



# Bayes Rule

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Lecture 5

STA 371G

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- The OraQuick ADVANCE Rapid HIV-1/2 Antibody Test has the following properties:
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  - If you **do not have HIV**, there is a 99.8% chance the test will show a **negative result**

## Example 1: HIV Testing

- HIV testing is important for public health, but HIV tests are not perfect
- The OraQuick ADVANCE Rapid HIV-1/2 Antibody Test has the following properties:
  - If you **have HIV**, there is a 99.3% chance the test will show a **positive result**
  - If you **do not have HIV**, there is a 99.8% chance the test will show a **negative result**
- 0.4% of people in the US are HIV-positive

## Example 1: HIV Testing

We know  $P(TP|HP) = 0.993$ , but we really want to know  $P(HP|TP)$ !



## Bayes Rule

For any events  $A$  and  $B$ ,

$$P(A|B) = \frac{P(\text{A and B})}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}.$$

## Bayes Rule

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Bayes Rule allows us to “reverse the conditioning” and find  $P(A|B)$  when we know  $P(B|A)$ .

## Example 1: HIV Testing

$$P(TP|HP) = 0.993, \quad P(TN|HN) = 0.998, \quad P(HP) = 0.004$$

What is:

- $P(HN) =$

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- $P(HN) = 1 - P(HP) = 0.996$
- $P(TN|HP) = 1 - P(TP|HP) = 0.007$
- $P(TP|HN) =$

## Example 1: HIV Testing

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What is:

- $P(HN) = 1 - P(HP) = 0.996$
- $P(TN|HP) = 1 - P(TP|HP) = 0.007$
- $P(TP|HN) = 1 - P(TN|HN) = 0.002$

## Example 1: HIV Testing

$$P(TP|HP) = 0.993, \quad P(TN|HN) = 0.998, \quad P(HP) = 0.004$$

$$P(TN|HP) = 0.007, \quad P(TP|HN) = 0.002, \quad P(HN) = 0.996$$

$$\begin{aligned} P(HP|TP) &= \frac{P(TP|HP)P(HP)}{P(TP|HP)P(HP) + P(TP|HN)P(HN)} \\ &= \frac{(0.993)(0.004)}{(0.993)(0.004) + (0.002)(0.996)} \\ &= 0.67 \end{aligned}$$

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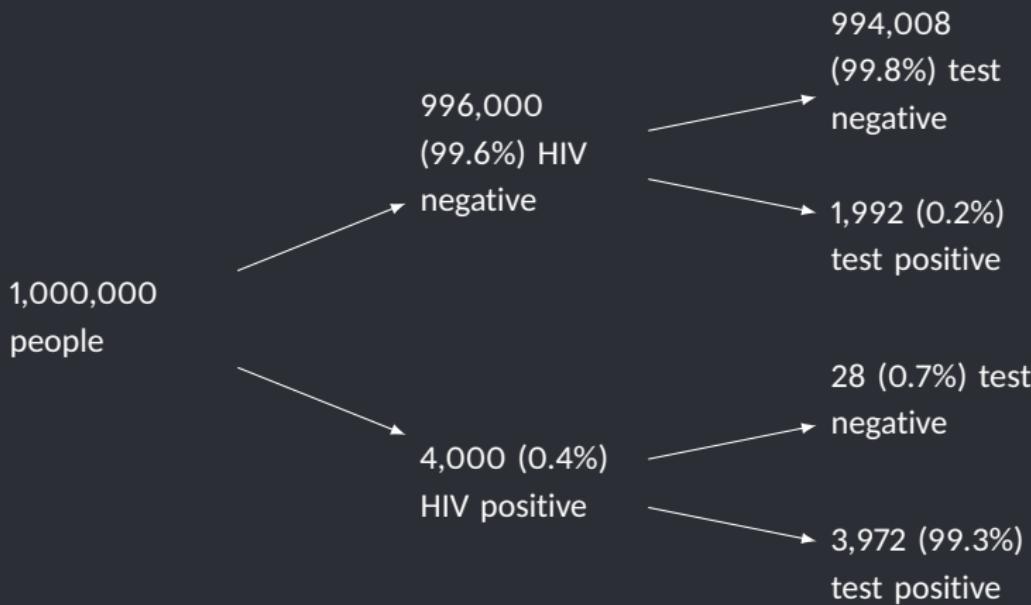
## Example 1: HIV Testing

- If you have HIV, there is a 99.3% chance the test will show a positive result
- If you do not have HIV, there is a 99.8% chance the test will show a negative result
- But if you test positive there is about a 1/3 chance you do NOT have HIV!

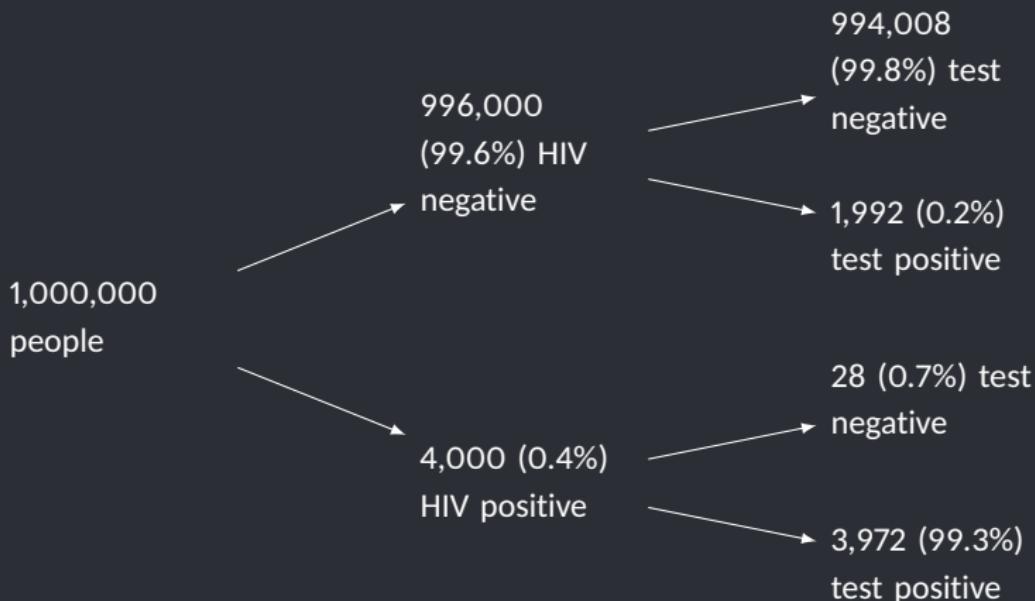
## Example 1: HIV Testing

- If you have HIV, there is a 99.3% chance the test will show a positive result
- If you do not have HIV, there is a 99.8% chance the test will show a negative result
- But if you test positive there is about a 1/3 chance you do NOT have HIV!
- This is counterintuitive — it's because of the way we are wired (it even has a name: “base rate fallacy”)

## Another way to look at it



## Another way to look at it



Of the  $3972 + 1992 = 5964$  people that tested positive, only 3972 (66.6%) are actually HIV positive!



Think of Bayes' Rule as a way to update our thinking based on new information:

$P(HP)$  ← Prior probability

$P(HP|TP)$  ← Posterior probability (includes new information)



## Magazine

# Do doctors understand test results?

By William Kremer  
BBC World Service

⌚ 7 July 2014

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Health Check

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⌚ 9 minutes ago

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### What really happened after this photo was taken

⌚ 7 hours ago



## Features



What really happened after this photo was taken 9/21

Just 21% of gynecologists got the right answer!

Just 21% of gynecologists got the right answer!

In other words, this is hard, and it goes against our intuition!

## Example 2: Supplier detective work

A smartphone company assembles their phones with screens from two different suppliers. Supplier A's screens are defective at a rate of 0.5%, and Supplier B's screens are defective at a rate of 0.3%. (55% of the screens are supplied by Supplier A.) Unfortunately, the shipments of screens from each supplier were not labeled and you can't tell which is which — and there are too many to count!

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Suppose we draw a screen at random from one shipment and find it is defective.

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Suppose we draw a screen at random from one shipment and find it is defective. How confident should we be that we drew from Supplier A?

## Example 2: Supplier detective work

$A =$  we drew from Supplier A's shipment

$D =$  we drew a defective screen

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- $P(D|A) =$

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What are we looking for?  $P(A|D)$

What is:

- $P(A) = 0.55$
- $P(D|A) = 0.005$
- $P(D|\bar{A}) =$

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- $P(D|\bar{A}) = 0.003$

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## Example 2: Supplier detective work

$$\begin{aligned} P(A|D) &= \frac{P(D|A)P(A)}{P(D|A)P(A) + P(D|\bar{A})P(\bar{A})} \\ &= \frac{(0.005)(0.55)}{(0.005)(0.55) + (0.003)(0.45)} \\ &= 0.67 \end{aligned}$$



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$$\begin{aligned} P(A|D) &= \frac{P(D|A)P(A)}{P(D|A)P(A) + P(D|\bar{A})P(\bar{A})} \\ &= \frac{(0.005)(0.55)}{(0.005)(0.55) + (0.003)(0.45)} \\ &= 0.67 \end{aligned}$$

We are 67% sure that the shipment we drew from is from Supplier A.



# LET'S MAKE A DEAL



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- There are three doors: two contain a goat, and one contains...a new car!
- You (the contestant) select one of three doors.
- Without revealing what is behind the selected door, Monty opens one of the other two doors to reveal a goat.

## Example 3: Monty Hall and *Let's Make a Deal*

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- There are three doors: two contain a goat, and one contains...a new car!
- You (the contestant) select one of three doors.
- Without revealing what is behind the selected door, Monty opens one of the other two doors to reveal a goat.
- Monty then gives you a choice: keep your original door, or switch to the other (unopened) door.







ZONK!!

## Example 3: Monty Hall and *Let's Make a Deal*

Do you have a better chance of getting the car by switching, or by keeping your original selection—or does it not matter?



## Example 3: Monty Hall and *Let's Make a Deal*

Let's suppose we open Door 1, and Monty shows us a goat behind Door 2. We have to decide: switch to Door 3, or keep our original choice of Door 1?

$D_3$  = The car is behind Door 3

$G_2$  = Monty shows us a goat behind Door 2

So we want to figure out  $P(D_3|G_2)$ . What is:

- $P(D_3) =$

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- $P(G_2|D_3) = 1$  (we already picked Door 1, and Monty can't show us the car, so he has to show us the goat behind Door 2)
- $P(G_2|\overline{D_3}) =$

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- $P(G_2|D_3) = 1$  (we already picked Door 1, and Monty can't show us the car, so he has to show us the goat behind Door 2)
- $P(G_2|\bar{D}_3) = 1/2$  (the car must be behind Door 1 or 2; we'll see the goat behind Door 2 only if the car is behind Door 1)

## Example 3: Monty Hall and *Let's Make a Deal*

$$\begin{aligned} P(D_3|G_2) &= \frac{P(G_2|D_3)P(D_3)}{P(G_2|D_3)P(D_3) + P(G_2|\overline{D_3})P(\overline{D_3})} \\ &= \frac{1 \cdot \frac{1}{3}}{\frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3}} \\ &= \frac{2}{3} \end{aligned}$$

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So it is *better to switch* — you have a 2/3 chance of winning if you switch!