THE UNIVERSITY OF TEXAS AT AUSTIN
McCOMBS SCHOOL OF BUSINESS

Bayes Rule

Lecture 5

STA 371G

Example 1: HIV Testing

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- The OraQuick ADVANCE Rapid HIV-1/2 Antibody Test has the following properties:
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  - If you **do not have HIV**, there is a 99.8% chance the test will show a **negative result**

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- HIV testing is important for public health, but HIV tests are not perfect
- The OraQuick ADVANCE Rapid HIV-1/2 Antibody Test has the following properties:
  - If you **have HIV**, there is a 99.3% chance the test will show a **positive result**
  - If you **do not have HIV**, there is a 99.8% chance the test will show a **negative result**
- 0.4% of people in the US are HIV-positive

Example 1: HIV Testing

We know $P(TP|HP) = 0.993$, but we really want to know $P(HP|TP)$!

Bayes Rule

For any events A and B,

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\overline{A})P(\overline{A})}.$$

Bayes Rule

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Bayes Rule allows us to “reverse the conditioning” and find $P(A|B)$ when we know $P(B|A)$.

Example 1: HIV Testing

$$P(TP|HP) = 0.993, \qquad P(TN|HN) = 0.998, \qquad P(HP) = 0.004$$

What is:

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- $P(HN) = 1 - P(HP) = 0.996$
- $P(TN|HP) =$

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What is:

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What is:

- $P(HN) = 1 - P(HP) = 0.996$
- $P(TN|HP) = 1 - P(TP|HP) = 0.007$
- $P(TP|HN) =$

Example 1: HIV Testing

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- $P(TP|HN) =$

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What is:

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- $P(TN|HP) = 1 - P(TP|HP) = 0.007$
- $P(TP|HN) = 1 - P(TN|HN) = 0.002$

Example 1: HIV Testing

$$P(TP|HP) = 0.993, \qquad P(TN|HN) = 0.998, \qquad P(HP) = 0.004$$

$$P(TN|HP) = 0.007, \qquad P(TP|HN) = 0.002, \qquad P(HN) = 0.996$$

$$\begin{aligned} P(HP|TP) &= \frac{P(TP|HP)P(HP)}{P(TP|HP)P(HP) + P(TP|HN)P(HN)} \\ &= \frac{(0.993)(0.004)}{(0.993)(0.004) + (0.002)(0.996)} \\ &= 0.67 \end{aligned}$$

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- If you have HIV, there is a 99.3% chance the test will show a positive result
- If you do not have HIV, there is a 99.8% chance the test will show a negative result
- But if you test positive there is about a 1/3 chance you do NOT have HIV!

Example 1: HIV Testing

- If you have HIV, there is a 99.3% chance the test will show a positive result
- If you do not have HIV, there is a 99.8% chance the test will show a negative result
- But if you test positive there is about a 1/3 chance you do NOT have HIV!
- This is counterintuitive — it's because of the way we are wired (it even has a name: "base rate fallacy")

Another way to look at it

1,000,000 people

- 996,000 (99.6%) HIV negative
  - 994,008 (99.8%) test negative
  - 1,992 (0.2%) test positive
- 4,000 (0.4%) HIV positive
  - 28 (0.7%) test negative
  - 3,972 (99.3%) test positive

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  - 3,972 (99.3%) test positive

Of the $3972 + 1992 = 5964$ people that tested positive, only 3972 (66.6%) are actually HIV positive!

Think of Bayes' Rule as a way to update our thinking based on new information:

$$P(HP) \longleftarrow \text{Prior probability}$$
$$P(HP|TP) \longleftarrow \text{Posterior probability (includes new information)}$$

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By William Kremer
BBC World Service

7 July 2014

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What really happened after this photo was taken

Just 21% of gynecologists got the right answer!

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In other words, this is hard, and it goes against our intuition!

Example 2: Supplier detective work

A smartphone company assembles their phones with screens from two different suppliers. Supplier A's screens are defective at a rate of 0.5%, and Supplier B's screens are defective at a rate of 0.3%. (55% of the screens are supplied by Supplier A.) Unfortunately, the shipments of screens from each supplier were not labeled and you can't tell which is which — and there are too many to count!

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Suppose we draw a screen at random from one shipment and find it is defective.

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Suppose we draw a screen at random from one shipment and find it is defective. How confident should we be that we drew from Supplier A?

Example 2: Supplier detective work

$A =$ we drew from Supplier A's shipment

$D =$ we drew a defective screen

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- $P(A) = 0.55$
- $P(D|A) =$

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$$D = \text{we drew a defective screen}$$

What are we looking for? $P(A|D)$

What is:

- $P(A) = 0.55$
- $P(D|A) = 0.005$
- $P(D|\overline{A}) =$

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What are we looking for? $P(A|D)$

What is:

- $P(A) = 0.55$
- $P(D|A) = 0.005$
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Example 2: Supplier detective work

$$
\begin{aligned}
P(A|D) &= \frac{P(D|A)P(A)}{P(D|A)P(A) + P(D|\overline{A})P(\overline{A})} \\
&= \frac{(0.005)(0.55)}{(0.005)(0.55) + (0.003)(0.45)} \\
&= 0.67
\end{aligned}
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P(A|D) &= \frac{P(D|A)P(A)}{P(D|A)P(A) + P(D|\bar{A})P(\bar{A})} \\
&= \frac{(0.005)(0.55)}{(0.005)(0.55) + (0.003)(0.45)} \\
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\end{aligned}
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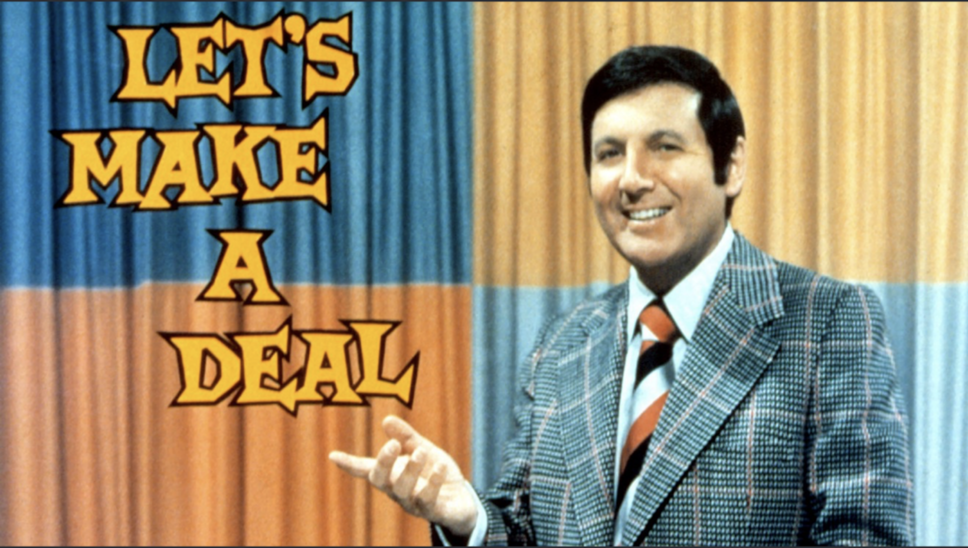
We are 67% sure that the shipment we drew from is from Supplier A.


LET'S
MAKE
A
DEAL

Example 3: Monty Hall and *Let's Make a Deal*

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- You (the contestant) select one of three doors.

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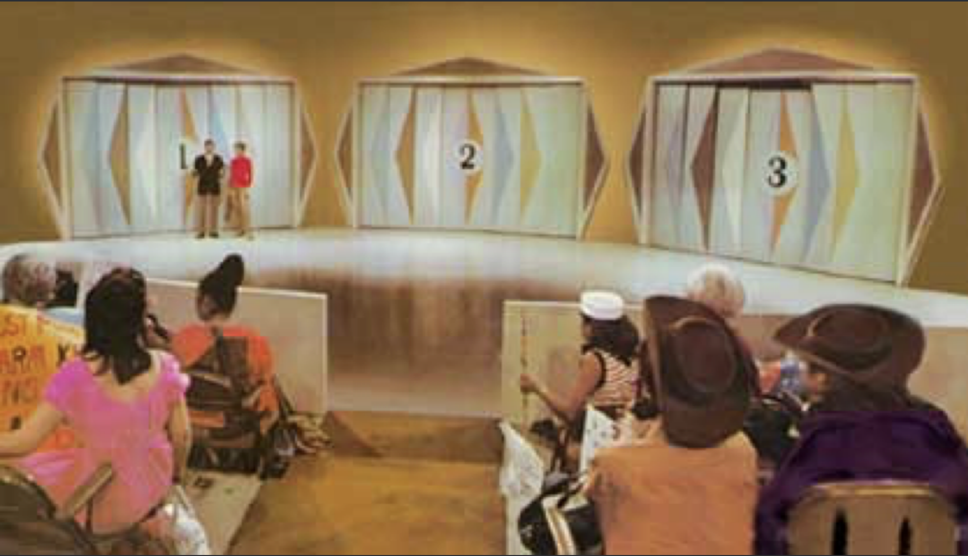
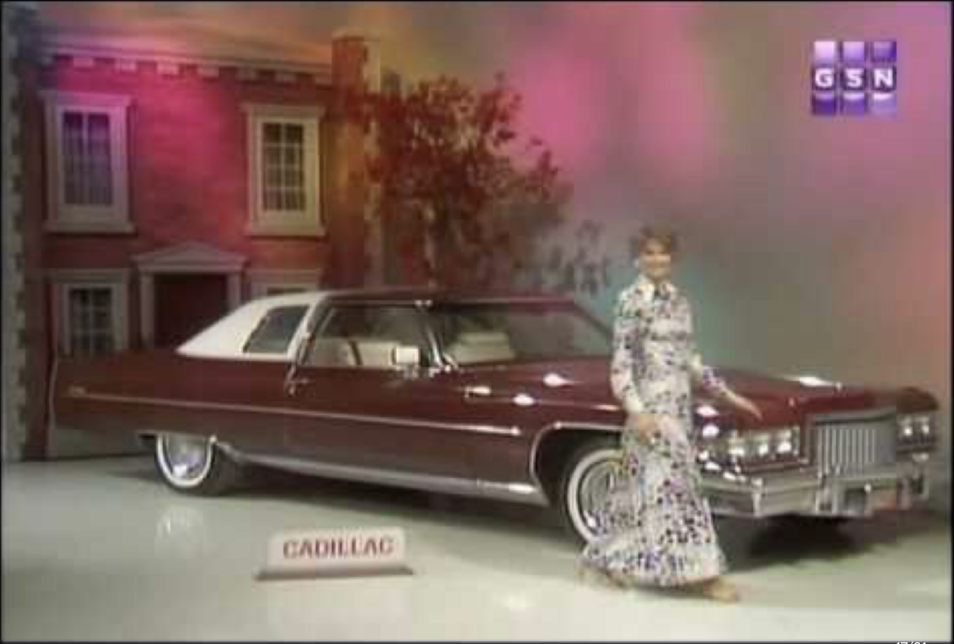
- Monty Hall used to host the game show *Let's Make a Deal*.
- There are three doors: two contain a goat, and one contains…a new car!
- You (the contestant) select one of three doors.
- Without revealing what is behind the selected door, Monty opens one of the other two doors to reveal a goat.

Example 3: Monty Hall and *Let's Make a Deal*

- Monty Hall used to host the game show *Let's Make a Deal*.
- There are three doors: two contain a goat, and one contains...a new car!
- You (the contestant) select one of three doors.
- Without revealing what is behind the selected door, Monty opens one of the other two doors to reveal a goat.
- Monty then gives you a choice: keep your original door, or switch to the other (unopened) door.


1
2
3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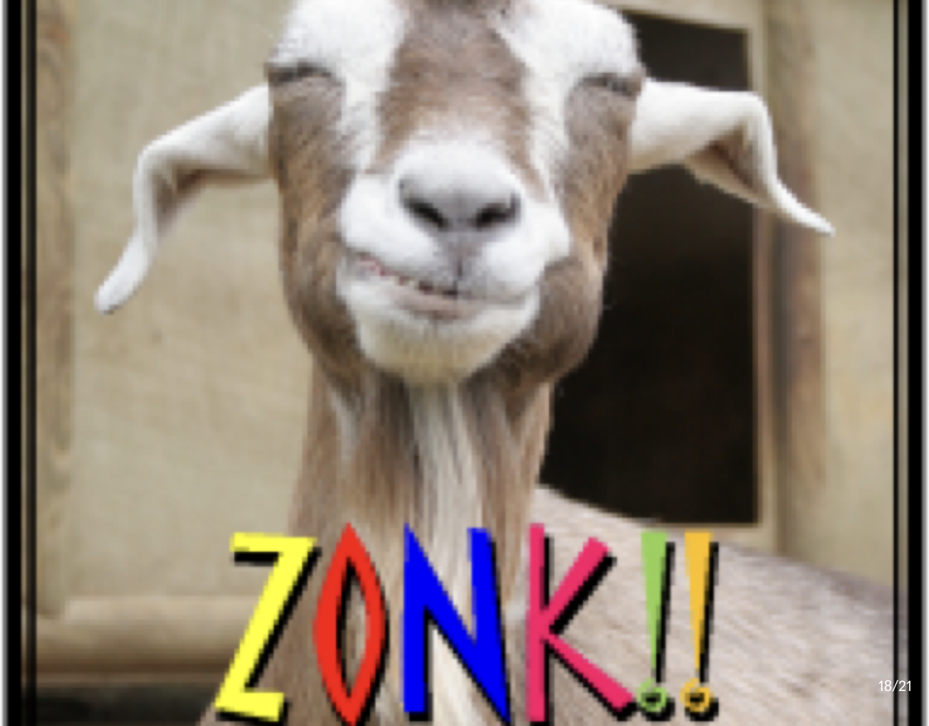
ZONK!!

Example 3: Monty Hall and *Let's Make a Deal*

Do you have a better chance of getting the car by switching, or by keeping your original selection—or does it not matter?

Example 3: Monty Hall and *Let's Make a Deal*

Let's suppose we pick Door 1, and Monty shows us a goat behind Door 2. We have to decide: switch to Door 3, or keep our original choice of Door 1?

$$D3 = \text{The car is behind Door 3}$$

$$G2 = \text{Monty shows us a goat behind Door 2}$$

So we want to figure out $P(D3|G2)$. What is:

- $P(D3) =$

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- $P(G2|\overline{D3}) =$

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- $P(D3) = 1/3$ (the car is placed randomly)
- $P(G2|D3) = 1$ (we already picked Door 1, and Monty can't show us the car, so he has to show us the goat behind Door 2)
- $P(G2|\overline{D3}) = 1/4$ (the car must be behind Door 1 or 2; we'll see the goat behind Door 2 only if the car is behind Door 1 *and* he chooses to open door 2 instead of 3)

Example 3: Monty Hall and *Let's Make a Deal*

$$
\begin{aligned}
P(D3|G2) &= \frac{P(G2|D3)P(D3)}{P(G2|D3)P(D3) + P(G2|\overline{D3})P(\overline{D3})} \\
&= \frac{1 \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{2}{3}} \\
&= \frac{2}{3}
\end{aligned}
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&= \frac{2}{3}
\end{aligned}
$$

So it is *better to switch* — you have a 2/3 chance of winning if you switch!