

# ADVANCEDMPE

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**ABSTRACT.** We consider the Newton iteration for a matrix polynomial equation which arises in stochastic problem. In this paper, we show that the elementwise minimal nonnegative solution of the matrix polynomial equation can be obtained using Newton's method if the equation satisfies the sufficient condition, and the convergence rate of the iteration is quadratic if the solution is simple. Moreover, we show that the convergence rate is at least linear if the solution is non-simple, but we can apply a modified Newton method whose iteration number is less than the pure Newton iteration number. Finally, we give a numerical experiment which is related with our issue.

## 1. INTRODUCTION

We consider a matrix polynomial equation(MPE) with  $n$ -degree defined by

$$(1.1) \quad P(X) = \sum_{k=0}^n A_k X^k = A_n X^n + A_{n-1} X^{n-1} + \cdots + A_1 X + A_0 = 0,$$

where the coefficient matrices  $A_k$ 's are  $m \times m$  matrices. Then, the unknown matrix  $X$  must be an  $m \times m$  matrix.

The MPE (1.1) often occurs in the theory of differential equations, system theory, network theory, stochastic theory, quasi-birth-and-death and other areas [1–4, 7, 13, 18–20].

Davis [5, 6] and Higham, Kim [14, 15] studied the Newton method for a quadratic matrix equation. Guo and Laub [11] considered a nonsymmetric algebraic Riccati equation, and they proposed iteration algorithms which converge to the minimal positive solution. In [8], Guo provided a sufficient condition for the existence of non-negative solutions of nonsymmetric algebraic Riccati equations. Kim [17] showed that the minimal positive solutions also can be found by the Newton method with the zero initial matrices in some different types of quadratic equations. Hautphenne, Latouche, and Remiche [12] studied the Newton method for the Markovian binary tree.

Seo and Kim [22, 24] studied the Newton iteration for a quadratic matrix equation and a matrix polynomial equation. Specially, in [22], they provided a relaxed Newton method whose convergence is faster than the pure one. Guo and Lancaster [10] analyzed and provided a modification about Newton's method for algebraic Riccati equations. They showed that the modification of Newton's method is better than the pure one if the minimal nonnegative solution is non-simple.

**Assumption 1.1.** For the MPE (1.1),

- 1) The coefficient matrices  $A_k$ 's are nonnegative except  $A_1$ .
- 2)  $-A_1$  is a nonsingular  $M$ -matrix.

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3)  $\sum_{k=0}^n A_k$  are irreducible.

Our goal is to propose a singular escaping Newton method for the MPE (1.1) which satisfies Assumption 1.1. This MPE is useful for stochastic theory, quasi-birth-and-death problem, and so on. In [10], Guo and Lancaster showed that  $\|Y_{i+1} - S\| < c\varepsilon$  for the modified iteration  $Y_{i+1}$ , the solution  $S$ , a constant  $c > 0$ , and small  $\varepsilon > 0$ . Similarly, Seo, Seo, and Kim [23] showed that the modified Newton iteration  $Y_{i+1}$  for the MPE is closer to the solution  $S$  than the pure Newton iteration  $X_{i+1}$ . But, in both of [10, 23], the authors showed that the modifications are better than the pure if the solution  $S$  is non-simple.

We start with some basic definitions.

**Definition 1.2.** Let a matrix  $A \in \mathbb{R}^{m \times m}$ .  $A$  is an *Z-matrix* if all its off-diagonal elements are nonpositive.

It is clear that any Z-matrix  $A$  can be written as  $sI - B$  with  $B \geq 0$  and  $s \in \mathbb{R}$ . Then M-matrix can be defined as follows.

**Definition 1.3.** A matrix  $A \in \mathbb{R}^{m \times m}$  is an *M-matrix* if  $A = rI - B$  for some nonnegative matrix  $B$  with  $r \geq \rho(B)$  where  $\rho$  is the spectral radius; it is a singular M-matrix if  $r = \rho(B)$  and a nonsingular M-matrix if  $r > \rho(B)$ .

The following result is well known and can be found in [9] and [21] for example.

**Theorem 1.4.** For a Z-matrix  $A$ , the following are equivalent:

- (1)  $A$  is a nonsingular M-matrix.
- (2)  $A^{-1}$  is nonnegative.
- (3)  $Av > 0$  for some vector  $v > 0$ .
- (4) All eigenvalues of  $A$  have positive real parts.

**Definition 1.5.** A positive solution  $S_1$  of the matrix equation  $P(X) = 0$  is the *elementwise minimal positive solution* and a positive solution  $S_2$  of  $P(X) = 0$  is the *elementwise maximal positive solution* if, for any positive solution  $S$  of  $P(X)$ ,

$$(1.2) \quad S_1 \leq S \leq S_2.$$

Similarly, if nonnegative solutions  $S_1$  and  $S_2$  satisfy (1.2) for any nonnegative solution  $S$ , then  $S_1$  is called the *elementwise minimal nonnegative solution* and  $S_2$  is called the *elementwise maximal nonnegative solution*.

**Definition 1.6.** [16, Definition 4.2.1, Definition 4.2.9] The *Kronecker product* of  $A = [a_{ij}] \in \mathbb{C}^{m \times n}$  and  $B = [b_{ij}] \in \mathbb{C}^{p \times q}$  is denoted by  $A \otimes B$  and is defined to be the block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp \times nq}.$$

The vec operator  $\text{vec} : \mathbb{C}^{m \times n} \rightarrow \mathbb{C}^{mn}$  is defined by

$$\text{vec}(A) = [\mathbf{a}_1^T \quad \mathbf{a}_2^T \quad \cdots \quad \mathbf{a}_n^T]^T,$$

where  $\mathbf{a}_i^T = [a_{1i} \quad a_{2i} \quad \cdots \quad a_{ni}]$ .

**Lemma 1.7.** [16, Lemma 4.3.1] Let  $A \in \mathbb{C}^{m \times n}$ ,  $B \in \mathbb{C}^{p \times q}$ , and  $C \in \mathbb{C}^{m \times q}$  be given and let  $X \in \mathbb{C}^{n \times p}$  be unknown. The matrix equation

$$(1.3) \quad AXB = C$$

is equivalent to the system of  $qm$  equations in  $np$  unknowns given by

$$(1.4) \quad (B^T \otimes A)\text{vec}(X) = \text{vec}(C),$$

that is,  $\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X)$ .

**Definition 1.8.** Let a matrix function  $F : \mathbb{C}^{m \times n} \rightarrow \mathbb{C}^{m \times n}$  be given, and let a matrix equation

$$(1.5) \quad F(X) = 0$$

be given. Then, a solution  $S \in \mathbb{C}^{m \times n}$  of (1.5) is called *simple* if the Fréchet derivative at  $S$  is nonsingular. For convenience, a solution is called *non-simple* if it is not simple.

For convenience, the notation  $\|\cdot\|$  is used instead of the Frobenius norm  $\|\cdot\|_F$  and  $\mathbb{N}_0$  is used as  $\mathbb{N} \cup \{0\}$  because the Frobenius norm and  $\mathbb{N}_0$  are used very frequently in this paper.

## 2. ANALYSIS FOR NEARLY-SINGULAR POINTS

**Assumption 2.1.** We assume that

- (1)  $F(s) = 0$ ;
- (2)  $N_1$  is eigenspace for the smallest eigenvalue;
- (3)  $X = N_1 \oplus X_1$  with a closed  $X_1$ ;
- (4)  $F'[s]X_1 = X_1$ ,  $F'[s]N_1 = N_1$ ;
- (5)  $P_{X_1}$  and  $P_{N_1}$ ;
- (6)  $W_{\rho,\theta}(s) = \{x \in X : \|x - s\| \leq \rho, \|P_{X_1}(x - s)\| \leq \theta\|P_{N_1}(x - s)\|\}$ .

**Definition 2.2.** (1)  $\gamma(M, N) = \inf\{\text{dist}(u, N) : u \in M, \|u\| = 1\}$ .

**Theorem 2.3.** Let

- (1)  $N_1 : 1\text{-dim.}$ ;
- (2)  $F''[s]N_1N_1 \cap X_1 = \{0\}$ ;
- (3)  $\|F''[s]nx\| \geq c_1\|n\|\|x\|$  for all  $n \in N_1, x \in X$  with  $c_1 > 0$ .

Then,

- (1)  $\exists \rho > 0, \theta > 0 : \exists F'[x]^{-1}$  for all  $x(\neq s) \in W_{\rho,\theta}(s)$ ;
- (2)  $\|F'[x]^{-1}\| \leq c_2\|x - s\|^{-1}$ ;
- (3)  $GW_{\rho,\theta}(s) \subset W_{\rho,\theta}(s)$  where  $Gx \equiv x - F'[x]^{-1}F(x)$ ;
- (4)  $F^{-1}(0) \cap B_\rho(s) = s$ ;
- (5)  $x_i = Gx_{i-1} \Rightarrow x_i \rightarrow s$  and  $\|P_{X_1}(x_i - s)\| \leq c\|x_{i-1} - s\|^2$ .

*Proof.* Let  $\theta > 0$  and  $\rho > 0$  be chosen. Since  $F(x) = F(s) + F'[s](x - s) + \beta_2(x)$ , then  $F'[x]^{-1}F(x) = F'[x]^{-1}F'[s]P_{X_1}(x - s) + \beta_1(x) = P_{X_1}(x - s) + \beta_1(x)$  for  $x \neq s$  in  $W$ . Thus  $Gx \rightarrow s$  as  $x \rightarrow s$  in  $W$ . Hence we define  $Gs = s$ .

$$F'[s]y = F'[x_0]y + F''[x_0](s - x_0)y + \frac{1}{2}F'''[x_0](s - x_0)^2y + \cdots$$

$$F''[x_0](x_0 - s)y = F'[x_0]y - F'[s]y + \frac{1}{2}F'''[x_0](s - x_0)^2y + \cdots$$

$$F''[x_0](x_0 - s)P_{N_1}(x_0 - s) = (F'[x_0] - F'[s])P_{N_1}(x_0 - s) + \beta_3(x_0)$$

Choose  $x_0 \in W$ . Then  $x_1 = Gx_0$ , the first Newton iterate, is defined and

$$F'[x_0](x_1 - s) = F'[x_0](x_0 - s) - F(x_0).$$

Now

$$F'[x_0]^{-1}F(x_0) = -(s - x_0) - F'[x_0]^{-1}\frac{F''[x_0]}{2}(s - x_0)^2 + \beta_2(x_0),$$

and so

**Another approach.**

$$\begin{aligned}
x_1 - s &= x_0 - s - F'[x_0]^{-1}F(x_0) \\
&= \frac{1}{2}F'[x_0]^{-1}F''[x_0](x_0 - s)^2 + \beta_2(x_0) \\
&= \frac{1}{2}F'[x_0]^{-1}\textcolor{red}{F''[x_0]}(\textcolor{red}{x_0 - s})(x_0 - s) + \beta_2(x_0) \\
&= \frac{1}{2}F'[x_0]^{-1}(\textcolor{red}{F'[x_0]} - F'[s])(x_0 - s) + \beta_2(x_0) \\
&= \frac{1}{2}(x_0 - s) + \frac{1}{2}F'[x_0]^{-1}F'[s](x_0 - s) + \beta_2(x_0) \\
&= \frac{1}{2}(x_0 - s) + \frac{1}{2}F'[x_0]^{-1}F'[s](P_{X_1} + P_{N_1})(x_0 - s) + \beta_2(x_0) \\
&= \frac{1}{2}(x_0 - s) + \frac{1}{2}F'[x_0]^{-1}F'[s]P_{X_1}(x_0 - s) + \frac{1}{2}F'[x_0]^{-1}F'[s]P_{N_1}(x_0 - s) + \beta_2(x_0) \\
&= \frac{1}{2}(x_0 - s) + \frac{1}{2}F'[x_0]^{-1}F'[s]P_{X_1}(x_0 - s) + \frac{1}{2}\lambda F'[x_0]^{-1}P_{N_1}(x_0 - s) + \beta_2(x_0)
\end{aligned}$$

$$P_{X_1}(x_1 - s) = \frac{1}{2}(P_{X_1}F'[x_0]^{-1})F''[x_0](x_0 - s)P_{X_1}(x_0 - s) - \frac{1}{2}\lambda P_{X_1}F'[x_0]^{-1}P_{N_1}(x_0 - s) + \beta_2(x_0)$$

$$\begin{aligned}
\|P_{X_1}(x_1 - s)\| &\leq c^*\|x_0 - s\|^2 + |\lambda|c^{**}\|x_0 - s\| \\
\frac{\|P_{X_1}(x_1 - s)\|}{\|x_0 - s\|} &\leq c^*\|x_0 - s\| + |\lambda|c^{**}
\end{aligned}$$

□

**Lemma 2.4.** (1) *There exists a constant  $c_3 > 0$  such that  $\|F'[x]y\| \geq c_3\|y\|$  holds for all  $y \in X$  and  $x \in B_\rho(s)$ .*

(2) *There exists a constant  $c_4 > 0$  so that  $\|(F'[x] - \lambda I)m\| \geq c_4\|x - s\|\|m\|$  holds for all  $m \in N_1$  and all  $x \in B_\rho(s)$ .*

*Proof.* (2)

$$\begin{aligned}
F'[x]m &= F'[s]m + F''[s]m(x - s) + \frac{1}{2}F'''[s]m(x - s)(x - s) + O(\|m\|\|x - s\|^3) \\
&= \lambda m + F''[s]m(x - s) + O(\|m\|\|x - s\|^2) \\
&= \lambda m + O(\|m\|\|x - s\|)
\end{aligned}$$

$$\begin{aligned}
F'[x]m - \lambda m &= O(\|m\|\|x - s\|) \\
(F'[x] - \lambda I)m &= O(\|m\|\|x - s\|)
\end{aligned}$$

□

**Lemma 2.5.** *There exist positive constants  $c_5$ ,  $\rho$ , and  $\theta$  such that  $\gamma(F'[x]N_1, F'[x]X_1) \geq c_5$  for  $x \neq s$  in  $W_{\rho, \theta}(s)$ .*

*Proof.* Fix  $\rho > 0$ . Suppose the conclusion is false. Then we have sequences  $\{x_i\} \in B_\rho(s)$ ,  $\{m_i\} \in N_1$ , and  $\{y_i\} \in X_1$  such that  $\|F'[x_i]m_i\| = 1$ ,  $\|F'[x_i]m_i - F'[x_i]y_i\| \equiv \varepsilon_i \rightarrow 0$ , and  $\|P_{X_i}(x_i - s)\| \leq \theta_i\|P_{N_1}(x_i - s)\|$  with  $\theta_i \rightarrow 0$ . **Note for  $\rho$  small  $F'[x]N_1 \neq \{0\}$ ,  $x \neq s$  in  $B_\rho(s)$ .** Anyway,  $\gamma(F'[x]N_1, F'[x]X_1)$  is defined. Note also that there exist  $a_1, a_2 > 0$  such that  $a_1 \leq \|y_i\| \leq a_2$ . **Since  $F''[s]N_1N_1$  is 1-dim. of  $X$  such that**

$F''[s]N_1N_1 \cap X_1 = \{0\}$ ,  $\|w - P_{X_1}w\| \geq \alpha\|w\|$  for some  $\alpha > 0$  and all  $w$  in  $F''[s]N_1N_1$ . We finally note that since  $F'[x_i]m_i = F'[s]m_i + F''[s](x_i - s)m_i + \beta_2(x_i)m_i$ , then

$$\begin{aligned} F'[x_i]m_i &= F'[s]m_i + F''[s](x_i - s)m_i + \beta_2(x_i)m_i \\ F'[x_i]m_i - \beta_2(x_i)m_i - \lambda m_i &= F''[s](x_i - s)m_i \\ \|F'[x_i]m_i - \beta_2(x_i)m_i - \lambda m_i\| &= \|F''[s](x_i - s)m_i\| \\ \|F'[x_i]m_i\| + c_6\|x_i - s\|^2\|m_i\| + |\lambda|\|m_i\| &\geq \|F'[x_i]m_i\| + \|\beta_2(x_i)m_i\| + |\lambda|\|m_i\| \\ &\geq \|F''[s](x_i - s)m_i\| \\ &\geq c_1\|x_i - s\|\|m_i\| \end{aligned}$$

$$(2.1) \quad 1 \geq c_1\|x_i - s\|\|m_i\| - c_6\|x_i - s\|^2\|m_i\| - |\lambda|\|m_i\|$$

and

$$(2.2) \quad 1 \leq |\lambda|\|m_i\| + \|F''[s]\|\|x_i - s\|\|m_i\| + c_6\|x_i - s\|^2\|m_i\|,$$

if  $\|F''[s]\|$  is bounded. Thus for some constants  $a_3$  and  $a_4$ ,

$$(2.3) \quad 0 < a_3 \leq \|x_i - s\| \cdot \|m_i\| \leq a_4$$

holds for  $\rho$  sufficiently small.

Define  $w_i = F''[s]P_{N_1}(x_i - s)m_i$ . Then,

$$\begin{aligned} \alpha c_1\|P_{N_1}(x_i - s)\| \cdot \|m_i\| &\leq \|w_i - P_{X_1}w_i\| \leq c\|w_i - F'[s]y_i\| \\ &\leq c\|F''[s](x_i - s)m_i - F'[s]y_i\| + c\|F''[s]P_{X_1}(x_i - s)m_i\| \\ &\leq c\|F''[s](x_i - s)m_i - F'[x_i]y_i\| + c\|F'[x_i]y_i - F'[s]y_i\| \\ (2.4) \quad &+ c\|F''[s]P_{X_1}(x_i - s)m_i\| \\ &\leq c\varepsilon_i + cc_6\|x_i - s\|^2\|m_i\| + \|\beta_1(x_i)\|a_2c \\ &+ c\|F''[s]\|\theta_i\|P_{N_1}(x_i - s)\| \cdot \|m_i\|, \end{aligned}$$

where  $\|P_{N_1}z\| \leq c\|z\|$ . Note that  $\|P_{X_1}(x_i - s)\| \leq c\theta_i\|x_i - s\|$  and  $\|P_{N_1}(x_i - s)\| \geq (1 - c\theta_i)\|x_i - s\|$ . Substituting into (2.4) and dividing by  $\|x_i - s\| \cdot \|n_i\|$ , we have

$$(2.5) \quad \alpha c_1(1 - c\theta_i) \leq c\varepsilon_i a_3^{-1} + cc_6\|x_i - s\| + c_7 a_3^{-1}\|x_i - s\| + c_5\theta_i.$$

Thus for  $i$  sufficiently large, we have  $\alpha c_1 \leq 2(cc_6 + c_7 a_3^{-1})\|x - s\|$ . Thus for  $\rho$  sufficiently small, we have a contradiction, completing the proof.  $\square$

### 3. NOTE

$$Q(X) = A_2X^2 + A_1X + A_0$$

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