Random Proofs

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1 Principle of Inclusion and Exclusion

1.1 2 Sets

We have two sets, A and B. Let's show that $|A \cup B| = |A| + |B| - |A \cap B|$. If we add $|A|$ and $|B|$, we overcount the overlap $|A \cap B|$, therefore we subtract it.

1.2 3 Sets

We have three sets, A, B, and C. Let's find $|A \cup B \cup C|$. $|A \cup B \cup C| \neq |A| + |B| + |C|$ as we overcounted overlaps of $A \cap B, B \cap C, C \cap A$. Let's subtract those values. $|A \cup B \cup C| \neq |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A|$ Oh no! We undercounted $A \cap B \cap C$. Let's just add it!

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

1.3 N sets

We have N sets, S_1, S_2, ..., S_{N-1}, and S_N. We wish to find $|\bigcup\limits_{i=1}^{N} S_i|$. Let's first add all sizes: $\sum\limits_{i=1}^{N} |S_i|$. Let's subtract each intersection of two sets. $\sum\limits_{i=1}^{N} |S_i| - \sum\limits_{1 \leq i < j \leq n}^{N} |S_i \cap S_j| + \sum\limits_{1 \leq i < j < k \leq n}^{N} |S_i \cap S_j \cap S_k| + ... + (-1)^{n-1} |S_1 \cap S_2 \cap S_3 \cap ... \cap S_N|$.

2 Derangements

2.1 Introduction

Let's say we have the set $A = \{1, 2, 3, 4, 5, 6\}$. How many ways can we permute the numbers such that every number in A isn't in its original position. An example of this kind of permutation is $\{2, 1, 4, 3, 6, 5\}$. How many such arrangements, aka **derangements**, can we make?

2.2 Answer

We know there are 6! total permutations. The total number of derangements is less than the total permutations. Let's just exclude some types of arrangements. Let N be the number of permutations where there is atleast 1 number in the original position. The total number of derangements would be $6!-N$. Let A_r be the set of permutations such that the rth object is in the right position. Then, $N = |\bigcup_{r=1}^{N} A_r|$. For generality, let $n = 6$ in this scenario, but keep in mind it can be any natural number. By PIE,

1. $\sum_{i=1}^{n} |A_i| = \binom{n}{1}(n-1)!$
2. $\sum_{i<j} |A_i \cap A_j| = \binom{n}{2}(n-2)!$
3. $N = |\bigcup_{r=1}^{n} A_r| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \sum_{i<j<k<l} |A_i \cap A_j \cap A_k \cap A_l| + ... + (-1)^{n+1}|A_1 \cap A_2 \cap ... \cap A_n| = \sum_{i=1}^{n} \binom{n}{i}(-1)^{n+1}(n-i)! = \sum_{i=1}^{n} (-1)^{n+1}\frac{n!}{i!}$
4. $6! - N = n! - \sum_{i=1}^{n} (-1)^{n+1}\frac{n!}{i!} = n! - n!\sum_{i=1}^{n} (-1)^{n+1}\frac{1}{i!} = n!(1 - \sum_{i=1}^{n} (-1)^{n+1}\frac{1}{i!}) = n!\sum_{i=0}^{n} (-1)^{n+1}\frac{1}{i!}$
5. The number of derangements is $\boxed{n!\sum_{i=0}^{n} \frac{(-1)^n}{i!}}$ or $\boxed{n![1 - 1 + \frac{1}{2} - \frac{1}{6} + ... + \frac{(-1)^n}{n!}]}$