ni\theta}} = r^n e^{ni(-\theta)} = (re^{i(-\theta)})^n = (\overline{z})^n$
6. If $f(z) = f(a + bi) = c + di$, and we want $f(\overline{z}) = \overline{f(z)}$, we can consider functions $f$ such that $f(a - bi) = c - di$ if $f(a + bi) = c + di$. Instead of $f : \mathbb{C} \to \mathbb{C}$, let's consider $f : \mathbb{R}^2 \to \mathbb{R}^2$. Now, we want $\overline{f(x, y)} = \overline{(a, b)} = (a, -b) = f(x, -y)$. Define $f$ as $\overline{f(x, y)} = \overline{(h(x), g(y))} = (h(x), -g(y)) = f(x, -y) = (h(x), g(-y))$. This means that odd functions $g$ satisfy the functional equation. Therefore, just define accordingly for complex numbers.

**Problem 2.2.** Let $\mathbb{F}$ be a finite field. Prove that there is no inner product on $\mathbb{F}^n$ over $\mathbb{F}$.

For $n = 1$, $\langle 0, 1 \rangle = \overline{\langle 1, 0 \rangle} = 0 = \langle 1, 0 \rangle$

**Problem 2.3.** Let $\mathbb{F}$ be an infinite field. Find three bases of $\mathbb{F}^3$ that are not orthonormal.

**Problem 2.4.** Let $V$ be an $n$-dimensional inner product space and assume that $S = \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a set of orthonormal vectors.

$S$ is a linearly independent set. Assume $\mathbf{v} = a_1 w_1 + \cdots + a_k w_k = \mathbf{0}$. Take the inner-product on both sides to get:

$$\langle \mathbf{v}, a_1 w_1 \rangle + \langle \mathbf{v}, a_2 w_2 \rangle + \cdots + \langle \mathbf{v}, a_k w_k \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$$

$$\overline{a_1} \langle \mathbf{v}, w_1 \rangle + \cdots + \overline{a_k} \langle \mathbf{v}, w_k \rangle = \overline{a_1} a_1 + \cdots + \overline{a_k} a_k = 0$$

Since for non-zero complex numbers $z$, $z\overline{z} > 0$, all of $\overline{a_i}a_i > 0$ and $\sum_i \overline{a_i}a_i > 0$ if $a_i \neq 0$. Since we have a contradiction, $a_i = 0$. Therefore $S$ is a linearly independent set of vectors.

If $k = n$, we still have $S$ a linearly independent set of vectors. Let $B$ be a basis for $V$. Since $B$ is a basis, it is a maximally linearly independent and minimally spanning set with size $n$. Since all maximally linearly independent sets have the same size, $S$ is also maximally linearly independent, which implies that it is a basis.

**Problem 2.5.** The norm on $V$ is defined by $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

Because $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ for non-zero $\mathbf{v}$, we have $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} > 0$. Since $\langle \mathbf{0}, \mathbf{0} \rangle = 0$, $\|\mathbf{0}\| = 0$. $\|(1/\|\mathbf{v}\|)\mathbf{v}\|^2 = \langle (1/\|\mathbf{v}\|)\mathbf{v}, (1/\|\mathbf{v}\|)\mathbf{v} \rangle = (1/\|\mathbf{v}\|^2)\langle \mathbf{v}, \mathbf{v} \rangle = (1/\|\mathbf{v}\|^2)\|\mathbf{v}\|^2 = 1$. Finally, due to the linearity of the inner product we have $\|\alpha\mathbf{v}\| = \sqrt{\langle \alpha\mathbf{v}, \alpha\mathbf{v} \rangle} = \sqrt{\alpha^2 \langle \mathbf{v}, \mathbf{v} \rangle} = |\alpha|\|\mathbf{v}\|$.

**Problem 2.6.** Apply the Gram-Schmidt process to turn $\{(1, 1, 0), (0, 1, 1)\}$ into an orthogonal basis for $\mathbb{R}^3$.

We calculate $e_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$ and $e_2 = (-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3})$. Now, we must find another vector $e_3 = (a, b, c)$ such that $\langle e_1, e_3 \rangle = \langle e_2, e_3 \rangle = 0$. Setting up a systems of equations and solving, we get $e_3 = (-b, b, -b)$. Since we want $\langle e_3, e_3 \rangle = 1$, we set $3b^2 = 1$. We have $b = \pm\frac{\sqrt{3}}{3}$.

**Problem 2.7.** Let $C = AB$. Consider the rows of a $n \times m$ matrix $A$ to be $A = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ with each $\mathbf{v}_i \in \mathbb{F}^m$ and the columns of a $m \times k$ matrix $B$ to be $B = (\mathbf{u}_1, \ldots, \mathbf{u}_k)$ with each $\mathbf{u}_i \in \mathbb{F}^m$. Show that $C(i, j) = \langle \mathbf{v}_i, \mathbf{u}_j \rangle$ if $\mathbb{F} = \mathbb{R}$ and $C(i, j) = \langle \mathbf{v}_i, \overline{\mathbf{u}_j} \rangle$ if $\mathbb{F} = \mathbb{C}$.

By the definition of matrix multiplication,

$$c_{ij} = \sum_{\ell=1}^{m} a_{i\ell} b_{\ell j}$$

We can see that $\mathbf{v}_i = (a_{i1}, \ldots, a_{im})$ and $\mathbf{u}_j = (b_{1j}, \ldots, b_{mj})$.
If $\mathbb{F} = \mathbb{R}$, we have $\langle \mathbf{v}_i, \mathbf{u}_j \rangle = a_{i1}b_{1j} + \cdots + a_{im}b_{mj} = c_{ij}$
If $\mathbb{F} = \mathbb{C}$, we have $\langle \mathbf{v}_i, \overline{\mathbf{u}_j} \rangle = \sum_{k=1}^{m} a_{ik}b_{kj} = c_{ij}$

**Problem 2.8.** Check that $e_1, e_2, e_1 + e_2$ forms a frame for $\mathbb{R}^2$.

We need to find bounds $A$ and $B$ such that

$$A(a^2 + b^2) \leq a^2 + b^2 + (a + b)^2 \leq B(a^2 + b^2)$$

$$A \leq 2 + \frac{2ab}{a^2 + b^2} \leq B$$

The minimum value of $\frac{2ab}{a^2+b^2}$ is $-1$ and the maximum value is 1. This means that $A = 1$ and $B = 3$ satisfy the bounds, showing that this is a frame.