Random Proofs

sxbuto

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1 Principle of Inclusion and Exclusion

1.1 2 Sets

We have two sets, A and B. Let's show that $|A \cup B| = |A| + |B| - |A \cap B|$. If we add $|A|$ and $|B|$, we overcount the overlap $|A \cap B|$, therefore we subtract it.

1.2 3 Sets

We have three sets, A, B, and C. Let's find $|A \cup B \cup C|$. $|A \cup B \cup C| \neq |A| + |B| + |C|$ as we overcounted overlaps of $A \cap B, B \cap C, C \cap A$. Let's subtract those values. $|A \cup B \cup C| \neq |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A|$ Oh no! We undercounted $A \cap B \cap C$. Let's just add it!

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

1.3 N sets

We have N sets, S_1, S_2, ..., S_{N-1}, and S_N. We wish to find $|\bigcup\limits_{i=1}^{N} S_i|$. Let's first add all sizes: $\sum\limits_{i=1}^{N} |S_i|$. Let's subtract each intersection of two sets. $\sum\limits_{i=1}^{N} |S_i| - \sum\limits_{1 \leq i < j \leq n}^{N} |S_i \cap S_j| + \sum\limits_{1 \leq i < j < k \leq n}^{N} |S_i \cap S_j \cap S_k| + ... + (-1)^{n-1} |S_1 \cap S_2 \cap S_3 \cap ... \cap S_N|$.

2 Derangements

2.1 Introduction

Let's say we have the set $A = \{1, 2, 3, 4, 5, 6\}$. How many ways can we permute the numbers such that every number in A isn't in its original position. An example of this kind of permutation is $\{2, 1, 4, 3, 6, 5\}$. How many such arrangements, aka **derangements**, can we make?

2.2 Answer

We know there are 6! total permutations. The total number of derangements is less than the total permutations. Let's just exclude some types of arrangements. Let N be the number of permutations where there is atleast 1 number in the original position. The total number of derangements would be $6! - N$. Let A_r be the set of permutations such that the rth object is in the right position. Then, $N = |\bigcup_{r=1}^{N} A_r|$. For generality, let $n = 6$ in this scenario, but keep in mind it can be any natural number. By PIE,

1. $\sum_{i=1}^{n} |A_i| = \binom{n}{1}(n-1)!$
2. $\sum_{i<j}^{n} |A_i \cap A_j| = \binom{n}{2}(n-2)!$
3. $N = |\bigcup_{r=1}^{n} A_r| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \sum_{i<j<k<l} |A_i \cap A_j \cap A_k \cap A_l| + ... + (-1)^{n+1}|A_1 \cap A_2 \cap ... \cap A_n| = \sum_{i=1}^{n} \binom{n}{i}(-1)^{n+1}(n-i)! = \sum_{i=1}^{n} (-1)^{n+1}\frac{n!}{i!}$
4. $6! - N = n! - \sum_{i=1}^{n} (-1)^{n+1}\frac{n!}{i!} = n! - n!\sum_{i=1}^{n} (-1)^{n+1}\frac{1}{i!} = n!(1 - \sum_{i=1}^{n} (-1)^{n+1}\frac{1}{i!}) = n!\sum_{i=0}^{n} (-1)^{n+1}\frac{1}{i!}$
5. The number of derangements is $\boxed{n!\sum_{i=0}^{n} \frac{(-1)^n}{i!}}$ or $\boxed{n![1 - 1 + \frac{1}{2} - \frac{1}{6} + ... + \frac{(-1)^n}{n!}]}$

3 Stars and Bars

3.1 Introduction

We know the basic stars and bars, counting the number of ways to put n stars in k sections. Let there be n stars, and we need to decide the way to place k bars. That means there is $n+k-1$ positions total, including the stars and bars. We have $n+k-1$ positions to place the bars to separate the stars.

3.1.1 Example 1

$a + b + c = 9$, count the number of ordered pairs of (a, b, c), with $a, b, c \geq 0$ and $a, b, c \in \mathbb{Z}$

$$\{* * * * * * * * *\}$$

Above are the stars, and we want to divide all the stars into three sections denoting the three variables. That means we want to place 2 bars to divide up the stars.

We can split up the stars like this, for example:
$\{* * \mid * * * * * * | *\}$ or $\{|| * * * * * * * **\}$

We can place the bars in $9-1+3 = 11$ positions. So let's choose the number of ways to choose 2 spots out of 11 total spots, which is just $\binom{11}{2} = 55$. That means the total number of ordered pairs (a, b, c) is **55**.

3.1.2 Restrictions

But what if we want to set a bound on what the variable can be? For example, we want to make it so $a \geq 1, b \geq 2, c \geq 3$. We can use this technique:

$a' = a - 1$, $b' = b - 2$, $c' = c - 3$, $a' + b' + c' = 3$ any ordered pair (a', b', c') that fulfils the above equation gives us the value of a, b, c which fulfils the $a + b + c = 9$ and the bounds.

now we can use the stars and bars technique to do $\binom{3-1+3}{2} = 10$, meaning that there are 10 ways for the equation and the bounds to be satisfied.