

Johnson-Lindenstrauss Lemma

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1 Johnson-Lindenstrauss Lemma

Lemma 1 let $\mathbf{y} = \frac{1}{\sqrt{k}} \mathbf{W} \mathbf{x}$, where $W_{i,j} \sim \mathcal{N}(0, 1)$:

$$\Pr((1 - \epsilon) \|\mathbf{x}\|^2 \leq \|\mathbf{y}\|^2 \leq (1 + \epsilon) \|\mathbf{x}\|^2) \geq 1 - 2 \exp \frac{(2\epsilon^3 - 3\epsilon^2)}{12} k \quad (1)$$

2 Proof of lemma 1

Let $W_{k \times d}$ be the random matrix where $W_{i,j} \stackrel{\text{i.i.d}}{\sim} \mathcal{N}(0, 1)$ and $\mathbf{x} \in \mathbb{R}^d$ and:

$$\begin{aligned} \mathbf{y} &= \frac{1}{\sqrt{k}} \mathbf{W} \mathbf{x} \\ \implies y_i &= \frac{1}{\sqrt{k}} \sum_{j=1}^d W_{i,j} x_j \end{aligned} \quad (2)$$

2.1 some helper properties

2.1.1 $\mathbb{E}[\|\mathbf{y}\|^2] = \|\mathbf{x}\|^2$

we know $\mathbb{E}[y_i] = \frac{1}{\sqrt{k}} \sum_{j=1}^d \mathbb{E}[W_{i,j}] x_j = 0$

$$\begin{aligned} \mathbb{E}[y_i^2] &= \frac{1}{k} \mathbb{E} \left[\left(\sum_{j=1}^d W_{i,j} x_j \right)^2 \right] \\ &= \frac{1}{k} \sum_{j=1}^d \mathbb{E}[W_{i,j}^2] x_j^2 \quad \text{all terms for } \mathbb{E}[W_{i,j} W_{i,j'}] = 0 \\ &= \frac{1}{k} \sum_{j=1}^d x_j^2 = \frac{1}{k} \|\mathbf{x}\|^2 \end{aligned} \quad (3)$$

This implies that $y_i \sim \mathcal{N}(0, \frac{1}{k} \|\mathbf{x}\|^2)$

$$\begin{aligned} \mathbb{E}[\|\mathbf{y}\|^2] &= \mathbb{E} \left[\sum_i y_i^2 \right] = \sum_i \mathbb{E}[y_i^2] \\ &= k \frac{1}{k} \|\mathbf{x}\|^2 \\ &= \|\mathbf{x}\|^2 \end{aligned} \quad (4)$$

2.1.2 introduce “normalized” $\hat{\mathbf{y}}$ such that $\hat{\mathbf{y}} \sim \mathcal{N}(0, \mathbf{I}_k)$:

$$\begin{aligned}
\text{let } \hat{\mathbf{y}} &= \frac{\sqrt{k}}{\|\mathbf{x}\|} \mathbf{y} \quad \text{be the “normalized” version of } \mathbf{y} \\
&= \frac{\sqrt{k}}{\|\mathbf{x}\|} \frac{1}{\sqrt{k}} \mathbf{W}\mathbf{x} = \frac{1}{\|\mathbf{x}\|} \mathbf{W}\mathbf{x} \\
\implies \hat{y}_i &= \frac{\sqrt{k}}{\|\mathbf{x}\|} y_i = \frac{1}{\|\mathbf{x}\|^2} W_{i,:} \mathbf{x} \\
&\sim \mathcal{N}(0, 1) \quad \text{using } y_i \sim \mathcal{N}\left(0, \frac{1}{k} \|\mathbf{x}\|^2\right)
\end{aligned} \tag{5}$$

one can interchange \mathbf{y} and $\hat{\mathbf{y}}$ (no need to use properties above):

$$\begin{aligned}
\hat{\mathbf{y}} &= \frac{\sqrt{k}}{\|\mathbf{x}\|} \mathbf{y} \\
\|\hat{\mathbf{y}}\|^2 &= \frac{k \mathbf{y}^\top \mathbf{y}}{\|\mathbf{x}\|^2} \\
&= \frac{k \|\mathbf{y}\|^2}{\|\mathbf{x}\|^2} \\
\implies \|\mathbf{y}\|^2 &= \frac{\|\hat{\mathbf{y}}\|^2 \|\mathbf{x}\|^2}{k}
\end{aligned} \tag{6}$$

2.2 prove $\Pr(\|\mathbf{y}\|^2 \geq (1 + \epsilon) \|\mathbf{x}\|^2) \leq \exp \frac{(2\epsilon^3 - 3\epsilon^2)}{12} k$

$$\begin{aligned}
&\Pr(\|\mathbf{y}\|^2 \geq (1 + \epsilon) \|\mathbf{x}\|^2) \quad \text{not in terms of } k \\
&= \Pr\left(\frac{\|\hat{\mathbf{y}}\|^2 \|\mathbf{x}\|^2}{k} \geq (1 + \epsilon) \|\mathbf{x}\|^2\right) \\
&= \Pr(\|\hat{\mathbf{y}}\|^2 \geq k(1 + \epsilon)) \quad \text{in terms of } k
\end{aligned} \tag{7}$$

using Chernoff bound for $\|\hat{\mathbf{y}}\|^2$:

$$\begin{aligned}
\Pr(\|\hat{\mathbf{y}}\|^2 \geq k(1 + \epsilon)) &\leq \frac{\mathbb{E}[\exp(\lambda \|\hat{\mathbf{y}}\|^2)]}{\exp(\lambda k(1 + \epsilon))} \\
&= \frac{\mathbb{E}[\exp(\lambda \sum_{i=1}^k \hat{y}_i^2)]}{\exp(\lambda k(1 + \epsilon))} \quad \sum_i \hat{y}_i^2 \sim \chi^2(k) \\
&= \frac{(1 - 2\lambda)^{-\frac{k}{2}}}{\exp(\lambda k(1 + \epsilon))} \\
&= \left(\frac{1}{\sqrt{1 - 2\lambda} \exp(\lambda(1 + \epsilon))} \right)^k \\
&= ((1 + \epsilon) \exp^{-\epsilon})^{\frac{k}{2}}
\end{aligned} \tag{8}$$

this is because, by solving for λ to minimize the bound which obtains $\lambda = \frac{\epsilon}{2(1 + \epsilon)}$ and then simplify, one obtains:

$$\begin{aligned}
& \frac{1}{\sqrt{1 - 2\frac{\epsilon}{2(1+\epsilon)} \exp\left(\frac{\epsilon}{2(1+\epsilon)}(1+\epsilon)\right)}} \\
&= \frac{1}{\sqrt{1 - \frac{\epsilon}{1+\epsilon} \exp(\frac{\epsilon}{2})}} \\
&= \frac{1}{\sqrt{\frac{1}{1+\epsilon}}} \exp^{-\frac{\epsilon}{2}} \\
&= (1+\epsilon)^{\frac{1}{2}} (\exp^{-\epsilon})^{\frac{1}{2}} \\
&= ((1+\epsilon) \exp^{-\epsilon})^{\frac{1}{2}}
\end{aligned} \tag{9}$$

Derive the fact:

$$\begin{aligned}
\log(1+x) &= \int_x \frac{1}{1+x} dx = \int_x (1+x)^{-1} dx \\
&= \int_x (1+0)^{-1} - (1+0)^{-2}x + \frac{2(1+0)^{-3}}{2!}x^2 - \frac{2 \times 3(1+0)^{-4}}{3!}x^3 + \dots \quad \because \text{Taylor expand about } x=0 \\
&= \int_x 1 - x + x^2 - x^3 + x^4 \\
&\leq \int_x 1 - x + x^2 \quad \because 0 < x < 1 \quad \text{and} \quad (-x^3 + x^4) < 0, (-x^5 + x^6) < 0 \dots \\
&= x - \frac{x^2}{2} + \frac{x^3}{3}
\end{aligned} \tag{10}$$

using the fact: $\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$:

$$\begin{aligned}
(1+\epsilon) &< \exp^{\epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3}} \\
\Rightarrow ((1+\epsilon) \exp^{-\epsilon})^{\frac{k}{2}} &\leq (\exp^{\epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3}} \exp^{-\epsilon})^{\frac{k}{2}} \\
&= (\exp^{-\frac{\epsilon^2}{2} + \frac{\epsilon^3}{3}})^{\frac{k}{2}} \quad \text{alternatively } \leq (\exp^{-\frac{\epsilon^2}{2} + \frac{\epsilon^3}{2}})^{\frac{k}{2}} = \exp^{-(\epsilon^2 - \epsilon^3) \frac{k}{4}} \\
&= (\exp^{\frac{2\epsilon^3 - 3\epsilon^2}{6}})^{\frac{k}{2}} \\
&= \exp^{\frac{(2\epsilon^3 - 3\epsilon^2)}{12} k}
\end{aligned} \tag{11}$$

so we proved that:

$$\begin{aligned}
\Pr(\|\mathbf{y}\|^2 \geq (1+\epsilon)\|\mathbf{x}\|^2) &= \Pr(\|\hat{\mathbf{y}}\|^2 \geq k(1+\epsilon)) \quad \text{from Eq.(7)} \\
&\leq ((1+\epsilon) \exp^{-\epsilon})^{\frac{k}{2}} \quad \text{from Eq.(8)} \\
&\leq \exp^{\frac{(2\epsilon^3 - 3\epsilon^2)}{12} k} \quad \text{from Eq.(11)}
\end{aligned} \tag{12}$$

2.3 prove $\Pr(\|\mathbf{y}\|^2 \leq (1 - \epsilon)\|\mathbf{x}\|^2) \leq (\exp^{-\frac{\epsilon^2}{2} + \frac{\epsilon^3}{3}})^{\frac{k}{2}}$

$$\begin{aligned}
& \Pr(\|\mathbf{y}\|^2 \leq (1 - \epsilon)\|\mathbf{x}\|^2) \\
&= \Pr\left(\frac{\|\hat{\mathbf{y}}\|^2 \|\mathbf{x}\|^2}{k} \leq (1 - \epsilon)\|\mathbf{x}\|^2\right) \\
&= \Pr(\|\hat{\mathbf{y}}\|^2 \leq k(1 - \epsilon)) \\
&= \Pr(\exp^{-\lambda \|\hat{\mathbf{y}}\|^2} \geq \exp^{-\lambda k(1 - \epsilon)}) \quad \lambda > 0
\end{aligned} \tag{13}$$

using Chernoff bound:

$$\begin{aligned}
\Pr(\|\hat{\mathbf{y}}\|^2 \leq k(1 - \epsilon)) &\leq \frac{\mathbb{E}[\exp^{-\lambda \|\hat{\mathbf{y}}\|^2}]}{\exp^{-\lambda k(1 - \epsilon)}} \quad \lambda > 0 \\
&= \frac{\mathbb{E}[\exp(-\lambda \sum_{i=1}^k \hat{y}_i^2)]}{\exp(-\lambda k(1 - \epsilon))} \quad \because \sum_i \hat{y}_i^2 \sim \chi^2(k) \\
&= \frac{(1 + 2\lambda)^{-\frac{k}{2}}}{\exp^{\lambda k(1 - \epsilon)}} \\
&= \left(\frac{1}{\sqrt{1 + 2\lambda} \exp^{-\lambda(1 - \epsilon)}} \right)^k \\
&= ((1 - \epsilon) \exp^{-\epsilon})^{\frac{k}{2}}
\end{aligned} \tag{14}$$

this is because, by solving for λ to minimize the bound which obtains $\lambda = \frac{\epsilon}{2(1 - \epsilon)}$ and then simplify, one obtains:

$$\begin{aligned}
& \frac{1}{\sqrt{1 + 2 \frac{\epsilon}{2(1 - \epsilon)}} \exp\left(\frac{-\epsilon}{2(1 - \epsilon)}(1 - \epsilon)\right)} \\
&= \frac{1}{\sqrt{1 - \frac{\epsilon}{1 - \epsilon}} \exp\left(\frac{-\epsilon}{2}\right)} \\
&= \frac{1}{\sqrt{\frac{1}{1 - \epsilon}}} \exp^{\frac{\epsilon}{2}} \\
&= (1 - \epsilon)^{\frac{1}{2}} (\exp^{\epsilon})^{\frac{1}{2}} \\
&= ((1 - \epsilon) \exp^{\epsilon})^{\frac{1}{2}}
\end{aligned} \tag{15}$$

Derive the fact:

$$\begin{aligned}
& \log(1 - x) < -x - \frac{x^2}{2} < -x - \frac{x^2}{2} - \frac{x^3}{3} \quad \text{for } x > 0 \\
& \implies (1 - \epsilon) < \exp^{-\epsilon - \frac{\epsilon^2}{2} - \frac{\epsilon^3}{3}}
\end{aligned} \tag{16}$$

the above inequality can be proven as:

$$\begin{aligned}
\log(1-x) &= \int_x \frac{-1}{1-x} dx = - \int_x (1-x)^{-1} dx \\
&= - \int_x (1-0)^{-1} + (1-0)^{-2}x + \frac{2(1-0)^{-3}}{2!}x^2 + \frac{2 \times 3(1-0)^{-4}}{3!}x^3 + \dots \quad \because \text{Taylor expand about } x=0 \\
&= \int_x -1 - x - x^2 - x^3 - x^4 - \dots \\
&\leq \int_x -1 - x - x^2 \quad \because 0 < x < 1 \\
&= -x - \frac{x^2}{2} - \frac{x^3}{3}
\end{aligned} \tag{17}$$

$$\begin{aligned}
\Pr(\|\mathbf{y}\|^2 \leq (1-\epsilon)\|\mathbf{x}\|^2) &\leq ((1-\epsilon)\exp^\epsilon)^{\frac{k}{2}} \\
&\leq \left(\exp^{-\epsilon - \frac{\epsilon^2}{2} - \frac{\epsilon^3}{3}} \exp^\epsilon\right)^{\frac{k}{2}} \\
&= \left(\exp^{-\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3}}\right)^{\frac{k}{2}} \\
&\leq \left(\exp^{-\frac{\epsilon^2}{2} + \frac{\epsilon^3}{3}}\right)^{\frac{k}{2}} \quad \because \epsilon > 0 \\
&\leq \exp^{-(\epsilon^2 - \epsilon^3)\frac{k}{4}} \quad \because \text{alternative version, see Eq.(11)}
\end{aligned} \tag{18}$$

2.4 Putting together:

$$\begin{aligned}
&\Pr(\|\mathbf{y}\|^2 \geq (1+\epsilon)\|\mathbf{x}\|^2 \cup \|\mathbf{y}\|^2 \leq (1-\epsilon)\|\mathbf{x}\|^2) \leq 2\exp^{\frac{(2\epsilon^3-3\epsilon^2)}{12}k} \\
\Rightarrow \Pr(\|\mathbf{y}\|^2 \leq (1+\epsilon)\|\mathbf{x}\|^2 \cap \|\mathbf{y}\|^2 \geq (1-\epsilon)\|\mathbf{x}\|^2) &\geq 1 - 2\exp^{\frac{(2\epsilon^3-3\epsilon^2)}{12}k} \\
\Rightarrow \Pr((1-\epsilon)\|\mathbf{x}\|^2 \leq \|\mathbf{y}\|^2 \leq (1+\epsilon)\|\mathbf{x}\|^2) &\geq 1 - 2\exp^{\frac{(2\epsilon^3-3\epsilon^2)}{12}k} \\
\Rightarrow \Pr((1-\epsilon)\|\mathbf{x}\|^2 \leq \|\mathbf{y}\|^2 \leq (1+\epsilon)\|\mathbf{x}\|^2) &\geq 1 - 2\exp^{-(\epsilon^2 - \epsilon^3)\frac{k}{4}} \quad \text{alternative version} \\
&\tag{19}
\end{aligned}$$

another alternative version:

$$\begin{aligned}
\Rightarrow \Pr((1-\epsilon)\|\mathbf{x}\|^2 \leq \|\mathbf{y}\|^2 \leq (1+\epsilon)\|\mathbf{x}\|^2) &\geq 1 - 2\exp^{-(\epsilon^2 - \epsilon^3)\frac{k}{4}} \quad \text{alternative version} \\
&\geq 1 - 2\exp^{-\epsilon^2\frac{k}{4}} \quad \text{looser bound}
\end{aligned} \tag{20}$$

2.4.1 alternative expression of J-L Lemma

let $\delta = \exp^{\frac{(2\epsilon^3-3\epsilon^2)}{12}k}$ or any other variants:

$$\begin{aligned}
&\Pr\left((1-\epsilon)\|\mathbf{x}\|^2 \leq \left\|\frac{1}{\sqrt{k}}W\mathbf{x}\right\|^2 \leq (1+\epsilon)\|\mathbf{x}\|^2\right) \geq 1 - 2\delta \\
\Rightarrow \Pr\left((1-\epsilon) \leq \frac{1}{k} \frac{\|W\mathbf{x}\|^2}{\|\mathbf{x}\|^2} \leq (1+\epsilon)\right) &\geq 1 - 2\delta \\
\Rightarrow \Pr\left(\left|\frac{1}{k} \frac{\|W\mathbf{x}\|^2}{\|\mathbf{x}\|^2} - 1\right| \leq \epsilon\right) &\geq 1 - 2\delta \\
\Rightarrow \Pr\left(\left|\frac{1}{k}\|W\mathbf{x}\|^2 - \|\mathbf{x}\|^2\right| \leq \epsilon\|\mathbf{x}\|^2\right) &\geq 1 - 2\delta
\end{aligned} \tag{21}$$

under special case where $\|\mathbf{x}\| = 1$

$$\implies \Pr\left(\left|\frac{1}{k}\|W\mathbf{x}\|^2 - 1\right| \leq \epsilon\right) \geq 1 - 2\delta \quad (22)$$

3 corollary of J-L lemma

3.1 norm preserving for n data points after dimension reduction

Corollary 1.1 *let $k \geq \left(\frac{24}{3\epsilon^2 - 2\epsilon^3} \log n\right)$ then: $\exists f : \mathbb{R}^d \rightarrow \mathbb{R}^k$, s.t.:*

$$(1 - \epsilon)\|\mathbf{x}_i - \mathbf{x}_j\|^2 \leq \|f(\mathbf{x}_i) - f(\mathbf{x}_j)\|^2 \leq (1 + \epsilon)\|\mathbf{x}_i - \mathbf{x}_j\|^2 \quad \forall \mathbf{x}_i, \mathbf{x}_j, \quad \forall 0 < \epsilon < 1 \quad (24)$$

3.1.1 choosing k in terms of n

looking at the (one side) J-L Lemma in Eq.(12). j is the only turnable parameter, so we need to choose an appropriate k . The clever choice is to let $k = \frac{24}{3\epsilon^2 - 2\epsilon^3} \log n$, so we have:

$$\begin{aligned} \Pr(\|\mathbf{y}\|^2 \geq (1 + \epsilon)\|\mathbf{x}\|^2) &\leq \exp^{\frac{(2\epsilon^3 - 3\epsilon^2)}{12} k} \\ &= \exp^{\frac{(2\epsilon^3 - 3\epsilon^2)}{12} \left(\frac{24}{3\epsilon^2 - 2\epsilon^3} \log n\right)} \quad \text{let } k = \frac{24}{3\epsilon^2 - 2\epsilon^3} \log n \quad (26) \\ &= \exp^{-2 \log n} \\ &= n^{-2} \end{aligned}$$

3.1.2 back to the dimension reduction case

let $\mathbf{y} = f(\mathbf{x}_i) - f(\mathbf{x}_j)$ $\mathbf{x} = \mathbf{x}_i - \mathbf{x}_j$:

$$\begin{aligned}
& \Pr(\|\mathbf{y}\|^2 \notin \{(1-\epsilon)\|\mathbf{x}\|^2 \cup (1+\epsilon)\|\mathbf{x}\|^2\}) \leq \frac{2}{n^2} \\
& \implies \Pr(\|f(\mathbf{x}_i) - f(\mathbf{x}_j)\|^2 \notin [(1-\epsilon)\|\mathbf{x}_i - \mathbf{x}_j\|^2 \cup (1+\epsilon)\|\mathbf{x}_i - \mathbf{x}_j\|^2]) \leq \frac{2}{n^2} \\
& \implies \Pr\left(\exists_{i,j} \left\{ \|f(\mathbf{x}_i) - f(\mathbf{x}_j)\|^2 \notin [(1-\epsilon)\|\mathbf{x}_i - \mathbf{x}_j\|^2 \cup (1+\epsilon)\|\mathbf{x}_i - \mathbf{x}_j\|^2] \right\}\right) \leq \binom{n}{k} \frac{2}{n^2} \quad \text{union bound} \\
& \quad = \frac{n(n-1)}{2} \frac{2}{n^2} = \frac{n-1}{n} \\
& \quad = 1 - \frac{1}{n} = 1 - \delta \\
& \implies \Pr\left(\forall_{i,j} \left\{ (1-\epsilon)\|\mathbf{x}_i - \mathbf{x}_j\|^2 \leq \|f(\mathbf{x}_i) - f(\mathbf{x}_j)\|^2 \leq (1+\epsilon)\|\mathbf{x}_i - \mathbf{x}_j\|^2 \right\}\right) \geq \frac{1}{n} \tag{27}
\end{aligned}$$

3.2 inner product

Corollary 1.2 Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ and that $\|\mathbf{u}\| \leq 1$ and $\|\mathbf{v}\| \leq 1$. Let $\mathbf{y}(\mathbf{x}) = \frac{1}{\sqrt{k}} \mathbf{W} \mathbf{x}$ where $W_{i,j} \sim \mathcal{N}(0, 1)$ (or $W_{i,j} \sim U(-1, +1)$), then:

$$\Pr(|\mathbf{u}^\top \mathbf{v} - \mathbf{y}(\mathbf{u})^\top \mathbf{y}(\mathbf{v})| \geq \epsilon) \leq 4 \exp^{-(\epsilon^2 - \epsilon^3) \frac{k}{4}} \tag{28}$$

3.2.1 proof

$$\begin{aligned}
& \Pr((1-\epsilon)\|\mathbf{x}\|^2 \leq \|\mathbf{y}\|^2 \leq (1+\epsilon)\|\mathbf{x}\|^2) \geq 1 - 2 \exp^{-(\epsilon^2 - \epsilon^3) \frac{k}{4}} \quad \text{alternative version} \\
& \implies \Pr((1-\epsilon)\|\mathbf{u} - \mathbf{v}\|^2 \leq \|\mathbf{u}' - \mathbf{v}'\|^2 \leq (1+\epsilon)\|\mathbf{u} - \mathbf{v}\|^2) \geq 1 - 2 \exp^{-(\epsilon^2 - \epsilon^3) \frac{k}{4}} \quad \text{let } \mathbf{x} \equiv \mathbf{u} - \mathbf{v} \text{ and } \mathbf{y} \equiv \mathbf{u}' - \mathbf{v}' \\
& \implies \Pr((1-\epsilon)\|\mathbf{u} + \mathbf{v}\|^2 \leq \|\mathbf{u}' + \mathbf{v}'\|^2 \leq (1+\epsilon)\|\mathbf{u} + \mathbf{v}\|^2) \geq 1 - 2 \exp^{-(\epsilon^2 - \epsilon^3) \frac{k}{4}} \quad \text{let } \mathbf{x} \equiv \mathbf{u} + \mathbf{v} \text{ and } \mathbf{y} \equiv \mathbf{u}' + \mathbf{v}' \tag{29}
\end{aligned}$$

therefore, for both of the above last two lines to satisfy simultaneously, we have (union of the complement) with probability of at least $1 - 4 \exp^{-(\epsilon^2 - \epsilon^3) \frac{k}{4}}$:

$$\begin{aligned}
& (1-\epsilon)\|\mathbf{u} - \mathbf{v}\|^2 \leq \|\mathbf{u}' - \mathbf{v}'\|^2 \leq (1+\epsilon)\|\mathbf{u} - \mathbf{v}\|^2 \quad \text{and} \\
& (1-\epsilon)\|\mathbf{u} + \mathbf{v}\|^2 \leq \|\mathbf{u}' + \mathbf{v}'\|^2 \leq (1+\epsilon)\|\mathbf{u} + \mathbf{v}\|^2 \tag{30}
\end{aligned}$$

when both of these satisfy, then the above implies:

3.2.2 Take left portion

$$\begin{aligned} (1 - \epsilon)\|\mathbf{u} - \mathbf{v}\|^2 &\leq \|\mathbf{u}' - \mathbf{v}'\|^2 \leq \frac{(1 + \epsilon)\|\mathbf{u} - \mathbf{v}\|^2}{(1 - \epsilon)\|\mathbf{u} + \mathbf{v}\|^2} \quad \text{and} \\ \underbrace{(1 - \epsilon)\|\mathbf{u} + \mathbf{v}\|^2}_{(1 - \epsilon)\|\mathbf{u} + \mathbf{v}\|^2} &\leq \|\mathbf{u}' + \mathbf{v}'\|^2 \leq (1 + \epsilon)\|\mathbf{u} + \mathbf{v}\|^2 \end{aligned} \quad (31)$$

$$\begin{aligned} 4\mathbf{u}'^\top \mathbf{v}' &= \|\mathbf{u}' + \mathbf{v}'\|^2 - \|\mathbf{u}' - \mathbf{v}'\|^2 \\ &\geq \underbrace{(1 - \epsilon)\|\mathbf{u} + \mathbf{v}\|^2}_{(1 - \epsilon)\|\mathbf{u} + \mathbf{v}\|^2} - \|\mathbf{u}' - \mathbf{v}'\|^2 \\ &\geq (1 - \epsilon)\|\mathbf{u} + \mathbf{v}\|^2 - \frac{(1 + \epsilon)\|\mathbf{u} - \mathbf{v}\|^2}{(1 - \epsilon)\|\mathbf{u} + \mathbf{v}\|^2} \\ &= (1 - \epsilon)(\|\mathbf{u}\|^2 + 2\mathbf{u}^\top \mathbf{v} + \|\mathbf{v}\|^2) - (1 + \epsilon)(\|\mathbf{u}\|^2 - 2\mathbf{u}^\top \mathbf{v} + \|\mathbf{v}\|^2) \\ &= (\|\mathbf{u}\|^2 + 2\mathbf{u}^\top \mathbf{v} + \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 + 2\mathbf{u}^\top \mathbf{v} - \|\mathbf{v}\|^2) - \epsilon(\|\mathbf{u}\|^2 + 2\mathbf{u}^\top \mathbf{v} + \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\mathbf{u}^\top \mathbf{v} + \|\mathbf{v}\|^2) \\ &= 4\mathbf{u}^\top \mathbf{v} - \epsilon(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2) \\ &= 4\mathbf{u}^\top \mathbf{v} - 4\epsilon \quad \because \|\mathbf{u}\| = 1 \text{ and } \|\mathbf{v}\| = 1 \\ \implies \mathbf{u}'^\top \mathbf{v}' &\geq \mathbf{u}^\top \mathbf{v} - \epsilon \end{aligned} \quad (32)$$

3.2.3 Combine with the right portion

we also use the two existing conditions in Eq.(30): (therefore no additional union bound is needed):