Assignment 4 Question 6 Report

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1 Part A

1.1 Proving $y^T Py \geq 0$

$$y^T Py = y^T A^T Ay$$

Here A is an $m \times n$ matrix. Hence y is an $n \times 1$ vector, meaning that Ay is a vector of size $m \times 1$.
Hence let $Ay = v$.
Therefore, $y^T Py = v^T v = x_1^2 + ... + x_n^2 \geq 0$
Above $x_1, ..., x_n$ are elements of v.
Hence proved.

1.2 Proving $z^T Pz \geq 0$

$$z^T Py = z^T AA^T z$$

This proof is similar to the above proof. Here A is an $m \times n$ matrix. Hence y is an $m \times 1$ vector, meaning that $A^T z$ is a vector of size $n \times 1$.
Hence let $A^T z = v$.
Therefore, $z^T Pz = v^T v = x_1^2 + ... + x_n^2 \geq 0$
Above $x_1, ..., x_n$ are elements of v.
Hence proved.

1.3 Eigen values of P and Q non-negative

Let $Pv = \lambda v$.
Hence $v^T P^T = \lambda v^T$
As $P^T = P$, $v^T P = \lambda v^T$
Multiplying on both sides by v,
$v^T Pv = \lambda v^T v \geq 0$ as we proved earlier.
But $v^T v \geq 0$ for all v as again we showed earlier.
Hence $\lambda \geq 0$, where λ is an eigen value of P.

Similarly for Q,
Let $Qv = \lambda v$.
Hence $v^T Q^T = \lambda v^T$
As $Q^T = Q$, $v^T P = \lambda v^T$
Multiplying on both sides by v,
$v^T Qv = \lambda v^T v \geq 0$ as we proved earlier.
But $v^T v \geq 0$ for all v as again we showed earlier.
Hence $\lambda \geq 0$, where λ is an eigen value of Q.

2 Part B

2.1 Proving Au is eigen vector of Q

If u is an eigen vector of P, we have,

$$A^T Au = \lambda U - (1)$$

Hence, we get,
$QAu = AA^T Au = \lambda Au$ (Substituting 1)
Therefore, we get $QAu = \lambda Au$, implying that Au is an eigen vector of Q.
Hence proved.
As A is $m \times n$ and Au is a valid multiplication, u must have n elements.

2.2 Proving $A^T v$ is eigen vector of P

If v is an eigen vector of P, we have,

$$AA^T v = \mu v - (1)$$

Hence, we get,
$PA^T v = A^T AA^T v = \mu A^T v$ (Substituting 1)
Therefore, we get $PA^T v = \mu A^T v$, implying that $A^T v$ is an eigen vector of P.
Hence proved.
As A is $m \times n$ and $A^T v$ is a valid multiplication, v must have m elements.

3 Part C

3.1 Proving existence of real non-negative γ_i such that $Au_i = \gamma_i v_i$

As v_i is an eigen vector of Q, we have,

$$AA^T v_i = \lambda v_i - (1)$$

Hence, we get that,
Au_i
$= \frac{AA^T v_i}{||A^T v_i||_2}$
$= \frac{\lambda v_i}{||A^T v_i||_2}$ (From 1)
Hence we can see that $\gamma_i = \frac{\lambda}{||A^T v_i||_2}$
Here the denominator is non-negative as it is the squared norm of a vector. The numerator is non-negative as λ is an eigen value of Q which we have proved is non-negative in **1.3**. Both the numerator and denominator are real.
Hence we have real, non-negative γ_i such that $Au_i = \gamma_i v_i$.
Hence proved.

4 Part D

Instead of proving straightaway that $A = U\Gamma V^T$, let us first prove that $AV = U\Gamma$.

If we multiply U and Γ as defined in the question, we get an $m \times m$ matrix of the form:-

$$[\gamma_1 v_1 \gamma_2 v_2 .... \gamma_m v_m]$$

If we multiply A and V as defined in the question, we get an $m \times m$ matrix of the form:-

$$[Au_1 Au_2 ... Au_m]$$

As proved in the earlier sub-part, $Au_i = \gamma_i v_i$
Hence if we compare the left hand side and the right hand side, we will get that they are equal.
Hence, $AV = U\Gamma$.
Let us multiply by V^T by both sides.
We get, $AVV^T = U\Gamma V^T$.
But V is a matrix of eigen vectors. Hence $v_i^T v_i = 1$ by definition and $v_i^T v_j = 0$ for j not equal to i as given in the question.
Hence we get that $VV^T = I$, which is an identity matrix.
Thus, we get $A = U\Gamma V^T$.
Hence Proved.

5 Conclusions

Hence we have proved all the results ending with the proof of singular value decompostion.