SOLVING NEARLY NON-SIMPLE MATRIX POLYNOMIAL EQUATIONS BY NEWTON'S METHOD WITH ADVANCED LINE SEARCHES

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Abstract.

1. Introduction

We consider a matrix polynomial equation(MPE) with n-degree defined by

$$P(X) = \sum_{k=0}^{n} A_k X^k = A_n X^n + A_{n-1} X^{n-1} + \cdots + A_1 X + A_0 = 0, \tag{1.1}$$

where the coefficient matrices A_k's are $m \times m$ matrices. Then, the unknown matrix X must be an $m \times m$ matrix.

The MPE (1.1) often occurs in the theory of differential equations, system theory, network theory, stochastic theory, quasi-birth-and-death and other areas [1–4, 9, 14, 20–22]. Specially, in quasi-birth-and-death and stochastic problems, finding the minimal nonnegative solution of a matrix equation is an important issue.

There are many researches to find the minimal nonnegative solution. Guo and Laub [13] considered a nonsymmetric algebraic Riccati equation, and they proposed iteration algorithms which converge to the minimal positive solution. In [10], Guo provided a sufficient condition for the existence of nonnegative solutions of nonsymmetric algebraic Riccati equations. Kim [19] showed that the minimal positive solutions also can be found by the Newton method with the zero initial matrices in some different types of quadratic equations. Seo and Kim [25, 28] studied the Newton iteration for a quadratic matrix equation and a matrix polynomial equation.

Newton's method is one of powerful tools to find solutions of nonlinear matrix equations. By Kantorovich theorem [16], the convergence rate of the method is quadratic if the derivative on the domain is Lipschitz continuous and at the solution is nonsingular. But, if the derivative at the solution is singular, then we cannot apply Kantrovich theorem, i.e., we cannot guarantee that the rate is quadratic. The followings are researches to analyze the problems with singular derivative at the solution and improve the method.

For general functions on Banach spaces, Reddien [24], Decker and Kelley [6, 7] gave analyses about Newton's method for singular problems. In [5], Decker, Keller, and Kelley provided an acceleration of Newton's method for singular problems and analyzed for the convergence rate of the method. Kelley and Suresh [18] suggested a new accelerated Newton's method at singular points. Decker and Kelley [8] showed an analysis for Newton's method at nearly singular roots. In [17], Kelley analyzed convergence rate of the method for functions whose high order derivatives at the

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solution are singular. For specific functions on $\mathbb{R}^{m\times n}$, Guo and Lancaster [12] analyzed and provided a modification about Newton's method for algebraic Riccati equations at singular roots. In [26], S-.H. Seo and J-.H. Seo suggested a modified Newton method for matrix polynomial equations with n-degree for singular problems.

In [5, 18], accelerations for Newton's method at singular roots was suggested for general functions on Banach spaces. For specific functions on $\mathbb{R}^{m\times n}$, modifications of Newton's method at singular roots was provided in [12, 27]. But, the main hypotheses of the papers are similar and that the solution is non-simple, i.e., the derivative at the solution is singular. It means that the accelerations for Newton's method cannot be applied and the accelerated iterations cannot be guaranteed converge to the solutions. Even if the iterations converge to the solutions, they cannot be guaranteed faster than the pure iterations.

Otherwise, in [8], the authors analyzed and suggested an acceleration for Newton's method about the case of nearly singular roots.

We give some basic definitions and lemmas for this paper.

Let $A, B \in \mathbb{R}^{m\times m}$ be matrices. If all elements of A is nonnegative, then we call that A is a *nonnegative* matrix and denote $A \geq 0$. In similar sense, we define $A \leq 0$, $A > 0$, and $A < 0$. If a matrix A can be written as $rI - B$ with $B \geq 0$ and $r \in \mathbb{R}$, we call that A is a *Z-matrix.* Moreover, if $A = rI - B$ is a Z-matrix and $r \geq \rho(B)$ then A is called an M-matrix. The following is a basic theorem for M-matrices.

Theorem 1.1. *For a Z-matrix A, the following are equivalent:*

(1) A is a nonsingular M-matrix.
(2) A^{-1} is nonnegative.
(3) $Av > 0$ for some vector $v > 0$.
(4) All eigenvalues of A have positive real parts.

Let a function $F : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ and an equation $F(X) = 0$ be given.

If nonnegative solutions S_1 and S_2 of $F(X) = 0$ are satisfy

$$S_1 \leq S \leq S_2, \tag{1.2}$$

for any nonnegative solution S of $F(X) = 0$, then they are called the *minimal nonnegative solution* and the *maximal nonnegative solution*, respectively. The *minimal positive solution* and the *maximal positive solution* also are can be defined, similarly. If the Fréchet derivative of F at a solution S is nonsingular, then S is called *simple.* Furthermore, we call that a not simple solution is *non-simple.*

Assumption 1.2. For the MPE (1.1),

1) The coefficient matrices A_k's are nonnegative except A_1.
2) $-A_1$ is a nonsingular M-matrix.
3) $\sum_{k=0}^{n} A_k$ is irreducible.
4) A_0, A_1, and $\sum_{k=2}^{n} A_k$ are irreducible.

For convenience, the notation $||\cdot||$ is used instead of the Frobenius norm $||\cdot||_F$ and $\mathbb{N}_0$ is used as $\mathbb{N} \cup \{0\}$ because the Frobenius norm and $\mathbb{N}_0$ are used very frequently in this paper.

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