Homework 1 : Graphs and Network Flows

Aashith Kamath Manjeshwar, Saikrishna Badrinarayanan

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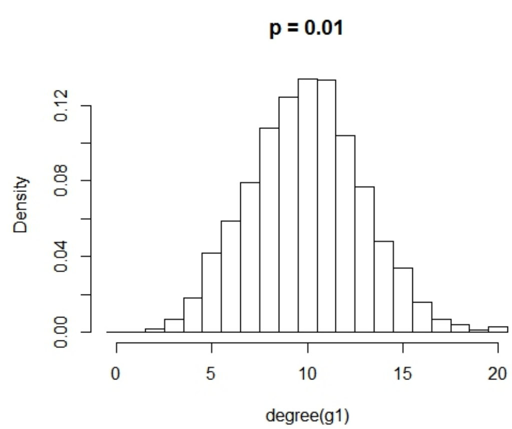
In this project, we use the igraph library to generate different kinds of networks and measure several properties of each of these networks. The project was coded in the language R.

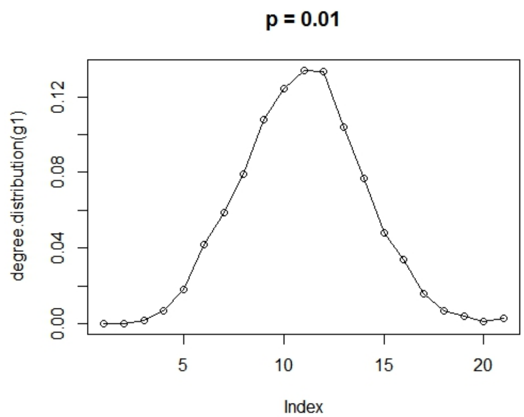
Problem 1 :

a)

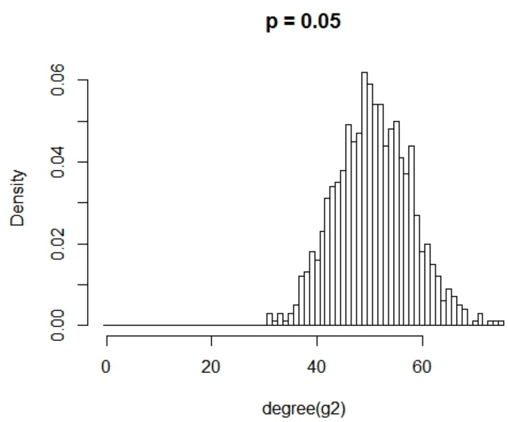
The first problem involved creating three undirected random networks with 1000 nodes each and a probability p that every pair of nodes has an edge between them. The values of p were taken as $0.01, 0.05$ and 0.1 respectively for each of the three graphs. After this, we plot the degree distribution for each of the graphs. The degree distribution is a plot containing the various degrees of the nodes in the graph on the x-axis and their corresponding density (frequency of that degree occuring divided by the total number of nodes) on the y-axis.

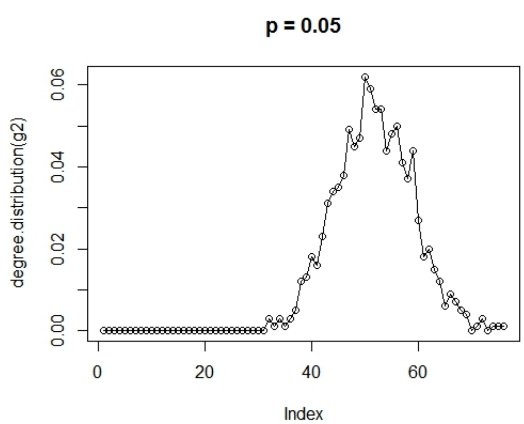
The degree distribution plot for the first graph is :



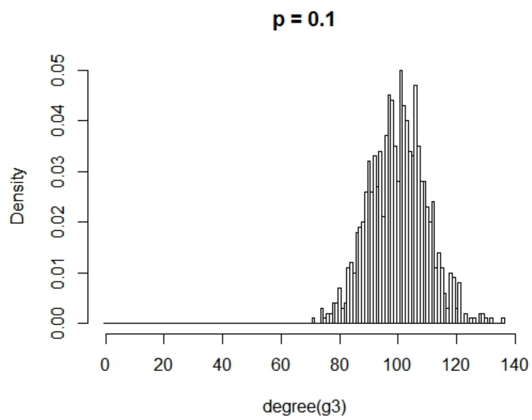


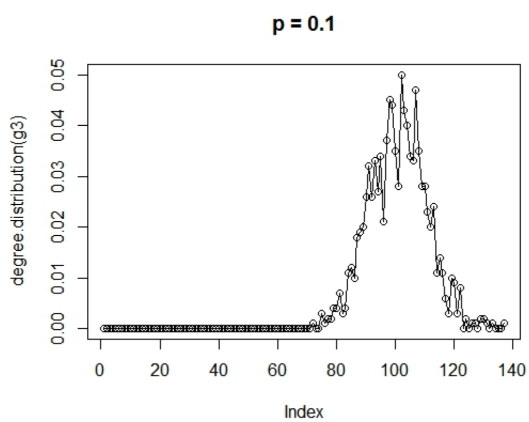
The degree distribution plot for the second graph is :





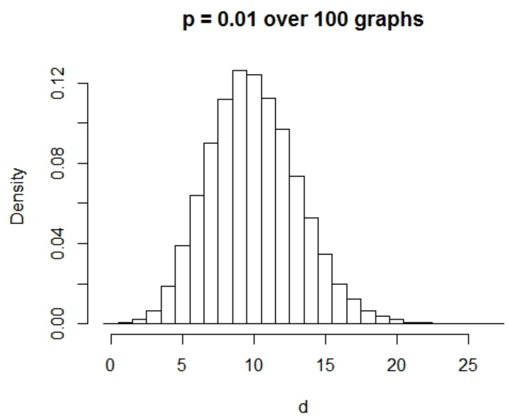
The degree distribution plot for the third graph is :

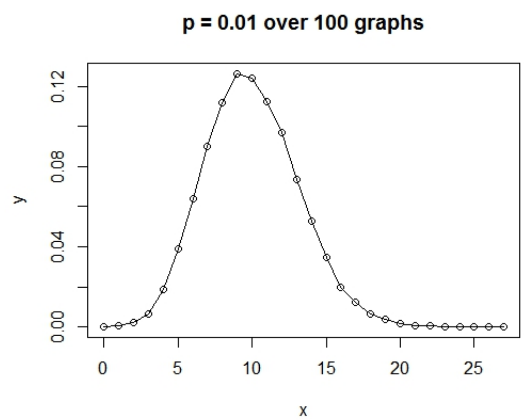




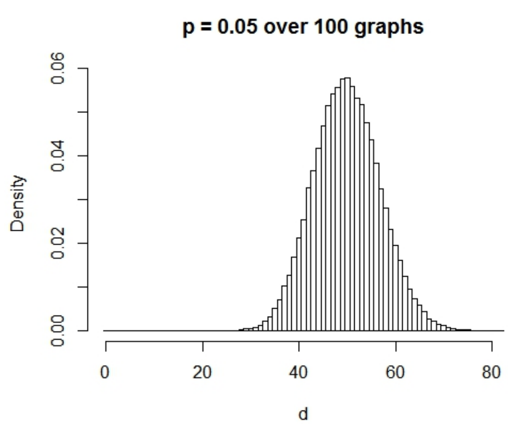
After this, we repeated the same process above 100 times and plot the average degree distribution over the 100 random graphs below.

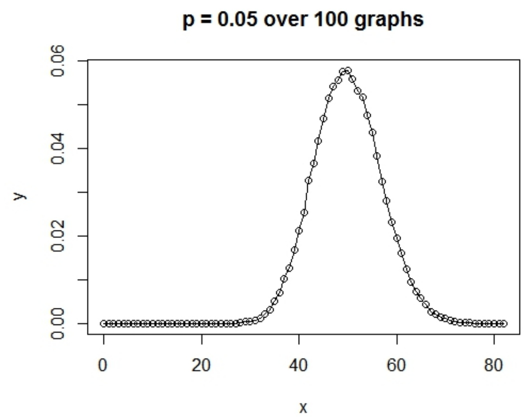
The average degree distribution for $p = 0.01$ is



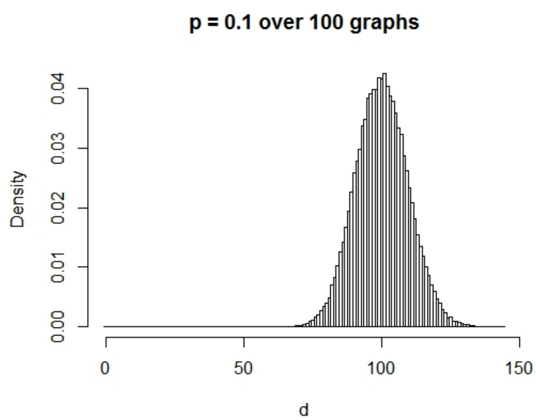


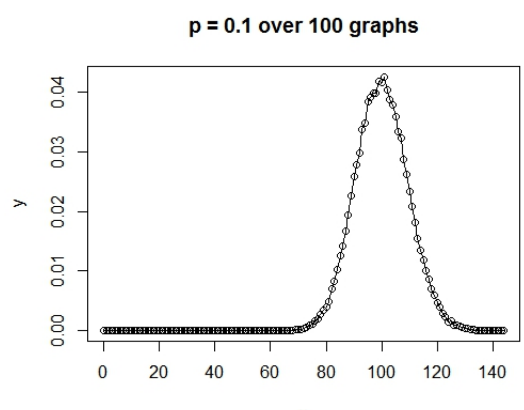
The average degree distribution for $p = 0.05$ is





The average degree distribution for $p = 0.1$ is





b)

This question asks whether the graphs generated above are connected or disconnected and to find out their diameters. In the rest of the question, let's denote the graph generated with $p = 0.01$ as the first graph, $p = 0.05$ as the second graph and $p = 0.1$ as the third graph. The diameter of a graph is the length of the shortest path between the nodes with the longest distance between them. That is, the length of the shortest path between the nodes that are farthest apart.

- The first graph is connected and it's diameter is 6.
- The second graph is connected and it's diameter is 3.
- The third graph is connected and it's diameter is 3.

c)

In this question, the aim is to numerically find a value p_c such that when $p < p_c$, the random networks we generate (with probability of every pair of nodes having an edge being p) is disconnected and when $p > p_c$, the generated random networks are connected. The way we do this is that we generate a random graph and check for different values of p whether the graph is connected or not. We thus obtain a minimum and maximum value of p_c, i.e beyond the maximum value, for all p, the graph is connected and below the minimum value, for all p, the graph is disconnected.

We note that since the graphs generated are random, the minimum and maximum values of p_c will change (slightly) if we run the procedure again. As a result, we repeat the process 100 times and take the average values. We observe that, after averaging, the minimum value of p_c is 0.0054 and the maximum value is 0.0072.

If we are required to give exactly 1 value of p_c such that the graph is disconnected for all $p < p_c$ and connected for all $p > p_c$, we can average the minimum and maximum values to obtain $p_c = 0.0063$

d)

In this question, we are expected to analytically calculate the value of p_c.

Let's first calculate the probability that a given node i is isolated. For a given node i, it is isolated only if it has no edges with any other nodes. The probability of it not having an edge with another particular node j is $(1 - p)$. Since the probability that i has an edge with any chosen node is exactly p and is independent of whether it has edges with other nodes, the probability that i has no edges with $(1 - p)^n$. Thus, the probability that a given node is isolated is $(1 - p)^n$. Since we expect the value of p to be small (i.e when $p = p_c$ we expect it to be small as for higher values of p, it's unlikely that the graph is disconnected), we can approximate $(1 - p)^n$ to e^{-pn}.

Now, let's calculate how many nodes are isolated. Rather, we will compute the expectation of the number of isolates nodes. Let's denote this quantity by $E(x)$. If the graph has n verticies $v_1, \dots v_n$, we can write $E(x) = E(x_1) + E(x_2) + \dots E(x_n)$, where $E(x_i)$ is the expectation that vertex v_i is isolated. From our

earlier calculation, $E(x_i) = 1.e^{-pn} + 0.(1 - e^{-pn}) = e^{-pn}\ \forall i$.
Thus, $E(x) = n.e^{-pn}$

We note that if $E(x) >= 1$, it means that the expected number of isolated nodes is atleast 1 and so the graph must be disconnected. However, if $E(x) < 1$, it just means that the expected number of isolated nodes is less than 1. The graph could still be disconnected in such a case. For example, a simple graph with 4 vertices in which the first two vertices have an edge between them, the third and fourth vertices have an edge between them. The graph is disconnected but no vertex is isolated. However, since we know that if $E(x) >= 1$, the graph is surely disconnected, we will use this measure to determine our value of p_c.
$E(x) >= 1 \Rightarrow n.e^{-pn} >= 1$
$\Rightarrow n >= e^{pn} \Rightarrow ln(n) >= pn$
$\Rightarrow p <= ln(n)/n$
Thus, whenever $p <= ln(n)/n$, the graph is disconnected. Hence the value of p_c is $ln(n)/n$.

For $n = 1000$, $ln(n)/n = 0.0069$. Thus, we observe that the numerically computed value in the previous question roughly matches with the analytically computed one for $n = 1000$.

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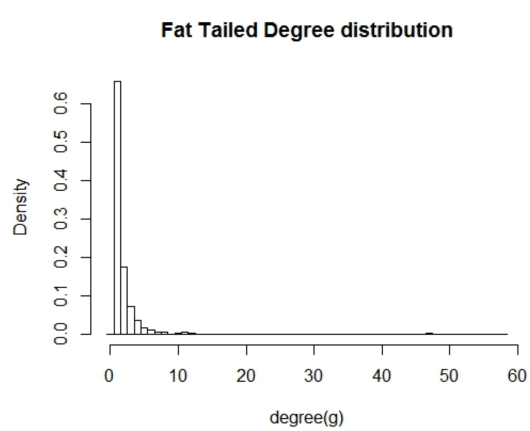
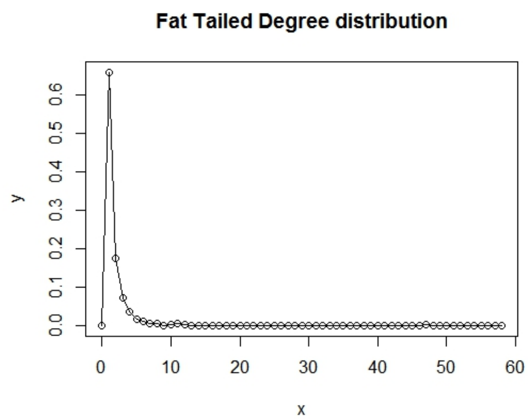
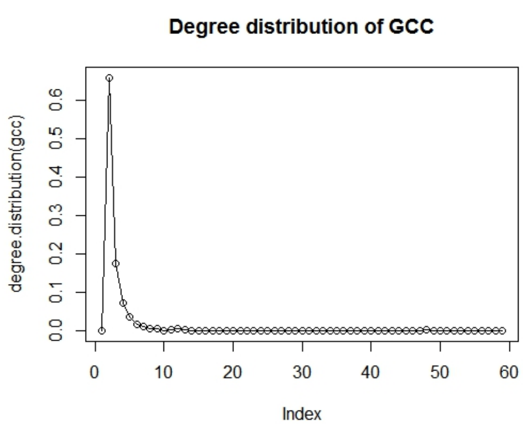
2) a)
This problem involves creating an undirected random network with 1000 nodes whose degree distribution is proportional to x^{-3} (a fat-tailed degree distribution). The degree distribution is a plot containing the various degrees of the nodes in the graph on the x-axis and their corresponding density (frequency of that degree occuring divided by the total number of nodes) on the y-axis.
We use the barabasi-albert model to generate the graph.
The degree distribution plot for the graph is :



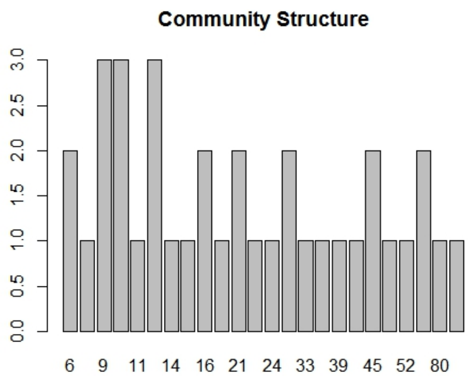


After that, we compute the diameter of the graph and it turns out to be 16.

b)
In this question, we first check if the graph is connected and it turns out that it indeed is.
We then find the Giant Connected Component (GCC). The GCC of a graph is a connected component that contains a constant fraction of the entire graph's vertices. We find the GCC by first computing the set of clusters in the graph and then the largest cluster among them. We then delete the vertices that do not belong to the largest cluster and this gives the GCC. We then plot the degree distribution of the GCC and it is shown below.

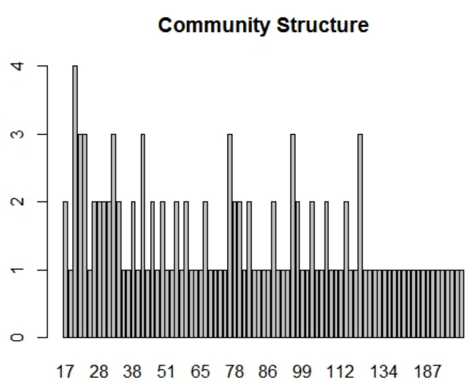


We then find the community structure using the fast greedy method on the GCC. A network is said to have a community structure if the nodes of the network can be easily grouped into (potentially overlapping) sets of nodes such that each set is densely connected internally. The community structure is :



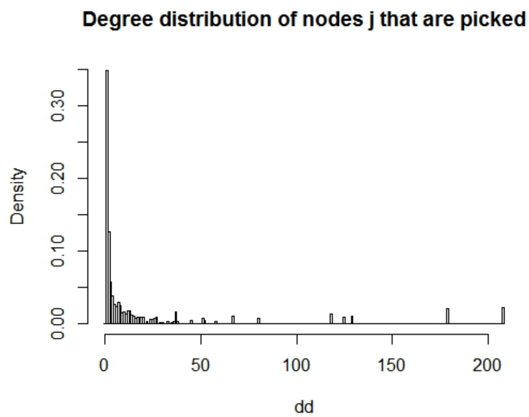
We then compute the modularity of the graph and it turns out to be 0.9184294 The modularity is so large because there is good connectivity within the modules and not very good community between the modules in the graph.

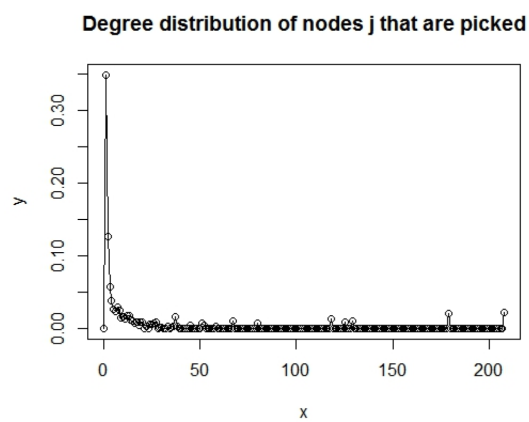
c)
In this problem, we create a larger network with 10000 nodes whose degree distribution is proportional to x^-3. We then compute the GCC and the community structure as in the previous question. The community structure is :



. After this, we measure the modularity of this graph and it turns out to be 0.9760596 which is larger than that in the case of the smaller graph.

d)
Here, we first randomly pick a node i and then randomly pick one of it's neighbours j. We repeat this process 100000 times and plot the degree distribution of the nodes j picked in this process. The degree distribution is :





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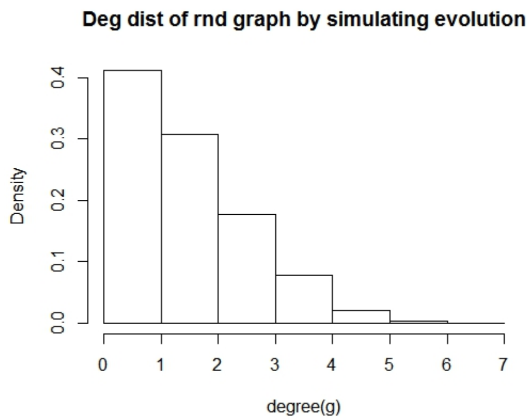
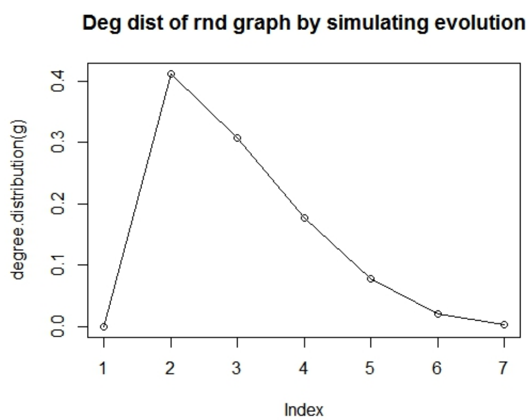
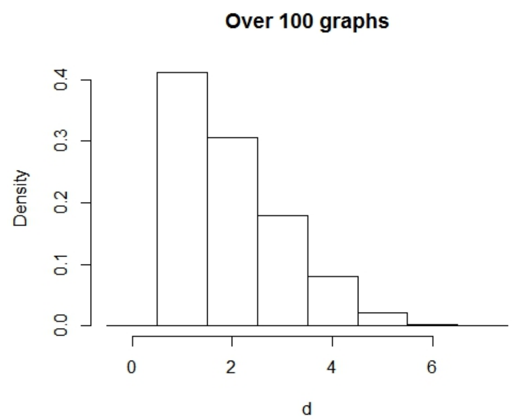
3) a)
In this problem, we create an undirected graph by simulating it's evolution. Each time a new vertex is added it creates a number of new links to old vertices and the probability that an old vertex is cited (included in the graph) depends on its in-degree and age. This process of adding the old vertex depending on it's high in-degree and age is called preferential attachment. We create such an undirected network with 1000 nodes and plot it's degree distribution

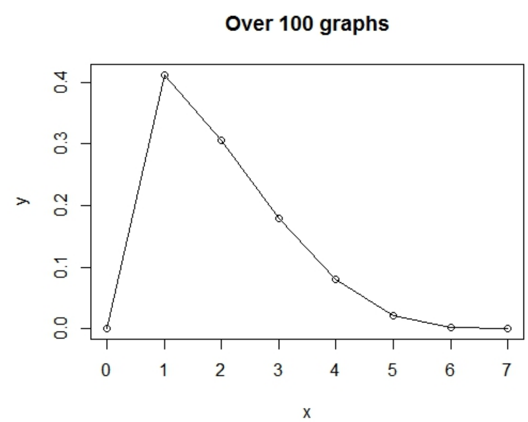




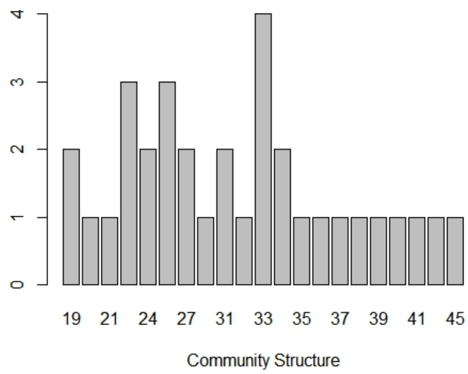
After this, we repeated the same process above 100 times and plot the average degree distribution over the 100 random graphs below.

The average degree distribution is





b)
As in problem 2, we find the GCC and use fast greedy method to compute the community structure. The community structure is :



We then measure the modularity and it is equal to 0.9359986.

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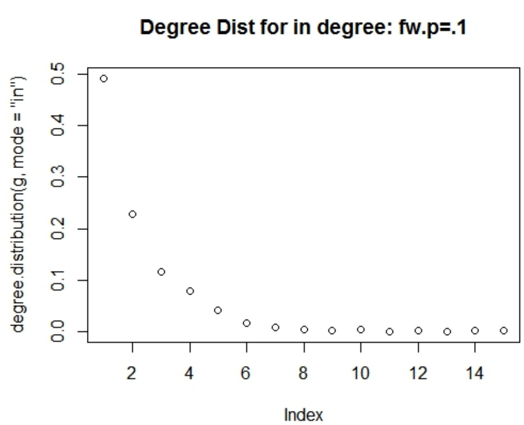
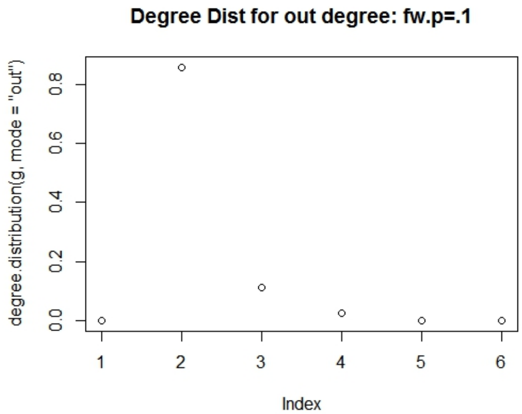
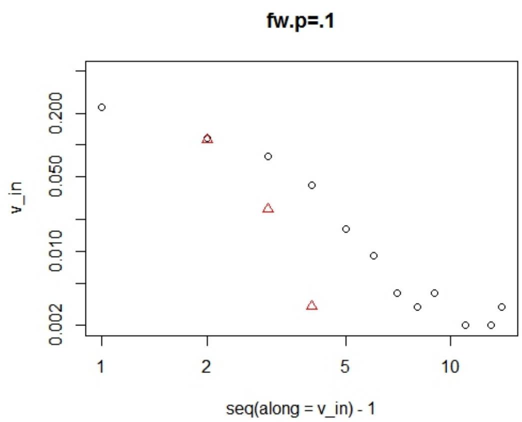
4) a)
In this problem, we create a graph by using the forest fire model. This is a growing network model, which resembles how the forest fire spreads by igniting trees close by. Unlike previous cases, it is a directed graph. We create several graphs with the forward burning probability ranging from 0.1 to 0.4 and the backward burning ratio set as 1. We then plot the in and out degree distributions. We define graph 1 as the one with forward burning probability as 0.1, graph 2 with 0.2, graph 3 with 0.27 and graph 4 with 0.37. Apart from the normal in-degree and out-degree plots, we additionally draw one more plot for each graph. We plot the density vs degree on a log log scale with both the in and out degrees. The "triangles" in the graph are for the out-degree and the "circles" are for the in-degree.

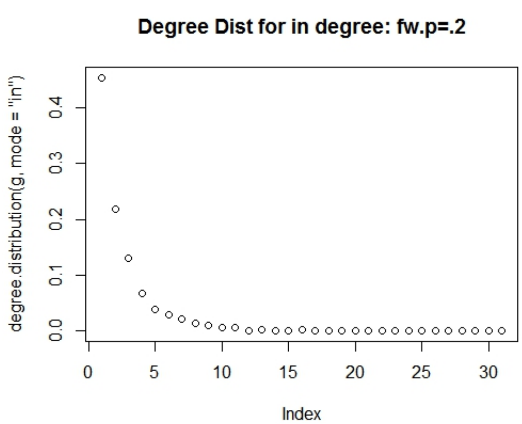
Graph 1

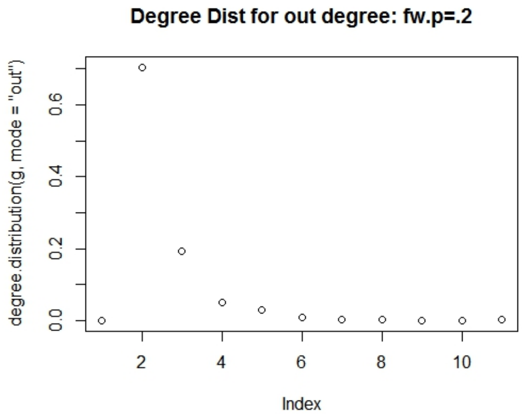



Degree Dist for out degree: fw.p=.1
degree.distribution(g, mode = "out")
0.0
0.2
0.4
0.6
0.8
1
2
3
4
5
6
Index

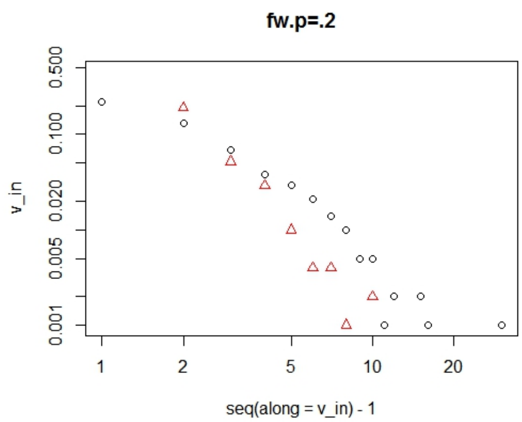

fw.p=.1
v_in
0.002
0.010
0.050
0.200
1
2
5
10
seq(along = v_in) - 1

Graph 2


Degree Dist for in degree: fw.p=.2
degree.distribution(g, mode = "in")
0.0
0.1
0.2
0.3
0.4
0
5
10
15
20
25
30
Index

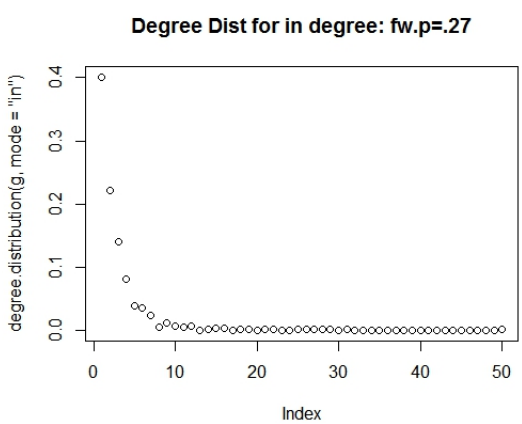


Degree Dist for out degree: fw.p=.2
degree.distribution(g, mode = "out")
0.0
0.2
0.4
0.6
2
4
6
8
10
Index

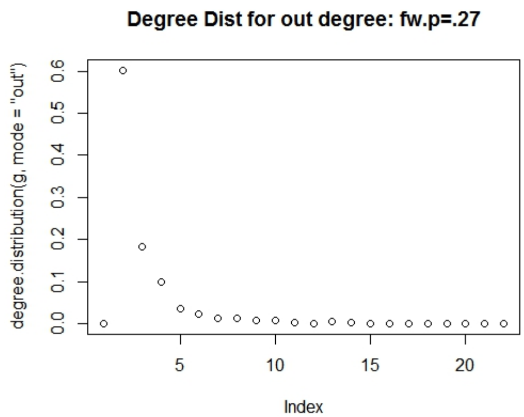


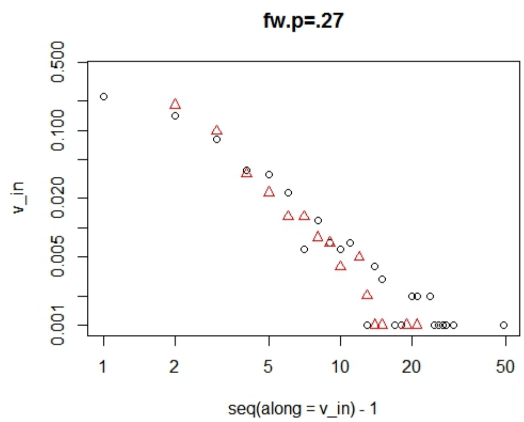
fw.p=.2
v_in
0.001
0.005
0.020
0.100
0.500
1
2
5
10
20
seq(along = v_in) - 1

Graph 3

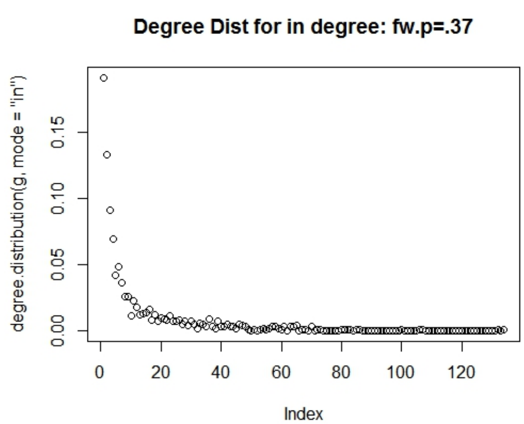


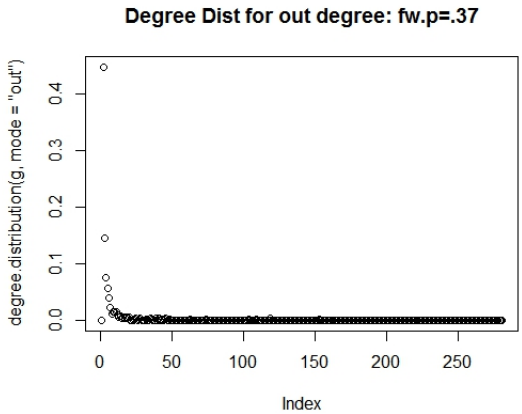
Degree Dist for in degree: fw.p=.27
degree.distribution(g, mode = "in")
0.0
0.1
0.2
0.3
0.4
0
10
20
30
40
50
Index

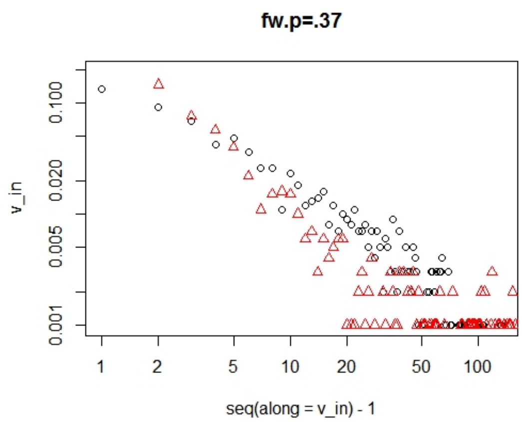

Degree Dist for out degree: fw.p=.27
degree.distribution(g, mode = "out")
0.0
0.1
0.2
0.3
0.4
0.5
0.6
5
10
15
20
Index


fw.p=.27
v_in
0.001
0.005
0.020
0.100
0.500
1
2
5
10
20
50
seq(along = v_in) - 1

Graph 4


Degree Dist for in degree: fw.p=.37
degree.distribution(g, mode = "in")
0.00
0.05
0.10
0.15
0
20
40
60
80
100
120
Index





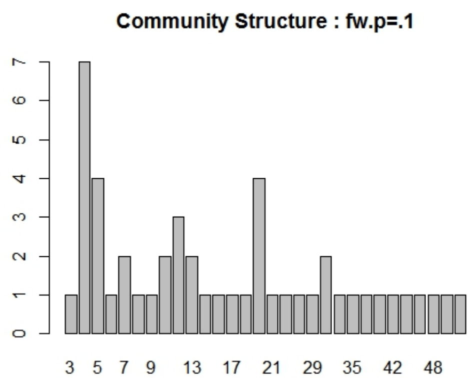
b)
We measure the diameters of the 4 graphs.

- Graph 1 : Diameter is 14.
- Graph 2 : Diameter is 15.
- Graph 3 : Diameter is 10.
- Graph 4 : Diameter is 12.

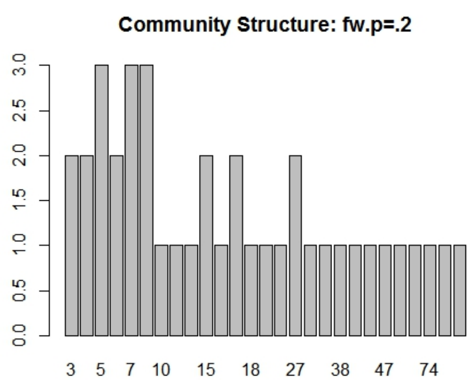
c)
In this problem, we find the community stucture of the 4 graphs. We use two approaches to do this.

In the first approach, since the fast greedy method only works on undirected graphs, we convert the directed GCC into an undirected one using the "as.indirected" function with the mode being "collapse" the igraph library. This converts the directed graph by creating one undirected edge for each pair of vertices which are connected with atleast one directed edge. It doesn't create multiple edges. There are other modes like "each" and "mutual" but we choose to work with the "collapse" mode (for no particular reason).

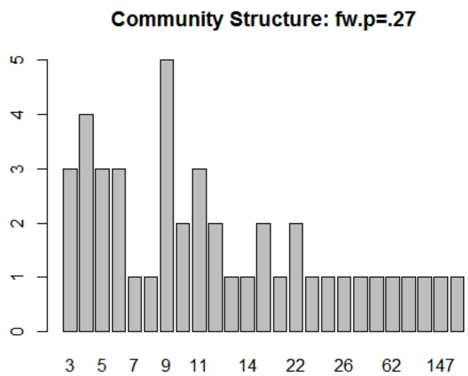
The community structure of the first graph is :



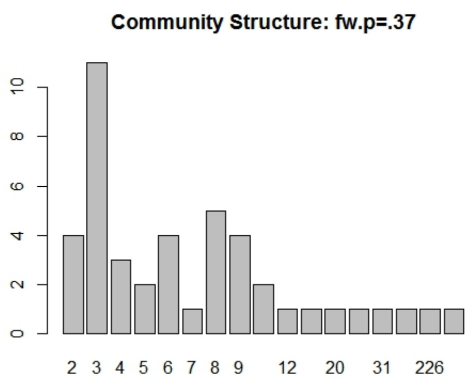
The community structure of the second graph is :



The community structure of the third graph is :



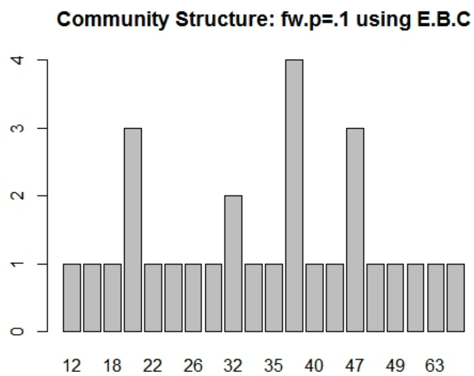
The community structure of the fourth graph is :



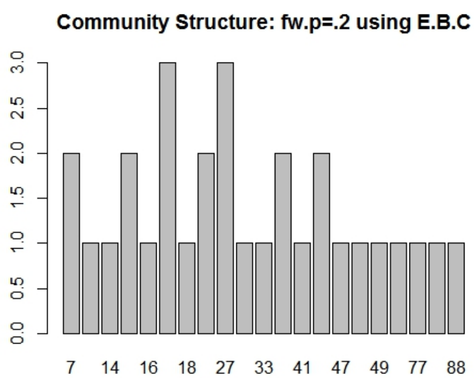
We then compute the modularity of the 4 graphs.

- Graph 1 : modularity is 0.9045291
- Graph 2 : modularity is 0.8848962
- Graph 3 : modularity is 0.7781591
- Graph 4 : modularity is 0.2152621

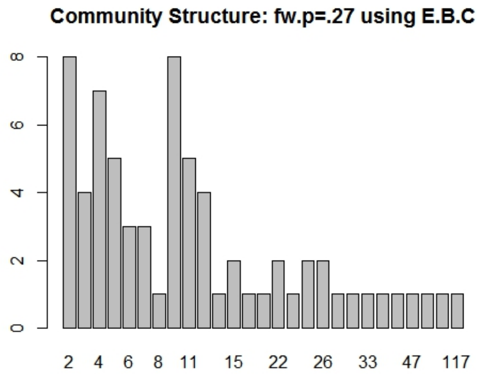
In the next approach, we find the community structure by using the "edge.betweenness.community" function on the directed GCC and then find the modularity as in the previous case.
The community structure of the first graph is :



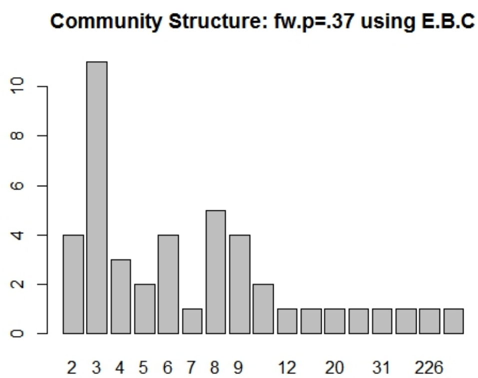
The community structure of the second graph is :



The community structure of the third graph is :



The community structure of the fourth graph is :



We then compute the modularity of the 4 graphs.

- Graph 1 : modularity is 0.9179842
- Graph 2 : modularity is 0.8978998
- Graph 3 : modularity is 0.7766938
- Graph 4 : modularity is 0.07439451

We conclude by noting that both the approaches seem to give roughly the same results with a few minor differences.