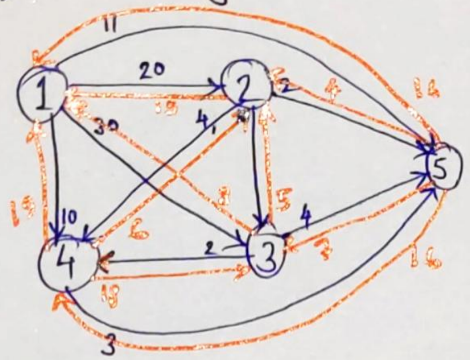
Travelling Sales Person Problem: Branch & Bound.

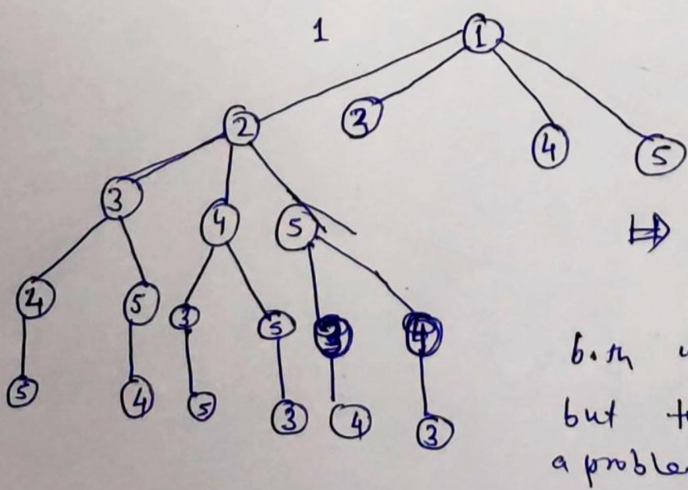
Here, first we will learn about the problem first, then the solution of it using branch & bound approach.



$$
\begin{array}{c|c}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} &
\begin{bmatrix}
\infty & 20 & 30 & 10 & 11 \\
15 & \infty & 16 & 4 & 2 \\
3 & 5 & \infty & 2 & 4 \\
19 & 6 & 18 & \infty & 3 \\
16 & 4 & 7 & 16 & \infty
\end{bmatrix}
\end{array}
$$

We have to find out the shortest tour such that it visits each vertex exactly once & returns to the original vertex. If we take '2' as the starting vertex then we have to travel other vertex '3' '4' '5' '1' once and return to the original vertex 2.

Let us consider this problem & try to understand how this problem is solved using branch & bound. If the salesman starting from vertex 1, then he can go visit the nodes in any order such that the cost is minimum.



⇐ This is the space tree which shows the all possible paths.

⇒ Backtracking & branch & bound are similar they both use the state-space tree but the approach towards solving a problem is different.

In TSP we do not want all possible solution we want minimal solution.

In branch & bound nodes are generated in level order, while generating the node for every node we compute the cost & for any ~~given~~ node if we are getting the cost grater than some minimum value then we will kill the node. Thus we will generate only those nodes which are fruitful. i.e. we will try to follow that path which will taking us to the optimized solution.

The problem can be solved by using matrixes.

1. Cost matrix.

2. Calculate minimum value for each row.

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\left[\begin{array}{ccccc}
\infty & 20 & 30 & 10 & 11 \\
15 & \infty & 16 & 4 & 2 \\
3 & 5 & \infty & 2 & 4 \\
19 & 6 & 18 & \infty & 3 \\
16 & 4 & 7 & 16 & \infty
\end{array}\right]
\end{array}
\begin{array}{c} \\ 10 \\ 2 \\ 2 \\ 3 \\ 4 \end{array}
$$

3. Subtract that minimum value from every element in the row, we will get a reduced matrix.

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\left[\begin{array}{ccccc}
\infty & 10 & 20 & 0 & 1 \\
13 & \infty & 14 & 2 & 0 \\
1 & 3 & \infty & 0 & 2 \\
16 & 3 & 15 & \infty & 0 \\
12 & 0 & 3 & 12 & \infty
\end{array}\right] \\
 & \begin{array}{ccccc} 1 & 0 & 3 & 0 & 0 \end{array}
\end{array}
\begin{array}{c} \\ 10 \\ 2 \\ 2 \\ 3 \\ 4 \\ 21 \end{array}
$$

4. rows are reduced write the minimum value of the columns to reduce the columns.

$= 4 + 21 = 25$

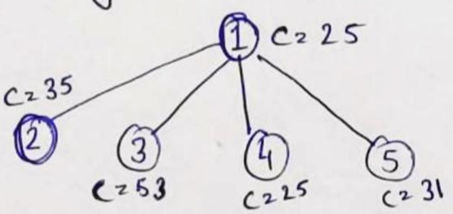
5

$$\begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} \infty & 10 & 17 & 0 & 1 \\ 12 & \infty & 11 & 2 & 0 \\ 0 & 3 & \infty & 0 & 2 \\ 15 & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & 12 & \infty \end{bmatrix} \end{array}$$

Columns are reduced & the total cost of reduction is $(21+4) = 25$.

Now the matrix is reduced, A matrix is called reduced if it have atleast one element in the every row/column whose value is zero.

Say I'm starting from vertex 1. then,-



→ we will be using the matrix as shown in 5.

→ we are going 1 to 2 so we will mark 1'st row & second column as infinite.

→ then once we are traversing from 1 to 2 then the reverse path is not allowed & we marked it as infinite as well. (2 to 1)

(1-2)

$$\begin{array}{c|c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} & \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & 2 & 0 \\ 0 & \infty & \infty & 0 & 2 \\ 15 & \infty & 12 & \infty & 0 \\ 11 & \infty & 0 & 12 & \infty \end{bmatrix} & \begin{array}{c} \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \\ & \begin{array}{ccccc} 0 & & 0 & 0 & 0 \end{array} & \end{array}$$

6 Check if the matrix beside is reduced or not

7 The matrix is reduced. & the cost of reduction can be calculated by the following formula,-

$$\text{Cost} = c(i, j) + r + \hat{r}$$

$c(i,j) \rightarrow$ cost of the edge.

$r \rightarrow$ cost of the reduction.

$\hat{r} \Rightarrow$ any more reduction done in the subsequent steps.

$\therefore$ Cost $= C(1,2) + r + \hat{r}$

$= 10 + 25 + 0$

$= 35$

(1-3)

$$\begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & \infty & \infty & \infty & \infty & \infty & \\ 2 & 12 & \infty & & 2 & 0 & 0 \\ 3 & \infty & 3 & \infty & 0 & 2 & 0 \\ 4 & 15 & 3 & & \infty & 0 & 0 \\ 5 & 11 & 0 & \infty & 12 & \infty & 0 \\ \hline & 11 & 0 & & 0 & 0 & \end{array} \xrightarrow{\text{reduce}} \begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \infty & \infty & \infty & \infty & \infty \\ 2 & 1 & \infty & \infty & 2 & 0 \\ 3 & \infty & 3 & \infty & 0 & 2 \\ 4 & 4 & 3 & \infty & \infty & 0 \\ 5 & 0 & 0 & \infty & 12 & \infty \end{array}$$

$\hat{r} \rightarrow$ cost of reduction = 11

Cost of vertex 3 = $C(1,3) + r + \hat{r}$

$= 17 + 25 + 11 = 53$

(1-4)

$$\begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & \infty & \infty & \infty & \infty & \infty & \\ 2 & 12 & \infty & 11 & \infty & 0 & 0 \\ 3 & 0 & 3 & \infty & \infty & 2 & 0 \\ 4 & \infty & 3 & 12 & \infty & 0 & 0 \\ 5 & 11 & 0 & 0 & \infty & \infty & 0 \\ \hline & 0 & 0 & 0 & & 0 & \end{array}$$

It is already reduced.

Cost of reduction = $C(1,4) + r + \hat{r}$

$= 0 + 25 + 0$

$= 25$

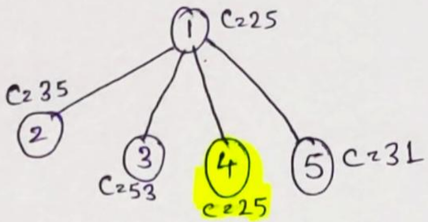
(1-5)

$$\begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & \infty & \infty & \infty & \infty & \infty & \\ 2 & 12 & \infty & 11 & 2 & \infty & 2 \\ 3 & 0 & 3 & \infty & 0 & \infty & 0 \\ 4 & 15 & 3 & 12 & \infty & \infty & 3 \\ 5 & \infty & 0 & 0 & 12 & \infty & 0 \\ \hline & 0 & 0 & 0 & 0 & & \end{array} \xrightarrow{\text{reduce}} \begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \infty & \infty & \infty & \infty & \infty \\ 2 & 10 & \infty & 9 & 0 & \infty \\ 3 & 0 & 3 & \infty & 0 & \infty \\ 4 & 12 & 0 & 10 & \infty & \infty \\ 5 & \infty & 0 & 0 & 12 & \infty \end{array}$$

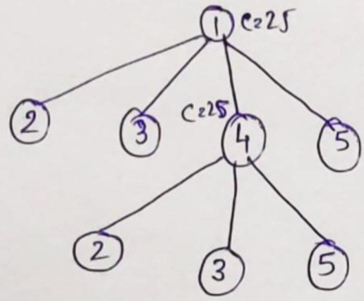
cost = 3+2
= 5

Cost of reduction $= C(1,5) + r + \hat{r}$
$= 1 + 25 + 5 = 31.$

From the computations, -



Now, the path '1-4' bears minimum cost, thus we will explore the options from the node 4.



$\Rightarrow$ we will be using the matrix (reduced) between node 1-4. (we obtained at page 4) As the base matrix.

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & 0 \\ 0 & 3 & \infty & \infty & 2 \\ \infty & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & \infty & \infty \end{bmatrix} \end{array}$$

Path. $\underline{(4-2)}$ we should mark 4'th row & 2'nd column as '∞' & 2-4 as '∞'.

$$\begin{array}{c|c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} & \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & 0 \\ 0 & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & 0 & \infty & \infty \end{bmatrix} & \begin{matrix} \\ 0 \\ 0 \\ \\ 0 \end{matrix} \\ & \begin{matrix} 0 & & 0 & & 0 \end{matrix} & \end{array}$$

It is already reduced $\therefore \hat{r} = 0.$

Cost of reduction

$= C(4,2) + r + \hat{r}$

$= 3 + 25 + 0$

$= 28.$

path (4-3) we should Mark 4'th row & 3'rd column infinite & position (3,4) as infinite.

$$\begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & \infty & \infty & \infty & \infty & \infty & \\ 2 & 12 & \infty & \infty & \infty & 0 & 0 \\ 3 & 0 & 3 & \infty & \infty & 2 & 0 \\ 4 & \infty & \infty & \infty & \infty & \infty & \\ 5 & 11 & 0 & \infty & \infty & \infty & 0 \\ \hline & 0 & 0 & & & 0 & \end{array}$$

It is already reduced ($\hat{r} = 0$)

$\therefore$ Cost of reduction

$= C(4,3) + r + \hat{r}$

$= 12 + 25 + 0 =$

$= 37$

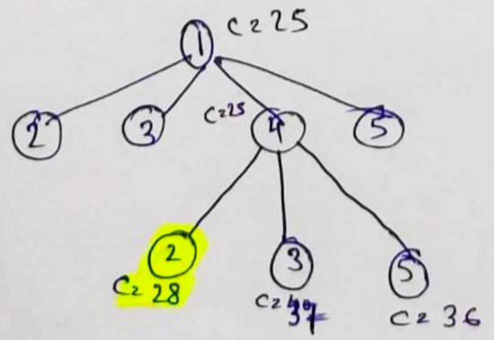
(4-5)

$$\begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & \infty & \infty & \infty & \infty & \infty & \\ 2 & 12 & \infty & 11 & \infty & \infty & 11 \\ 3 & 0 & 3 & \infty & \infty & \infty & 0 \\ 4 & \infty & \infty & \infty & \infty & \infty & \\ 5 & 11 & 0 & 0 & \infty & \infty & 0 \\ \hline & 0 & 0 & 0 & & & 11 \end{array} \xRightarrow{\text{reduce}} \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \infty & \infty & \infty & \infty & \infty \\ 2 & 1 & \infty & 0 & \infty & \infty \\ 3 & 0 & 3 & \infty & \infty & \infty \\ 4 & \infty & \infty & \infty & \infty & \infty \\ 5 & 11 & 0 & 0 & \infty & \infty \end{array}$$

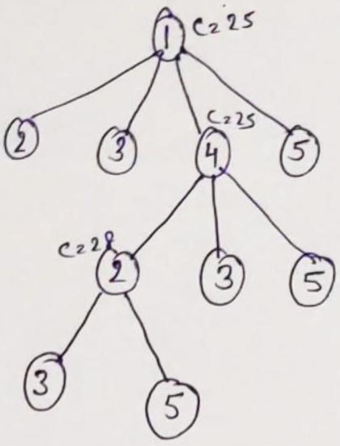
Cost of reduction $\Rightarrow \hat{r} = 11$.

$\therefore$ cost of reduction $= C(4,5) + r + \hat{r}$

$= 0 + 25 + 11 = 36$.



Now the path '4-2' provides the minimum cost. So we will explore the options from node 2.



⇒ we will be using the matrix (reduced) between node 4-2 (obtained at page 5) as the base matrix.

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \infty & \infty & \infty & \infty & \infty \\ 2 & 12 & \infty & 11 & \infty & 0 \\ 3 & 0 & \infty & \infty & \infty & 2 \\ 4 & \infty & \infty & \infty & \infty & \infty \\ 5 & 11 & \infty & 0 & \infty & \infty \end{array}$$

Path (2-3) we will mark 2nd row & 3rd column as '∞' & (3, 2) as '∞'.

$$\begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & \infty & \infty & \infty & \infty & \infty & \\ 2 & \infty & \infty & \infty & \infty & \infty & \\ 3 & 0 & \infty & \infty & \infty & 2 & 0 \\ 4 & \infty & \infty & \infty & \infty & \infty & \\ 5 & 11 & \infty & \infty & \infty & \infty & 11 \\ \hline & 0 & & & & 2 & \end{array} \xrightarrow{\text{reduce}} \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \infty & \infty & \infty & \infty & \infty \\ 2 & \infty & \infty & \infty & \infty & \infty \\ 3 & 0 & \infty & \infty & \infty & 0 \\ 4 & \infty & \infty & \infty & \infty & \infty \\ 5 & 0 & \infty & \infty & & \infty \end{array}$$

reduction cost ⇒ $\hat{r} = 11 + 2 = 13$.

Cost of reduction $= c(2,3) + r + \hat{r}$

$= 11 + 28 + 13 = 52.$

Path (2-5)

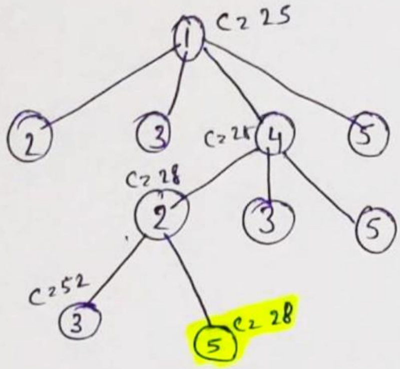
$$\begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & \infty & \infty & \infty & \infty & \infty & \\ 2 & \infty & \infty & \infty & \infty & \infty & \\ 3 & 0 & \infty & \infty & \infty & \infty & 0 \\ 4 & \infty & \infty & \infty & \infty & \infty & \\ 5 & 11 & \infty & 0 & \infty & \infty & 0 \\ \hline & 0 & & 0 & & & \end{array}$$

It is already reduced $\therefore \hat{r} = 0$.

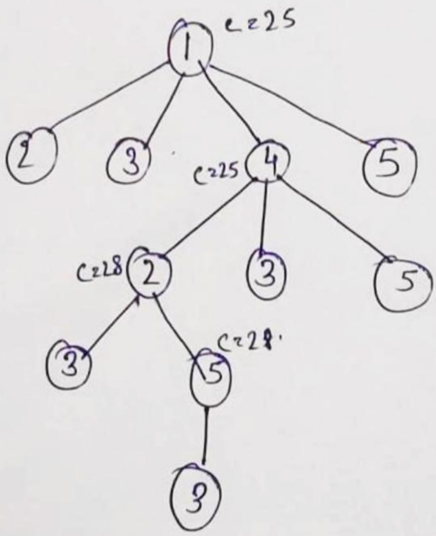
Cost of reduction $= c(2,5) + r + \hat{r}$

$= 0 + 28 + 0$

$= 28.$



Now the path 2-5 provides the minimum cost. So we will explore the options from node 5.



⇒ We will be using the matrix (reduced) between node 2-5 (we obtained at page 7) as the base matrix.

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \alpha & \alpha & \alpha & \alpha & \alpha \\ 2 & \alpha & \alpha & \alpha & \alpha & \alpha \\ 3 & 0 & \alpha & \alpha & \alpha & \alpha \\ 4 & \alpha & \alpha & \alpha & \alpha & \alpha \\ 5 & 11 & \alpha & 0 & \alpha & \alpha \end{array}$$

Path (5-3) we will mark 5th row & 3rd column as 'α' & (3,5) as 'α'.

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \alpha & \alpha & \alpha & \alpha & \alpha \\ 2 & \alpha & \alpha & \alpha & \alpha & \alpha \\ 3 & 0 & \alpha & \alpha & \alpha & \alpha \\ 4 & \alpha & \alpha & \alpha & \alpha & \alpha \\ 5 & \alpha & \alpha & \alpha & \alpha & \alpha \\ \hline & 0 & & & & \end{array} \cdot$$

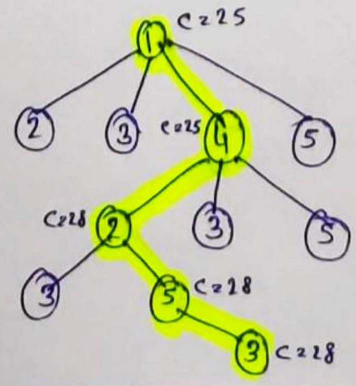
It is already reduced $\therefore \hat{r} = 0$

⇒ cost of reduction $= c(5,3) + r + \hat{r}$

$= 0 + 28 + 0$

$= 28$.

∴ Final path:



Travelling cost = 28.

Algorithm:

Step1: Create the adjacency matrix for the given graph.

Step 2: reduce the adjacency matrix

- if the minimum no. in each row is non zero.
  ~~remove min_r~~
  - Subtract min_r from each element in the row
- if the minimum no. in each colum is not zero
  - Subtract min_c from each element in the colum.
- reduction cost = $\sum_{r=1}^{\text{no. of row}} min_r + \sum_{c=1}^{\text{no. of col}^n} min_c = r_{adj}$

Step 3: – Explore all the options from the starting node

- cost of reduction at root = reduction cost of the adjacency matrix = r_{adj}.
- if we are exploring node V from node 'U'
  - mark U row & V colum as '∞' in the reduced adj matrix or base matrix for the subsequent steps.
  - mark position (V,U) as '∞'
  - reduce the matrix using step2.
- calculate reduction cost = $\hat{r}_n$
- calculate total cost of reduction $= C(U,V) + r + \hat{r}$

Step 4: Select the least reduction cost node as the base node & repeat step 3 until we obtained a solution.