

Random Proofs

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1 Principle of Inclusion and Exclusion

1.1 2 Sets

We have two sets, A and B . Let's show that $|A \cup B| = |A| + |B| - |A \cap B|$. If we add $|A|$ and $|B|$, we overcount the overlap $|A \cap B|$, therefore we subtract it.

1.2 3 Sets

We have three sets, A , B , and C . Let's find $|A \cup B \cup C|$. $|A \cup B \cup C| \neq |A| + |B| + |C|$ as we overcounted overlaps of $A \cap B$, $B \cap C$, $C \cap A$. Let's subtract those values. $|A \cup B \cup C| \neq |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A|$ Oh no! We undercounted $A \cap B \cap C$. Let's just add it!

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

1.3 N sets

We have N sets, S_1, S_2, \dots, S_{N-1} , and S_N . We wish to find $|\bigcup_{i=1}^N S_i|$. Let's first

add all sizes: $\sum_{i=1}^N |S_i|$. Let's subtract each intersection of two sets. $\sum_{i=1}^N |S_i| - \sum_{1 \leq i < j \leq N} |S_i \cap S_j| + \sum_{1 \leq i < j < k \leq N} |S_i \cap S_j \cap S_k| + \dots + (-1)^{n-1} |S_1 \cap S_2 \cap S_3 \cap \dots \cap S_N|$.

2 Derangements

2.1 Introduction

Let's say we have the set $A = \{1, 2, 3, 4, 5, 6\}$. How many ways can we permute the numbers such that every number in A isn't in its original position. An example of this kind of permutation is $\{2, 1, 4, 3, 6, 5\}$. How many such arrangements, aka **derangements**, can we make?

2.2 Answer

We know there are $6!$ total permutations. The total number of derangements is less than the total permutations. Let's just exclude some types of arrangements. Let N be the number of permutations where there is atleast 1 number in the original position. The total number of derangements would be $6! - N$. Let A_r be the set of permutations such that the r th object is in the right position. Then, $N = |\bigcup_{r=1}^N A_r|$. For generality, let $n = 6$ in this scenario, but keep in mind it can be any natural number. By PIE,

$$\begin{aligned}
 & 1. \sum_{i=1}^n |A_i| = \binom{n}{1}(n-1)! \\
 & 2. \sum_{i < j}^n |A_i \cap A_j| = \binom{n}{2}(n-2)! \\
 & 3. N = |\bigcup_{r=1}^n A_r| = \sum_{i=1}^n |A_i| - \sum_{i < j} |A_i \cap A_j| + \sum_{i < j < k} |A_i \cap A_j \cap A_k| - \\
 & \quad \sum_{i < j < k < l} |A_i \cap A_j \cap A_k \cap A_l| + \dots + (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n| = \\
 & \quad \sum_{i=1}^n \binom{n}{i} (-1)^{n+1} (n-i)! = \sum_{i=1}^n (-1)^{n+1} \frac{n!}{i!} \\
 & 4. 6! - N = n! - \sum_{i=1}^n (-1)^{n+1} \frac{n!}{i!} = n! - n! \sum_{i=1}^n (-1)^{n+1} \frac{1}{i!} = \\
 & \quad n! \left(1 - \sum_{i=1}^n (-1)^{n+1} \frac{1}{i!} \right) = n! \sum_{i=0}^n (-1)^{n+1} \frac{1}{i!} \\
 & 5. \text{ The number of derangements is } \boxed{n! \sum_{i=0}^n \frac{(-1)^i}{i!}} \text{ or} \\
 & \quad \boxed{n! \left[1 - 1 + \frac{1}{2} - \frac{1}{6} + \dots + \frac{(-1)^n}{n!} \right]}
 \end{aligned}$$