

ANALYSIS OF NEWTON METHOD FOR A MATRIX POLYNOMIAL EQUATION

SANG-HYUP SEO, JONG-HYEON SEO, AND HYUN-MIN KIM[†]

ABSTRACT. We consider the Newton iteration for a matrix polynomial equation which arises in stochastic problem. In this paper, we show that the elementwise minimal nonnegative solution of the matrix polynomial equation can be obtained using Newton's method if the equation satisfies the sufficient condition, and the convergence rate of the iteration is quadratic if the solution is simple. Moreover, we show that the convergence rate is at least linear if the solution is non-simple, but we can apply a modified Newton method whose iteration number is less than the pure Newton iteration number. Finally, we give a numerical experiment which is related with our issue.

1. INTRODUCTION

We consider a matrix polynomial equation(MPE) with n -degree defined by

$$(1.1) \quad P(X) = \sum_{k=0}^n A_k X^k = A_n X^n + A_{n-1} X^{n-1} + \cdots + A_1 X + A_0 = 0,$$

where the coefficient matrices A_k 's are $m \times m$ matrices. Then, the unknown matrix X must be an $m \times m$ matrix.

The MPE (1.1) often occurs in the theory of differential equations, system theory, network theory, stochastic theory, quasi-birth-and-death and other areas [1–4, 7, 13, 18–20].

Davis [5, 6] and Higham, Kim [14, 15] studied the Newton method for a quadratic matrix equation. Guo and Laub [11] considered a nonsymmetric algebraic Riccati equation, and they proposed iteration algorithms which converge to the minimal positive solution. In [8], Guo provided a sufficient condition for the existence of nonnegative solutions of nonsymmetric algebraic Riccati equations. Kim [17] showed that the minimal positive solutions also can be found by the Newton method with the zero initial matrices in some different types of quadratic equations. Hautphenne, Latouche, and Remiche [12] studied the Newton method for the Markovian binary tree.

Seo and Kim [22, 23] studied the Newton iteration for a quadratic matrix equation and a matrix polynomial equation. Specially, in [22], they provided a relaxed Newton method whose convergence is faster than the pure one. Guo and Lancaster [10] analyzed and provided a modification about Newton's method for algebraic Riccati equations. They showed that the modification of Newton's method is better than the pure one if the minimal nonnegative solution is non-simple.

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[†]Corresponding Author.

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Assumption 1.1. For the MPE (1.1),

- 1) The coefficient matrices A_k 's are nonnegative except A_1 .
- 2) $-A_1$ is a nonsingular M -matrix.
- 3) $\sum_{k=0}^n A_k$ are irreducible.

The goal of this paper is to propose a modified Newton method of the MPE (1.1) which satisfies Assumption 1.1. This MPE is useful for stochastic theory, quasi-birth-and-death area, and so on. The modified Newton method is better than the pure Newton's method if the elementwise minimal positive solution is non-simple. The idea of the modified Newton method is from the modification of Newton's method for algebraic Riccati equations of [10]. In [10], Guo and Lancaster showed that $\|Y_{i+1} - S\| < c\varepsilon$ for the modified iteration Y_{i+1} , the solution S , a constant $c > 0$, and small $\varepsilon > 0$. On the other hand, we show that the modified Newton iteration Y_{i+1} for the MPE is closer to the solution S than the pure Newton iteration X_{i+1} .

We start with some basic definitions.

Definition 1.2. Let a matrix $A \in \mathbb{R}^{m \times m}$. A is an Z -matrix if all its off-diagonal elements are nonpositive.

It is clear that any Z -matrix A can be written as $sI - B$ with $B \geq 0$ and $s \in \mathbb{R}$. Then M -matrix can be defined as follows.

Definition 1.3. A matrix $A \in \mathbb{R}^{m \times m}$ is an M -matrix if $A = rI - B$ for some nonnegative matrix B with $r \geq \rho(B)$ where ρ is the spectral radius; it is a singular M -matrix if $r = \rho(B)$ and a nonsingular M -matrix if $r > \rho(B)$.

The following result is well known and can be found in [9] and [21] for example.

Theorem 1.4. For a Z -matrix A , the following are equivalent:

- (1) A is a nonsingular M -matrix.
- (2) A^{-1} is nonnegative.
- (3) $Av > 0$ for some vector $v > 0$.
- (4) All eigenvalues of A have positive real parts.

Definition 1.5. A positive solution S_1 of the matrix equation $P(X) = 0$ is the *elementwise minimal positive solution* and a positive solution S_2 of $P(X) = 0$ is the *elementwise maximal positive solution* if, for any positive solution S of $P(X)$,

$$(1.2) \quad S_1 \leq S \leq S_2.$$

Similarly, if nonnegative solutions S_1 and S_2 satisfy (1.2) for any nonnegative solution S , then S_1 is called the *elementwise minimal nonnegative solution* and S_2 is called the *elementwise maximal nonnegative solution*.

Definition 1.6. [16, Definition 4.2.1, Definition 4.2.9] The *Kronecker product* of $A = [a_{ij}] \in \mathbb{C}^{m \times n}$ and $B = [b_{ij}] \in \mathbb{C}^{p \times q}$ is denoted by $A \otimes B$ and is defined to be the block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp \times nq}.$$

The vec operator $\text{vec} : \mathbb{C}^{m \times n} \rightarrow \mathbb{C}^{mn}$ is defined by

$$\text{vec}(A) = [\mathbf{a}_1^T \quad \mathbf{a}_2^T \quad \cdots \quad \mathbf{a}_n^T]^T,$$

where $\mathbf{a}_i^T = [a_{1i} \quad a_{2i} \quad \cdots \quad a_{ni}]$.

Lemma 1.7. [16, Lemma 4.3.1] *Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times q}$, and $C \in \mathbb{C}^{m \times q}$ be given and let $X \in \mathbb{C}^{n \times p}$ be unknown. The matrix equation*

$$(1.3) \quad AXB = C$$

is equivalent to the system of qm equations in np unknowns given by

$$(1.4) \quad (B^T \otimes A)\text{vec}(X) = \text{vec}(C),$$

that is, $\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X)$.

Definition 1.8. Let a matrix function $F : \mathbb{C}^{m \times n} \rightarrow \mathbb{C}^{m \times n}$ be given, and let a matrix equation

$$(1.5) \quad F(X) = 0$$

be given. Then, a solution $S \in \mathbb{C}^{m \times n}$ of (1.5) is called *simple* if the Fréchet derivative at S is nonsingular.

For convenience, the notation $\|\cdot\|$ is used instead of the Frobenius norm $\|\cdot\|_F$ and \mathbb{N}_0 is used as $\mathbb{N} \cup \{0\}$ because the Frobenius norm and \mathbb{N}_0 are used very frequently in this paper.

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SANG-HYUP SEO, DEPARTMENT OF MATHEMATICS, PUSAN NATIONAL UNIVERSITY, BUSAN, 46241, REPUBLIC OF KOREA

Email address: saibie1677@gmail.com

JONG-HYEON SEO, CHUBU UNIVERSITY ACADEMY OF EMERGING SCIENCE, KASUGAI, 487-0027, JAPAN

Email address: hyeonni94@gmail.com

HYUN-MIN KIM, DEPARTMENT OF MATHEMATICS, PUSAN NATIONAL UNIVERSITY, BUSAN, 46241, REPUBLIC OF KOREA

Email address: hyunmin@pusan.ac.kr