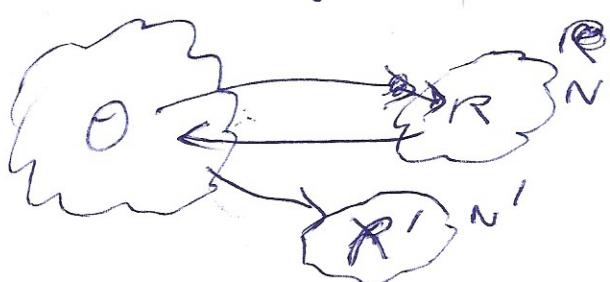


Analysis of problem:

Suppose you pulled a slot machine 5 times, and won 1 time. What's the probability estimate for next pull being a win?

Provides insight into:

for flat prior, MAP and ML estimators are $\frac{1}{5}$. But if we consider full posterior, you could do more things like taking average of ~~posterior~~ values based on posterior distribution



Version 1

Facts:

$$\text{Beta function} \quad B(x, y) = \frac{(x-1)! (y-1)!}{(x+y-1)!}$$

$$\text{Beta dist} \quad \pi(\theta; n, y) = \frac{1}{B(n, y)} \theta^{n-1} (1-\theta)^{y-1}$$

useful Integral: $\int_{\theta=0}^1 \theta^r (1-\theta)^{n-r} d\theta = \frac{r! (n-r)!}{(n+1)!}$

Slot machine has bernoulli model for one try and hence binomial for n trials. ~~If~~ \rightarrow are success, then let θ be # of success.

Let's use Beta dist $\pi(\theta; \alpha, \beta)$ as prior.

$$f_\theta(\theta) = \pi(\theta; \alpha, \beta)$$

$$f_\theta(\theta) = \pi(\theta; \alpha, \beta) = n_r \theta^r (1-\theta)^{n-r}$$

~~P~~ $P(X|\theta) = n_r \theta^r (1-\theta)^{n-r}$
 To understand this consider θ taking a discrete set of values with some probability distribution

θ

$P_\theta(\theta)$

0.2	0.1
0.5	0.2
0.7	0.7
0.9	0.0

(2)

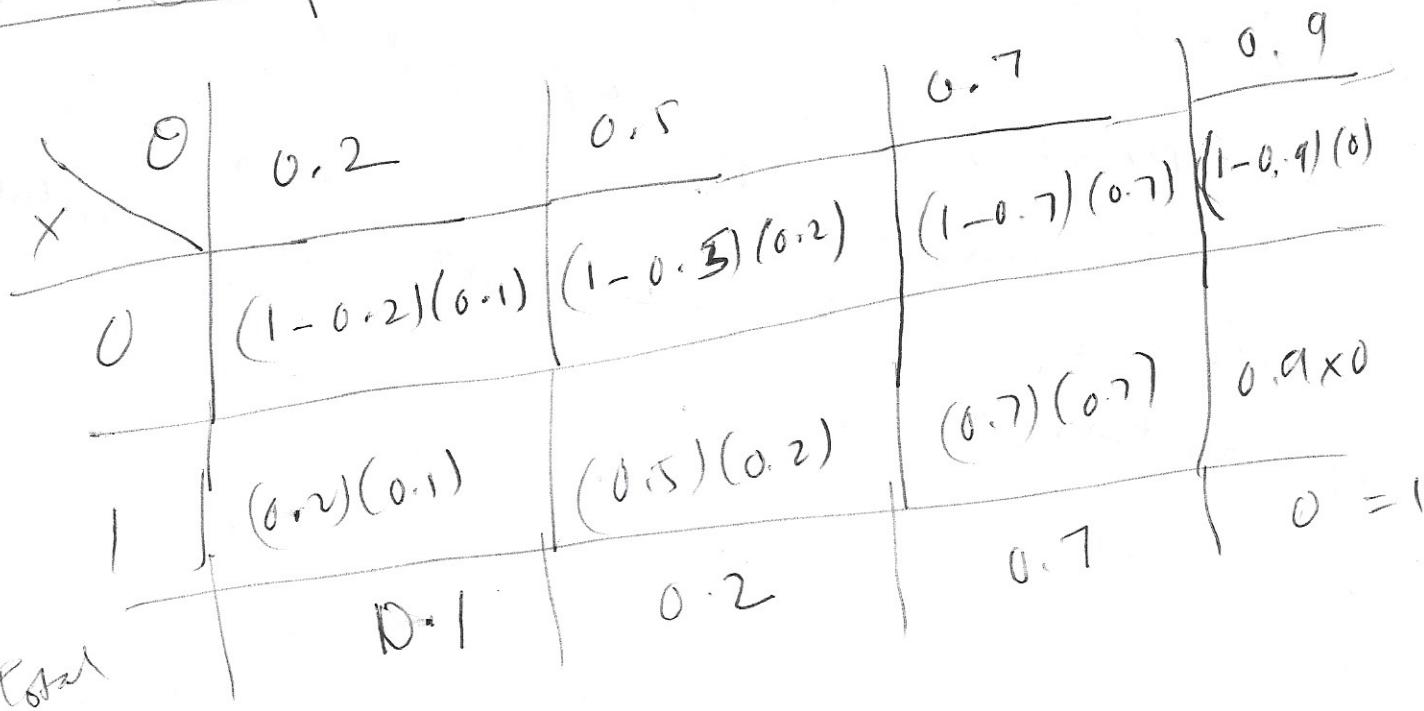
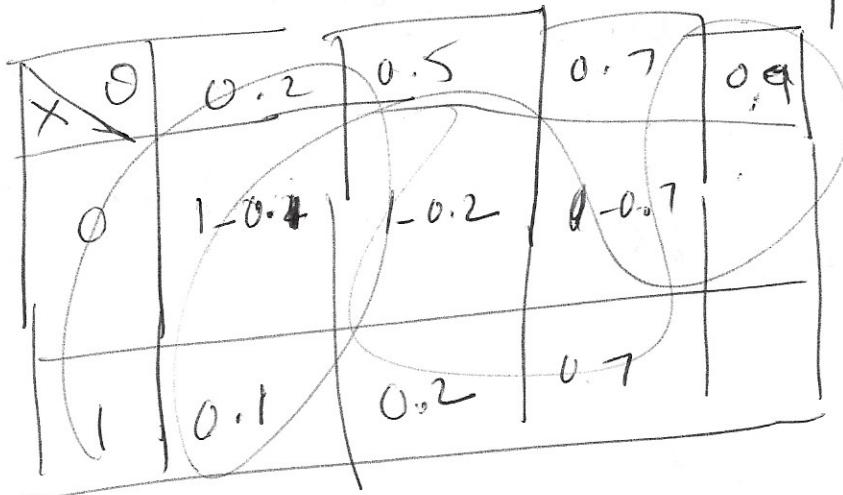
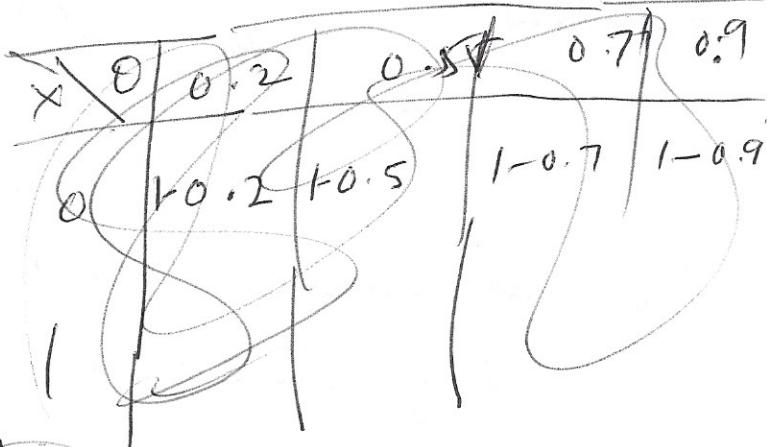
JPT for Bernoulli

* $x = 1$ win
 $= 0$ lose



win

θ $P(\theta)$
0.2 (0.1)



so for one attempt, success is (3pt) | failure is

$$P(r=1|\theta) = \theta f_\theta(\theta) \quad P(r=0|\theta) = (1-\theta) f_\theta(\theta)$$

$$P_{r|\theta} \uparrow$$

For r attempts

$$P(r|\theta) = \theta^r (1-\theta)^{n-r} f_\theta(\theta)$$

So, likelihood for r wins

$$P(r|\theta) = {}^n C_r \theta^r (1-\theta)^{n-r} f_\theta(\theta)$$

$$\begin{aligned} L(\theta) &= {}^n C_r \frac{\int_0^\infty \theta^r (1-\theta)^{n-r} f_\theta(\theta) d\theta}{B(\alpha, \beta)} \\ &= {}^n C_r \frac{\int_0^\infty \theta^r (1-\theta)^{n-r} d\theta}{B(\alpha, \beta)} \\ &= {}^n C_r \frac{1}{B(\alpha, \beta)} \int_{\theta=0}^1 \theta^{r+\alpha-1} (1-\theta)^{n-r+\beta-1} d\theta \\ &= {}^n C_r \frac{(r+\alpha-1)! (n-r+\beta-1)!}{B(\alpha, \beta) (n+\alpha+\beta-1)!} \end{aligned}$$

$$\begin{aligned}
 \textcircled{4} \quad f_{\theta|R}(r|\theta) &= \frac{f_{\theta|R}(r|\theta)}{f_R(r)} = \frac{\theta^r (1-\theta)^{n-r}}{\Gamma(\alpha+\beta)} \frac{\theta^{\alpha-1} (1-\theta)^{\beta-1}}{\Gamma(\alpha) \Gamma(\beta)} \\
 &= \frac{n_r \theta^r (1-\theta)^{n-r}}{\Gamma(\alpha+\beta)} \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \frac{(r+\alpha-1)! (n-r+\beta-1)!}{(n+\alpha+\beta-1)!} \\
 &= \theta^{r+\alpha-1} (1-\theta)^{n-r+\beta-1} \frac{(n+\alpha+\beta-1)!}{(r+\alpha-1)! (n-r+\beta-1)!} \\
 &= \frac{\theta^{x-1} (1-\theta)^{y-1}}{(x-1)! (y-1)!} \frac{(x+y-1)!}{(x+y-1)!} \quad \text{where } x = r+\alpha \\
 &\quad y = n-r+\beta
 \end{aligned}$$

$$= {}_T(r+\alpha, n-r+\beta)$$

Pg 5

$$\begin{aligned}
 p(\tilde{x}|R) &= \int_{\theta} f_{R,\theta}(\tilde{x}, \theta | x) d\theta \\
 &= \int_{\theta} f_{R|\theta}(\tilde{x} | \theta, R) f_{\theta|R}(\theta | R) d\theta \\
 &= \int_{\theta} \cancel{f_{\theta|R}} f_{R|\theta}(\tilde{x} | \theta) f_{\theta|R}(\theta | R) d\theta \quad \text{markov assumption} \\
 &= \int_{\theta} \theta \pi(\gamma + \alpha, n - r + \beta) d\theta \\
 &= E[\theta | R] = \frac{\gamma + \alpha}{\gamma + \alpha + n - r + \beta} \\
 &= \frac{\gamma + \alpha}{n + \alpha + \beta}
 \end{aligned}$$

For uniform prior $\alpha = \beta = 1$

$$p(\tilde{x}|R) = \frac{\gamma + 1}{n + 2} = \frac{1 + 1}{5 + 2} = \frac{2}{7}$$

Version 2

First some facts:
Beta function

$$B(x, y) = \frac{(x-1)! (y-1)!}{(x+y-1)!}$$

Beta distribution

$$\pi(\theta; x, y) = \frac{1}{B(x, y)} \theta^{x-1} (1-\theta)^{y-1}$$

Useful Integral

$$\int_{\theta=0}^1 \theta^x (1-\theta)^{n-x} d\theta = \frac{x! (n-x)!}{(n+1)!}$$

Model:

A single trial of slot machine is a bernoulli trial with θ = probability of success/winning so n pulls/trials with π successes/wins is a binomial distribution.

Before pulling / collecting data of trials, we can assume probability as driven by a beta distribution with α, β chosen to

represent our beliefs.

set $R = 1$ single trial success/win
 $= 0$ single trial loss/failure

Let's say we observed n of them

R_1, R_2, \dots, R_n . Each is i.i.d
 $R_i \sim \text{bernoulli}(\theta)$ set R represent any of them.

50

(2)

Prior distribution of θ

$$f_{\theta}(\theta) = \pi(\theta; \alpha, \beta) \quad \text{--- (1)}$$

Conditional

$$f_{R|\theta}(r|\theta) = {}^n C_r \theta^r (1-\theta)^{n-r} \quad \text{--- (2)}$$

Here $r=0$ lose in one trial
 $=1$ win

To understand (2) Consider a situation where θ takes discrete probability distribution values with some

θ	$f_{\theta}(\theta)$
0.2	0.1
0.5	0.2
0.7	0.7
0.9	0.0

$$\text{JPT for } R, \theta \quad P(R=1 | \theta=0.2) = \frac{0.2}{(0.2)(0.1) + (1-0.2)(0.1)} = 0.2 = \theta$$

(3)

R	θ	0.2	0.5	0.7	0.9
0		$(1-0.2)f_{\theta}(0)$	$(1-0.5)f_{\theta}(0)$	$(1-0.7)f_{\theta}(0)$	$(1-0.9)f_{\theta}(0)$
1		$\frac{(0.2)f_{\theta}(1)}{\theta}$	$\frac{(0.5)f_{\theta}(1)}{\theta}$	$\frac{(0.7)f_{\theta}(1)}{\theta}$	$\frac{(0.9)f_{\theta}(1)}{\theta}$
	Total	0.1	0.2	0.7	0.0

so for one pull, win is characterized by given $\theta = 0.2$

$$f_{R|\theta}(r=1 | \theta) = \theta \quad f_{R|\theta}(r=0 | \theta) = 1 - \theta$$

$$P(R=1 | \theta=0.2) = 0.2$$

lose is characterized by

$$f_{R|\theta}(r=0 | \theta) = (1-\theta)$$

Hence for 'g' wins,

$$f_{R|\theta}(r | \theta) = {}^n C_r \theta^r (1-\theta)^{n-r}$$

[think of addition of probabilities]

(4)

Now

$$F_R(r) = \int_{\theta_1} f_{R,\theta}(r, \theta_1) d\theta_1$$

$$\begin{aligned}
 &= \int_{\theta_1} f_{R/\theta}(r|\theta_1) f_{\theta_1}(\theta_1) d\theta_1 \quad \text{Integrand is same as} \\
 &= \int_{\theta_1} {}^n C_r \theta_1^{r-1} (1-\theta_1)^{n-r} \frac{1}{B(\alpha, \beta)} \theta_1^{\alpha-1} (1-\theta_1)^{\beta-1} d\theta_1 \\
 &= {}^n C_r \frac{1}{B(\alpha, \beta)} \int_{\theta_1=0}^1 \theta_1^{r+\alpha-1} (1-\theta_1)^{\beta-1} d\theta_1 \\
 &= {}^n C_r \frac{1}{B(\alpha, \beta)} \frac{(r+\alpha-1)! (n-r+\beta-1)!}{(n+\alpha+\beta-1)!} \tag{4}
 \end{aligned}$$

Using $\theta = \theta_1$ to illustrate that : all we are doing is integrating across values of θ in its set Θ and what variable name we use is irrelevant and not the same as θ used in other places in this document.

Pg 5

$$f_{\theta|R}(\theta|r) = \frac{f_{R|\theta}(r|\theta) f_\theta(\theta)}{f_R(r)} \quad \text{--- (5)}$$

Notice, denominator is

numerator $\propto \theta$ as expected

From ①, ②, ④ plugging in ⑤

$$f_{\theta|R}(\theta|r) = \frac{n_r \theta^r (1-\theta)^{n-r} \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}}{\frac{1}{B(\alpha, \beta)} \frac{(r+\alpha-1)! (n-r+\beta-1)!}{(n+\alpha+\beta-1)!}} \quad \text{--- (6a)}$$

$$= \theta^{r+\alpha-1} (1-\theta)^{n-r+\beta-1} \frac{(n+\alpha+\beta-1)!}{(r+\alpha-1)! (n-r+\beta-1)!}$$

$$= \frac{\theta^{x-1} (1-\theta)^{y-1}}{(x-1)! (y-1)!} \quad \text{where } x = r+\alpha \\ y = n-r+\beta$$

$$= \pi(\alpha; x, y) = \pi(\theta; r+\alpha, n-r+\beta) \quad \text{--- (6)}$$

To predict next outcome $\tilde{r} = 1$ win
 $\text{OR } \tilde{r} = 0$ loose 6

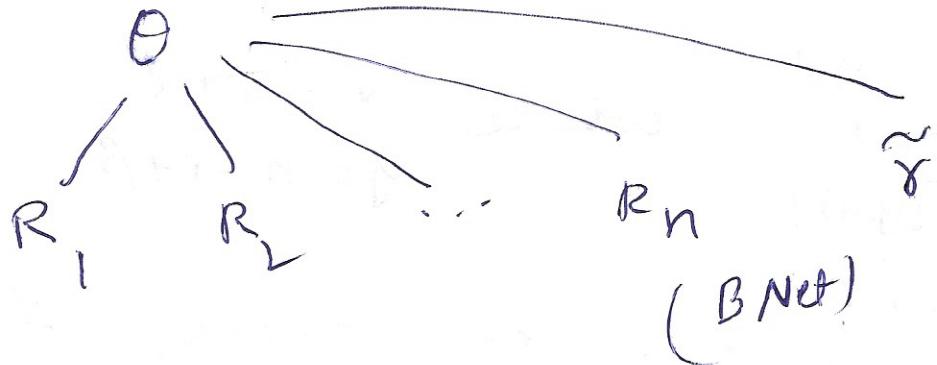
$f_{\theta|n}$

$$f_{\tilde{r}|R}(\tilde{r}|R) = \int_{\theta} f_{R|\theta}(\tilde{r}, \theta|R) d\theta$$

$$= \int_{\theta} f_{R|\theta}(\tilde{r}|\theta) f_{\theta|R}(\theta|r) d\theta$$

$$= \int_{\theta} f_{R|\theta}(\tilde{r}|\theta) f_{\theta|R}(\theta|r) d\theta \quad \text{--- (7)}$$

In (7) we used the fact that each data/outcome is independent of other given θ (markov property)



$$\begin{aligned} P(\tilde{r}|\theta, R_1, R_2, \dots, R_n) \\ = P(\tilde{r}|\theta) \end{aligned}$$

(7)

$$f_{\tilde{\gamma}|R}(\tilde{\gamma}|R) =$$

Also

$$f_{\tilde{\gamma}|R}(\tilde{\gamma}|\theta) = \theta^{\tilde{\gamma}} (1-\theta)^{1-\tilde{\gamma}} \quad (8)$$

Here since we are considering next pull
 only ~~or~~ $\tilde{\gamma} = 1$

$$f_{\tilde{\gamma}|R}(\tilde{\gamma}=1|\theta) = \theta \quad (8a)$$

This is nothing but
 expectation $E[R]$
 in $P(\cdot)$ is ~~is~~
 just the prob. dist. of R
 see eqn 6

From (7)

$$f_{\tilde{\gamma}|R}(\tilde{\gamma}=1|R) = \int_0^\infty \theta^1 \pi(r+\alpha, n-r+\beta) d\theta$$

$$= E[\theta|R] = \frac{r+\alpha}{r+\alpha+n-r+\beta}$$

$$= \frac{r+\alpha}{n+\alpha+\beta} \quad (9)$$

For uniform prior on θ , $\alpha = \beta = 1$

$$f_{\tilde{\gamma}|R}(\tilde{\gamma}=1|R) = \frac{r+1}{n+2} \quad (10)$$

(8)

Since

$$n = 5$$

$$\gamma = 1$$

The probability of next pull being
winner, given we got only one win in the last 5 pulls

$$= \frac{1+1}{5+2} = \frac{2}{7}$$

plugging in eqn (10)

Incidentally equation (7) can be used

Other ways:

For max. likelihood

$$f_{\tilde{\gamma}|R}(\tilde{\gamma}|R) = \int_0^R f_{\tilde{\gamma}|\theta}(\tilde{\gamma}|\theta) f_{\theta|R}(\theta|r) d\theta$$

$$\approx \int_0^R f_{\tilde{\gamma}|\hat{\theta}_{MC}}(\tilde{\gamma}|\hat{\theta}_{MC}) f_{\theta|R}(\theta|r) d\theta$$

$$= f_{\tilde{\gamma}|\theta}(\tilde{\gamma}|\hat{\theta}_{MC}) \int_0^R f_{\theta|R}(\theta|r) d\theta$$

$= 1$

$$= f_{\tilde{\gamma}|\theta}(\tilde{\gamma}|\hat{\theta}_{MC})$$

Eqn ① can be used other ways too.

For ML estimate

$$f_{\tilde{y}|R}(\tilde{y}|R) = \int_{\theta} f_{R|\theta}(\tilde{y}|\theta) f_{\theta|R}(\theta|r) d\theta$$

$$\approx \int_{\theta} f_{R|\theta}(\tilde{y}|\hat{\theta}_{ML}) f_{\theta|R}(\theta|r) d\theta$$

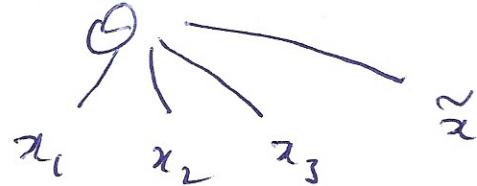
$$= f_{\tilde{y}|\theta}(\tilde{y}|\hat{\theta}_{ML}) \underbrace{\int_{\theta} f_{\theta|R}(\theta|r) d\theta}_{=1}$$

Or for MAP, the same

$$f_{\tilde{y}|R}(\tilde{y}|R) \approx f_{R|\theta}(\tilde{y}|\hat{\theta}_{MAP})$$

Note: In Eqn ① we said markov assumption (which is essentially markov property). But since all data is conditionally independent given θ

$$P(x|\theta, D_n) = P(x|\theta)$$



(a)

For MAP from ⑦

$$f_{\tilde{\gamma} | R}(\tilde{\gamma} | R) \underset{R | \theta}{\sim} f_{\theta}(\tilde{\gamma} | \hat{\theta}_{MAP})$$

So, you just take $\hat{\theta}_{ML}$ or

and plug in ⑧a

$\hat{\theta}_{MAP}$

For example for ML ,

$$\text{prob. of next pull being winner (ML)} = \hat{\theta}_{ML} = \frac{r}{n} = \frac{1}{5}$$

prob. of next pull being winner (MAP) = $\hat{\theta}_{MAP} = \frac{r}{n} = \frac{1}{5}$

(for flat prior)
 $ML = MAP$

cont'd →

10

If prior is not flat, MAP
is calculated as below (Holy trinity)

$$L = n \sum_r \theta^r (1-\theta)^{n-r} \frac{\theta^{\alpha-1}}{\beta(\alpha, \beta)} (1-\theta)^{\beta-1}$$

from eqn 6a needs to be maximized

Log likelihood

$$\log L = \dots + (\alpha+r-1) \log \theta + (n-r+\beta-1) \log(1-\theta)$$

$$\frac{\partial \log L}{\partial \theta} = 0 \Rightarrow$$

$$\frac{\alpha+r-1}{\theta} = \frac{n-r+\beta-1}{1-\theta}$$

$$\text{or see } \frac{1-\theta}{\theta} = \frac{n-r+\beta-1}{r+\alpha-1}$$

$$\text{or } \hat{\theta} = \frac{\alpha+r-1}{n+\alpha+\beta-2} \left[\begin{array}{l} \frac{r}{n} \text{ for flat} \\ \text{prior} \\ \alpha = \beta = 1 \end{array} \right]$$

(11)

What about the expected
payoff of next slot machine
pull? i.e., ~~$E[R]$~~ ~~$E[\tilde{R}]$~~ ~~$E[\tilde{R}|R]$~~

$$E[\tilde{R}|R] = ? \quad ; R = 0 \text{ or } 1$$

$$E[\tilde{R}|R] = 1 \cdot f(\tilde{r}=1|R)$$

$$+ 0 \cdot f(\tilde{r}=0|R) \quad (\text{using } \tilde{R}=\tilde{r})$$

$$= \frac{r+1}{n+2}$$

$$= \frac{2}{7}$$