Photometric Stereo (Part 2)

Computer Vision I

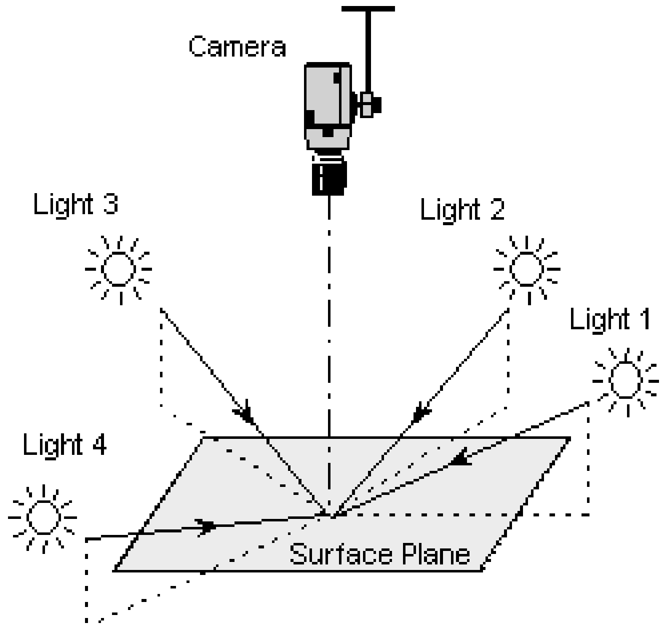
CSE 252A

Lecture 5

Announcements

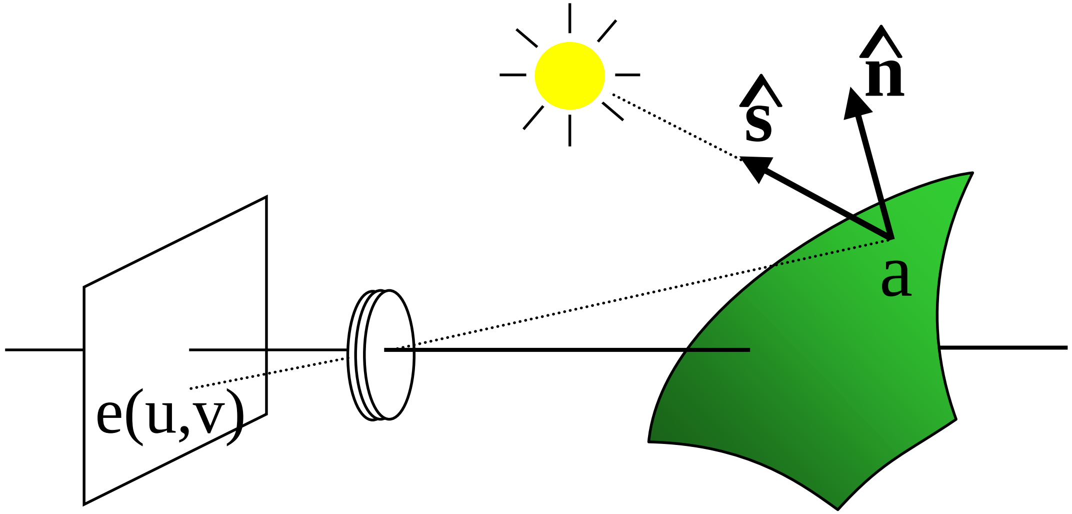
- Homework 2 is due Oct 22, 11:59 PM
- Reading:
  - What Is the Set of Images of an Object under All Possible Illumination Conditions?

Photometric stereo



- Single viewpoint, multiple images under different lighting.
  1. Arbitrary known BRDF, known lighting
  2. Lambertian BRDF, known lighting
  3. Lambertian BRDF, unknown lighting.

Lambertian Surface



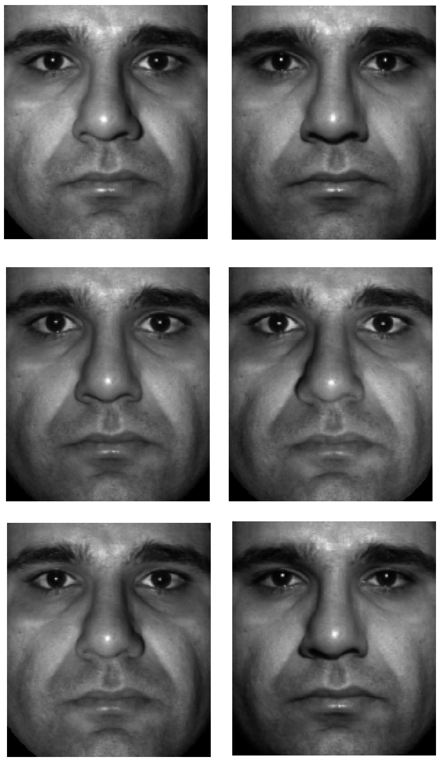
At image location (u,v), the intensity of a pixel x(u,v) is:

$$e(u,v) = [a(u,v)\,\hat{\mathbf{n}}(u,v)] \cdot [s_0 \hat{\mathbf{s}}]$$
$$= \mathbf{b}(u,v) \cdot \mathbf{s}$$

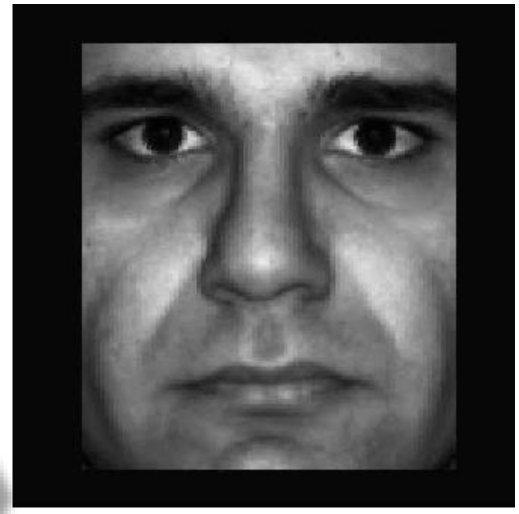
where

- a(u,v) is the albedo of the surface projecting to (u,v).
- $\hat{\mathbf{n}}(u,v)$ is the direction of the surface normal.
- s_0 is the light source intensity.
- $\hat{\mathbf{s}}$ is the direction to the light source.

Lambertian Photometric Stereo



Images with known associated light sources



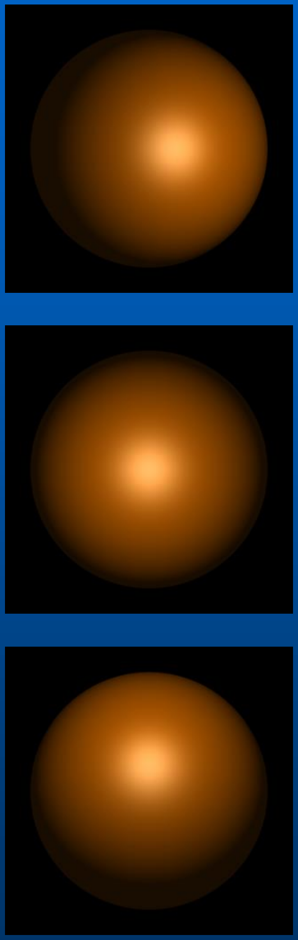
Albedo

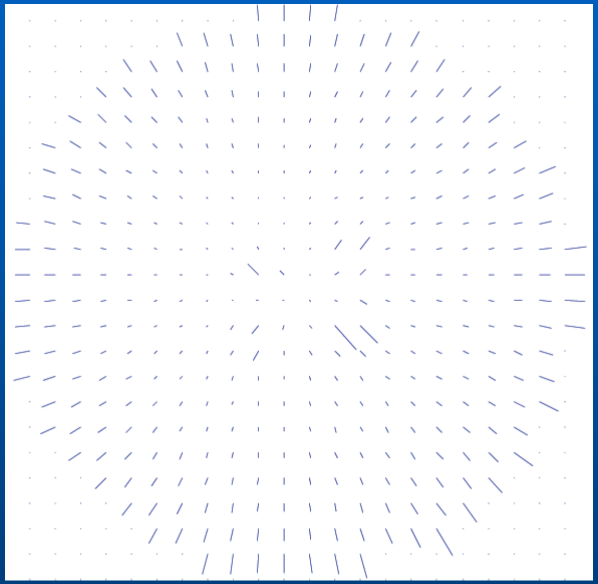
Surface (from normals)

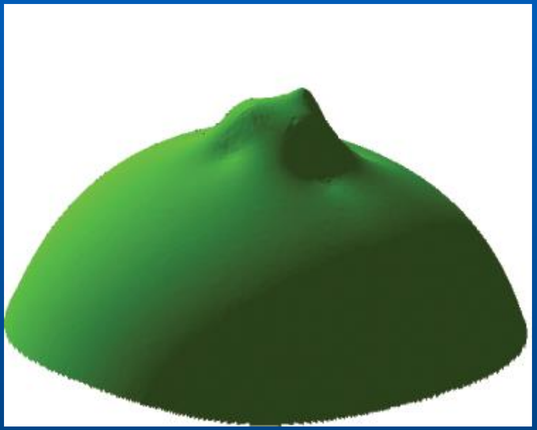
Surface (albedo texture map)

SUV Color Space for Photometric Stereo of Glossy Objects

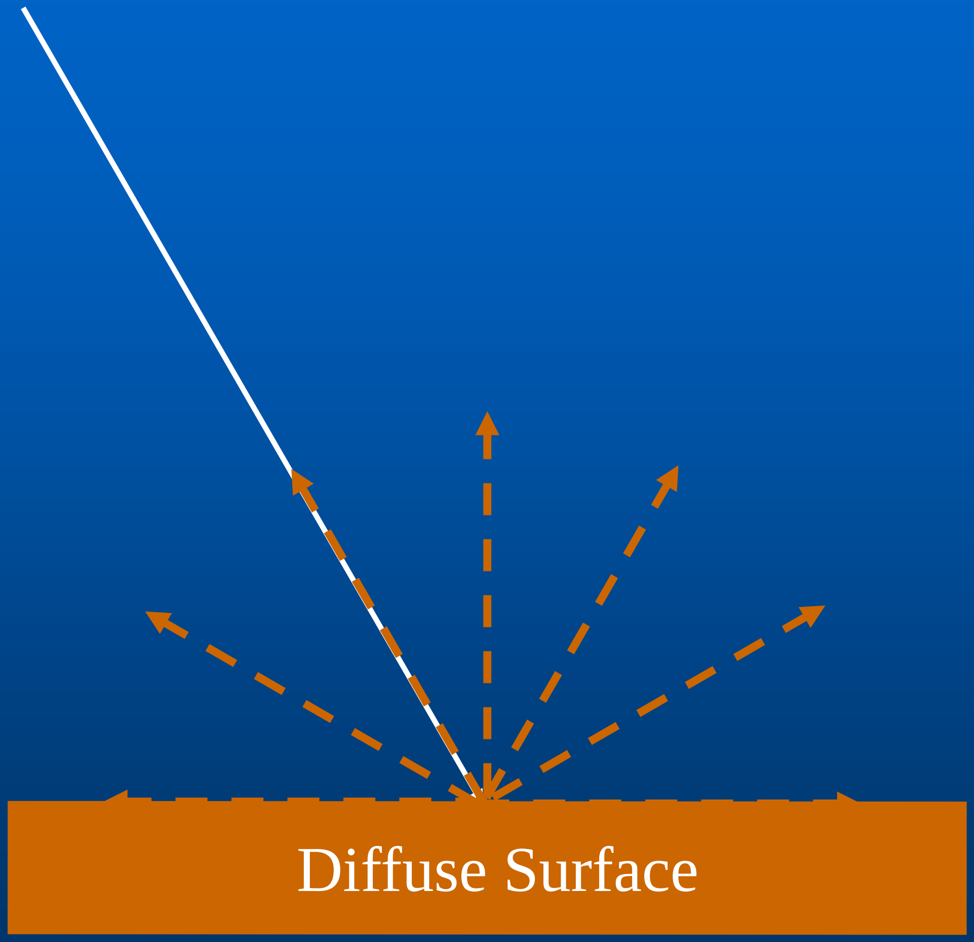
Motivation: Lambertian Algorithm Applied to non-Lambertian Surface: Photometric Stereo

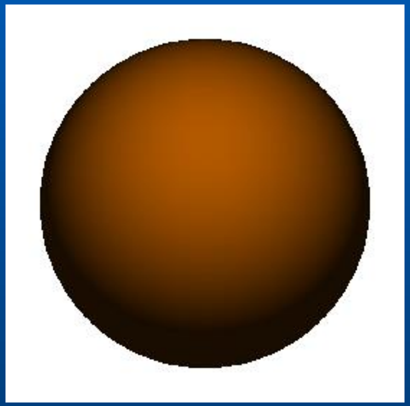






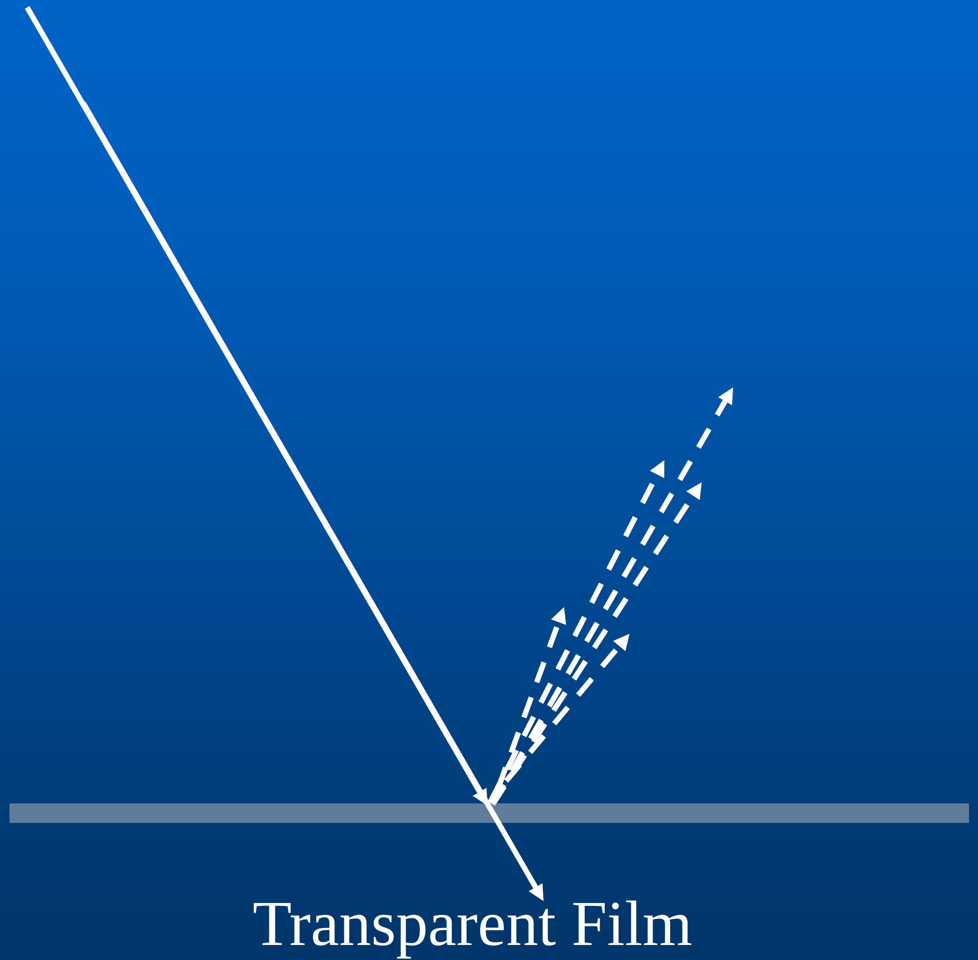
Dichromatic Reflection Model

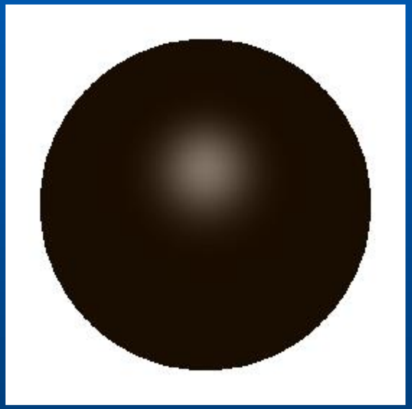




Color depends on light source color and diffuse color

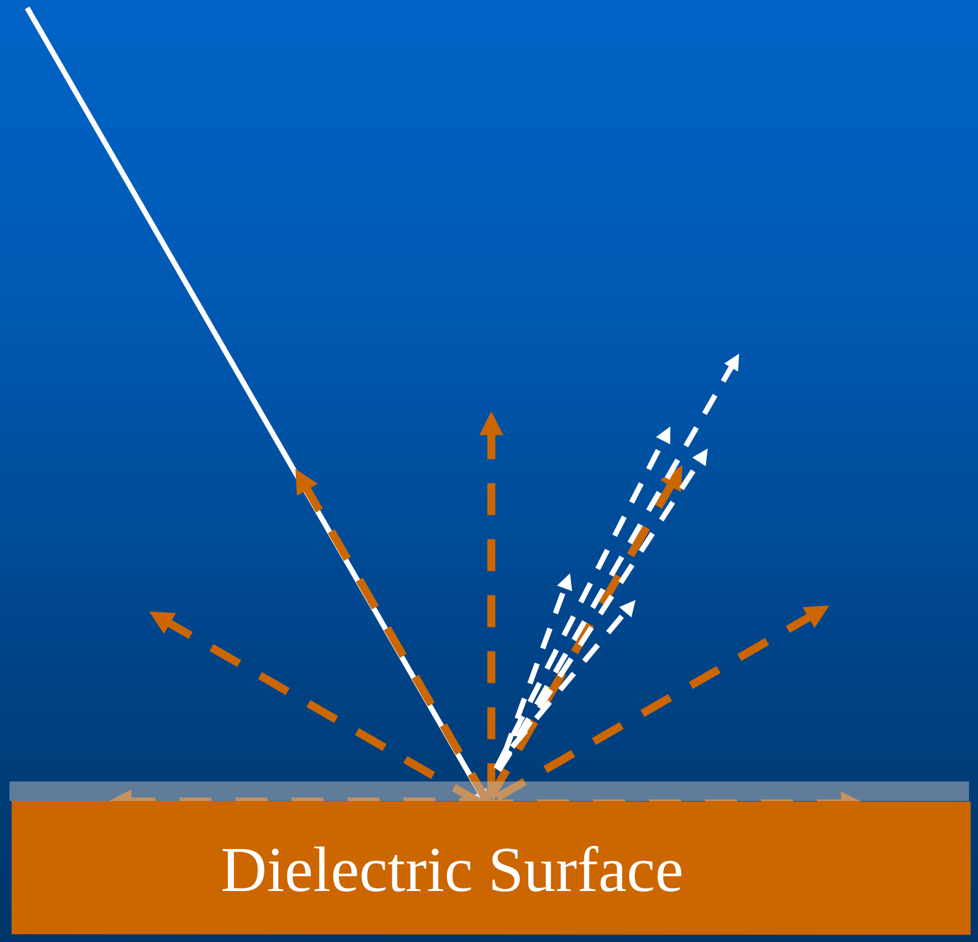
Dichromatic Reflection Model





Color of light source

Dichromatic Reflection Model



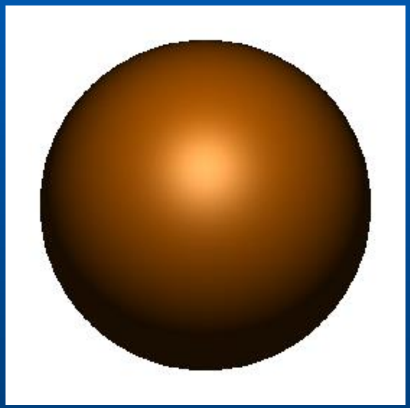


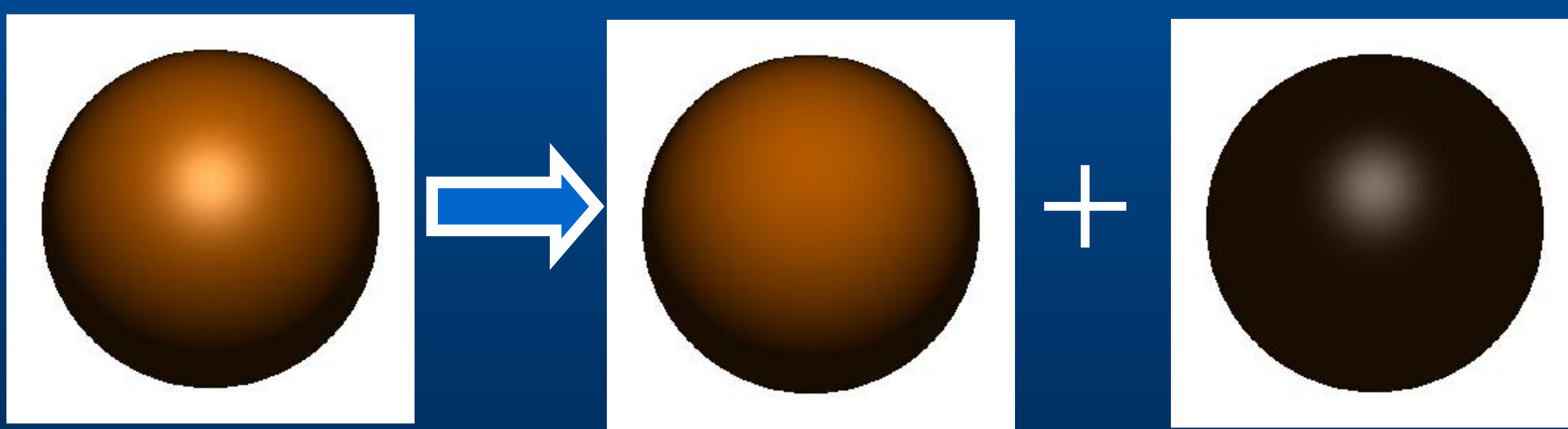
Image formation: Color Channel *k*

$$
\begin{aligned}
L(\lambda) &= \text{Spectral Power Distribution of light source} \\
C_k(\lambda) &= \text{Camera Sensitivity} \\
S_k &= \int C_k(\lambda) L(\lambda) d\lambda. \\
D_k &= \int C_k(\lambda) L(\lambda) g_d(\lambda) d\lambda
\end{aligned}
$$

Specular Color

Diffuse Color

$$I_k = (D_k f_d + S_k f_s(\theta))\, \hat{\mathbf{n}} \cdot \hat{\mathbf{l}},$$



Where f_d and f_s are the diffuse and specular BRDF

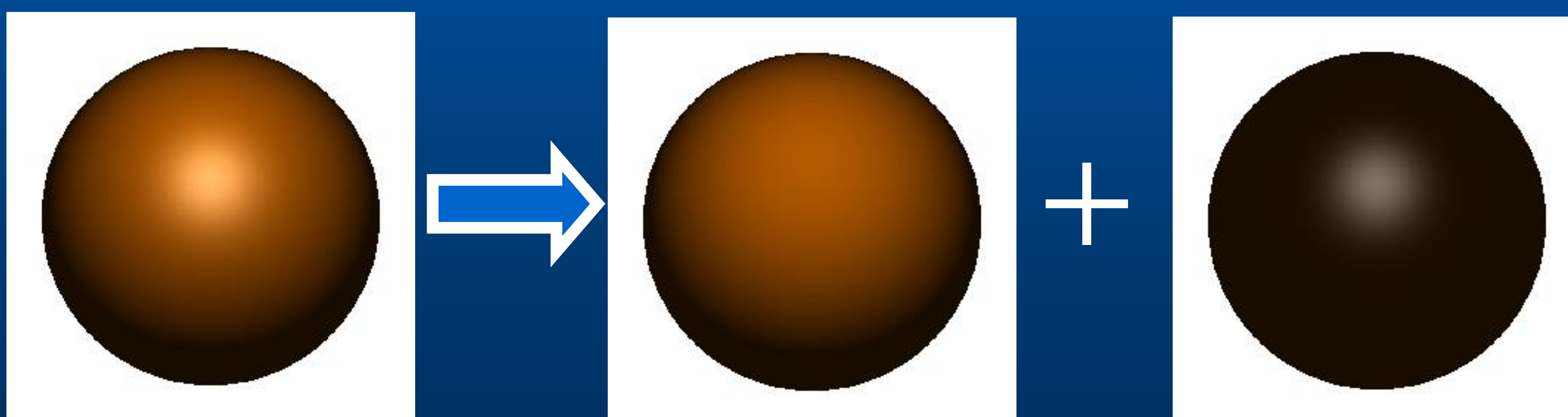
Image formation: 3 color channels

$$I_k = (D_k f_d + S_k f_s(\boldsymbol{\theta}))\, \hat{\mathbf{n}} \cdot \hat{\mathbf{l}},$$

$$\begin{bmatrix} I_r \\ I_g \\ I_b \end{bmatrix} = \left(f_d \hat{\mathbf{n}} \cdot \hat{l}\right) \begin{bmatrix} D_r \\ D_g \\ D_b \end{bmatrix} + \left(f_s(q)\hat{\mathbf{n}} \cdot \hat{l}\right) \begin{bmatrix} S_r \\ S_g \\ S_b \end{bmatrix}$$

Image color lies in span of diffuse color ***D*** and specular color ***S***

$$\mathbf{I} = (f_d \hat{n} \times \hat{l})\mathbf{D} + (f_s(q)\hat{n} \times \hat{l})\mathbf{S}$$

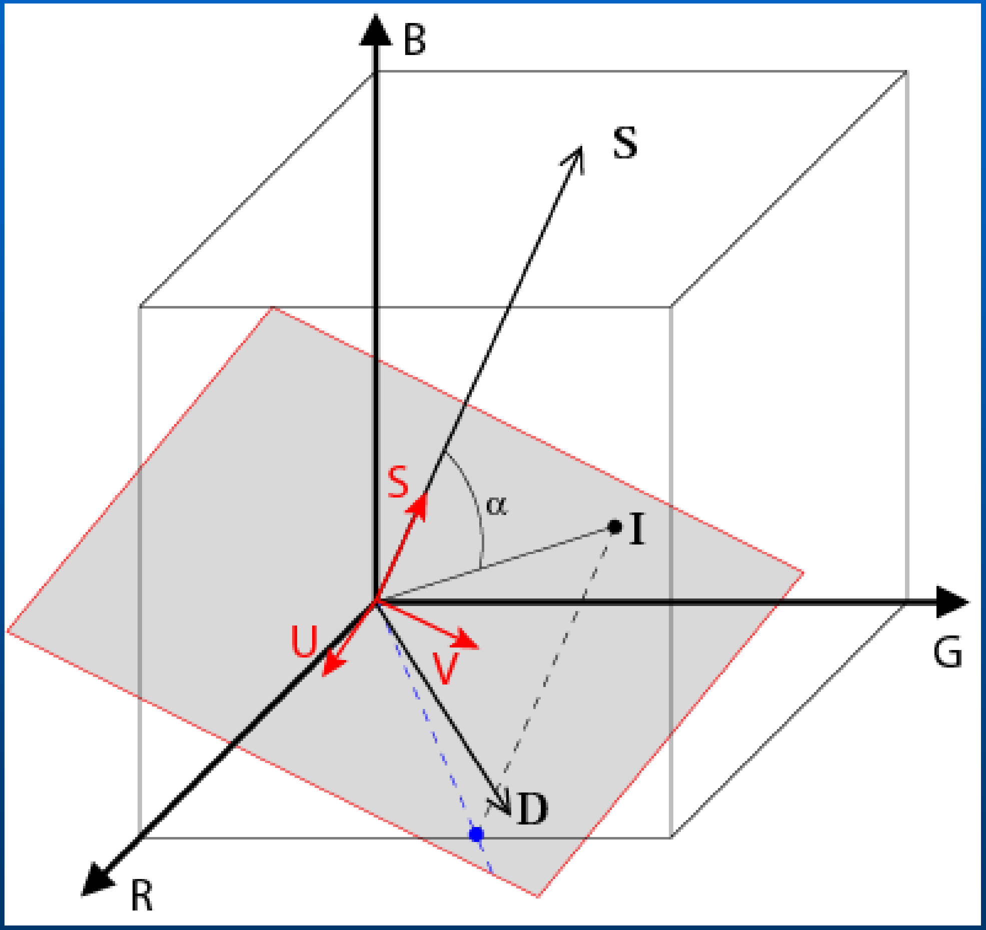


Varying diffuse color

Note:

- Diffuse color D varies over the image
- Specular color is just color of light source

Data-dependent SUV Color Space



$$\mathbf{I}_{SUV} = [R]\mathbf{I}_{RGB}$$

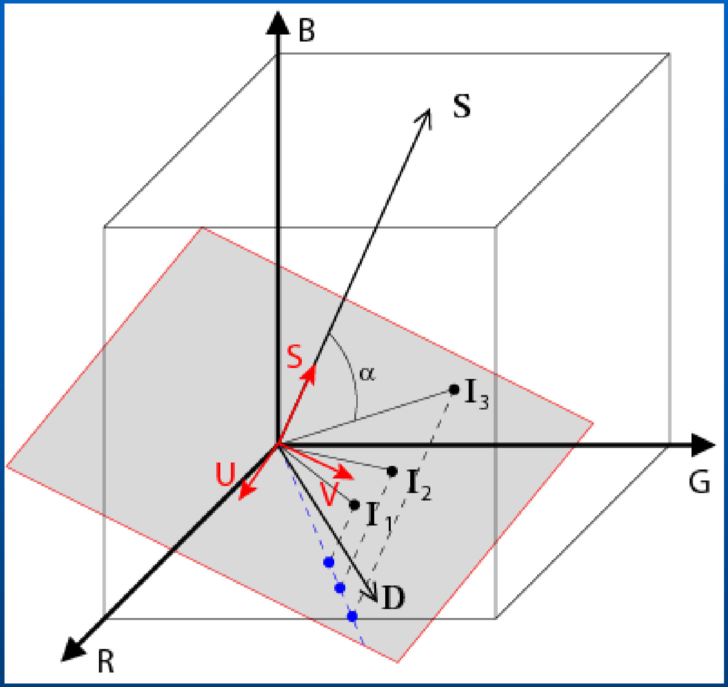
$$[R] = [\ \mathbf{S} \mid \mathbf{U} \mid \mathbf{V}]^t$$

$$[R] \in SO(3)$$

First row of R is specular color S. Other rows are orthogonal to S

U,V spans a plane orthogonal to **S**

Properties of SUV



- Data-dependent.
- Rotational (hence, linear) Transformation.
- The S channel encodes the entire specular component and an unknown amount of diffuse component.
- Shading information is preserved in u and v channels.

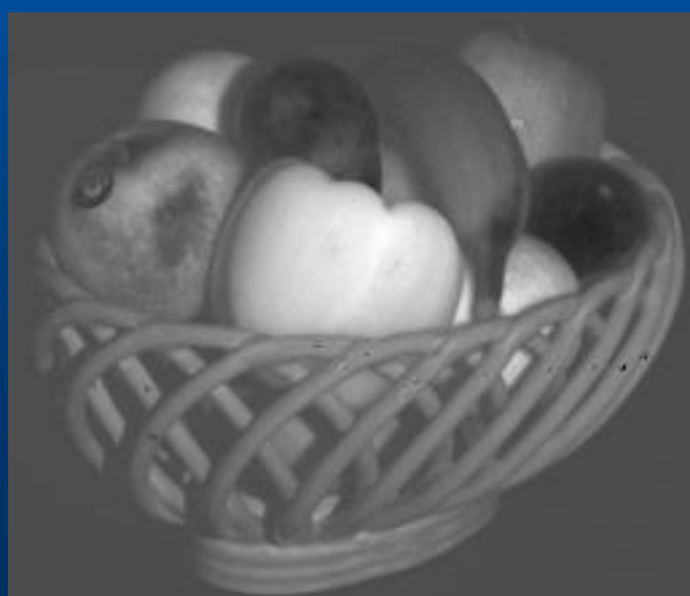
$$I_U = \mathbf{r}_2^\top \mathbf{D} f_d \hat{\mathbf{n}} \cdot \hat{\mathbf{l}}$$
$$I_V = \mathbf{r}_3^\top \mathbf{D} f_d \hat{\mathbf{n}} \cdot \hat{\mathbf{l}}.$$

Example

RGB

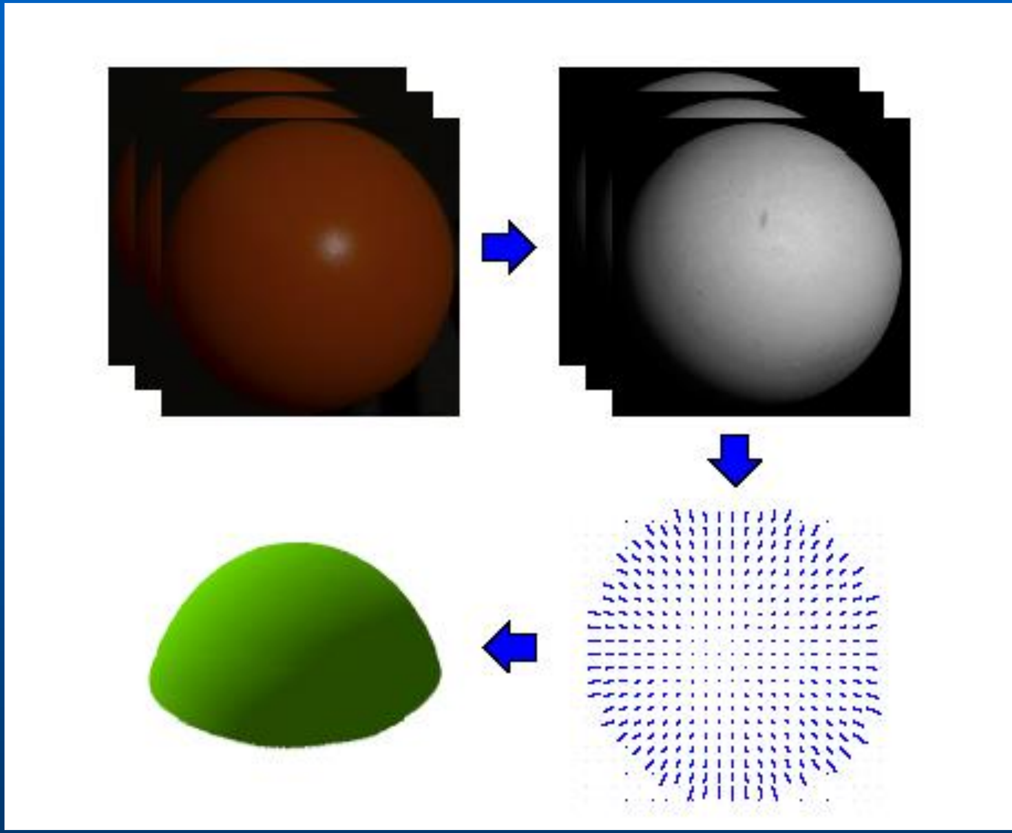
S channel

U channel



V channel

Multi-channel Photometric Stereo



Multi-channel Photometric Stereo

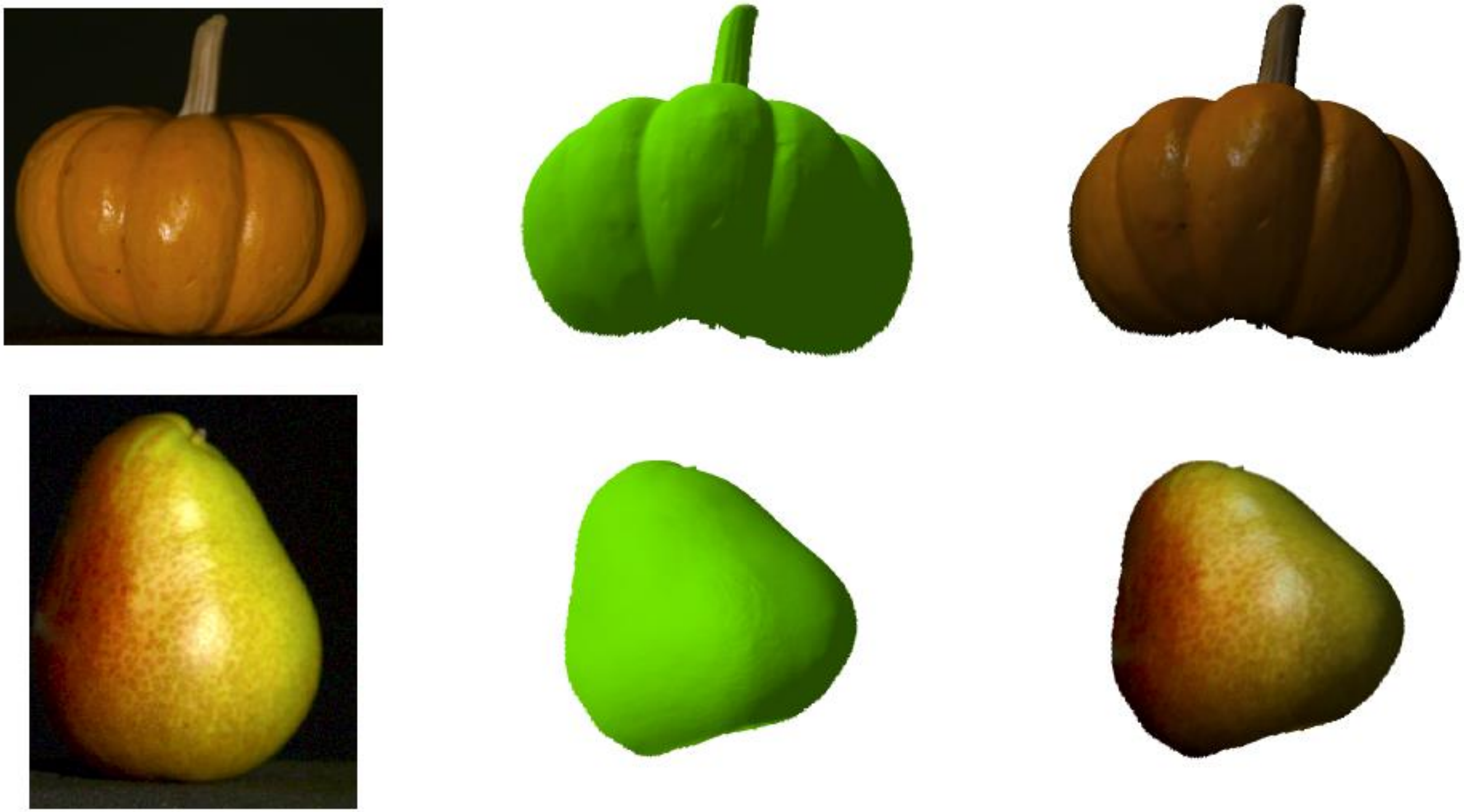
$\mathbf{J} = [I_U \quad I_V]^\top$
$\mathbf{J}^k$: 2-channel color vector under the k^{th} light source.
$\hat{\mathbf{l}}^k$: The k^{th}three light source directions.
$\boldsymbol{\rho}$: 2-channel UV albedo.

$$\mathbf{J}^k = \left[I_U^k, I_V^k\right]^\top = (\hat{\mathbf{n}} \cdot \hat{\mathbf{l}}^k)\boldsymbol{\rho},$$

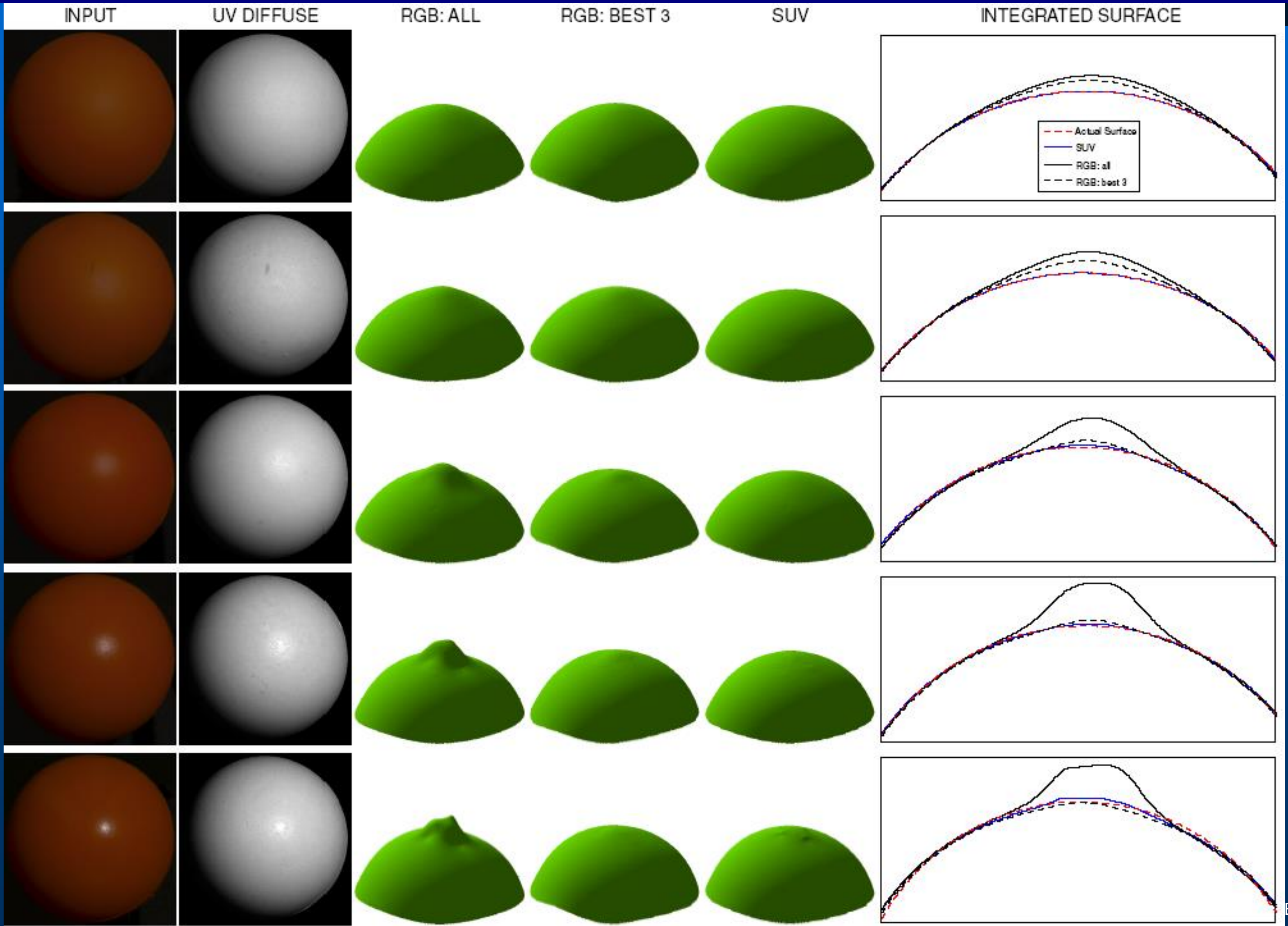
$$\begin{array}{llll}
\text{Shading vector} & =\mathbf{F} & = \left[f^1, f^2, f^3\right]^\top & =[\hat{\mathbf{l}}^1 \ \hat{\mathbf{l}}^2 \ \hat{\mathbf{l}}^3]^\top \hat{\mathbf{n}} \\
\text{Intensity matrix} & =[J] & = \begin{bmatrix} J_1^1 & J_2^1 \\ J_1^2 & J_2^2 \\ J_1^3 & J_2^3 \end{bmatrix} & = \begin{bmatrix} f^1\rho_U & f^1\rho_V \\ f^2\rho_U & f^2\rho_V \\ f^3\rho_U & f^3\rho_V \end{bmatrix} \quad =\mathbf{F}\boldsymbol{\rho}^\top.
\end{array}$$

The least squares estimate of the shading vector $\mathbf{F}$ is the principal eigenvector of $[J][J]^\top$. Once the shading vector is known, the surface normal is found by solving the matrix equation $\mathbf{F} = [\hat{\mathbf{l}}^1 \ \hat{\mathbf{l}}^2 \ \hat{\mathbf{l}}^3]^\top \hat{\mathbf{n}}$.

Qualitative Results



Quantitative Results

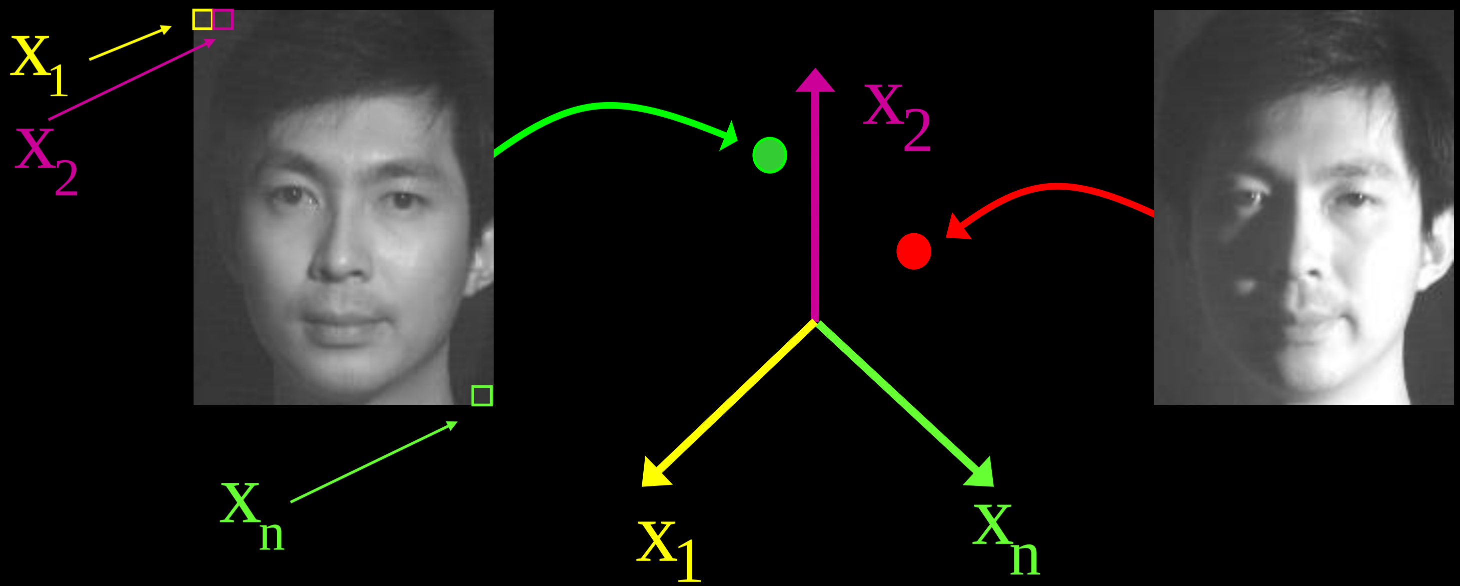


3. Photometric Stereo with unknown lighting and Lambertian surfaces

What is the set of images of an object under all possible lighting conditions?

In answering this question, we'll arrive at a photometric stereo method for reconstructing surface shape with unknown lighting.

The Space of Images



- Consider an n-pixel image to be a point in an n-dimensional space, $\mathbf{x} \in \mathbf{R}^n$.
- Each pixel value is a coordinate of $\mathbf{x}$.

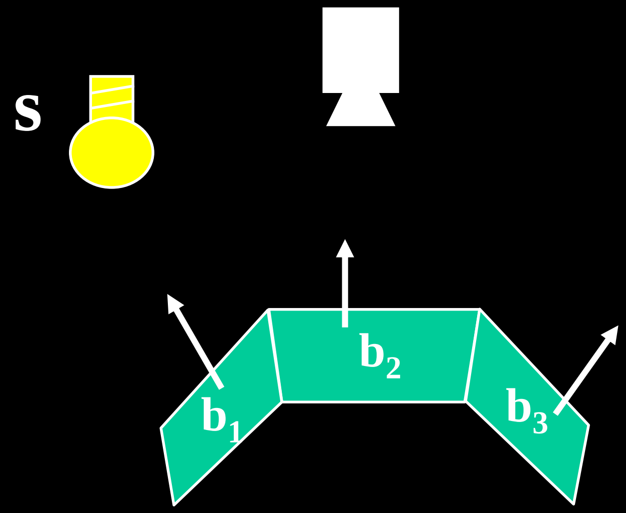
Assumptions

For discussion, we assume:

- Lambertian reflectance functions.
- Objects have convex shape.
- Light sources at infinity.
- Orthographic projection.

- Note: many of these can be relaxed….

Model for Image Formation



Lambertian Assumption with shadowing:

$$\mathbf{x} = \max(\mathbf{B}\,\mathbf{s}, 0) \qquad \mathbf{B} = \begin{bmatrix} \mathbf{b}^T_1 \\ \mathbf{b}^T_2 \\ \vdots \\ \mathbf{b}^T_n \end{bmatrix}_{n \times 3}$$

where

- **x** is an *n*-pixel image vector
- **B** is a matrix whose rows are unit normals scaled by the albedos
- $\mathbf{s} \in \mathbf{R}^3$ is vector of the light source direction scaled by intensity

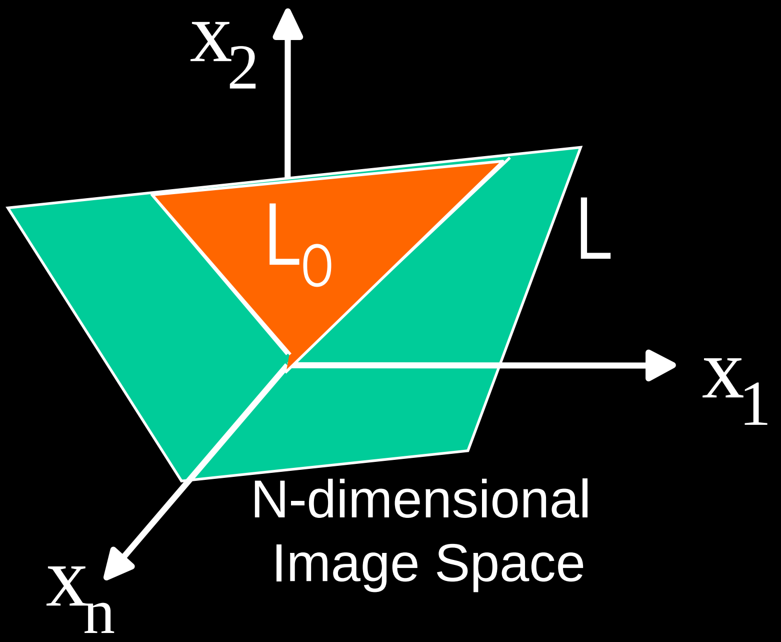
3-D Linear subspace

The set of images of a Lambertian surface with no shadowing is a subset of 3-D linear subspace.

[Moses 93], [Nayar, Murase 96], [Shashua 97]

$$\mathcal{L} = \{\mathbf{x} \mid \mathbf{x} = \mathrm{B}\mathbf{s},\ \forall \mathbf{s} \in \mathrm{R}^3\}$$

where B is a *n* by 3 matrix whose rows are product of the surface normal and Lambertian albedo



Still Life

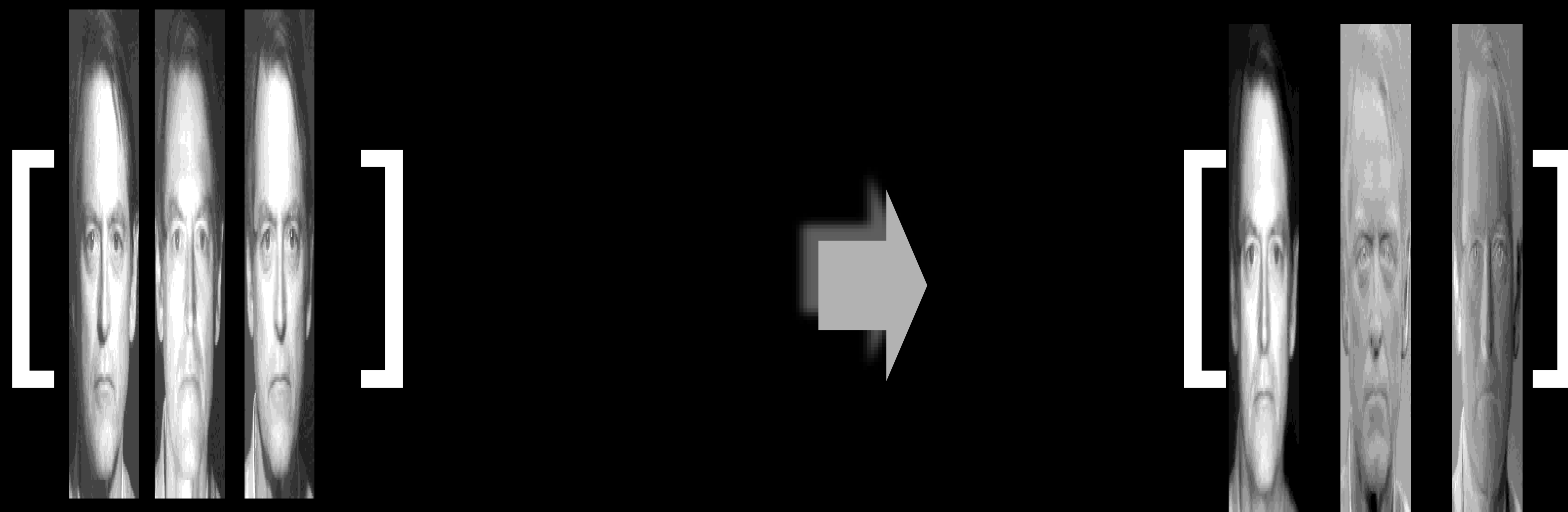
Original Images

Basis Images

Rendering Images: $\Sigma_i \max(\mathbf{B}\mathbf{s}_i, \mathbf{0})$

1 Light | **2 Lights** | **3 Lights**

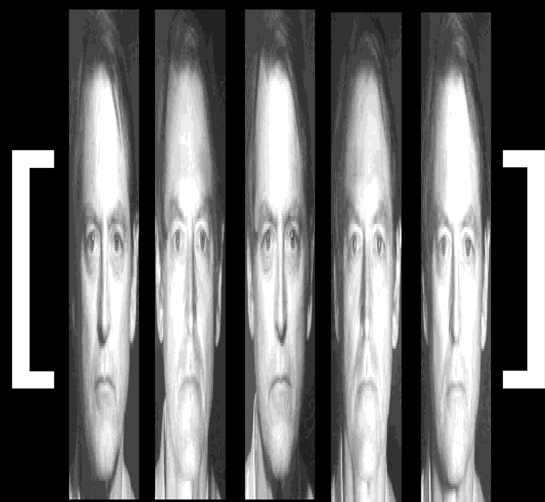
How do you construct subspace?



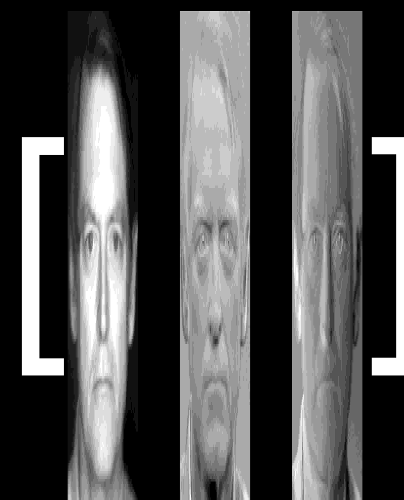
[$\mathbf{x}_1$ $\mathbf{x}_2$ $\mathbf{x}_3$] **B**

- Any three images without shadows taken under different lighting span L
- Not orthogonal
- Orthogonalize with Grahm-Schmidt

How do you construct subspace?



$[\ \mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3\ \mathbf{x}_4\ \mathbf{x}_5\ ]$



B

With more than three images, perform least squares estimation of **B** using Singular Value Decomposition (SVD)

Singular Value Decomposition

Excellent ref: "Matrix Computations," Golub and Van Loan

- Any m by n matrix $\boldsymbol{A}$ may be factored such that

$$\boldsymbol{A} = \boldsymbol{U\Sigma V}^T$$

$$[m \times n] = [m \times m][m \times n][n \times n]$$

- $\boldsymbol{U}$: m by m, orthogonal matrix
  - Columns of **U** are the eigenvectors of $\boldsymbol{AA}^T$
- $\boldsymbol{V}$: n by n, orthogonal matrix,
  - columns are the eigenvectors of $\boldsymbol{A}^T\boldsymbol{A}$
- $\boldsymbol{\Sigma}$**:** m by n, diagonal with non-negative entries (σ_1, σ_2, …, σ_s) with s=min(m,n) are called the called the singular values
  - Singular values are the square roots of eigenvalues of both $\boldsymbol{AA}^T$ and $\boldsymbol{A}^T\boldsymbol{A}$
  - *Result of SVD algorithm***:** $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_s$

SVD Properties

- In MATLAB: [U S V] = svd(A), and you can verify that A=U*S*V'
- r = rank(***A***) is the number of nonzero singular values.
- ***U***, ***V*** give us orthonormal bases for the subspaces of ***A***:
  - 1st *r* columns of ***U***: Column space of ***A***
  - Last *m* - *r* columns of ***U***: Left nullspace of ***A***
  - 1st *r* columns of ***V***: Row space of ***A***
  - last *n* - *r* columns of ***V***: Nullspace of ***A***
- *For d<= r, the first d column of U provide the best d-dimensional basis for columns of A in least squares sense.*

SVD, economy decomposition

- For an m by n matrix $\boldsymbol{A}$, *where $m>=n$,* one can view $\boldsymbol{\Sigma}$ as $\begin{bmatrix} \Sigma' \\ 0 \end{bmatrix}$
- Where $\boldsymbol{\Sigma}'$ =diag($\sigma_1, \sigma_2, \ldots, \sigma_n$) and lower ($m$-$n$) by n submatrix is of zeros.
- Economy decomposition

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$$

$$[m \times n] = [m \times n][n \times n][n \times n]$$

- V is an orthogonal matrix
- Columns of U are orthogonal
- In MATLAB: [U S V] = svd(A,'econ')

Estimating B with SVD

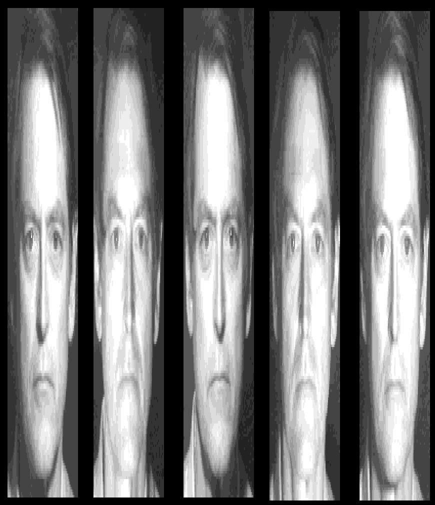
1. Construct data matrix

$$\mathbf{D} = [\, \mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3 \ldots \mathbf{x}_n \,]$$

2. [**U S V**] = svd(D,’econ’)

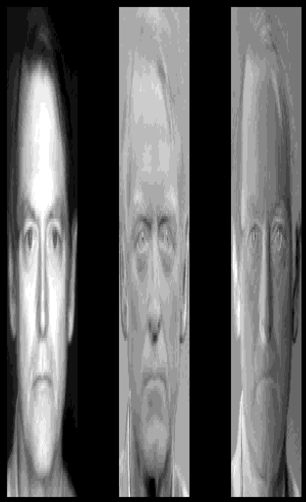
- If data has no noise, then rank(D)=3, and the first three singular values diag(S) would be positive and rest would be zero.
- Take first three columns of **U** as **B.**

D = [



]

B = [



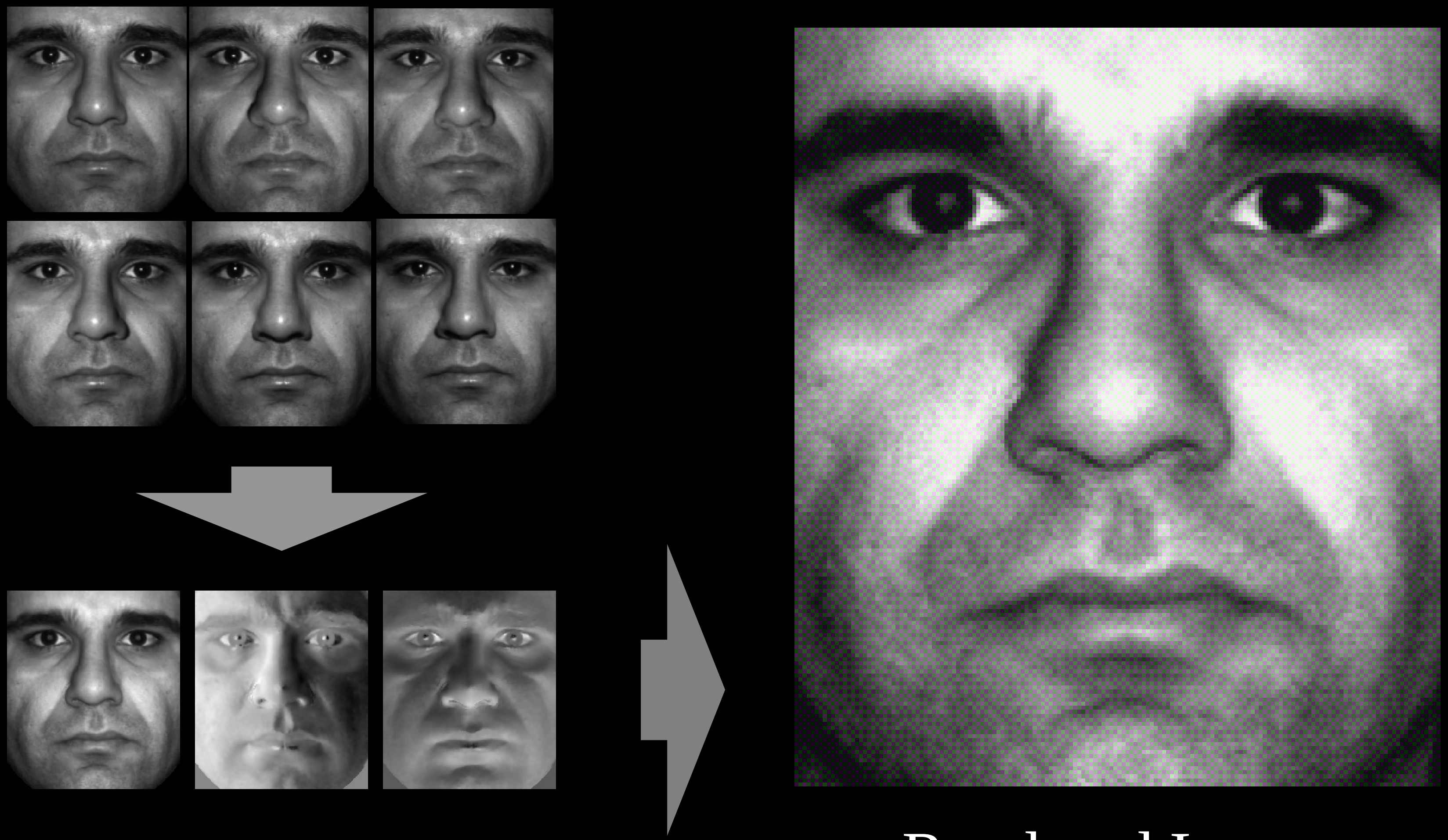
]

Face Basis

Original Images

Basis Images

Render Image from Basis Images



Rendered Image
(Single Light Source)

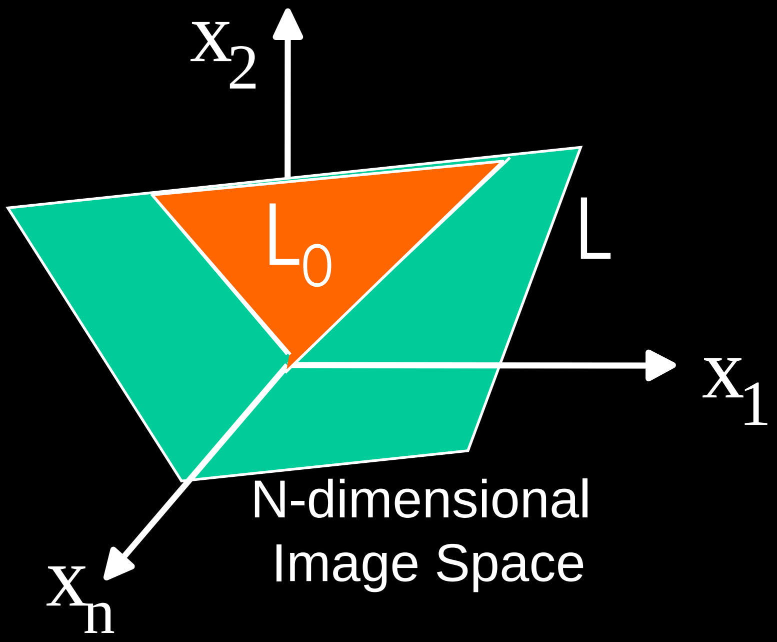
3-D Linear subspace

The set of images of a Lambertian surface with no shadowing is a subset of 3-D linear subspace.

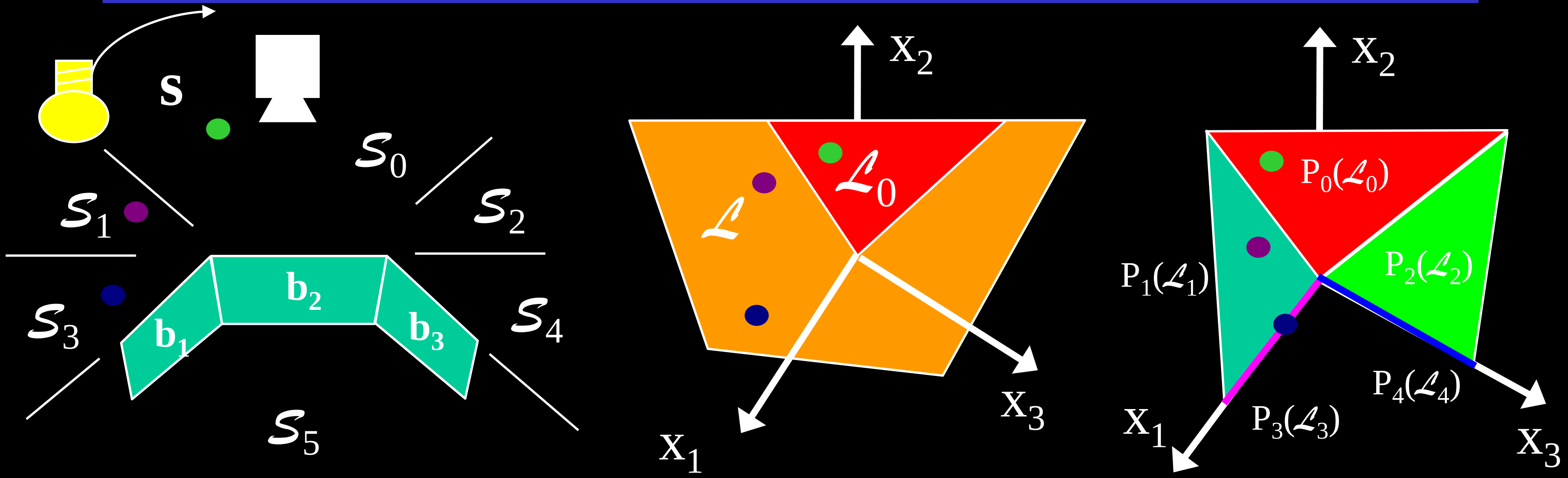
[Moses 93], [Nayar, Murase 96], [Shashua 97]

$$\mathcal{L} = \{\mathbf{x} \mid \mathbf{x} = B\mathbf{s}, \forall \mathbf{s} \in R^3\}$$

where B is a *n* by 3 matrix whose rows are product of the surface normal and Lambertian albedo



Set of Images from a Single Light Source

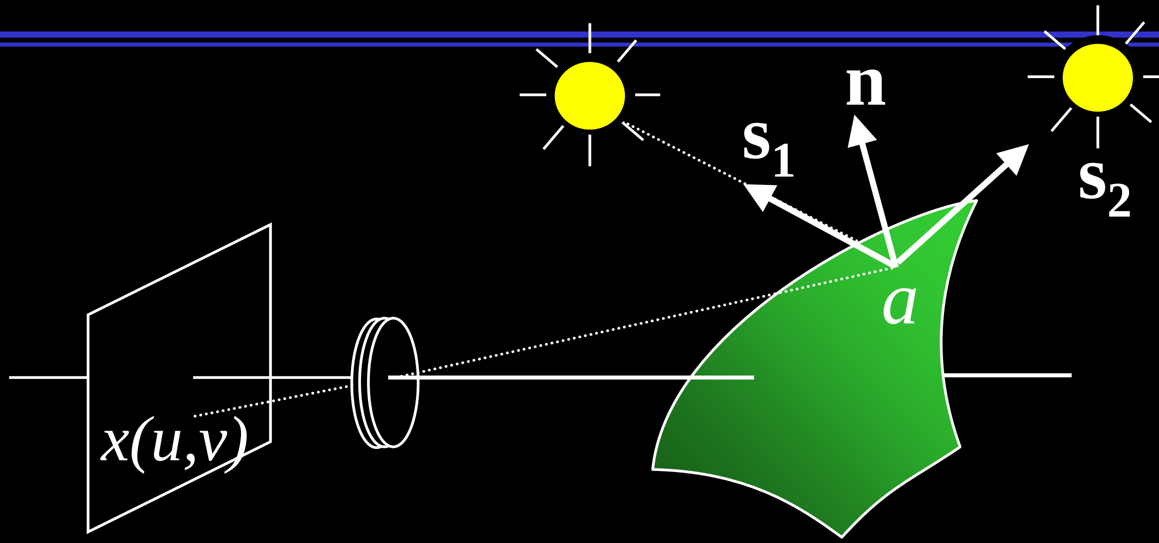


- Let L_i denote the intersection of L with an orthant *i* of R^n.
- Let P_i (L_i) be the projection of L_i onto a "wall" of the positive orthant given by max(**x**, **0**).

Then, the set of images of an object produced by a single light source is:

$$\boxed{\mathcal{U} = \bigcup_{i=0}^{M} P_i\,(\mathcal{L}_i)}$$

Lambertian, Shadows and Multiple Lights



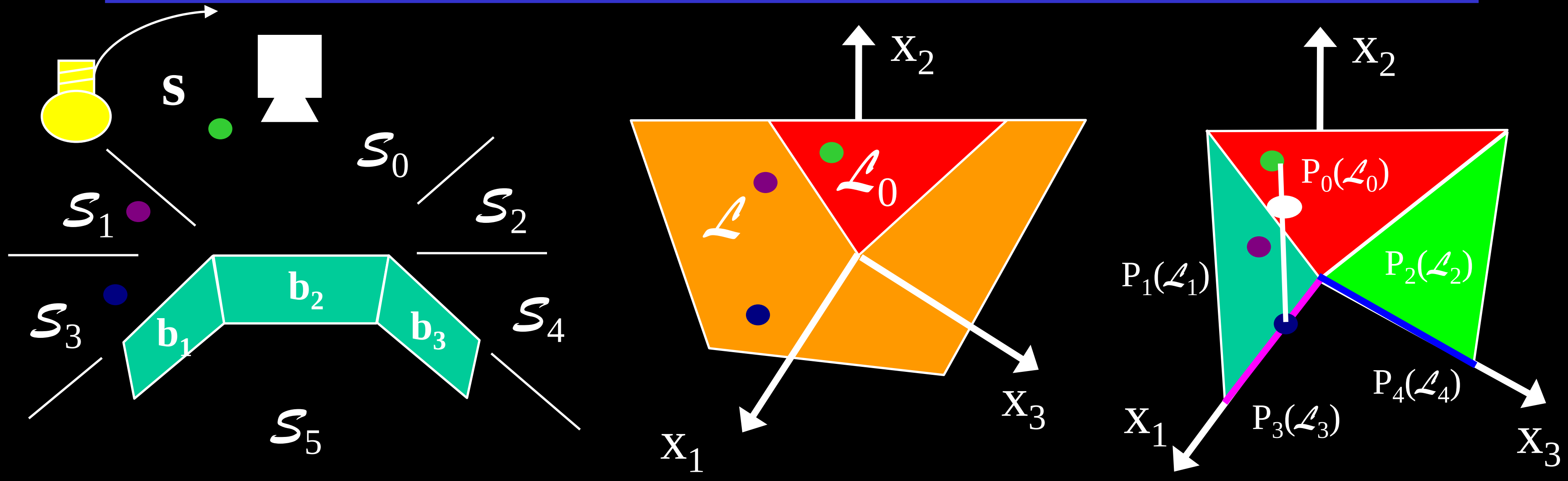
The image **x** produced by multiple light sources is

$$\mathbf{x} = \sum_i \max(\mathbf{B}\,\mathbf{s}_i, 0)$$

where

- **x** is an *n*-pixel image vector.
- **B** is a matrix whose rows are unit normals scaled by the albedo.
- $\mathbf{s}_i$ is the direction and strength of the light source *i*.

Set of Images from Multiple Light Sources

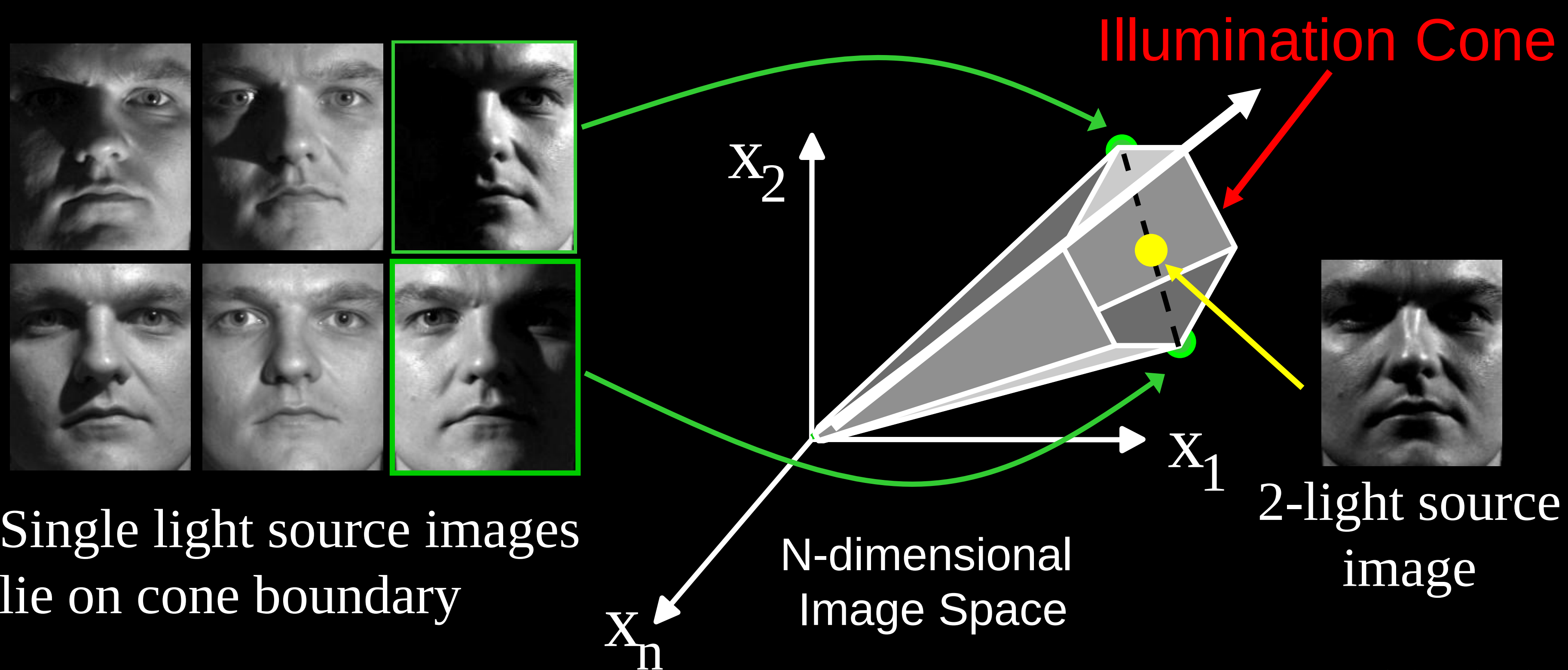


- With two lights on, resulting image along line segment between single source images: superposition of images, non-negative lighting
- For all numbers of sources, and strengths, rest is convex hull of U.

The Illumination Cone

Theorem: *The set of images of any object in fixed posed, but under all lighting conditions, is a* ***convex cone*** *in the image space.*

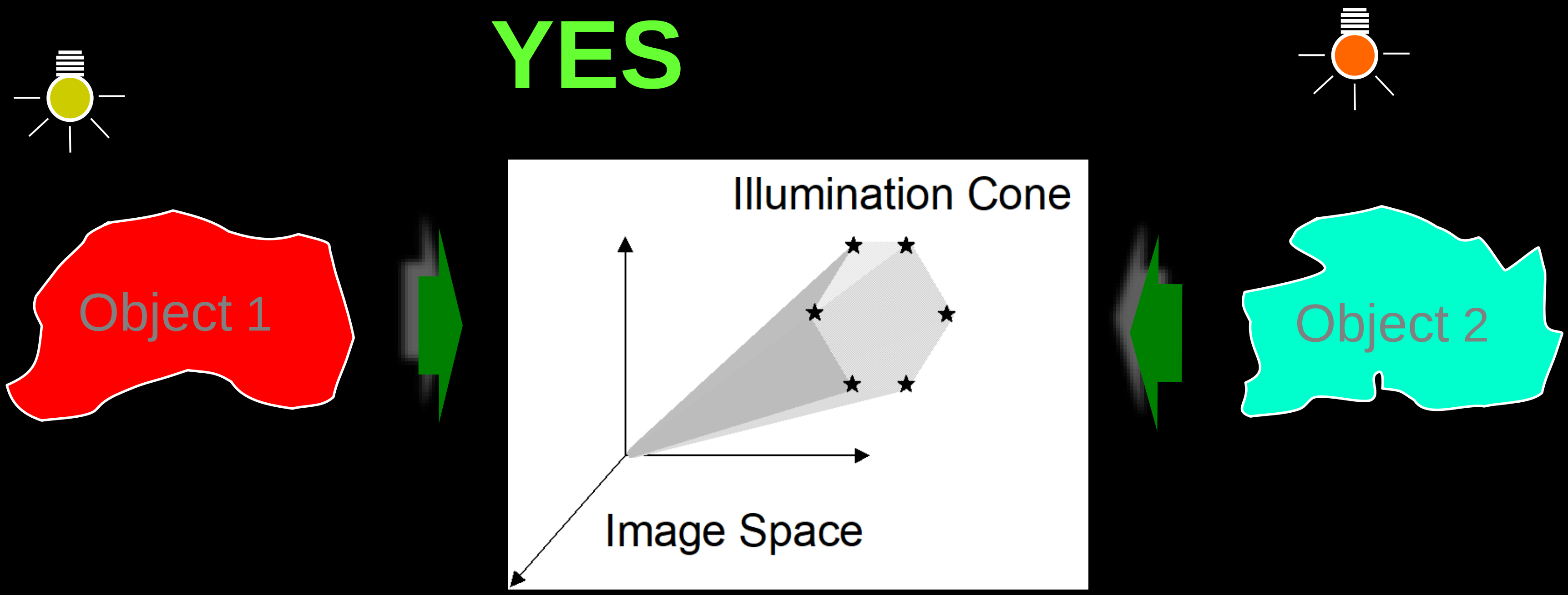
(Belhumeur and Kriegman, IJCV, 98)



Do Ambiguities Exist?

Can two objects of differing shapes produce the same illumination cone?

YES



Do Ambiguities Exist? Yes

- Cone is determined by linear subspace **L**
- The columns of **B** span **L**
- For any **A** $\in$ GL(3), **B* = BA** also spans **L.**
- For any image of **B** produced with light source **S**, the same image can be produced by lighting **B*** with $\mathbf{S^* = A^{-1}S}$ because

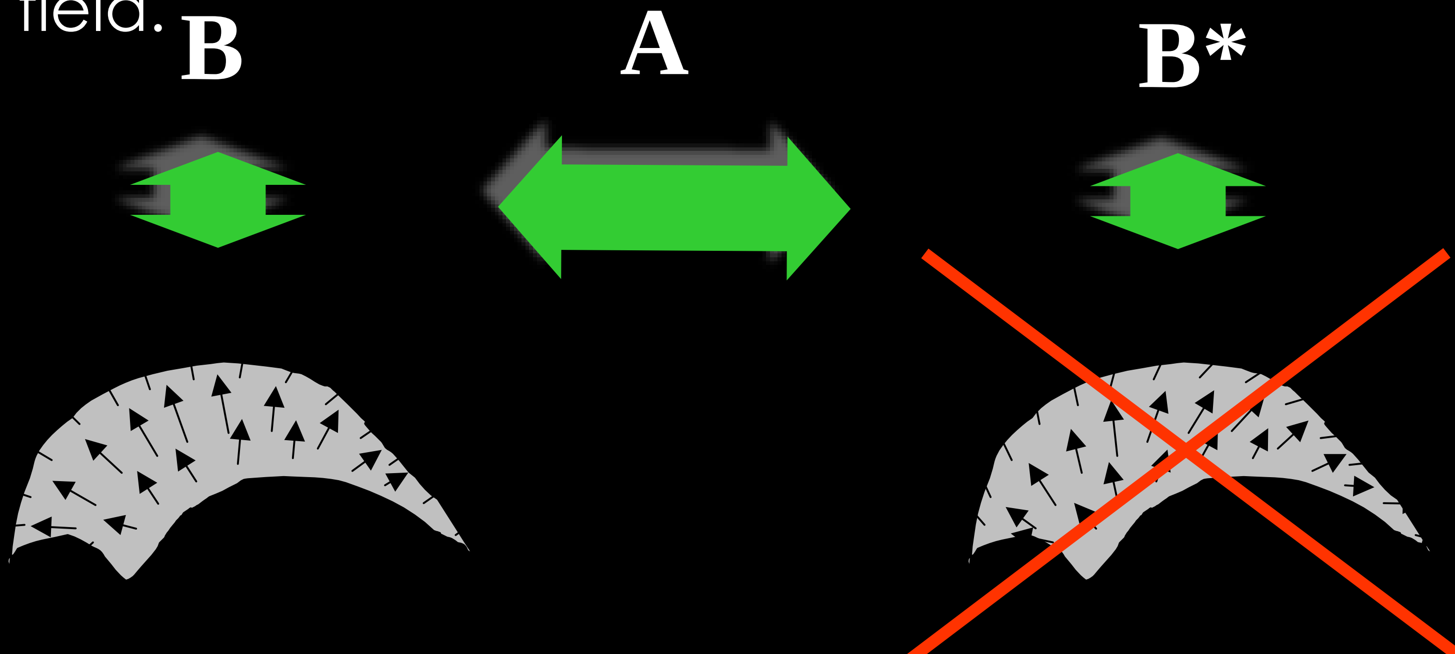
$$\mathbf{X = B^*S^* = B\,AA^{-1}S = BS}$$

- When we estimate B using SVD, the rows are NOT generally normal * albedo.

Surface Integrability

In general, $\mathbf{B}^*$ does not have a corresponding surface.

Linear transformations of the surface normals in general do not produce an integrable normal field.

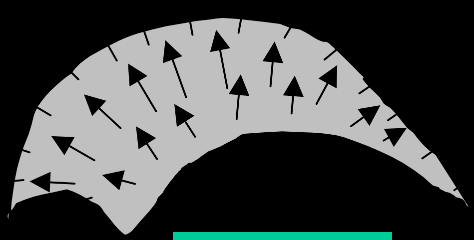


GBR Transformation

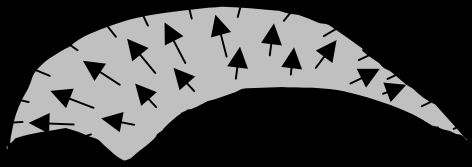
Only Generalized Bas-Relief transformations satisfy the integrability constraint:

$$\mathbf{A} = \mathbf{G}^T = \begin{bmatrix} \lambda & 0 & -\mu \\ 0 & \lambda & -\nu \\ 0 & 0 & 1 \end{bmatrix}^T$$

$\mathbf{B}$ $\quad \xleftrightarrow{\mathbf{G}^T} \quad$ $\overline{\mathbf{B}}$

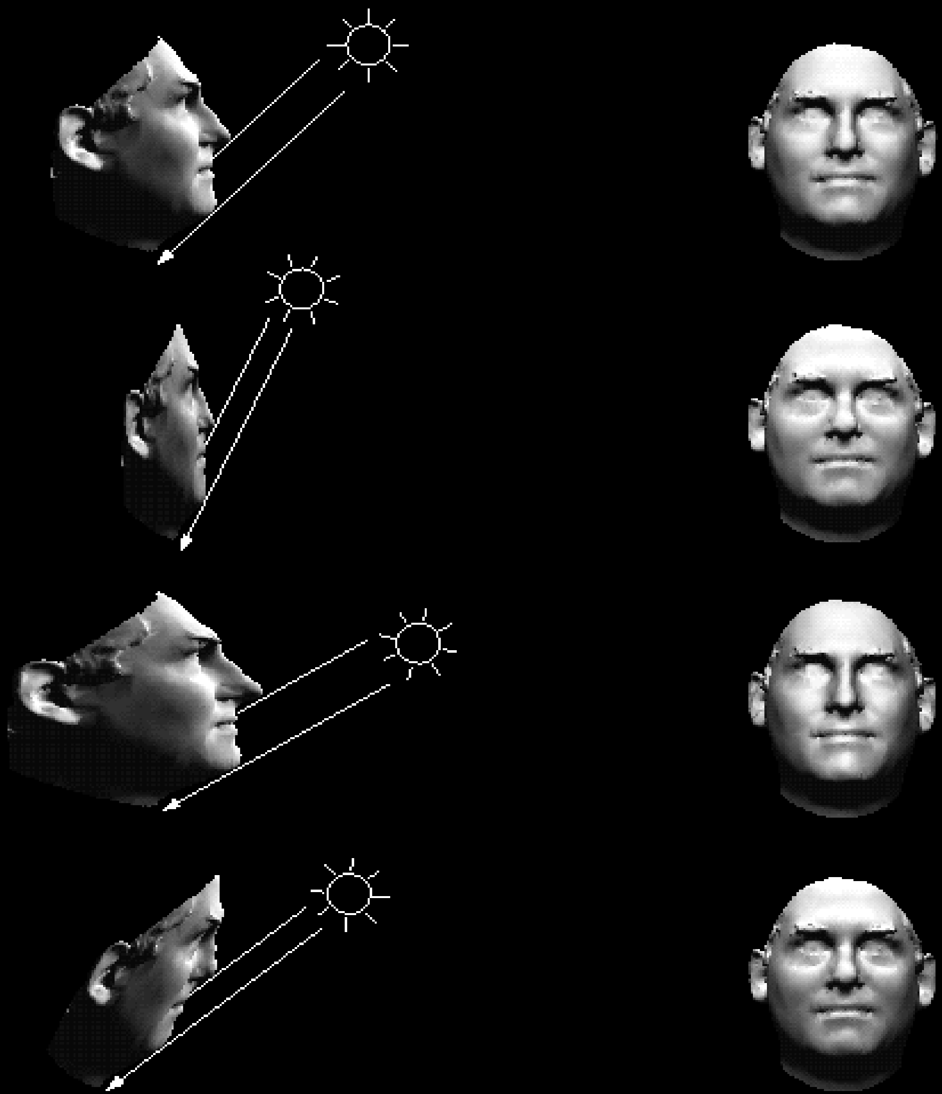


$f(x, y)$



$$\boxed{\bar{f}(x, y) = \lambda f(x, y) + \mu x + \nu y}$$

Generalized Bas-Relief Transformations



Objects differing by a GBR have the same illumination cone.

Without knowledge of light source location, one can only recover surfaces up to GBR transformations.

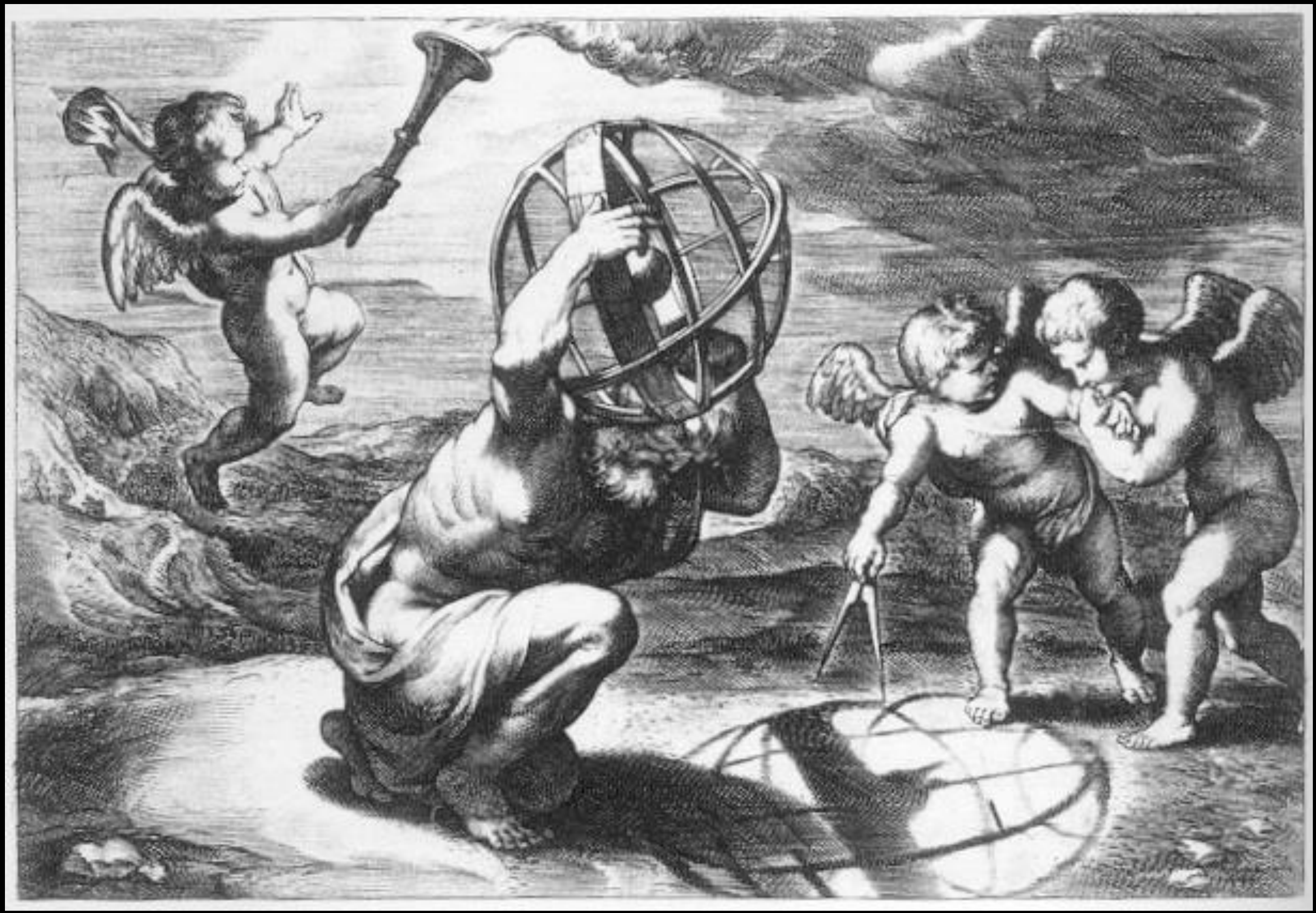
Uncalibrated photometric stereo

1. Take n images as input, perform SVD to compute B*.
2. Find some A such that B*A is close to integrable.
3. Integrate resulting gradient field to obtain height function f*(x,y).

Comments:

- f*(x,y) differs from f(x,y) by a GBR.
- Can use specularities to resolve GBR for non-Lambertian surface.

What about cast shadows for nonconvex objects?



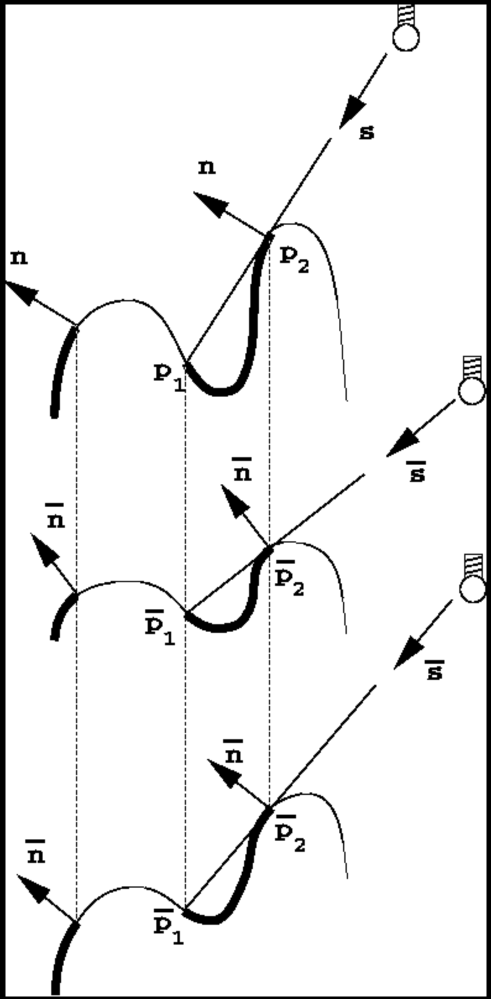
P.P. Reubens in Opticorum Libri Sex, 1613

GBR Preserves Shadows

Given a surface f and a GBR transformed surface f' then for every light source **s** which illuminates f there exists a light source **s'** which illuminates f' such that the **attached** and **cast shadows** are identical.

GBR is the only transform that preserves shadows.

[Kriegman, Belhumeur 2001]



Bas-Relief Sculpture

Codex Urbinas

As far as light and shade are concerned low relief fails both as sculpture and as painting, because the shadows correspond to the low nature of the relief, as for example in the shadows of foreshortened objects, which will not exhibit the depth of those in painting or in sculpture in the round.

Leonardo da Vinci

Treatise on Painting (Kemp)

Uncalibrated Photometric Stereo

- For calibrated photometric stereo, we estimated the n by 3 matrix **B** of surface normals scaled by albedo using lighting.
- Uncalibrated Input: Only images. No lighting info.
- Without shadowing, all images lie in 3D subspace of the n-pixel image space spanned by columns of an *n by 3* matrix **B***.
- From 3 or more images, SVD can be used to estimate **B***.
- The n by 3 matrix **B** of surface normals scaled by albedo differs from **B*** by a 3x3 linear transformation **B=AB***.
- After enforcing integrability, one can only estimate shape and albedo (**B**) up to a Generalized Bas Relief (GBR) transformation which has 3 parameters (depth scaling, tilt)

Next Lecture

- Early vision
  - Linear filters
- Reading:
  - Chapter 4: Linear filters