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Keywords (separated by '-')	Order-convex function - Nonlinear matrix equation - Newton method - Relaxed Newton method

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Convergence of relaxed Newton method for order-convex matrix equations

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Abstract
In this paper, we suggest singular monotone Newton theorem and a relaxed Newton method for a nonlinear matrix equation $F(X) = 0$ where F is an order-convex matrix function. At first, we introduce monotone Newton theorem from Ortega and Rheinboldt (SIAM, 2000), and we show singular monotone Newton theorem which shows that the Newton sequence converges to a solution of $F(X) = 0$ even if the Fréchet derivative at the solution is singular. At last, we provide a relaxed Newton method for $F(X) = 0$ which is better than the pure Newton method and we give some numerical experiments for the relaxed Newton method.

Keywords Order-convex function · Nonlinear matrix equation · Newton method · Relaxed Newton method

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1 Introduction

We introduce relevant definitions. A matrix A is called a *nonnegative (or positive) matrix* if all entries of A are nonnegative (or positive) and we say that $A \geq B$ (or $A > B$) if $A - B$ is a nonnegative (or positive) matrix. We say that A and B are *comparable* if $A \geq B$ or $A \leq B$, and that A and B are *strictly comparable* if $A > B$ or $A < B$. A matrix C is called a Z-*matrix* if there exist a real number r and a nonnegative matrix N such that $C = rI - N$. C is called an M-*matrix* if $r \geq \rho(N)$ where $\rho(N)$ is the spectral radius of N. The linear operator vec : $\mathbb{R}^{m\times n} \to \mathbb{R}^{mn}$ is defined by

$$\mathrm{vec}(A) = \left[a_1^T \; a_2^T \; \cdots \; a_n^T\right]^T \quad \text{for } A = \left[\, a_1 | a_2 | \cdots | a_n \,\right], \tag{1.1}$$

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where a_k's are column vectors of A.

In many applications, finding a solution of a nonlinear matrix equation is one of the important issues. Nonsymmetric algebraic Riccati equations occur in applied probability and transportation theory and have been extensively studied. See the details in Rogers (1994), Juang (1995), Benner and Byers (1998), Guo and Laub (2000), Guo et al. (2006), Guo and Higham (2007). Markovian binary trees are found in biology, epidemiology, and telecommunication systems (Haccou et al. 2005; Kimmel and Axelrod 2002; Hautphenne et al. 2006). Matrix polynomial equations occur in the theory of differential equations, system theory, network theory, stochastic theory, quasi-birth-and-death and other areas (Alfa 2003; Bean et al. 1997; Geoffrey et al. 1985; Gohberg et al. 1982; Lancaster 1966; Lancaster and Tismenetsky 1985; He and Neuts 2001; Dario 2005; Latouche and Ramaswami 1999).

These equations have the order-convex properties in common. Order-convex functions have been studied in Vandergraft (1967), Ortega and Rheinboldt (2000). Ortega and Rheinboldt (2000) showed that the convergence of the Newton method for a nonlinear equation $F(X) = 0$ if F is order convex.

Davis (1981, 1983) and Higham and Kim (2000, 2001) considered the Newton method for a quadratic matrix equation. Guo (2001) considered a nonsymmetric algebraic Riccati equation, and they proposed iteration algorithms which converge to the minimal positive solution. In some different types of quadratic equations, (Kim 2008) showed that the minimal positive solutions also can be found by the Newton method with zero starting matrix. Hautphenne et al. (2008) studied the Newton method for the Markovian binary tree. Seo et al. (2013); Seo and Kim (2014) studied about the Newton iteration for a quadratic matrix equation and a matrix polynomial equation. In Seo and Kim (2014), they provided a relaxed Newton method whose convergence is faster than the pure one. The definition of the minimal nonnegative or positive solution is given as the following.

Definition 1.1 Let S_1 and S_2 be nonnegative solutions of a matrix equation $F(X) = 0$. Then, S_1 is called the *minimal nonnegative solution* of $F(X) = 0$ and S_2 is called the *maximal nonnegative solution* of $F(X) = 0$ if

$$S_1 \leq S \leq S_2, \tag{1.2}$$

for any nonnegative solution S of $F(X) = 0$. Similarly, S_1 and S_2 are called the *minimal positive solution* and the *maximal positive solution*, respectively, if positive solutions S_1 and S_2 satisfy (1.2) for any positive solution S.

Some properties of nonnegative matrices and M-matrices are given in the following theorem.

Theorem 1.2 (Guo and Higham 2007, Theorem 2.1) (Poole and Boullion 1974, Theorem 2.1) *For a Z-matrix A, the following are equivalent:*

(1) *A is a nonsingular M-matrix.*
(2) *A^{-1} is nonnegative.*
(3) *$Av > 0$ for some vector $v > 0$.*
(4) *All eigenvalues of A have positive real parts.*

Theorem 1.3 (Roger 1985, Theorem 8.1.18) *Let $A, B \in \mathbb{R}^{n\times n}$. If $|A| \leq B$, then $\rho(A) \leq \rho(|A|) \leq \rho(B)$ where $|A|_{ij} = |A_{ij}|$.*

Theorem 1.4 (Guo and Higham 2007, Lemma 2.2) *Let $A \in \mathbb{R}^{m\times m}$ be a nonsingular M-matrix. If B is a Z-matrix and $B \geq A$, then B is also a nonsingular M-matrix.*

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The outline of the paper is as follows. In Sect. 2, we introduce order-convex functions, and review its properties. In Sect. 3, we introduce (Ortega and Rheinboldt 2000, monotone Newton theorem) and provide a modified theorem which shows that the Newton sequence converges to a solution of $F(X) = 0$ where F is the order convex even if the Fréchet derivative at the solution is singular. We provide a relaxed Newton method for an order-convex matrix equation in Sect. 4. Numerical experiments are given in Sect. 4.1.

2 Order-convex functions

For a matrix function, the Gâteaux and Fréchet derivative are defined as the followings:

Definition 2.1 (Higham 2008) Let $F : \mathbb{C}^{m\times n} \to \mathbb{C}^{p\times q}$ be a matrix function. The *Gâteaux derivative* of F at X in the direction E is defined by

$$G_F(X, E) = \lim_{t\to 0} \frac{F(X + tE) - F(X)}{t}. \tag{2.1}$$

Assume that $\|\cdot\|_{m\times n} : \mathbb{C}^{m\times n} \to \mathbb{R}$ and $\|\cdot\|_{p\times q} : \mathbb{C}^{p\times q} \to \mathbb{R}$ are norms, and a linear transformation $L_{F(X)} : \mathbb{C}^{m\times n} \to \mathbb{C}^{p\times q}$ satisfies

$$F(X + H) - F(X) = L_{F(X)}(H) + o(H), \tag{2.2}$$

where $\|o(H)\|_{p\times q}/\|H\|_{m\times n} \to 0$ as $\|H\|_{m\times n} \to 0$. Then, $L_{F(X)}(H)$ is called the *Fréchet derivative* of F at X in the direction H.

If the Fréchet derivative of f exists at X, then it is equal to the Gâteaux derivative (Higham 2008). In this paper, we deal with Fréchet differentiable functions. So, we denote the Gâteaux and Fréchet derivative of F at X in the direction H as $F'[X](H)$. $F'[X]$ is a linear operator from $\mathbb{C}^{m\times n}$ to $\mathbb{C}^{p\times q}$. Thus, $\mathrm{vec} \circ F'[X] \circ \mathrm{vec}^{-1}$ is a $pq \times mn$ matrix. For convenience, we denote $\mathcal{F}'[X]$ this matrix.

The followings are the definition and an important theorem of order-convex functions.

Definition 2.2 (Vandergraft 1967; Ortega and Rheinboldt 2000) Let $F : \mathfrak{D} \subset \mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$ be a function. Then, F is *order convex* on a convex subset $\mathfrak{D}_0 \subset \mathfrak{D}$ if

$$F(tX + (1 - t)Y) \le tF(X) + (1 - t)F(Y), \tag{2.3}$$

where $X, Y \in \mathfrak{D}_0$ are comparable and $t \in (0, 1)$.

Theorem 2.3 (Ortega and Rheinboldt 2000, Theorem 13.3.2.) *Let $F : \mathfrak{D} \subset \mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$ be Gâteaux differentiable on the convex set $\mathfrak{D}_0 \subset \mathfrak{D}$. Then, the following statements are equivalent.*

(1) *F is order convex on $\mathfrak{D}_0$;*
(2) *$F(Y) - F(X) \ge F'[X](Y - X)$, for comparable $X, Y \in \mathfrak{D}_0$;*
(3) *$\big(F'[Y] - F'[X]\big)(Y - X) \ge 0$, for comparable $X, Y \in \mathfrak{D}_0$.*

Let a matrix equation

$$F(X) = 0 \tag{2.4}$$

be given, where $F : \mathfrak{D} \subset \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$. If F is order convex on $\mathfrak{D}_0 \subset \mathfrak{D}$, then we say that (2.4) is an *order-convex matrix equation* on $\mathfrak{D}_0$, for the convenience.

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Consider a fixed point iteration

$$X_{i+1} = G(X_i), \tag{2.5}$$

where G is a Gâteaux differentiable and order-convex matrix function on $\mathfrak{D}_0$.

Lemma 2.4 *Let $\{X_i\}$ be a sequence defined by (2.5) with a given $X_0 \in \mathfrak{D}_0$. If X_j and X_l are comparable and in $\mathfrak{D}_0$ for all $j, l \in \mathbb{N}_0$ where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then*

$$G'[X_{l-1}](X_{j-1} - X_{l-1}) \le X_j - X_l \le G'[X_{j-1}](X_{j-1} - X_{l-1}). \tag{2.6}$$

Furthermore, if $\{X_i\}$ converges to $S \in \mathfrak{D}_0$, then

$$G'[X_{j-1}](S - X_{j-1}) \le S - X_j \le G'[S](S - X_{j-1}). \tag{2.7}$$

Proof Since X_j and X_l are comparable for all $j, l \in \mathbb{N}_0$, we obtain that

$$\begin{aligned} X_j - X_l &= G(X_{j-1}) - G(X_{l-1}) \\ &\ge G'[X_{l-1}](X_{j-1} - X_{l-1}), \end{aligned} \tag{2.8}$$

by Theorem 2.3.

On the other hand, we obtain that

$$X_j - X_l = G(X_{j-1}) - G(X_{l-1}) \le G'[X_{j-1}](X_{j-1} - X_{l-1}). \tag{2.9}$$

Therefore, (2.6) holds for all $j, l \in \mathbb{N}_0$.

Obviously, S is comparable with X_j for all $j \in \mathbb{N}_0$ and $S = G(S)$. Thus

$$\begin{aligned} G'[X_{j-1}](S - X_{j-1}) &\le G(S) - G(X_{j-1}) \\ &\le G'[S](S - X_{j-1}). \end{aligned}$$

Since $S - X_j = G(S) - G(X_{j-1})$, (2.7) holds. □

Theorem 2.5 *Let $\{X_i\}$ be a sequence defined in Lemma 2.4. If $\{X_i\}$ is monotone nondecreasing, converges to $S \in \mathfrak{D}_0$, and $\mathcal{G}'[S] \ge 0$, then*

$$\limsup_{i\to\infty} \sqrt[i]{\|X_i - S\|_F} \le \rho(\mathcal{G}'[S]). \tag{2.10}$$

Proof By Lemma 2.4, we get the inequality

$$\begin{aligned} S - X_i &\le G'[S](S - X_{i-1}) \\ &\le (G'[S] \circ G'[S])(S - X_{i-2}) \\ &\;\vdots \\ &\le \underbrace{(G'[S] \circ \cdots \circ G'[S])}_{i\text{-times}}(S - X_0). \end{aligned} \tag{2.11}$$

Applying 2-norm and Frobenius norm, we obtain that

$$\begin{aligned} \|X_i - S\|_F &= \|\mathrm{vec}(S - X_i)\|_2 \\ &\le \|(\mathcal{G}'[S])^i \cdot \mathrm{vec}(S - X_0)\|_2 \\ &\le \|(\mathcal{G}'[S])^i\|_2 \|\mathrm{vec}(S - X_0)\|_2 \\ &= \|(\mathcal{G}'[S])^i\|_2 \|S - X_0\|_F. \end{aligned} \tag{2.12}$$

Therefore

$$\begin{aligned}\limsup_{i\to\infty}\sqrt[i]{\|X_i - S\|_F} &\le \limsup_{i\to\infty}\sqrt[i]{\|(\mathcal{G}'[S])^i\|_2\|S - X_0\|_F}\\ &= \limsup_{i\to\infty}\sqrt[i]{\|(\mathcal{G}'[S])^i\|_2}\\ &= \rho(\mathcal{G}'[S]),\end{aligned}$$

by (Roger 1985, Corollary 5.6.14). □

Corollary 2.6 *Let $\{X_i\}$ be a sequence in Lemma 2.4. Suppose that the following conditions hold.*

(1) *$\{X_i\}$ is monotone nondecreasing and converges to S,*
(2) *$S - X_0 > 0$,*
(3) *$\mathcal{G}'[X_0]$ is a nonnegative matrix such that $\mathcal{G}'[X_0]\mathbf{1} > 0$,*
(4) *$\{\mathcal{G}'[X_i]\}$ is monotone nondecreasing,*
(5) *$\mathcal{G}'[\,\cdot\,]$ is continuous on $\mathfrak{D}_0$.*

Then

$$\limsup_{i\to\infty}\sqrt[i]{\|X_i - S\|_F} = \rho(\mathcal{G}'[S]). \tag{2.13}$$

Proof Note that $\lim_{i\to\infty}\mathcal{G}'[X_i] = \mathcal{G}'[S]$. So, for any $\varepsilon > 0$, there exists an integer l such that

$$\rho(\mathcal{G}'[X_l]) \ge \rho(\mathcal{G}'[S]) - \varepsilon.$$

From the inequality

$$\begin{aligned}\operatorname{vec}(S - X_i) &\ge \operatorname{vec}(G'[X_{i-1}](S - X_{i-1}))\\ &= \mathcal{G}'[X_{i-1}]\cdot\operatorname{vec}(S - X_{i-1})\\ &\ge \mathcal{G}'[X_{i-1}]\cdot\operatorname{vec}(G'[X_{i-2}](S - X_{i-2}))\\ &= \mathcal{G}'[X_{i-1}]\mathcal{G}'[X_{i-2}]\cdot\operatorname{vec}(S - X_{i-2})\\ &\ \ \vdots\\ &\ge \mathcal{G}'[X_{i-1}]\mathcal{G}'[X_{i-2}]\cdots\mathcal{G}'[X_0]\cdot\operatorname{vec}(S - X_0),\end{aligned} \tag{2.14}$$

we obtain that

$$\begin{aligned}\|X_i - S\|_F &\ge \|\mathcal{G}'[X_{i-1}]\mathcal{G}'[X_{i-2}]\cdots\mathcal{G}'[X_0]\cdot\operatorname{vec}(S - X_0)\|_2\\ &\ge \|\mathcal{G}'[X_l]^{i-l}\mathcal{G}'[X_0]^l\cdot\operatorname{vec}(S - X_0)\|_2.\end{aligned} \tag{2.15}$$

Since $\mathcal{G}'[X_0]^l\cdot\operatorname{vec}(S - X_0) > 0$, there exist $c > 0$ and a nonnegative vector v such that

$$\begin{aligned}\|\mathcal{G}'[X_l]^{i-l}\mathcal{G}'[X_0]^l\cdot\operatorname{vec}(S - X_0)\|_2 &\ge \|\mathcal{G}'[X_l]^{i-l}cv\|_2\\ &= c\|\mathcal{G}'[X_l]^{i-l}v\|_2\\ &= c\|\mathcal{G}'[X_l]^{i-l}\|_2.\end{aligned}$$

Therefore

$$\begin{aligned}\limsup_{i\to\infty}\sqrt[i]{\|X_i - S\|_F} &\ge \limsup_{i\to\infty}\sqrt[i]{\|\mathcal{G}'[X_l]^{i-l}\mathcal{G}'[X_0]^l\cdot\operatorname{vec}(S - X_0)\|_2}\\ &\ge \limsup_{i\to\infty}\sqrt[i]{c\|\mathcal{G}'[X_l]^{i-l}\|_2}\\ &= \limsup_{i\to\infty}\sqrt[i]{\|\mathcal{G}'[X_l]^{i-l}\|_2}\\ &= \rho(\mathcal{G}'[X_l]) \ge \rho(\mathcal{G}'[S]) - \varepsilon.\end{aligned}$$

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Since $\varepsilon > 0$ is arbitrary, we have

$$\limsup_{i\to\infty} \sqrt[i]{\|X_i - S\|_F} = \rho(\mathcal{G}'[S]),$$

by Theorem 2.5. □

For example, let the fixed point iteration

$$X_{i+1} = \mathcal{L}^{-1}(X_i C X_i + X_i D_2 + A_2 X_i + B) \tag{2.16}$$

be given, where $A_2, D_2 \geq 0$, $B, C > 0$ and $\mathcal{L}(X) = A_1 X + X D_1$ such that A_1 and D_1 are M-matrices. If X_0 is the solution of the equation $A_1 X_0 + X_0 D_1 = B$, and we define a matrix function G by

$$G(X) = \mathcal{L}^{-1}(XCX + XD_2 + A_2 X + B), \tag{2.17}$$

then we can easily verify that G is order convex on $\mathbb{R}_+^{m\times n} := \{X \in \mathbb{R}^{m\times n} | X \geq 0\}$, and $\{X_i\}$ satisfies the conditions of Corollary 2.6. Therefore

$$\limsup_{i\to\infty} \sqrt[i]{\|X_i - S\|_F} = \rho(\mathcal{G}'[S]), \tag{2.18}$$

where $G'[S]$ is the derivative of G at S and $\mathcal{G}'[S] = \mathrm{vec} \circ G'[S] \circ \mathrm{vec}^{-1}$. In (Guo and Laub 2000, Theorem 3.2), Guo and Laub also proved (2.18) with direct calculations.

3 The Newton method for an order-convex function

The Newton method for solving a matrix equation $F(X) = 0$ is defined by a given initial matrix X_0 and the iteration

$$X_{i+1} = X_i - F'[X_i]^{-1}(F(X_i)), \quad i = 0, 1, 2, \ldots, \tag{3.1}$$

where $F : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is a matrix function and $F'[X]$ is the Fréchet derivative of F at X. If $F'[X_i]$ is nonsingular, then (3.1) is equivalent to

$$\begin{cases} -F'[X_i](H_i) = F(X_i), \\ X_{i+1} = X_i + H_i, \end{cases} \quad i = 0, 1, 2, \ldots. \tag{3.2}$$

To get H_i in the equation $-F'[X_i](H_i) = F(X_i)$, we apply the operator vec in both sides of the equation. Then, we obtain the following equation:

$$-\mathcal{F}'[X_i]\mathrm{vec}(H_i) = \mathrm{vec}\,(F(X_i))\,. \tag{3.3}$$

Since $-\mathcal{F}'[X_i]$ is a nonsingular $mn \times mn$ matrix, we can obtain the unique H_i for each $i = 0, 1, 2, \ldots$.

Theorem 3.1 (Ortega and Rheinboldt 2000, monotone Newton theorem) *Let $F : \mathfrak{D} \subset \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$, and assume that there exist $X_0, Y_0 \in \mathfrak{D}$ such that*

$$X_0 \leq Y_0, \quad [X_0, Y_0] \subset \mathfrak{D}, \quad F(X_0) \leq 0 \leq F(Y_0), \tag{3.4}$$

where $[X_0, Y_0] = \{A | X_0 \leq A \leq Y_0\}$.

Suppose that F is continuous, Gâteaux differentiable, order convex on $[X_0, Y_0]$, and that, for each $X \in [X_0, Y_0]$, $\mathcal{F}'[X]^{-1}$ exists and is nonnegative. Then, the Newton iteration defined by

$$Y_{k+1} = Y_k - F'[Y_k]^{-1}(F(Y_k)), \quad k = 0, 1, 2, \ldots, \tag{3.5}$$

converges to $S \in [X_0, Y_0]$ *and is monotone nonincreasing. Furthermore, if* $\mathcal{F}'[\,\cdot\,]$ *is either continuous at* S *or monotone nondecreasing on* $[X_0, Y_0]$*, then* S *is the unique solution of* $F(X) = 0$ *in* $[X_0, Y_0]$*.*

Corollary 3.2 *Let* $F : \mathfrak{D} \subset \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$*, and suppose that there exist* $X_0, Y_0 \in \mathfrak{D}$ *such that*

$$X_0 \leq Y_0, \quad [X_0, Y_0] \subset \mathfrak{D}, \quad F(X_0) \geq 0 \geq F(Y_0). \tag{3.6}$$

Suppose that F *is continuous, Gâteaux differentiable, order convex on* $[X_0, Y_0]$*, and that, for each* $X \in [X_0, Y_0]$*,* $-\mathcal{F}'[X]^{-1}$ *exists and is nonnegative. Then, the Newton iteration*

$$X_{i+1} = X_i - F'[X_i]^{-1}(F(X_i)), \quad i = 0, 1, 2, \ldots, \tag{3.7}$$

converges to $S \in [X_0, Y_0]$ *and is monotone nondecreasing. Furthermore, if* $\mathcal{F}'[\,\cdot\,]$ *is either continuous at* S *or monotone nonincreasing on* $[X_0, Y_0]$*, then* S *is the unique solution of* $F(X) = 0$ *in* $[X_0, Y_0]$*.*

Proof Define $F_1 : [X_0, Y_0] \to [X_0, Y_0]$ by $F_1(X) = X_0 + Y_0 - X$ and $F_2(X) = (F \circ F_1)(X)$. Since F_1 is Fréchet differentiable in $[X_0, Y_0]$, F_2 is Gâteaux differentiable $[X_0, Y_0]$ in and

$$\begin{aligned} F_2'[X](H) &= \left(F'[F_1(X)] \circ F_1'[X]\right)(H) \\ &= -F'[X_0 + Y_0 - X](H), \end{aligned}$$

by (Ortega and Rheinboldt 2000, 3.1.7. Chain Rule.). Furthermore, for comparable $X, Y \in [X_0, Y_0]$,

$$\begin{aligned} F_2(Y) - F_2(X) &= F(X_0 + Y_0 - Y) - F(X_0 + Y_0 - X) \\ &\geq F'[X_0 + Y_0 - X](X_0 + Y_0 - Y - X_0 - Y_0 + X) \\ &= -F'[X_0 + Y_0 - X](Y - X) \\ &= F_2'[X](Y - X). \end{aligned}$$

Since $-\mathcal{F}'[X]^{-1}$ exists and is nonnegative for all $X \in [X_0, Y_0]$, $\mathcal{F}_2'[X]^{-1} = -\mathcal{F}'[X_0 + Y_0 - X]^{-1}$ exists and is nonnegative for all $X \in [X_0, Y_0]$. At last,

$$F_2(X_0) = F(Y_0) \leq 0 \leq F(X_0) = F_2(Y_0).$$

Therefore, by Theorem 3.1, the Newton iteration

$$Y_{i+1} = Y_i - F_2'[Y_i]^{-1}(F_2(Y_i)), \quad i = 0, 1, 2, \ldots,$$

converges to $S_2 \in [X_0, Y_0]$ and is monotone nonincreasing.

Put $X_k = X_0 + Y_0 - Y_k$. Then

$$\begin{aligned} X_{i+1} &= X_0 + Y_0 - Y_{i+1} \\ &= X_0 + Y_0 - Y_i + F_2'[Y_i]^{-1}(F_2(Y_i)) \\ &= X_i - F'[X_0 + Y_0 - Y_i]^{-1}(F(X_0 + Y_0 - Y_i)) \\ &= X_i - F'[X_i]^{-1}(F(X_i)). \end{aligned}$$

So, $\{X_i\}$ is also the Newton sequence and monotone nondecreasing, and converges to $S = X_0 + Y_0 - S_2$.

Finally, since $\mathcal{F}'[X] = -\mathcal{F}_2'[X_0 + Y_0 - X]$, $\mathcal{F}'[\,\cdot\,]$ is monotone nonincreasing on $[X_0, Y_0]$ if and only if $\mathcal{F}_2'[\,\cdot\,]$ is monotone nondecreasing on $[X_0, Y_0]$. Moreover, $\mathcal{F}'[\,\cdot\,]$ is continuous

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at S if and only if $\mathcal{F}_2'[\cdot]$ is continuous at S_2 since $\|X - S\| = \|X - X_0 - Y_0 + S_2\| = \|X_0 + Y_0 - X - S_2\|$ and

$$\begin{aligned}\|\mathcal{F}'[X] - \mathcal{F}'[S]\| &= \|\mathcal{F}'[X] - \mathcal{F}'[X_0 + Y_0 - S_2]\|\\ &= \| - \mathcal{F}_2[X_0 + Y_0 - X] + \mathcal{F}_2'[S_2]\|\\ &= \|\mathcal{F}_2[X_0 + Y_0 - X] - \mathcal{F}_2'[S_2]\|,\end{aligned}$$

for any norm $\|\cdot\|$. □

Suppose that a matrix function F and a matrix $X_0 = 0$ satisfy the hypothesis of Corollary 3.2. Then, the corollary shows that the Newton sequence converges to a nonnegative solution S. Moreover, if S is the unique solution in $[X_0, Y_0]$, then S is the minimal nonnegative solution. All of these are based on the hypothesis that $\mathcal{F}'[S]$ is nonsingular. But, sometimes, the derivative is singular.

For example, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(\mathbf{x}) = \begin{bmatrix}(x_1 - 1)^2 + (x_2 - 1)^2\\ \frac{1}{2}(x_1 - 1)^2 + (x_2 - 1)^2\end{bmatrix}, \quad \text{where } \mathbf{x} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{3.8}$$

Then, f is differentiable and order convex on $\mathbb{R}^2$, $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^2$, and

$$f'[\mathbf{x}] = \begin{bmatrix}2x_1 - 2 & 2x_2 - 2\\ x_1 - 1 & 2x_2 - 2\end{bmatrix}.$$

If $\mathbf{x}_0 = 0$ and $\mathbf{y}_0 = [1\ 1]^T$, then $f(\mathbf{y}_0) = 0$ and $f'[\mathbf{y}_0] = 0$ is singular. So, f satisfies the conditions of Corollary 3.2 except that $-f'[\mathbf{y}_0]^{-1}$ exists. But, the Newton sequence $\{\mathbf{x}_i\}$ converges to $[1\ 1]^T$.

So, we provide a modified monotone Newton theorem for the case that the derivative at S is singular. Before we do that, we introduce the definition of strictly order convex and relevant properties.

Definition 3.3 Let $F : \mathfrak{D} \subset \mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$ be a function. Then, F is *strictly order convex* on a convex subset $\mathfrak{D}_0 \subset \mathfrak{D}$ if

$$F(tX + (1-t)Y) < tF(X) + (1-t)F(Y), \tag{3.9}$$

where $X, Y \in \mathfrak{D}_0$ are strictly comparable and $t \in (0, 1)$.

Theorem 3.4 (cf. (Ortega and Rheinboldt 2000, Theorem 3.2.2)) *Suppose that $f : D \subset \mathbb{R}^n \to \mathbb{R}^1$ is Gâteaux differentiable at each point of a convex set $D_0 \subset D$. Then, for any two points $x, y \in D_0$, there is a $t \in (0, 1)$ such that*

$$f(y) - f(x) = f'[x + t(y - x)](y - x).$$

Theorem 3.5 *Let $\mathfrak{D}$ be open, $\mathfrak{D}_0 \subset \mathfrak{D}$ be a convex set, and $F : \mathfrak{D} \subset \mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$ be Gâteaux differentiable on $\mathfrak{D}$. Then, the following statements are equivalent.*

(1) *F is strictly order convex on $\mathfrak{D}_0$;*
(2) *$F(Y) - F(X) > F'[X](Y - X)$, for strictly comparable $X, Y \in \mathfrak{D}_0$;*
(3) *$[F'[Y] - F'[X]](Y - X) > 0$, for strictly comparable $X, Y \in \mathfrak{D}_0$.*

Proof Let $X, Y \in \mathfrak{D}_0$ be strictly comparable.

Suppose that (2) holds. Let $Z = tX + (1-t)Y$ for some $t \in (0, 1)$. Then, Z is strictly comparable with X and Y. So, we obtain that

$$F(X) - F(Z) > F'[Z](X - Z), \tag{3.10}$$

$$F(Y) - F(Z) > F'[Z](Y - Z). \tag{3.11}$$

Multiplying t to (3.10) and $1-t$ to (3.11), and adding these, we obtain that

$$tF(X) + (1-t)F(Y) - F(Z) > F'[Z](tX + (1-t)Y - Z) = 0, \tag{3.12}$$

i.e., (1) holds.

Conversely, suppose that (1) holds. Let $Z_1 = (1-t_1)X + t_1 Y$ for some $t_1 \in (0, 1)$. Then

$$\begin{aligned}
&(1-t_1)F(X) + t_1 F(Y) > F(Z_1) \\
&\quad \Leftrightarrow t_1\,[F(Y) - F(X)] > F(Z_1) - F(X) \\
&\quad \Leftrightarrow F(Y) - F(X) > (1/t_1)\,[F(Z_1) - F(X)]\,.
\end{aligned}$$

Since X and $X + t_2(Z_1 - X)$ are strictly comparable for any $t_2 \in (0, 1)$, we obtain that

$$F(Z_1) - F(X) > \frac{1}{t_2}\,[F(X + t_2(Z_1 - X)) - F(X)]\,.$$

In the view of Gâteux differentiable of F,

$$F(Z_1) - F(X) \geq F'[X](Z_1 - X)$$

follows as $t_2 \to 0$.

By the way, $Z_1 - X = t_1(Y - X)$. Therefore

$$\begin{aligned}
F(Y) - F(X) &> (1/t_1)\,[F(Z_1) - F(X)] \\
&\geq (1/t_1)F'[X](Z_1 - X) \\
&= F'[X](Y - X).
\end{aligned}$$

Suppose that (2) holds. Then, adding the following two inequalities:

$$F(Y) - F(X) > F'[X](Y - X) \quad \text{and} \quad F(X) - F(Y) > F'[Y](X - Y),$$

we easily obtain (3).

Conversely, if (3) holds, then

$$(F'_{ij}[Y] - F'_{ij}[X])(Y - X) > 0, \quad i = 1, \ldots, p, \quad j = 1, \ldots, q,$$

where F_{ij} is the (i, j)-entry function of F. By Theorem 3.4, there is $t_{ij} \in (0, 1)$ such that

$$F_{ij}(Y) - F_{ij}(X) = F'_{ij}[Z^{ij}](Y - X),$$

where $Z^{ij} = X + t_{ij}(Y - X)$. Since Z^{ij} is strictly comparable with X and Y,

$$(F'_{ij}[Z^{ij}] - F'_{ij}[X])(Y - X) = (1/t_{ij})(F'_{ij}[Z^{ij}] - F'_{ij}[X])(Z^{ij} - X) > 0,$$

and, so,

$$F_{ij}(Y) - F_{ij}(X) = F'_{ij}[Z^{ij}](Y - X) > F'_{ij}[X](Y - X),$$

for all $i = 1, \ldots, p$ and $j = 1, \ldots, q$. □

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Lemma 3.6 *The following statements hold:*

(1) *Let A and B be nonnegative matrices. If $A \leq B$ and $B - A$ is irreducible, then $\rho(A) < \rho(B)$.*
(2) *Let $C \in \mathbb{R}^{n\times n}$ be a singular M-matrix. If D is a Z-matrix and $D - C$ is nonnegative and irreducible, then D is a nonsingular M-matrix.*

Proof To prove (1), suppose that $0 \leq A \leq B$, $B - A$ is irreducible, and $\rho(A) = \rho(B)$. By Perron–Frobenius Theorem, there exists a vector $x > 0$ such that $(B - A)x = \rho(B - A)x$ and $\rho(B - A) > 0$. Clearly, $\rho(B) < \rho(B) + \rho(B - A)$. By (Roger 1985, Theorem 8.3.2), we obtain that $By < [\rho(B) + \rho(B - A)]\, y$ for any nonzero $y \geq 0$. Therefore

$$\begin{aligned} Ax &= Bx - \rho(B - A)x \\ &< (\rho(B) + \rho(B - A))x - \rho(B - A)x \\ &= \rho(B)x = \rho(A)x. \end{aligned}$$

But, it contradicts to (Roger 1985, Corollary 8.1.29) which shows that $Ax < \beta x$ implies $\rho(A) < \beta$ for any $A \geq 0$ and $x > 0$. So, $\rho(A) < \rho(B)$.

To show (2), put $s = \max\{D_{ii} | 1 \leq i \leq n\}$. Since $s \geq C_{ii}$ for all $i = 1, 2, \ldots, n$, $N_D = sI - D$ and $N_C = sI - C$ are nonnegative and $N_C - N_D = D - C$ is nonnegative and irreducible. By (1) and that C is singular, $\rho(N_D) < \rho(N_C) = s$. Therefore, D is a nonsingular M-matrix. □

From Theorem 3.5 and Lemma 3.6, we yield the following theorem.

Theorem 3.7 *Let $\mathfrak{D} \subset \mathbb{R}^{m\times n}$ be open and $F : \mathfrak{D} \subset \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ be a matrix function which satisfies the following conditions:*

(1) *There exist X_0 and Y such that $X_0 < Y$ and $F(X_0) > 0 \geq F(Y)$,*
(2) *$\mathcal{F}'[X]$ exists for all $X \in [X_0, Y]$, and $A < B$ implies that $\mathcal{F}'[B] - \mathcal{F}'[A]$ is nonnegative and irreducible for all $A, B \in [X_0, Y]$, and*
(3) *$-\mathcal{F}'[X_0]$ is a Z-matrix and $-\mathcal{F}'[Y]$ is a singular or nonsingular M-matrix.*

Then, $X_0 < X_1 < Y$, $F(X_1) > 0 \geq F(Y)$, and $-\mathcal{F}'[X_1]$ is a Z-matrix, where

$$X_1 = X_0 - F'[X_0]^{-1}(F(X_0)).$$

Proof By the conditions (1) and (2), $\mathcal{F}'[Y] - \mathcal{F}'[X_0]$ is nonnegative and irreducible. So, $-\mathcal{F}'[X_0]$ is a nonsingular M-matrix by Lemma 3.6.

From the condition (2), for any strictly comparable $A, B \in [X_0, Y]$,

$$(F'[B] - F'[A])(B - A) > 0.$$

By Theorem 3.5, F is strictly order convex on $[X_0, Y]$. Thus, we get that

$$\begin{aligned} F(X_1) &> F(X_0) + F'[X_0](X_1 - X_0) \\ &= F(X_0) + F'[X_0]\left(-F'[X_0]^{-1}(F(X_0))\right) \\ &= 0. \end{aligned} \tag{3.13}$$

Moreover

$$\begin{aligned} -F'[X_0](Y - X_1) &= -F'[X_0](Y - X_0) + F'[X_0](X_1 - X_0) \\ &> F(X_0) - F(Y) + F'[X_0](X_1 - X_0) \\ &\geq F(X_0) + F'[X_0](X_1 - X_0) \\ &= 0, \end{aligned} \tag{3.14}$$

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and this inequality is equivalent to

$$-\mathcal{F}'[X_0]\mathrm{vec}(Y - X_1) > 0. \tag{3.15}$$

By the definition of X_1 and the condition (1), we obtain that

$$-\mathcal{F}'[X_0]\mathrm{vec}(X_1 - X_0) = \mathrm{vec}(F(X_0)) > 0. \tag{3.16}$$

Since $-\mathcal{F}'[X_0]$ is a nonsingular M-matrix, $X_0 < X_1 < Y$ from (3.15), (3.16), and (Seo et al. 2013, Theorem 2.6).

Finally, $\mathcal{F}'[X_1] - \mathcal{F}'[X_0] \geq 0$ since $X_0 < X_1$. Hence, all off-diagonal entries of $\mathcal{F}'[X_1]$ are nonnegative, i.e., $-\mathcal{F}'[X_1]$ a Z-matrix. □

From Theorem 3.7, we obtain the following main theorem.

Theorem 3.8 (Singular monotone Newton Theorem) *Let $\mathfrak{D} \subset \mathbb{R}^{m\times n}$ be open and $F : \mathfrak{D} \subset \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$, and suppose that there exist $X_0, Y_0 \in \mathfrak{D}$ such that*

$$X_0 < Y_0, \quad [X_0, Y_0] \subset \mathfrak{D}, \quad F(X_0) > 0 \geq F(Y_0). \tag{3.17}$$

Suppose that F is Fréchet differentiable on $[X_0, Y_0]$, $A < B$ implies that $\mathcal{F}'[B] - \mathcal{F}'[A]$ is nonnegative and irreducible for all $A, B \in [X_0, Y_0]$, $-\mathcal{F}'[X_0]$ is a Z-matrix, and $-\mathcal{F}'[Y_0]$ is a singular or nonsingular M-matrix. Then, the Newton iteration

$$X_{i+1} = X_i - F'[X_i]^{-1}(F(X_i)), \qquad i = 0, 1, 2, \ldots, \tag{3.18}$$

converges to $S \in [X_0, Y_0]$ and is monotone increasing.

4 Relaxed Newton method for order-convex equations

Benner and Byers (1998) provided an exact line search method along the Newton direction for solving generalized continuous-time algebraic Riccati equations. Higham and Kim (2001) improved Newton's method for solving quadratic matrix equations with line searches. Seo and Ki (2008) also provided Newton's method with exact line searches for the matrix polynomial equations. The idea of exact line search methods for matrix equations is to find λ which minimizes the merit function

$$p(\lambda) = ||F(X_i + \lambda H_i)||^2. \tag{4.1}$$

But, we may not apply the method to order-convex equations since we can not guarantee that $\mathcal{F}'[X_i + \lambda H_i]^{-1}$ exists and is nonnegative. By the way, (Seo and Kim 2014) provided the relaxed Newton method for a matrix polynomial equation(MPE) arising in stochastic models. The MPE under the condition in Seo and Kim (2014) is also an order-convex matrix equation on $\{X \in \mathbb{R}^{m\times m} : X \geq 0\}$. So, in this section, we extend the proof of the relaxed Newton method for the order-convex matrix equations.

Theorem 4.1 *Let $F : \mathfrak{D} \subset \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$, and suppose that there exist $Y_0 \in \mathfrak{D}$ such that*

$$X_0 \leq Y_0, \quad [X_0, Y_0] \subset \mathfrak{D}, \quad F(X_0) \geq 0 \geq F(Y_0). \tag{4.2}$$

Suppose that F is continuous, Gâteaux differentiable, order convex on $[X_0, Y_0]$, $\mathcal{F}'[\,\cdot\,]$ is monotone nondecreasing on $[X_0, Y_0]$, and that, for each $X \in [X_0, Y_0]$, $-\mathcal{F}'[X]^{-1}$ exists and is nonnegative. Put X_1 be the next Newton step for F of X_0. If we define that

$$X_\lambda = X_0 - \lambda F'[X_0]^{-1}(F(X_0)), \tag{4.3}$$

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where

$$\lambda = \min\left\{ \frac{[F(X_0)]_{jk} + [F(X_1)]_{jk}}{[F(X_0)]_{jk}} \middle| [F(X_0)]_{jk} > 0,\ 1 \le j \le m,\ 1 \le k \le n \right\}, \tag{4.4}$$

then $X_1 \le X_\lambda \le Y_0$ *and* $F(X_\lambda) \ge 0$.

Proof From Corollary 3.2, $\{X_i\}$ is well defined and monotone nondecreasing and we can easily check that $F(X_i) \ge 0$ for $i = 1, 2, \ldots$. Thus, it is clear that $\lambda \ge 1$. From $\lambda \ge 1$,

$$X_1 = X_0 - F'[X_0]^{-1}(F(X_0)) \le X_0 - \lambda F'[X_0]^{-1}(F(X_0)) = X_\lambda. \tag{4.5}$$

On the other hand, since $\mathcal{F}'[\,\cdot\,]$ is monotone nondecreasing on $[X_0, Y_0]$,

$$\begin{aligned}
-F'[X_0](X_2 - X_\lambda) &= -F'[X_0]\left(X_1 - F'[X_1]^{-1}(F(X_1)) - X_\lambda\right)\\
&= -F'[X_0]\left((\lambda-1)F'[X_0]^{-1}(F(X_0)) - F'[X_1]^{-1}(F(X_1))\right)\\
&= (1-\lambda)F(X_0) + F'[X_0]\left(F'[X_1]^{-1}(F(X_1))\right)\\
&\ge (1-\lambda)F(X_0) + F'[X_1]\left(F'[X_1]^{-1}(F(X_1))\right)\\
&= F(X_0) + F(X_1) - \lambda F(X_0) \ge 0,
\end{aligned}$$

where X_2 is the next Newton step of X_1. Since $-\mathcal{F}'[X_0]^{-1}$ is nonnegative, $X_\lambda \le X_2$. So, $X_\lambda \le Y_0$. Since $X_\lambda \in [X_0, Y_0]$ and $\mathcal{F}'[\,\cdot\,]$ is monotone nondecreasing on $[X_0, Y_0]$,

$$\begin{aligned}
F(X_\lambda) &\ge F(X_1) + F'[X_1](X_\lambda - X_1)\\
&= F(X_1) + (\lambda-1)F'[X_1]\left(-F'[X_0]^{-1}(F(X_0))\right)\\
&\ge F(X_1) + (\lambda-1)F'[X_0]\left(-F'[X_0]^{-1}(F(X_0))\right)\\
&= F(X_1) + F(X_0) - \lambda F(X_0) \ge 0.
\end{aligned}$$

□

Theorem 4.2 *Let* $F : \mathfrak{D} \subset \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$, *and suppose that there exist* $Y_0 \in \mathfrak{D}$ *such that*

$$X_0 \le Y_0, \quad [X_0, Y_0] \subset \mathfrak{D}, \quad F(X_0) \ge 0 \ge F(Y_0). \tag{4.6}$$

Suppose that F *is continuous, Gâteaux differentiable, order convex on* $[X_0, Y_0]$, $\mathcal{F}'[\,\cdot\,]$ *is monotone nondecreasing on* $[X_0, Y_0]$, *and that, for each* $X \in [X_0, Y_0]$, $-\mathcal{F}'[X]^{-1}$ *exists and is nonnegative. Then, for the relaxed Newton iteration*

$$X_{i+1} = X_i - \lambda_i F'[X_i]^{-1}(F(X_i)), \tag{4.7}$$

the sequence $\{X_i\}$ *is well defined and monotone nondecreasing, and converges to* $S \in [X_0, Y_0]$ *where*

$$\lambda_i = \min\left\{ \frac{[F(X_i)]_{jk} + [F(X_{i+1})]_{jk}}{[F(X_i)]_{jk}} \middle| [F(X_i)]_{jk} > 0,\ 1 \le j \le m,\ 1 \le k \le n \right\}.$$

4.1 Numerical experiments

In this subsection, we give numerical experiments to compare the relaxed Newton method with the pure one. The tolerance of the algorithms is $m \times 10^{-16}$ and we stop the algorithms if the relative residuals are less than the tolerance. The relative residuals are given by

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$$\frac{\|R(X_i)\|_F}{\|C\|_F\|X_i\|_F^2 + \|D\|_F\|X_i\|_F + \|A\|_F\|X_i\|_F + \|B\|_F}, \tag{4.8}$$

in Example 4.3, and

$$\frac{\|M(x_i)\|_F}{\|x_i\|_F + \|a\|_F + \|B\|_F\|x_i\|_F^2}, \tag{4.9}$$

in Example 4.4. The following experiments were executed by MATLAB R2015a (8.5.0.197613) 64-bit with Celeron dual processor at 1.80GHz with 4GB of RAM.

Example 4.3 We consider the nonsymmetric algebraic Riccati equation

$$R(X) = XCX - XD - AX + B = 0, \tag{4.10}$$

where the matrix

$$K = \begin{bmatrix} D & -C \\ -B & A \end{bmatrix} \tag{4.11}$$

is a nonsingular or singular irreducible M-matrix. Guo (2001) proved the existence of the minimal nonnegative solution of (4.10) with (4.11). Xue et al. (2012) called this equation as an M-matrix algebraic Riccati equation(MARE). Let A, B, C, and D be given by

$$\begin{aligned} A &= I_3 - (1-\delta)\left[\frac{3}{8}\left(\mathbf{1}_{3\times 3} - I_3\right)\right], \qquad & B &= \frac{3+\delta}{3}\left[\frac{1}{20}\mathbf{1}_{3\times 5}\right], \\ C &= (1-\delta)\left[\frac{1}{12}\mathbf{1}_{5\times 3}\right], \qquad & D &= I_5 - \frac{3+\delta}{3}\left[\frac{3}{16}\left(\mathbf{1}_{5\times 5} - I_5\right)\right], \end{aligned} \tag{4.12}$$

for $0 \le \delta < 1$, where $\mathbf{1}_{m\times n}$ is an $m \times n$ matrix whose all of elements equal to 1. For this case, K is a singular irreducible M-matrix and (4.10) is also an order-convex matrix equation. If $\delta = 0$, then the minimal nonnegative solution S of (4.10) equals to $(1/5)\mathbf{1}_{3\times 5}$ and the Fréchet derivative at S is singular.

Table 1 reports the numbers of the pure and relaxed Newton iterations for given δ. Figure 1 shows the relative residuals of the iterations when $\delta = 10^{-4}$ and $\delta = 10^{-16}$, respectively. From them, we see that the relaxed Newton iteration is less than the pure one.

Table 2 and Fig. 2 show the average of total computation time and its efficiency of the pure and relaxed Newton methods for 300 trials with given δ. The results show that the relaxed Newton method reduces computation time about 15–20% than the pure one.

Table 1 The numbers of iterations for Example 4.3

δ	Pure	Relaxed	δ	Pure	Relaxed
10^{-1}	6	5	10^{-9}	19	13
10^{-2}	8	6	10^{-10}	20	14
10^{-3}	10	7	10^{-11}	21	15
10^{-4}	11	8	10^{-12}	23	16
10^{-5}	13	9	10^{-13}	24	17
10^{-6}	14	10	10^{-14}	24	18
10^{-7}	16	11	10^{-15}	25	18
10^{-8}	17	12	10^{-16}	25	18

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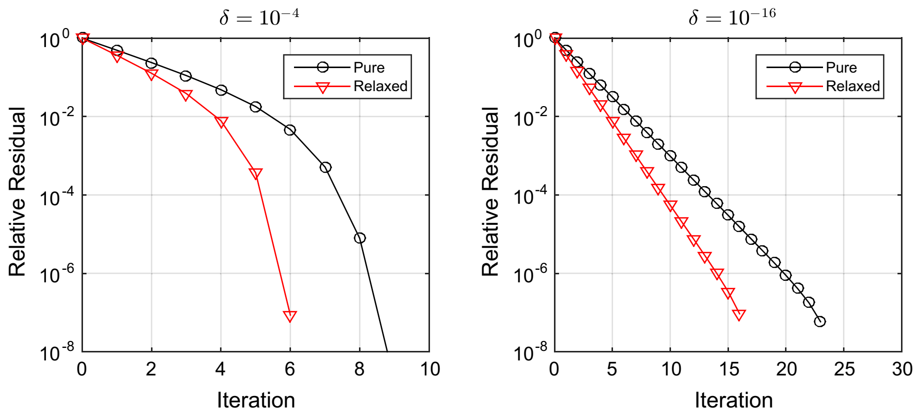

Fig. 1 The convergence of the relaxed and pure Newton methods in Example 4.3

Table 2 The averages of total computation time for 300 trials in Example 4.3

δ	Pure	Relaxed	Efficiency (%)	δ	Pure	Relaxed	Efficiency (%)
10^{-1}	0.00205	0.00197	3.91	10^{-9}	0.00706	0.00571	23.64
10^{-2}	0.00291	0.00252	15.37	10^{-10}	0.00744	0.00618	20.44
10^{-3}	0.00363	0.00295	23.15	10^{-11}	0.00784	0.00670	16.94
10^{-4}	0.00391	0.00337	16.54	10^{-12}	0.00863	0.00716	20.59
10^{-5}	0.00467	0.00382	22.26	10^{-13}	0.00898	0.00765	17.39
10^{-6}	0.00510	0.00431	18.20	10^{-14}	0.00902	0.00810	11.33
10^{-7}	0.00587	0.00477	23.21	10^{-15}	0.00941	0.00815	15.58
10^{-8}	0.00627	0.00525	19.37	10^{-16}	0.00940	0.00810	16.12

Example 4.4 Let a Markovian binary tree equation(MBTE)

$$M(x) = B(x \otimes x) - x + a = 0 \tag{4.13}$$

be given, where $a \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n \times n^2}$ such that $0 \leq a \leq 1, a \neq 0, B \geq 0$, and $a + B\mathbf{1} = \mathbf{1}$. The MBTE is called subcritical, supercritical, or critical if $\rho(M)$ of the nonnegative matrix

$$M = B(\mathbf{1} \otimes I + I \otimes \mathbf{1}) = B(\mathbf{1} \oplus \mathbf{1}) \tag{4.14}$$

is strictly less than one, strictly greater than one, or equals to one, respectively. In the subcritical and critical cases, the solution s of (4.13) equals to $\mathbf{1}$, while in the supercritical case $s \leq \mathbf{1}, s \neq \mathbf{1}$. Thus, the supercritical cases is the only interesting case. For the details, see (Hautphenne et al. 2008, 2011).

We apply the pure and relaxed Newton method for the example in Hautphenne et al. (2008). Let D_0, D_1, P_0, P_1, and $\mathbf{d}$ be defined by Fig. 3.

The diagonal of D_0 is such that $D_0\mathbf{1} + D_1\mathbf{1} + \mathbf{d} = 0$. Define that $a = -D_0^{-1}\mathbf{d}$ and $B = -D_0^{-1}R$, where $R_{i,9(j-1)+k} = [D_1]_{ii}[P_1]_{ij}[P_0]_{ik}$ for $i, j, k = 1, 2, \ldots, 9$. Then, the MBTE (4.13) is an order-convex equation on $\mathbb{R}^n$. Moreover, according to Hautphenne et al. (2008), (4.13) is supercritical if δ is greater than about 0.85. So, we experiment with $\delta \geq 0.85$.

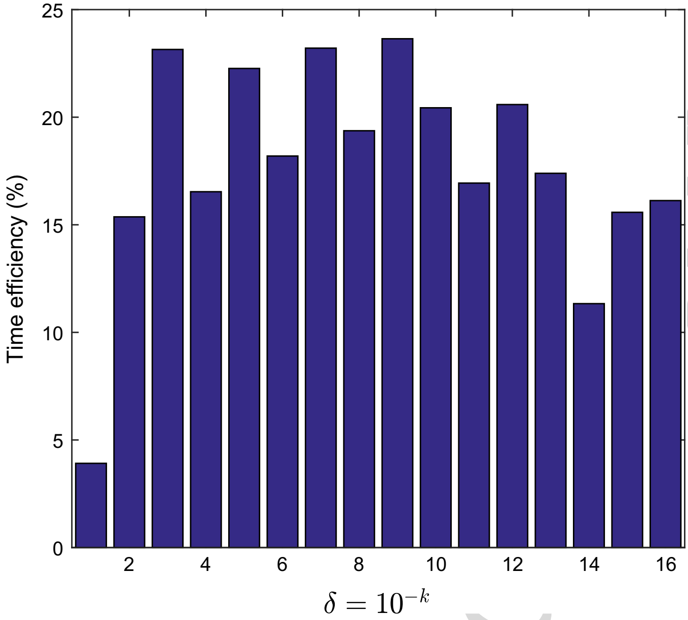

Fig. 2 The time efficiency of the relaxed Newton method in Example 4.3

$$D_0 = 10^{-3}\left[\begin{array}{cccc|c|cccc} \cdot & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdot & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & 6 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & \cdot & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \cdot & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & 6 \\ 1 & 0 & 0 & 0 & 1 & 6 & 0 & 0 & \cdot \end{array}\right]; \; D_1 = 10^{-2}\cdot \mathrm{diag}\left[\begin{array}{c} \delta \\ \delta \\ \delta \\ \delta \\ \hline 5 \\ \hline 4 \\ 4 \\ 4 \\ 4 \end{array}\right]; \; \mathbf{d} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right];$$

$$P_1 = \left[\begin{array}{cccc|c|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0.1 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]; \; P_0 = \left[\begin{array}{cccc|c|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0.1 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Fig. 3 Parameter for a family of unbalanced MBTs (Hautphenne et al. 2008)

Figure 4 shows the relative residuals of the pure and relaxed Newton iterations when $\delta = 1.85$ and $\delta = 0.85 + 10^{-7}$, respectively. Table 3 reports the numbers of two iterations for each δ. From the experiments, we also see that the relaxed Newton iteration is less than the pure one. The efficiency on the iteration of the relaxed one is about 20%.

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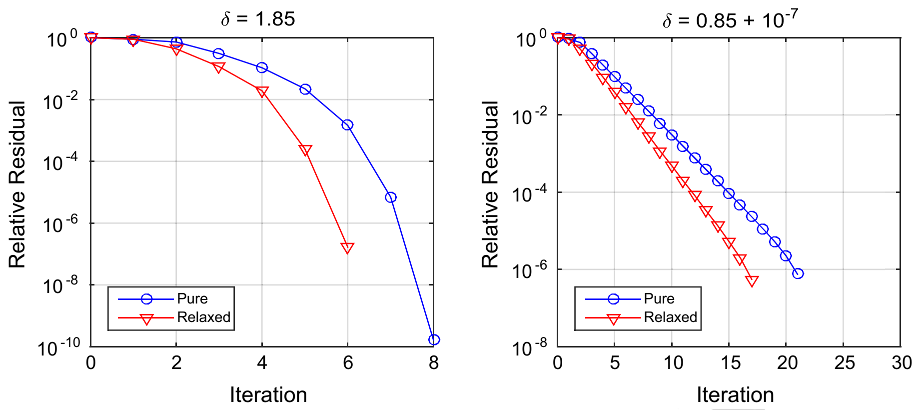

Fig. 4 The convergence of the relaxed and pure Newton methods in Example 4.4

Table 3 The numbers of iterations for Example 4.4

δ	Pure	Relaxed	δ	Pure	Relaxed
$0.85 + 10^{-0}$	10	8	$0.85 + 10^{-4}$	21	17
$0.85 + 10^{-1}$	13	11	$0.85 + 10^{-5}$	23	18
$0.85 + 10^{-2}$	16	13	$0.85 + 10^{-6}$	23	19
$0.85 + 10^{-3}$	18	15	$0.85 + 10^{-7}$	23	19

5 Conclusion

For an order-convex matrix equation (2.4), it was proved that the convergence of the Newton sequence if the Fréchet derivative at the solution of F is nonsingular. We have proved the convergence even if the Fréchet derivative is singular. On the other hand, we have shown that the relaxed Newton method can be applied for not only matrix polynomial equations but also order-convex matrix equations. We also have shown that the relaxed Newton method for order-convex matrix equations is more efficient than the pure one on the iterations.

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