<u>Unit-I</u>

Multiple Integrals

Introduction:

When a function $f(x)$ is integrated with respect to x between the limits a and b, we get the definite integral $\int_a^b f(x)\,dx$.

If the integrand is a function $f(x, y)$ and if it is integrated with respect to x and y repeatedly between the limits x_0 and x_1 (for x) and between the limits y_0 and y_1 (for y), we get a double integral that is denoted by the symbol

$$\int_{y_0}^{y_1} \left(\int_{x_0}^{x_1} f(x, y)\,dx \right) dy$$

Extending the concept of double integral one step further, we get the triple integral.

$$\int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y, z)\,dx\,dy\,dz$$

Evaluation of double and triple integrals

To evaluate $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\,dx\,dy$, we first integrate $f(x,y)$ with respect to x partially, i.e treating y as a constant temporarily, between x_0 and x_1. The resulting function got after the inner integration and substitution of limits will be a function of y. Then we integrate this function of y with respect to y between the limits y_0 and y_1 as usual.

Eg: Verify that $\int_1^2\int_0^1 (x^2+y^2)\,dx\,dy$

$$= \int_0^1\int_1^2 (x^2+y^2)\,dy\,dx$$

$$LHS = \int_1^2 \left[\int_0^1 (x^2+y^2)\,dx\right] dy$$

$$= \int_1^2 \left[\frac{x^3}{3} + y^2x\right]_0^1 dy \quad = \int_1^2 \left(\frac{1}{3} + y^2\right) dy$$

$$= \left(\frac{1}{3}y + \frac{y^3}{3}\right)_1^2$$

$$= \left(\frac{2}{3} + \frac{8}{3} - \frac{1}{3} - \frac{1}{3} \right)$$

$$= \frac{10}{3} - \frac{2}{3}$$

$$= 8/3 .$$

$$R.HS = \int_0^1 \boxed{\int_1^2 (x^2+y^2)\, dy}\, dx$$

$$= \int_0^1 \left(x^2 y + y^3/3 \right)_1^2 dx$$

$$= \int_0^1 \left(2x^2 + 8/3 - x^2 - 1/3 \right) dx$$

$$= \int_0^1 \left(x^2 + 7/3 \right) dx$$

$$= \left(\frac{x^3}{3} + \frac{7}{3} x \right)_0^1$$

$$= \frac{1}{3} + 7/3 - 0$$

$$= 8/3 \;//$$

$\therefore$ L.H.S = R.H.S.

Note: From the above problem we note the following fact:

If the limits of integration in a double integral are constants, then the order of integration is immaterial, provided the relevant limits are taken for the concerned variable and the integrand is continuous in the region of integration. This result holds good for a triple integral also.

H.W

1) Evaluate $\displaystyle\int_0^2 \int_0^1 4xy \, dx \, dy$ (Ans: 4)

2) Evaluate $\displaystyle\int_1^b \int_1^a \frac{dx\,dy}{xy}$ (Ans: $\log a \log b$)

3) Evaluate $\displaystyle\int_0^{\pi/2} \int_0^{\pi/2} \sin(\theta + \phi) \, d\phi \, d\cdot\phi$ (Ans: 2)

4) Evaluate $\displaystyle\int_0^{\pi} \int_0^{\sin\theta} r \, dr \, d\theta$ (Ans: $\pi/4$)

5) Evaluate $\displaystyle\int_0^1 \int_0^2 \int_0^3 xyz \, dx \, dy \, dz$ (Ans: $9/2$)

soln:

Verification:

$$\int_0^a \int_0^b f(x,y)\,dy\,dx \neq \int_0^b \int_0^a f(x,y)\,dy\,dx$$

For eg, Consider $\int_0^2 \int_0^1 x^3 y \, dy\, dx$

$$\text{S.T} \int_0^2 \int_0^1 x^3 y\, dy\, dx \neq \int_0^1 \int_0^2 x^3 y \, dy\, dx$$

$$\text{L.H.S} = \int_0^2 \boxed{\int_0^1 x^3 y \, dy}\, dx$$

$$= \int_0^2 \left[\frac{x^3 y^2}{2}\right]_0^1 dx$$

$$= \int_0^2 \left[\frac{x^3}{2} - 0\right] dx$$

$$= \left[\frac{x^4}{8}\right]_0^2$$

$$= \frac{16}{8} - 0$$

$$= 2$$

$$\text{R.H.S} = \int_0^1 \int_0^2 x^3 y\, dy\, dx$$

$$= \int_0^1 \left[\frac{x^3 y^2}{2}\right]_0^2 dx$$

$$= \int_0^1 \left[x^3 \times 2 - 0\right] dx$$

$$= \left[\frac{2x^4}{4}\right]_0^1$$

$$= \frac{2}{4} - 0$$

$$= 1/2$$

$\therefore$ L.H.S $\neq$ R.H.S

Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy(x+y)\,dx\,dy$

<u>soln</u>:

Since the limits for the inner integral are functions of x, the variable of inner integration should be y. Effecting this change, the given integral I becomes.

$$I = \int_0^1 \left[\int_x^{\sqrt{x}} xy(x+y)\,dy \right] dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left[\frac{x^2 \times x}{2} + \frac{x^2\sqrt{x}}{3} - \frac{x^4}{2} - \frac{x^4}{\cancel{3}} \right] dx$$

$$= \int_0^1 \left[\frac{x^3}{2} + \frac{x^{5/2}}{3} - \frac{x^4}{2} - \frac{x^4}{3} \right] dx$$

$$= \left[\frac{x^4}{8} + \frac{7}{6} - \frac{x^5}{10} - \frac{x^5}{15} \right]_0^1$$

Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy(x+y)\, dx\, dy$

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$$= \int_0^1 \left[\frac{x^2 \times x}{2} + \frac{x^2\sqrt{x}}{3} - \frac{x^4}{2} - \frac{x^4}{3} \right] dx$$

$$= \int_0^1 \left[\frac{x^3}{2} + \frac{x^{5/2}}{3} - \frac{x^4}{2} - \frac{x^4}{3} \right] dx$$

$$= \left[\frac{x^4}{8} + \frac{x^{7/2} \times 2}{21} - \frac{x^5}{10} - \frac{x^5}{15} \right]_0^1 \qquad \left(x^{5/2+1} = \frac{x^{7/2}}{7/2} \right.$$

$$= \frac{1}{8} + \frac{2}{21} - \frac{1}{10} - \frac{1}{15} = \frac{21+16}{21 \times 8} - \left[\frac{25}{150}\right]$$

$$= \frac{37}{168} - \frac{1}{\cancel{30}\,6}$$

$$= \frac{1}{8} + \frac{2}{21}$$

$$= \frac{37}{168} - \frac{1}{6}$$

$$= \frac{37 - 28}{168}$$

$$= \frac{9}{168}$$

$$= \frac{3}{56} \,//$$

1) Evaluate $\displaystyle\int_0^{2\pi}\int_0^{\pi}\boxed{\int_0^{a} r^4 \sin\phi \, dr}\, d\phi \, d\theta$

The given integral $= \displaystyle\int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^{a} r^4 \sin\phi \, dr$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \left(\frac{r^5}{5}\right)_0^a \sin\phi \; d\phi$$

$$= \int_0^{2\pi} d\theta \left[\frac{a^5}{5} \int_0^{\pi} \sin\phi \, d\phi\right]$$

$$= \int_0^{2\pi} d\theta \left[\frac{a^5}{5} \left(-\cos\phi\right)_0^{\pi}\right]$$

$$= \frac{-a^5}{5} \int_0^{2\pi} d\theta \,(-1 - 1)$$

$$= \frac{-a^5}{5} \int_0^{2\pi} -2 \, d\theta$$

$$= \frac{2a^5}{5} \int_0^{2\pi} d\theta$$

$$= \frac{2a^5}{5} \left(\theta - 0\right)_0^{2\pi}$$

$$= \frac{2a^5}{5} \times 2\pi$$

$$= \frac{4\pi a^5}{5} \;//$$

<u>Note</u>: (Interesting Result)

$$\int \log x \; d(\log x) = \frac{(\log x)^2}{2} + c$$

$$\left[\because \int x \, dx = \frac{x^2}{2} + c\right]$$

This formula is true for all functions.

i.e $$\int \log x . \frac{1}{x} \, dx = \frac{(\log x)^2}{2} + c$$

i.e $$\int \frac{\log x}{x} \, dx = \frac{(\log x)^2}{2} + c$$

$$\int \sin x \; d(\sin x) = \frac{\sin^2 x}{2} + c$$

i.e $$\int \sin x \cos x \, dx = \frac{\sin^2 x}{2} + c$$

2) Evaluate $\int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z} \, dx \, dy \, dz$

Since the upper limit for the innermost integration is a function of x, & y, the corresponding variable of integration should be z. Since the upper limit for the middle integration is a function of x, the corresponding variable of integration should be y. The variable of integration for the outermost integration is then x.

Then the given triple integral I becomes.

$$I = \int_0^{\log 2} \int_0^{x} \boxed{\int_0^{x+y} e^{x+y+z} \, dz} \, dy \, dx$$

$$= \int_0^{\log 2} \int_0^{x} \left[\int_0^{x+y} e^{x+y} . e^{z} \, dz \right] dy . dx$$

$$= \int_0^{\log 2} \int_0^{x} \left[e^{x+y} . e^{z} \right]_0^{x+y} dy \, dx$$

$$= \int_0^{\log 2} \int_0^{x} \left[e^{2(x+y)} - e^{x+y} \right] dy \, dx$$

$$= \int_0^{\log 2} \left\{ \int_0^{x} \left[e^{2x} \cdot e^{2y} - e^{x} \cdot e^{y} \right] dy \right\} \cdot dx$$

$$= \int_0^{\log 2} \left[\frac{e^{2y}}{2} \cdot e^{2x} - e^{x} \cdot e^{y} \right]_0^{x} dx$$

$$= \int_0^{\log 2} \left[\frac{e^{2x} \cdot e^{2x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x} \right] dx$$

$$= \left[\frac{e^{4x}}{4 \times 2} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^{x} \right]_0^{\log 2}$$

$$= \frac{e^{4 \times \log 2}}{8} - \frac{e^{2 \log 2}}{2} - \frac{e^{2 \times \log 2}}{4} + e^{\log 2}$$

$$- \left[\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1 \right]$$

$$= \frac{16}{8} - \frac{4}{2} - \frac{4}{4} + 2 - \left[\frac{1 - 4 - 2 + 8}{8} \right]$$

$$= 2 - 2 - 1 + 2 - \left[\frac{3}{8} \right]$$

$$= 1 - 3/8 = 5/8$$

H.W

1) Evaluate $\int_0^1 \int_0^{1-z} \int_0^{1-y-z} xyz \, dx \, dy \, dz$

(Ans: $\frac{1}{720}$)

2) Evaluate $\int_0^1 \int_0^{\sqrt{1+y^2}} \frac{dx \, dy}{1+x^2+y^2}$.

(Ans: $\frac{\pi}{4} \log(1+\sqrt{2})$)

~~region~~

Evaluate $\int_0^{\pi/2} \int_0^{\pi/2} \sin(\theta + \phi) \, d\theta \, d\phi$

soln:

$$\int_0^{\pi/2} \left[\int_0^{\pi/2} \left(\sin\theta \cos\phi + \cos\theta \sin\phi \right) d\theta \right] d\phi$$

$$= \int_0^{\pi/2} \Big[-\cos\theta \cos\phi + \sin\theta \sin\phi \Big]_0^{\pi/2} d\phi$$

$$= \int_0^{\pi/2} \Big[0 + \sin\phi + \cos\phi - 0 \Big] d\phi$$

$$= \int_0^{\pi/2} (\sin\phi + \cos\phi) \, d\phi$$

$$= \Big[-\cos\phi + \sin\phi \Big]_0^{\pi/2}$$

$$= 0 + 1 + 1 + 0 = 2 \,//$$

2) Evaluate $\iint_R xy \, dx \, dy$, where R is the regi

bounded by the line $x + 2y = 2$, lying in the first quadrant.

soln:

$$x + 2y = 2$$

$$\Rightarrow \quad \frac{x}{2} + y = 1$$

We draw a rough sketch of the boundaries of R and identify R.

$\frac{x}{2} + y = 1$

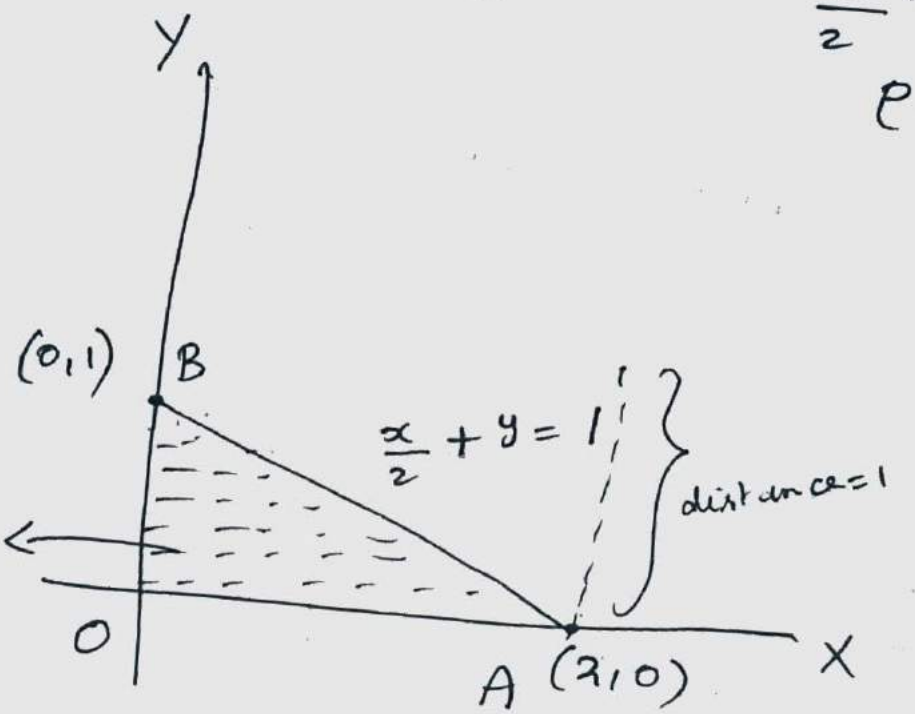
Put $y = 0$

$x = 2$

$(2,0) \in R$

~~$(1,0) \in R$~~

$(0,1) \in R$



Since the limits of the variables of integration are not given in the problem, we can choose order of integration arbitrarily.

Let us integrate with respect to x first and then with respect to y. Then the integral I becomes.

$$I = \int \left[\int_R xy \, dx \right] dy$$

First we perform inner integration w. r. t. x
Treat y = constant

To <u>find</u> the <u>limits for</u> x:

G.T. $x + 2y = 2$

$$x = 2 - 2y$$

$\therefore$ limits of x are 0 and $2-2y$.

Since y is treated as constant,

limits of y are 0 and 1.

$$I = \int_0^1 \boxed{\int_0^{2-2y} xy \, dx} \, dy$$

$$= \int_0^1 \left(\frac{x^2}{2} y \right)_0^{2-2y} dy = \int_0^1 \left[\frac{(2-2y)^2 y}{2} \right] dy$$

$$= \int_0^1 \frac{(4 + 4y^2 - 8y) y}{2} dy$$

$$= \int_0^1 \left(2y + 2y^3 - 4y^2\right) dy$$

$$= \left[\frac{2y^2}{2} + \frac{2y^4}{4} - \frac{4y^3}{3}\right]_0^1$$

$$= 1 + \frac{1}{2} - \frac{4}{3}$$

$$= 1 + \frac{3-8}{6}$$

$$= 1 - \frac{5}{6}$$

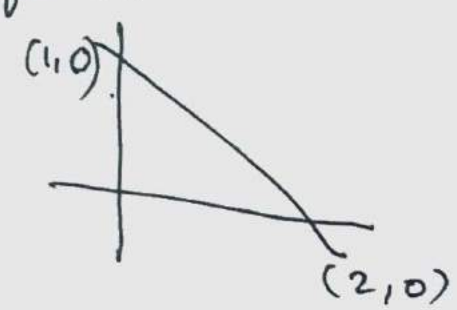
$$= \frac{1}{6} \,//$$

Alternate method:

Suppose we integrate w.r.t. y first.

Then to find the limits for y:

Treat $x =$ constant.



G.T $\quad x + 2y = 2$

$$2y = 2 - x$$

$$y = \frac{1}{2}(2-x)$$

$\therefore$ limits of y are 0 and $\frac{1}{2}(2-x)$

Limits of x are 0 and 2

~~H.W~~

~~Evaluate~~

H.W

1) Evaluate $\int_0^4 \int_{y^2/4}^{y} \frac{y \, dx \, dy}{x^2+y^2}$ and also

(Ans. 2 log 2)

sketch the region of integlation roughly

2) Evaluate $\int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y \, dx \, dy$ and also

$(a^3/6)$ Ans

sketch the region of integlation roughly.

3) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2} \, dx \, dy$

$(\pi a^3/6)$ (Ans)

4) Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz \, dx \, dy \, dz$

$(1/720)$ Ans:

~~Change of order of integ~~

~~double integral~~

17) S.T $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}} = \frac{\pi^2}{8}$

soln:

Limits of z : 0 and $\sqrt{1-x^2-y^2}$ (Inner most)

Limits of y : 0 and $\sqrt{1-x^2}$ (middle)

Limits of x : 0 and 1 (outer integral)

Let $a^2 = 1-x^2-y^2$, $a = \sqrt{1-x^2-y^2}$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-z^2}}$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left\{\sin^{-1}\left(\frac{z}{a}\right)\right\}_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$\left(\because \int \frac{dz}{\sqrt{a^2-z^2}} = \sin^{-1}(z/a) \right)$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left\{ \sin^{-1}\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}} \right) \cancel{dy\,dx} - \sin^{-1}(0) \right\} dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0) \right] dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\pi/2 - 0 \right] dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} (\pi/2)\, dy\,dx$$

$$= \pi/2 \int_0^1 \boxed{\int_0^{\sqrt{1-x^2}} dy}\, dx$$

$$= \pi/2 \int_0^1 \left(y \right)_0^{\sqrt{1-x^2}} dx$$

$$= \pi/2 \int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x \right]_0^1$$

$$= \pi/2 \left[\frac{\pi}{4} - 0 \right] = \frac{\pi^2}{8} //$$

1) Evaluate $\iint_R xy\, dx\, dy$ where R is the region bdd by the parabola $y^2 = x$ and the lines $y = 0$ and $x + y = 2$, lying in the first quadrant.

$x+y=2$

$(2,0)$

$(0,2)$

$y^2+y=2$

$y(y+1)=$

$y^2+y-2=$

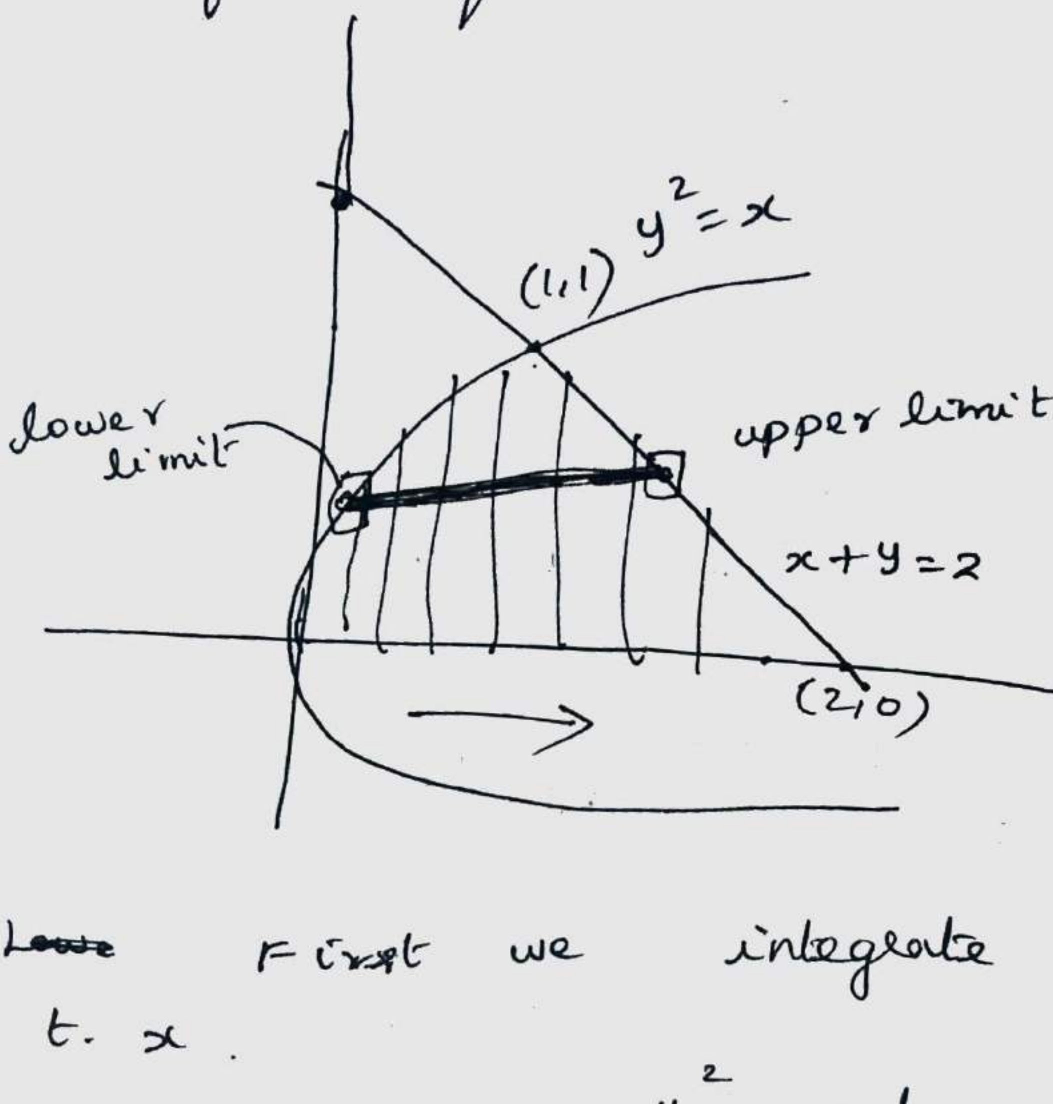
$y=-2$

$y=1$

$\boxed{(1,1)}$

$(4,-2)$

soln:



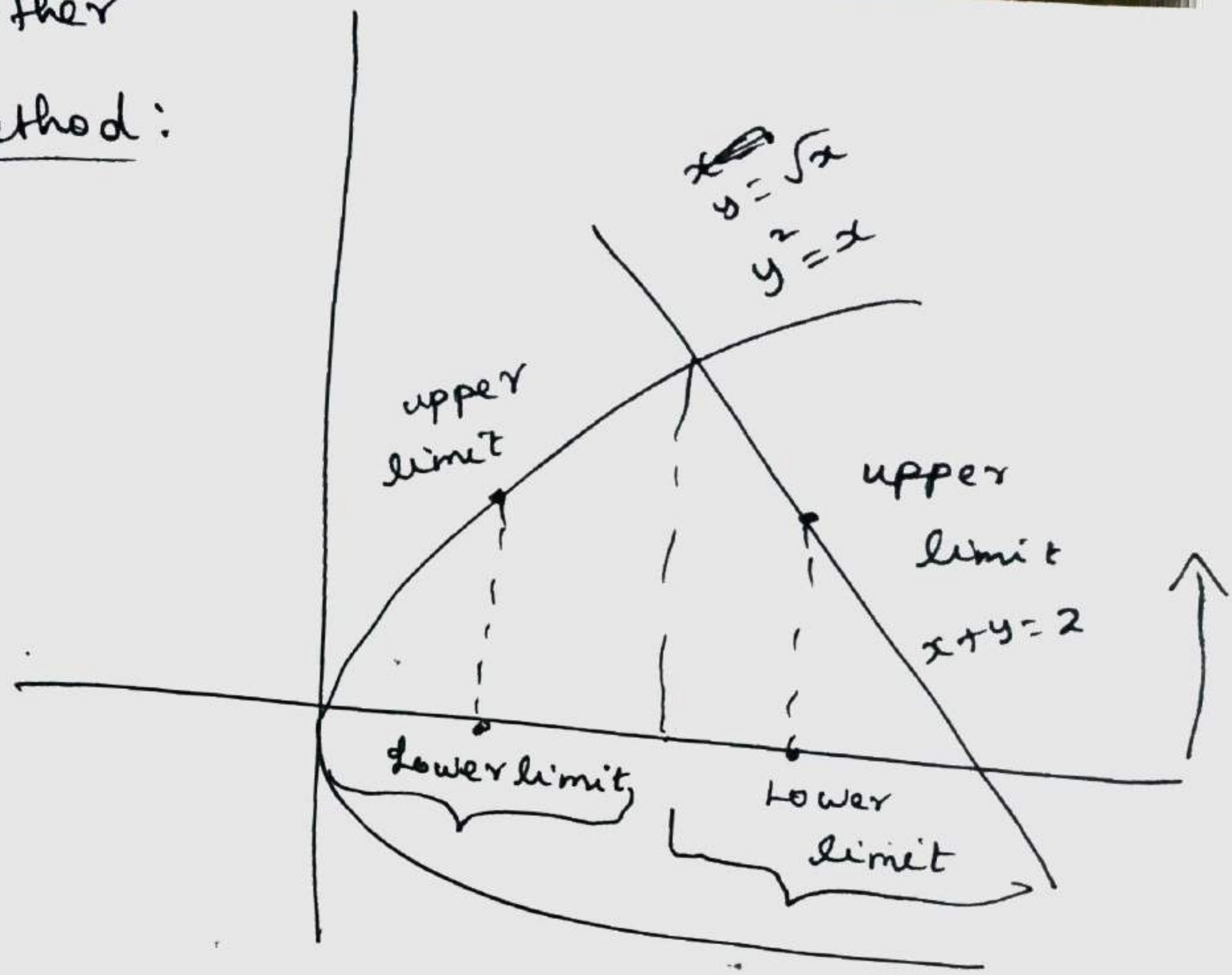
~~Lowe~~ First we integrate w.r.t. x.

Limits of x are y^2 and $2-y$

Limits of y are 0 and 1

$$I = \int_0^1 \int_{y^2}^{2-y} xy\, dx\, dy$$

Another method:



suppose we integrate w.r.t. y first

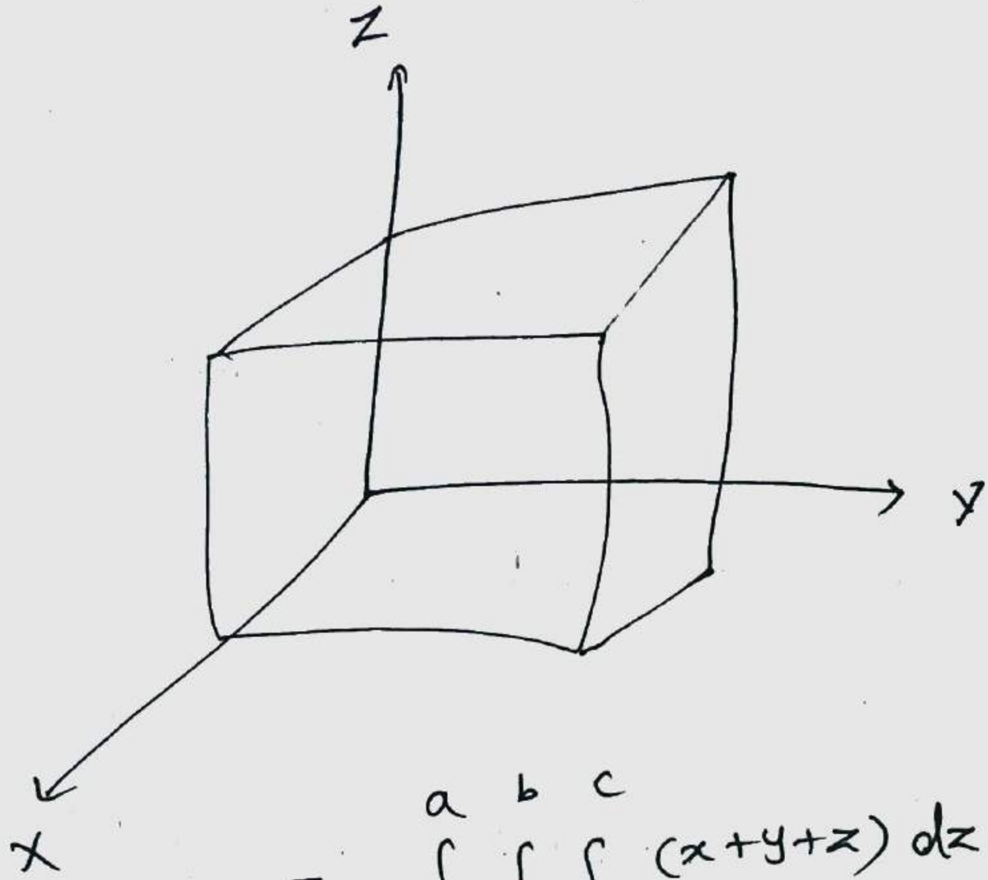
~~Limits of y~~

$$I = \int_0^1 \int_0^{\sqrt{x}} xy\, dy\, dx + \int_1^2 \int_0^{2-x} xy\, dy\, dx$$

(check Ans : $3/8$)

2) Evaluate $\iiint_V (x+y+z)\,dx\,dy\,dz$ where V is the volume of the rectangular parallelopiped bounded by $x=0$, $x=a$, $y=0$, $y=b$, $z=0$ and $z=c$.

Soln:



$$I = \int_0^a \int_0^b \int_0^c (x+y+z)\,dz\,dy\,dx$$

$$= \frac{abc}{2}(a+b+c)$$

1) Evaluate $\iiint_V (x+y+z)\,dx\,dy\,dz$ where

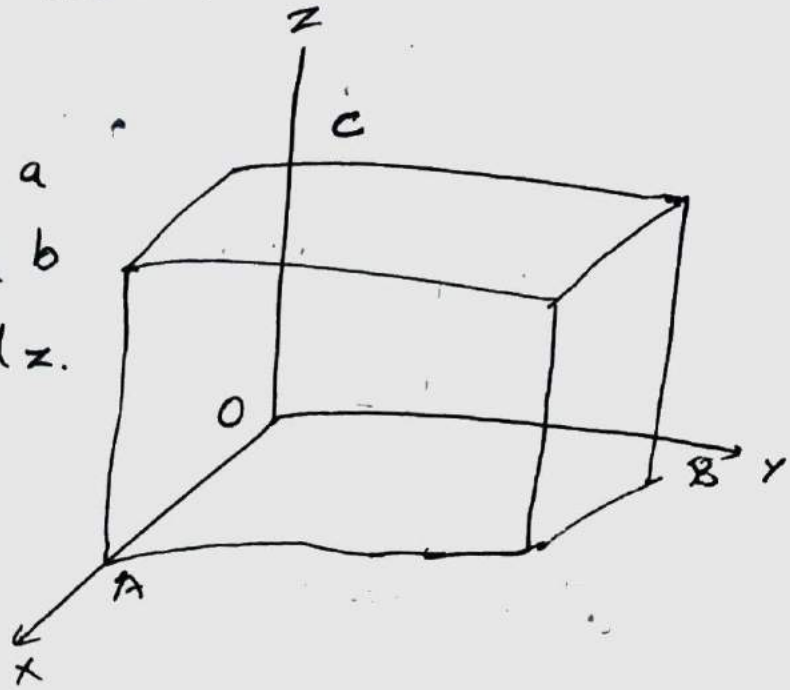
V is the volume of the rectangular parallelopiped bdd by $x=0$, $x=a$, $y=0$, $y=b$, $z=0$ and $z=c$.

soln:

Limits of x are 0 and a

Limits of y are 0 and b

Limits of z are 0 and z.



since the limits are not given, we can choose the order of integration arbitrarily.

$$I = \int_0^a \int_0^b \boxed{\int_0^c (x+y+z)\,dz}\,dy\,dx$$

$$= \int_0^a \int_0^b \left\{(x+y)z + \frac{z^2}{2}\right\}_{z=0}^{c} dy\,dx$$

$$= \int_0^a \int_0^b \left[c(x+y) + \frac{c^2}{2}\right] dy\,dx$$

$$= \int_0^a \left[cxy + \frac{cy^2}{2} + \frac{c^2}{2}y\right]_0^b dx$$

$$= \int_0^a \left[bcx + \frac{b^2c}{2} + \frac{bc^2}{2} \right] dx$$

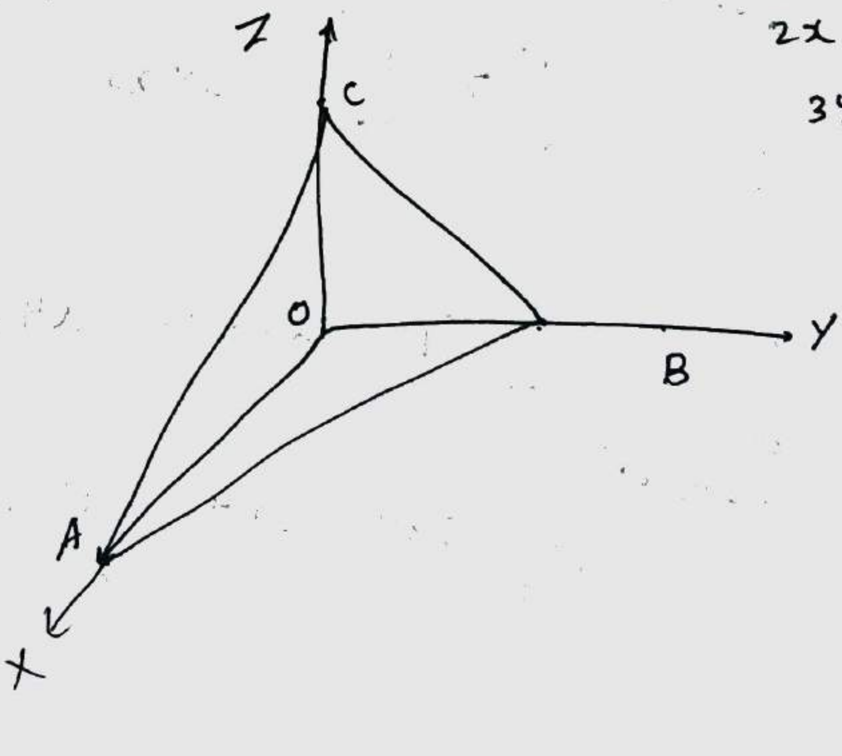
$$= \left[bc\frac{x^2}{2} + \frac{b^2cx}{2} + \frac{bc^2x}{2} \right]_0^a$$

$$= \frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}$$

$$= \frac{abc}{2}(a+b+c) //.$$

2) Evaluate $\iiint_V dx\, dy\, dz$, where V is the finite region of space (tetra-hedron) formed by the planes $x=0$, $y=0$, $z=0$ and $2x+3y+4z=12$. (plane eqn)

soln:



$$2x+3y=12$$

$$3y = 12-2x$$

$$y = \frac{1}{3}(12-2x)$$

soln:

let $I = \iiint_V dz\, dy\, dx$

$2x+3y+4z=12 \implies z = \frac{1}{4}(12-2x-3y)$

Limits of z are 0 and $\frac{1}{4}(12-2x-3y)$

~~Limits~~

Limits of y are 0 and $\frac{1}{3}(12-2x)$

Limits of x are 0 and 6

$\frac{1}{4}(12-2x-3y)=0$

$-3y=-12+2x$

$y=\frac{1}{3}(12-2x)$

$\frac{1}{3}(12-2x)=0$

$12=2x$

$\boxed{x=6}$

$$I = \int_0^6 \int_0^{\frac{1}{3}(12-2x)} \int_0^{\frac{1}{4}(12-2x-3y)} dz\, dy\, dx$$

$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} \left[z \right]_0^{\frac{1}{4}(12-2x-3y)} dy\, dx$$

$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} \frac{1}{4}(12-2x-3y)\, dy\, dx$$

$$= \int_0^6 \left\{ \frac{1}{4}\left(12y - 2xy - \frac{3y^2}{2}\right) \right\}_0^{\frac{1}{3}(12-2x)} dx$$

$$= \frac{1}{4} \int_0^6 \left\{ (12 - 2x)y - \frac{3y^2}{2} \right\}_0^{\frac{1}{3}(12-2x)} dx$$

$$= \frac{1}{4} \int_0^6 \left[\frac{(12-2x)^2}{3} - \frac{(12-2x)^2}{2} \right] dx$$

$$= \frac{1}{24} \int_0^6 \left(\frac{(12-2x)^2}{\cancel{6}} \right) dx$$

$$= \frac{1}{24} \int_0^6 4 \frac{(6-x)^2}{\cancel{6}} dx$$

$$= \frac{1}{6} \int_0^6 (6-x)^2 \, dx$$

$$= \frac{1}{6} \left[\frac{(6-x)^3}{-3} \right]_0^6$$

$$= 0 - \left(- \frac{6^3}{3 \times 6} \right)$$

$$= 12 \ //$$

3) Evaluate $\iiint_V \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$. where V is the region of space bdd by the co-ordinate planes and the sphere $x^2+y^2+z^2=1$ and contained in the positive octant.

soln:

$\cancel{\sqrt{1-x^2-y^2-z^2} = 0}$

$$x^2+y^2+z^2=1$$

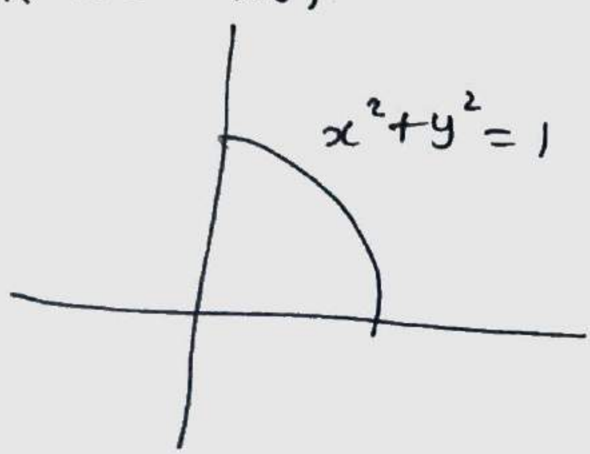
$$z^2 = 1-x^2-y^2$$

$$z = \sqrt{1-x^2-y^2}$$

Limits of z are 0 and $\sqrt{1-x^2-y^2}$

$\cancel{\sqrt{1-x^2-y^2}}$

After performing the inner most integration, the resulting double integral is evaluated over the orthogonal projection of the spherical surface on the xy plane



$$x^2 + y^2 = 1$$

$$y^2 = 1 - x^2$$

$$y = \sqrt{1-x^2}$$

limits of y are 0 and $\sqrt{1-x^2}$

limits of x are 0 and 1.

$$I = \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{(1-x^2-y^2)-z^2}}$$

$$= \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \left(\sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\right)_{z=0}^{z=\sqrt{1-x^2-y^2}}$$

$$= \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \left(\sin^{-1} 1 - \sin^{-1} 0\right)$$

$$= \pi/2 \int_0^1 dx \, (y)_0^{\sqrt{1-x^2}}$$

$$= \pi/2 \int_0^1 dx \left[\sqrt{1-x^2}\right]$$

$$= \frac{\pi}{2}\left(\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right)_0^1 = \pi^2/8.$$

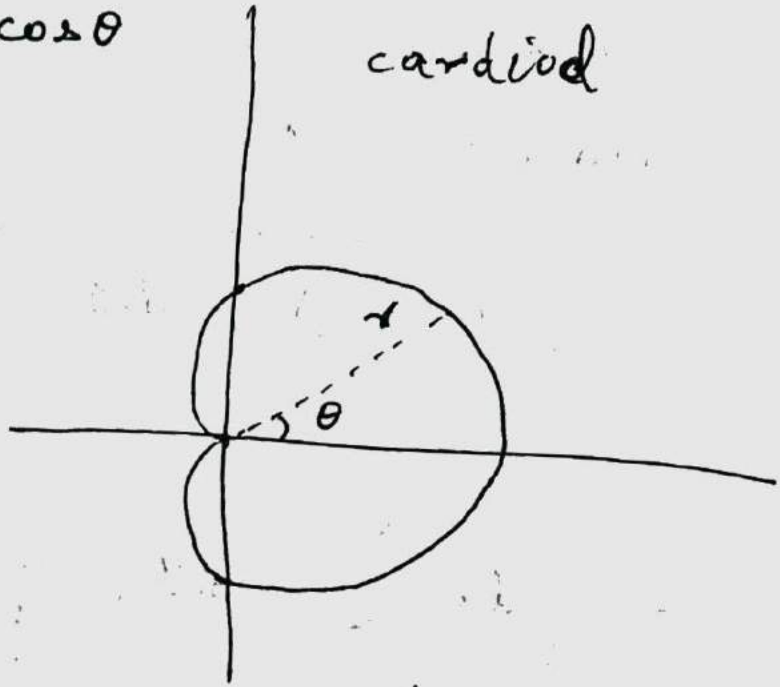
1) Find the area ~~that~~ of the cardiod $r = a(1+\cos\theta)$ by using double integration.

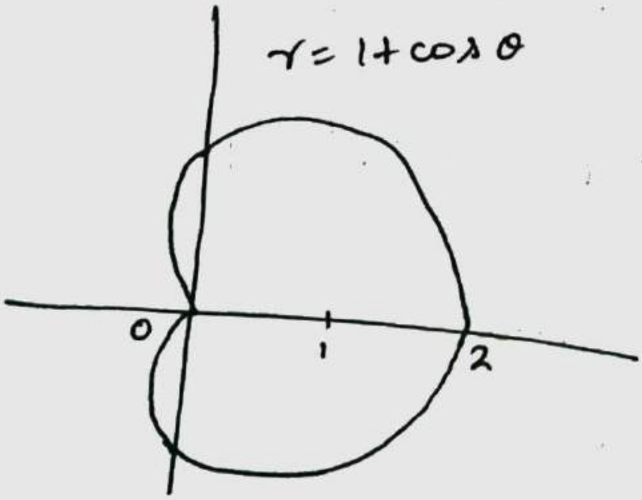
Gin: $r = a(1+\cos\theta)$

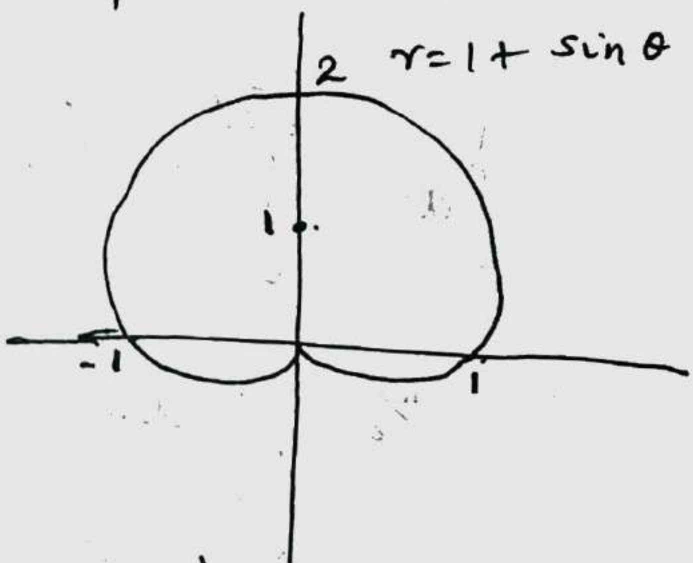
$= a + a\cos\theta$

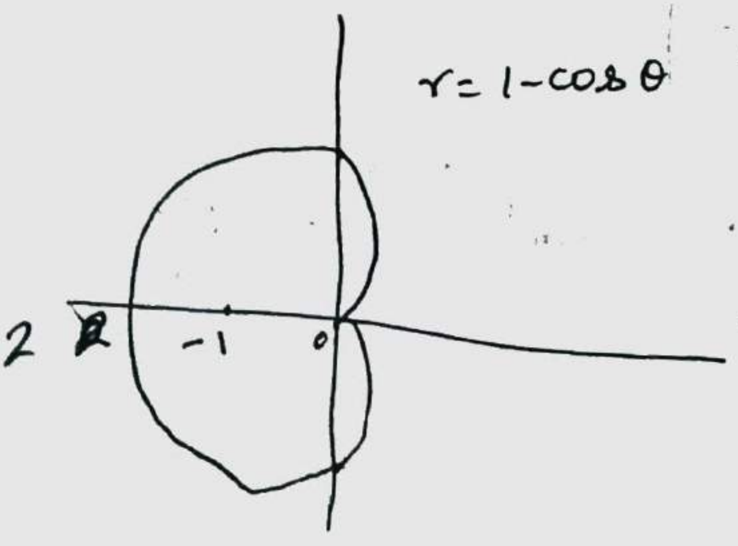
When $\theta = 0$, $r = 2a$

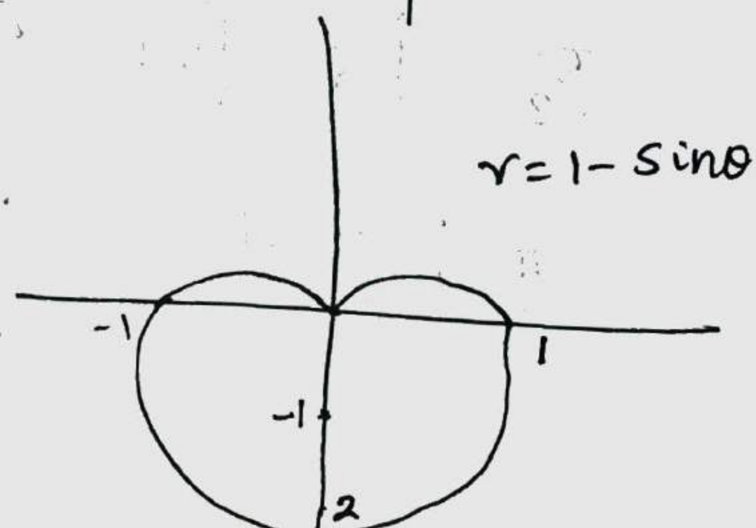
$\theta = \pi$, $r = 0$











The cardiod $r = a(1+\cos\theta)$ is symmetrical about the initial line.

~~The~~

~~Limits~~ First we integrate w.r.t r.

Limits of r are 0 to $a(1+\cos\theta)$

Limits of θ are 0 to π

$$\text{Area} = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= 2\int_0^{\pi}\frac{a^2(1+\cos\theta)^2}{2}\,d\theta$$

$$= \int_0^{\pi} a^2(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= a^2\int_0^{\pi}\left[1+2\cos\theta+\left(\frac{1+\cos 2\theta}{2}\right)\right]$$

$$= a^2\int_0^{\pi}\left[1+\tfrac{1}{2}+2\cos\theta+\frac{\cos 2\theta}{2}\right]d\theta$$

$$= a^2 \int_0^{\pi} \left[\frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2} \right] d\theta$$

$$= a^2 \left[\frac{3}{2}\theta - 2\sin\theta + \frac{\sin 2\theta}{4} \right]_0^{\pi}$$

$$= a^2 \left[\frac{3\pi}{2} - 0 + 0 - (0) \right]$$

$$= \frac{3\pi a^2}{2} //.$$

Plane Area as a Double integral

(i) $A = \iint_R dx\, dy$ (cartesian form)

(ii) $A = \iint_R r\, dr\, d\theta$ (Polar form)

Volume as triple integral

Volume $V = \iiint_V dx\, dy\, dz$ (cartesian form)

$V = \iiint_V r\, dr\, d\theta\, dz$ (cylindrical form)

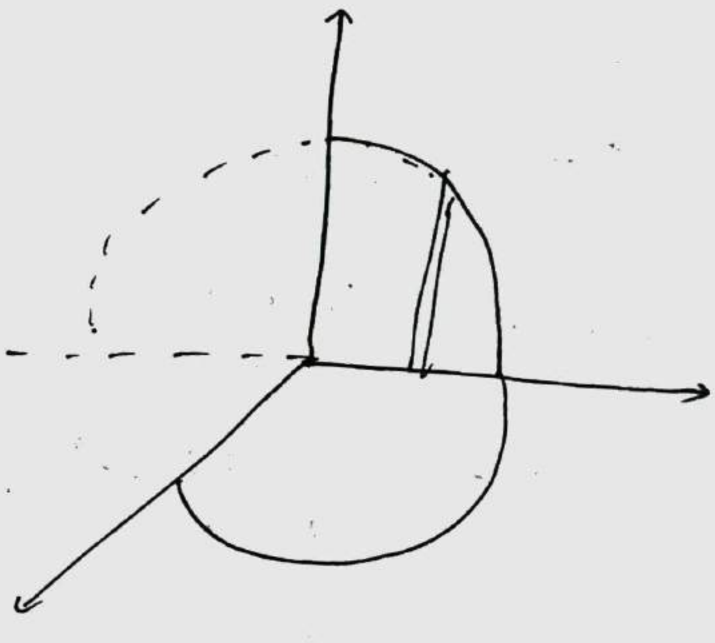
1) Find the volume of the ellipsoid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, by triple integrals.

Soln:

Gn. surface is the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

The ellipse in 3D is an ellipsoid.



G.T

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$z^2 = c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

$$z = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

Limits of z are 0 to $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

Put $z=0$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Longrightarrow \quad y = \frac{b}{a}\sqrt{a^2-x^2}$$

Limits of y are 0 to $\frac{b}{a}\sqrt{a^2-x^2}$

Put $z=0,\ y=0$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$\therefore \quad x^2 = a^2$$

$$x = a$$

$$\text{Volume} = 8 \times \iiint dx\,dy\,dz$$

$$= 8\iiint_V dz\ dy\ dx$$

$$c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} = 0$$

$$\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} = 0$$

$$1-\frac{x^2}{a^2}-\frac{y^2}{b^2} = 0$$

$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$$

$$y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$$

$$y = \frac{b}{a}\sqrt{a^2-x^2}$$

$$= 8 \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$

$$= 8 \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \Big[z \Big]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy\, dx$$

$$= 8 \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy\, dx$$

$$= 8c \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \sqrt{\frac{a^2b^2-b^2x^2-a^2y^2}{a^2b^2}}\, dy\, dx$$

$$= \frac{8c}{b} \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \frac{\sqrt{b^2(a^2-x^2)-a^2y^2}}{\sqrt{a^2}}\, dy\, dx$$

$$= \frac{8c}{b} \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \sqrt{\frac{b^2(a^2-x^2)}{a^2}-y^2}\, dy\, dx$$

$$= \frac{8c}{b} \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \sqrt{b^2\left(1-\frac{x^2}{a^2}\right) - y^2} \; dy\, dx$$

$$= \left(\because \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right)$$

$$\frac{b}{a}\sqrt{(a^2-x^2)} = a, \quad a^2 = \frac{b^2}{a^2}(a^2-x^2)$$
$$= b^2\left(1-\frac{x^2}{a^2}\right)$$

$$= \frac{8c}{b} \int_0^a \left\{ \frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right) - y^2} + \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2} \sin^{-1}\frac{y}{a} \right\}_0^{\frac{b}{a}\sqrt{a^2-x^2}}$$

$$= \frac{8c}{2b} \int_0^a \left\{ 0 + b^2\left(1-\frac{x^2}{a^2}\right) \sin^{-1}\left(\frac{\frac{b}{a}\sqrt{a^2-x^2}}{\frac{b}{a}\sqrt{a^2-x^2}}\right) \right\} dx$$

$$= \frac{8c}{2b} \int_0^a \left\{ b^2\left(1-\frac{x^2}{a^2}\right) \sin^{-1} 1 - \cancel{\sin 0} \right\} dx$$

$$= \frac{8c}{2b} \int_0^a b^2\left(1-\frac{x^2}{a^2}\right) \frac{\pi}{2}$$

$$= \frac{8\pi c}{2\times 2b}\left[\int_0^a \frac{b^2(a^2-x^2)}{a^2}\,dx\right.$$

$$= \frac{2\cancel{4}\pi c}{b}\left[b^2x - \frac{b^2x^3}{3a^2}\right]_0^a$$

$$= \frac{2\cancel{4}\pi c}{b}\left[ab^2 - \frac{a^3b^2}{3a^2}\right]$$

$$= \frac{2\cancel{4}\pi c}{b}\left[ab^2 - \frac{ab^2}{3}\right]$$

$$= \frac{2\cancel{4}\pi c}{b}\times b^2\left(a - \frac{a}{3}\right)$$

$$= 2\cancel{4}\pi bc\left(\frac{2a}{3}\right)$$

$$= \frac{4\cancel{8}\pi abc}{3} \quad //$$

H.W

1) Change the order of integration

$$\int_0^a \int_y^a \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$$ and then evaluate it.

(Ans: $\frac{a^2}{2} \log(1+\sqrt{2})$)

2) $$\int_0^1 \int_x^1 \frac{x \, dx\, dy}{x^2+y^2}$$ and then evaluate it.

Ans: $\frac{1}{2} \log 2$

3) Change the order of integration

in $$\int_0^b \int_0^{\frac{a}{b}(b-y)} xy \, dx\, dy$$ and then evaluate

it.

(Ans: $\frac{a^2 b^2}{24}$)

4) Change the order of integration in

$$\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2 \, dy\, dx$$ and then integrate it.

5) Change the order of integration

$$\int_{0}^{a}\int_{a-y}^{\sqrt{a^2-y^2}} y\,dx\,dy$$ and then evaluate it.

(Ans: $a^3/6$)

6) Change the order of integration in

$$\int_{0}^{a}\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2+y^2}} xy\,dx\,dy$$ and then

evaluate it. (Ans: $\frac{2}{3}a^4$)

7) Find the area between the circle $x^2+y^2=a^2$ and the line $x+y=a$ lying in the first quadrant, by double integration.

(Ans: $(\pi-2)\frac{a^2}{4}$)

8) Find the area that lies inside the cardioid $r=a(1+\cos\theta)$ and outside the circle $r=a$ by double integration.

(Ans: $\frac{a^2}{4}(\pi+8)$)

Vector Calculus:

In Vector Algebra, we mostly deal with constant vectors, viz, vectors which are constant in magnitude and fixed in direction. In vector calculus we deal with variable vectors, i.e vectors which are varying in magnitude or direction or both.

Eg:

Temperature, Electric Potential are examples of scalar point functions.

Velocity and Gravitational force are examples of vector point functions.

Vector Differential Operator ∇ (to be read as del)

The operator ∇ is defined as

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

∇ is called the vector differential operator, as it behaves like a vector (though not a vector) with $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ as coeffts of $\vec{i}, \vec{j}, \vec{k}$ respectively.

Gradient of a Scalar Point function

Let $\varphi(x, y, z)$ be a scalar point function defined in a certain region of space.

Then vector point function is given by

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \vec{i}\frac{\partial \varphi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$$

is defined as the gradient of ϕ and denoted as grad ϕ

Note: (i) $\nabla\phi$ is a vector pt. fn

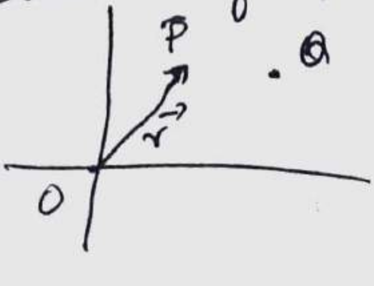
(ii) $\nabla\phi$ should be written as $\phi\nabla$

(iii) Dont put either '.' or 'x' in between ∇ & ϕ.

(iv) If ϕ is a constant, $\nabla\phi = 0$ (i.e $\vec{0}$)

Directional Derivative of a Scalar Pt. function $\phi(x,y,z)$

$\frac{d\phi}{dr}$ gives the rate of change of ϕ with respect to the distance measured in the direction of $\vec{r}$ ($\vec{r} = \vec{OP}$, position vector)



In particular, $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$ are the directional derivatives of ϕ at $P(x, y, z)$ in the directions of the co-ordinate axes.

Geometrical interpretation of $\nabla \phi$:

$\nabla \phi$ is a vector whose magnitude is the greatest directional derivative of ϕ and whose direction is that outward drawn normal to the level surface $\phi = c$.

Maximum directional derivative of ϕ is $\frac{d\phi}{dn}$, i.e the directional derivative ϕ in the direction of $\hat{n}$.

i.e Maximum value of directional derivative of ϕ is $|\nabla \phi|$ (i.e magnitude)

Line integral of vector point functions

Let $\vec{F}(x,y,z)$ be a vector point function defined at all points in some region of space and let C be a curve in that region.

Defn: $\int_C \vec{F}\cdot d\vec{r}$ is defined as the line integral of $\vec{F}$ along the curve C.

Physically, $\int_A^B \vec{F}\cdot d\vec{r}$ denotes the total work done by the force $\vec{F}$ in displacing a particle from A to B along the curve C.

Note:

1) If the value of $\int_A^B \vec{F}\cdot d\vec{r}$ does not depend on the curve C, but only on the terminal points A and B, $\vec{F}$ is called a conservative vector.

2) If the path of integration C is a closed curve, then line integral is

$$\oint_C \vec{F}\cdot d\vec{r}$$

Condition for $\vec{F}$ to be Conservative:

If $\vec{F}$ is an irrotational vector, it is conservative.

Note: If $\vec{F}$ is irrotational and c is a closed curve, then $\oint_c \vec{F} \cdot d\vec{r} = 0$.

Surface Integral of Vector Point function

Let S be a two sided surface, one side of which is considered arbitrarily as the positive side.

Let $\vec{F}$ be a vector point function defined at all points of S.

Let $\hat{n}$ be the unit vector normal to the surface S at (x, y, z) drawn in the positive side (or outward direction).

Surface integral of $\vec{F}$ over S is denoted by $\int_S \vec{F} \cdot \hat{n} \, ds$

1) Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (\sin y)\vec{i} + x(1+\cos y)\vec{j} + z\vec{k}$ and C is the circle $x^2 + y^2 = a^2$ in the xy-plane.

5, 6, 7, 8

12, 13, 14, 15,

<u>soln:</u>

$$\oint_C \vec{F} \cdot d\vec{r} = \int_C \left[(\sin y)\vec{i} + x(1+\cos y)\vec{j} + z\vec{k}\right] \cdot \left[dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}\right]$$

$$= \int_{\substack{x^2+y^2=a^2 \\ z=0}} \left[\sin y\,dx + x(1+\cos y)\,dy + z\,dz\right]$$

$$= \int_{x^2+y^2=a^2} \left[\sin y\,dx + x(1+\cos y)\,dy\right]$$

Since C is a closed curve, we use the parametric equations of C, namely $x = a\cos\theta$, $y = a\sin\theta$, where θ is the variable of integration.

θ varies from 0 to 2π.

$$\oint_C \vec{F} \cdot d\vec{r} = \int \left\{\left[\sin y\,dx + x\cos y\,dy\right] + x\,dy\right\}$$

$$= \int d(x\sin y) + x\,dy$$

$$= \int_0^{2\pi} \left\{ \int d\left[a\cos\theta .\ \sin(a\sin\theta)\right] + a^2\cos^2\theta \, d\theta \right\}$$

$$= \left[a\cos\theta .\ \sin(a\sin\theta) + \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) \right]_0^{2\pi}$$

$$\left[\because \int dx = x \right]$$

$$= \pi a^2 \,//.$$

Integral theorems

The following three theorems in Vector Calculus are of importance from theoretical and practical considerations.

1) Green's theorem in a plane
2) Stoke's theorem
3) Gauss Divergence theorem.

Gauss Divergence theorem

If S is a closed surface enclosing a region of space with volume V and if $\vec{F}$ is a vector point function with continuous first order partial derivatives in V, then

$$\iint_S \vec{F} \cdot d\vec{s} = \iiint_V (\operatorname{div} \vec{F})\, dv$$

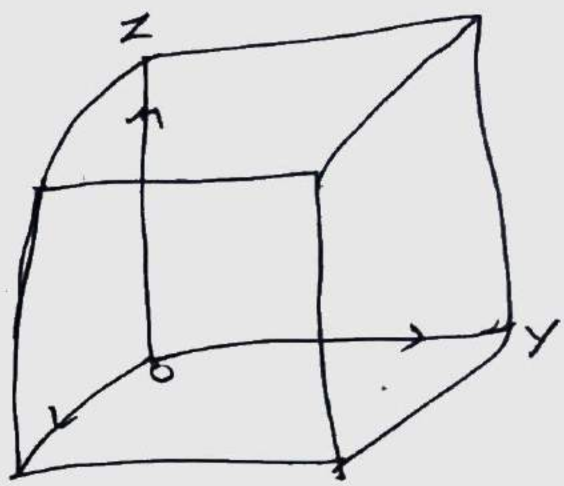
1) Verify Gauss divergence theorem for $\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{K}$ where S is the surface of the cuboid formed by the planes $x=0$, $x=a$, $y=0$, $y=b$, $z=0$, $z=c$.

soln :

Div. theorem is

$$\iint_S \vec{F}\cdot d\vec{s} = \iiint_V (\operatorname{div}\vec{F})\, dv \quad \text{(1)}$$

S is made up of six plane surfaces.



$\hat{n} \rightarrow$ unit vector outward normal

[outward means negative direction]

$x=0$	$x=a$	$y=0$	$y=b$	$z=0$	$z=c$
yz plane	IIel to yz Plane	xz Plane	IIel to yz plane	xy plane	IIel to xy Plane
$\hat{n}=-i$	$\hat{n}=\vec{i}$	$\hat{n}=-\vec{j}$	$\hat{n}=\vec{j}$	$\hat{n}=-\vec{k}$	$\hat{n}=\vec{k}$

$$\text{L.H.S of (1)} = \iint_{\substack{x=0\\ \hat{n}=-i}} + \iint_{\substack{x=a\\ \hat{n}=i}} + \iint_{\substack{y=0\\ \hat{n}=-j}} + \iint_{\substack{y=b\\ \hat{n}=j}}$$

$$+ \iint_{\substack{z=0\\ \hat{n}=-\vec{k}}} + \iint_{\substack{z=c\\ \hat{n}=\vec{k}}} (x^2\vec{i} + y^2\vec{j} + z^2\vec{k}) \cdot \hat{n}\, ds$$

$$= \iint_{x=0} -x^2\, ds + \iint_{x=a} x^2\, ds + \iint_{y=0} -y^2\, ds$$

$$+ \iint_{y=b} y^2\, ds - \iint_{z=0} z^2\, ds + \iint_{z=c} z^2\, ds$$

$$= 0 + \int_0^c \int_0^b a^2\, dy\, dz + 0 + b^2 \int_0^a \int_0^c dz\, dx$$

$$+ 0 + c^2 \int_0^b \int_0^a dx\, dy$$

$$= a^2 \int_0^c (y)_0^b\, dz + b^2 \int_0^a (z)_0^c\, dx$$

$$+ c^2 \int_0^b (x)_0^a\, dy$$

$$= a^2 b \int_0^c dz + b^2 c \int_0^a dx + c^2 a \int_0^b dy$$

$$= a^2 b (z)_0^c + b^2 c (x)_0^a + c^2 a (y)_0^b$$

$$= a^2 bc + b^2 ca + c^2 ab$$

$$= abc(a+b+c) \;//$$

R.H.S of ① $= \iiint_V (\operatorname{div} \vec{F})\, dv$

$$\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$$

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

$$\iiint_V (\operatorname{div} \vec{F})\, dv = \iiint_V (2x + 2y + 2z)\, dV$$

$$= \int_0^c \int_0^b \left(\int_0^a (2x + 2y + 2z)\, dx \right) dy\, dz$$

$$= \int_0^c \int_0^b \left(x^2 + 2yx + 2zx \right)_0^a dy\, dz$$

$$= \int_0^c \int_0^b \left(a^2 + 2ay + 2az \right) dy\, dz$$

$$= \int_0^c \left[a^2 y + a y^2 + 2az \cdot y \right]_0^b dz$$

$$= \int_0^c \left(a^2b + ab^2 + 2abz\right) dz$$

$$= \left(a^2bz + ab^2z + abz^2\right)_0^c$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc\,(a+b+c)$$

Defn: [Analytic function]

A function defined at a point z_0 is said to be analytic at z_0 if it has a derivative at z_0 and at every point in some nbd of z_0.

Cauchy- Riemann Eqns:

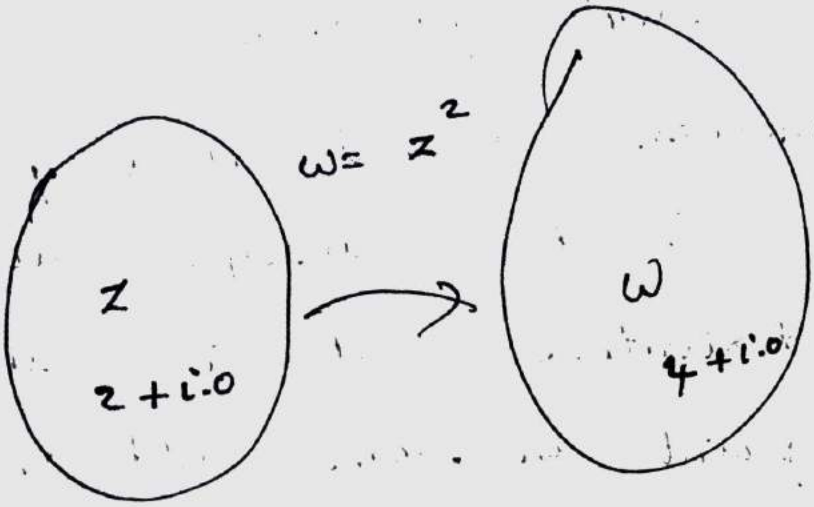
Necessary condition for a complex function $f(z) = u(x,y) + iv(x,y)$ to be analytic is :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y}.$$

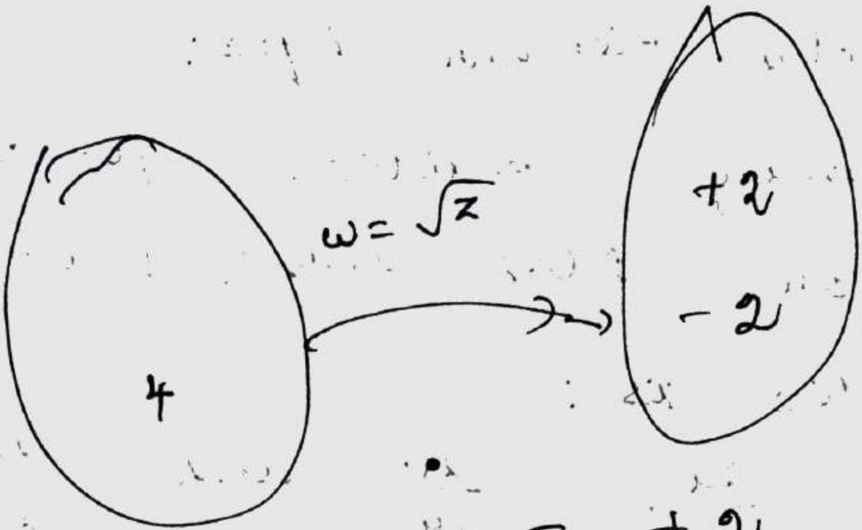
~~Defn:~~ Milne Thompson method is used to construct an analytic function whose real or imaginary part is given.

Singular point:

A point, at which a function $f(z)$ is not analytic is called a singular point or singularity of $f(z)$.



single valued



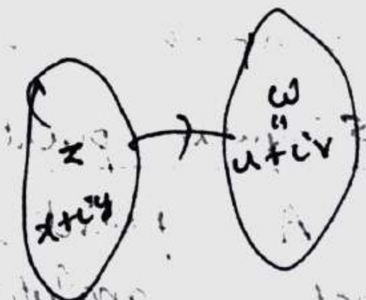
multi-valued.

$$\sqrt{4} = \pm 2$$

Note:

$$z = x + iy$$

$$f(z) = w = u + iv$$



$$f'(z) = u_x + i v_x$$

or

$$f'(z) = v_y - i u_y$$

(By C-R-eqns)

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial}{\partial x}(u+iv) = \frac{\partial w}{\partial x}$$

$$\frac{dw}{dz} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= -i\frac{\partial w}{\partial y}$$

C-R Eqns in Polar Co-ordinates:

$z = re^{i\theta}$, $f(z) = u(r,\theta) + iv(r,\theta)$,

then C-R eqns in polar form

$$\boxed{\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \\ \frac{\partial v}{\partial r} &= \frac{-1}{r}\frac{\partial u}{\partial \theta} \end{aligned}}$$

H.W

1) P.T the following fns are analytic.

(i) z^3 (ii) e^{-x} (iii) $\sin z$ (iv) $\cosh z$

(v) z^n (vi) $\log z$.

Find where each of the following functions ceases to be analytic.

(i) $\dfrac{z}{(z^2-1)}$ (ii) $\dfrac{z^2-4}{z^2+1}$ (iii) $\dfrac{z+1}{(z-i)^2}$

(iv) z^3-4z-1.

soln :

(i) $f(z)$ is not analytic at $z = \pm 1$

(ii) $f(z)$ is not analytic at $z = \pm i$

~~(iv)~~
(iii) $$f(z) = \frac{z+1}{(z-i)^2}$$

$$f'(z) = \frac{(z-i)^2 \cdot 1 - (z+i) \cdot 2(z-i)}{(z-i)^4}$$

$$= \frac{-(z+3i)}{(z-i)^3} \rightarrow \infty \quad \text{at } z = i$$

$\therefore f(z)$ is not analytic at $z = i$.

(iv) Let $f(z) = z^3 - 4z - 1$

$$f'(z) = 3z^2 - 4$$

$\therefore f(z)$ is analytic everywhere.

Construction of an Analytic function, when its Real or Imaginary Part is known.

Method 1: Let $u(x, y)$, the real part of the analytic function $f(z) = u(x, y) + iv(x, y)$ be known.

In this method, we first find $v(x, y)$ and then find $u(x, y) + iv(x, y)$ as a function of z.

$$\boxed{\begin{array}{l} M\,dx + N\,dy \text{ is an exact differential if} \\ \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \end{array}}$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \quad \text{(Total differential)} \quad \text{--- (1)}$$

R.H.S of (1) is an exact differential.

since $$\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Milne Thomson method :

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \quad \text{(Total differential)} \quad \text{---} (1)$$

is

R.H.S of (1) is an exact differential

since $\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

$$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$$

$$\Longrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Milne. Thomson method:

Let $u(x,y)$ be the real part of the analytic function $f(z) = u(x,y) + i\,v(x,y)$. In this method, we first find $f'(z)$ as a function of z and then $f(z)$ by ordinary integration. The imaginary part of $f(z)$ gives $v(x,y)$. Since $u(x,y)$ is given, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ can be found out.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

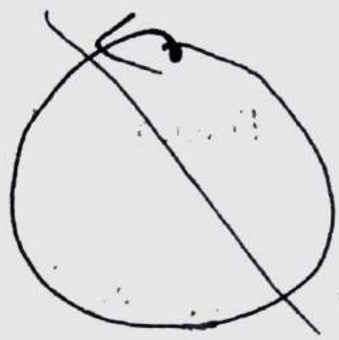
$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

Complex

$$= u_x(x,y) - i\,u_y(x,y)$$

$$= u_x(z,0) - i\,u_y(z,0), \text{ By}$$

Milne-Thomson Rule.



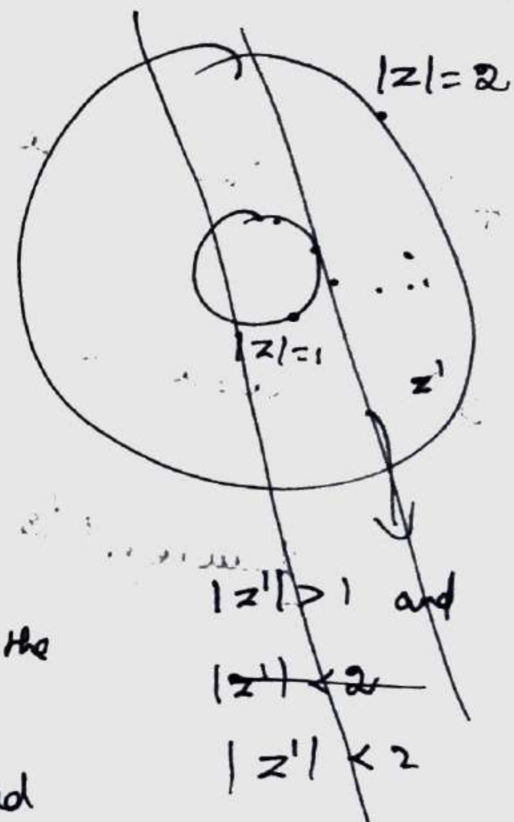
$$\therefore f(z) = \int \left[u_x(z,0) - i\,u_y(z,0)\right] dz + c$$

H.W

1) P.T the following function

$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic. Also find the conjugate harmonic function v and the corresponding analytic fn $(u+iv)$

(Ans : $z^3 + 3z^2 + 1 + ic$)



$|z'| > 1$ and

$\cancel{|z'| < 2}$

$|z'| < 2$

$\left|\frac{2}{z}\right| < 1$

2) In a two dimensional fluid flow, find if $xy(x^2 - y^2)$ can represent the stream function. If so, find the corresponding velocity potential and also the complex potential.

[Gn : $\psi = xy(x^2-y^2)$ find (i) ϕ [velocity potential]
(ii) $f(z)$ [complex potential] harmonic conjugate]

3) Determine the analytic function $f(z) = u + iv$, given that $3u + 2v = y^2 - x^2 + 16xy$.

Power series formulae:

1) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

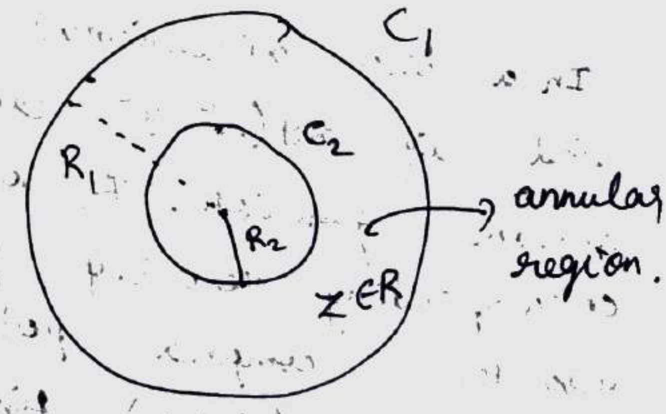
2) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$

3) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

4) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

5) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

Laurent's series:



Laurent's series:

let C_1, C_2 be two concentric circles

$|z-a| = R_1$ and $|z-a| = R_2$ where $R_2 < R_1$.

let $f(z)$ be analytic on C_1 and C_2 and in the annular region R between them.

Then for any point z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

$$= a_0 + a_1 (z-a) + a_2 (z-a)^2 + a_3 (z-a)^3 + \dots$$

$$\dots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \frac{b_3}{(z-a)^3} + \dots$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{n+1}} \, dz$$ and

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{-n+1}} \, dz.$$

WORKED EXAMPLE 5(b)

Example 5.1 Change the order of integration in $\int_0^a\int_y^a \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$ and then evaluate it.

The region of integration R is defined by $y \le x \le a$ and $0 \le y \le a$.

i.e. it is bounded by the lines $x = y$, $x = a$, $y = 0$ and $y = a$.

The rough sketch of the boundaries and the region R is given in Fig. 5.19.

After changing the order of integration, the given integral I becomes

$$I = \iint_R \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$$

The limits of inner integration are found by treating x as a constant, i.e. by drawing a line parallel to the y-axis in the region of integration as explained in the previous section.

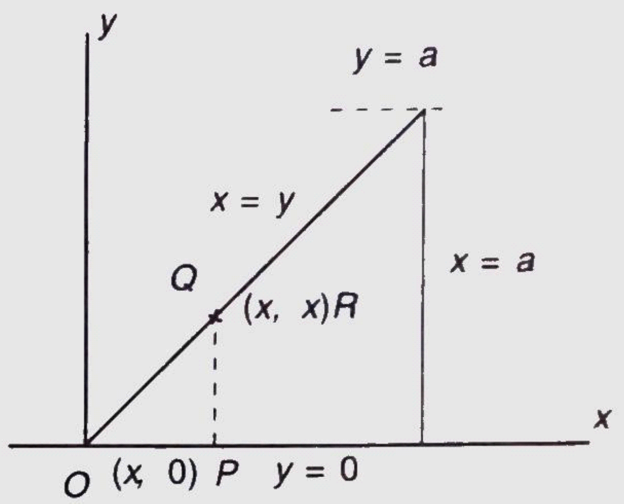


Fig. 5.19

Thus

$$I = \int_0^a\int_0^x \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$$

$$= \int_0^a x\left\{\log\left(y+\sqrt{y^2+x^2}\right)\right\}_{y=0}^{y=x} dx$$

$$= \int_0^a x\,[\log(x+x\sqrt{2}) - \log x]\,dx$$

$$= \log(1+\sqrt{2})\cdot\left(\frac{x^2}{2}\right)_0^a = \frac{a^2}{2}\log(1+\sqrt{2})$$

Example 5.2 Change the order of integration in $\int_0^1\int_x^1 \frac{x\,dx\,dy}{x^2+y^2}$ and then evaluate it.

Note ☑ Since the limits of inner integration are x and 1, the corresponding variable of integration should be y. So we rewrite the given integral I in the corrected form first.

$$I = \int_0^1\int_x^1 \frac{x\,dy\,dx}{x^2+y^2}$$

The region of integration R is bounded by the lines $x = 0$, $x = 1$, $y = x$ and $y = 1$ and is given in Fig. 5.20.

The limits for the inner integration (after changing the order of integration) with respect to x are fixed as usual, by drawing a line parallel to x-axis (y = constant)

$\therefore$

$$I = \int_0^1\int_0^y \frac{x\,dx\,dy}{x^2+y^2}$$

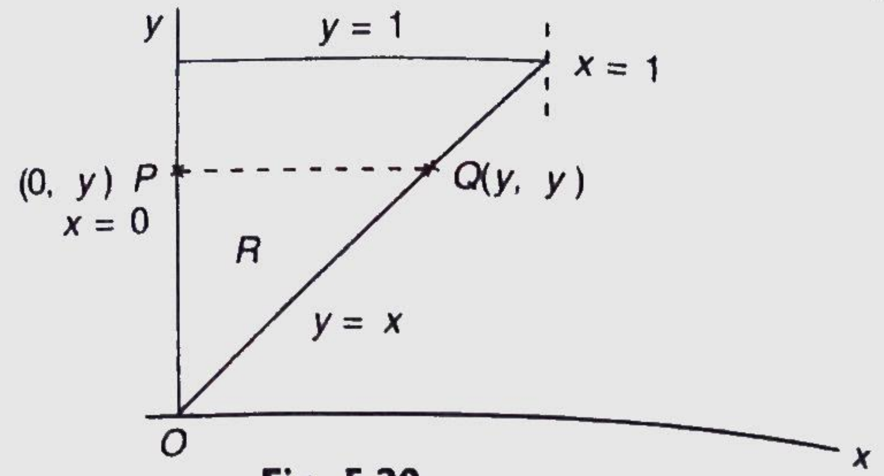


Fig. 5.20

$$= \int_0^1 \left[\frac{1}{2} \log (x^2 + y^2) \right]_{x=0}^{x=y} dy$$

$$= \frac{1}{2} \int_0^1 \log \left(\frac{2y^2}{y^2} \right) dy$$

$$= \frac{1}{2} \log 2.$$

Example 5.3 Change the order of integration in $\int_0^b \int_0^{\frac{a}{b}(b-y)} xy \, dx \, dy$ and then evaluate it.

The region of integration R is bounded by the lines $x = 0$, $x = \frac{a}{b}(b - y)$ or $\frac{x}{a} + \frac{y}{b} = 1$, $y = 0$ and $y = b$ and is shown in Fig. 5.21.

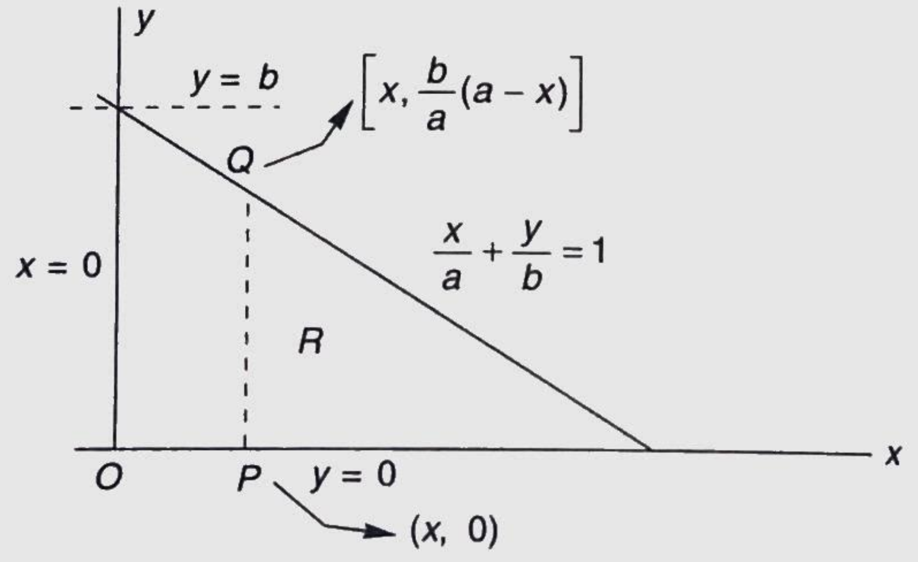


Fig. 5.21

After changing the order of integration, the integral becomes $I = \iint_R xy \, dy \, dx$.

The limits are fixed as usual.

$$I = \int_0^a \int_0^{\frac{b}{a}(a-x)} xy \, dy \, dx$$

$$= \int_0^a x \left(\frac{y^2}{2}\right)_0^{\frac{b}{a}(a-x)} dx$$

$$= \frac{b^2}{2a^2} \int_0^a x\,(a-x)^2\, dx$$

$$= \frac{b^2}{2a^2}\left[a^2\frac{x^2}{2} - 2a\frac{x^3}{3} + \frac{x^4}{4}\right]_0^a$$

$$= \frac{a^2b^2}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

$$= \frac{a^2b^2}{24}$$

Example 5.4 Change the order of integration in $\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2\, dy\, dx$ and then integrate it.

The region of integration R is bounded by the lines $x = 0$, $x = a$, $y = 0$ and the curve $y = \frac{b}{a}\sqrt{a^2 - x^2}$ i.e. the curve $\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$, i.e. the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and is shown in Fig. 5.22.

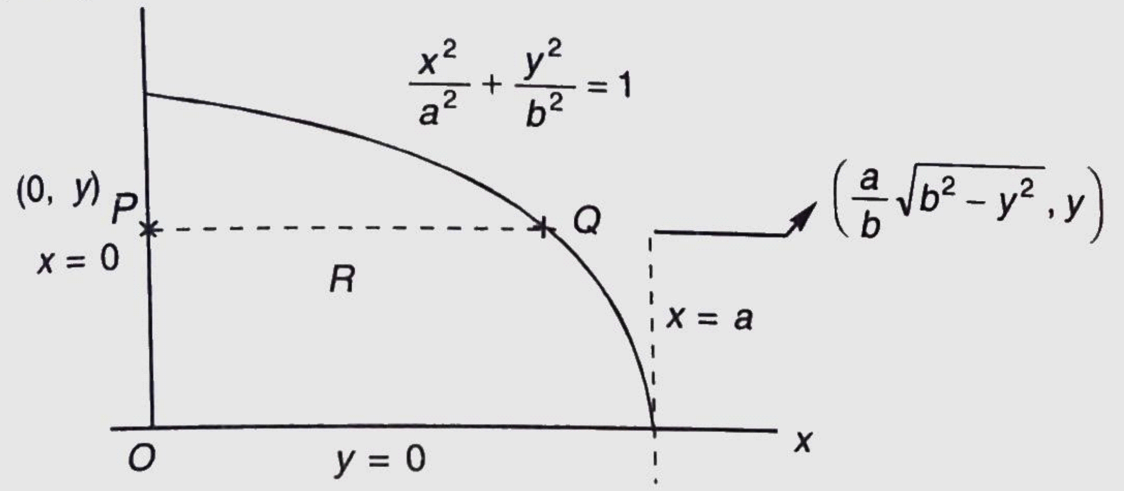


Fig. 5.22

After changing the order of integration, the integral becomes

$$I = \iint_R x^2\, dx\, dy.$$

The limits are fixed as usual.

$$I = \int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} x^2\, dx\, dy$$

$$= \int_0^b \left(\frac{x^3}{3}\right)_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= \frac{a^3}{3b^3} \int_0^b (b^2 - y^2)^{\frac{3}{2}}\, dy$$

$$= \frac{a^3}{3b^3} \int_0^{\pi/2} b^4 \cos^4\theta\, d\theta \qquad \text{(on putting } y = b \sin\theta\text{)}$$

$$= \frac{a^3 b}{3} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi}{16} a^3 b$$

Example 5.5 Change the order of integration in $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y\, dx\, dy$ and then evaluate it.

The region of integration R is bounded by the line $x = a - y$, the curve $x = \sqrt{a^2 - y^2}$, the lines $y = 0$ and $y = a$.

i.e. the line $x + y = a$, the circle $x^2 + y^2 = a^2$ and the lines $y = 0$, $y = a$. R is shown in Fig. 5.23.

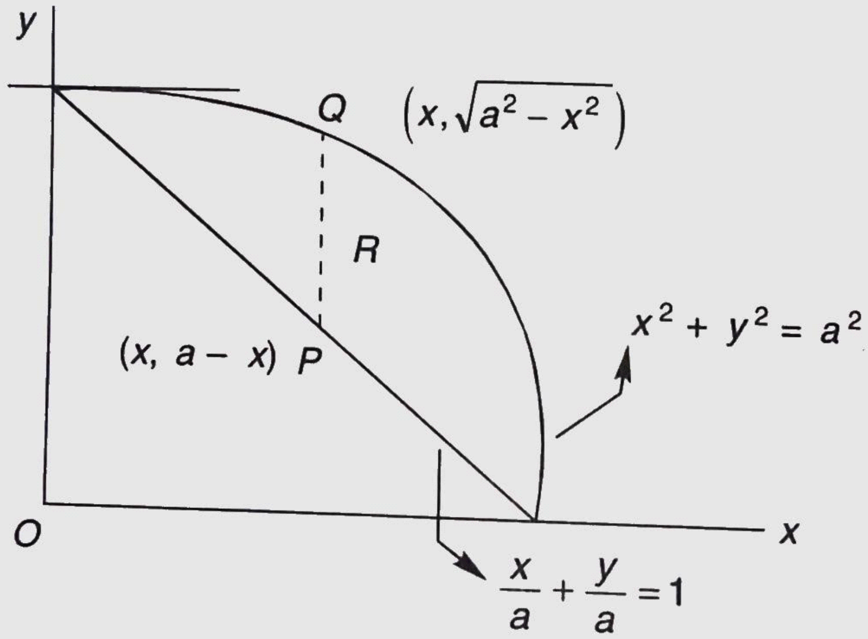


Fig. 5.23

After changing the order of integration, the integral I becomes,

$$I = \iint_R y\, dy\, dx$$

$$= \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y\, dy\, dx$$

$$= \int_0^a \left(\frac{y^2}{2}\right)_{a-x}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a (2ax - 2x^2)\, dx$$

VECTOR CALCULUS

In Vector Algebra we mostly deal with constant vectors namely, vectors which are constant in magnitude and fixed in direction.

In Vector Calculus we deal with variable vectors ie vectors which are varying in magnitude or direction or both.

Fundamental Results :

1) If $\vec{P} = P_1\vec{i} + P_2\vec{j} + P_3\vec{k}$ then the magnitude of a vector $\vec{P}$, $P = |\vec{P}| = \sqrt{P_1^2 + P_2^2 + P_3^2}$ unit vector in the direction of $\hat{P}$ is $\dfrac{\vec{P}}{|\vec{P}|}$

2) $\vec{P}\cdot\vec{Q} = PQ\cos\theta$ defines the dot or scalar product of the vectors $\vec{P}$ and $\vec{Q}$, θ is the angle between them

(i) $\vec{P}\cdot\vec{Q} = 0$, if $\vec{P}$ & $\vec{Q}$ are $\perp$er.

(ii) $\vec{P}\cdot\vec{P} = P^2$

(iii) $\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$.

(iv) $\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{i} = 0$

(v) $\vec{P}\cdot\vec{Q} = \vec{Q}\cdot\vec{P}$.

3) $\vec{P}\times\vec{Q} = PQ\sin\theta\,\hat{n}$ defines the cross or vector product of the vectors $\vec{P}$ and $\vec{Q}$ where θ is the angle and $\hat{n}$ is a unit vector in the direction of perpendicular to the plane of $\vec{P}$ and $\vec{Q}$.

(i) $\vec{P} \times \vec{Q} = -\vec{Q} \times \vec{P}$

(ii) $\vec{P} \times \vec{P} = +\vec{0}$

(iii) $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{K} \times \vec{K} = 0$

(iv) $\vec{i} \times \vec{j} = \vec{K}, \quad \vec{j} \times \vec{K} = \vec{i}, \quad \vec{K} \times \vec{i} = \vec{j}$.

(v) $\vec{P} \times \vec{Q} = \vec{0}$ if $\vec{P}, \vec{Q}$ are parallel.

Differential operators. ∇ (del)

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{K}\frac{\partial}{\partial z}$$ is a vector differential operator possessing properties similar to those of vectors. [It can act on a scalar (or) a vector function.]

Gradient

If $\phi(x, y, z)$ is a scalar point function continuously differentiable in a given region of space then the gradient of ϕ is defined by

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{K}\frac{\partial}{\partial z}\right)\phi$$

$$= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{K}\frac{\partial\phi}{\partial z} \quad [\text{grad}\,\phi = \nabla\phi]$$

Note:

(i) $\nabla\phi$ defines a vector field, whereas ϕ is a scalar field.

(ii) If ϕ is a constant, then $\nabla\phi = 0$.

Directional Derivatives:

The directional derivatives of a scalar point function $\phi(x, y, z)$ in a given direction is the rate of change of $\phi(x, y, z)$ in that direction. It is given by the component of $\nabla\phi$ in that direction.

If $\hat{a}$ is the unit vector in the given direction then the directional derivative of ϕ is given by $\nabla\phi \cdot \hat{a}$.

Maximum value of directional derivative of ϕ is $|\nabla\phi|$. Unit normal vector to the surface ϕ is $\dfrac{\nabla\phi}{|\nabla\phi|}$

Divergence

If $\vec{F}(x, y, z)$, $\vec{F} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ is a vector point function continuously differential in a given region of a space, then the divergence of $\vec{F}$ is defined by

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(f_1\vec{i} + f_2\vec{j} + f_3\vec{k}\right)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \text{ which is a scalar}$$

~~Divergence~~ Divergence of $\vec{F}$ (or) $\nabla \cdot \vec{F}$ (or) del dot $\vec{F}$

Note: (i) $\nabla \cdot \vec{F} \neq \vec{F} \cdot \nabla$ $\because$ $\nabla \cdot \vec{F}$ is a scalar by $\vec{F} \cdot \nabla$ is only an operator.

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(ii) $\nabla \cdot \vec{f} = 0$, if $\vec{f}$ is a constant vector and conversely.

SOLENOIDAL VECTOR

A vector $\vec{f}$ is called Solenoidal if $\operatorname{div} \vec{f} = 0$

ie $\nabla \cdot \vec{f} = 0$

CURL

If $\vec{f}(x, y, z)$ is a differentiable vector field then the curl of $\vec{f}$ is defined by

$$\nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} \quad \text{where } \vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

$$= \vec{i}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] - \vec{j}\left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right] + \vec{k}\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$$

curl of $\vec{f}$ or curl $\vec{f}$ or $\nabla \times \vec{f}$ or rotation of $\vec{f}$.

IRROTATIONAL VECTOR

A vector $\vec{f}$ is called irrotational, if curl $\vec{f} = 0$

ie $\nabla \times \vec{f} = \vec{0}$.

Problems:

1) If $\phi(x, y, z) = 3x^2y - y^3z^2$. Find grad ϕ at the point $(1, -2, -1)$.

Soln:

grad $\phi = \nabla\phi = \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right]\left[3x^2y - y^3z^2\right]$

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$= \vec{i}(6xy) + \vec{j}(3x^2 - 3y^2z^2) + \vec{k}(-2y^3z)$

$\nabla\phi_{(1,-2,-1)} = \vec{i}(-12) + \vec{j}(3-12) + \vec{k}(-16)$

$= -12\vec{i} - 9\vec{j} - 16\vec{k}$.

Q.B2.)

2) Show that the vector $\vec{F} = 5y^4z^3\vec{i} + 8xz^2\vec{j} - y^2x\vec{k}$ is Solenoidal.

Soln:

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(5y^4z^3\vec{i} + 8xz^2\vec{j} - y^2x\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(5y^4z^3) + \frac{\partial}{\partial y}(8xz^2) + \frac{\partial}{\partial z}(-y^2x)$$

$$= 0 + 0 + 0 = 0.$$

$\Rightarrow \nabla\cdot\vec{F} = 0$.

$\therefore$ The vector $\vec{F} = 5y^4z^3\vec{i} + 8xz^2\vec{j} - y^2x\vec{k}$ is Solenoidal.

3) Find a, $\vec{F} = (3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.

Q.B4

Soln. $\nabla\cdot\vec{F} = 0$

$$\Rightarrow \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0.$$

$\Rightarrow 3 + a + 2 = 0 \Rightarrow \boxed{a = -5}$

4) Find the directional derivative of $\phi = xyz - xy^2z^3$ at $(1, 2, -1)$ in the direction $\vec{i} - \vec{j} - 3\vec{k}$. Find also its maximum value.

Soln: $\phi = xyz - xy^2z^3$.

$$\nabla\phi = \vec{i}\frac{\partial}{\partial x}(xyz - xy^2z^3) + \vec{j}\frac{\partial}{\partial y}(xyz - xy^2z^3) + \vec{k}\frac{\partial}{\partial z}(xyz - xy^2z^3)$$

$$= (yz - y^2z^3)\vec{i} + (xz - 2xz^3)\vec{j} + (xy - 3xy^2z^2)\vec{k}.$$

$\nabla\phi$ at $(1,2,-1) = 2\vec{i} + 3\vec{j} - 10\vec{k}$

unit vector in the direction $\vec{i} - \vec{j} - 3\vec{k}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i} - \vec{j} - 3\vec{k}}{\sqrt{1+1+9}} = \frac{\vec{i} - \vec{j} - 3\vec{k}}{\sqrt{11}}$$

Directional Derivative of ϕ in the given ~~equation~~ direction is

$$\nabla\phi \cdot \hat{a} = (2\vec{i} + 3\vec{j} - 10\vec{k}) \cdot \frac{\vec{i} - \vec{j} - 3\vec{k}}{\sqrt{11}}$$

$$= \frac{2 - 3 + 30}{\sqrt{11}} = \frac{29}{\sqrt{11}}$$

Maximum value of directional derivative of ϕ is $|\nabla\phi|$

ie $|\nabla\phi| = |2\vec{i} + 3\vec{j} - 10\vec{k}| = \sqrt{4+9+100} = \sqrt{113}$

4, B(3)

5) Show that the vector $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrotational

Soln:
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \vec{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$= \vec{i}\,[x-x] - \vec{j}\,[y-y] + \vec{k}\,[z-z] = \vec{0}$. 7

Hence $\nabla \times \vec{F} = 0$ is irrotational.

QB5. QB.12 (i).

6) Find the value of a, if the vector $\vec{F} = (axy - z^3)\vec{i} + (a-2)x^2\vec{j} + (1-a)xz^2\vec{k}$ is irrotational

Soln. ~~Find~~ the vector $\vec{F}$ is Irrotational $\Rightarrow \nabla \times \vec{f} = \vec{0}$.

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = \vec{0}.$$

$\Rightarrow \vec{i}\,(0) - \vec{j}\left[(1-a)z^2 + 3z^2\right] + \vec{k}\left[2x(a-2) - ax\right] = \vec{0}$.

$\Rightarrow -\vec{j}\left[(1-a)z^2 + 3z^2\right] + \vec{k}\left[2ax - 4x - ax\right] = \vec{0}$.

$\Rightarrow -\vec{j}\left[z^2 - az^2 + 3z^2\right] + \vec{k}\left[ax - 4x\right] = 0\vec{i} + 0\vec{j} + 0\vec{k}$.

Equating the corresponding coefficients

$\Rightarrow ax - 4x = 0 \Rightarrow ax = 4x \Rightarrow \boxed{a = 4}$

QB6

7) If $\nabla\phi = yz\vec{i} + zx\vec{j} + xy\vec{k}$, find ϕ

Soln: $\nabla\phi = yz\vec{i} + zx\vec{j} + xy\vec{k}$.

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = yz\vec{i} + zx\vec{j} + xy\vec{k}.$$

Equating the corresponding coefficients we get

$$\frac{\partial\phi}{\partial x} = yz;\quad \frac{\partial\phi}{\partial y} = zx,\quad \frac{\partial\phi}{\partial z} = xy.$$

Integrating wrt x, y, z resply, we get

$\phi = xyz + c, \quad \phi = xyz + c, \quad \phi = xyz + c.$

Hence the possible form of ϕ is $xyz + c$

8) prove that curl (curl $\vec{f}$) $= \nabla(\nabla \cdot \vec{f}) - \nabla^2 \vec{f}$.

soln.

$$\text{curl } \vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] - \vec{j}\left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right] + \vec{k}\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$$

$$\text{curl}(\text{curl } \vec{f}) = \nabla \times (\nabla \times \vec{f}).$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} & \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} & \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{vmatrix}$$

$$= \Sigma \vec{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right]$$

$$= \Sigma \vec{i}\left[\frac{\partial^2 f_2}{\partial x \partial y} - \frac{\partial^2 f_1}{\partial y^2} - \frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_3}{\partial z \partial x}\right]$$

$$= \Sigma \vec{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) - \left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)\right]$$

$$= \Sigma \vec{i}\left[\frac{\partial}{\partial x}(\nabla \cdot \vec{f}) - \nabla^2 f_1\right]$$

$$= \nabla(\nabla \cdot \vec{f}) - \nabla^2 \vec{f}$$

$$\text{curl}(\text{curl } \vec{f}) = \nabla(\nabla \cdot \vec{f}) - \nabla^2 \vec{f}.$$

H/W.

1.) Prove that $\vec{f} = 3y^4z^2\vec{i} + 4x^4z^2\vec{j} - 3x^2y^2\vec{k}$ is solenoidal

2) Show that $\vec{f} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ is irrotational

3) Find the value of a if $(x + 3y)\vec{i} + (y - 2z)\vec{j} + (x + az)\vec{k}$ is solenoidal.

4) Find the scalar point function ϕ whose gradient is $yz^2\vec{i} + (xz^2 - 1)\vec{j} + 2(xyz - 1)\vec{k}$.

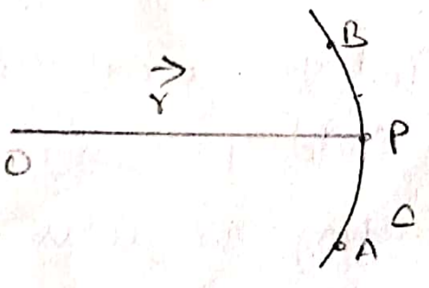
5) Find the directional derivative of $\phi = xy + yz + zx$ at $(1, 2, 0)$ in the direction $\vec{i} + 2\vec{j} + 2\vec{k}$. Find also its maximum value.

Line Integral.

An integral which is evaluated along a curve is called a line integral.

Let C be the given curve. Let A and B be two points on the curve. Then the line integral from A to B is given by

$$\int_C \vec{f} \cdot d\vec{r} = \int_A^B \vec{f} \cdot d\vec{r}$$



$$\vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}.$$

① Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = x^2y^2\vec{i} + y\vec{j}$ and C is
QB 12 (ii).
the curve $y^2 = 4x$ in the xy plane from (0,0) to (4,4)

Soln: $\vec{F} = x^2y^2\vec{i} + y\vec{j}$.

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$.

$\vec{F}\cdot d\vec{r} = x^2y^2\,dx + y\,dy$.

Given $y^2 = 4x$.

$\therefore \vec{F}\cdot d\vec{r} = 4x^3\,dx + y\,dy$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_{(0,0)}^{(4,4)} 4x^3\,dx + y\,dy$$

$$= \int_0^4 4x^3\,dx + \int_0^4 y\,dy$$

$$= 4\left[\frac{x^4}{4}\right]_0^4 + \left[\frac{y^2}{2}\right]_0^4$$

$$= 256 + \frac{16}{2} = 256 + 8 = 264$$

QB 7.

$\phi = xyz$

$\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}$

$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$.

$= yz\vec{i} + xz\vec{j} + xy\vec{k}$.

$\nabla\phi_{(1,1,1)} = \vec{i} + \vec{j} + \vec{k}$.

$\therefore$ Directional derivative is $\nabla\phi\cdot\hat{a}$

$= \vec{i}+\vec{j}+\vec{k} \cdot \frac{(\vec{i}+\vec{j}+\vec{k})}{\sqrt{3}}$

$= \frac{1+1+1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$.

2) Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} - z\vec{k}$ from $t=0$ to $t=1$ along the curve $x = 2t^2$, $y = t$, $z = 4t^3$.

Soln: Workdone $= \int_C \vec{F}\cdot d\vec{r}$.

$\vec{F}\cdot d\vec{r} = 3x^2\,dx + (2xz - y)\,dy - z\,dz$.

$x = 2t^2$, $y = t$, $z = 4t^3$.

$dx = 4t\,dt$, $dy = dt$, $dz = 12t^2\,dt$.

$$\vec{F}\cdot d\vec{r} = 3(2t^2)^2 4t\,dt + [2(2t^2)(4t^3) - t]\,dt - 4t^3(12t^2)dt$$

$$= \cancel{48t^5}\,dt + 16t^5\,dt - \cancel{48t^5\,dt} - t\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (16t^5 - t)\,dt$$

$$= \left[16\frac{t^6}{6} - \frac{t^2}{2}\right]_0^1 = \frac{16}{6} - \frac{1}{2} = \frac{16-3}{6} = \frac{13}{6}.$$

$$\text{workdone} = \int_C \vec{F}\cdot d\vec{r} = \frac{13}{6} \text{ units}.$$

H/W.

3) Find the total WD in moving a particle in a force field given by $\vec{F} = 3xy\vec{i} - 5z\vec{j} + 10x\vec{K}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$. from $t=1$ to $t=2$.

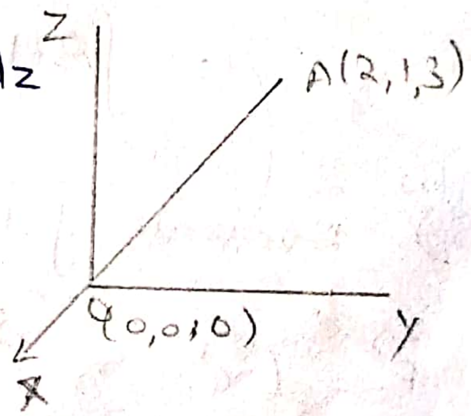
Soln: 303 units.

4.) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{K}$.

along the straight line c from (0,0,0) to (2,1,3).

Soln:

$$\vec{F}\cdot d\vec{r} = 3x^2dx + (2xz - y)dy + z\,dz$$



Consider the Integral $\int_{C=OA} \vec{F}\cdot d\vec{r}$.

Along OA,

$x = 0$ to $x = 2$

$y = 0$ to $y = 1$

$z = 0$ to $z = 3$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}$$

$\Rightarrow x = 2t,\ y = t,\ z = 3t$.

$\Rightarrow dx = 2dt,\ dy = dt,\ dz = 3dt$.

Q.B 8)

Unit normal vector to the Surface $\phi = \frac{\nabla\phi}{|\nabla\phi|}$

$\phi = x^2 - y^2 + z = 2$

$\nabla\phi = 2x\vec{i} - 2y\vec{j} + \vec{k}$.

$\nabla\phi(1,-1,2) = 2\vec{i} + 2\vec{j} + \vec{k}$.

$|\nabla\phi| = \sqrt{4+4+1} = 3$.

$\therefore$ unit normal vector $= \frac{2\vec{i}+2\vec{j}+\vec{k}}{3}$.

$t=0$ and $t=1$ corresponds to the points $(0,0,0)$ and $(2,1,3)$ on the path.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C 3x^2 dx + (2xz - y)dy + z\,dz.$$

$$= \int_0^1 3(2t)^2 2dt + [2(2t)(3t) - t]dt + (3t)3dt.$$

$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)\,dt.$$

$$= \int_0^1 36t^2 + 8t\,dt = \left[36\frac{t^3}{3} + 8\frac{t^2}{2}\right]_0^1 = 12 + 4 = 16.$$

H/w.

5) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where

$\vec{F} = (x^2+y^3)\vec{i} + (x^3 - y^2)\vec{j}$ along the straight line joining $(1,0)$ & $(0,1)$.

Soln: ~~7/4~~. $-2/3$.

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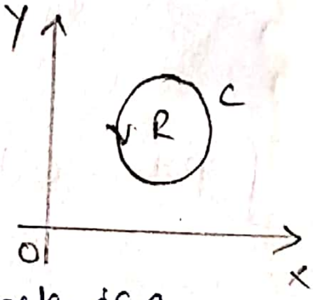
GREENS THEOREM IN THE PLANE.

If $P(x,y)$ and $Q(x,y)$ are continuous function with continuous partial derivatives in a region R of the plane and on its boundary c which is a simple closed curve, then



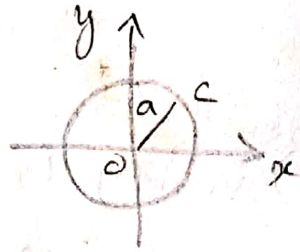
$$\oint_c (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where 'C' is described in the anticlockwise direction.

Note: $\iint_R dx\,dy =$ Area of the region R.

1) Using Green's theorem, Evaluate $\int_c (2x-y)dx + (x+y)dy$ where C is the boundary of the circle $x^2+y^2=a^2$.

Soln: Here $P = 2x-y$, $Q = x+y$.



$$\frac{\partial P}{\partial y} = -1, \quad \frac{\partial Q}{\partial x} = 1$$

$$\therefore \int_c P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\Rightarrow \int_c (2x-y)dx + (x+y)dy = \iint_R (1+1)\,dx\,dy$$

$$= 2\iint_R dx\,dy$$

$$= 2 \times \text{Area of the region } R$$

$$= 2\pi a^2.$$

H/W

2) Evaluate $\int_C y(2xy-1)\,dx + x(2xy+1)\,dy$ where C is the circle $x^2+y^2=4$ using Green's Theorem.

Soln: Here $P = 2xy^2 - y$ $\qquad Q = 2x^2y + x$

$$\frac{\partial P}{\partial y} = 4xy - 1 \qquad \frac{\partial Q}{\partial x} = 4xy + 1.$$

$$\int_C P\,dx + Q\,dy = \iint_R (4xy+1-4xy+1)\,dx\,dy$$

$$= \iint_R 2\,dx\,dy.$$

$$= 2 \times \text{Area of the region } R.$$

$$= 2 \times \pi (2)^2$$

$$= 8\pi.$$

Q.B (14)

3) Verify Green's theorem in the plane for $\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where C is the boundary of the region defined by $y=\sqrt{x}$ and $y=x^2$.

$x = \sqrt{x}$

$x^4 = x$

Soln: Given: $C = C_1 + C_2$.

The point of intersection are $(0,0)$ and $(1,1)$.

$x(x^3-1)=0$

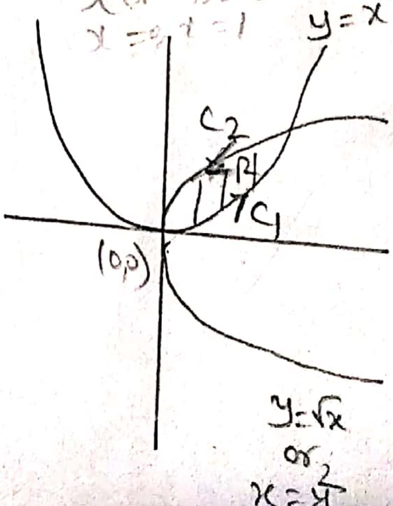
$x=0,1$

Along C_1 : LHS

$$y = x^2$$

$dy = 2x\,dx$, x varies from 0 to 1

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



$$\int_C P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy.$$

$$\int_{C_1} P\,dx + Q\,dy = \int_0^1 (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy.$$

$$= \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)\,2x\,dx$$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx.$$

$$= \left[\frac{3x^3}{3} - 8\frac{x^5}{5} + 8\frac{x^4}{4} - 12\frac{x^5}{5} \right]_0^1$$

$$= \left[x^3 - \frac{20}{5}x^5 + \frac{8x^4}{4} \right]_0^1$$

$$= 1 - \frac{\cancel{20}^{\,4}}{\cancel{5}} + \frac{\cancel{8}^{\,2}}{\cancel{4}} = 1 - 4 + 2 = -1$$

Along C_2: $y = \sqrt{x}$

$\Rightarrow x = y^2$

$\Rightarrow dx = 2y\,dy$; y varies from 1 to 0.

$$\therefore \int_C P\,dx + Q\,dy = \int_{C_2} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy.$$

$$= \int_1^0 (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)\,dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)\,dy$$

$$= \left[6 \cdot \frac{y^6}{6} - 22\frac{y^4}{4} + 4\frac{y^2}{2} \right]_1^0$$

$$= \left[-1 + \frac{11}{2} - 2 \right] = \frac{11-6}{2} = 5/2.$$

$$\therefore \int_C P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = -1 + 5/2$$
$$= 3/2.$$

RHS. $P = 3x^2 - 8y^2$, $Q = 4y - 6xy$

$$\frac{\partial P}{\partial y} = -16y \qquad \frac{\partial Q}{\partial x} = -6y.$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint_R (-6y + 16y)\,dx\,dy$$

$$= 10 \int_0^1 \int_{x^2}^{\sqrt{x}} y\,dx\,dy.$$

$$= 10 \int_0^1 \left[\frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx.$$

$$= \frac{10}{2} \int_0^1 (x - x^4)\,dx.$$

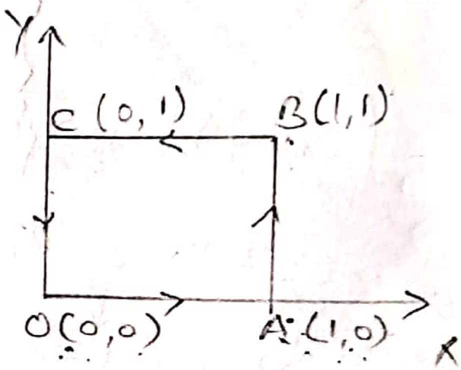
$$= 5 \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = 5 \left[\frac{1}{2} - \frac{1}{5} \right]$$

$$= 5 \left[\frac{3}{10} \right] = 3/2.$$

LHS = RHS.

Hence Green's theorem is verified.

4) Q,B(13) Verify Green's theorem for the integral $\int (x^2+y)\,dx - xy^2\,dy$ taken around the boundary of the square whose vertices are $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.



Soln. Here $P = x^2+y$, $Q = -xy^2$

$$\frac{\partial P}{\partial y} = 1 \quad , \quad \frac{\partial Q}{\partial x} = -y^2$$

$\therefore$ By Greens theorem

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy .$$

Now RHS

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = -\int_0^1\int_0^1 (y^2+1)\,dx\,dy .$$

$$= -\int_0^1 \left[y^2x + x\right]_0^1 dy = -\left[\frac{y^3}{3} + y\right]_0^1 = -\left[\frac{1}{3}+1\right]$$

$$= -4/3 .$$

LHS. Line integral along $\overrightarrow{OA}$, $y=0$, $dy=0$.
x varies from 0 to 1.

$$\therefore \int_{C_1} (x^2+y)\,dx - xy^2\,dy = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = 1/3 .$$

Along $\overrightarrow{AB}$, $x=1$, $dx=0$, y varies from 0 to 1.

$$\therefore \int_C (x^2+y)\,dx - xy^2\,dy = \int_0^1 -y^2\,dy = \left[-\frac{y^3}{3}\right]_0^1 = -1/3 .$$

Along $\overrightarrow{BC}$, $y=1$, $dy=0$, x varies from 1 to 0.

$$\therefore \int_C (x^2+y)\,dx - xy^2\,dy = \int_1^0 (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_1^0 = -4/3$$

Along $\vec{CO}$, $x=0$, $dx=0$, y varies from 1 to 0

$$\therefore \int_C (x^2+y)\,dx - xy^2\,dy = 0$$

$$\therefore \int_C (x^2+y)\,dx - xy^2\,dy = 1/3 - 1/3 - 4/3 + 0 = -4/3$$

$\therefore$ LHS = RHS.

$$\therefore \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Hence proved.

5) Show by applying Green's theorem that the area bounded by a simple closed curve C is $\frac{1}{2}\int_C (x\,dy - y\,dx)$, and hence find the area of the Ellipse $x^2/a^2 + y^2/b^2 = 1$

Soln: By Green's theorem,

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Here $P = -y/2$, $Q = x/2$

$$\frac{\partial P}{\partial y} = -1/2, \quad \frac{\partial Q}{\partial x} = 1/2, \text{ ~~we get~~}$$

$$\frac{1}{2}\int_C x\,dy - y\,dx = \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dx\,dy$$

$$= \iint_R dx\,dy$$

= Area of the region R Enclosed by C.

For the Ellipse C, let us take the parametric Equation, $x = a\cos\theta$, $y = b\sin\theta$ $(0 \le \theta \le 2\pi)$.

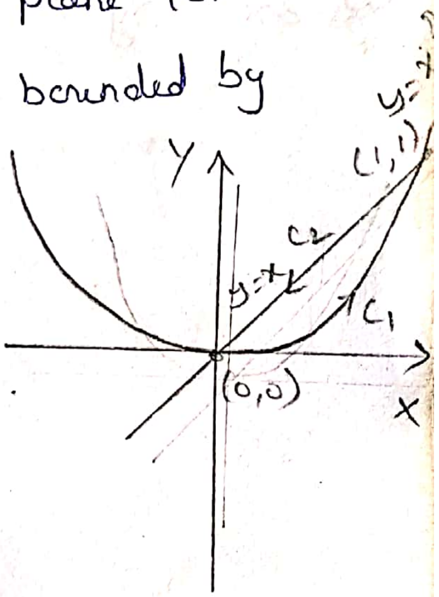
$\therefore$ Area of Ellipse $= \frac{1}{2}\int_0^{2\pi} a\cos\theta \, b\cos\theta \, d\theta - b\sin\theta(-a\sin\theta)\, d\theta$.

$$= \frac{1}{2}\int_0^{2\pi} (ab\cos^2\theta + ab\sin^2\theta)\, d\theta.$$

$$= \frac{1}{2}ab\int_0^{2\pi} d\theta = \frac{1}{2}ab\left[\theta\right]_0^{2\pi} = \pi ab.$$

H/W

6) verify Green's theorem in the plane for $\int_C (xy+y^2)\,dx + x^2\,dy$ where C is bounded by $y = x$ and $y = x^2$.



Soln: Point of intersection of $y=x$ and $y=x^2$ are at $(0,0)$ and $(1,1)$.

Along C_1: $y = x^2$, $dy = 2x\,dx$

& x varies from 0 to 1.

$$\therefore \int_{C_1} (xy+y^2)\,dx + x^2\,dy = \int_0^1 (x^3+x^4)\,dx + x^2(2x)\,dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5}$$

$$= \frac{15+4}{20}$$

$$= \frac{19}{20}$$

Along C_2 : $y = x$, $dy = dx$ and $x = 1$ to 0.

$$\therefore \int_{C_2} (xy + y^2)\,dx + x^2\,dy = \int_1^0 (x^2 + x^2)\,dx + x^2\,dx$$

$$= \left[3 \cdot \frac{x^3}{3}\right]_1^0 = -1$$

$$\therefore \int_C (xy + y^2)\,dx + x^2\,dy = \frac{19}{20} - 1 = -\frac{1}{20}.$$

Consider RHS.

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dx\,dy$$

$$= \int_0^1 \int_{x^2}^{x} (2x - x - 2y)\,dx\,dy.$$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)\,dx\,dy$$

$$= \int_0^1 \left[xy - 2\frac{y^2}{2}\right]_{x^2}^{x} dx$$

$$= \int_0^1 (\cancel{x^2} - \cancel{x^2} - x^3 + x^4)\,dx.$$

$$= \left[-\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 = + \left[-\frac{1}{4} + \frac{1}{5}\right]$$

$$= -1/20.$$

$\therefore$ LHS = RHS.

Hence Green's theorem is verified.

7) H/W

Verify Greens theorem for $\int_C (x-2y)dx + x\,dy$ taken around the unit circle $x^2+y^2=1$.

Soln: $P = x-2y$, $Q = x$

$\frac{\partial P}{\partial y} = -2$ $\quad \frac{\partial Q}{\partial x} = 1$

For the circle, let us take the parametric Equation

$x = \cos\theta$, $y = \sin\theta$ $\quad (0 \le \theta \le \pi)$

$\therefore$ Area of the circle $= \int_0^{2\pi} (x-2y)\,dx + x\,dy$.

$$= \int_0^{2\pi} (\cos\theta - 2\sin\theta)(-\sin\theta)\,d\theta + \cos\theta\,(\cos\theta)\,d\theta.$$

$$= \int_0^{2\pi} \left[-\sin\theta\cos\theta + 2\sin^2\theta + \cos^2\theta\right] d\theta$$

$$= \int_0^{2\pi} \left[-\sin\theta\cos\theta + \sin^2\theta + 1\right] d\theta.$$

$$= \int_0^{2\pi} \left[-\frac{\sin 2\theta}{2} + \left(\frac{1-\cos 2\theta}{2}\right) + 1\right] d\theta$$

$$= \left[\frac{\cos 2\theta}{4} + \frac{1}{2}\theta - \frac{\sin 2\theta}{4} + \theta\right]_0^{2\pi}$$

$$= \frac{1}{4} + \frac{1}{2}\cdot 2\pi + 2\pi - \frac{1}{4} = 3\pi$$

RHS

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_R (1+2)\,dx\,dy$$

$$= 3\iint_R dx\,dy$$

$= 3 \times$ area of the circle

$$= 3\pi (1)^2$$

$$= 3\pi.$$

LHS = RHS.

Hence Green's theorem is verified.

GAUSS DIVERGENCE THEOREM.

QB(9) If S is a closed surface enclosing a region of space with volume V and if $\vec{F}$ is a vector point function with continuous first order partial derivatives in V, then

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

where $\hat{n}$ is a unit normal vector to surface (S)

QB(15)

1) Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

Soln: The Gauss Divergence theorem is

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

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RHS.

$$\nabla\cdot\vec{f} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right).$$

$$= \frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz).$$

$$= 4z - 2y + y = 4z - y.$$

$$\iiint_V \nabla\cdot\vec{f}\, dv = \int_0^1\int_0^1\int_0^1 (4z - y)\, dx\, dy\, dz.$$

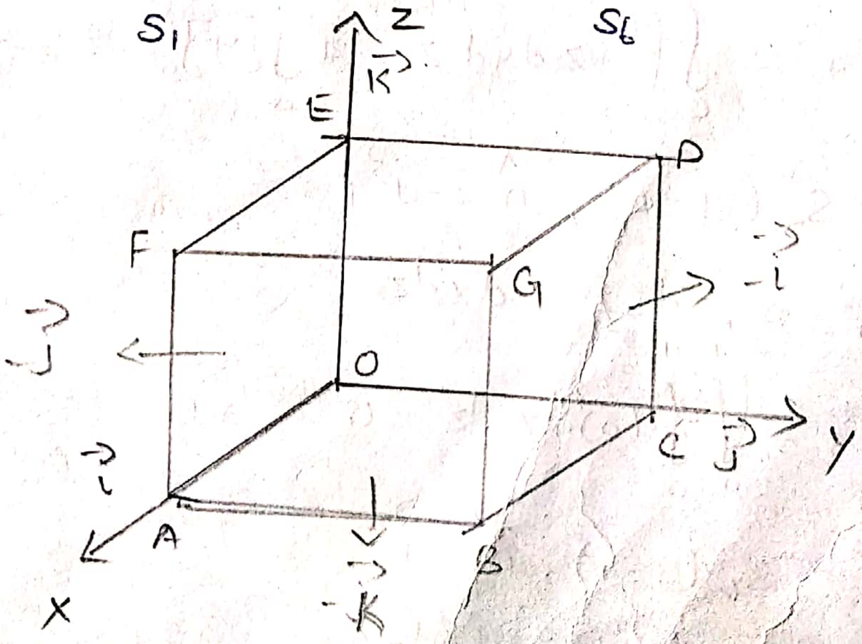
$$= \int_0^1\int_0^1 (4xz - xy)_0^1\, dy\, dz = \int_0^1\int_0^1 (4z - y)\, dy\, dz$$

$$= \int_0^1 \left[4zy - \frac{y^2}{2}\right]_0^1 dz = \int_0^1 \left(4z - \frac{1}{2}\right) dz$$

$$= \left[4\cdot\frac{z^2}{2} - \frac{1}{2}z\right]_0^1 = \frac{4}{2} - \frac{1}{2} = 3/2.$$

LHS.

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} \vec{f}\cdot\hat{n}\, ds_1 + \cdots + \iint_{S_6} \vec{F}\cdot\hat{n}\, ds_6$$



(i) OABC $S_1 (z=0)$, $\hat{n} = -\vec{k}$.

$$\iint_{S_1} \vec{F}\cdot\hat{n}\, ds_1 = \iint_{S_1} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{k})\, dx\, dy$$

$$= \iint_{S_1} -yz\, dx\, dy = 0.$$

(ii) DEFG $S_2 (z=1)$, $\hat{n} = \vec{k}$; $\vec{F}\cdot\hat{n} = yz = y$, $ds_2 = dx\,dy$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\, ds_2 = \int_0^1\int_0^1 y\, dx\, dy = \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx = \frac{1}{2}[x]_0^1 = 1/2.$$

(iii) OCDE : $S_3 (x=0)$, $\hat{n} = -\vec{i}$.

$\vec{F}\cdot\hat{n} = -4xz = 0$ $(\because x=0)$; $ds_3 = dy\,dz$.

$$\iint_{S_3} \vec{F}\cdot\hat{n}\, ds_3 = \iint_{S_3} (0)\, dy\,dz = 0.$$

(iv) ABGF : $S_4 (x=1)$, $\hat{n} = \vec{i}$

$\vec{F}\cdot\hat{n} = 4xz = 4z$ $(\because x=1)$; $ds_4 = dy\,dz$.

$$\iint_{S_4} \vec{F}\cdot\hat{n}\, ds_4 = \int_0^1\int_0^1 4z\, dy\, dz = 4\int_0^1 [y]_0^1 z\, dz = 4\left[\frac{z^2}{2}\right]_0^1 = 2.$$

(v) OAFE ; $S_5 (y=0)$; $\hat{n} = -\vec{j}$.

$\vec{F}\cdot\hat{n} = y^2 = 0$, $ds_5 = dx\,dz$.

$$\iint_{S_5} \vec{F}\cdot\hat{n}\, ds_5 = \int_0^1\int_0^1 (0)\, dx\, dz = 0.$$

(vi) BCDG : $S_6 (y=1)$, $\hat{n} = \vec{j}$

$\vec{F}\cdot\hat{n} = -y^2 = -1$, $ds_6 = dx\,dz$.

$$\iint_{S_6} \vec{F}\cdot\hat{n}\, ds_6 = \int_0^1\int_0^1 (-1)\, dx\, dz = -\int_0^1 [x]_0^1 dz$$

$$= -[z]_0^1 = -1.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = \frac{1}{2} + 2 - 1 = 3/2.$$

Hence the theorem is verified.

2) verify divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$.

<u>soln.</u>

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$= 2x + 2y + 2z = 2(x+y+z).$$

<u>RHS.</u>

$$\iiint_V \nabla\cdot\vec{F}\, dv = 2\int_0^c\int_0^b\int_0^a (x+y+z)\, dx\, dy\, dz.$$

$$= 2\int_0^c\int_0^b \left[\frac{x^2}{2} + yx + zx\right]_0^a dy\, dz.$$

$$= 2\int_0^c\int_0^b \left(\frac{a^2}{2} + ay + az\right) dy\, dz.$$

$$= 2\int_0^c \left[\frac{a^2}{2}y + a\frac{y^2}{2} + azy\right]_0^b dz.$$

$$= 2\int_0^c \left(\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right) dz$$

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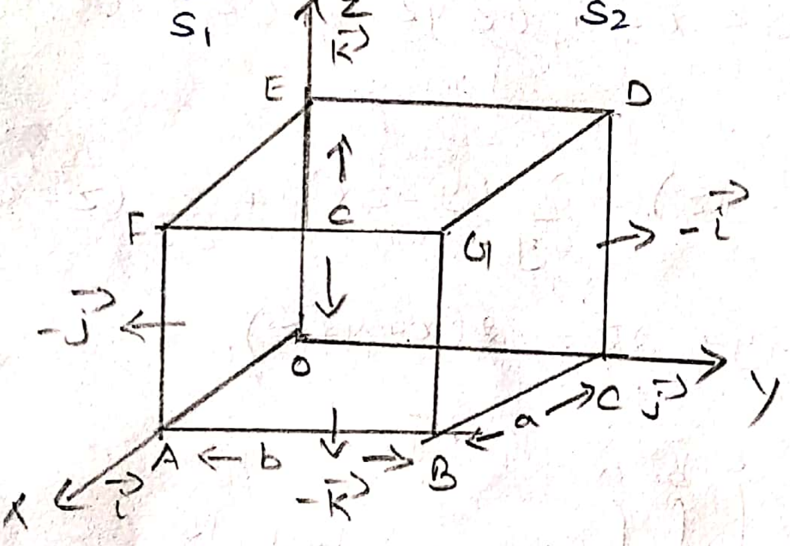
$$= 2\left[\frac{a^2b\,z}{2} + \frac{ab^2\,z}{2} + ab\frac{z^2}{2}\right]_0^c$$

$$= 2\left[\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right].$$

$$= 2\left[\frac{abc}{2}(a+b+c)\right] = abc(a+b+c).$$

LHS:

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \vec{F}\cdot\hat{n}\,ds_1 + \cdots + \iint_{S_2} \vec{F}\cdot\hat{n}\,ds_6.$$



(i) OABC S_1 $(z=0)$, $\hat{n} = -\vec{k}$.

$$\vec{F}\cdot\hat{n} = -(z^2 - xy)\ ;\ ds_1 = dx\,dy.$$
$$= xy.$$

$$\therefore \iint_{S_1} \vec{F}\cdot\hat{n}\,ds_1 = \int_0^a\int_0^b xy\,dx\,dy = \int_0^a x\left[\frac{y^2}{2}\right]_0^b dx = \frac{b^2}{2}\left[\frac{x^2}{2}\right]_0^a$$
$$= \frac{a^2b^2}{4}$$

ii) DEFG, S_2 $(z=c)$, $\hat{n} = \vec{k}$.

$$\vec{F}\cdot\hat{n} = z^2 - xy = c^2 - xy,\quad ds_2 = dx\,dy.$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds_2 = \int_0^a\int_0^b (c^2 - xy)\,dx\,dy = \int_0^a \left[c^2y - \frac{xy^2}{2}\right]_0^b dx$$

$$= \int_0^a \left(c^2 b - \frac{x b^2}{2}\right) dx = \left[c^2 b x - \frac{b^2}{2} \cdot \frac{x^2}{2}\right]_0^a$$

$$= abc^2 - \frac{a^2 b^2}{4}$$

(iii) OCDE S_3 $(x=0)$, $\hat{n} = -\vec{i}$

$$\vec{F} \cdot \hat{n} = (yz - x^2) = yz, \quad ds_3 = dy\,dz.$$

$$\iint_{S_3} \vec{F} \cdot \hat{n}\, ds_3 = \int_0^b \int_0^c yz\, dy\, dz = \int_0^b \left[\frac{yz^2}{2}\right]_0^c dy$$

$$= \frac{c^2}{2} \left[\frac{y^2}{2}\right]_0^b = \frac{b^2 c^2}{4}.$$

(iv) ABGF S_4 $(x=a)$, $\hat{n} = \vec{i}$

$$\vec{F} \cdot \hat{n} = x^2 - yz, \quad ds_4 = dy\,dz.$$

$$= a^2 - yz.$$

$$\iint_{S_4} \vec{F} \cdot \hat{n}\, ds_4 = \int_0^b \int_0^c (a^2 - yz)\, dy\, dz = \int_0^b \left[a^2 z - \frac{yz^2}{2}\right]_0^c dy$$

$$= \int_0^b \left(a^2 c - \frac{yc^2}{2}\right) dy = \left[a^2 c y - \frac{y^2 c^2}{4}\right]_0^b = a^2 bc - \frac{b^2 c^2}{4}$$

(v) OAFE S_5 $(y=0)$, $\hat{n} = -\vec{j}$.

$$\vec{F} \cdot \hat{n} = -(y^2 - zx) = zx, \quad ds_5 = dx\,dz.$$

$$\iint_{S_5} \vec{F} \cdot \hat{n}\, ds_5 = \int_0^a \int_0^c zx\, dx\, dz = \int_0^a \left[\frac{z^2}{2} x\right]_0^c dx = \frac{c^2}{2} \left[\frac{x^2}{2}\right]_0^a$$

$$= \frac{a^2 c^2}{4}.$$

(vi) BCDG S_6 $(y=b)$; $\hat{n}=\vec{j}$.

$\vec{F}\cdot\hat{n} = y^2 - zx = b^2 - zx$, $ds_6 = dx\,dz$.

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds_6 = \int_0^a\int_0^c (b^2 - zx)\,dx\,dz = \int_0^a \left[b^2 z - \frac{z^2 x}{2}\right]_0^c dx$$

$$= \left[cb^2 x - \frac{c^2}{2}\cdot\frac{x^2}{2}\right]_0^a = ab^2c - \frac{a^2c^2}{4}.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \frac{a^2b^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4} + \frac{a^2c^2}{4} + ab^2c - \frac{a^2c^2}{4}$$

$$= abc^2 + a^2bc + ab^2c$$

$$= abc(a+b+c).$$

LHS = RHS.

Hence the theorem is verified.

3) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = 2xy\vec{i} + yz^2\vec{j} + xz\vec{k}$ and S is the surface of the parallelopiped bounded by $x=0$, $y=0$, $z=0$, $x=2$, $y=1$, $z=3$.

Soln: By Divergence theorem.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(xz)$$

$$= 2y + z^2 + x.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{3} (2y + z^2 + x)\,dx\,dy\,dz$$

$$= \int_0^2\int_0^1 \left(6y + \frac{27}{3} + 3x\right) dx\, dy$$

$$= \int_0^2 \left(3y^2 + 9y + 3xy\right)_0^1 dx$$

$$= \int_0^2 (3 + 9 + 3x)\, dx = \left[12x + \frac{3x^2}{2}\right]_0^2 = 12(2) + \frac{3(2)^2}{2}$$

$$= 24 + 6 = 30$$

H/W.

1) verify divergence theorem for $\vec{f} = x^2\vec{i} + 2y^2\vec{j} + 3z^2\vec{k}$ taken over the cube bounded by the planes $x = 0, 1$; $y = 0, 1$; $z = 0, 1$.

2) $\vec{f} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$; $x = 0, 1$; $y = 0, 1$; $z = 0, 1$ ~~&~~.

3) $\vec{f} = (x^3 - yz)\vec{i} - 2x^2y\vec{j} + 2\vec{k}$, $x = 0$, $y = 0$, $z = 0$, $x = 1$, $y = 1$, $z = 1$.

STOKE'S THEOREM : Q.B (10)

The surface integral of normal component of curl of a vector $\vec{F}$ over a open surface 'S' is equal to tangential component of $\vec{f}$ over the curve C enclosing the surface 'S' ie $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = \int_C \vec{F}\cdot d\vec{r}$

1) Verify stoke's theorem for $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ for the rectangular region by $x = 0$, $x = 2$, $y = 0$, $y = 2$ on the xy plane.

Soln: stoke's theorem

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = \int_C \vec{F}\cdot d\vec{r}$$

LHS

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(2y+2y) = 4y\vec{k}.$$

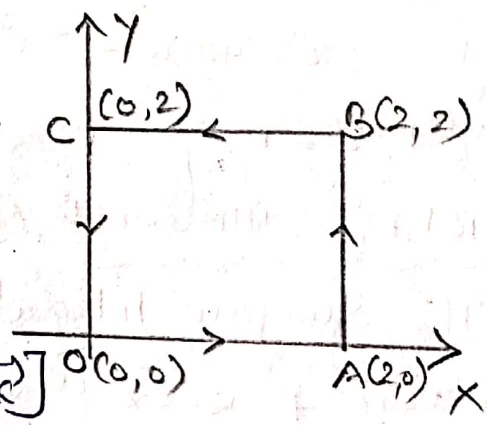
[on the xy plane we know $z=0$ & $\vec{k}$ be the unit normal vector to the xy plane]

$$(\nabla\times\vec{F})\cdot\hat{n} = 4y\vec{k}\cdot\vec{k} = 4y \quad [ds = dx\,dy]$$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \int_0^2\int_0^2 4y\,dx\,dy = 4\int_0^2 \left[\frac{y^2}{2}\right]_0^2 dx$$

$$= 4\left[2x\right]_0^2 = 16.$$

RHS $\int_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$



Along $\vec{OA}$: $y=0$, $x = 0$ to 2.

$$\vec{F}\cdot d\vec{r} = \left[(x^2-y^2)\vec{i} + 2xy\vec{j}\right]\cdot\left[dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}\right]$$

$$= (x^2-y^2)dx + 2xy\,dy.$$

$$= x^2dx \quad [\because y=0].$$

$$\therefore \int_{OA}\vec{F}\cdot d\vec{r} = \int_0^2 x^2dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}.$$

Along $\vec{AB}$: $y = 0$ to 2, $x=2$, $dx=0$.

$$\vec{F}\cdot d\vec{r} = 4y\,dy$$

$$\therefore \int_{AB}\vec{F}\cdot d\vec{r} = \int_0^2 4y\,dy = \left[\frac{4y^2}{2}\right]_0^2 = 4\cdot\frac{2^2}{2} = 8$$

Along $\vec{BC}$: $y = 2,\ dy = 0,\ x = 2$ to 0.

$\vec{F} \cdot d\vec{r} = (x^2 - 4)\,dx$.

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_2^0 (x^2 - 4)\,dx = \left[\frac{x^3}{3} - 4x\right]_2^0 = \left[0 - \left(\frac{8}{3} - 8\right)\right] = \frac{16}{3}$$

Along $\vec{CO}$: $x = 0,\ dx = 0,\ y = 2$ to 0

$\vec{F} \cdot d\vec{r} = 0$

$$\therefore \int_{CO} \vec{F} \cdot d\vec{r} = 0.$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \frac{8}{3} + 8 + \frac{16}{3} + 0 = \frac{24}{3} + 8 = 16.$$

Hence the theorem is verified.

2) Verify stoke's theorem for $\vec{F} = (y-z)\vec{i} + yz\vec{j} - xz\vec{k}$ where S is the surface of the cube bounded by the plane $x = 0, y = 0, z = 0$ and $x = 1, y = 1, z = 1$ above xy plane.

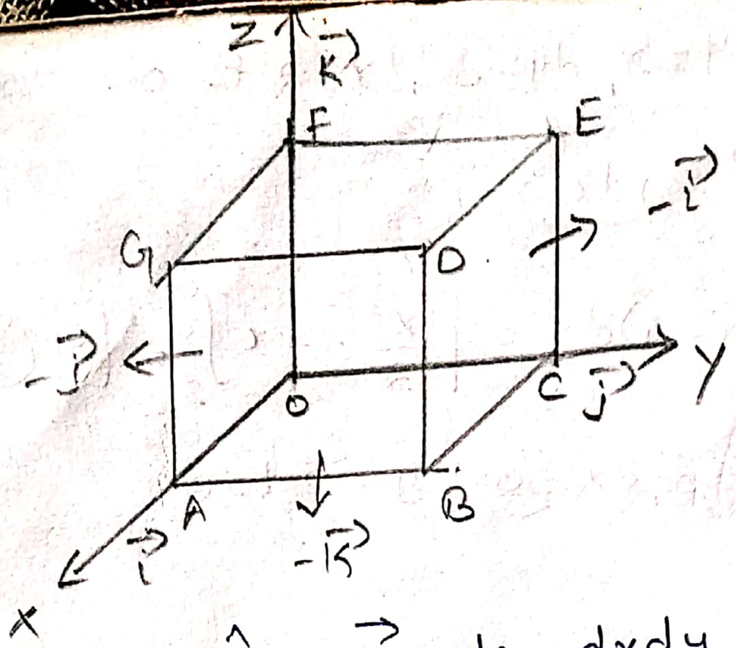
Soln: LHS: $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & yz & -xz \end{vmatrix}$

$= \vec{i}\,(0 - y) - \vec{j}\,(-z + 1) + \vec{k}\,(0 - 1)$

$= -y\vec{i} + (z-1)\vec{j} - \vec{k}$.

~~RHS~~

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_{GDEF} + \iint_{OCEF} + \iint_{GABD} + \iint_{OAGF} +$$



Along GDEF, $\hat{n} = +\vec{k}$, $ds = dx\,dy$.

$\therefore (\nabla \times \vec{F}) \cdot \hat{n} = -1$

$$\therefore \iint_{GDEF} (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \int_0^1\int_0^1 (-1)\,dx\,dy = -1.$$

Along OCEF, $\hat{n} = -\vec{i}$, $ds = dy\,dz$

$(\nabla \times \vec{F}) \cdot \hat{n} = y$.

$$\therefore \iint_{OCEF} (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \int_0^1\int_0^1 y\,dy\,dz = \left[\frac{y^2}{2}\right]_0^1 [z]_0^1 = \frac{1}{2}.$$

Along GABD, $\hat{n} = \vec{i}$, $ds = dy\,dz$.

$(\nabla \times \vec{F}) \cdot \hat{n} = -y$.

$$\therefore \iint_{GABD} (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \int_0^1\int_0^1 -y\,dy\,dz = -\frac{1}{2}.$$

Along OAGF, $\hat{n} = -\vec{j}$, $ds = dx\,dz$.

$(\nabla \times \vec{F}) \cdot \hat{n} = -(z-1)$.

$$\therefore \iint_{OAGF} (\nabla \times \vec{F}) \cdot \hat{n}\, ds = -\int_0^1\int_0^1 (z-1)\,dx\,dz = -\int_0^1 \left(\frac{z^2}{2} - z\right)dx$$

$$= \frac{1}{2}[x]_0^1 = \frac{1}{2}$$

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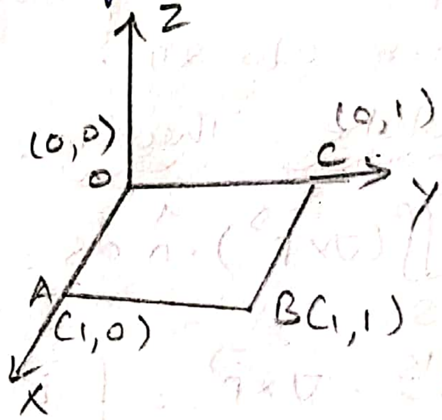
Along BDEC , $\hat{n} = \vec{j}$, $ds = dx\,dz$.

$(\nabla \times \vec{F}) \cdot \hat{n} = (z-1)$.

$\therefore \iint\limits_{BDEC} \nabla \times \vec{F} \cdot \hat{n} = \int_0^1 \int_0^1 (z-1)\, dx\, dz = -\frac{1}{2}$.

$\therefore \iint\limits_{S} (\nabla \times \vec{F}) \cdot \hat{n}\, ds = -1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -1$.

RHS : The boundary of C is the square in the xy- plane , $\therefore z = 0$.



$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

(i) Along $\vec{OA}$, $y = 0$, $dy = 0$, $x = 0$ to 1

$\vec{F} \cdot d\vec{r} = \left[(y-z)\vec{i} + yz\vec{j} - xz\vec{k} \right] \cdot \left[dx\vec{i} + dy\vec{j} + dz\vec{k} \right]$

$= (y-z)\,dx + yz\,dy - xz\,dz$.

$\Rightarrow \vec{F} \cdot d\vec{r} = 0 \quad (\because z = 0)$

$\therefore \int_{OA} \vec{F} \cdot d\vec{r} = 0$.

(ii) Along $\vec{AB}$, $x = 1$, $dx = 0$, $y = 0$ to 1.

$\vec{F} \cdot d\vec{r} = 0 \quad \therefore \int_{AB} \vec{F} \cdot d\vec{r} = 0$.

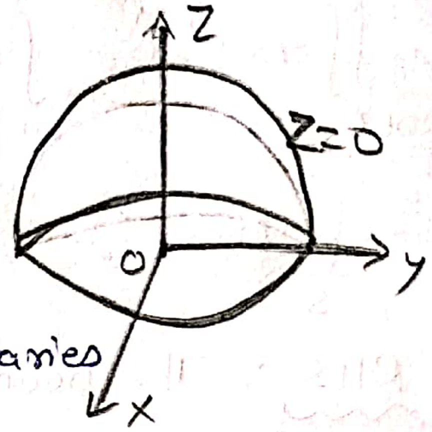
(iii) Along $\vec{BC}$ $y = 1$, $dy = 0$, $x = 1$ to 0.

$\vec{F} \cdot d\vec{r} = dx \quad \therefore \int_{BC} \vec{F} \cdot d\vec{r} = \int_1^0 dx = [x]_1^0 = -1$

(iv) Along $\vec{CO}$, $x = 0$, $dx = 0$, $y = 1$ to 0 $\vec{F} \cdot d\vec{r} = 0 \therefore \int_C \vec{F} \cdot d\vec{r} = 0$

$\Rightarrow$ LHS = RHS .

3) Verify Stoke's theorem for $\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is the circular boundary on $z=0$ plane.



Soln: The boundary curve C is the circle $x^2+y^2+z^2=1$, $z=0 \Rightarrow x^2+y^2=1$

The parametric Equations of the sphere is $x = \cos\theta$, $y = \sin\theta$, θ varies from 0 to 2π.

Stoke's Theorem:

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = \int_C \vec{F}\cdot d\vec{r}.$$

LHS: $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$

$$= \vec{i}\,[-2yz-(-2yz)] - \vec{j}\,[0-0] + \vec{k}\,[0-(-1)]$$

$$= \vec{i}\,(0) + \vec{k} = \vec{k}.$$

$$(\nabla\times\vec{F})\cdot\hat{n} = \vec{k}\cdot\vec{k} = 1.$$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = \iint_S dx\, dy = \pi(1)^2 = \pi.$$

RHS: $\vec{F}\cdot d\vec{r} = (2x-y)\,dx - yz^2\,dy - y^2z\,dz.$

$x = \cos\theta$, $y = \sin\theta$, $z = 0$.

$dx = -\sin\theta$, $dy = \cos\theta$, $dz = 0$.

$$\vec{F}\cdot d\vec{r} = [2\cos\theta - \sin\theta][-\sin\theta]\,d\theta.$$

$$= [-2\sin\theta\cos\theta + \sin^2\theta]\,d\theta$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (\sin^2\theta - 2\sin\theta\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} \left[\left(\frac{1-\cos 2\theta}{2}\right) - \sin 2\theta\right] d\theta$$

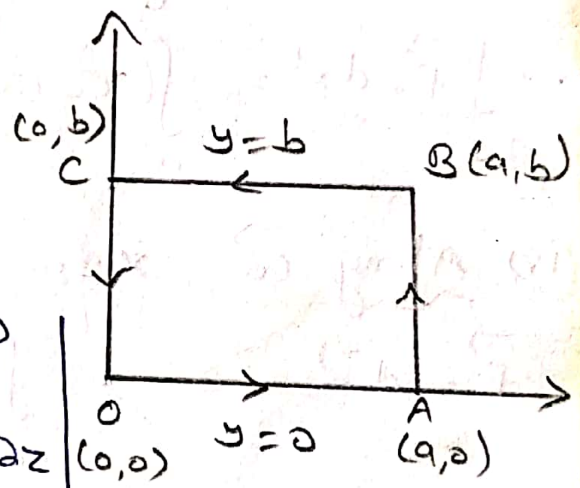
$$= \left[\frac{1}{2}\theta - \frac{\sin 2\theta}{4} + \frac{\cos 2\theta}{2}\right]_0^{2\pi} = \pi.$$

Hence the theorem is verified.

H/W. 4) Verify stoke's theorem for $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$ in the Rectangular region in the $z=0$ plane whose sides are along the lines $x=0$, $x=a$, $y=0$ and $y=b$.

Soln: Stoke's theorem is

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = \int_C \vec{F}\cdot d\vec{r}.$$



$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(2y+2y) = 4y\vec{k}.$$

$$(\nabla\times\vec{F})\cdot\hat{n} = 4y\vec{k}\cdot\vec{k} = 4y.$$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = 4\int_0^b\int_0^a y\, dx\, dy = 4[x]_0^a \left[\frac{y^2}{2}\right]_0^b$$

$$= \frac{4ab^2}{2} = 2ab^2.$$

Now $\int_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$.

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(i) Along $\vec{OA}$; $y=0,\ dy=0,\ x=0$ to a.

$$\vec{F}\cdot d\vec{r} = x^2\,dx \quad \therefore \int_{OA}\vec{F}\cdot d\vec{r} = \int_0^a x^2\,dx = \frac{a^3}{3}.$$

(ii) Along $\vec{AB}$, $x=a,\ dx=0,\ y=0$ to b.

$$\vec{F}\cdot d\vec{r} = 2xy\,dy = 2ay\,dy.$$

$$\therefore \int_{AB}\vec{F}\cdot d\vec{r} = \int_0^b 2ay\,dy = 2a\left[\frac{y^2}{2}\right]_0^b = \frac{2ab^2}{2} = ab^2.$$

(iii) Along $\vec{BC}$, $y=b,\ dy=0,\ x=a$ to 0.

$$\vec{F}\cdot d\vec{r} = (x^2-y^2)\,dx = (x^2-b^2)\,dx.$$

$$\therefore \int_{BC}\vec{F}\cdot d\vec{r} = \int_a^0 (x^2-b^2)\,dx = \left[\frac{x^3}{3}-b^2x\right]_a^0 = -\frac{a^3}{3}+ab^2.$$

(iv) Along $\vec{CO}$, $x=0,\ dx=0,\ y=0$ to b.

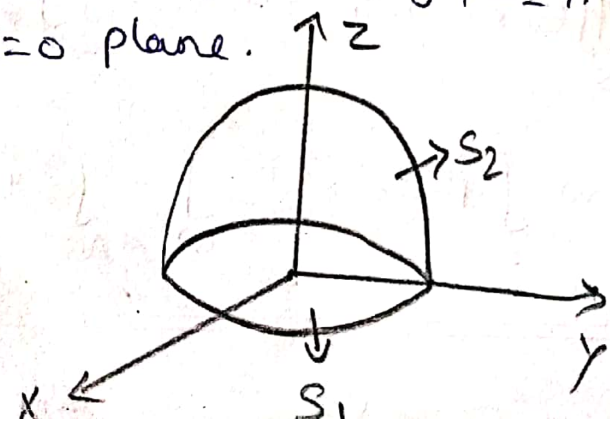
$$\vec{F}\cdot d\vec{r} = 0 \quad \therefore \int_{CO}\vec{F}\cdot d\vec{r} = 0.$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \cancel{\frac{a^3}{3}} + ab^2 - \cancel{\frac{a^3}{3}} + ab^2 = 2ab^2.$$

Hence the theorem is verified.

H/W.

5) Verify Stoke's theorem for $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$. and c is its boundary on $z=0$ plane.



Soln: Stoke's theorem is

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds.$$

But by Divergence theorem.

$$\iint_S (\nabla\times\vec{f})\cdot\hat{n}\, ds = \iiint_V \nabla\cdot(\nabla\times\vec{f})\,dv = 0$$
$$[\because \nabla\cdot(\nabla\times\vec{f}) = 0].$$

$$\therefore \iint_{S_1} (\nabla\times\vec{f})\cdot\hat{n}\, ds_1 + \iint_{S_2} (\nabla\times\vec{f})\cdot\hat{n}\, ds_2 = 0.$$

$$\Rightarrow \iint_{S_1} (\nabla\times\vec{f})\cdot\hat{n}\, ds_1 = -\iint_{S_2} (\nabla\times\vec{f})\cdot\hat{n}\, ds_2 \quad -(1)$$

S_1 is circle in the xy plane $(z=0)$.

$\hat{n} = -\vec{k}$.

$$\nabla\times\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = \vec{i}(-1) - \vec{j}(1) + \vec{k}(-1)$$
$$= -(\vec{i}+\vec{j}+\vec{k}).$$

$$(\nabla\times\vec{f})\cdot\hat{n} = -(\vec{i}+\vec{j}+\vec{k})\cdot(-\vec{k}) = 1.$$

$$\iint_{S_1} (\nabla\times\vec{f})\cdot\hat{n}\, ds_1 = \iint_{S_1} dx\,dy = \pi(1)^2 = \pi \quad [S_1 : \text{area of the } \odot]$$

$$\therefore \iint_{S_2} (\nabla\times\vec{f})\cdot\hat{n}\, ds_2 = -\pi \quad [\because \text{from } (1)].$$

$\therefore$ By Stoke's theorem $\iint_S (\nabla\times\vec{f})\cdot\hat{n}\, ds = \int_C \vec{f}\cdot d\vec{r}$.

where C is the circle in the xy plane $x=\cos\theta,\ y=\sin\theta$; $dx=-\sin\theta,\ dy=\cos\theta$

$$\vec{f}\cdot d\vec{r} = (y\vec{i}+z\vec{j}+x\vec{k})\cdot(dx\vec{i}+dy\vec{j}+dz\vec{k}) = y\,dx + z\,dy + x\,dz.$$
$$= y\,dx \quad [\because \text{In } xy \text{ plane } z=0].$$

$$\therefore \int_C \vec{f}\cdot d\vec{r} = \int_C y\,dx = \int_0^{2\pi} \sin\theta(-\sin\theta)\,d\theta = -\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= -\left[\frac{1}{2}\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = -\left[\frac{1}{2}\cdot 2\pi\right] = -\pi.$$

HW

6) Evaluate $\int \vec{F}\cdot d\vec{r}$ by Stoke's theorem where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$ and c is the boundary of the $\Delta(0,0,0)$, $(1,0,0)$ & $(1,1,0)$

VECTOR CALCULUS

In Vector Algebra we mostly deal with constant vectors namely, vectors which are constant in Magnitude and fixed in direction.

In Vector Calculus we deal with variable vectors i.e vectors which are varying in magnitude or direction or both.

Fundamental Results :

1) If $\vec{P} = P_1\vec{i} + P_2\vec{j} + P_3\vec{k}$ then the magnitude of a vector $\vec{P}$, $P = |\vec{P}| = \sqrt{P_1^2 + P_2^2 + P_3^2}$ unit vector in the direction of $\vec{P}$ is $\dfrac{\vec{P}}{|\vec{P}|}$

2) $\vec{P}\cdot\vec{Q} = PQ\cos\theta$ defines the dot or scalar product of the vectors $\vec{P}$ and $\vec{Q}$, θ is the angle between them

(i) $\vec{P}\cdot\vec{Q} = 0$, if $\vec{P}$ & $\vec{Q}$ are $\perp$er.

(ii) $\vec{P}\cdot\vec{P} = P^2$

(iii) $\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$.

iv) $\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{i} = 0$

v) $\vec{P}\cdot\vec{Q} = \vec{Q}\cdot\vec{P}$.

3) $\vec{P}\times\vec{Q} = PQ\sin\theta\,\hat{n}$ defines the cross or vector product of the vectors $\vec{P}$ and $\vec{Q}$ where θ is the angle and $\hat{n}$ is a unit vector in the direction of perpendicular to the plane of $\vec{P}$ and $\vec{Q}$.

(i) $\vec{P} \times \vec{Q} = -\vec{Q} \times \vec{P}$

(ii) $\vec{P} \times \vec{P} = +\vec{0}$

(iii) $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{K} \times \vec{K} = 0$

(iv) $\vec{i} \times \vec{j} = \vec{K}$, $\vec{j} \times \vec{K} = \vec{i}$, $\vec{K} \times \vec{i} = \vec{j}$.

(v) $\vec{P} \times \vec{Q} = \vec{0}$ if $\vec{P}$, $\vec{Q}$ are parallel.

Differential operators. ∇ (del)

$\nabla = \vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{K}\dfrac{\partial}{\partial z}$ is a vector differential operator possessing properties similar to those of vectors. [It can act on a scalar (or) a vector function.]

Gradient

If $\phi(x, y, z)$ is a scalar point function continuously differentiable in a given region of space then the gradient of ϕ is defined by

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{K}\frac{\partial}{\partial z}\right)\phi$$

$$= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{K}\frac{\partial\phi}{\partial z} \quad [\text{grad}\,\phi = \nabla\phi]$$

Note:

(i) $\nabla\phi$ defines a vector field, whereas ϕ is a scalar field.

(ii) If ϕ is a constant, then $\nabla\phi = 0$.

Directional Derivatives:

The directional derivatives of a scalar point function $\phi(x, y, z)$ in a given direction is the rate of change of $\phi(x, y, z)$ in that direction. It is given by the component of $\nabla\phi$ in that direction.

If $\hat{a}$ is the unit vector in the given direction then the directional derivative of ϕ is given by $\nabla\phi \cdot \hat{a}$.

Maximum value of directional derivative of ϕ is $|\nabla\phi|$. Unit normal vector to the surface ϕ is $\frac{\nabla\phi}{|\nabla\phi|}$

Divergence

If $\vec{f}(x, y, z)$, $\vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ is a vector point function continuously differential in a given region of a space, then the divergence of $\vec{F}$ is defined by

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(f_1\vec{i} + f_2\vec{j} + f_3\vec{k}\right)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$ which is a scalar

~~Divergence~~ Divergence of $\vec{F}$ (or) $\nabla \cdot \vec{F}$ (or) del dot f

Note: (i) $\nabla \cdot \vec{F} \neq \vec{F} \cdot \nabla$ $\because$ $\nabla \cdot \vec{F}$ is a scalar by $\vec{f} \cdot \nabla$ is only an operator.

(ii) $\nabla \cdot \vec{f} = 0$, if $\vec{f}$ is a constant vector and conversely.

SOLENOIDAL VECTOR

A vector $\vec{f}$ is called Solenoidal if $\text{div}\vec{f} = 0$

ie $\nabla \cdot \vec{f} = 0$

CURL

If $\vec{f}(x, y, z)$ is a differentiable vector field then the curl of $\vec{f}$ is defined by

$$\nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} \quad \text{where } \vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

$$= \vec{i}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] - \vec{j}\left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right] + \vec{k}\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$$

curl of $\vec{f}$ or curl $\vec{f}$ or $\nabla \times \vec{f}$ or rotation of $\vec{f}$.

IRROTATIONAL VECTOR

A vector $\vec{f}$ is called irrotational, if curl $\vec{f} = 0$

ie $\nabla \times \vec{f} = \vec{0}$.

Problems:

1) If $\phi(x, y, z) = 3x^2y - y^3z^2$. Find grad ϕ at the point $(1, -2, -1)$.

AB1'

Soln:

$$\text{grad}\,\phi = \nabla\phi = \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right]\left[3x^2y - y^3z^2\right]$$

5

$= \vec{i}(6xy) + \vec{j}(3x^2 - 3y^2z^2) + \vec{k}(-2y^3z)$

$\therefore \nabla\phi_{(1,-2,-1)} = \vec{i}(-12) + \vec{j}(3-12) + \vec{k}(-16)$

$= -12\vec{i} - 9\vec{j} - 16\vec{k}$.

Q.B2.)

2) Show that the vector $\vec{F} = 5y^4z^3\vec{i} + 8xz^2\vec{j} - y^2x\vec{k}$ is Solenoidal.

Soln:

$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(5y^4z^3\vec{i} + 8xz^2\vec{j} - y^2x\vec{k}\right)$

$= \frac{\partial}{\partial x}(5y^4z^3) + \frac{\partial}{\partial y}(8xz^2) + \frac{\partial}{\partial z}(-y^2x)$

$= 0 + 0 + 0 = 0$.

$\Rightarrow \nabla\cdot\vec{f} = 0$.

$\therefore$ The vector $\vec{f} = 5y^4z^3\vec{i} + 8xz^2\vec{j} - y^2x\vec{k}$ is Solenoidal.

3) Find a, $\vec{f} = (3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is Solenoidal.

Q.B4

Soln. $\nabla\cdot\vec{f} = 0$

$\Rightarrow \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0$.

$\Rightarrow 3 + a + 2 = 0 \Rightarrow \boxed{a = -5}$

4) Find the directional derivative of $\phi = xyz - xy^2z^3$ at $(1, 2, -1)$ in the direction $\vec{i} - \vec{j} - 3\vec{k}$. Find also its maximum value.

Soln: $\phi = xyz - xy^2z^3$.

$$\nabla\phi = \vec{i}\frac{\partial}{\partial x}(xyz - xy^2z^3) + \vec{j}\frac{\partial}{\partial y}(xyz - xy^2z^3) + \vec{k}\frac{\partial}{\partial z}(xyz - xy^2z^3).$$

$$= (yz - y^2z^3)\vec{i} + (xz - 2xz^3)\vec{j} + (xy - 3xy^2z^2)\vec{k}.$$

$\nabla\phi$ at $(1,2,-1) = 2\vec{i} + 3\vec{j} - 10\vec{k}$

unit vector in the direction $\vec{i} - \vec{j} - 3\vec{k}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i} - \vec{j} - 3\vec{k}}{\sqrt{1+1+9}} = \frac{\vec{i} - \vec{j} - 3\vec{k}}{\sqrt{11}}$$

Directional Derivative of ϕ in the given ~~equation~~ direction. is

$$\nabla\phi \cdot \hat{a} = (2\vec{i} + 3\vec{j} - 10\vec{k}) \cdot \frac{\vec{i} - \vec{j} - 3\vec{k}}{\sqrt{11}}$$

$$= \frac{2 - 3 + 30}{\sqrt{11}} = \frac{29}{\sqrt{11}}$$

Maximum value of directional derivative of ϕ is $|\nabla\phi|$

ie $|\nabla\phi| = |2\vec{i} + 3\vec{j} - 10\vec{k}| = \sqrt{4+9+100} = \sqrt{113}$.

A, B(3)

5) Show that the vector $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrotational

Soln:
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \vec{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

7

$= \vec{i}\,[x - x] - \vec{j}\,[y-y] + \vec{k}\,[z-z] = \vec{0}.$

Hence $\nabla \times \vec{F} = 0$ is irrotational.

QB5. QB. 12 (i).

6) Find the value of a, if the vector $\vec{F} = (axy - z^3)\vec{i} + (a-2)x^2\vec{j} + (1-a)xz^2\vec{k}$ is irrotational

Soln. ~~Find the value~~ $\vec{F}$ is Irrotational $\Rightarrow \nabla \times \vec{f} = \vec{0}$.

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = \vec{0}.$$

$\Rightarrow \vec{i}(0) - \vec{j}\left[(1-a)z^2 + 3z^2\right] + \vec{k}\left[2x(a-2) - ax\right] = \vec{0}.$

$\Rightarrow -\vec{j}\left[(1-a)z^2 + 3z^2\right] + \vec{k}\left[2ax - 4x - ax\right] = \vec{0}.$

$\Rightarrow -\vec{j}\left[z^2 - az^2 + 3z^2\right] + \vec{k}\left[ax - 4x\right] = 0\vec{i} + 0\vec{j} + 0\vec{k}.$

Equating the corresponding coefficients

$\Rightarrow ax - 4x = 0 \Rightarrow ax = 4x \Rightarrow \boxed{a = 4}$

QB6

7) If $\nabla\phi = yz\vec{i} + zx\vec{j} + xy\vec{k}$, find ϕ

Soln: $\nabla\phi = yz\vec{i} + zx\vec{j} + xy\vec{k}.$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = yz\vec{i} + zx\vec{j} + xy\vec{k}.$$

Equating the corresponding coefficients we get

$$\frac{\partial\phi}{\partial x} = yz;\quad \frac{\partial\phi}{\partial y} = zx,\quad \frac{\partial\phi}{\partial z} = xy.$$

Integrating wrt x, y, z resply, we get

$\phi = xyz + c, \quad \phi = xyz + c, \quad \phi = xyz + c.$

Hence the possible form of ϕ is $xyz + c$

8) prove that curl (curl $\vec{f}$) $= \nabla(\nabla \cdot \vec{f}) - \nabla^2 \vec{f}$.

soln.

$$\text{curl } \vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] - \vec{j}\left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right] + \vec{k}\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$$

$$\text{curl }(\text{curl } \vec{f}) = \nabla \times (\nabla \times \vec{f}).$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} & \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} & \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{vmatrix}$$

$$= \Sigma \vec{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right]$$

$$= \Sigma \vec{i}\left[\frac{\partial^2 f_2}{\partial x \partial y} - \frac{\partial^2 f_1}{\partial y^2} - \frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_3}{\partial z \partial x}\right]$$

$$= \Sigma \vec{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) - \left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)\right]$$

$$= \Sigma \vec{i}\left[\frac{\partial}{\partial x}(\nabla \cdot \vec{f}) - \nabla^2 f_1\right]$$

$$= \nabla(\nabla \cdot \vec{f}) - \nabla^2 \vec{f}.$$

$$\text{curl }(\text{curl } \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}.$$

H/W.

1.) Prove that $\vec{f} = 3y^4z^2\vec{i} + 4x^4z^2\vec{j} - 3x^2y^2\vec{k}$ is solenoidal

2) Show that $\vec{f} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ is irrotational

3) Find the value of a if $(x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}$ is solenoidal.

4) Find the scalar point function ϕ whose gradient is $yz^2\vec{i} + (xz^2 - 1)\vec{j} + 2(xyz - 1)\vec{k}$.

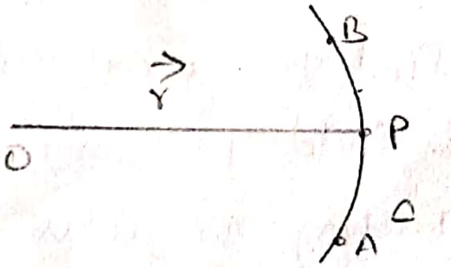
5) Find the directional derivative of $\phi = xy + yz + zx$ at $(1, 2, 0)$ in the direction $\vec{i} + 2\vec{j} + 2\vec{k}$. Find also its maximum value.

Line Integral.

An integral which is evaluated along a curve is called a line integral.

Let C be the given curve let A and B be two points on the curve. Then the line integral from A to B is given by

$$\int_C \vec{f} \cdot d\vec{r} = \int_A^B \vec{f} \cdot d\vec{r}$$



$$\vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}.$$

① Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = x^2y^2\vec{i} + y\vec{j}$ and C is the curve $y^2 = 4x$ in the xy plane from $(0,0)$ to $(4,4)$

QB 12 (ii).

<u>soln:</u> $\vec{F} = x^2y^2\vec{i} + y\vec{j}$.

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}.$$

$$\vec{F}\cdot d\vec{r} = x^2y^2\,dx + y\,dy.$$

Given $y^2 = 4x$.

$$\therefore \vec{F}\cdot d\vec{r} = 4x^3\,dx + y\,dy.$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{(0,0)}^{(4,4)} 4x^3\,dx + y\,dy$$

$$= \int_0^4 4x^3\,dx + \int_0^4 y\,dy$$

$$= 4\left[\frac{x^4}{4}\right]_0^4 + \left[\frac{y^2}{2}\right]_0^4$$

$$= 256 + \frac{16}{2} = 256 + 8 = 264$$

<u>QB 7.</u>

$\phi = xyz$

$\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}.$$

$$= yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}.$$

$$\nabla\phi_{(1,1,1)} = \vec{i} + \vec{j} + \vec{k}.$$

$\therefore$ Directional derivative is $\nabla\phi\cdot\hat{a}$

$$= \vec{i} + \vec{j} + \vec{k} \cdot \frac{(\vec{i}+\vec{j}+\vec{k})}{\sqrt{3}}$$

$$= \frac{1+1+1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

2) Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} - z\vec{k}$ from $t=0$ to $t=1$ along the curve $x = 2t^2$, $y = t$, $z = 4t^3$.

<u>soln:</u> Workdone $= \int_C \vec{F}\cdot d\vec{r}$.

$$\vec{F}\cdot d\vec{r} = 3x^2\,dx + (2xz - y)\,dy - z\,dz.$$

$$x = 2t^2,\quad y = t,\quad z = 4t^3.$$

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$dx = 4t\,dt, \quad dy = dt, \quad dz = 12t^2\,dt.$

$$\vec{F}\cdot d\vec{r} = 3(2t^2)^2 4t\,dt + [2(2t^2)(4t^3) - t]dt - 4t^3(12t^2)dt$$

$$= \cancel{48t^5\,dt} + 16t^5\,dt - \cancel{48t^5\,dt} - t\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (16t^5 - t)dt$$

$$= \left[16\frac{t^6}{6} - \frac{t^2}{2}\right]_0^1 = \frac{16}{6} - \frac{1}{2} = \frac{16-3}{6} = \frac{13}{6}.$$

workdone $= \int_C \vec{F}\cdot d\vec{r} = \frac{13}{6}$ units.

H/W.

3) Find the total WD in moving a particle in a force field given by $\vec{F} = 3xy\vec{i} - 5z\vec{j} + 10x\vec{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$. from $t=1$ to $t=2$.

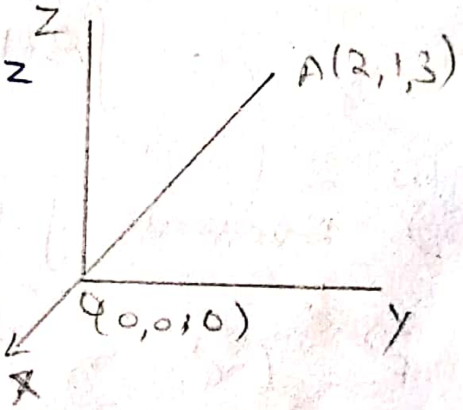
Soln: 303 units.

4.) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$.

along the straight line c from $(0,0,0)$ to $(2,1,3)$.

Soln:

$$\vec{F}\cdot d\vec{r} = 3x^2dx + (2xz - y)dy + zdz$$



Consider the Integral $\int_{C=OA} \vec{F}\cdot d\vec{r}$.

Along OA,

$x = 0$ to $x = 2$.

$y = 0$ to $y = 1$

$z = 0$ to $z = 3$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}$$

$\Rightarrow x = 2t,\ y = t,\ z = 3t.$

$\Rightarrow dx = 2dt,\ dy = dt,\ dz = 3dt.$

$t=0$ and $t=1$ corresponds to the points $(0,0,0)$ and $(2,1,3)$ on the path.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C 3x^2dx + (2xz-y)dy + zdz.$$

$$= \int_0^1 3(2t)^2 2dt + [2(2t)(3t) - t]dt + (3t)3dt.$$

$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)\,dt.$$

$$= \int_0^1 36t^2 + 8t\,dt = \left[36\frac{t^3}{3} + 8\frac{t^2}{2}\right]_0^1 = 12+4 = 16.$$

Q.B 8)

Unit normal vector to the Surface $\phi = \frac{\nabla\phi}{|\nabla\phi|}$

$\phi = x^2 - y^2 + z = 2$

$\nabla\phi = 2x\vec{i} - 2y\vec{j} + \vec{k}.$

$\nabla\phi(1,-1,2) = 2\vec{i} + 2\vec{j} + \vec{k}.$

$|\nabla\phi| = \sqrt{4+4+1} = 3.$

$\therefore$ unit normal vector $= \frac{2\vec{i}+2\vec{j}+\vec{k}}{3}.$

H/w.

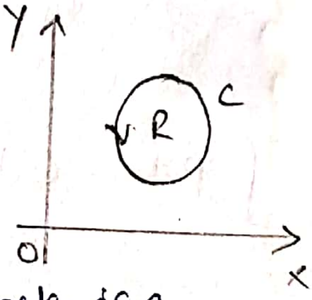
5) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where

$\vec{F} = (x^2+y^3)\vec{i} + (x^3-y^2)\vec{j}$ along the straight line joining $(1,0)$ & $(0,1)$.

Soln: ~~7/4~~. $-2/3$.

GREENS THEOREM IN THE PLANE.

If $P(x,y)$ and $Q(x,y)$ are continuous function with continuous partial derivatives in a region R of the plane and on its boundary c which is a simple closed curve, then



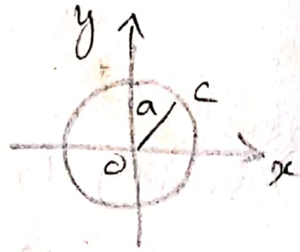
$$\oint_c (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where 'C' is described in the anticlockwise direction.

Note: $\iint_R dx\,dy =$ Area of the region R.

1) Using Green's theorem, Evaluate $\int_C (2x-y)dx + (x+y)dy$ where C is the boundary of the circle $x^2+y^2=a^2$.

Soln: Here $P = 2x-y$, $Q = x+y$.



$$\frac{\partial P}{\partial y} = -1, \quad \frac{\partial Q}{\partial x} = 1$$

$$\therefore \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\Rightarrow \int_c (2x-y)dx + (x+y)dy = \iint_R (1+1)\,dx\,dy$$

$$= 2\iint_R dx\,dy$$

$= 2 \times$ Area of the region R.

$= 2\pi a^2$.

H/W

2) Evaluate $\int_C y(2xy-1)\,dx + x(2xy+1)\,dy$ where C is the circle $x^2+y^2=4$ using Green's Theorem.

<u>soln</u>: Here $P = 2xy^2 - y \qquad Q = 2x^2y + x$

$$\frac{\partial P}{\partial y} = 4xy - 1 \qquad \frac{\partial Q}{\partial x} = 4xy + 1.$$

$$\int_C P\,dx + Q\,dy = \iint_R (4xy+1-4xy+1)\,dx\,dy$$

$$= \iint_R 2\,dx\,dy.$$

$= 2 \times$ Area of the region R.

$$= 2 \times \pi (2)^2$$

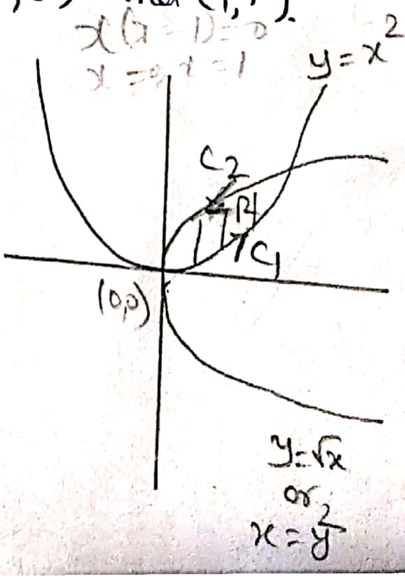
Q.B (14)

$$= 8\pi.$$

3) Verify Green's theorem in the plane for $\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where C is the boundary of the region defined by $y=\sqrt{x}$ and $y=x^2$.

<u>soln</u>: Given: $C = C_1 + C_2$.

The point of intersection are $(0,0)$ and $(1,1)$.



Along C_1 LHS

$$y = x^2$$

$dy = 2x\,dx$, x varies from 0 to 1

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$$\int_C P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy.$$

$$\int_{C_1} P\,dx + Q\,dy = \int_0^1 (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy.$$

$$= \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)\,2x\,dx$$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx.$$

$$= \left[\frac{3x^3}{3} - 8\frac{x^5}{5} + 8\frac{x^4}{4} - 12\frac{x^5}{5}\right]_0^1$$

$$= \left[x^3 - \frac{20}{5}x^5 + \frac{8x^4}{4}\right]_0^1$$

$$= 1 - \frac{\cancel{20}^{\,4}}{\cancel{5}} + \frac{\cancel{8}^{\,2}}{\cancel{4}} = 1 - 4 + 2 = -1$$

Along C_2: $y = \sqrt{x}$

$\Rightarrow x = y^2$

$\Rightarrow dx = 2y\,dy$; y varies from 1 to 0.

$$\therefore \int_C P\,dx + Q\,dy = \int_{C_2} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy.$$

$$= \int_1^0 (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)\,dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)\,dy$$

$$= \left[6 \cdot \frac{y^6}{6} - 22\frac{y^4}{4} + 4\frac{y^2}{2} \right]_1^0$$

$$= \left[-1 + \frac{11}{2} - 2 \right] = \frac{11-6}{2} = 5/2.$$

$$\therefore \int_C P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = -1 + 5/2$$

$$= 3/2.$$

<u>RHS.</u> $P = 3x^2 - 8y^2$, $Q = 4y - 6xy$

$$\frac{\partial P}{\partial y} = -16y \qquad \frac{\partial Q}{\partial x} = -6y.$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_R (-6y + 16y)\,dx\,dy$$

$$= 10 \int_0^1 \int_{x^2}^{\sqrt{x}} y\,dx\,dy.$$

$$= 10 \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx.$$

$$= \frac{10}{2} \int_0^1 (x - x^4)\,dx.$$

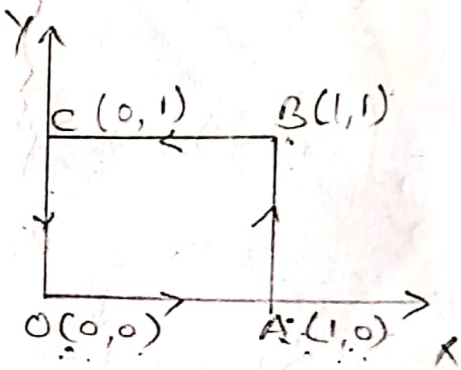
$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5\left[\frac{1}{2} - \frac{1}{5}\right]$$

$$= 5\left[\frac{3}{10}\right] = 3/2.$$

LHS = RHS.

Hence Green's theorem is verified.

4) Verify Green's theorem for the integral
Q,B (13)
$\int (x^2+y)\,dx - xy^2dy$ taken around the boundary of the square whose vertices are $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.



Soln. Here $P = x^2+y$, $Q = -xy^2$

$$\frac{\partial P}{\partial y} = 1 \quad , \quad \frac{\partial Q}{\partial x} = -y^2$$

∴ By Greens theorem

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Now RHS $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = -\int_0^1\int_0^1 (y^2+1)\,dx\,dy.$

$$= -\int_0^1 \left[y^2x + x\right]_0^1 dy = -\left[\frac{y^3}{3} + y\right]_0^1 = -\left[\frac{1}{3}+1\right]$$

$$= -4/3.$$

LHS. Line integral along $\overrightarrow{OA}$, $y=0$, $dy=0$
x varies from 0 to 1.

$$\therefore \int_{C_1} (x^2+y)\,dx - xy^2dy = \int_0^1 x^2dx = \left[\frac{x^3}{3}\right]_0^1 = 1/3.$$

Along $\overrightarrow{AB}$, $x=1$, $dx=0$, y varies from 0 to 1.

$$\therefore \int_C (x^2+y)\,dx - xy^2dy = \int_0^1 -y^2dy = \left[-\frac{y^3}{3}\right]_0^1 = -1/3.$$

Along $\overrightarrow{BC}$, $y=1$, $dy=0$, x varies from 1 to 0.

$$\therefore \int_C (x^2+y)\,dx - xy^2dy = \int_1^0 (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_1^0 = -4/3$$

Along $\overrightarrow{CO}$, $x = 0$, $dx = 0$, y varies from 1 to 0

$$\therefore \int_C (x^2+y)\,dx - xy^2\,dy = 0$$

$$\therefore \int_C (x^2+y)\,dx - xy^2\,dy = 1/3 - 1/3 - 4/3 + 0 = -4/3$$

$\therefore$ LHS = RHS.

$$\therefore \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Hence proved.

5) Show by applying Green's theorem that the area bounded by a simple closed curve C is $\frac{1}{2}\int_C (x\,dy - y\,dx)$, and hence find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$

Soln: By Green's theorem,

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Here $P = -y/2$, $Q = x/2$

$$\frac{\partial P}{\partial y} = -1/2, \quad \frac{\partial Q}{\partial x} = 1/2, \text{ ~~we get~~}$$

$$\frac{1}{2}\int_C x\,dy - y\,dx = \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dx\,dy$$

$$= \iint_R dx\,dy$$

= Area of the region R enclosed by C.

19

For the Ellipse C, let us take the parametric Equation, $x = a\cos\theta$, $y = b\sin\theta$ $(0 \le \theta \le 2\pi)$.

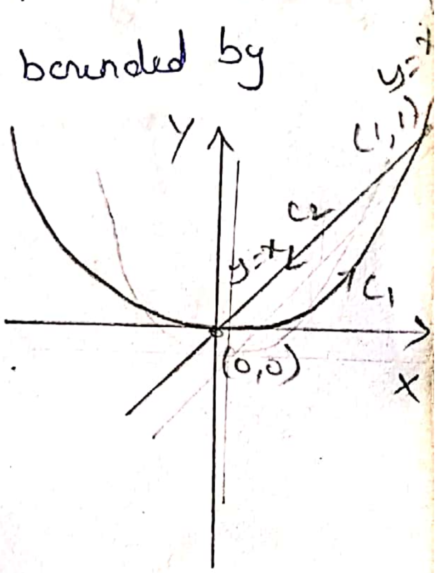
$\therefore$ Area of Ellipse $= \frac{1}{2}\int_0^{2\pi} a\cos\theta \, b\cos\theta \, d\theta - b\sin\theta(-a\sin\theta)\, d\theta$.

$$= \frac{1}{2}\int_0^{2\pi} (ab\cos^2\theta + ab\sin^2\theta)\, d\theta.$$

$$= \frac{1}{2}ab\int_0^{2\pi} d\theta = \frac{1}{2}ab\left[\theta\right]_0^{2\pi} = \pi ab.$$

H/W

6) verify Green's theorem in the plane for $\int_C (xy+y^2)\,dx + x^2\,dy$ where C is bounded by $y = x$ and $y = x^2$.



Soln: Point of intersection of $y=x$ and $y=x^2$ are at $(0,0)$ and $(1,1)$.

Along C_1: $y = x^2$, $dy = 2x\,dx$
& x varies from 0 to 1.

$$\therefore \int_{C_1} (xy+y^2)\,dx + x^2\,dy = \int_0^1 (x^3 + x^4)\,dx + x^2(2x)\,dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5}$$

$$= \frac{15+4}{20}$$

$$= \frac{19}{20}$$

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Along C_2 : $y = x$, $dy = dx$ and $x = 1$ to 0.

$$\therefore \int_{C_2} (xy + y^2)\,dx + x^2 dy = \int_1^0 (x^2 + x^2)\,dx + x^2 dx$$

$$= \left[3 \cdot \frac{x^3}{3}\right]_1^0 = -1$$

$$\therefore \int_C (xy + y^2)\,dx + x^2 dy = \frac{19}{20} - 1 = -\frac{1}{20}.$$

Consider RHS.

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dx\,dy$$

$$= \int_0^1 \int_{x^2}^{x} (2x - x - 2y)\,dx\,dy.$$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)\,dx\,dy$$

$$= \int_0^1 \left[xy - 2\frac{y^2}{2}\right]_{x^2}^{x} dx$$

$$= \int_0^1 (\cancel{x^2} - \cancel{x^2} - x^3 + x^4)\,dx.$$

$$= \left[-\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 = + \left[-\frac{1}{4} + \frac{1}{5}\right]$$

$$= -1/20.$$

$\therefore$ LHS = RHS.

Hence Green's theorem is verified.

7) H/W
Verify Greens theorem for $\int_C (x-2y)dx + xdy$ taken around the unit circle $x^2+y^2=1$.

Soln: $P = x-2y$, $Q = x$

$\frac{\partial P}{\partial y} = -2$ $\frac{\partial Q}{\partial x} = 1$

For the circle, let us take the parametric Equation

$x = \cos\theta$, $y = \sin\theta$ $(0 \leq \theta \leq \pi)$

∴ Area of the circle $= \int_0^{2\pi} (x-2y)\,dx + x\,dy$.

$= \int_0^{2\pi} (\cos\theta - 2\sin\theta)(-\sin\theta)\,d\theta + \cos\theta(\cos\theta)\,d\theta$.

$= \int_0^{2\pi} \left[-\sin\theta\cos\theta + 2\sin^2\theta + \cos^2\theta\right] d\theta$

$= \int_0^{2\pi} \left[-\sin\theta\cos\theta + \sin^2\theta + 1\right] d\theta$.

$= \int_0^{2\pi} \left[-\frac{\sin 2\theta}{2} + \left(\frac{1-\cos 2\theta}{2}\right) + 1\right] d\theta$

$= \left[\frac{\cos 2\theta}{4} + \frac{1}{2}\theta - \frac{\sin 2\theta}{4} + \theta\right]_0^{2\pi}$

$= \frac{1}{4} + \frac{1}{2}\cdot 2\pi + 2\pi - \frac{1}{4} = 3\pi$

RHS

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_R (1+2)\,dx\,dy$$

$$= 3\iint_R dx\,dy$$

$= 3 \times$ area of the circle

$$= 3\pi(1)^2$$

$$= 3\pi$$

LHS = RHS.

Hence Green's theorem is verified.

GAUSS DIVERGENCE THEOREM.

QB(9) If S is a closed surface enclosing a region of space with volume V and if $\vec{F}$ is a vector point function with continuous first order partial derivatives in V, then

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

where $\hat{n}$ is a unit normal vector to surface (S)

QB(15)

1) Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$.

over the cube $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

Soln: The Gauss Divergence theorem is

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

RHS.

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right).$$

$$= \frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz).$$

$$= 4z - 2y + y = 4z - y.$$

$$\iiint_V \nabla\cdot\vec{F}\,dV = \int_0^1\int_0^1\int_0^1 (4z - y)\,dx\,dy\,dz.$$

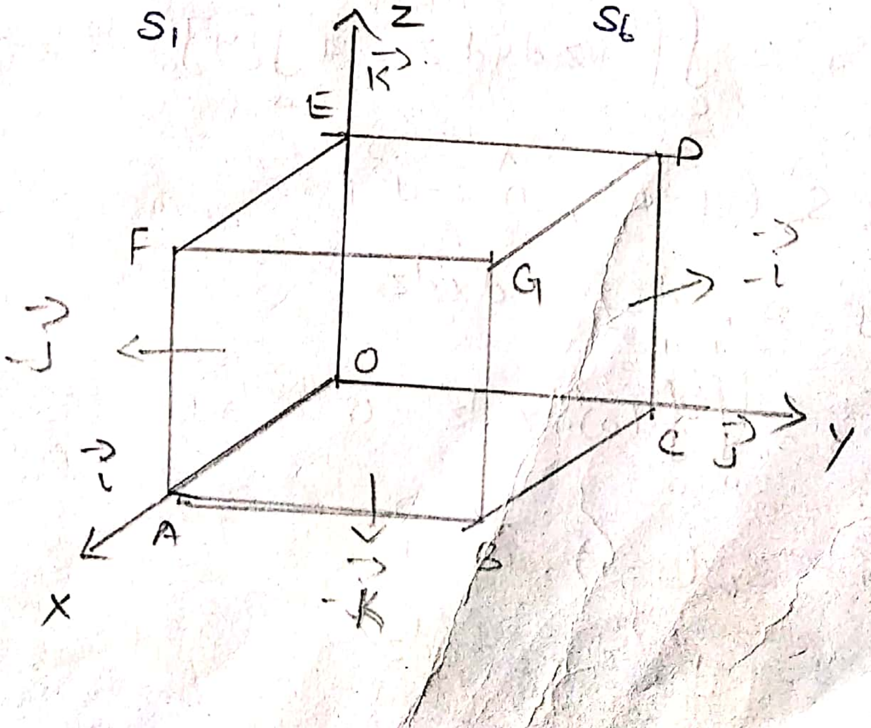
$$= \int_0^1\int_0^1 (4xz - xy)_0^1\,dy\,dz = \int_0^1\int_0^1 (4z - y)\,dy\,dz$$

$$= \int_0^1 \left[4zy - \frac{y^2}{2}\right]_0^1 dz = \int_0^1 \left(4z - \frac{1}{2}\right)dz$$

$$= \left[4\cdot\frac{z^2}{2} - \frac{1}{2}z\right]_0^1 = \frac{4}{2} - \frac{1}{2} = 3/2.$$

LHS.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \vec{F}\cdot\hat{n}\,ds_1 + \cdots + \iint_{S_6} \vec{F}\cdot\hat{n}\,ds_6$$



(i) OABC $S_1 (z=0)$, $\hat{n} = -\vec{k}$.

$$\iint_{S_1} \vec{F}\cdot\hat{n}\, ds_1 = \iint_{S_1} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot (-\vec{k})\, dx\,dy$$

$$= \iint_{S_1} -yz\, dx\,dy = 0.$$

(ii) DEFG $S_2 (z=1)$, $\hat{n} = \vec{k}$; $\vec{F}\cdot\hat{n} = yz = y$, $ds_2 = dx\,dy$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\, ds_2 = \int_0^1\int_0^1 y\,dx\,dy = \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx = \frac{1}{2}[x]_0^1 = 1/2.$$

(iii) OCDE : $S_3 (x=0)$, $\hat{n} = -\vec{i}$.

$\vec{F}\cdot\hat{n} = -4xz = 0$ $(\because x=0)$; $ds_3 = dy\,dz$.

$$\iint_{S_3} \vec{F}\cdot\hat{n}\, ds_3 = \iint_{S_3} (0)\,dy\,dz = 0.$$

(iv) ABGF : $S_4 (x=1)$, $\hat{n} = \vec{i}$

$\vec{F}\cdot\hat{n} = 4xz = 4z$ $(\because x=1)$; $ds_4 = dy\,dz$.

$$\iint_{S_4} \vec{F}\cdot\hat{n}\, ds_4 = \int_0^1\int_0^1 4z\,dy\,dz = 4\int_0^1 [y]_0^1 z\,dz = 4\left[\frac{z^2}{2}\right]_0^1 = 2.$$

(v) OAFE : $S_5 (y=0)$; $\hat{n} = -\vec{j}$.

$\vec{F}\cdot\hat{n} = y^2 = 0$, $ds_5 = dx\,dz$.

$$\iint_{S_5} \vec{F}\cdot\hat{n}\, ds_5 = \int_0^1\int_0^1 (0)\,dx\,dz = 0.$$

(vi) BCDG : $S_6 (y=1)$, $\hat{n} = \vec{j}$

$\vec{F}\cdot\hat{n} = -y^2 = -1$, $ds_6 = dx\,dz$.

$$\iint_{S_6} \vec{F}\cdot\hat{n}\, ds_6 = \int_0^1\int_0^1 (-1)\, dx\, dz = -\int_0^1 [x]_0^1 dz$$

$$= -[z]_0^1 = -1.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = \frac{1}{2} + 2 - 1 = 3/2.$$

Hence the theorem is verified.

2) verify divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$.

Soln.

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$= 2x + 2y + 2z = 2(x+y+z).$$

RHS.

$$\iiint_V \nabla\cdot\vec{F}\, dv = 2\int_0^c\int_0^b\int_0^a (x+y+z)\, dx\, dy\, dz.$$

$$= 2\int_0^c\int_0^b \left[\frac{x^2}{2} + yx + zx\right]_0^a dy\, dz.$$

$$= 2\int_0^c\int_0^b \left(\frac{a^2}{2} + ay + az\right) dy\, dz.$$

$$= 2\int_0^c \left[\frac{a^2}{2}y + a\frac{y^2}{2} + azy\right]_0^b dz.$$

$$= 2\int_0^c \left(\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right) dz$$

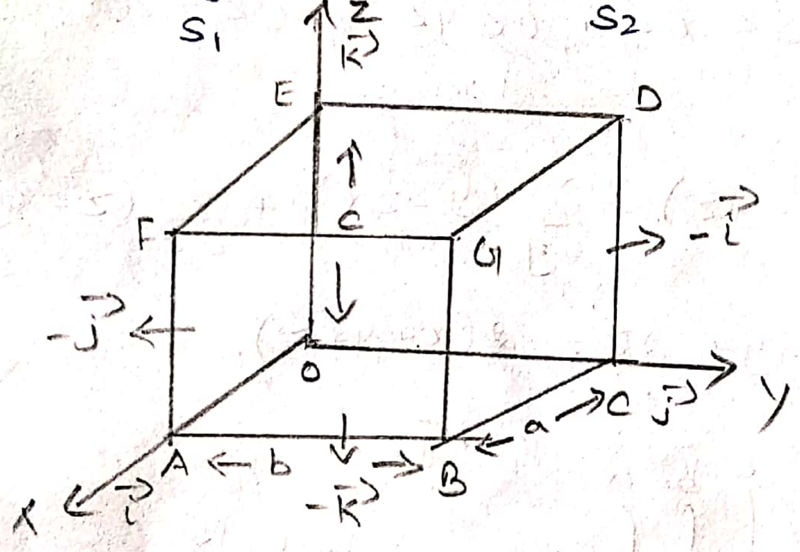
$$= 2\left[\frac{a^2b\,z}{2} + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right]_0^c$$

$$= 2\left[\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right].$$

$$= 2\left[\frac{abc}{2}(a+b+c)\right] = abc(a+b+c).$$

LHS:

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \vec{F}\cdot\hat{n}\,ds_1 + \cdots + \iint_{S_2} \vec{F}\cdot\hat{n}\,ds_6.$$



(i) OABC S_1 $(z=0)$, $\hat{n} = -\vec{k}$.

$$\vec{F}\cdot\hat{n} = -(z^2 - xy);\quad ds_1 = dx\,dy.$$

$$= xy.$$

$$\therefore \iint_{S_1} \vec{F}\cdot\hat{n}\,ds_1 = \int_0^a\int_0^b xy\,dx\,dy = \int_0^a x\left[\frac{y^2}{2}\right]_0^b dx = \frac{b^2}{2}\left[\frac{x^2}{2}\right]_0^a$$

$$= \frac{a^2b^2}{4}$$

ii) DEFG, S_2 $(z=c)$, $\hat{n} = \vec{k}$.

$$\vec{F}\cdot\hat{n} = z^2 - xy = c^2 - xy,\quad ds_2 = dx\,dy.$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds_2 = \int_0^a\int_0^b (c^2 - xy)\,dx\,dy = \int_0^a \left[c^2y - \frac{xy^2}{2}\right]_0^b dx$$

$$= \int_0^a \left(c^2 b - \frac{x b^2}{2}\right) dx = \left[c^2 bx - \frac{b^2}{2}\cdot\frac{x^2}{2}\right]_0^a$$

$$= abc^2 - \frac{a^2 b^2}{4}$$

(iii) OCDE $S_3\ (x=0)$, $\hat{n} = -\vec{i}$

$$\vec{F}\cdot\hat{n} = (yz - x^2) = yz, \quad ds_3 = dy\,dz.$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\, ds_3 = \int_0^b\int_0^c yz\, dy\, dz = \int_0^b \left[\frac{yz^2}{2}\right]_0^c dy$$

$$= \frac{c^2}{2}\left[\frac{y^2}{2}\right]_0^b = \frac{b^2c^2}{4}.$$

(iv) ABGF $S_4\ (x=a)$, $\hat{n} = \vec{i}$

$$\vec{F}\cdot\hat{n} = x^2 - yz, \quad ds_4 = dy\,dz.$$

$$= a^2 - yz.$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\, ds_4 = \int_0^b\int_0^c (a^2 - yz)\, dy\, dz = \int_0^b \left[a^2 z - \frac{yz^2}{2}\right]_0^c dy$$

$$= \int_0^b \left(a^2 c - \frac{yc^2}{2}\right) dy = \left[a^2 cy - \frac{y^2c^2}{4}\right]_0^b = a^2bc - \frac{b^2c^2}{4}$$

(v) OAFE $S_5\ (y=0)$, $\hat{n} = -\vec{j}$.

$$\vec{F}\cdot\hat{n} = -(y^2 - zx) = zx, \quad ds_5 = dx\,dz.$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\, ds_5 = \int_0^a\int_0^c zx\, dx\, dz = \int_0^a \left[\frac{z^2}{2}x\right]_0^c dx = \frac{c^2}{2}\left[\frac{x^2}{2}\right]_0^a$$

$$= \frac{a^2c^2}{4}.$$

(vi) BCDG S_6 $(y=b)$, $\hat{n}=\vec{j}$.

$\vec{F}\cdot\hat{n} = y^2 - zx = b^2 - zx$, $ds_6 = dx\,dz$.

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds_6 = \int_0^a\int_0^c (b^2 - zx)\,dx\,dz = \int_0^a \left[b^2 z - \frac{z^2 x}{2}\right]_0^c dx$$

$$= \left[cb^2 x - \frac{c^2}{2}\cdot\frac{x^2}{2}\right]_0^a = ab^2c - \frac{a^2c^2}{4}.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \frac{a^2b^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4} + \frac{a^2c^2}{4} + ab^2c - \frac{a^2c^2}{4}$$

$$= abc^2 + a^2bc + ab^2c$$

$$= abc(a+b+c).$$

LHS = RHS.

Hence the theorem is verified.

3) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = 2xy\vec{i} + yz^2\vec{j} + xz\vec{k}$ and S is the surface of the parallelopiped bounded by $x=0$, $y=0$, $z=0$, $x=2$, $y=1$, $z=3$.

Soln: By Divergence theorem.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(xz)$$

$$= 2y + z^2 + x.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{3} (2y + z^2 + x)\,dx\,dy\,dz$$

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$$= \int_0^2 \int_0^1 \left(6y + \frac{27}{3} + 3x\right) dx\, dy$$

$$= \int_0^2 \left(3y^2 + 9y + 3xy\right)_0^1 dx$$

$$= \int_0^2 (3 + 9 + 3x)\, dx = \left[12x + \frac{3x^2}{2}\right]_0^2 = 12(2) + \frac{3(2)^2}{2}$$

$$= 24 + 6 = 30$$

H/W.

1) verify divergence theorem for $\vec{f} = x^2\vec{i} + 2y^2\vec{j} + 3z^2\vec{k}$ taken over the cube bounded by the planes $x = 0, 1$; $y = 0, 1$; $z = 0, 1$.

2) $\vec{f} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$; $x = 0, 1$; $y = 0, 1$; $z = 0, 1$.

3) $\vec{f} = (x^3 - yz)\vec{i} - 2x^2y\vec{j} + 2\vec{k}$, $x = 0$, $y = 0$, $z = 0$, $x = 1$, $y = 1$, $z = 1$.

STOKE'S THEOREM : Q.B (10)

The Surface Integral of normal component of curl of a vector $\vec{F}$ over a open surface 'S' is equal to tangential component of $\vec{f}$ over the curve C Enclosing the surface 'S' ie $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \int_C \vec{F} \cdot d\vec{r}$

1) Verify stoke's theorem for $\vec{f} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ for the rectangular region by $x = 0$, $x = 2$, $y = 0$, $y = 2$ on the xy plane.

Soln: stoke's theorem

$$\iint_S (\nabla \times \vec{f}) \cdot \hat{n}\, ds = \int_C \vec{f} \cdot d\vec{r}$$

LHS

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(2y+2y) = 4y\vec{k}.$$

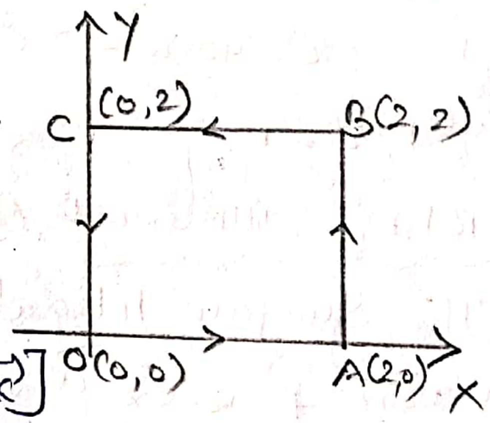
[on the xy plane we know $z=0$ & $\vec{k}$ be the unit normal vector to the xy plane]

$$(\nabla\times\vec{F})\cdot\hat{n} = 4y\vec{k}\cdot\vec{k} = 4y \quad [ds = dx\,dy]$$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \int_0^2\int_0^2 4y\,dx\,dy = 4\int_0^2 \left[\frac{y^2}{2}\right]_0^2 dx$$

$$= 4\,[2x]_0^2 = 16.$$

RHS $\int_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$



Along $\vec{OA}$: $y=0$, $x = 0$ to 2.

$$\vec{F}\cdot d\vec{r} = [(x^2-y^2)\vec{i} + 2xy\vec{j}]\cdot[dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}]$$

$$= (x^2-y^2)dx + 2xy\,dy.$$

$$= x^2dx \quad [\because y=0].$$

$$\therefore \int_{OA} \vec{F}\cdot d\vec{r} = \int_0^2 x^2dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}.$$

Along $\vec{AB}$: $y = 0$ to 2, $x=2$, $dx=0$.

$$\vec{F}\cdot d\vec{r} = 4y\,dy$$

$$\therefore \int_{AB} \vec{F}\cdot d\vec{r} = \int_0^2 4y\,dy = \left[\frac{4y^2}{2}\right]_0^2 = 4\cdot\frac{2^2}{2} = 8$$

Along $\vec{BC}$: $y=2$, $dy=0$, $x=2$ to 0.

$\vec{F}\cdot d\vec{r} = (x^2-4)\,dx$.

$$\int_{BC}\vec{F}\cdot d\vec{r} = \int_2^0 (x^2-4)\,dx = \left[\frac{x^3}{3}-4x\right]_2^0 = \left[0-\left(\frac{8}{3}-8\right)\right] = \frac{16}{3}$$

Along $\vec{CO}$: $x=0$, $dx=0$, $y=2$ to 0

$\vec{F}\cdot d\vec{r} = 0$

$$\therefore \int_{CO}\vec{F}\cdot d\vec{r} = 0.$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \frac{8}{3}+8+\frac{16}{3}+0 = \frac{24}{3}+8 = 16.$$

Hence the theorem is verified.

2) Verify stoke's theorem for $\vec{F}=(y-z)\vec{i}+yz\vec{j}-xz\vec{k}$ where S is the surface of the cube bounded by the plane $x=0, y=0, z=0$ and $x=1, y=1, z=1$ above xy plane.

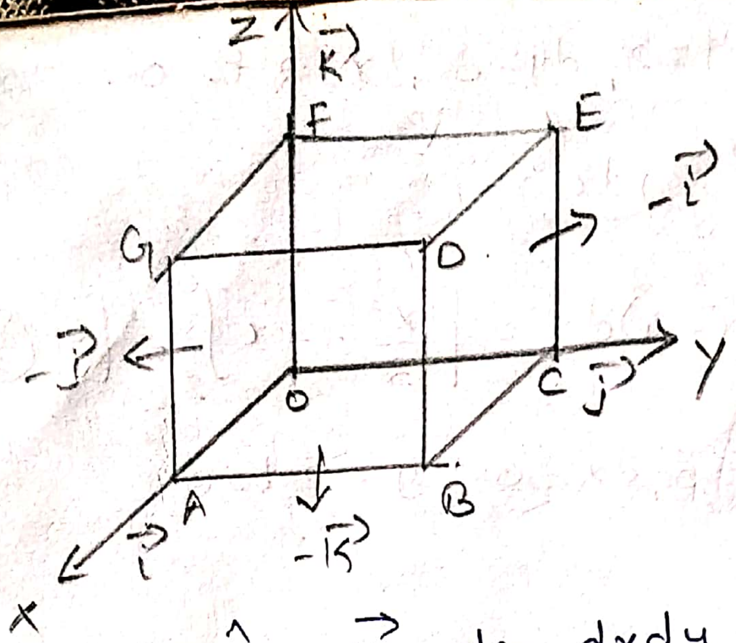
Soln: LHS: $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & yz & -xz \end{vmatrix}$

$= \vec{i}\,(0-y) - \vec{j}\,(-z+1) + \vec{k}\,(0-1)$

$= -y\vec{i} + (z-1)\vec{j} - \vec{k}$.

~~RHS~~

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint_{GDEF} + \iint_{OCEF} + \iint_{GABD} + \iint_{OAGF} +$$



Along GDEF, $\hat{n} = +\vec{k}$, $ds = dx\,dy$.

$\therefore (\nabla \times \vec{f}) \cdot \hat{n} = -1$

$$\therefore \iint_{GDEF} (\nabla \times \vec{f}) \cdot \hat{n}\, ds = \int_0^1\int_0^1 (-1)\, dx\,dy = -1.$$

Along OCEF, $\hat{n} = -\vec{i}$, $ds = dy\,dz$

$(\nabla \times \vec{f}) \cdot \hat{n} = y$.

$$\therefore \iint_{OCEF} (\nabla \times \vec{f}) \cdot \hat{n}\, ds = \int_0^1\int_0^1 y\, dy\,dz = \left[\frac{y^2}{2}\right]_0^1 [z]_0^1 = \frac{1}{2}.$$

Along GABD, $\hat{n} = \vec{i}$, $ds = dy\,dz$.

$(\nabla \times \vec{f}) \cdot \hat{n} = -y$.

$$\therefore \iint_{GABD} (\nabla \times \vec{f}) \cdot \hat{n}\, ds = \int_0^1\int_0^1 -y\, dy\,dz = -\frac{1}{2}.$$

Along OAGF, $\hat{n} = -\vec{j}$, $ds = dx\,dz$.

$(\nabla \times \vec{f}) \cdot \hat{n} = -(z-1)$.

$$\therefore \iint_{OAGF} (\nabla \times \vec{f}) \cdot \hat{n}\, ds = -\int_0^1\int_0^1 (z-1)\, dx\,dz = -\int_0^1 \left(\frac{z^2}{2} - z\right) dx$$

$$= \frac{1}{2}[x]_0^1 = \frac{1}{2}$$

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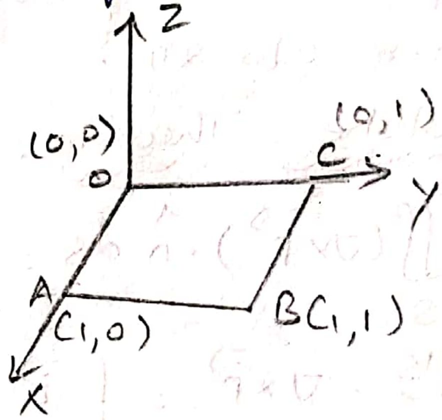
Along BDEC, $\hat{n} = \vec{j}$, $ds = dx\,dz$.

$(\nabla \times \vec{F}) \cdot \hat{n} = (z-1)$.

$\therefore \iint_{BDEC} \nabla \times \vec{F} \cdot \hat{n} = \int_0^1 \int_0^1 (z-1)\,dx\,dz = -\frac{1}{2}$.

$\therefore \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,ds = -1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -1$.

RHS: The boundary of C is the square in the xy-plane, $\therefore z = 0$.



$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

(i) Along $\vec{OA}$, $y = 0$, $dy = 0$, $x = 0$ to 1

$\vec{F} \cdot d\vec{r} = \left[(y-z)\vec{i} + yz\vec{j} - xz\vec{k}\right] \cdot \left[dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}\right]$

$= (y-z)dx + yz\,dy - xz\,dz$.

$\Rightarrow \vec{F} \cdot d\vec{r} = 0 \quad (\because z = 0)$

$\therefore \int_{OA} \vec{F} \cdot d\vec{r} = 0$.

(ii) Along $\vec{AB}$, $x = 1$, $dx = 0$, $y = 0$ to 1.

$\vec{F} \cdot d\vec{r} = 0 \quad \therefore \int_{AB} \vec{F} \cdot d\vec{r} = 0$.

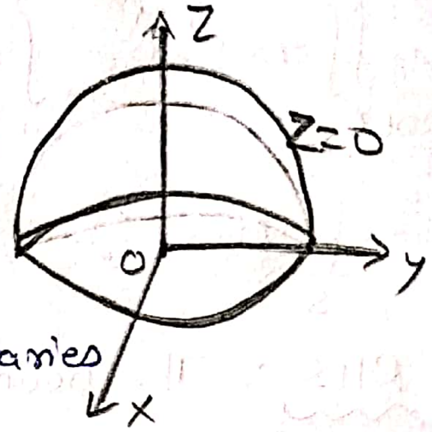
(iii) Along $\vec{BC}$ $y = 1$, $dy = 0$, $x = 1$ to 0.

$\vec{F} \cdot d\vec{r} = dx \quad \therefore \int_{BC} \vec{F} \cdot d\vec{r} = \int_1^0 dx = [x]_1^0 = -1$

(iv) Along $\vec{CO}$, $x = 0$, $dx = 0$, $y = 1$ to 0 $\quad \vec{F} \cdot d\vec{r} = 0 \quad \therefore \int_C \vec{F} \cdot d\vec{r} = 0$

$\Rightarrow$ LHS = RHS.

3) Verify Stoke's theorem for $\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is the circular boundary on $z=0$ plane.



Soln: The boundary curve C is the circle $x^2+y^2+z^2=1$, $z=0 \Rightarrow x^2+y^2=1$

The parametric equations of the sphere is $x = \cos\theta$, $y = \sin\theta$, θ varies from 0 to 2π.

Stoke's Theorem:

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \int_C \vec{F}\cdot d\vec{r}.$$

LHS: $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$

$$= \vec{i}\,[-2yz-(-2yz)] - \vec{j}\,[0-0] + \vec{k}\,[0-(-1)]$$

$$= \vec{i}\,(0) + \vec{k} = \vec{k}.$$

$$(\nabla\times\vec{F})\cdot\hat{n} = \vec{k}\cdot\vec{k} = 1.$$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint_S dx\,dy = \pi(1)^2 = \pi.$$

RHS: $\vec{F}\cdot d\vec{r} = (2x-y)\,dx - yz^2\,dy - y^2z\,dz.$

$x = \cos\theta$, $y = \sin\theta$, $z=0$.

$dx = -\sin\theta$, $dy = \cos\theta$, $dz = 0$.

$$\vec{F}\cdot d\vec{r} = [2\cos\theta - \sin\theta][-\sin\theta]\,d\theta.$$

$$= [-2\sin\theta\cos\theta + \sin^2\theta]\,d\theta$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (\sin^2\omega - 2\sin\omega\cos\omega)\, d\omega.$$

$$= \int_0^{2\pi} \left[\left(\frac{1-\cos 2\omega}{2}\right) - \sin 2\omega\right] d\omega$$

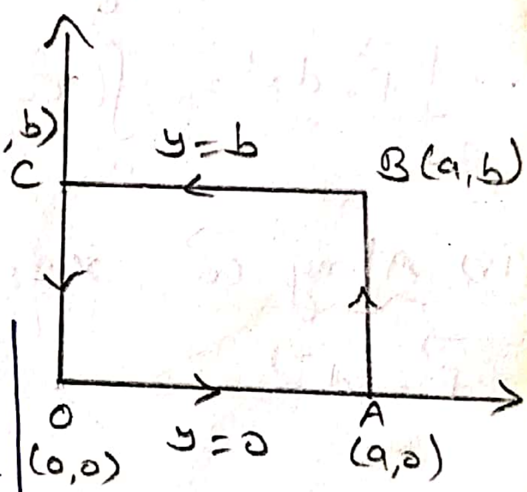
$$= \left[\frac{1}{2}\omega - \frac{\sin 2\omega}{4} + \frac{\cos 2\omega}{2}\right]_0^{2\pi} = \pi.$$

Hence the theorem is verified.

__H/W:__ 4) Verify stoke's theorem for $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$ in the Rectangular region in the $z=0$ plane whose sides are along the lines $x=0$, $x=a$, $y=0$ and $y=b$.

__Soln:__ Stoke's theorem is

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = \int_C \vec{F}\cdot d\vec{r}.$$



$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(2y+2y) = 4y\vec{k}.$$

$$(\nabla\times\vec{F})\cdot\hat{n} = 4y\vec{k}\cdot\vec{k} = 4y.$$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = 4\int_0^b\int_0^a y\, dx\, dy = 4[x]_0^a \left[\frac{y^2}{2}\right]_0^b$$

$$= \frac{4ab^2}{2} = 2ab^2.$$

Now $\int_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$.

(i) Along $\overrightarrow{OA}$; $y=0$, $dy=0$, $x=0$ to a.

$$\vec{F}\cdot d\vec{r} = x^2\,dx \quad \therefore \int_{OA} \vec{F}\cdot d\vec{r} = \int_0^a x^2\,dx = \frac{a^3}{3}.$$

(ii) Along $\overrightarrow{AB}$, $x=a$, $dx=0$, $y=0$ to b.

$$\vec{F}\cdot d\vec{r} = 2xy\,dy = 2ay\,dy.$$

$$\therefore \int_{AB} \vec{F}\cdot d\vec{r} = \int_0^b 2ay\,dy = 2a\left[\frac{y^2}{2}\right]_0^b = \frac{2ab^2}{2} = ab^2.$$

(iii) Along $\overrightarrow{BC}$, $y=b$, $dy=0$, $x=a$ to 0.

$$\vec{F}\cdot d\vec{r} = (x^2-y^2)\,dx = (x^2-b^2)\,dx.$$

$$\therefore \int_{BC} \vec{F}\cdot d\vec{r} = \int_a^0 (x^2-b^2)\,dx = \left[\frac{x^3}{3} - b^2x\right]_a^0 = -\frac{a^3}{3} + ab^2.$$

(iv) Along $\overrightarrow{CO}$, $x=0$, $dx=0$, $y=0$ to b.

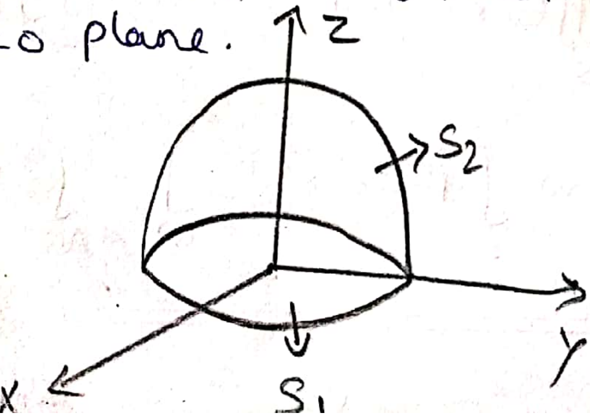
$$\vec{F}\cdot d\vec{r} = 0 \quad \therefore \int_{CO} \vec{F}\cdot d\vec{r} = 0.$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \cancel{\frac{a^3}{3}} + ab^2 - \cancel{\frac{a^3}{3}} + ab^2 = 2ab^2.$$

Hence the theorem is verified.

H/W.

5) Verify Stoke's theorem for $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$. and c is its boundary on $z=0$ plane.



Soln: Stoke's theorem is

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds.$$

But by Divergence theorem.

$$\iint_S (\nabla\times\vec{f})\cdot\hat{n}\, ds = \iiint_V \nabla\cdot(\nabla\times\vec{f})\,dv = 0$$
$$[\because \nabla\cdot(\nabla\times\vec{f}) = 0].$$

$$\therefore \iint_{S_1} (\nabla\times\vec{f})\cdot\hat{n}\, ds_1 + \iint_{S_2} (\nabla\times\vec{f})\cdot\hat{n}\, ds_2 = 0.$$

$$\Rightarrow \iint_{S_1} (\nabla\times\vec{f})\cdot\hat{n}\, ds_1 = -\iint_{S_2} (\nabla\times\vec{f})\cdot\hat{n}\, ds_2 \quad -\ 1$$

S_1 is circle in the xy plane $(z=0)$.

$\hat{n} = -\vec{K}$.

$$\nabla\times\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{K} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = \vec{i}(-1) - \vec{j}(1) + \vec{K}(-1)$$
$$= -(\vec{i}+\vec{j}+\vec{K}).$$

$$(\nabla\times\vec{F})\cdot\hat{n} = -(\vec{i}+\vec{j}+\vec{K})\cdot(-\vec{K}) = 1.$$

$$\iint_{S_1} (\nabla\times\vec{f})\cdot\hat{n}\, ds_1 = \iint_{S_1} dx\,dy = \pi(1)^2 = \pi \quad [S_1 : \text{area of the } \odot]$$

$$\therefore \iint_{S_2} (\nabla\times\vec{f})\cdot\hat{n}\, ds_2 = -\pi \quad [\because \text{from } 1]$$

$\therefore$ By Stoke's theorem $\iint_S (\nabla\times\vec{f})\cdot\hat{n}\, ds = \int_C \vec{f}\cdot d\vec{r}$.

where C is the circle in the xy plane $x=\cos\theta,\ y=\sin\theta$; $dx=-\sin\theta,\ dy=\cos\theta$

$$\vec{F}\cdot d\vec{r} = (y\vec{i}+z\vec{j}+x\vec{K})\cdot(dx\vec{i}+dy\vec{j}+dz\vec{K}) = y\,dx + z\,dy + x\,dz.$$
$$= y\,dx \quad [\because \text{In } xy \text{ plane } z=0].$$

$$\therefore \int_C \vec{f}\cdot d\vec{r} = \int_C y\,dx = \int_0^{2\pi} \sin\theta(-\sin\theta)\,d\theta = -\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= -\left[\frac{1}{2}\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = -\left[\frac{1}{2}\cdot 2\pi\right] = -\pi.$$

HW

6) Evaluate $\int \vec{F}\cdot d\vec{r}$ by Stoke's theorem where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{K}$ and C is the boundary of the Δ $(0,0,0)$, $(1,0,0)$ & $(1,1,0)$

ANALYTIC FUNCTIONS

SINGLE VALUED FUNCTION : $f(z) = \sqrt{x}$

$f(z) = \sqrt{z}(4) = \pm 2$

Let $w = f(z) = u + iv$ is a function of complex variable $w = f(z)$ is called single valued if for every value of z there exist a single value of w.

Eg: $w = z$, $w = z^2$ (single valued).

$w = \sqrt{z}$ is a multivalued function [since every value of z we've 2 values for w.]

ANALYTIC FUNCTION : QB ①

A single valued function $f(z)$ which is differentiable at $z = z_0$ and at every point in some neighbourhood of z_0, is said to be analytic at point $z = z_0$.

SINGULAR POINT :

The point at which the function is not differentiable.

QB(3).

Necessary condition for a Function To be Analytic :

(The necessary condition for a function $f(z) = u + iv$ to be analytic are

(i) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ (ii) $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

(or) $u_x = v_y$ $u_y = -v_x$.

provided $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ exists)

These Equations are known as Cauchy - Riemann Equations [C-R].

QB② Properties of Analytic fun's

(i) If $f(z) = u + iv$, is anal., then u & v are harmonic i.e $\nabla^2 u = 0$, $\nabla^2 v = 0$.

(ii). Sums, Products, and compositions of analytic fun's are analytic

PROBLEMS:

QB 11: check whether $w = z^2$ is analytic fn. or not.

1) Using C-R Equation show that $f(z) = z^2$ is analytic in the entire z-plane.

Soln: $f(z) = z^2$ $\quad (z = x+iy)$

$u+iv = (x+iy)^2$

$u+iv = x^2-y^2+i2xy$.

$\Rightarrow u = x^2-y^2, \quad v = 2xy$

$u_x = 2x$ — ① $\quad v_x = 2y$ — ③ $\quad$ from ① & ④ $u_x = v_y$

$u_y = -2y$ — ② $\quad v_y = 2x$ — ④ $\quad$ from ② & ③ $u_y = -v_x$.

$\Rightarrow u_x, u_y, v_x, v_y$ exists and satisfy C-R equations

$\therefore f(z)$ is analytic.

QB 5

2) $f(z) = \bar{z}$, Check whether $w = \bar{z}$ is AF or not.

Soln. $u+iv = x-iy \Rightarrow u = x$ & $v = -y$

$u_x = 1 \quad v_x = 0$

$u_y = 0 \quad v_y = -1$

$\Rightarrow u_x \neq v_y$ & $u_y \neq -v_x$.

$\therefore f(z)$ is not analytic.

3) Show that $f(z) = z^3$ is analytic.

Soln: $f(z) = z^3 = (x+iy)^3 = x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$

$= x^3 - 3xy^2 + i3x^2y - iy^3$.

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

$= u+iv$.

$\Rightarrow u = x^3 - 3xy^2, \quad v = 3x^2y - y^3$.

$u_x = 3x^2 - 3y^2$ — ①, $\quad v_x = 6xy$ — ③

$u_y = -6xy$ — ②, $\quad v_y = 3x^2 - 3y^2$ — ④

From ① & ④, $u_x = v_y$. $\therefore$ The C-R equations are satisfied

From ② & ③, $u_y = -v_x$ $\quad \therefore f(z)$ is analytic.

4) Show that the function $e^x(\cos y + i\sin y)$ is an analytic function. Find its derivative.

Soln: Let $f(z) = e^x(\cos y + i\sin y)$

$$u + iv = e^x \cos y + ie^x \sin y.$$

$\Rightarrow u = e^x \cos y \qquad v = e^x \sin y$

$u_x = e^x \cos y \qquad v_x = e^x \sin y$

$u_y = -e^x \sin y \qquad v_y = e^x \cos y.$

$\Rightarrow u_x = v_y \quad \& \quad u_y = -v_x.$

$\therefore f(z)$ is analytic.

$f(z) = u + iv.$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y$$
$$= e^x(\cos y + i\sin y)$$
$$= e^x . e^{iy} = e^{x+iy} = e^z.$$

$\Rightarrow f'(z) = e^z.$

5) Examine the analyticity of the function $f(z) = xy^2 + iyx^2$

Soln: $f(z) = xy^2 + iyx^2.$

$u = xy^2 \qquad v = yx^2$

$u_x = y^2 \qquad v_x = 2xy$

$u_y = 2xy \qquad v_y = x^2.$

$\Rightarrow u_x \neq v_y \quad \& \quad u_y \neq -v_x.$

C-R Equations are not satisfied. $\therefore f(z)$ is not analytic

Test whether the following functions are analytic or not.

1) $f(z) = xy + iy$ Soln: Not Analytic.

2) $f(z) = z\bar{z}$ Soln: Not Analytic.

3) $f(z) = \sin x \cosh y + i\cos x \sinh y$. Soln Analytic.

Definition: $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is known as Laplace's Equation in two dimensions.

Q.B(6)

Harmonic function: Any function which satisfies the Laplace's Eqn. is known as a harmonic function.

Method To find The conjugate function:

For the function $f(z) = u + iv$, if u is known, then the conjugate function of u is v and is given by

$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy.$

$\Rightarrow v = \int \frac{\partial v}{\partial x} dx + \int \frac{\partial v}{\partial y} dy = -\int \frac{\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x} dy.$

1) Show that the function $u = 2x(1-y)$ is harmonic.

P.T QB(7): $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic

Soln: $u = 2x - 2yx$

$u_x = 2 - 2y.$ $\quad u_y = -2x$

$u_{xx} = 0$ $\quad u_{yy} = 0.$

Hence $u_{xx} + u_{yy} = 0 \Rightarrow u$ is harmonic function.

$u_x = 6xy + 4x$, $u_y = 3x^2 - 3y^2 - 4y$

$u_{xx} = 6y + 4$, $u_{yy} = -6y - 4$

$u_{xx} + u_{yy} = 6y + 4 - 6y - 4 = 0.$

$\therefore u$ is harmonic fn.

2) Let $f(z)$ be an analytic function. If $u = 2x - x^3 + 3xy^2$ find the harmonic conjugate function.

Soln: $u = 2x - x^3 + 3xy^2.$

$u_x = 2 - 3x^2 + 3y^2$, $\quad u_y = 6xy.$

$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy.$

$v = -\int \frac{\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x} dy$

$= -\int 6xy \, dx + \int (2 - 3x^2 + 3y^2) dy.$

$= -6 \cdot \frac{x^2}{2} y + 2y - 3x^2 y + 3y^3/3$

$\therefore v = -6x^2y + 2y + y^3 = y^3 + 2y - 6x^2y.$

3) Prove the following function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic. Also find the conjugate harmonic function.

Soln: $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

$u_x = 3x^2 - 3y^2 + 6x$ $\qquad u_y = -6xy - 6y$

$u_{xx} = 6x + 6$ $\qquad u_{yy} = -6x - 6$.

$u_{xx} + u_{yy} = 6x + 6 - 6x - 6 = 0$.

$\Rightarrow u$ is harmonic.

By Milne's Thom method

$f'(z) = u_x - i u_y$

$= 3x^2 - 3y^2 + 6x - i(-6xy - 6y)$

$x \to z,\ y \to 0$.

$f'(z) = 3z^2 + 6z$.

$\int f'(z)dz = f(z) = z^3 + 3z^2 + c$

$u + iv = (x+iy)^3 + 3(x+iy)^2 + c_1 + ic_2$.

Since v is the conjugate harmonic equation, $u + iv$ is analytic.

$\therefore$ By C-R Equations, $u_x = v_y$ & $u_y = -v_x$.

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$v = -\int \frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy$$

$$= -\int(-6xy - 6y)dx + \int(3x^2 - 3y^2 + 6x)dy$$

$$= \frac{6x^2y}{2} + 6xy + 3x^2y - \frac{3y^3}{3} + 6xy$$

$$= 3x^2y + 6xy + 3x^2y - y^3 + 6xy$$

$$v = 6x^2y + 12xy - y^3 + c.$$

H/w.

1) Show that $u = xy(x^2 - y^2)$ is harmonic.

2) If $f(z)$ is analytic function and $u = 3x^2y + 2x^2 - y^3 - 2y^2$ Then find the harmonic conjugate function.

MILNE THOMSON METHOD.

Let $u(x,y)$ be the real part of the analytic function $f(z) = u(x,y) + iv(x,y)$.

Since $u(x,y)$ is given $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$ can be found out.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (By C-R Equation)

$= u_x(x,y) - i u_y(x,y)$

Replace x by z and y by 0.

$= u_x(z,0) - i u_y(z,0)$ by Milne-Thomson Method.

$f(z) = \int [u_x(z,0) - i u_y(z,0)]\,dz + c.$

If $v(x,y)$ is given,

$f(z) = \int [v_y(z,0) + i v_x(z,0)]\,dz + c.$

1) Show that the function $u = 2x - x^3 + 3xy^2$ is harmonic and find the corresponding analytic function, $f(z) = u + iv$.

Soln $u = 2x - x^3 + 3xy^2$.

$u_x = 2 - 3x^2 + 3y^2$ $\quad u_y = 6xy$

$u_{xx} = -6x$ $\quad u_{yy} = 6x$.

$\therefore u_{xx} + u_{yy} = -6x + 6x = 0.$

$u(x,y)$ satisfies Laplace Equation.

$\therefore$ u is harmonic.

$u_x(x,y) = 2 - 3x^2 + 3y^2$, $u_y(x,y) = 6xy$.

By Milne Thomson Method, Replace x by z and y by 0.

$u_x(z,0) = 2 - 3z^2$, $u_y(z,0) = 0$.

Then $f(z) = \int [u_x(z,0) - i u_y(z,0)]\,dz + c.$

$= \int (2 - 3z^2)\,dz + c.$

$\Rightarrow f(z) = 2z - z^3 + c.$

2) Construct the analytic function of $f(z)$ of which the real part is $e^x \cos y$.

Soln: Given $u = e^x \cos y$

$u_x = e^x \cos y \qquad u_y = -e^x \sin y$

$\therefore$ By Milne's Thomson method Replace x by z, y by 0.

$u_x(z,0) = e^z \cos 0 = e^z, \quad u_y(z,0) = -e^z \sin 0 = 0.$

$f(z) = \int [u_x(z,0) - i u_y(z,0)] dz = \int e^z dz = e^z + C.$

$\Rightarrow f(z) = e^z + C.$

3) Find the analytic function whose imaginary parts is given by $v = e^y \cos x$.

Soln. Given $v = e^y \cos x$.

$v_x = -e^y \sin x, \quad v_y = e^y \cos x$

By Milne Thomson Method, Replace x by z & y by 0.

$v_x(z,0) = -e^0 \sin z = -\sin z, \quad v_y(z,0) = \cos z.$

$\therefore f(z) = \int [v_y(z,0) + i v_x(z,0)] dz + C.$

$= \int (\cos z - i \sin z) dz + C.$

$= \sin z + i \cos z + C.$

$= i[\cos z - i \sin z] + C = i e^{-iz} + C.$

H/W.

1) $u = 4xy - 3x + 2$ Soln: $-3z - i2z^2 + C.$

2) $v = -\sin x \sinh y$ Soln: $\cos z + C.$

4) Construct the analytic function $f(z) = u + iv$ given $u - v = e^x(\cos y - \sin y)$.

Soln: Method (I):

$f(z) = u + iv.$

$i f(z) = i(u + iv) = iu - v.$

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$f(z) + i f(z) = u + iv + iu - v.$

$(1+i) f(z) = (u-v) + i(u+v).$

$= U + iV.$

where $U = u - v$, $V = u + v$.

$U = u - v = e^x(\cos y - \sin y).$

$U_x = e^x(\cos y - \sin y)$, $U_y = e^x(-\sin y - \cos y)$

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy.$$

$$dV = -\frac{\partial U}{\partial y} dx + \frac{\partial U}{\partial x} dy.$$

$$dV = e^x(\sin y + \cos y)dx + e^x(\cos y - \sin y)dy.$$

[To evaluate the $\int M dx + N dy$, where $M dx + N dy$ is an Exact ~~differentiable~~ differential in which N does not contain any term independent of x, it is enough to evaluate just $\int M dx$ treating y as constant].

$\therefore V = \int e^x(\sin y + \cos y) dx = e^x(\sin y + \cos y) + C.$

$\therefore (1+i) f(z) = U + iV.$

$\Rightarrow (1+i) f(z) = e^x(\cos y - \sin y) + i[e^x(\cos y + \sin y) + C]$

$= e^x \cos y - e^x \sin y + i e^x \cos y + i e^x \sin y + iC.$

$= e^x(\cos y + i \sin y) + i e^x(\cos y + i \sin y) + iC.$

$= e^x \cdot e^{iy} + i e^x \cdot e^{iy} + iC.$

$= e^{x+iy} + i e^{x+iy} + iC.$

$= e^z + i e^z + iC.$

$= (1+i) e^z + iC.$

$f(z) = e^z + \frac{i}{1+i} c = e^z + c_1.$

Method ②: $f(z) = u + iv.$

$if(z) = i(u+iv) = iu - v.$

$f(z) + if(z) = u + iv + iu - v.$

$(1+i)f(z) = (u-v) + i(u+v). = U + iV.$

where $U = u - v, = e^x(\cos y - \sin y)$

$V = u + v.$

Considering the derivative of $(1+i) f(z).$

$(1+i)f'(z) = U_x + iV_x = U_x - iU_y \quad \because U_y = -V_x.$

But $U_x = e^x(\cos y - \sin y),\ U_y = e^x(-\sin y - \cos y).$

$(1+i) f'(z) = e^x(\cos y - \sin y) + ie^x(\sin y + \cos y).$

$= e^z(\cos 0 - \sin 0) + ie^z(\sin 0 + \cos 0)$ by M.T Meth

$= e^z + ie^z = e^z(1+i).$

$\Rightarrow f'(z) = e^z.$

integrating wrt z, $f(z) = e^z + c.$

5) find the real part of the analytic function whose Imaginary part is $e^{2x}[x \cos 2y - y \sin 2y]$.

Soln: $v = e^{2x} x \cos 2y - e^{2x} y \sin 2y.$

~~$v_x = -2e^{2x} x \sin 2y$~~

$v_x = e^{2x} \cdot \cos 2y + 2e^{2x} x \cos 2y - 2e^{2x} y \sin 2y.$

$v_y = -2e^{2x} \cdot x \sin 2y - e^{2x} \sin 2y - 2e^{2x} y \cos 2y.$

Replace x by z and y by 0.

$v_x = e^{2z}(1) + 2e^{2z} \cdot z(1) - 0 = e^{2z} + 2e^{2z} \cdot z.$

$v_y = 0 - 0 - 0 = 0.$

$$\therefore f(z) = \int (v_y + i v_x)\, dz = \int 0 + i(e^{2z} + 2ze^{2z})\, dz + c.$$

$$= i\int (e^{2z} + 2ze^{2z})\, dz + c.$$

$$= i\left[\frac{e^{2z}}{2} + 2\left(\frac{ze^{2z}}{2} - \int \frac{e^{2z}}{2}\, dz\right)\right] + c.$$

$$= i\left[\frac{e^{2z}}{2} + ze^{2z} - 2\cdot\frac{e^{2z}}{4}\right] + c.$$

$$\therefore f(z) = i z e^{2z} + c.$$

$$u + iv = i(x+iy)\, e^{2(x+iy)} + c.$$

$$= (ix - y)\, e^{2x} \cdot e^{2iy} + c.$$

$$= (ix e^{2x} - y e^{2x})(\cos 2y + i \sin 2y) + c.$$

$$= ix e^{2x} \cos 2y - x e^{2x} \sin 2y - y e^{2x} \cos 2y - i y e^{2x} \sin 2y + c$$

$$= -\left[e^{2x}(x \sin 2y + y \cos 2y) + c\right] + i e^{2x}\left[x \cos 2y - y \sin 2y\right] + c$$

Hence the real part is $-e^{2x}\left[x \sin 2y + y \cos 2y\right] + c.$

6) Construct the analytic function $f(z) = u + iv$ given $u = e^{-x}(x \cos y + y \sin y)$ subject to the condition $f(z) = 1$ at origin.

Soln: $u = e^{-x}(x \cos y + y \sin y).$

$$u_x = e^{-x} \cos y - e^{-x} x \cos y - e^{-x} y \sin y.$$

$$= e^{-x} \cos y - e^{-x}(x \cos y + y \sin y).$$

$$u_x(z, 0) = e^{-z} - z e^{-z}$$

$$u_y = -e^{-x} x \sin y + e^{-x} \sin y + e^{-x} \cdot y \cos y.$$

$$u_y(z, 0) = 0.$$

$$f(z) = \int (u_x - i u_y)\, dz + c = \int (e^{-z} - z e^{-z})\, dz + c.$$

$$= -e^{-z} - \left[-z e^{-z} - e^{-z}\right] + c = z e^{-z} + c.$$

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Given $f(z)=1$ at origin ie $f(0)=1$, $z=0$.

$f(0)=0e^0+c \Rightarrow 1=c$

$\therefore f(z)=ze^{-z}+1$.

7) find the analytic function whose imaginary part is $e^x(x\sin y+y\cos y)$.

Soln: $v=e^x(x\sin y+y\cos y)$

$v=e^x.x\sin y+e^x.y\cos y$.

$v_x=e^x.x\sin y+e^x\sin y+e^x.y\cos y$.

$v_y=e^x.x\cos y+e^x\cos y+e^x.y(-\sin y)$.

$v_x(z,0)=e^z.z\sin 0+e^z\sin 0+e^z.0.\cos 0=0$.

$v_y(z,0)=ze^z+e^z=(1+z)e^z$.

$f(z)=\int[v_y(z,0)+iv_x(z,0)]dz+c$.

$=\int(z+1)e^z dz+c$.

$=(1+z)e^z-e^z+c \quad = ze^z+c$.

8) Construct the analytic function whose imaginary part is $e^{-x}(x\cos y+y\sin y)$ and which equals 1 at the origin.

Soln: $v=e^{-x}(x\cos y+y\sin y)$

$v=e^{-x}x\cos y+e^{-x}y\sin y$.

$v_x=-e^{-x}x\cos y+e^{-x}\cos y-e^{-x}y\sin y$.

$v_y=-e^{-x}x\sin y+e^{-x}\sin y+e^{-x}y\cos y$.

By M.T method, Replace x by z and y by 0.

$v_x(z,0)=-e^{-z}.z+e^{-z}$; $v_y(z,0)=0$.

$f(z)=\int[v_y(z,0)+iv_x(z,0)]dz+c$.

$=\int i[-e^{-z}z+e^{-z}]dz+c$

$= i\left[\int e^{-z}dz + \int -e^{-z} \frac{z}{z} dz\right] + c$

$= i\left[-e^{-z} - \int z\, e^{-z}dz\right] + C = i\left[-e^{-z} + z e^{-z} + e^{-z}\right] + C.$

$= i z e^{-z} + c$

$f(z) = 1$ at $z = 0$ $\therefore$ $c = 1$

$\therefore$ $f(z) = i z e^{-z} + 1.$

9) Prove that $u = x^2 - y^2$, $v = \frac{-y}{x^2+y^2}$ both u and v Satisfy Laplace's Equation but that $u + iv$ is not a regular fn. of z.

Soln: Let $f(z) = u + iv$, $u = x^2 - y^2$, $v = \frac{-y}{x^2+y^2}$.

$u_x = 2x$, $u_y = -2y$

$u_{xx} = 2$ $u_{yy} = -2$.

$u_{xx} + u_{yy} = 2 - 2 = 0$ $\Rightarrow$ u satisfies Laplace equation.

$v_x = -y\left[-(x^2+y^2)^{-2}\right] \cdot 2x = 2xy / (x^2+y^2)^2.$

$$v_{xx} = \frac{(x^2+y^2)^2 \cdot 2y - 2xy \cdot 2(x^2+y^2) \cdot 2x}{(x^2+y^2)^4}.$$

$$= \frac{2y(x^2+y^2)^2 - 8x^2y(x^2+y^2)}{(x^2+y^2)^4} = \frac{2y\left[x^2+y^2\right]\left[x^2+y^2-4x^2\right]}{(x^2+y^2)^4}.$$

$$= \cancel{2y(x^2+y} \;\; \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}.$$

$$v_y = -\left[\frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}\right] = -\frac{(x^2-y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

$$v_{yy} = \frac{(x^2+y^2)\,2y - (y^2-x^2)\,2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2)\left[x^2+y^2-2(y^2-x^2)\right]}{(x^2+y^2)^4} = \frac{2y(-y^2+3x^2)}{(x^2+y^2)^3}.$$

$$= \frac{-2y(y^2-3x^2)}{(x^2+y^2)^3}.$$

$v_{xx} + v_{yy} = 0 \Rightarrow v$ satisfies Laplace Equation.

But $u_x \neq v_y$ & $u_y \neq -v_x$.

$\Rightarrow$ C-R Equations are not satisfied by u and v.

Hence $f(z) = u+iv$ is not a regular analytic fn. of z.

Orthogonal Curves [Product of the Slopes $m_1 m_2 = -1$].

Two curves are said to be orthogonal if they intersect at right angles.

1) If $f(z) = u+iv$ is an analytic function prove that the families of curves $u(x,y) = c_1$ and $v(x,y) = c_2$ are orthogonal.

Soln: Given $u(x,y) = c_1$ — (1)

Differentiating Eqn. (1)

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0.$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = 0.$$

$$m_1 = \frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y}$$

$v(x,y) = c_2$ — (2)

Differentiating Eqn (2)

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\cdot\frac{dy}{dx} = 0.$$

$$m_2 = \frac{dy}{dx} = \frac{-\partial v/\partial x}{\partial v/\partial y}.$$

$$m_2 = \frac{\partial u/\partial y}{\partial u/\partial x}$$ using C-R Eqn.

$u_y = -v_x$

$u_x = v_y$.

Then $$m_1 m_2 = \frac{-\partial u/\partial x}{\partial u/\partial y}\cdot\frac{\partial u/\partial y}{\partial u/\partial x} = -1.$$

$\therefore$ $m_1 m_2 = -1$, the given curves are orthogonal.

2) With usual notation S.T $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\left(\frac{\partial^2}{\partial z \partial \bar{z}}\right)$.

soln: $z = x+iy$ — (1)

$\bar{z} = x-iy$ — (2)

(1) + (2) $\Rightarrow 2x = z+\bar{z}$ (1) − (2) $\Rightarrow 2iy = z-\bar{z}$

$\Rightarrow x = \frac{1}{2}(z+\bar{z})$ $\Rightarrow y = \frac{1}{2i}(z-\bar{z})$.

$\Rightarrow y = -\frac{i}{2}(z-\bar{z})$.

$\frac{\partial x}{\partial z} = \frac{1}{2}, \quad \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \quad \frac{\partial y}{\partial z} = -\frac{i}{2}, \quad \frac{\partial y}{\partial \bar{z}} = \frac{i}{2}$.

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x}\right) - \frac{i}{2}\left(\frac{\partial}{\partial y}\right)$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right] \quad \text{— (3)}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \quad \text{— (4)}$$

(3) × (4) $\Rightarrow$
$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

3) If $f(z) = u+iv$ is an analytic function prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

soln: We know that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\left(\frac{\partial^2}{\partial z \partial \bar{z}}\right)$.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\left(\frac{\partial^2}{\partial z \partial \bar{z}}\right)|f(z)|^2.$$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}\left[f(z)\,f(\bar{z})\right]$$

$$= 4 \frac{\partial}{\partial z} f(z) \cdot f'(\bar{z})$$

$$= 4 f'(z) \cdot f'(\bar{z})$$

$$= 4 |f'(z)|^2.$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4 |f'(z)|^2.$$

C-R Equation In Polar form:

Let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$.

$$f(z) = f(re^{i\theta}) \Rightarrow u + iv = f(re^{i\theta}).$$

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \cdot \frac{\partial u}{\partial \theta}.$$

1) prove that $f(z) = z^n$ is analytic function.

Soln: Let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$.

$$f(z) = (re^{i\theta})^n = r^n e^{in\theta}.$$

$$\Rightarrow u + iv = r^n(\cos n\theta + i\sin n\theta).$$

$$\Rightarrow u = r^n \cos n\theta, \quad v = r^n \sin n\theta.$$

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta, \quad \frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta.$$

$$\frac{\partial u}{\partial \theta} = -r^n \cdot n\sin n\theta, \quad \frac{\partial v}{\partial \theta} = r^n \cdot n\cos n\theta.$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \cdot \frac{\partial u}{\partial \theta}.$$

C-R equations are satisfied

$\therefore f(z)$ is analytic function.

CONFORMAL REPRESENTATION.

ISOGONAL MAPPING :

Let $f(z)$ be a single-valued function defined in a region D. Then the mapping $w = f(z)$ is said to be isogonal if it preserves the magnitude of the angle between every two curves. It is immaterial whether the sense of orientation of the angle of is preserved or not.

CONFORMAL MAPPING :

The mapping $w = f(z)$ is said to be conformal if it preserves the magnitude of the angle between every two curves and also if it preserves the sense of orientation of the angle.

SOME TRANSFORMATION :

1) $w = z + c$

Given $w = z + c$ where c is a complex constant.

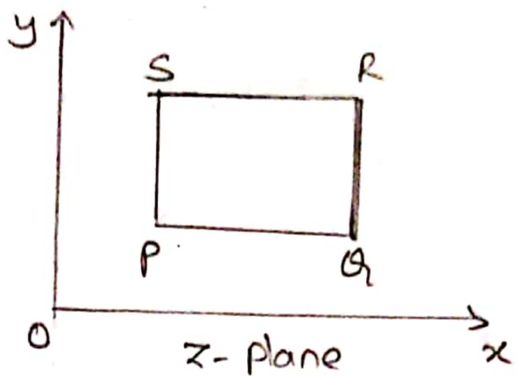
Thus, if $z = x + iy$, $c = a + ib$ then.

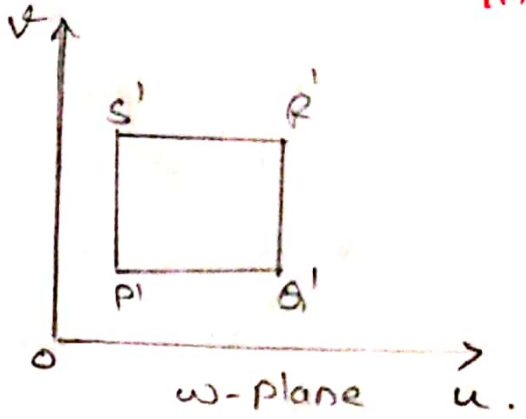
$$w = (x+iy) + (a+ib)$$

$$\Rightarrow u + iv = (x+a) + i(y+b).$$

$$\Rightarrow u = x+a, \quad v = y+b$$

The point $A(x, y)$ in the z-plane is mapped onto the point $A'(x+a, y+b)$ in the w-plane. Every point in any region of the z-plane is mapped onto the w-plane it is clear that if the w-plane is superposed on the z-plane, the figure is shifted through a distance given by the vector c. The two regions have the same shape and size.





1.) Determine the region in the w-plane in which the rectangle bounded by the lines $x=0$, $y=0$, $x=2$ & $y=1$. is mapped under the transformation $w = z+2+3i$

Soln: Given $w = z+2+3i$

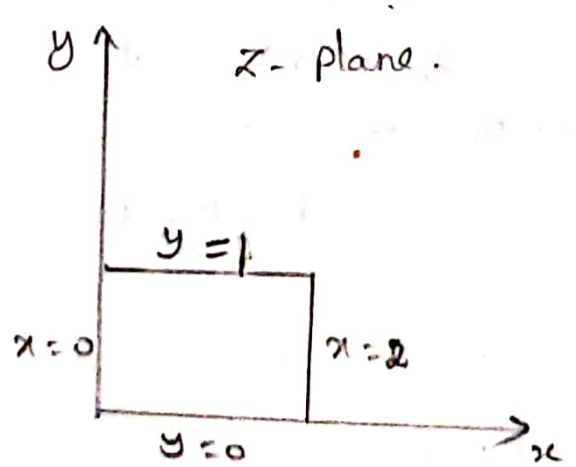
$$u+iv = x+iy+2+3i$$

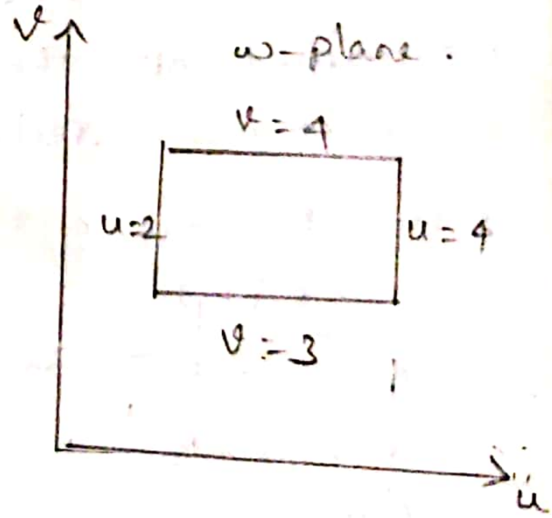
$$u+iv = (x+2)+i(y+3).$$

$\Rightarrow$ $u = x+2$, $v = y+3$.

$x=0 \Rightarrow u=2$, $y=0 \Rightarrow v=3$

$x=2 \Rightarrow u=4$, $y=1 \Rightarrow v=4$.





Hence the lines $x=0$, $x=2$, $y=0$ and $y=1$ are mapped into the lines $u=2$, $v=3$, $u=4$, $v=4$ resply. which forms a rectangle in the w-plane.

2) Find the region in the w-plane in which the rectangle bounded by lines $x=0$, $y=0$, $x=1$, and $y=2$ is mapped under the transformation $w = z+(2-i)$.

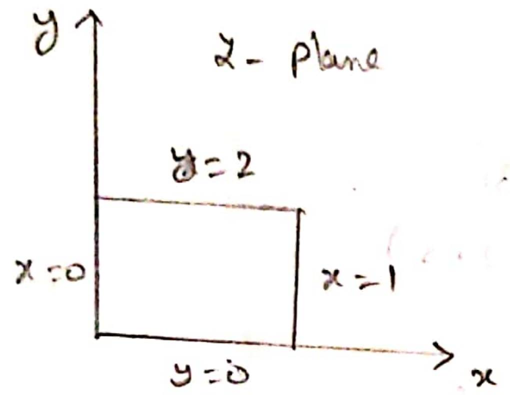
Soln: Given $w = z+(2-i)$.

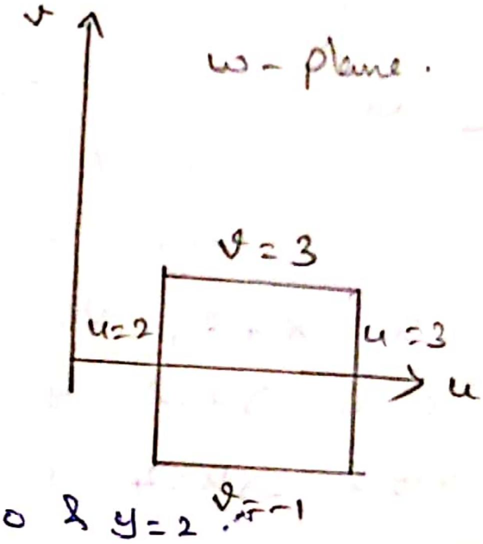
$$u+iv = x+iy+2-i$$

$$\Rightarrow \quad u = x+2, \quad v = y-1.$$

$x=0 \Rightarrow u=2$, $y=0 \Rightarrow v=-1$

$x=1 \Rightarrow u=3$, $y=2 \Rightarrow v=1$.





Hence the lines $x=0$, $x=1$, $y=0$ & $y=2$ are mapped into the lines $u=2$, $u=3$, $v=1$, $v=-1$ resply which forms a rectangle in the w- plane.

3) Find the image of the circle $|z|=4$ by the transformation $w = z+3+2i$

Soln Given $w = z+3+2i \Rightarrow u+iv = x+iy+3+2i$

$\Rightarrow \quad u = x+3$, $v = y+2$.

(ii) Given $|z|=4$ i.e $\sqrt{x^2+y^2} = 4 \Rightarrow x^2+y^2=16$.

$\Rightarrow (u-3)^2+(v-2)^2 = 16$.

Hence the circle $x^2+y^2=16$ mapped into

$$(u-3)^2+(v-2)^2 = 16.$$

$\therefore$ w-plane is also a circle with centre at $(3,2)$ and radius 4.

Scanned by CamScanner

2) $w = cz$

This transformation is known as magnification and rotation.

Given : $w = cz$ — ①, c, z, w all are complex no's

$c = ae^{i\alpha}$, $z = re^{i\theta}$, $w = Re^{i\phi}$

Put these values in ①, we've.

$$Re^{i\phi} = (ae^{i\alpha})(re^{i\theta}) = are^{i(\theta+\alpha)}.$$

$\Rightarrow R = ar$ and $\phi = \theta + \alpha$.

Thus we see that the transformation $w = cz$ correspond to a rotation, together with magnification.

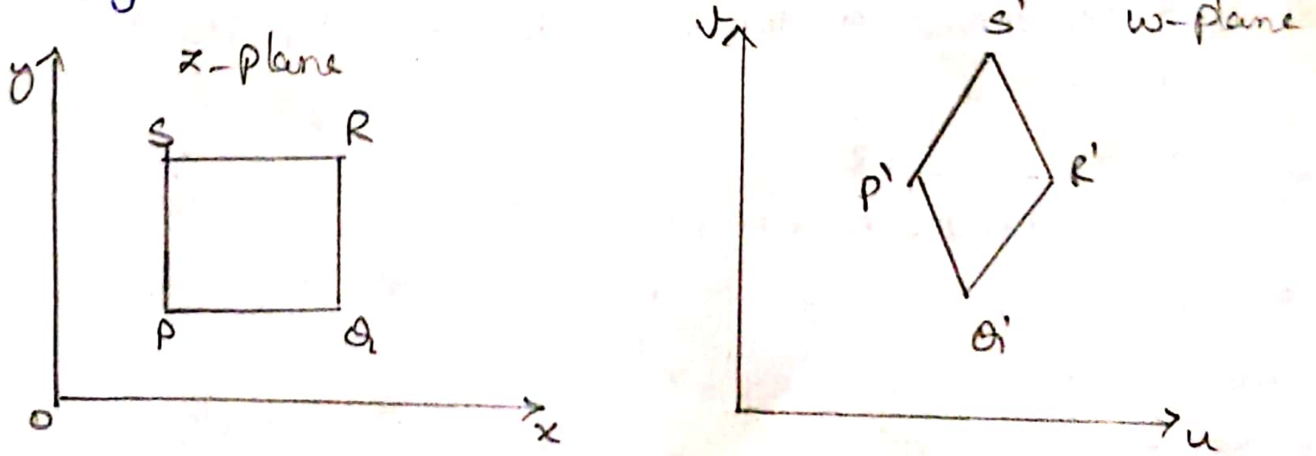
Algebraically, $w = cz$

$$u + iv = (a+ib)(x+iy)$$

$$u + iv = ax + iay + ibx - by.$$

$\Rightarrow u = ax - by$, $v = ay + bx$

on solving these equations we can get the values of x & y.



on putting these values in the equation of the curve to be transformed we get the equation of the image.

1. Determine the region R' of the w-plane into which the triangular region R enclosed by the lines $x=0$, $y=0$, $x+y=2$ is transformed under the transformation $w=z$.

Soln: Given $w=z$.

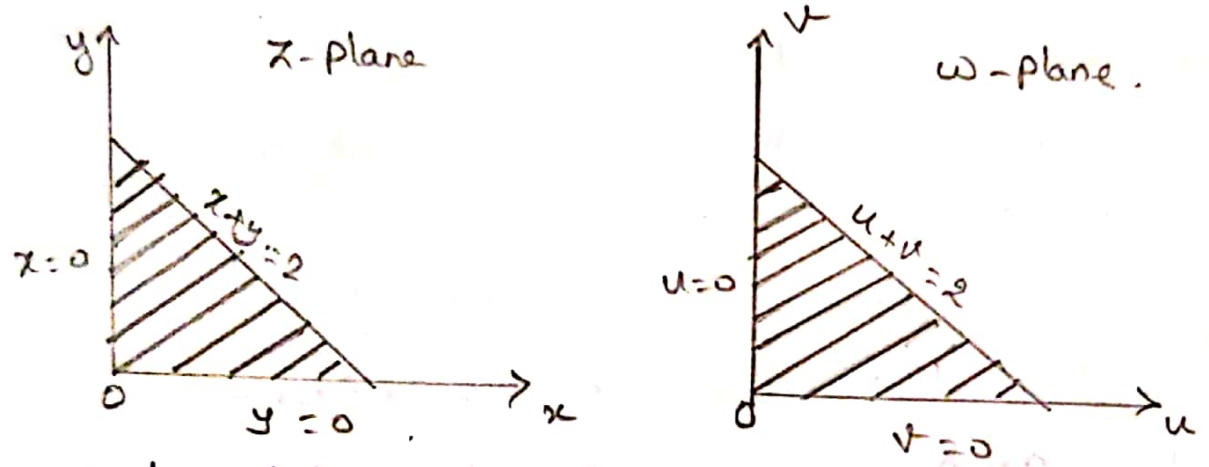
$$u+iv = x+iy.$$

$$\Rightarrow u=x \quad , \quad v=y.$$

when $x=0$, $u=0$

$y=0$ $v=0$.

Now $x+y=2 \Rightarrow u+v=2$.



$\therefore$ The line (R) $x+y=2$ in z-plane is transformed into the line R' $u+v=2$ in the w-plane.

2. Find the image of the circle $|z|=r$ under the transformation $w=5z$.

Soln Given $w=5z$.

$$u+iv = 5(x+iy)$$

$$\Rightarrow u=5x \, , \; v=5y \Rightarrow x=u/5 \, , \; y=v/5 \quad -(1)$$

Given $|z|=r \Rightarrow x^2+y^2=r^2 \quad -(2)$

Substituting (1) in (2), we get $(u/5)^2+(v/5)^2=r^2$.

$$\Rightarrow u^2+v^2=25r^2 \Rightarrow u^2+v^2=(5r)^2.$$

which is a circle in w-plane whose centre is at $(0,0)$ and radius is $5r$.

Hence $|z|=r$ is transformed into $|w|=5r$.

3. Determine the region R' of the w-plane into which the triangular region R enclosed by the lines $x=0$, $y=0$, $x+y=1$ is transformed under the transformation $w = 2z$.

Soln: $w = u+iv$, $z = x+iy$.

Given $w = 2z$.

$$u+iv = 2(x+iy)$$

$$u = 2x, \quad v = 2y.$$

when $x=0$, $u=0$.

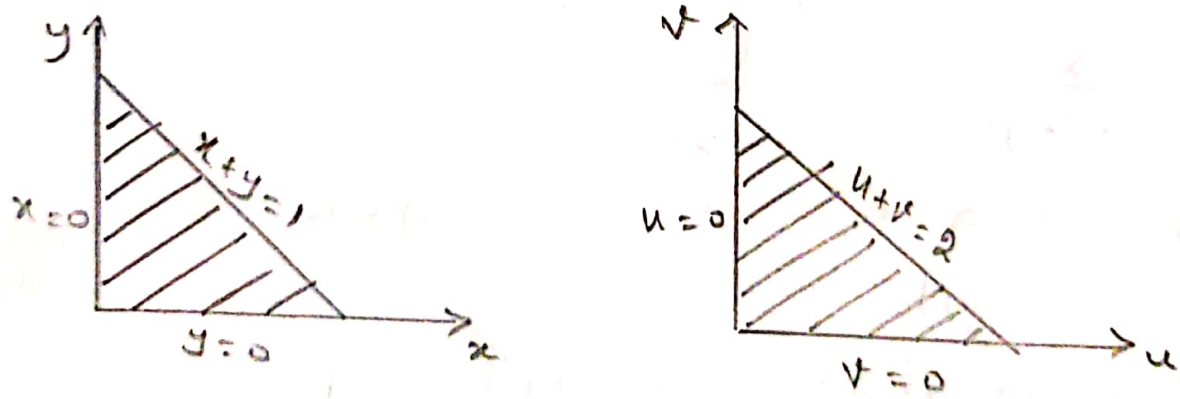
∴ The line $x=0$ is transformed into the line $u=0$ in w-plane.

when $y=0$, $v=0$.

∴ The line $y=0$ is transformed into the line $v=0$ in w-plane.

when $x+y=1$, we get $\frac{u}{2}+\frac{v}{2}=1 \Rightarrow u+v=2$.

∴ The line $x+y=1$ is transformed into the line $u+v=2$ in the w-plane.



3. $w = 1/z$.

Given $w = 1/z$ or $z = 1/w$.

$$z = 1/w \Rightarrow x+iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$\Rightarrow x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

The general equation of the circle in the z-plane is

$x^2+y^2+2gx+2fy+c=0$ — ①

on substituting the values of x and y in ① we get

$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+2g\left(\frac{u}{u^2+v^2}\right)+2f\left(\frac{-v}{u^2+v^2}\right)+c=0$$

ie. $2gu-2fv+1+c(u^2+v^2)=0$.

This is the equation of the circle in w-plane.

This shows that a circle in z-plane transforms to another circle in w-plane.

But a (circle through origin transforms into a straight line).

1.) Find the image of the circle $|z-1|=1$ in the complex plane under the mapping $w=1/z$.

Soln: Given $w=1/z$.

$$u+iv=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}$$

$$\Rightarrow u=\frac{x}{x^2+y^2},\quad v=\frac{-y}{x^2+y^2}$$

The equation of the circle is $|z-1|=1$

ie. $|x+iy-1|=1$.

$$|(x-1)+iy|=1 \Rightarrow (x-1)^2+y^2=1.$$

$$\Rightarrow x^2-2x+1+y^2=1 \Rightarrow x^2+y^2=2x.$$

$$\Rightarrow \frac{x}{x^2+y^2}=\frac{1}{2} \quad \text{ie } u=1/2.$$

$$\Rightarrow 2u-1=0,$$

which is a straight line in w-plane.

(2) Find the image of $|z-2i|=2$ under the mapping $w = 1/z$. QB (14)

Soln: Given $w = 1/z$

$$u+iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$\Rightarrow u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}$$

The given curve is $|z-2i|=2$.

$$|x+iy-2i| = 2.$$

$$\Rightarrow x^2+(y-2)^2 = 4.$$

$$\Rightarrow x^2+y^2-4y+4 = 4.$$

$$\Rightarrow x^2+y^2-4y = 0 \Rightarrow x^2+y^2 = 4y$$

$$\Rightarrow \frac{y}{x^2+y^2} = \frac{1}{4} \quad \text{ie} \quad -v = \frac{1}{4} \Rightarrow -4v-1=0.$$

$\Rightarrow 4v+1=0$, a straight line which is the required image.

3. Find the image of $|z-3|=5$, under the mapping $w = 1/z$.

Soln: Given $w = 1/z$.

$$u+iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$\Rightarrow u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}.$$

The given curve is $|z-3|=5$

$$\cancel{\Rightarrow} |x+iy-3| = 5 \Rightarrow (x-3)^2+y^2 = 25$$

$$\Rightarrow x^2-6x+9+y^2 = 25$$

$$\Rightarrow x^2+y^2-6x-16 = 0.$$

which is the Equation of the Circle in z-plane

$x = \frac{u}{u^2+v^2}$, $y = -\frac{v}{u^2+v^2}$.

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} - 6 \cdot \frac{u}{u^2+v^2} - 16 = 0.$$

$$\frac{u^2+v^2}{(u^2+v^2)^2} - \frac{6u}{u^2+v^2} - 16 = 0.$$

$\Rightarrow 1 - 6u - 16(u^2+v^2) = 0$ which is the Eqn. of the circle in w-plane.

BILINEAR TRANSFORMATION.

Definition: The transformation $w = \frac{az+b}{cz+d}$, where a, b, c, d are complex numbers is called a bilinear transformation. It is also called Mobius or linear fractional transformation. The inverse transformation is $z = \frac{-dw+b}{cw-a}$ which is also bilinear transformation.

CROSS RATIO: If z_1, z_2, z_3, z_4 are 4 points in the z-plane then $\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$ is called cross ratio of these points.

CROSS RATIO PROPERTY OF A BILINEAR TRANSFORMATION:

The bilinear transformation that maps z_1, z_2, z_3 of the z-plane into w_1, w_2, w_3 of the w-plane is

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}.$$

1) Find the bilinear transformation transforming $z = -1, 0, 1$ into $w = 0, i, 3i$.

Soln.

$$\frac{(w-0)(i-3i)}{(0-i)(3i-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

$$\frac{2iw}{i(3i-w)} = \frac{z+1}{1-z}.$$

$$2w(1-z) = (3i-w)(z+1)$$

$$2w - 2wz = 3iz + 3i - wz - w.$$

$$2w + w - 2wz + zw = 3i(z+1)$$

$$3w - wz = 3i(z+1)$$

$$(3-z)w = 3i(z+1)$$

$$w = \frac{-3i(z+1)}{z-3}$$

2) $z = 0, 1, -1;\quad w = i, \infty, 0.$

Soln. $$\frac{(w-w_1)\, w_2\,(1 - w_3/w_2)}{w_2(w_1/w_2 - 1)(w_3 - w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}.$$

$$\frac{(w-i)(1-0)}{(0-1)(0-w)} = \frac{(z-0)(1+1)}{(0-1)(-1-z)} \Rightarrow \frac{w-i}{w} = \frac{2z}{1+z}.$$

$$\Rightarrow (w-i)(1+z) = 2wz.$$

$$\Rightarrow w + wz - i - iz = 2wz.$$

$$\Rightarrow w - wz - i - iz = 0.$$

$$\Rightarrow w(1-z) - i(1+z) = 0$$

$$\Rightarrow w = \frac{i(1+z)}{1-z}.$$

3) Find the bilinear transformation $z = 1, i, -1$ into $w = 0, 1, \infty$. Show that this transformation maps the interior of the unit circle of the z-plane onto the upper half of the w-plane

Soln: $$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}.$$

$$\frac{(w-w_1)\,w_3\left(\frac{w_2}{w_3}-1\right)}{(w_1-w_2)\,w_3\left(1-\frac{w}{w_3}\right)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(0-1)}{(0-1)(1-0)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$w = \frac{(z-1)(i+1)}{(z+1)(i-1)} \times \frac{(-i-1)}{(-i-1)}$$

$$= \frac{(z-1)}{(z+1)} \frac{(1-i-i-1)}{(1-i+i+1)} = \frac{-2i(z-1)}{2(z+1)}$$

$$\Rightarrow w = \frac{i(1-z)}{1+z}.$$

$\Rightarrow w(1+z) = i - iz.$

$\Rightarrow w + wz = i - iz \Rightarrow w + wz - i + iz = 0$

$\Rightarrow (w - i) + z(w+i) = 0.$

$\Rightarrow z(w+i) = -(w-i)$

$$\Rightarrow z = \frac{i-w}{i+w}$$

The interior of the unit circle $|z|=1$ is given by.

$|z|<1 \Rightarrow \left|\frac{i-w}{i+w}\right| < 1 \Rightarrow |i-w| < |i+w|.$

$\Rightarrow |i-u-iv| < |i+u+iv|$, $\quad |x+iy| = \sqrt{x^2+y^2}$

$|-u+i(1-v)| < |u+i(1+v)|$

$\Rightarrow \sqrt{u^2+(1-v)^2} < \sqrt{u^2+(1+v)^2}$

$\Rightarrow \cancel{u^2} + \cancel{1} - 2v + \cancel{v^2} < \cancel{u^2} + \cancel{1} + 2v + \cancel{v^2}.$

$\Rightarrow -4v < 0 \Rightarrow 4v > 0 \Rightarrow v > 0.$

$\therefore |z|<1$ is transformed into $v>0$. Thus the interior ~~trans~~ of the unit circle in the z-plane is transformed into the upper half of the w-plane.

4) Show that the transformation $w = \frac{z-1}{z+1}$ maps the unit circle in the w-plane into the imaginary axis in the z-plane.

Soln: The image of the unit circle $|w|=1$ is given by $\left|\frac{z-1}{z+1}\right|=1. \Rightarrow |z-1| = |z+1|$.

$\Rightarrow |x+iy-1| = |x+iy+1| \Rightarrow (x-1)^2+y^2 = (x+1)^2+y^2$.

$\Rightarrow -2x = 2x$.

$\Rightarrow x=0$ which is imaginary axis.

5) Show that the transformation $w = \frac{1+z}{1-z}$ transforms $|w| \le 1$ into the half plane $Re(z) \le 0$.

Soln. $|w| \le 1$

$\Rightarrow \left|\frac{z+1}{1-z}\right| \le 1 \Rightarrow |1+z| \le |1-z|$.

$\Rightarrow |x+iy+1| \le |1-x-iy|$

$\Rightarrow (x+1)^2+y^2 \le (1-x)^2+y^2$.

$\Rightarrow 2x \le -2x \Rightarrow 4x \le 0$.

$\Rightarrow x \le 0 \Rightarrow Re(z) \le 0$.

Thus the transformation $w = 1+z/1-z$ transforms $|w| \le 1$ into the half plane $Re(z) \le 0$.

6) Find the bilinear transformation which maps the points $0, 1, \infty$ of the z-plane into the points $i, 1, -i$ of the w-plane.

Soln: $$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)z_3\left(\frac{z_2}{z_3}-1\right)}{(z_1-z_2)z_3\left(1-\frac{z}{z_3}\right)}$$

$$\frac{(w-i)(1+i)}{(i-1)(-i-w)} = \frac{(z-0)(0-1)}{(0-1)(1-0)}$$

$$\Rightarrow \frac{(w-i)(1+i)}{(-i-w)(i-1)} = \frac{z}{1} \Rightarrow \frac{w-i}{-i-w} = \frac{z(i-1)}{1+i} = \frac{iz-z}{1+i}$$

Using Componendo & Dividendo.

$$\frac{a}{b} = \frac{c}{d} \text{ then } \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

$$\frac{(w-i)+(-i-w)}{(w-i)-(-i-w)} = \frac{(iz-z)+(1+i)}{(iz-z)-(1+i)}$$

$$-\frac{1}{i} = i$$

$$\frac{-\cancel{2}i}{\cancel{2}w} = \frac{z(i-1)+(1+i)}{z(i-1)-(1+i)}$$

$$w = \frac{-i[z(i-1)-(1+i)]}{z(i-1)+(1+i)}.$$

$$w = \frac{z(1+i)+(i-1)}{z(i-1)+(1+i)}.$$

$$= \frac{z(1+i)+i(1+i)}{zi(1+i)+(1+i)} = \frac{(1+i)(z+i)}{(1+i)(iz+1)}.$$

$$\Rightarrow w = \frac{z+i}{iz+1}$$

H/W.

1) $z = 1, 0, -1$; $w = 0, -1, \infty$ soln: $w = z-1/z+1$

2) $z = 0, 1, \infty$; $w = -5, -1, 3$ soln: $w = 3z-5/z+1$

3) $z = i, -1, 1$; $w = 0, 1, \infty$ soln: $w = (0-i)(z-i)/z-1$

4) $z = -i, 0, i$; $w = -1, i, 1$ soln: $w = -(z-1)/z+1$ (or) $1-z/z+1$.

5) Show that the transformation $w = z-1/z+1$ transforms $|w| \leq 1$ into the half plane $Re(z) \geq 0$.

1. $v = xy$ soln: $f(z) = z^2/2 + c$.

2. $v = 3x^2y - y^3$ soln: $f(z) = z^3 + c$.

unit - IV.

COMPLEX INTEGRATION

CONTOUR :

A continuous curve made up of a finite no: of arcs is called a contour.

(CAUCHY INTEGRAL FORMULA : QB(2)

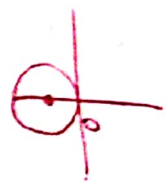
If $f(z)$ is analytic within and on closed contour c and a is the point within c then

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}\,dz$$ where integration around c is in the positive sense)

Problems,

1) Using Cauchy's integral formula, evaluate $\int_c \frac{3z^2+7z+1}{z+1}\,dz$ where c is the circle $|z+1| = 1$

soln: $|z+1| = 1 \Rightarrow |z-(-1)| = 1$ is the circle with centre -1 and radius 1.



Consider the function $f(z) = \frac{3z^2+7z+1}{z+1}$

$z+1 = 0 \Rightarrow z = -1$

The above function is analytic at all points except at $z=-1$ which lies inside c.

By Cauchy's integral formula,

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a) \quad \text{--- (1)}$$

Take $f(z) = 3z^2+7z+1$.

We see that $f(z)$ is analytic on and inside c, with $z=-1$ inside c. and $a=-1$.

Then $f(a) = f(-1) = 3(-1)^2+7(-1)+1 = -3$.

Substituting these values in (1) we get.

$$\int \frac{3z^2+7z+1}{z+1}\,dz = 2\pi i \times (-3) = -6\pi i$$

2) Using Cauchy Integral formula, Evaluate $\int_c \frac{3z^2+z}{z^2-1}\,dz$ where c is the circle $|z-1|=1$

Soln: $|z-1|=1$ is the circle with centre 1 and radius 1

consider the function $f(z) = \frac{3z^2+z}{z^2-1} = \frac{3z^2+z}{(z-1)(z+1)}$

$z^2-1=0 \Rightarrow (z+1)(z-1)=0 \Rightarrow z=1,-1.$

The above function is analytic at all points except at $z=1$ which lies inside c and $z=-1$ which lies outside c.

By Cauchy's Integral formula.

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a) \quad - \text{①}$$

Take $f(z) = \frac{3z^2+z}{z+1}$.

we see that $f(z)$ is analytic on and inside c with $z=1$ inside c.

$a=1$. Then $f(a) = f(1) = \frac{3(1)^2+1}{1+1} = \frac{4}{2} = 2.$

Substituting these values in ①, we get

$$\int_c \frac{3z^2+z}{z+1} \Big/ z-1 \; dz = 2\pi i\,(2) = 4\pi i$$

$$\therefore \int_c \frac{3z^2+z}{z^2-1}\,dz = 4\pi i$$

3) Using Cauchy's integral formula Evaluate $\int_c \frac{z+4}{z^2+2z+5}\,dz$ where c is the circle $|z+1-i|=2$.

Soln: $|z+1-i|=2 \Rightarrow |z-(-1+i)|=2$ is the circle with centre at $-1+i$ and radius 2.

Consider the function $f(z) = \frac{z+4}{z^2+2z+5}$

$z^2+2z+5=0 \Rightarrow z = \frac{-2\pm\sqrt{4-20}}{2} = -1\pm 2i$

$\Rightarrow z = -1+2i$, $z = -1-2i$

$\therefore \dfrac{z+4}{z^2+2z+5} = \dfrac{z+4}{[z-(-1+2i)][z-(-1-2i)]}$.

The above function is analytic at all points except at $z = -1+2i$ which lies inside c and $z = -1-2i$ which lies outside c.

$\therefore$ By Cauchy's Integral formula:

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a) \quad -(1)$$

Take $f(z) = \dfrac{z+4}{z-(-1-2i)} = \dfrac{z+4}{z+1+2i}$

We see that $f(z)$ is analytic on and inside c with $z = -1+2i$ inside c.

$a = -1+2i$. Then $f(a) = f(-1+2i) = \dfrac{-1+2i+4}{-1+2i+1+2i} = \dfrac{2i+3}{4i}$

Substituting these values in (1) we get.

$$\int_c \frac{\left(\frac{z+4}{z-(-1-2i)}\right)}{z-(-1+2i)}\,dz = 2\pi i \cdot \frac{2i+3}{4i} = \frac{\pi}{2}(2i+3)$$

$$\therefore \int_c \frac{z+4}{z^2+2z+5}\,dz = \pi/_2\,(2i+3).$$

QB (11)(i)

4) Evaluate $\int_c \dfrac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz$ where c is $|z|=3$ using Cauchy integral formula.

Soln: $|z|=3$ is a circle with centre at the origin and radius 3 units

Consider $\dfrac{1}{(z-1)(z-2)} = \dfrac{A}{z-1} + \dfrac{B}{z-2}$

$$\Rightarrow \quad 1 = A(z-2) + B(z-1).$$

put $z=1 \Rightarrow A=-1$

put $z=2 \Rightarrow B=1.$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}.$$

$$\therefore \int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz = -\int_C \frac{\sin\pi z^2+\cos\pi z^2}{z-1}\,dz + \int_C \frac{\sin\pi z^2+\cos\pi z^2}{z-2}\,dz$$

Since $z=1$ and $z=2$ lies inside c and $f(z)=\sin\pi z^2+\cos\pi z^2$ is analytic on and inside c.

By Cauchy Integral formula:

$$\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}\,dz = -2\pi i \times f(1) + 2\pi i \times f(2)$$

$$= -2\pi i(\sin\pi+\cos\pi) + 2\pi i(\sin 4\pi + \cos 4\pi)$$

$$= -2\pi i(0-1) + 2\pi i(0+1)$$

$$= 2\pi i + 2\pi i = 4\pi i$$

Derivatives of an Analytic Function At Interior Points.

Since $f(z)$ is analytic and a is any interior points.

we've $f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz.$

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz.$$

$$f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-a)^4}\,dz$$

$$\vdots$$

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz.$$

$$\Rightarrow \int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!}f^n(a).$$

1) Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz$ where C is the circle $|z-i| = 2$.

Soln: $|z-i| = 2$ is the circle with centre at i and radius 2 units.

Consider the function $f(z) = \frac{z-1}{(z+1)^2(z-2)}$

$(z+1)^2(z-2) = 0 \Rightarrow z = -1, 2.$

The above function is analytic at all points except at $z=-1$ which lies inside C and $z=2$ which lies outside C.

By Cauchy's integral formula for derivatives

$$\int_C \frac{f(z)}{(z-a)^2}\,dz = 2\pi i\, f'(a) \quad - (1)$$

$|-1-i| = \sqrt{1+1} = \sqrt{2}$

$|2-i| = \sqrt{4+1} = \sqrt{5}$

Take $f(z) = \frac{z-1}{z-2}$.

We see that $f(z)$ is analytic on and inside C with $z=-1$ inside C.

Take $a=-1$ Then $f'(-1) = -1/9$.

$f(z) = \frac{z-1}{z-2}$.

$f'(z) = \frac{(z-2)-(z-1)}{(z-2)^2}$.

$f'(-1) = -1/9$

Substituting these values in (1) we get

$$\therefore \int_C \frac{\left(\frac{z-1}{z-2}\right)}{(z+1)^2}\,dz = \frac{2\pi i}{1!}\left(-\frac{1}{9}\right) = -\frac{2\pi i}{9}$$

2) Evaluate $\int_C \frac{e^{2z}}{(z+1)^4}\,dz$, where C is $|z| = 2$ using C.I.F

Soln: $|z| = 2$ is the circle with centre at origin and radius 2 units.

Here $f(z) = e^{2z}$.

Clearly, $z=-1$ lies inside C.

By Cauchy Integral formula for derivatives.

$$\int_c \frac{f(z)}{(z-a)^4}\,dz = 2\pi i\,\frac{f'''(a)}{3!}.$$

$f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}.$

$\Rightarrow f'''(-1) = 8e^{-2}.$

$$\therefore \int_c \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i \cdot 8e^{-2}}{6} = \frac{8\pi i e^{-2}}{3}$$

H/W

1) $\int_c \frac{z+1}{z^2+2z+4}\,dz$, $|z+1+i| = 2$. soln: πi

2) $\int_c \frac{\cos\pi z^2}{(z-1)(z-2)}\,dz$, $|z| = 3$ soln: $4\pi i$

3) $\int_c \frac{ze^z}{(z-2)^3}\,dz$, where $z=2$ lies inside the closed curve C.

LAURENT'S SERIES.

If a function $f(z)$ is analytic in the annulus between two concentric circles C and C' with centre at $z=a$ and radius R, R', $\boxed{R' < R}$ then at any point in the annulus.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=0}^{\infty} b_n (z-a)^{-n}.$$

where $a_n = \frac{1}{2\pi i}\int_c \frac{f(z')}{(z'-a)^{n+1}}\,dz'$ and

$$b_n = \frac{1}{2\pi i}\int_{c'} \frac{f(z')}{(z'-a)^{-n+1}}\,dz'$$

Note: $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called integral part and $\sum_{n=0}^{\infty} b_n (z-a)^{-n}$ is called principal part of the Laurent's series

1) Find the Laurent's series of $f(z) = \dfrac{1}{(z-1)(z-2)}$ in $|z| > 2$.

Soln: $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{A}{z-1} + \dfrac{B}{z-2}$.

$1 = A(z-2) + B(z-1)$. put $z = 2 \Rightarrow B = 1$.
$z = 1 \Rightarrow A = -1$

$\therefore f(z) = \dfrac{-1}{z-1} + \dfrac{1}{z-2}$.

Region $|z| > 2$: for all z in the region $|z| > 2$, z is such that $1 < |z|$, $2 < |z|$.

$\therefore \left|\dfrac{1}{z}\right| < 1$, $\left|\dfrac{2}{z}\right| < 1$.

So rewriting $f(z)$ as one which is Expandable as binomial series,

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2} = \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{z\left(1-\frac{2}{z}\right)}.$$

$$= \frac{-1}{z}\left[1-\frac{1}{z}\right]^{-1} + \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}.$$

$$= -\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right] + \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right]$$

$$f(z) = -\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right] + \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right]$$

The first series is valid for $\left|\frac{1}{z}\right| < 1$ ie $1 < |z|$

The second series is valid for $\left|\frac{2}{z}\right| < 1$ ie $2 < |z|$

$\therefore$ The whole Expansion is valid when $|z| > 2$.

2) Find the Laurent's Expansion of $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ in $2 < |z| < 3$.

Soln: $f(z) = \dfrac{z^2-1}{(z+2)(z+3)} = A + \dfrac{B}{z+2} + \dfrac{C}{z+3}$.

$z^2 - 1 = A(z+2)(z+3) + B(z+3) + C(z+2)$

put $z=-3$, $C=-8$ & $z=-2$, $B=3$.

Equating the Coefficients of z^2, $A=1$.

$\therefore$ $f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$.

Region: $2<|z|<3$ for all z in the region $2<|z|<3$, z is such that $2<|z|$, $|z|<3$

$\left|\frac{2}{z}\right|<1$, $\left|\frac{z}{3}\right|<1$.

So rewriting $f(z)$ as one which is Expandable as binomial series.

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}.$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$f(z) = 1 + \frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2 \cdots\right] - \frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2 \cdots\right]$$

In the above expansion, the first series is valid when $\left|\frac{2}{z}\right|<1$ ie $|2|<|z|$.

The second series is valid when $\left|\frac{z}{3}\right|<1$ ie $|z|<3$.

$\therefore$ The whole Expansion is valid when $2<|z|<3$.

3) Find the Laurent's Expansion of the function $f(z) = \frac{7z-2}{z(z+1)(z-2)}$ in the annulus $1<|z+1|<3$.

Soln: Region $1<|z+1|<3$. for all z in the region $1<|z+1|<3$.

setting $z+1=u \Rightarrow 1<|u|<3$.

$1 < |u|$, $|u| < 3$.

$\left|\frac{1}{u}\right| < 1$ $\left|\frac{u}{3}\right| < 1$.

$$\therefore f(z) = \frac{7(u-1)-2}{(u-1)(u-1+1)(u-1-2)} = \frac{7u-9}{u(u-1)(u-3)}$$

$$\frac{7u-9}{u(u-1)(u-3)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u-3}.$$

$7u-9 = A(u-1)(u-3) + Bu(u-3) + Cu(u-1)$.

put $u=0 \Rightarrow -9 = 3A. \Rightarrow A = -3$.

put $u=1 \Rightarrow -2 = -2B \Rightarrow B = 1$

put $u=3 \Rightarrow 12 = 6C \Rightarrow C = 2$.

$$f(z) = \frac{-3}{u} + \frac{1}{u-1} + \frac{2}{u-3}.$$

$$= -\frac{3}{u} + \frac{1}{u\left(1-\frac{1}{u}\right)} + \frac{2}{-3\left(1-\frac{u}{3}\right)}$$

$$= -\frac{3}{u} + \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$= -\frac{3}{u} + \frac{1}{u}\left(1+\frac{1}{u}+\frac{1}{u^2}+\cdots\right) - \frac{2}{3}\left(1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\cdots\right)$$

$$f(z) = -\frac{3}{u} + \left(\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\cdots\right) - \frac{2}{3}\left(1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\cdots\right)$$

The first series is valid for $\left|\frac{1}{u}\right| < 1$ ie $1 < |u|$.

The second series is valid for $\left|\frac{u}{3}\right| < 1$ ie $|u| < 3$.

$\therefore$ The whole Expansion is valid when $1 < |u| < 3$.

$$\therefore f(z) = \left[\frac{-2}{z+1} + \frac{1}{(z+1)^2} + \cdots\right] - \frac{2}{3}\left[1 + \frac{z+1}{3} + \left(\frac{z+1}{3}\right)^2 + \cdots\right]$$

is valid in the region $1 < |z+1| < 3$.

<u>H/W</u>. (1) Q.B 11(i) Expand $f(z) = \frac{1}{z^2-3z+2}$ in L.S valid in $1 \le |z| \le 2$.

2) $f(z) = \frac{z^2-1}{z^2+5z+6}$, $2 < |z| < 3$ 3) $f(z) = \frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$, $3 < |z+2| < 5$

POLES AND RESIDUES.

POLES: A pt. $z=a$ is said to be a pole of $f(z)$ of order n if we can find a +ve int. $\ni$ $\lim_{z\to a}(z-a)^n f(z) \neq 0$

Consider the function.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=0}^{\infty} b_n (z-a)^{-n}$$

If there are finite number of term in the principal part i.e if the principal part is $b_1(z-a)^{-1} + b_2(z-a)^{-2} + \dots + b_m(z-a)^{-m}$ then $z=a$ is a pole of order m.

If $m=1$, i.e if there is only one term in the principal part then $z=a$ is a simple pole of $f(z)$ or pole of order one.

Calculation of Residues:

1.) Let $z=a$ be a simple pole of $f(z)$. Then residue of $f(z)$ at $z=a$ is $\lim_{z\to a}(z-a)f(z)$.

2) If $f(z) = \frac{\phi(z)}{\psi(z)}$, $\phi(a) \neq 0$ and $\psi(a) = 0$ then residue of $f(z)$ at $z=a$ is $\frac{\phi(a)}{\psi'(a)}$.

3) Let $z=a$ be a pole of order m then residue of $f(z)$ at $z=a$ is $\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)$.

1.) Find the poles and residues of $f(z) = \frac{z}{z^2-3z+2}$

Soln: $f(z) = \frac{z}{z^2-3z+2}$

To find the poles put $Dr=0$.

ie $z^2 - 3z + 2 = 0$.

$(z-1)(z-2) = 0 \Rightarrow z = 1, z = 2$ are 2 simple poles of $f(z)$.

Residues:

Res. of $f(z)$ at $z=1$ $= \lim_{z \to 1} (z-1) \cdot \dfrac{z}{(z-1)(z-2)}$

$= \dfrac{1}{1-2} = -1$.

Res. of $f(z)$ at $z=2$ $= \lim_{z \to 2} (z-2) \cdot \dfrac{z}{(z-1)(z-2)}$

$= \dfrac{2}{2-1} = 2$.

2) Find the poles and residues of $f(z) = \dfrac{z^3}{(z-a)^2}$

Soln: $f(z) = \dfrac{z^3}{(z-a)^2}$

To find the poles put $Dr = 0$.

$(z-a)^2 = 0 \Rightarrow z = a$ is a pole of order 2.

Residues:

Res. of $f(z)$ at $z=a$ is $= \dfrac{1}{(2-1)!} \lim_{z \to a} \dfrac{d}{dz^{2-1}} (z-a)^2 \cdot \dfrac{z^3}{(z-a)^2}$

$= \lim_{z \to a} \dfrac{d}{dz}(z^3) = \lim_{z \to a} 3z^2 = 3a^2$.

3) Find the poles and residues of $f(z) = \cot z$.

Soln: $f(z) = \cot z = \dfrac{\cos z}{\sin z}$

This is of the form $f(z) = \dfrac{\phi(z)}{\psi(z)}$

poles, $\sin z = 0 \Rightarrow z = n\pi$.

$\phi(a) = \cos n\pi \neq 0$.

$\psi(a) = \sin n\pi = 0$

$\therefore$ Res of $f(z) = \dfrac{\cos z}{\frac{d}{dz}(\sin z)} = \dfrac{\cos z}{\cos z} = 1.$

4) Find the pole and residue of $f(z) = \dfrac{ze^z}{(z-1)^3}$

Soln: $z=1$ is a pole of order 3 for $f(z) = \dfrac{ze^z}{(z-1)^3}$

Res. of $f(z)$ at $z=1$ $= \dfrac{1}{(3-1)!} \lim\limits_{z\to 1} \dfrac{d^2}{dz^{3-1}} (z-1)^3 \cdot \dfrac{ze^z}{(z-1)^3}$

$$= \frac{1}{2!} \lim_{z\to 1} \frac{d^2}{dz^2} ze^z$$

$$= \frac{1}{2!} \lim_{z\to 1} \frac{d}{dz} (ze^z + e^z).$$

$$= \frac{1}{2} \lim_{z\to 1} (ze^z + e^z + e^z). = \frac{1}{2}(e+e+e) = \frac{3e}{2}.$$

5) Find the residue and pole of $f(z) = \dfrac{e^z}{z}$.

Soln: $\dfrac{e^z}{z}$ has a simple pole of at $z=0$.

Res. of $f(z)$ at $z=0$ $= \lim\limits_{z\to 0} (z-0) \cdot \dfrac{e^z}{z}$.

$$= \lim_{z\to 0} \frac{ze^z}{z} = e^0 = 1.$$

6) Find the poles and residues of $f(z) = \dfrac{\sin z}{z^2}$

Soln: $\dfrac{\sin z}{z^2}$ has a double pole at $z=0$.

Res. of $f(z)$ at $z=0$ $= \dfrac{1}{1!} \lim\limits_{z\to 0} \dfrac{d}{dz}\left(z^2 \cdot \dfrac{\sin z}{z^2}\right)$

$$= \lim_{z\to 0} \frac{d}{dz}(\sin z).$$

$$= \lim_{z\to 0} (\cos z) = 1.$$

CAUCHY'S RESIDUE THEOREM Q.B 9

Let $f(z)$ be single valued and analytic within and on a closed contour C expect at a finite number of poles $z_1, z_2, \ldots z_n$ and let $R_1, R_2, \ldots, R_n$ be resply the residues of $f(z)$ of these poles then

$$\int_C f(z)\,dz = 2\pi i \text{ (Sum of the residues of } f(z) \text{ at the poles within } C)$$

Q.B 14

1) Using Cauchy's Residue theorem Evaluate $\int_C \frac{3z^2+2}{(z-1)(z^2+9)}dz$ where C is $|z-2| = 2$.

Soln: $|z-2| = 2$ is the circle with centre at 2 and radius 2 units.

$(z-1)(z^2+9) = 0 \Rightarrow z = 1, \pm 3i$

$z=1$ is a simple pole and $z = \pm 3i$ are two simple poles.

$\therefore z=1$ is the only pole which lies inside C.

Res. of $f(z)$ at $z=1$ is $\lim_{z \to 1} (z-1) \cdot \frac{3z^2+2}{(z-1)(z^2+9)} = \frac{3+2}{1+9} = \frac{1}{2}$.

By Cauchy's Residue theorem.

$$\int_C f(z)\,dz = 2\pi i \text{ (Sum of the residues of } f(z) \text{ at the poles which lies inside } C)$$

$$= 2\pi i \cdot \frac{1}{2} = \pi i$$

$$\int_C \frac{3z^2+2}{(z-1)(z^2+9)}dz = \pi i$$

2) Determine poles and residues of $f(z) = \frac{z}{(1-z)^2(z+2)}$ and hence Evaluate $\int_C f(z)\,dz$ where c is the curve $|z| = 5/2$.

Soln $|z| = 5/2$ is the circle with centre at origin and radius $5/2$ units.

$(1-z)^2(z+2) = 0 \Rightarrow z=1$ is a pole of order 2.
and $z=-2$ is a simple pole.
poles $z=1, z=-2$ are lying inside c.

Res. of $f(z)$ at $z=1 = \dfrac{1}{(2-1)!} \lim_{z\to 1} \dfrac{d^{2-1}}{dz^{2-1}} (1-z)^2 \cdot \dfrac{z}{(1-z)^2(z+2)}$

$$= \lim_{z\to 1} \frac{d}{dz} (1-z)^2 \cdot \frac{z}{(1-z)^2(z+2)}$$

$$= \lim_{z\to 1} \frac{(z+2)-z}{(z+2)^2} = \frac{3-1}{9} = \frac{2}{9}$$

Res. of $f(z)$ at $z=-2 = \lim_{z\to -2} (z+2) \cdot \dfrac{z}{(1-z)^2(z+2)}$

$$= \frac{-2}{(1+2)^2} = \frac{-2}{9}.$$

$\therefore$ By Cauchy's Residue theorem.

$\int_C f(z)\,dz = 2\pi i$ (sum of the residues of $f(z)$ at poles within C)

$$= 2\pi i \left(\frac{2}{9} - \frac{2}{9}\right) = 0.$$

3) Evaluate $\int_C \dfrac{e^z \cos z}{z}\,dz$ where C is $|z|=1$.

Soln $|z|=1$ is a circle with centre at origin and radius 1 unit.

$\dfrac{e^z \cos z}{z}$ has a simple pole at $z=0$ which is inside the circle $|z|=1$.

Res. of $f(z)$ at $z=0 = \lim_{z\to 0} z \cdot \dfrac{e^z \cos z}{z} = \lim_{z\to 0} e^z \cos z = e^0 \cos 0 = 1$

By Cauchy's residue theorem,

$\int_C \dfrac{e^z \cos z}{z}\,dz = 2\pi i \times$ residue at $z=0$.

$$= 2\pi i \times 1 = 2\pi i$$

4) Evaluate $\int_C \frac{e^{-z}}{z^2}\,dz$ where c is the circle $x^2+y^2=4$

Soln: $x^2+y^2=4$ is a circle with centre at $(0,0)$ and radius 2 units.

$\frac{e^{-z}}{z^2}$ has a double pole at $z=0$ which is inside c.

$$\text{Res of } f(z) \text{ at } z=0 = \frac{1}{1!} \lim_{z\to 0} \frac{d}{dz}\left(z^2 \cdot \frac{e^{-z}}{z^2}\right).$$

$$= \lim_{z\to 0} \frac{d}{dz}\left(e^{-z}\right) = \lim_{z\to 0}\left(-e^{-z}\right) = -1$$

$\therefore$ By Cauchy's residue theorem.

$$\int_C \frac{e^{-z}}{z^2}\,dz = 2\pi i \times (-1) = -2\pi i.$$

5) Evaluate $\int_C f(z)\,dz$, $f(z) = \frac{z-3}{(z+1)^2(z-2)}$ c is the circle $|z-i| = 2$.

6) Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$ where c is the circle $|z| = 3/2$

TAYLOR'S SERIES.

Let $f(z)$ be analytic at all points within a circle c with centre at a and radius R. Then at each point z inside c.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ where } a_n = \frac{f^n(a)}{n!}$$

$$= f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots$$

This is known as Taylor's Series for the function $f(z)$.

1.) Expand $\log(1+z)$ as a Taylor's Series about $z=0$.

Soln: $f(z) = \log(1+z)$ $\qquad f(0)=0$

$f'(z) = \dfrac{1}{1+z}$ $\qquad f'(0)=1$

$f''(z) = \dfrac{-1}{(1+z)^2}$ $\qquad f''(0) = -1$

$f'''(z) = \dfrac{2}{(1+z)^3}$ $\qquad f'''(0)=2$

$f^{IV}(z) = \dfrac{-6}{(1+z)^4}$ $\qquad f^{IV}(0) = 6$.

So the Taylor's Series for $\log(1+z)$ about $z=0$ is

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

2) Find the Taylor's series for $1/z$ about $z=1$.

Soln: $f(z) = 1/z$ $\qquad f(1)=1$.

$f'(z) = -1/z^2$ $\qquad f'(1) = -1$

$f''(z) = 2/z^3$ $\qquad f'''(1) = 2$

$f'''(z) = -6/z^4$ $\qquad f^{IV}(1) = -6$.

The required Taylor's series about $z=1$ is

$$f(z) = f(1) + \frac{f'(1)}{1!}(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \cdots$$

$$\frac{1}{z} = 1 + (-1)(z-1) + \frac{2}{2!}(z-1)^2 + \frac{(-6)}{3!}(z-1)^3 + \cdots$$

$$\frac{1}{z} = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots$$

QB.5 $\qquad$ sinz

3) Expand cos z in a Taylor's Series about $z=0$.

Soln: $f(z) = \cos z$ sin z $\qquad f(0) = 1$ 0

$f'(z) = -\sin z$ · cos z $\qquad f'(0) = 0$ 1.

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$f''(z) = -\cos z$ $-\sin z$ $f''(0) = -1$ 0

$f'''(z) = \sin z$ $-\cos z$ $f'''(0) = 0$ -1

$f^{IV}(z) = \cos z$ $\sin z$ $f^{IV}(0) = 1.$ 0

$$f(z) = 0 + \frac{1}{1!} z - \frac{1}{3!} z^3 + \cdots$$
$$= \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

So the Taylor's series for $\cos z$ about $z=0$ is

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \cdots$$

$$\cos z = 1 + 0 \cdot z - \frac{1}{2!} z^2 + \frac{0}{3!} z^3 + \frac{1}{4!} z^4 + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

4) Expand e^z in a Taylor's series about $z=0$.

Soln: $f(z) = e^z$ $f(0) = e^0 = 1$

$f'(z) = e^z$ $f'(0) = 1$

$f''(z) = e^z$ $f''(0) = 1$

$f'''(z) = e^z$ $f'''(0) = 1$

$f^{IV}(z) = e^z$ $f^{IV}(0) = 1$

So the Taylor's series for e^z about $z=0$ is

$$f(z) = f(0) + \frac{f'(0)}{1!}(z-0) + \frac{f''(0)}{2!}(z-0)^2 + \frac{f'''(0)}{3!} z^3 + \cdots$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

CONTOUR INTEGRATION

Integration Around the Unit Circle.

An integral of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$ where the integrand is a rational function of $\cos\theta$ and $\sin\theta$ can be evaluated by putting $e^{i\theta} = z$.

Then $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$.

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right)$.

Also $e^{i\theta} i\,d\theta = dz \Rightarrow d\theta = \frac{dz}{iz}$

Hence $\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta = \int_C F(z)\,dz$.

where $F(z)$ is a rational function of z and C is the unit circle $|z|=1$.

But By Cauchy's Residue theorem.

$$\int_C F(z)\,dz = 2\pi i \,\Sigma R.$$

where ΣR denotes the sum of the residues of $F(z)$ at its poles inside C.

QB(15)

1. Using Method of contour integration evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

Soln: Let $z=e^{i\theta}$ Then $dz = e^{i\theta}\cdot i\,d\theta$.

$\Rightarrow d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{z+1/z}{2}$

$\therefore \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_C \frac{dz/iz}{2+\left(\frac{z+1/z}{2}\right)}$ where c is the unit circle $|z|=1$

$$= \int_C \frac{dz/iz}{2+\frac{1}{2}\left(\frac{z^2+1}{z}\right)} = \int_C \frac{dz/iz}{\frac{4z+z^2+1}{2z}}$$

$$= \int_C \frac{dz}{iz} \times \frac{2z}{z^2+4z+1} = \frac{2}{i}\int_C \frac{dz}{z^2+4z+1}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\int_C \frac{dz}{z^2+4z+1} = \frac{2}{i}\int_C f(z)\,dz$$

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$\therefore$ By Cauchy's Residue Theorem

$= \frac{2}{i} \cdot 2\pi i$ (sum of the residue of $f(z)$ at its poles inside C)

$= 4\pi$ (sum of the residue of $f(z)$ at its poles inside C)

The poles of $f(z)$ are given by.

$z^2+4z+1=0$.

$\Rightarrow z = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$.

$\Rightarrow z = -2+\sqrt{3}$ & $z = -2-\sqrt{3}$. $\qquad \sqrt{3} = 1.732$

$\alpha = -2+\sqrt{3}$ & $\beta = -2-\sqrt{3}$.

$z=\alpha$ lies inside c.

$\therefore$ Residue of $f(z)$ at $z=\alpha = \lim_{z\to\alpha} (z-\alpha) \cdot \frac{1}{(z-\alpha)(z-\beta)}$

$= \lim_{z\to\alpha} \frac{1}{z-\beta} = \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{2\sqrt{3}}$

$\therefore \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = 4\pi \cdot \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$.

2) Using Contour Integration, Evaluate $\int_0^{2\pi} \frac{d\theta}{5-4\sin\theta}$

Soln: Let $z = e^{i\theta}$, $0 \le \theta \le 2\pi$

Then $dz = e^{i\theta} \cdot i\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$ & $\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$.

$\therefore \int_0^{2\pi} \frac{d\theta}{5-4\sin\theta} = \int_c \frac{dz/iz}{5-4\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]} = \int_c \frac{dz/iz}{5-\cancel{4}^{2}\left[\frac{1}{\cancel{2}iz}(z^2-1)\right]}$

$= \int_c \frac{dz/iz}{\frac{5iz-2z^2+2}{iz}} = \int_c \frac{dz}{iz} \times \frac{iz}{-2z^2+5iz+2}$

$= \int_c \frac{dz}{-2z^2+5iz+2} = -\int_c \frac{dz}{(z-2i)(z+2i)}$

$$\therefore \int_0^{2\pi} \frac{d\omega}{5-4\sin\omega} = -\cancel{\frac{1}{2}} \int_C \frac{dz}{(z-2i)(z-\frac{i}{2})}$$

$\therefore$ By Cauchy's Residue Theorem.

$$= -\cancel{\frac{1}{2}} \left[2\pi i \text{ (Sum of the residues of } f(z) \text{ at its poles inside } C)\right]$$

The poles are given by $(z-2i)(2z-i) = 0$.

$$\Rightarrow z = 2i, \; z = i/2$$

$$\alpha = 2i, \; \beta = i/2.$$

$z = \beta$ is a simple pole which lies inside C.

Res of $f(z)$ at $z=\beta$ is $= \lim_{z \to \beta} (z-\beta) \cdot \dfrac{1}{(z-\alpha)(z-\beta)}$

$$= \frac{1}{\beta - \alpha} = \frac{1}{\frac{i}{2} - 2i} = \frac{1}{\frac{i-4i}{2}} = \frac{2}{-3i}$$

$$\therefore \int_0^{2\pi} \frac{d\omega}{5-4\sin\omega} = -2\pi\cancel{i} \left(\frac{2}{-3\cancel{i}}\right) = \frac{\cancel{2}\pi}{2} = \frac{4\pi}{3}$$

Evaluation of $\int_{-\infty}^{\infty} f(x)\,dx$ (No Poles on the Real Axis)

If the function $f(z)$ is such that it has no poles on the real axis and possibly has some poles in the upper half of the z-plane then we can evaluate $\int_{-\infty}^{\infty} f(x)\,dx$ by considering the integral of $f(z)$ taken around a closed contour C consisting of the upper half of the circle $|z| = R$, and part of the real axis from $-R$ to R provided that the integral round the semi circle C_R tends to a limit as $R \to \infty$.

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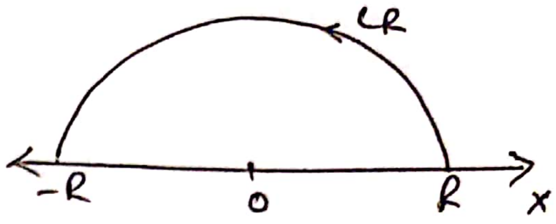
Cauchy's Lemma :

If $f(z)$ is continuous and $\lim_{z\to\infty} z f(z) \to 0$ Then

$\int_{C_R} f(z)dz \to 0$ as $|z| = R \to \infty$, where $C_R : |z| = R$.



Note: In particular, if $f(z) = \frac{p(z)}{q(z)}$ such that the degree of the denominator is greater than that of the numerator by atleast two and $q(z)$ has no poles on the real axis, then we have the following important result :

If $p(x)$ and $q(x)$ are real polynomials such that the degree of $q(x)$ is greater than that of $p(x)$ by atleast two and if $q(x) = 0$ has no real roots, then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx = 2\pi i \times \left(\text{sum of the residue of } \frac{p(z)}{q(z)} \text{ at its poles in the upper half of the z-plane}\right)$$

1. Use the method of contour integration to Evaluate

$$\int_0^{\infty} \frac{dx}{(1+x^2)^2}$$

Soln: Consider $\int_C \frac{dz}{(1+z^2)^2} = \int_C f(z)dz$.

where C is the closed contour consisting of part of the real axis from $-R$ to R and upper half of the large circle $|z| = R$.

By Cauchy Residue Theorem, we have

$$\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$$

$= 2\pi i$ (Sum of the residue of $f(z)$ at a poles inside c).

Since $\lim_{z\to\infty} z\, f(z) = \lim_{z\to\infty} z \cdot \frac{1}{(1+z^2)^2}$

$$= \lim_{z\to\infty} z \cdot \frac{1}{z^4\left(1+\frac{1}{z^2}\right)^2}$$

$$= \lim_{z\to\infty} \frac{1}{z^3\left(1+\frac{1}{z^2}\right)^2} = 0.$$

Then By Cauchy's Lemma $\int_{C_R} \frac{dz}{(1+z^2)^2} = 0.$

Also as $R \to \infty$ we have

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

Hence $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi i$ (Sum of the residue of $f(z)$ in the upper half plane).

The poles of $f(z)$ are given by

$(1+z^2)^2 = 0 \Rightarrow z = \pm i$ is pole of degree 2.

$z = +i$ is the only pole which lie inside c.

Res. of $f(z)$ at $z = i$

$$= \frac{1}{1!} \lim_{z\to i} \frac{d}{dz} (z-i)^2 \cdot \frac{1}{(z+i)^2 (z-i)^2}$$

$$= \lim_{z\to i} \frac{0 - 2(z+i)}{(z+i)^4} = \frac{-2(2i)}{(2i)^4} = \frac{-4i}{16} = \frac{-i}{4}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi i(-i/4) = \pi/2.$$

$$\therefore 2\int_{0}^{\infty} \frac{dx}{(1+x^2)^2} = \pi/2 \Rightarrow \int_{0}^{\infty} \frac{dx}{(1+x^2)^2} = \pi/4.$$

QB(3) Evaluate $\int_C \frac{dz}{z+4}$ where C is the circle $|z|=2$.

$z+4=0 \Rightarrow z=-4$ which lies outside C, $|z|=2$

$\therefore \frac{1}{z+4}$ is Anal. inside and on C,

$\therefore$ By Cauchy's Integral thm. $\int_C \frac{1}{z+4}\,dz = 0$.

QB(4): Evaluate $\int_C \frac{\cos \pi z}{z-1}\,dz$ if C is $|z|=2$.

Singular point is given $z-1=0 \Rightarrow z=1$.

Let $f(z)=\cos \pi z$ is Anal. inside & on C.

$\therefore \int_C \frac{\cos\pi z}{z-1}\,dz = \int_C \frac{f(z)}{z-1}\,dz = 2\pi i\, f(1)$.

$= 2\pi i \cos \pi$

$= -2\pi i$.

QB(6): Singular point.

A point $z=a$ is a singular point of $f(z)$, if $f(z)$ is not analytic at $z=a$.

QB(7): 4 types of Singularities.

Poles, Essential singularity, Non isolated singularity, Removable singularity.

$$\boxed{\begin{array}{l} \bullet\ \text{Critical points} \quad \frac{dw}{dz}=0. \\ \\ \bullet\ \text{Invariant or fixed points} : w=z \end{array}}$$

unit - IV

find the poles of $\frac{z^2+4}{(z^2-1)(z^2+1)}$

QB(8) To find poles put $dr=0$.

$(z^2-1)(z^2+1)=0$

$(z-1)\quad (z-1)(z+1)=0 \Rightarrow (z-1)^2(z+1)=0$

$z=1$ is a pole of order 2

$z=-1$ is a ~~pol~~ simple pole.

① find the critical points of the transformation.

Soln: $w = z^2$, $\frac{dw}{dz} = 2z$. $\because z^2$ is differential everywhere, the critical points are given by $dw/dz = 0$

$\Rightarrow 2z = 0 \Rightarrow z = 0$ is only critical path.

② $w = z + \frac{1}{z}$.

$\frac{dw}{dz} = 1 - \frac{1}{z^2} = \frac{z^2-1}{z^2} = 0 \Rightarrow z^2 - 1 = 0 \Rightarrow z = \pm 1$

$\therefore$ The critical points are $z = \infty, \pm 1$.

③ find the invariant points of the transformation

$w = 2 - \frac{2}{z}$.

Soln. The invariant points are given by $w = z$.

$z = 2 - \frac{2}{z} \Rightarrow 2z - 2 = z^2 \Rightarrow z^2 - 2z + 2 = 0$.

$\therefore z = \frac{2 \pm 2i}{2} = 1 \pm i$.

The invariant points are $z = 1+i$, & $z = 1-i$

④ find the fixed points of the transformation

QB(9) $w = \frac{6z-9}{z}$.

Soln. fixed points are given by $w = z$.

$\Rightarrow z = \frac{6z-9}{z} \Rightarrow z^2 - 6z + 9 = 0 \Rightarrow (z-3)^2 = 0$

$\Rightarrow z = 3$:

H/W. ⑤ find the invariant points of the transformation $w = \frac{2z+4i}{iz+1}$

Soln. $z = 4, i, -i$

UNIT - IV LAPLACE COMPLEX INTEGRATION

PART - B (16 marks)

(11) (ii) Expand $\dfrac{1}{z^2-3z+2}$ in the region $1 \le |z| \le 2$ in Laurent's series.

Soln. Let $f(z) = \dfrac{1}{z^2-3z+2} = \dfrac{1}{(z-1)(z-2)}$.

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}.$$

$$1 = A(z-2) + B(z-1).$$

Put $z=1$, $\therefore 1 = A(1-2) \Rightarrow \boxed{A=-1}$

$z=2$, $\therefore 1 = B(2-1) \Rightarrow \boxed{B=1}$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}.$$

$$\therefore f(z) = -\frac{1}{z-1} + \frac{1}{z-2}.$$

If $1 \le |z| \le 2$, $|z| \le 2 \Rightarrow \left|\frac{z}{2}\right| \le 1.$

and $1 \le |z| \Rightarrow \frac{1}{|z|} \le 1.$

So the Laurent's series is

$$f(z) = \frac{-1}{z-1} + \frac{1}{z-2}.$$

$$= -\frac{1}{z\left[1-\frac{1}{z}\right]} + \frac{1}{-2\left[1-\frac{z}{2}\right]}$$

$$= -\frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} - \frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$$

$$= -\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right] - \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right]$$

(12) (i) Evaluate $\int_C \frac{z^2+1}{z^2-1}\,dz$ where C is $|z-1|=1$ using Cauchy's integral formula.

Soln. We know that

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)}.$$

$z^2=1$
$z=\pm 1$
$z=1,\ z=-1$

Since C is $|z-1|=1$, $z=-1$ is outside C.

$$\therefore \int_C \frac{z^2+1}{z^2-1}\,dz = \int_C \frac{z^2+1}{(z-1)(z+1)}\,dz = \int_C \frac{\left(\frac{z^2+1}{z+1}\right)}{z-1}\,dz.$$

$\therefore$ By Cauchy's integral formula.

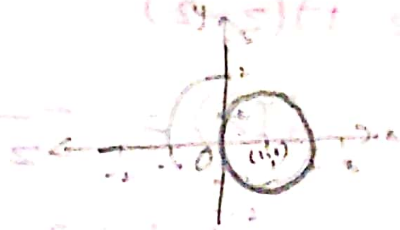
$$\int_C \frac{f(z)\,dz}{z-z_0} = 2\pi i\, f(z_0) \quad \text{here } f(z) = \frac{z^2+1}{z+1}$$

$$z_0 = 1$$

$$\therefore \int_C \frac{z^2+1}{z^2-1}\,dz = 2\pi i\, f(1)$$

$$= 2\pi i\,(1)$$

$$= 2\pi i.$$



(12) (ii) Expand $f(z) = \frac{z}{(z-1)(z-3)}$ as Laurent's series in the region $1<|z|<3$.

Soln. Now $\frac{z}{(z-1)(z-3)} = \frac{A}{(z-1)} + \frac{B}{z-3}$

$$z = A(z-3) + B(z-1).$$

Put $z=3$, $3 = B(3-1) \Rightarrow 2B = 3 \Rightarrow B = 3/2$

$z=1, \quad 1=-2A \Rightarrow \boxed{A=-1/2}$

If $1<|z|<3$, $\quad 1<|z| \Rightarrow \frac{1}{|z|}<1$.

$|z|<3 \Rightarrow \left|\frac{z}{3}\right|<1$.

So Laurent's series is,

$$\frac{z}{(z-1)(z-3)} = \frac{-1/2}{z-1} + \frac{3/2}{z-3}$$

$$= -\frac{1}{2}\cdot\frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{3}{2}\cdot\frac{1}{-3\left(1-\frac{z}{3}\right)}.$$

$$= \frac{-1}{2z}\left[1-\frac{1}{z}\right]^{-1} - \frac{1}{2}\left[1-\frac{z}{3}\right]^{-1}.$$

$$= -\frac{1}{2z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right] - \frac{1}{2}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\cdots\right].$$

---

(13) Find the residues at the poles of

$$f(z) = \frac{z+2}{(z+1)^2(z-2)}.$$

Soln. The poles are given by $(z+1)^2(z-2)=0$.

$(z+1)^2=0 \Rightarrow z=-1,-1$

$z-2=0 \Rightarrow z=2$.

$\therefore$ $z=-1$ is a pole of order 2.

$z=2$ is a pole of order 1 ie simple pole.

$$R(2) = \lim_{z \to 2} (z-2) \frac{(z+2)}{(z+1)^2 (z-2)}$$

$$= \lim_{z \to 2} \frac{z+2}{(z+1)^2} = \frac{2+2}{(2+1)^2} = \frac{4}{9}.$$

$$R(-1) = \frac{1}{(2-1)!} \lim_{z \to -1} \frac{d}{dz} \left[(z+1)^2 \cdot \frac{z+2}{(z+1)^2 (z-2)} \right].$$

[Using the formula, if a is a pole of order m.

$$R(a) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz} (z-a)^m f(z) \Big]$$

$$\therefore R(-1) = \lim_{z \to -1} \frac{d}{dz} \left(\frac{z+2}{z-2} \right)$$

$$= \lim_{z \to -1} \left[\frac{(z-2) - (z+2)}{(z-2)^2} \right]$$

$$= \frac{-3-1}{3^2} = \frac{-4}{9}$$

UNIT - V. STATISTICS. PART - B (16 Marks).

(15) A Certain drug was administrated to 500 people out of a total of 800 included in the sample to test its efficiency against typhoid. The Results are given below:

	Typhoid	No Typhoid
Drug	200	300
No Drug	280	20.

(Given for 5% level of significance.)

On the basis of these data can it be concluded that the drug is effective in Preventing typhoid. (✱).

Soln. Null Hypothesis : H_0 - Drug and Typhoid are independent. Alternative Hypothesis : H_1 - Drug is effective

observed frequency :

	Typhoid	No Typhoid	Total
Drug	200	300	500
No Drug	280	20	300
Total	480	320	800.

Expected frequency : $E_i = \frac{\text{Row total} \times \text{Column total}}{\text{Grand total}}$.

	Typhoid	No Typhoid	Total
Drug	$\frac{500 \times 480}{800} = 300$	$\frac{500 \times 320}{800} = 200$	500.
No Drug	$\frac{300 \times 480}{800} = 180$	$\frac{300 \times 320}{800} = 120$	300.
total	480	320	800.

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{\overset{33.3}{100^2}}{300} + \frac{\overset{50}{100^2}}{200} + \frac{\overset{55.5}{100^2}}{180} + \frac{\overset{83.33}{100^2}}{120} = 630. \quad = 222.16$$

Calculated value = 630. 222.16.

Degrees of freedom = $(r-1)(s-1) = (2-1)(2-1) = 1$.

Tabulated value = 3.84.

Calculated value > Tabulated value.

H_0 is rejected.

Conclusion : Drug is Effective.

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UNIT - I . ODE .

Part - B . 16 marks Questions .

(1) Solve $(D^2-4D+4)y = e^{-4x} + 5\cos 3x$.

Soln. A.E is $m^2-4m+4=0$.

$(m-2)^2=0 \Rightarrow m=2,2$.

$\therefore$ C.f $= (C_1+C_2x)e^{2x}$.

$$P.I_1 = \frac{e^{-4x}}{D^2-4D+4} = \frac{e^{-4x}}{(-4)^2-4(4)+4} = \frac{e^{-4x}}{36}.$$

$$P.I_2 = \frac{5\cos 3x}{D^2-4D+4} = \frac{5\cos 3x}{-9-4D+4} = \frac{5\cos 3x}{-5-4D}.$$

$$= \frac{-5\cos 3x\,(5-4D)}{(5+4D)(5-4D)} = \frac{-25\cos 3x + 20D(\cos 3x)}{25-16D^2}.$$

$$= \frac{-25\cos 3x + 20(-3.\sin 3x)}{25-(16)(-9)}$$

$$= \frac{-25\cos 3x - 60\sin 3x}{25+144}$$

$$= \frac{-5}{169}\left[5\cos 3x + 12\sin 3x\right].$$

So $y = C.f + P.I_1 + P.I_2$.

$$y = (C_1+C_2x)e^{2x} + \frac{e^{-4x}}{36} - \frac{5}{169}\left[5\cos 3x + 12\sin 3x\right].$$

(12) Solve by method of variation of parameters
$(D^2+4)y = \tan 2x$.

Soln. The A.E is $m^2+4=0$
$m^2=-4 \Rightarrow m = \pm 2i$.

$\therefore$ C.F $= C_1 \cos 2x + C_2 \sin 2x$.

C.F $= C_1 y_1 + C_2 y_2$.

$\Rightarrow y_1 = \cos 2x$, $y_2 = \sin 2x$, here $r = \tan 2x$.

$y_1' = -2\sin 2x$, $y_2' = 2\cos 2x$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x.$$

$$= 2(\sin^2 2x + \cos^2 2x)$$

$$= 2(1) = 2.$$

$$u = -\int \frac{r y_2}{W} \, dx + k_1.$$

$$= -\int \frac{\sin 2x \cdot \tan 2x}{2} \, dx + k_1.$$

$$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} \, dx + k_1 = \frac{-1}{2}\int \left(\frac{1-\cos^2 2x}{\cos 2x}\right) dx + k_1$$

$$= -\frac{1}{2}\left[\int \frac{1}{\cos 2x}\, dx - \int \cos 2x \, dx\right] + k_1$$

$$= -\frac{1}{2}\left[\int \sec 2x \, dx - \int \cos 2x \, dx\right] + k_1$$

$$= -\frac{1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right] + k_1$$

$$u = -\frac{1}{4}\left[\log(\sec 2x + \tan 2x) - \sin 2x\right] + k_1.$$

$$v = \int \frac{r y_1}{W}\,dx + k_2$$

$$= \int \frac{\cos 2x \cdot \tan 2x}{2}\,dx + k_2 = \int \frac{\sin 2x}{2}\,dx + k_2$$

$$= \frac{1}{2}\left(-\frac{\cos 2x}{2}\right) + k_2 = -\frac{1}{4}\cos 2x + k_2.$$

$$\therefore v = -\frac{1}{4}\cos 2x + k_2.$$

$$\therefore P.I = u y_1 + v y_2$$

$$= -\frac{1}{4}\left[\log(\sec 2x + \tan 2x) - \sin 2x\right] \cdot \cos 2x$$

$$-\frac{1}{4}\cos 2x \sin 2x.$$

$$= -\frac{1}{4}\log(\sec 2x + \tan 2x)\cdot\cos 2x + \cancel{\frac{1}{4}\sin 2x \cos 2x} - \cancel{\frac{1}{4}\sin 2x \cos 2x}$$

$$= -\frac{1}{4}\log(\sec 2x + \tan 2x)\cdot\cos 2x + k_1$$

(13) Solve : $(x^2D^2 + xD + 1)y = 4\sin\log x$.

<u>Soln.</u> Put $x = e^z$, $\theta = \frac{d}{dz}$, $xD = \theta$

$\log x = z$. $\quad x^2D^2 = \theta(\theta - 1)$.

∴ Eqn. becomes

$[\theta(\theta-1) + \theta + 1]y = 4\sin z$.

$(\theta^2 - \theta + \theta + 1)y = 4\sin z$.

$(\theta^2 + 1)y = 4\sin z$.

A.E is $m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i$

C.f $= c_1\cos z + c_2\sin z$

P.I $= \frac{4\sin z}{\theta^2+1} = \frac{4\sin z}{-1+1} = \frac{4}{0}\sin z$

∴ P.I $= \frac{\cancel{4}^{2} z\sin z}{\cancel{2}\theta} = 2z\cdot\frac{1}{\theta}(\sin z)$.

$= 2z\int \sin z\, dz$

$\cancel{= 2\theta \int z\sin z\, dz}$.

$= 2z(-\cos z) + c$.

$\cancel{= 2\theta\,[z(-\cos z) - (1)(\sin z)]}$

$\cancel{= 2[-z\cos z - \sin z]}$

$\cancel{= -2[\sin z + z\cos z]}$.

∴ $y = P.I + C.f = -2\log x\cos(\log x) + c_1\cos(\log x) + c_2\sin(\log x)$

$\cancel{= c_1\cos(\log x) + c_2\sin(\log x) - 2\sin(\log x) - 2\log x\cos(\log x)}$.

(5) Solve the Simultaneous Equations :

$$\frac{dx}{dt} + 2x - 3y = 5e^{-t}, \quad \frac{dy}{dt} - 3x + 2y = 0,$$

Soln. Equation can be written as

$$(D+2)x - 3y = 5e^{-t} \quad - (1)$$

$$-3x + (D+2)y = 0 \quad - (2)$$

$$(1)\times(D+2) \quad (D+2)^2 x - 3(D+2)y = 5(D+2)e^{-t}.$$

$$(2)\times 3: \quad -9x + 3(D+2)y = 0.$$

$$[(D+2)^2 - 9]x = 5(D+2)e^{-t}.$$

$$(D^2+4D-5)x = 5[D(e^{-t}) + 2e^{-t}]$$

$$= 5(-e^{-t} + 2e^{-t}) = 5e^{-t}.$$

$$\Rightarrow (D^2+4D-5)x = 5e^{-t}.$$

A.E is $m^2 + 4m - 5 = 0$.

$(m+5)(m-1) = 0$.

$\Rightarrow m = 1, -5$

C.F $x = c_1 e^{t} + c_2 e^{-5t}$.

$$P.I = \frac{5e^{-t}}{D^2+4D-5} = \frac{5e^{-t}}{1-4-5} = \frac{-5}{8}e^{-t}.$$

$$\therefore x = c_1 e^{t} + c_2 e^{-5t} - \frac{5}{8}e^{-t}.$$

$$(1)\times 3 : \quad 3(D+2)x - 9y = 15e^{-t}.$$

$$(2)\times(D+2) : \quad -3(D+2)x + (D+2)^2 y = 0.$$

$$[(D+2)^2 - 9]y = 15e^{-t}.$$

$(D^2+4D-5)y = 15e^{-t}$.

C.f is $y = c_1e^{t} + c_2e^{-5t}$.

$$P.I = \frac{15e^{-t}}{D^2+4D-5} = \frac{-15}{8}e^{-t}.$$

$$y = c_1e^{t} + c_2e^{-5t} - \frac{15}{8}e^{-t}.$$

UNIT - II.

VECTOR CALCULUS.

Part - B (16 marks)

(1) Show that the $\vec{F} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$ is irrotational and hence find the scalar potential ϕ.

Soln. If $\nabla\times\vec{F} = 0$, then $\vec{F}$ is irrotational.

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$= \vec{i}(-1+1) - \vec{j}(3z^2-3z^2) + \vec{k}(6x-6x)$$

$$= \vec{0}.$$

$\therefore \vec{F}$ is irrotational, (If $\nabla\times\vec{F} = 0$, then $\vec{F}$ is conservative vector field)

Hence $\vec{F} = \nabla\phi$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}.$$

$$(6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}.$$

$$6xy+z^3 = \frac{\partial\phi}{\partial x} \Rightarrow \phi = 6y.\frac{x^2}{2} + xz^3.$$
$$= 3x^2y + xz^3.$$

$$(3x^2-z) = \frac{\partial\phi}{\partial y} \Rightarrow \phi = 3x^2y - zy.$$

$$(3xz^2-y) = \frac{\partial\phi}{\partial z} \Rightarrow \phi = 3x.\frac{z^3}{3} - yz = xz^3 - yz.$$

$$\therefore \phi = 3x^2y + xz^3 - yz + C.$$

UNIT - III. ANALYTICAL FUNCTIONS.

PART - B (16 marks)

(4) Show that the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic. find $f(z)$ and hence find the harmonic conjugate fn of u.

Soln. $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1.$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \qquad \frac{\partial u}{\partial y} = -3x \cdot 2y - 6y = -6xy - 6y$$

$$\frac{\partial^2 u}{\partial x^2} = 6x+6 \qquad \frac{\partial^2 u}{\partial y^2} = -6x-6.$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \therefore u \text{ is harmonic}$$

By M.i.h

$f'(z) = u_x - iu_y$ $\qquad$ $\cancel{f(z)} = u_x(z,0) = 3z^2 + 6z$

$u_y(z,0) = 0$

$$= 3x^2 - 3y^2 + 6x - i(-6xy - 6y)$$

$x \to z$, $y \to 0$. (By Milne Thomson method)

$f'(z) = 3z^2 + 6z$. $f(z) = \int [u_x(z,0) - i u_y(z,0)]\,dz + C$

Integrating $\underline{f(z) = z^3 + 3z^2 + C}$. $\int (3z^2 + 6z)\,dz + C$

$u + iv = (x+iy)^3 + 3(x+iy)^2 + C_1 + iC_2$. $\frac{3z^3}{3} + \frac{6z^2}{2} + C$

$u + iv = x^3 + 3ix^2y - 3xy^2 - iy^3 + 3x^2 + 6xyi - 3y^2 + C_1 + iC_2$.

$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + C_1$ $\quad f(z) = z^3 + 3z^2 + C$

$v = 3x^2y - y^3 + 6xy + C_2$. $\quad u + iv = (x$

(12) Given $u = e^x(x\cos y - y\sin y)$ is the real part of $f(z) = u + iv$.

$u_x = e^x(\cos y) + (x\cos y - y\sin y)e^x$.

$= e^x[\cos y + x\cos y - y\sin y]$.

$\therefore u_{xx} = e^x(\cos y) + (\cos y + x\cos y - y\sin y)e^x$.

$= e^x[2\cos y + x\cos y - y\sin y]$.

$u_y = e^x[-x\sin y - y\cos y - \sin y]$.

$u_{yy} = e^x[-x\cos y + y\sin y - \cos y - \cos y]$

$= e^x[-x\cos y + y\sin y - 2\cos y]$.

$\therefore u_{xx} + u_{yy} = 0$.

$\therefore u$ satisfies Laplace's Equation.

Replacing x by z and y by 0.

we get

$u_x(z,0) = e^z(\cos 0 + z\cos 0 - 0)$

$= e^z(1+z)$.

$u_y(z,0) = e^z \cdot 0 = 0$.

By Milne Thomson method.

$f'(z) = u_x(z,0) - i u_y(z,0)$

$= (1+z)e^z - i(0) = (1+z)e^z$.

Integrating,

$f(z) = \int (1+z)e^z dz$.

$= (1+z)e^z - e^z + c$.

$f(z) = ze^z + c$.

Q.12. P.T the function $u = e^x(x\cos y - y\sin y)$ satisfies Laplace's equation and find the corresponding analytic function $f(z)$

13 (i) Find the image of the circle $|z| = 1$ under the map $w = z + (3+2i)$.

Soln. $w = u + iv$.

$w = z + (3+2i)$

$u + iv = w = x + iy + (3+2i)$

$u + iv = (x+3) + i(y+2)$.

$u = x+3$, $v = y+2$.

$x = u-3$, $y = v-2$.

Now $|z|=1$, $x^2+y^2=1$.

$(u-3)^2+(v-2)^2=1$.

Hence the circle ~~xxx~~ $x^2+y^2=1$ is mapped into a ⊙ with centre $(3,2)$ & radius 1.

13) (ii) Find the Image of the circle $|z|=2$ under the transformation $w=5z$.

Soln. $z=x+iy$, $w=u+iv$.

$$w=5z$$

$$u+iv=5(x+iy)=5x+i5y.$$

$$u=5x, \quad v=5y.$$

$$x=\frac{u}{5}, \quad y=\frac{v}{5}.$$

$$|z|=2 \Rightarrow x^2+y^2=4 \Rightarrow \left(\frac{u}{5}\right)^2+\left(\frac{v}{5}\right)^2=4.$$

$$\Rightarrow u^2+v^2=100.$$

Hence the circle $x^2+y^2=4$ is mapped into a circle with centre origin and radius 10 units.

15. (i) Find the bilinear transformation which maps the points $-2, 0, 2$ into the points $0, i, -i$.

Soln. Given $z_1=-2$, $z_2=0$, $z_3=2$.

$w_1=0$, $w_2=i$, $w_3=-i$.

Substituting these values in

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(2i)}{(w+i)(i)} = \frac{(z+2)(-2)}{(z-2)(2)}.$$

$$\frac{2w}{w+i} = -\frac{(z+2)}{(z-2)}$$

$$2wz - 4w = -wz - iz - 2w - 2i.$$

$$2wz + wz - 4w + 2w = -i(z+2)$$

$$3wz - 2w = -i(z+2)$$

$$w(3z-2) = -i(z+2).$$

$$w = -\frac{i(z+2)}{(3z-2)}.$$

(ii) Find the bilinear transformation which maps the points $0, 1, i$ in z-plane onto the points $\infty, 1, -i$ in w-plane.

Soln. Given: $z_1 = 0$, $z_2 = 1$, $z_3 = i$

$w_1 = \infty$, $w_2 = 1$, $w_3 = -i$

$$\frac{(w-\infty)(1+i)}{(w+i)(1-\infty)} = \frac{z(1-i)}{(z-i)}$$

omitting factors of ∞.

$$\frac{1+i}{w+i} = \frac{z(1-i)}{z-i}$$

$$w+i = \frac{(z-i)(1+i)}{z(1-i)} = \frac{z+iz-i+1}{z-iz}$$

$$w = \frac{z+iz-i+1}{z-iz} - i = \frac{z+iz-i+1-iz-z}{z-iz}$$

$$w = \frac{1-i}{z(1-i)} = \frac{1}{z}.$$

$$\boxed{w = \frac{1}{z}}$$

(10) find the invariant points of $w = \frac{z-1}{z+1}$.

Soln. $w = z$.

$$z = \frac{z-1}{z+1} \Rightarrow z(z+1) = z-1$$

$$\Rightarrow z^2 + z - z + 1 = 0$$

$$\Rightarrow z^2 + 1 = 0.$$

$\Rightarrow z = \pm i$. are invariant pts.

(8) find the critical points of $w = z^4 - 1$.

Soln. $\frac{dw}{dz} = 0$.

$4z^3 = 0$.

$\Rightarrow \boxed{z = 0}$ is the critical point.

LAPLACE TRANSFORM UNIT- V.

Definition: Let $f(t)$ be a function of the variable t which is defined for all positive values of t. Let s be a real constant. If the integral $\int_0^\infty e^{-st} f(t)\,dt$ exists, and equal to $F(s)$. Then $F(s)$ is called the Laplace Transform of $f(t)$ and it is denoted by $L[f(t)]$.

i.e. $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$.

EXPONENTIAL ORDER: A function $f(t)$ is said to be of exponential order if $\lim_{t\to\infty} e^{-st} f(t) = 0$.

Example: The function t^2 is of exponential order.

$$\lim_{t\to\infty} e^{-st} t^2 = \lim_{t\to\infty} \frac{t^2}{e^{st}} \quad \left(= \frac{\infty}{\infty} \text{ form}\right).$$

$$= \lim_{t\to\infty} \frac{2t}{se^{st}} \quad \left(= \frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{t\to\infty} \frac{2}{s^2 e^{st}} = 0 \quad \left(\because \frac{2}{\infty} = 0\right).$$

$\therefore \lim_{t\to\infty} e^{-st} t^2 = 0$. Hence t^2 is of exponential order

CONDITIONS FOR THE EXISTENCE OF THE LAPLACE TRANSFORM

Laplace Transform :

(i) $f(t)$ is continuous or piece-wise continuous in the closed interval $[a, b]$ where $a > 0$.

(ii) $f(t)$ is of Exponential order.

LAPLACE TRANSFORM OF SOME BASIC FUNCTIONS :

	$f(t)$	$L[f(t)]$.
1.	1	$1/s$
2.	e^{at}	$1/s-a$
3.	e^{-at}	$1/s+a$
4.	$\cos at$	$\frac{s}{s^2+a^2}$.
5.	$\sin at$	$\frac{a}{s^2+a^2}$.
6.	$\cosh at$	$\frac{s}{s^2-a^2}$.
7.	$\sinh at$	$\frac{a}{s^2-a^2}$.
8.	t^n	$\frac{n!}{s^{n+1}}$

PROBLEMS :

1. Find the Laplace Transform of $f(t) = 3e^{5t} + 5\cos t + t^3$

Soln:

$$L[3e^{5t} + 5\cos t + t^3] = 3L[e^{5t}] + 5L[\cos t] + L[t^3]$$

$$= 3 \cdot \frac{1}{s-5} + 5 \cdot \frac{s}{s^2+1} + \frac{3!}{s^4}$$

2. find $L[\sin^2 2t]$.

Soln:

$$L[\sin^2 2t] = L\left[\frac{1-\cos 4t}{2}\right] \qquad \left[\because \sin^2 t = \frac{1-\cos 2t}{2}\right]$$

$$= L\left[\frac{1}{2}\right] - L\left[\frac{\cos 4t}{2}\right]$$

$$= \frac{1}{2}L[1] - \frac{1}{2}L[\cos 4t]$$

$$= \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{s}{s^2+4^2} = \frac{1}{2s} - \frac{1}{2} \cdot \frac{s}{s^2+16}$$

$L[1] = \int_0^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$

3) find $L[\sin 3t \cos t]$.

$$\left(\because \sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]\right)$$

Soln:

$$L[\sin 3t \cos t] = L\left[\frac{1}{2}(\sin 4t + \sin 2t)\right]$$

$$= \frac{1}{2}[L(\sin 4t) + L(\sin 2t)]$$

$$= \frac{1}{2}\left[\frac{4}{s^2+16} + \frac{2}{s^2+4}\right]$$

4) find $L[\cos^2 3t]$.

Soln:

$$L[\cos^2 3t] = L\left[\frac{1+\cos 6t}{2}\right] \qquad \left[\because \cos^2 t = \frac{1+\cos 2t}{2}\right]$$

$$= \frac{1}{2}\left[L[1] + L[\cos 6t]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right].$$

$$= \frac{1}{2s} + \frac{s}{2(s^2+36)}.$$

1.) $L[e^{-2t} + \sin 3t]$

5) find $L[\cos^3 2t]$.

$\left[\cos^3 t = \frac{\cos 3t + 3\cos t}{4}\right]$.

Soln:

$$L[\cos^3 2t] = L\left[\frac{\cos 6t + 3\cos 2t}{4}\right].$$

$$= \frac{1}{4}\left[L(\cos 6t) + 3L(\cos 2t)\right].$$

$$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{3s}{s^2+4}\right].$$

6) find $L[\sin 3t \cdot \sin 4t]$.

Soln.

$$L[\sin 3t \cdot \sin 4t] = L\left[\frac{1}{2}(\cos t - \cos 7t)\right]$$

$$= \frac{1}{2} L[\cos t - \cos 7t]$$

$$= \frac{1}{2}\left[L(\cos t) - L(\cos 7t)\right].$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+49}\right].$$

H/W.

7) $L[e^{5t} + 5e^{-6t}]$. Soln: $\frac{1}{s-5} + 5 \cdot \frac{1}{s+6}$.

8) $L[\cosh 3t + 2e^{-3t} + \sin 2t]$

soln: $\frac{s}{s^2-9} + \frac{2}{s+3} + \frac{2}{s^2+4}$

9) $L[\sin^2 3t]$. Soln: $\frac{1}{2s} - \frac{s}{2(s^2+36)}$

10) $L[\cos^2 4t]$. Soln: $\frac{1}{2s} + \frac{s}{2(s^2+64)}$

11) $L[\sin^3 2t]$. Soln: $\frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right]$.

12) $L[\cos^3 3t]$. Soln: $\frac{s}{4}\left[\frac{1}{s^2+81} + \frac{3}{s^2+9}\right]$.

13) $L[\sin 2t \cos 3t]$ Soln: $\frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$.

14) $L[\cos 3t \cos 4t]$ Soln: $\frac{1}{2}\left[\frac{s}{s^2+49}\right] + \frac{1}{2}\left[\frac{s}{s^2+1}\right]$.

15) $L[\sin 2t \sin 3t]$ Soln: $\frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right]$.

16) $L[7e^{2t} + 2e^{-2t} + 5\cos 2t + 8t^4 + 5\sin 2t + 5]$.

Soln: $\frac{7}{s-2} + \frac{2}{s+2} + \frac{5s}{s^2+4} + 8\cdot\frac{4!}{s^5} + 5\cdot\frac{2}{s^2+4} + \frac{5}{s}$.

17) $L[t^3+2t+3]$. Soln: $\frac{6}{s^4} + \frac{2}{s^2} + \frac{3}{s}$.

SHIFTING THEOREM:

~~1) $L[t^2 e^{-at}$~~

If $L[f(t)] = F(s)$, then

$L[e^{at} f(t)] = F(s-a)$.

$L[e^{-at} f(t)] = F(s+a)$.

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Problems:

1) find $L[t^2 e^{-2t}]$

Soln: $f(t) = t^2$.

$L[f(t)] = \frac{2!}{s^3}$.

$\therefore L[e^{-2t} t^2] = \frac{2}{(s+2)^3}$.

$f(t) = t^2$

$L[f(t)] = L[t^2] = \frac{2!}{s^3}$.

$\therefore L[t^2 e^{-2t}] = \left[\frac{2!}{s^3}\right]_{s \to s+2}$

$= \frac{2}{(s+2)^3}$

2) Find $L[\cosh t \cdot \sin 2t]$.

Soln:

$L[\cosh t \cdot \sin 2t] = L\left[\left(\frac{e^t + e^{-t}}{2}\right) \cdot \sin 2t\right]$

$= \frac{1}{2}\left[L(e^t \sin 2t) + L(e^{-t} \sin 2t)\right]$

$= \frac{1}{2}\left[L(\sin 2t)_{s \to s-1} + L(\sin 2t)_{s \to s+1}\right]$

$= \frac{1}{2}\left[\left(\frac{2}{s^2+4}\right)_{s \to s-1} + \left(\frac{2}{s^2+4}\right)_{s \to s+1}\right]$

$= \frac{1}{2}\left[\frac{2}{(s-1)^2+4} + \frac{2}{(s+1)^2+4}\right]$.

H/W

3) find $L[e^t(\cosh 2t + \frac{1}{2}\sinh 2t)]$.

Soln: $\frac{s-1}{(s-1)^2-4} + \frac{1}{(s-1)^2-4}$

4) find $L[\cosh t \cos 2t]$

soln: $\frac{1}{2} \frac{s-1}{(s-1)^2+4} + \frac{1}{2} \cdot \frac{s+1}{(s+1)^2+4}$

5) $L[e^{-3t}\sin 2t]$: Soln: $\frac{2}{(s+3)^2+4}$.

6) $L[t^3 e^{5t}]$ soln: $\frac{6}{(s-5)^4}$. $L[e^{-at}\sin bt]$.

$f(t) = \sin bt$

$L[f(t)] = \frac{b}{s^2+b^2}$

$L[e^{-at}\sin bt) = \left(\frac{b}{s^2+b^2}\right)_{s \to s+a}$

$= \frac{b}{(s+a)^2+b^2}$ //.

7) $L[e^{-t}(3\sinh 2t - 5\cosh 2t)]$.

Soln:

$L[e^{-t}(3\sinh 2t - 5\cosh 2t)]$

$= L[3\sinh 2t - 5\cosh 2t]_{s \to s+1}$

$= \left[3 \cdot \frac{2}{s^2-4} - 5 \cdot \frac{s}{s^2-4}\right]_{s \to s+1}$.

$= \frac{6}{(s+1)^2-4} - \frac{5 \cdot (s+1)}{(s+1)^2-4}$

~~IMPROTANT~~

IMPORTANT RESULTS:

1.) $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} L[f(t)]$.

2.) $L\left[\frac{f(t)}{t^n}\right] = \int_s^\infty \int_s^\infty \cdots \int_s^\infty L[f(t)](ds)^n$.

PROBLEMS:

1) Find $L[t\cos 3t]$.

Soln:

$L[t\cos 3t] = -\frac{d}{ds} L[\cos 3t]$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+9}\right].$$

$$= -\left[\frac{(s^2+9) - s(2s)}{(s^2+9)^2}\right].$$

$$= -\left[\frac{s^2+9-2s^2}{(s^2+9)^2}\right] = \frac{s^2-9}{(s^2+9)^2}.$$

2) Find $L[t^2e^{-4t}]$. $\times L[t^2e^{-at}]$

$(s+4)^{-1}$

$-1(s+4)^{-2}$

$-\frac{1}{(s+4)^2}$

Soln:

$$L[t^2e^{-4t}] = \frac{d^2}{ds^2}L[e^{-4t}] = \frac{d^2}{ds^2}\left[\frac{1}{s+4}\right].$$

$$= \frac{d}{ds}\left[-\frac{1}{(s+4)^2}\right] = \frac{2}{(s+4)^3}.$$

$-(s+4)^{-2}$

$-(-2)(s+4)^{-3}$

$2(s+4)^{-3}$

3) $L[t^2\sin 2t]$, Soln: $\frac{12s^2-16}{(s^2+4)^3}$.

4) $L[te^{-4t}\sin 3t]$.

Soln: $L[te^{-4t}\sin 3t] = L[t\sin 3t]_{s\to s+4}$. — ①

$$L[t\sin 3t] = -\frac{d}{ds}L[\sin 3t]$$

$$= -\frac{d}{ds}\left[\frac{3}{s^2+9}\right].$$

$$= \frac{3(2s)}{(s^2+9)^2} = \frac{6s}{(s^2+9)^2}$$

$\therefore$ ① $\Rightarrow L[te^{-4t}\sin 3t] = \left[\frac{6s}{(s^2+9)^2}\right]_{s\to s+4}$

$$= \frac{6(s+4)}{[(s+4)^2+9]^2} = \frac{6(s+4)}{(s^2+8s+25)^2}.$$

5) find $L[te^{-t}\cosh t]$.

Soln: $\frac{(s+1)^2+1}{[(s+1)^2-1]^2} = \frac{s^2+2s+2}{(s^2+2s)^2}.$

6) $L[te^{-t}\sin t]$

Soln: $\frac{2(s+1)}{[(s+1)^2+1]^2}.$

7) find $L\left[\frac{1-e^t}{t}\right].$ $\qquad \frac{1}{\infty}=0$

Soln:

$$L\left[\frac{1-e^t}{t}\right] = \int_s^\infty L(1-e^t)\,ds = \int_s^\infty [L(1)-L(e^t)]\,ds.$$

$$= \int_s^\infty \left(\frac{1}{s}-\frac{1}{s-1}\right)ds = \left[\log s - \log(s-1)\right]_s^\infty.$$

$$= \left[\log\frac{s}{s-1}\right]_s^\infty = \left[\log\left(\frac{1}{1-1/s}\right)\right]_s^\infty.$$

$$= \log 1 - \log\frac{1}{1-1/s}$$

$$= 0 - \log\frac{s}{s-1}.$$

$$= \log\left(\frac{s-1}{s}\right).$$

8) Find $L\left[\frac{1-\cos 2t}{t}\right]$.

Soln:

$$L\left[\frac{1-\cos 2t}{t}\right] = \int_s^\infty L\left[1-\cos 2t\right] ds.$$

$$= \int_s^\infty \left[L(1) - L(\cos 2t)\right] ds.$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds.$$

$$= \left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty.$$

$$= \left[\log \frac{s}{\sqrt{s^2+4}}\right]_s^\infty = \log \frac{\sqrt{s^2+4}}{s}.$$

9) Find $L\left[\frac{e^{at} - \cos 6t}{t}\right]$.

Soln:

$$L\left[\frac{e^{at}-\cos 6t}{t}\right] = \int_s^\infty L\left[e^{at} - \cos 6t\right] ds.$$

$$= \int_s^\infty \left[L(e^{at}) - L(\cos 6t)\right] ds$$

$$= \int_s^\infty \left(\frac{1}{s-a} - \frac{s}{s^2+36}\right) ds$$

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$$= \left[\log (s-a) - \frac{1}{2}\log (s^2+36)\right]_s^\infty$$

$$= \left[\log (s-a) - \log \sqrt{s^2+36}\right]_s^\infty$$

$$= \left[\log \frac{s-a}{\sqrt{s^2+36}}\right]_s^\infty$$

$$= \log \frac{\sqrt{s^2+36}}{s-a}.$$

10) Find $L\left[\frac{\sin^2 t}{t}\right]$.

Soln:

$$L\left[\frac{\sin^2 t}{t}\right] = \int_s^\infty L(\sin^2 t)\, ds.$$

$$= \int_s^\infty L\left(\frac{1-\cos 2t}{2}\right) ds.$$

$$= \frac{1}{2}\int_s^\infty \left[L(1) - L(\cos 2t)\right] ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds.$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log (s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \frac{1}{2}\log\frac{\sqrt{s^2+4}}{s}.$$

11) Evaluate $L\left[\frac{\cos 4t \sin 2t}{t}\right]$.

Soln:

$$L\left[\frac{\cos 4t \sin 2t}{t}\right] = \int_s^\infty L[\cos 4t \sin 2t]\,ds.$$

$$= \int_s^\infty L\left[\frac{\sin 6t - \sin 2t}{2}\right] ds.$$

$$= \frac{1}{2}\int_s^\infty [L(\sin 6t) - L(\sin 2t)]\,ds$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{6}{s^2+36} - \frac{2}{s^2+4}\right] ds.$$

$$= \frac{1}{2}\left[\cancel{6}\cdot\frac{1}{\cancel{6}}\tan^{-1}\left(\frac{s}{6}\right) - \cancel{2}\cdot\frac{1}{\cancel{2}}\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[(\tan^{-1}\infty - \tan^{-1}\infty) - \tan^{-1}\left(\frac{s}{6}\right) + \tan^{-1}\left(\frac{s}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{6}\right) + \tan^{-1}\left(\frac{s}{2}\right)\right].$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{2}\right) - \tan^{-1}\left(\frac{s}{6}\right)\right].$$

12) Find $L\left[\frac{1-\cos t}{t}\right]$ soln: $\frac{1}{2}\log\frac{s^2+1}{s^2}$ or $\log\frac{\sqrt{s^2+1}}{s}$.

13) Find $L\left[\frac{e^{at} - \cos bt}{t}\right]$ soln: $\frac{1}{2}\log\frac{s^2+b^2}{(s-a)^2}$.

INITIAL VALUE THEOREM :

If $L[f(t)] = f(s)$, then

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s f(s)$$

FINAL VALUE THEOREM:

If $L[f(t)] = f(s)$, then

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s f(s).$$

1.) Verify the initial and final value theorem for

$f(t) = 1 + e^{-t}(\sin t + \cos t)$.

soln: $f(s) = L[f(t)] = L[1 + e^{-t}\sin t + e^{-t}\cos t]$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$= \frac{1}{s} + \frac{1}{s^2+2s+2} + \frac{s+1}{s^2+2s+2}$$

$$\Rightarrow s f(s) = 1 + \frac{s}{s^2+2s+2} + \frac{s(s+1)}{s^2+2s+2}$$

Initial value theorem:

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s f(s).$$

LHS

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} [1 + e^{-t}\sin t + e^{-t}\cos t].$$

$$= 2$$

RHS $\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[1+\frac{s}{s^2+2s+2}+\frac{s(s+1)}{s^2+2s+2}\right].$

$$= \lim_{s\to\infty}\left[1+\frac{s^2(1/s)}{s^2\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}+\frac{s^2(1+1/s)}{s^2\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}\right]$$

$$= 1+0+1 = 2.$$

LHS = RHS, thus verified.

Final value Theorem:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

LHS.

$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left[1+e^{-t}\sin t+e^{-t}\cos t\right]$

$$= 1+0+0 = 1.$$

RHS

$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[1+\frac{s}{s^2+2s+1}+\frac{s(s+1)}{s^2+2s+1}\right].$

$$= 1+0+0 = 1$$

LHS = RHS, thus verified.

2) verify the Initial and final value theorem for the function $f(t) = e^{-2t}\sin 3t$.

Soln: Given $f(t) = e^{-2t}\sin 3t$.

$F(s) = L[f(t)] = L[e^{-2t}\sin 3t]$

$$= L[\sin 3t]_{s\to s+2}$$

$$= \left[\frac{3}{s^2+9}\right]_{s \to s+2}.$$

$$= \frac{3}{(s+2)^2+9} = \frac{3}{s^2+4s+13}.$$

Initial Value Theorem:

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s F(s)$$

$$\text{LHS} = \lim_{t \to 0} f(t) = \lim_{t \to 0} e^{-2t} \sin 3t = 0.$$

$$\text{RHS} = \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \frac{3s}{s^2+4s+13}.$$

$$= \lim_{s \to \infty} \frac{3}{s\left[1+\frac{4}{s}+\frac{13}{s^2}\right]}$$

$$= 0.$$

$\therefore$ LHS = RHS

Hence initial value theorem verified.

Final value theorem:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s).$$

$$\text{LHS} = \lim_{t \to \infty} f(t) = \lim_{t \to \infty} e^{-2t} \sin 3t = 0.$$

$$\text{RHS} = \lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{3s}{s^2+4s+13} = 0.$$

LHS = RHS

Hence final value theorem verified

H/W

3) verify I.V.T for $f(t) = 2+3\cos t$, [soln: 5].

4) verify F.V.T for $f(t) = t^2 e^{-3t}$, [soln : 0]

5) verify the I.V.T for $f(t) = a\cos bt$.

soln:

The initial value theorem is

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Now $f(t) = a\cos bt$ and $F(s) = a \cdot \dfrac{s}{s^2+b^2}$

So $\lim_{t\to 0} f(t) = \lim_{t\to 0} a\cos bt = a\times 1 = a$ — ①

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left(\frac{as}{s^2+b^2}\right).$$

$$= \lim_{s\to\infty} \frac{a}{1+b^2/s^2} = a \quad — ②$$

The Equality of the limits ① & ② verifies the theorem.

6) verify the I.V.T for $f(t) = ae^{-bt}$.

Soln: $f(t) = ae^{-bt}$.

The I.V.T is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$.

Now $f(t) = ae^{-bt}$ & $F(s) = a \cdot \dfrac{1}{s+b}$.

So $\lim_{t\to 0} f(t) = \lim_{t\to 0} ae^{-bt} = a$.

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} a \cdot \frac{s}{s+b} = \lim_{s\to\infty} a \cdot \frac{1}{1+b/s} = a.$$

Hence initial value theorem is verified.

LAPLACE TRANSFORM OF PERIODIC FUNCTION:

Let $f(t)$ be a periodic function with period 'a'

then $L[f(t)] = \dfrac{1}{1-e^{-as}} \displaystyle\int_0^a e^{-st} f(t)\, dt.$

ie $f(t) = f(t+a) = f(t+2a) = f(t+3a) = \ldots$

PROBLEMS:

1) Find the Laplace Transform of the periodic function $f(t) = t$ for $0 < t < 4$ and $f(t) = f(t+4)$

Soln: The given function is a periodic function with period 4.

$$\therefore L[f(t)] = \frac{1}{1-e^{-4s}} \int_0^4 t e^{-st}\, dt.$$

$$= \frac{1}{1-e^{-4s}} \left[t \cdot \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^4$$

$$= \frac{1}{1-e^{-4s}} \left[-\frac{4e^{-4s}}{s} - \frac{e^{-4s}}{s^2} + \frac{1}{s^2} \right].$$

$$= \frac{1-4se^{-4s} - e^{-4s}}{(1-e^{-4s}) \cdot s^2}.$$

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2) Find the Laplace Transform of

$$f(t) = \begin{cases} 1 & , 0 < t < a/2 \\ -1 & , a/2 < t < a \end{cases}, \quad f(a+t) = f(t)$$

Soln: The given function is a periodic function with period a.

$$\therefore L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a f(t)\, e^{-st}\, dt.$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} 1 \cdot e^{-st} dt + \int_{a/2}^{a} (-1) e^{-st}\, dt \right].$$

$$= \frac{1}{1-e^{-as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^{a/2} - \left(\frac{e^{-st}}{-s} \right)_{a/2}^{a} \right].$$

$$= \frac{1}{1-e^{-as}} \left[\frac{e^{-as/2}}{-s} + \frac{1}{s} + \frac{e^{-as}}{s} - \frac{e^{-as/2}}{s} \right].$$

$$= \frac{1}{1-e^{-as}} \left[\frac{1+e^{-as} - 2e^{-as/2}}{s} \right].$$

$$= \frac{1}{1-e^{-as}} \left[\frac{1+e^{-as} - 2e^{-as/2}}{s} \right].$$

3.) Find the Laplace transform of the periodic function

$$f(t) = \begin{cases} \sin t & , 0 < t < \pi \\ 0 & , \pi < t < 2\pi . \end{cases}$$

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Soln: The given function is a periodic function with period 2π.

$$L[f(t)] = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} f(t)\, e^{-st}\, dt.$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} \sin t\, e^{-st}\, dt + \int_{\pi}^{2\pi} (0)\, e^{-st}\, dt \right].$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}}{s^2+1} \left(-s\sin t - \cos t\right) \right]_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-s\pi}}{s^2+1} \left(-s\sin\pi - \cos\pi\right) - \frac{1}{s^2+1}(-1) \right].$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1} \right] \qquad \left[\begin{array}{l} \because \sin\pi = 0 \\ \cos\pi = -1 \end{array} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{1+e^{-s\pi}}{s^2+1} \right]$$

$$= \frac{\cancel{(e^{-s\pi}+1)}}{(1-e^{-s\pi})\cancel{(1+e^{-s\pi})}(s^2+1)} = \frac{1}{(1-e^{-s\pi})(s^2+1)}.$$

$$\therefore L[f(t)] = \frac{1}{(1-e^{-\pi s})(s^2+1)}.$$

4) Find the Laplace Transform of the periodic function $f(t) = \begin{cases} t & , 0 < t < 1 \\ 0 & , 1 < t < 2 \end{cases}$ & $f(t+2) = f(t)$

Soln: $\dfrac{1 - e^{-s}(s+1)}{s^2(1-e^{-2s})}$.

5) Find the Laplace transform of the periodic function

$$f(t) = \begin{cases} t & , \quad 0 < t < a \\ 2a - t & , \quad a < t < 2a. \end{cases}$$

Soln: The given function is a periodic function with Period 2a.

$$L[f(t)] = \frac{1}{1-e^{-2as}}\left[\int_0^a t e^{-st}\,dt + \int_a^{2a} (2a-t)e^{-st}\,dt\right].$$

$$= \frac{1}{1-e^{-2as}}\left[\left(\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right)_0^a + \left((2a-t)\frac{e^{-st}}{-s} - (-1)\frac{e^{-st}}{s^2}\right)_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{ae^{-as}}{-s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1 - 2e^{-as} + e^{-2as}}{s^2}\right].$$

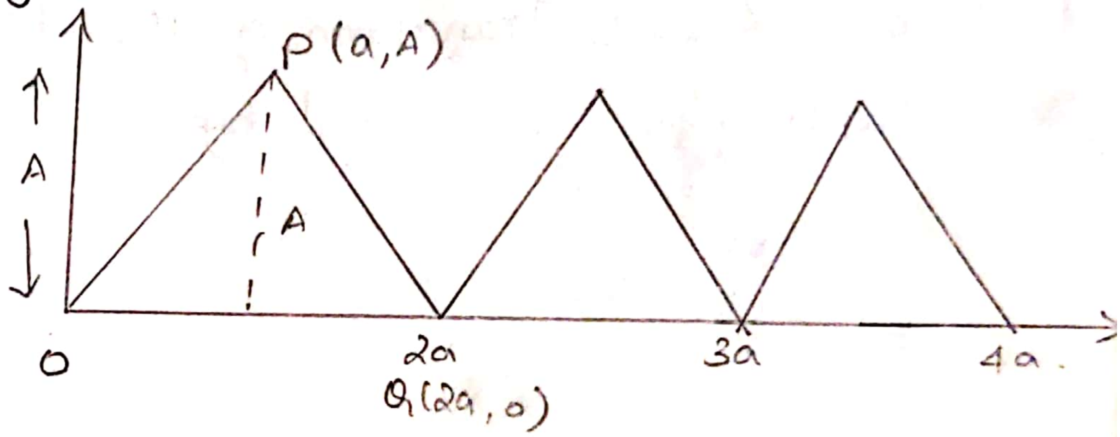
$$= \frac{1}{1-e^{-2as}} \frac{(1-e^{-as})^2}{s^2} = \frac{(1-e^{-as})^{\cancel{2}}}{(\cancel{1-e^{-as}})(1+e^{-as}).s^2}$$

$$= \frac{1-e^{-as}}{(1+e^{-as})} \cdot \frac{1}{s^2}.$$

$$= \frac{1}{s^2} \cdot \frac{1-e^{-2as/2}}{1+e^{-2as/2}} \qquad \left(\because \tanh x = \frac{1-e^{-2x}}{1+e^{-2x}} \right).$$

$$= \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$$

6) find the transform of the triangle wave represented by the figure given below.



Soln:

Equation of OP (0,0), (a,A).

$$\frac{x-0}{a-0} = \frac{y-0}{A-0}$$

$$\Rightarrow y = \frac{A}{a}x \quad - (1)$$

Equation of PQ.

$$\frac{y-A}{0-A} = \frac{x-a}{2a-a}$$

$$\Rightarrow \frac{y-A}{-A} = \frac{x-a}{a}.$$

$$\rightarrow y = \frac{A}{a}(a-x) + A \quad - (2)$$

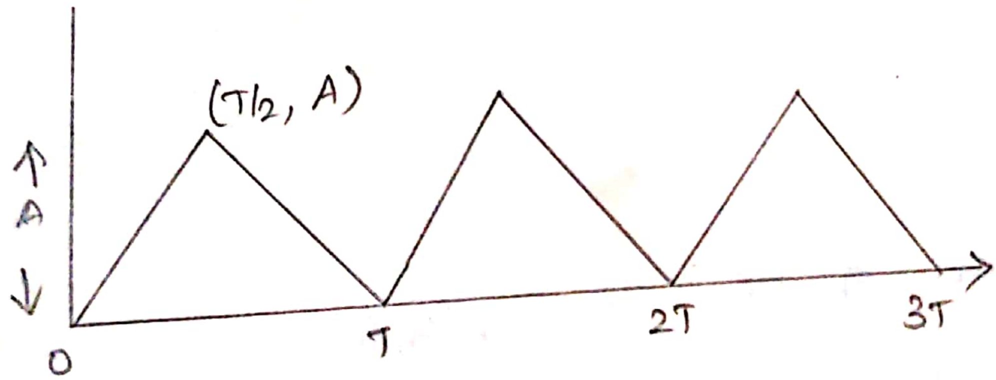
$\Rightarrow y = A - \frac{xA}{a} + A = 2A - \frac{Ax}{a}$.

$\therefore y = \frac{A}{a}(2a - x)$, $a < x < 2a$

$$f(t) = \begin{cases} \frac{At}{a} & , 0 < t < a. \\ \frac{A}{a}(2a-t) & , a < t < 2a. \end{cases}$$

Soln: $L[f(t)] = \frac{A}{as^2} \tanh\left(\frac{as}{2}\right)$.

7) obtain Laplace Transform of the periodic Saw tooth wave represented by.



Soln: $L[f(t)] = \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$.

INVERSE LAPLACE TRANSFORM:

Definition:

If the laplace Transform of a function $f(t)$ is $F(s)$ ie $L[f(t)] = F(s)$, then $f(t)$ is called an Inverse laplace Transform of $F(s)$. It is denoted by $f(t) = L^{-1}[F(s)]$.

Here L^{-1} is called the inverse laplace Transform operator.

Inverse Laplace Transform of some Basic functions:

1.) $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

2.) $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

3) $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

4) $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$

5) $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

6) $L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at.$

7) $L^{-1}\left[\frac{1}{s}\right] = 1.$

8) $L^{-1}\left[\frac{1}{s^2}\right] = t$

9) $L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$.

10) $L^{-1}\left[\frac{1}{(s-a)^2}\right] = t e^{at}$.

11) Shifting property : We know that if

$L[f(t)] = f(s)$, then

$L[e^{-at} f(t)] = f(s+a)$

Hence $L^{-1}[f(s+a)] = e^{-at} f(t)$.

$= e^{-at} L^{-1}[f(s)]$.

1) find $L^{-1}\left[\frac{1}{s-3} + \frac{1}{s} + \frac{s}{s^2-4}\right]$.

Soln:

$L^{-1}\left[\frac{1}{s-3} + \frac{1}{s} + \frac{s}{s^2-4}\right] = L^{-1}\left[\frac{1}{s-3}\right] + L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{s}{s^2-4}\right]$

$= e^{3t} + 1 + \cosh 2t$.

2) find $L^{-1}\left[\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9}\right]$.

Soln:

$L^{-1}\left[\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9}\right] = L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s+4}\right] +$

$L^{-1}\left[\frac{1}{s^2+4}\right] + L^{-1}\left[\frac{s}{s^2-9}\right]$.

$= t + e^{-4t} + \frac{1}{2}\sin 2t + \cosh 3t$.

3) Find $L^{-1}\left[\frac{s-3}{(s-3)^2+4}\right]$ (Shifting property)

Soln:

$$L^{-1}\left[\frac{s-3}{(s-3)^2+4}\right] = e^{3t} L^{-1}\left[\frac{s}{s^2+4}\right] = e^{3t}\cos 2t.$$

4) $L^{-1}\left[\frac{s}{s^2+4s+5}\right]$.

QB ⑫ X 2 marks.

$$L^{-1}\left[\frac{3s+2}{s^2-4}\right] = L^{-1}\left[\frac{3s}{s^2-4}\right] + L^{-1}\left[\frac{2}{s^2-4}\right]$$

$$= 3L^{-1}\left[\frac{s}{s^2-4}\right] + L^{-1}\left[\frac{2}{s^2-4}\right]$$

$$= 3\cosh 2t + \sinh 2t$$

Soln:

$$L^{-1}\left[\frac{s}{s^2+4s+5}\right]$$

$$L^{-1}\left[\frac{s}{s^2+4s+5}\right] = L^{-1}\left[\frac{s}{(s+2)^2+1}\right].$$

$$= L^{-1}\left[\frac{(s+2)-2}{(s+2)^2+1}\right].$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] - 2L^{-1}\left[\frac{1}{(s+2)^2+1}\right].$$

$$= e^{-2t}L^{-1}\left[\frac{s}{s^2+1}\right] - 2e^{-2t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= e^{-2t}\cos t - 2e^{-2t}\sin t.$$

RESULTS:

1) $L^{-1}[s\,f(s)] = \frac{d}{dt} L^{-1}[f(s)]$.

2) $L^{-1}\left[\frac{f(s)}{s}\right] = \int_0^t L^{-1}[f(s)]\,dt$.

1) find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$.

Soln:

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = L^{-1}\left[s \cdot \frac{1}{(s+2)^2}\right].$$

$$= \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2}\right].$$

$$= \frac{d}{dt} e^{-2t} L^{-1}\left[\frac{1}{s^2}\right]$$

$$= \frac{d}{dt}(e^{-2t} \cdot t)$$

$$= e^{-2t} + te^{-2t}(-2).$$

$$= e^{-2t}(1-2t).$$

QB qmdv.

2) Find $L^{-1}\left[\frac{s-3}{s^2+4s+13}\right]$

Soln:

$$L^{-1}\left[\frac{s-3}{s^2+4s+13}\right] = L^{-1}\left[\frac{s}{s^2+4s+13}\right] - L^{-1}\left[\frac{3}{s^2+4s+13}\right].$$

$$= \frac{d}{dt} L^{-1}\left[\frac{1}{s^2+4s+13}\right] - 3\, L^{-1}\left[\frac{1}{s^2+4s+13}\right].$$

$$= \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2+9}\right] - 3\, L^{-1}\left[\frac{1}{(s+2)^2+9}\right].$$

$$= \frac{d}{dt} e^{-2t} L^{-1}\left[\frac{1}{s^2+9}\right] - 3\, e^{-2t} L^{-1}\left[\frac{1}{s^2+9}\right].$$

$$= \frac{d}{dt} e^{-2t} \cdot \frac{\sin 3t}{3} - 3e^{-2t} \cdot \frac{\sin 3t}{3}.$$

$$= \frac{1}{3}\left[3e^{-2t}\cos 3t - 2\sin 3t\, e^{-2t}\right] - e^{-2t}\sin 3t.$$

$$= e^{-2t}\cos 3t - \frac{5}{3}\sin 3t \cdot e^{-2t}.$$

3) find $L^{-1}\left[\frac{s^2}{(s-2)^3}\right]$.

Soln:

$$L^{-1}\left[\frac{s^2}{(s-2)^3}\right] = L^{-1}\left[s \cdot \frac{s}{(s-2)^3}\right]$$

$$= \frac{d}{dt} L^{-1}\left[\frac{s}{(s-2)^3}\right].$$

$$= \frac{d}{dt} \cdot \frac{d}{dt} L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$= \frac{d^2}{dt^2} \cdot e^{2t} L^{-1}\left[\frac{1}{s^3}\right].$$

$$= \frac{d^2}{dt^2} \cdot e^{2t} \cdot \frac{t^2}{2}.$$

$$= \frac{d}{dt}\left[e^{2t} \cdot (t^2 + t)\right].$$

$$= (2t^2 + 4t + 1)e^{2t}.$$

4) find $L^{-1}\left[\frac{s}{(s+2)^2+4}\right]$.

Soln:

$$L^{-1}\left[\frac{s}{(s+2)^2+4}\right] = \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2+4}\right]$$

$$= \frac{d}{dt} e^{-2t} L^{-1}\left[\frac{1}{s^2+2^2}\right].$$

$$= \frac{d}{dt} e^{-2t} \cdot \frac{1}{2}\sin 2t.$$

$$= \frac{1}{2}\left[2e^{-2t}\cos 2t - 2e^{-2t}\sin 2t\right].$$

$$= e^{-2t}(\cos 2t - \sin 2t).$$

5) find $L^{-1}\left[\frac{s^2}{(s-1)^4}\right]$.

Soln:

$$L^{-1}\left[\frac{s^2}{(s-1)^4}\right] = L^{-1}\left[s\cdot\frac{s}{(s-1)^4}\right] = \frac{d}{dt} L^{-1}\left[\frac{s}{(s-1)^4}\right].$$

$$= \frac{d}{dt} L^{-1}\left[\frac{(s-1)+1}{(s-1)^4}\right].$$

$$= \frac{d}{dt} \cdot L^{-1}\left[\frac{s-1}{(s-1)^4} + \frac{1}{(s-1)^4}\right].$$

$$= \frac{d}{dt} L^{-1}\left[\frac{1}{(s-1)^3} + \frac{1}{(s-1)^4}\right].$$

$$= \frac{d}{dt}\left[L^{-1}\left[\frac{1}{(s-1)^3}\right] + L^{-1}\left[\frac{1}{(s-1)^4}\right]\right].$$

$$= \frac{d}{dt}\left[e^t L^{-1}\left(\frac{1}{s^3}\right) + e^t L^{-1}\left(\frac{1}{s^4}\right)\right].$$

$$= \frac{d}{dt}\left[e^t \cdot \frac{t^2}{2} + e^t \cdot \frac{t^3}{6}\right].$$

$$= \frac{1}{2}\left[e^t \cdot 2t + t^2 e^t\right] + \frac{1}{6}\left[e^t 3t^2 + t^3 \cdot e^t\right].$$

$$= te^t + e^t \cdot t^2 + \frac{t^3 \cdot e^t}{6}.$$

$$= \frac{d^2}{dt^2} e^t L^{-1}\left[\frac{1}{s^4}\right].$$

$$= \frac{d^2}{dt^2} e^t \cdot \frac{t^3}{6} = \frac{1}{6}\left[t^3 e^t + 3t^2 e^t\right]$$

$$= \frac{1}{6} e^t\left[3t^2 + 6t\right] + \frac{1}{6}\left(t^3 + 3t^2\right)e^t$$

$$= \frac{1}{6} e^t\left[t^3 + 6t^2 + 6t\right]$$

QB.

1) Find $L^{-1}\left[\frac{1}{s(s+3)}\right]$.

2 mark.

soln:

$$L^{-1}\left[\frac{1}{s(s+3)}\right] = L^{-1}\left[\frac{1}{s} \cdot \frac{1}{s+3}\right]$$

$$= \int_0^t L^{-1}\left[\frac{1}{s+3}\right] dt.$$

$$= \int_0^t e^{-3t} dt = \left[\frac{e^{-3t}}{-3}\right]_0^t = \frac{e^{-3t}}{-3} + \frac{1}{3}.$$

$$= \frac{1 - e^{-3t}}{3}.$$

2) find $L^{-1}\left[\frac{1}{s^2(s+a)}\right]$.

soln:

$$L^{-1}\left[\frac{1}{s^2(s+a)}\right] = \int_0^t L^{-1}\left[\frac{1}{s(s+a)}\right] dt. \quad — (1)$$

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left[\frac{1}{s+a}\right] dt$$

$$= \int_0^t e^{-at} dt$$

$$=\left[\frac{e^{-at}}{-a}\right]_0^t = \frac{1-e^{-at}}{a}.$$

$$\therefore \text{①} \Rightarrow L^{-1}\left[\frac{1}{s^2(s+a)}\right] = \int_0^t \left(\frac{1-e^{-at}}{a}\right) dt.$$

$$= \frac{1}{a}\left[t + \frac{e^{-at}}{a}\right]_0^t.$$

$$= \frac{1}{a}\left[t + \frac{e^{-at}}{a} - \frac{1}{a}\right] = \frac{1}{a^2}\left[at + e^{-at} - 1\right].$$

3) find $L^{-1}\left[\frac{1}{s(s+2)^3}\right]$

Soln:

$$L^{-1}\left[\frac{1}{s(s+2)^3}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{(s+2)^3}\right]$$

$$= \int_0^t L^{-1}\left[\frac{1}{(s+2)^3}\right] dt.$$

$$= \int_0^t e^{-2t} L^{-1}\left[\frac{1}{s^3}\right] dt.$$

$$= \int_0^t e^{-2t}\cdot\frac{t^2}{2}\, dt.$$

$$= \frac{1}{2}\int_0^t t^2 e^{-2t}\, dt.$$

$$= \frac{1}{2}\left[t^2\left(\frac{e^{-2t}}{-2}\right) - 2t\left(\frac{e^{-2t}}{4}\right) + 2\left(\frac{e^{-2t}}{-8}\right)\right]_0^t$$

$$= \frac{1}{2}\left[-\frac{t^2e^{-2t}}{2} - \frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + \frac{1}{4}\right].$$

$$= \frac{1}{8}\left[-2t^2e^{-2t} - 2te^{-2t} - e^{-2t} + 1\right].$$

4) $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

Soln:

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+a^2}\right].$$

$$= \int_0^t L^{-1}\left[\frac{1}{s^2+a^2}\right]dt.$$

$$= \int_0^t \frac{\sin at}{a}\,dt.$$

$$= \frac{1}{a}\left[-\frac{\cos at}{a}\right]_0^t = \frac{1}{a^2}\left[-\cos at + 1\right]$$

$$= \frac{1}{a^2}\left[1 - \cos at\right]$$

5) find $L^{-1}\left[\frac{1}{s(s^2-2s+5)}\right]$.

Soln:

$$L^{-1}\left[\frac{1}{s(s^2-2s+5)}\right] = L^{-1}\left[\cdot\frac{1}{s}\cdot\frac{1}{(s^2-2s+5)}\right]$$

$$= \int_0^t L^{-1}\left[\frac{1}{s^2-2s+5}\right]dt$$

$$= \int_0^t L^{-1}\left[\frac{1}{(s-1)^2+2^2}\right] dt.$$

$$= \int_0^t e^t L^{-1}\left[\frac{1}{s^2+2^2}\right] dt.$$

$$= \int_0^t e^t \cdot \frac{\sin 2t}{2}\, dt.$$

$$= \frac{1}{2}\int_0^t e^t \sin 2t\, dt.$$

$$= \frac{1}{2}\left[\frac{e^t}{1^2+2^2}(\sin 2t - 2\cos 2t)\right]_0^t.$$

$$= \frac{1}{10}\left[e^t \sin 2t - 2e^t \cos 2t\right]_0^t.$$

$$= \frac{1}{10}\left[e^t \sin 2t - 2e^t \cos 2t - 0 + 2\right].$$

$$= \frac{1}{10}\left[e^t \sin 2t - 2e^t \cos 2t + 2\right].$$

6) find $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$.

Soln:

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s(s^2+a^2)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s}\cdot\frac{s}{(s^2+a^2)^2}\right]$$

$$= \int_0^t L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]dt.$$

$$= \int_0^t t\,\frac{\sin at}{2a}\,dt \qquad \left[\because L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = t\,\frac{\sin at}{2a}\right].$$

$$= \frac{1}{2a}\int_0^t t\sin at\,dt$$

$$= \frac{1}{2a}\left[t\left(-\frac{\cos at}{a}\right) - \left(\frac{-\sin at}{a^2}\right)\right]_0^t$$

$$= \frac{1}{2a}\left[-\frac{t\cos at}{a} + \frac{\sin at}{a^2}\right]$$

$$= \frac{1}{2a^3}\left[-at\cos at + \sin at\right].$$

RESULT :

3) If $f(s)$ is interms of log, $\tan^{-1}$, $\cot^{-1}$ then

$$L^{-1}[f(s)] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}f(s)\right]$$

QB 8 marks.

1) Find the Inverse Laplace Transform of $\log\left(\frac{s+a}{s+b}\right)$.

X.

Soln: $f(s) = \log\left(\frac{s+a}{s+b}\right)$.

$$L^{-1}[f(s)] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\frac{s+a}{s+b}\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left[\log(s+a) - \log(s+b)\right]\right].$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right].$$

$$= -\frac{1}{t}\left[e^{-at} - e^{-bt}\right] = \frac{e^{-bt} - e^{-at}}{t}.$$

2) find $L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right]$.

Soln:

$$L^{-1}[F(s)] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(\frac{1+s}{s^2}\right)\right].$$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left[\log(1+s) - \log s^2\right]\right].$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{1+s} - \frac{2s}{s^2}\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{1+s} - \frac{2}{s}\right].$$

$$= -\frac{1}{t}\left[e^{-t} - 2\right] = \frac{2 - e^{-t}}{t}.$$

3) Find $L^{-1}\left[\tan^{-1}\left(\frac{s+3}{2}\right)\right]$ ✓

Soln:

$$L^{-1}[F(s)] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\tan^{-1}\left(\frac{s+3}{2}\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{2}\cdot\frac{1}{1+\frac{(s+3)^2}{4}}\right]$$

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$$= -\frac{1}{t} L^{-1}\left[\frac{1}{2}\cdot\frac{4}{4+(s+3)^2}\right]$$ $\left[\because \frac{d}{dx}\tan^{-1}(x/a) = \frac{1}{a}\cdot\frac{1}{1+(x/a)^2}\right]$

$$= -\frac{1}{t} L^{-1}\left[\frac{2}{(s+3)^2+2^2}\right]$$

$$= -\frac{1}{t}\cdot e^{-3t} L^{-1}\left[\frac{2}{s^2+2^2}\right]$$

$$= -\frac{1}{t} e^{-3t}\sin 2t.$$

4) Find $L^{-1}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]$.

Soln:

$$L^{-1}\left[\log\frac{s^2+a^2}{s^2+b^2}\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\frac{s^2+a^2}{s^2+b^2}\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]\right].$$

$$= -\frac{1}{t} L^{-1}\left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right]$$

$$= -\frac{1}{t}\left[2 L^{-1}\left(\frac{s}{s^2+a^2}\right) - 2 L^{-1}\left(\frac{s}{s^2+b^2}\right)\right]$$

$$= -\frac{1}{t}\left[2\cos at - 2\cos bt\right]$$

$$= \frac{2(\cos bt - \cos at)}{t}$$

5) find $L^{-1}\left[\log\left(1+\frac{\omega^2}{s^2}\right)\right]$.

Soln:

$$L^{-1}\left[\log\left(1+\frac{\omega^2}{s^2}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\left(\frac{s^2+\omega^2}{s^2}\right)\right].$$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left(\log(s^2+\omega^2) - \log s^2\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{2s}{s^2+\omega^2} - \frac{2s}{s^2}\right]$$

$$= -\frac{1}{t}\cdot 2\, L^{-1}\left[\frac{s}{s^2+\omega^2} - \frac{1}{s}\right]$$

$$= -\frac{1}{t}\cdot 2(\cos\omega t - 1) = \frac{2(1-\cos\omega t)}{t}.$$

6) $L^{-1}\left[\log\frac{s}{(s^2+4)^2}\right]$ soln: $\frac{1}{t}(4\cos 2t - 1)$.

8 marks.

7) $L^{-1}\left[\log\frac{s^2+1}{(s-1)^2}\right]$

*AB

Soln:

$$L^{-1}\left[\log\frac{s^2+1}{(s-1)^2}\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\frac{s^2+1}{(s-1)^2}\right].$$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left(\log(s^2+1) - \log(s-1)^2\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\cancel{\frac{d}{ds}}\ \frac{2s}{s^2+1} - \frac{d}{ds}\left(2\log(s-1)\right)\right].$$

$$= -\frac{1}{t}L^{-1}\left[\frac{2s}{s^2+1} - \frac{2}{s-1}\right]$$

$$= -\frac{1}{t}\left(2\cos t - 2e^{t}\right).$$

$$= \frac{2\left(e^{t} - \cos t\right)}{t}.$$

8) $L^{-1}\left[\tan^{-1}\frac{a}{s}\right]$

Soln:

$$L^{-1}\left[\tan^{-1}\frac{a}{s}\right] = -\frac{1}{t}\, L^{-1}\left[\frac{d}{ds}\tan^{-1}\frac{a}{s}\right].$$

$$= -\frac{1}{t}\, L^{-1}\left[\frac{1}{1+(a/s)^2}\left(-a/s^2\right)\right].$$

$$= +\frac{1}{t}\, L^{-1}\left[\frac{a}{s^2+a^2}\right].$$

$$= \frac{\sin at}{t}.$$

9) $L^{-1}\left[\cot^{-1}\frac{s}{a}\right]$

Soln:

$$L^{-1}\left[\cot^{-1}\frac{s}{a}\right] = -\frac{1}{t}\, L^{-1}\left[\frac{d}{ds}\cot^{-1}\frac{s}{a}\right]$$

$$= -\frac{1}{t}\, L^{-1}\left[\frac{-1}{1+(s/a)^2}\cdot 1/a\right].$$

$$= \frac{1}{t}\, L^{-1}\left[\frac{a}{s^2+a^2}\right].$$

$$= \frac{\sin at}{t}.$$

10) Find the inverse Laplace transform of $\tan^{-1}(a/s) + \cot^{-1}(s/b)$.

Q.B 8 marks

Soln:

$$L^{-1}\left[\tan^{-1}\frac{a}{s} + \cot^{-1}\frac{s}{b}\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left(\tan^{-1}\frac{a}{s} + \cot^{-1}\frac{s}{b}\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{1+(a/s)^2}(-a/s^2) - \frac{1}{1+(s/b)^2}\cdot(1/b)\right]$$

$$= +\frac{1}{t} L^{-1}\left[\frac{a}{s^2+a^2} + \frac{b}{s^2+b^2}\right].$$

$$= \frac{1}{t}(\sin at + \cancel{\cos}\ \sin bt)$$

11) $L^{-1}\left[\cot^{-1}\frac{a}{s+b}\right]$

Soln:

$$L^{-1}\left[\cot^{-1}\frac{a}{s+b}\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\cot^{-1}\frac{a}{s+b}\right].$$

$$= -\frac{1}{t} L^{-1}\left[-\frac{1}{1+(a/s+b)^2}\cdot -a/(s+b)^2\right].$$

$$= -\frac{1}{t} L^{-1}\left[\frac{(s+b)^2}{(s+b)^2+a^2}\cdot\frac{a}{(s+b)^2}\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{a}{(s+b)^2+a^2}\right]$$

$$= -\frac{1}{t} \cdot e^{-bt} L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$= -\frac{1}{t} e^{-bt}. \sin at.$$

12) $L^{-1}\left[\cot^{-1}\frac{s+b}{a}\right]$ soln: $\frac{e^{-bt}\sin at}{t}$

13) $L^{-1}\left[\tan^{-1}\frac{a}{s+b}\right]$ soln: $\frac{e^{-bt}\sin at}{t}$.

METHOD OF PARTIAL FRACTIONS

1) Find $L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]$.

Soln:

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}.$$

$$5s^2-15s-11 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1).$$

$s=-1 \Rightarrow -27A = 9 \Rightarrow \boxed{A = -1/3}$

$s=2 \Rightarrow 3D = 21 \Rightarrow \boxed{D=-7}$

Equating the coefficient of s^3, $A+B=0 \Rightarrow \boxed{B = 1/3}$

Equating the constant, $-8A+4B-2C+D=-11$.

$$8/3 + 4/3 - 2C - 7 = -11$$

$$\Rightarrow \boxed{C=4}$$

$$\therefore \frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{-1/3}{s+1} + \frac{1/3}{s-2} + \frac{4}{(s-2)^2} - \frac{7}{(s-2)^3}$$

$$= -\frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s-2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2}\right] - 7L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}t - 7e^{2t}\cdot t^2/2.$$

2) find $L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right]$

Soln:

$$\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4s+13}.$$

$$\Rightarrow 1-s = A(s^2+4s+13) + (Bs+C)(s+1).$$

put $s=-1 \Rightarrow 2=10A \Rightarrow \boxed{A=1/5}$

Equating the coefficient of s^2.

$0=A+B \Rightarrow B=-A \Rightarrow \boxed{B=-1/5}$

Equating the constant coefficient

$1=13A+C \Rightarrow C=1-13A = 1-\frac{13}{5} = -8/5$

$\Rightarrow \boxed{C=-8/5}$

$$\therefore \frac{1-s}{(s+1)(s^2+4s+13)} = \frac{1/5}{s+1} + \frac{-1/5\,s - 8/5}{s^2+4s+13}$$

$$L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right] = \frac{1}{5}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{5}L^{-1}\left[\frac{s+8}{s^2+4s+13}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}L^{-1}\left[\frac{s+2+6}{(s+2)^2+9}\right].$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}L^{-1}\left[\frac{s+6}{s^2+9}\right].$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}L^{-1}\left[\frac{s}{s^2+9} + \frac{6}{s^2+9}\right].$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\left[\cos 3t + \frac{6}{3}\sin 3t\right].$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos 3t - \frac{2}{5}e^{-2t}\sin 3t.$$

3) find $L^{-1}\left[\frac{1}{(s+1)(s+3)}\right]$.

soln.

$$\frac{1}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}.$$

$$1 = A(s+3) + B(s+1).$$

$s=-1 \Rightarrow A = 1/2.$

$s=-3 \Rightarrow B = -1/2$

$$\therefore \frac{1}{(s+1)(s+3)} = \frac{1/2}{s+1} + \frac{-1/2}{s+3}.$$

$$L^{-1}\left[\frac{1}{(s+1)(s+3)}\right] = \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{2}\left(e^{-t} - e^{-3t}\right)$$

4) find $L^{-1}\left[\frac{4s+5}{(s-1)^2 \cdot (s+2)}\right]$

Soln:

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}.$$

$$4s+5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2.$$

put $s=1 \Rightarrow 9 = 3B \Rightarrow \boxed{B=3}$

put $s=-2 \Rightarrow -3 = 9C \Rightarrow \boxed{C=-1/3}$

Compare the coefficient of s^2.

$$0 = A + C \Rightarrow A = -C \Rightarrow \boxed{A = 1/3}$$

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{1/3}{s-1} + \frac{3}{(s-1)^2} + \frac{-1/3}{s+2}$$

$$\therefore L^{-1}\left[\frac{4s+5}{(s-1)^2(s+2)}\right] = \frac{1}{3}L^{-1}\left[\frac{1}{s-1}\right] + 3L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3}L^{-1}\left[\frac{1}{s+2}\right].$$

$$= \frac{1}{3}e^{t} + 3e^{t}L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{3}e^{-2t}$$

$$= \frac{1}{3}e^{t} + 3e^{t}.t - \frac{1}{3}e^{-2t}.$$

CONVOLUTION OF TWO FUNCTIONS :

If $f(t)$ and $g(t)$ are given functions, then the convolution of $f(t)$ and $g(t)$ is defined as.

$\int_0^t f(u)\, g(t-u)\, du$ and it is denoted by $f(t) * g(t)$.

ie $f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du = L^{-1}[F(s) \cdot G(s)]$.

CONVOLUTION THEOREM :

Laplace transform of convolution of two functions is product of their Laplace transforms

ie $L[f(t) * g(t)] = L[f(t)] \cdot L[g(t)]$.

NOTE :

Let $L[f(t)] = F(s)$. $L[g(t)] = G(s)$

Then $L[f(t) * g(t)] = L[f(t)]\, L[g(t)]$.

$= F(s) \cdot G(s)$

$\Rightarrow L^{-1}[F(s) \cdot G(s)] = f(t) * g(t)$.

$= L^{-1}[F(s)] * L^{-1}[G(s)]$

PROBLEMS :

1) Evaluate $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ using convolution theorem.

X QB 16 marks.

Soln :

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \cos at * \frac{\sin at}{a}.$$

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du.$$

$$= \frac{1}{a} \int_0^t \cos au \sin a(t-u)\, du.$$

$$= \frac{1}{a} \int_0^t \cos au \sin (at-au)\, du.$$

$$\left\{ \sin A \cos B = \frac{1}{2} \left[\sin (A+B) + \sin(A-B) \right] \right\} \rightarrow \text{formula.}$$

$$= \frac{1}{2a} \int_0^t \left[\sin (at-au+au) + \sin t\,(at-au-au) \right] du$$

$$= \frac{1}{2a} \int_0^t \left[\sin at + \sin(at-2au) \right] du$$

$$= \frac{1}{2a} \left[\left(\sin at \cdot u \right)_0^t + \left(\frac{-\cos(at-2au)}{-2a} \right)_0^t \right]$$

$$= \frac{1}{2a} \left[t \sin at + \frac{\cos(at-2at)}{2a} - \frac{\cos(at)}{2a} \right]$$

$$= \frac{1}{2a} \left[t \sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a} \right]$$

$$= \frac{t \sin at}{2a}$$

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2) Find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$ using convolution theorem.

Soln:

$$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\frac{1}{s+a}\right] * L^{-1}\left[\frac{1}{s+b}\right].$$

$$= e^{-at} * e^{-bt}.$$

$$f(t) * g(t) = \int_0^t e^{-au}\, e^{-b(t-u)}\, du.$$

$$= \int_0^t e^{-au} \cdot e^{-bt} \cdot e^{bu}\, du.$$

$$= e^{-bt} \int_0^t e^{-(a-b)u}\, du.$$

$$= e^{-bt} \left[\frac{e^{-(a-b)u}}{-(a-b)}\right]_0^t.$$

$$= \frac{e^{-bt}}{-(a-b)}\left[e^{-(a-b)t} - 1\right].$$

$$= \frac{-e^{-bt}}{a-b}\left[e^{-at} \cdot e^{bt} - 1\right].$$

$$= \frac{-e^{-at} + e^{-bt}}{a-b}.$$

$$= \frac{e^{-bt} - e^{-at}}{a-b} \quad \therefore = \frac{1}{a-b}\left(e^{-bt} - e^{-at}\right).$$

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3) Find $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ using convolution theorem.

Soln:

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \cos at * \cos bt.$$

$$= \int_0^t \cos au \cos b(t-u)\, du.$$

$$= \int_0^t \cos au \cos(bt-bu)\, du.$$

$$\left[\because \cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]\right].$$

$$= \frac{1}{2}\int_0^t \left[\cos(au+bt-bu) + \cos(au-bt+bu)\right] du.$$

$$= \frac{1}{2}\int_0^t \left[\cos\left((a-b)u+bt\right) + \cos\left((a+b)u-bt\right)\right] du.$$

$$= \frac{1}{2}\left[\frac{\sin\left[(a-b)u+bt\right]}{(a-b)} + \frac{\sin\left[(a+b)u-bt\right]}{(a+b)}\right]_0^t.$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{(a+b)\sin at + (a-b)\sin at - (a+b)\sin bt + (a-b)\sin bt}{(a-b)(a+b)}\right]$$

$$= \frac{1}{2}\left[\frac{a\sin at + \cancel{b\sin at} + a\sin at - \cancel{b\sin at} - \cancel{a\sin bt} + b\sin bt + \cancel{a\sin bt} - b\sin bt}{a^2 - b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at - 2b\sin bt}{a^2 - b^2}\right].$$

4) find $L^{-1}\left[\frac{1}{s^2(s+1)^2}\right]$ using convolution theorem.

Soln:

$$L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = L^{-1}\left[\frac{1}{s^2}\right] * L^{-1}\left[\frac{1}{(s+1)^2}\right].$$

$$= t * e^{-t} L^{-1}\left[\frac{1}{s^2}\right].$$

$$= t * te^{-t}.$$

$$= te^{-t} * t.$$

$$= \int_0^t u e^{-u}(t-u)\,du.$$

$$= \int_0^t (ut - u^2)e^{-u}\,du.$$

$$= \left[(ut - u^2)\frac{e^{-u}}{-1} - (t - 2u)\frac{e^{-u}}{1} + (-2)\frac{e^{-u}}{-1}\right]_0^t$$

$$= [te^{-t} + 2e^{-t} + t - 2]$$

$$= t(e^{-t}+1) + 2(e^{-t}-1).$$

5) Using Convolution theorem, find $L^{-1}\left[\frac{2}{(s+1)(s^2+4)}\right]$.

Soln:

$$f(t) * g(t) = L^{-1}[F(s) * G(s)] = \int_0^t f(u)\, g(t-u)\, du.$$

$$L^{-1}\left[\frac{2}{(s+1)(s^2+4)}\right] = L^{-1}\left[\frac{2}{s^2+4}\right] * L^{-1}\left[\frac{1}{s+1}\right].$$

$$= \sin 2t * e^{-t}.$$

$$= \int_0^t \sin 2u\, e^{-(t-u)}\, du.$$

$$= e^{-t} \int_0^t e^{u} \sin 2u\, du.$$

$$= e^{-t}\left[\frac{e^{u}}{1+4}(\sin 2u - 2\cos 2u)\right]_0^t$$

$$= e^{-t}\left[\frac{e^{t}}{5}(\sin 2t - 2\cos 2t) - \frac{1}{5}(-2)\right]$$

$$= e^{-t}\left[\frac{e^{t}}{5}(\sin 2t - 2\cos 2t) + \frac{2}{5}\right]$$

$$= \frac{1}{5}(\sin 2t - 2\cos 2t + 2e^{-t}).$$

6) Find $L^{-1}\left[\frac{1}{(s^2+4)^2}\right]$ using Convolution theorem.

Soln:

$$L^{-1}\left[\frac{1}{(s^2+4)^2}\right] = L^{-1}\left[\frac{1}{s^2+4} \cdot \frac{1}{s^2+4}\right].$$

$$= L^{-1}\left[\frac{1}{s^2+4}\right] * L^{-1}\left[\frac{1}{s^2+4}\right].$$

$$= \frac{1}{2}\sin 2t * \frac{1}{2}\sin 2t.$$

$$= \frac{1}{4}\int_0^t \sin 2u \sin 2(t-u)\, du.$$

$$\left[\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]\right].$$

$$= \frac{1}{4}\int_0^t \frac{\cos(2u-2t+2u) - \cos(2u+2t-2u)}{2}\, du$$

$$= \frac{1}{8}\int_0^t \left[\cos 2(2u-t) - \cos 2t\right] du$$

$$= \frac{1}{8}\left[\frac{\sin(4u-2t)}{4} - u\cos 2t\right]_0^t.$$

$$= \frac{1}{8}\left[\frac{\sin 2t}{4} - t\cos 2t + \frac{\sin 2t}{4}\right].$$

$$= \frac{1}{8}\left[\frac{\sin 2t}{2} - t\cos 2t\right].$$

7) Using Convolution theorem find $L^{-1}\left[\frac{1}{s(s^2+1)}\right]$

Soln:

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right] = L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{s^2+1}\right].$$

$$= 1 * \sin t.$$

$$= \int_0^t \sin(t-u)\,du.$$

$$= \left[\frac{\cos(t-u)}{1}\right]_0^t$$

$$= 1 - \cos t.$$

8) Write the value of $1 * e^t$.

Soln: By definition of convolution, we have

$$1 * e^t = \int_0^t 1 \cdot e^{t-u}\,du = \left[-e^{t-u}\right]_0^t = -1 + e^t = e^t - 1.$$

9) Write the value of $t * e^t$.

Soln: $t * e^t = \int_0^t t e^{t-u}\,du.$

$$= \left[t\left(-e^{t-u}\right) - \left(e^{t-u}\right)\right]_0^t$$

$$= -t - 1 + e^t$$

$$= e^t - t - 1.$$

TRANSFORM OF DERIVATIVES :

$L[y'(t)] = s\,L[y(t)] - y(0).$

$L[y''(t)] = s^2 L[y(t)] - s\,y(0) - y'(0).$

...

In general,

$L[y^n(t)] = s^n L[y(t)] - s^{n-1} y(0) - s^{n-2} y'(0) \cdots y^{n-1}(0)$

$L\left[\int_0^t y(t)\,dt\right] = \frac{1}{s} L[y(t)].$

1) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$, with $y = \frac{dy}{dx} = 1$ at $x=0$
(y')
$y(0)=1$

Soln:

$y'' - 2y' + 2y = 0.$

Taking Laplace transform on both sides.

$L[y''] - 2L[y'] + 2L[y] = 0.$

$[s^2 L[y] - s\,y(0) - y'(0)] - 2[s\,L(y) - y(0)] + 2L[y] = 0$

$\cancel{s^2 - 2s + 2}$

$[s^2 L[y] - s(1) - (1)] - 2[s\,L(y) - 1] + 2L[y] = 0.$

$(s^2 - 2s + 2) L(y) - s - 1 + 2 = 0$

$L(y) = \frac{s-1}{s^2 - 2s + 2}$

$y = L^{-1}\left[\frac{s-1}{s^2 - 2s + 2}\right]$

$$y = L^{-1}\left[\frac{s-1}{(s-1)^2+1}\right].$$

$$y = e^{x} L^{-1}\left[\frac{s}{s^2+1}\right].$$

$$y = e^{x}\cos x$$

2) Solve $y'' + 2y' + y = te^{-t}$, $y=1$, $y'=-2$, when $t=0$.

Soln:-

$$L[y''] + 2L[y'] + L[y] = L[te^{-t}].$$

$$[s^2 L(y) - s y(0) - y'(0)] + 2[sL(y) - y(0)] + L(y)$$

$$= L[t]_{s\to s+1}$$

$$s^2 L(y) - s - (-2) + 2[sL(y) - 1] + L(y) = \left[\frac{1}{s^2}\right]_{s\to s+1}$$

$$(s^2+2s+1)L(y) + \cancel{2} - \cancel{2} = \frac{1}{(s+1)^2} + s.$$

$$(s^2+2s+1)L(y) = \frac{1}{(s+1)^2} + s.$$

$$L(y) = \frac{1+s(s+1)^2}{(s+1)^2(s^2+2s+1)}.$$

$$= \frac{1+s(s+1)^2}{(s+1)^2(s+1)^2}.$$

$$= \frac{1+s(s+1)^2}{(s+1)^4}$$

$$L(y) = \frac{1}{(s+1)^4} + \frac{s}{(s+1)^2}.$$

$$y = L^{-1}\left[\frac{1}{(s+1)^4}\right] + L^{-1}\left[\frac{s}{(s+1)^2}\right].$$

$$= e^{-t} L^{-1}\left[\frac{1}{s^4}\right] + L^{-1}\left[\frac{(s+1)-1}{(s+1)^2}\right].$$

$$= \frac{e^{-t}}{3!} L^{-1}\left[\frac{3!}{s^4}\right] + L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$= \frac{e^{-t}}{3!} t^3 + e^{-t} - e^{-t} L^{-1}\left[\frac{1}{s^2}\right].$$

$$= \frac{e^{-t} t^3}{3!} + e^{-t} - e^{-t}.\, t$$

$$y = e^{-t}\left[\frac{t^3}{6} - t + 1\right].$$

3) Solve $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$, $y = 4, y' = -2$ at $t = 0$.

Soln :

The given Equation can be written as

$$y'' + y' = t^2 + 2t.$$

Taking Laplace Transform on both sides we get.

$$L[y''] + L[y'] = L[t^2] + L[2t]$$

$$s^2L(y) - sy(0) - y'(0) + sL(y) - y(0) = L[t^2] + 2L[t]$$

$$s^2L(y) - 4s + 2 + sL(y) - 4 = \frac{2}{s^3} + 2 \cdot \frac{1}{s^2}.$$

$$(s^2+s)L(y) - 4s = \frac{2(s+1)}{s^3} + 2.$$

$$(s^2+s)L(y) = \frac{2(s+1)}{s^3} + 4s + 2.$$

$$L(y) = \left[\frac{2(s+1)}{s^3} + 4s + 2\right] \Big/ s(s+1).$$

$$L(y) = \frac{2(s+1)}{s^3(s+1)s} + \frac{4s}{s(s+1)} + \frac{2}{s(s+1)}.$$

$t^3 = \frac{3!}{s^4}.$

$$= \frac{2}{s^4} + \frac{4}{s+1} + \frac{2}{s(s+1)}.$$

$$= \frac{2}{s^4} + \frac{4s+2}{s(s+1)}.$$

$$\frac{4s+2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$\Rightarrow 4s+2 = A(s+1) + Bs$

put $s=0$, $A=2$.

$s=-1$, $B=2$.

$$\therefore \frac{4s+2}{s(s+1)} = \frac{2}{s} + \frac{2}{s+1}$$

$2 \cdot \frac{1}{s^4} + 4 \cdot \frac{1}{s+1} + 2 \cdot \frac{1}{s} \cdot \frac{1}{s+1}$

$2 \cdot \frac{1}{3!} t^3 + 4e^{-t} + 2\int_0^t \left[L^{-1}\left(\frac{1}{s+1}\right)\right] ds$

$2 \cdot \frac{1}{6} t^3 + 4e^{-t} + 2\int_0^t e^{-t} ds$

$\frac{1}{3} t^3 + 4e^{-t} + 2\left[\frac{e^{-t}}{-1}\right]_0^t$

$\frac{t^3}{3} + 4e^{-t} + 2(e^{-t} + 1)$

$\frac{t^3}{3} + 2e^{-t} + 2$

$$L(y) = \frac{2}{s^4} + \frac{2}{s} + \frac{2}{s+1}$$

$$= \frac{2}{s^4} + 2\left[\frac{1}{s} + \frac{1}{s+1}\right].$$

$$y = L^{-1}\left[\frac{2}{s^4}\right] + 2L^{-1}\left[\frac{1}{s} + \frac{1}{s+1}\right].$$

$$= 2 \cdot \frac{t^3}{3!} + 2 + 2e^{-t}.$$

$$= \frac{t^3}{3} + 2 + 2e^{-t}.$$

H/W.

4) Solve $(D^2 + 4D + 13)y = e^{-t}\sin t$ given that $y = 0$ and $\frac{dy}{dt} = 0$ when $t = 0$.

PERIODIC FUNCTION:

Defn: A function $f(t)$ is said to have period T or to be periodic with period T if for all t, $f(t+T) = f(t)$ where T is a positive constant. The least value of $T > 0$ is called the period of $f(t)$.

Eg: ① Consider $f(t) = \sin t$.

$$f(t+2\pi) = \sin(t+2\pi) = \sin \theta.$$

$$\text{ie } f(t) = f(t+2\pi) = \sin t.$$

Eg: 2: $\tan t$ is a periodic function with period π.

Since $\tan(t+\pi) = \tan t$.

Convolution Theorem: If $f(t)$ and $g(t)$ are functions defined for $t \geq 0$, then

$$L[f(t) * g(t)] = L[f(t)] \cdot L[g(t)].$$

Using this theorem we get,

$$L^{-1}[F(s) \cdot G(s)] = f(t) * g(t)$$
$$= L^{-1}[F(s)] * L^{-1}[G(s)].$$

.. Problems -

1. find $L^{-1}[\cot^{-1} as]$.

Soln: We know that

$$L^{-1}[F(s)] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds} F(s)\right].$$

$$= -\frac{1}{t} L^{-1}\left[-a \cdot \frac{1}{1+(as)^2}\right].$$

$$= \frac{1}{at} L^{-1}\left[\frac{a}{s^2+(1/a)^2} \cdot \frac{1}{a^2}\right]$$

$$= \frac{1}{at} L^{-1}\left[\frac{1}{s^2+(1/a)^2}\right].$$

$$= \frac{1}{1/a} \cdot \frac{1}{at} L^{-1}\left[\frac{1/a}{s^2+(1/a)^2}\right]$$

$$= a \cdot \frac{1}{at} L^{-1}\left[\frac{1/a}{s^2+(1/a)^2}\right].$$

$$= \frac{1}{t} \sin(1/a)t.$$

(2) find $L^{-1}\left[\cot^{-1}\frac{2}{s+1}\right]$.

soln:

$$L^{-1}\left[\cot^{-1}\frac{2}{s+1}\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\cot^{-1}\frac{2}{s+1}\right].$$

$$= -\frac{1}{t} L^{-1}\left[-\frac{1}{1+\left(\frac{2}{s+1}\right)^2} \times \frac{-2}{(s+1)^2}\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{2}{\cancel{(s+1)^2}} \times \frac{\cancel{(s+1)^2}}{(s+1)^2+4}\right].$$

$$= -\frac{1}{t} L^{-1}\left[\frac{2}{(s+1)^2+4}\right]$$

$$= -\frac{1}{2} e^{-t} L^{-1}\left[\frac{2}{s^2+2^2}\right]$$

$$= -\frac{1}{2} e^{-t} \sin 2t.$$

Multiple Integrals

Double Integration:

1) Evaluate : $\int_0^3 \int_0^2 xy(x+y)\, dx\, dy$

Soln:
$$\int_0^3 \int_0^2 (x^2y + xy^2)\, dx\, dy$$

$$\int_0^3 \left[\frac{x^3 y}{3} + \frac{x^2 y^2}{2} \right]_0^2 dy = \int_0^3 \left[\frac{8y}{3} + \frac{4}{2} y^2 \right] dy$$

$$= \left[\frac{8y^2}{6} + \frac{4y^3}{6} \right]_0^3 = \left[\frac{8(9)}{6} + \frac{4(27)}{6} \right] - (0)$$

$$= 12 + 18 = 30$$

2) Evaluate $\int_0^1 \int_1^2 x(x+y)\, dy\, dx.$

Soln
$$\int_0^1 \int_1^2 (x^2 + xy)\, dy\, dx.$$

$$\int_0^1 \left[x^2 y + \frac{xy^2}{2} \right]_1^2 dx = \int_0^1 \left[\left(2x^2 + \frac{4x}{2} \right) - \left(x^2 + \frac{x}{2} \right) \right] dx$$

$$= \int_0^1 \left(x^2 + \frac{3x}{2} \right) dx = \left[\frac{x^3}{3} + \frac{3x^2}{4} \right]_0^1$$

$$= \left[\frac{1}{3} + \frac{3}{4} - (0) \right]$$

$$= \underline{4+9} = \underline{13}$$

3 . S.T $\int_0^a \int_0^b (x+y)\,dx\,dy = \int_0^b \int_0^a (x+y)\,dy\,dx$

Soln: L.H.S $\int_0^a \left[\frac{x^2}{2} + yx \right]_0^b dy$ $\quad \int y\,dx$

$$= \int_0^a \left[\frac{b^2}{2} + yb \right] dy$$

$$= \left[\frac{b^2 y}{2} + \frac{by^2}{2} \right]_0^a$$

$$= \frac{b^2 a}{2} + \frac{ba^2}{2} = \frac{ab}{2}\left[b+2\right]$$

R.H.S $\int_0^b \int_0^a (x+y)\,dy\,dx$

$$= \int_0^b \left[xy + \frac{y^2}{2} \right]_0^a dx = \int_0^b \left[xa + \frac{a^2}{2} \right] dx$$

$$= \left[\frac{x^2}{2} a + \frac{a^2}{2} x \right]_0^b = \frac{b^2 a}{2} + \frac{a^2 b}{2}$$

$$= \frac{ab}{2}\left[b+a\right]$$

$\therefore$ L.H.S = R.H.S

4) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dy\,dx}{\sqrt{a^2-x^2}}$

Soln:- $\int_0^a \left[\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx = \int_0^a \left(\left[\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\right]-0\right]dx$

$\cancel{\int_0^a \left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^{\sqrt{a^2-x^2}} dy}$

$= [x]_0^a = a$

Triple Integration:-

1) S.T $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}} = \pi^2/8$

Soln

Let $a^2 = 1-x^2-y^2$, then $a=\sqrt{1-x^2-y^2}$

$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-z^2}}$

$\cancel{\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \sin^{-1}(z/a)\, dz}$

$\Rightarrow \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx.$

$\left(\because \int \frac{dz}{\sqrt{a^2-z^2}} = \sin^{-1}(z/a)\right)$

$\Rightarrow \int_0^1 \int_0^{\sqrt{1-x^2}} \sin^{-1}\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}}\right) - \sin^{-1}(0)\Big]\, dy\,dx$

$\Rightarrow \int_0^1 \int_0^{\sqrt{1-x^2}} [\sin^{-1}(1) - \sin^{-1}(0)]\, dy\,dx$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \pi/2 \, dy\,dx = \pi/2 \int_0^1 [y]_0^{\sqrt{1-x^2}} dx$$

$$\Rightarrow \pi/2 \int_0^1 \sqrt{1-x^2}\, dx. \qquad \left(\because \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \right)$$

$$\Rightarrow \pi/2 \left[\frac{x}{2}\sqrt{1-x^2} + 1/2 \sin^{-1}\left(\frac{x}{1}\right) \right]_0^1$$

$$= \pi/2 \left[1/2\,(0) + 1/2 \sin^{-1}(1) - 0 \right]$$

$$= \pi/2 \left[1/2 \cdot \pi/2 \right] = \pi^2/8.$$

2) S.T $\int_0^2 \int_1^3 \int_1^2 xy^2 z \, dz\,dy\,dx$

Soln:

$$\int_0^2 \int_1^3 xy^2 \left[\frac{z^2}{2} \right]_1^2 dy\,dx$$

$$\Rightarrow \int_0^2 \int_1^3 \frac{xy^2}{2} [4-1] = \int_0^2 \int_1^3 \frac{3}{2} xy^2\, dy\,dx$$

$$\Rightarrow \int_0^2 \frac{3}{2} x \left[\frac{y^3}{3} \right]_1^3 dx = \frac{3}{2} \int_0^2 \frac{x}{3} (27-1)\, dx$$

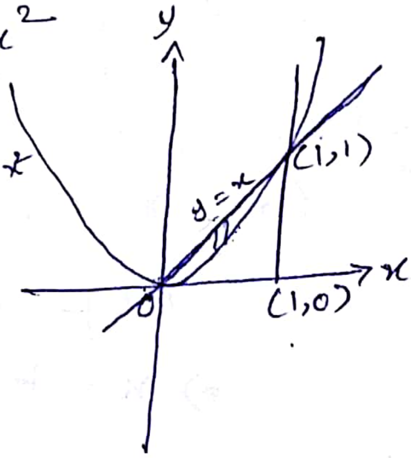
$$\Rightarrow \frac{26}{2} \int_0^2 x\, dx = \frac{26}{2} \left[\frac{x^2}{2} \right]_0^2$$

$$= \frac{26}{\cancel{2}} \left[\frac{\cancel{4}}{\cancel{2}} \right] = 26.$$

3) Evaluate $\iint \cancel{xy}\,(x+y)\,dy\,dx$ over the area between $y = x^2$ & $y = x^2$

Soln,

Gn., $y = x$ and $y = x^2$



$x^2 = x$

$\Rightarrow x(x-1) = 0$

$x = 0 \quad x = 1$

$\therefore y = 0, y = 1$

$\therefore$ point of intersection are $O(0,0)$ & $A(1,1)$

$$\therefore \iint (x-y)\,d\cancel{x}\,dy = \int_0^1 \int_{x^2}^{x} (x-y)\,dy\,dx$$

$$= \int_0^1 \left[xy - y^2/2\right]_{x^2}^{x} dx$$

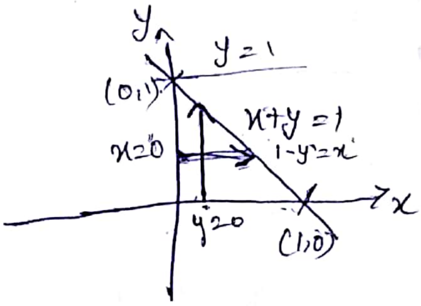
$$= \int_0^1 \left(x^2 - x^2/2\right) - \left(x^3 - \frac{x^4}{2}\right) dx$$

$$= \int_0^1 \left(x^2/2 - x^3 + x^4/2\right) dx$$

$$= \left[\frac{x^3}{6} - \frac{x^4}{4} + \frac{x^5}{10}\right]_0^1$$

$$= 1/6 - 1/4 + 1/10 - 0 = \frac{10-15+6}{60} = 1/60$$

4) Evaluate $\iint xy\,dx\,dy$ over the region in the +ve quadrant for which $x+y=1$



Soln

Gn., $x=0$,

$x+y=1$

$\Rightarrow x = 1-y$

$x+y=1$

$x=0 \Rightarrow y=1$

$y=0 \Rightarrow x=1$

$\therefore x \to 0$ to $1-y$

$y \to 0$ to 1

$$\iint xy\,dx\,dy = \int_0^1 \int_0^{1-y} (xy\,dx)\,dy$$

$$= \int_0^1 \left[\frac{yx^2}{2}\right]_0^{1-y} dy \Rightarrow \int_0^1 \frac{y(1-y)^2}{2}\,dy$$

$$= \frac{1}{2}\int_0^1 y(1-2y+y^2)\,dy$$

$$= \frac{1}{2}\int_0^1 (y-2y^2+y^3)\,dy = \frac{1}{2}\left[\frac{y^2}{2}-\frac{2y^3}{3}+\frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right] = \frac{1}{2}\left[\frac{6-8+3}{12}\right]$$

$$= \frac{1}{24}[9-8] = \frac{1}{24}$$

5) Evalu

bound

x+y+

Soln.,

Gn.,

is a

the

x+y

Here

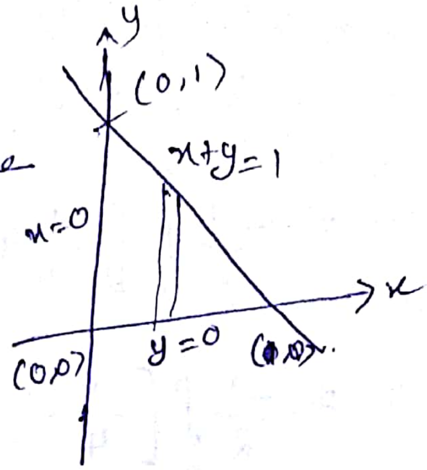
Z=0

y va

x va

5) Evaluate $\iiint \frac{dz\,dy\,dx}{(x+y+z+1)^3}$ over the region bounded by the planes $x=0, y=0, z=0,$ $x+y+z=1$

Soln :



Gn., region in the xy plane is a triangle bounded by the lines $x=0, y=0$ & $x+y=1$

7x

Here z varies from $z=0$ to $z=1-x-y$

y varies from $y=0$ to $y=1-x$

$\left[\, x^n dx = \frac{x^{n+1}}{n+1}\right.$

x varies from $x=0$ to $x=1$

$$\iiint \frac{dz\,dy\,dx}{(x+y+z+1)^3} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dz\,dy\,dx}{(x+y+z+1)^3}$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(x+y+z+1)^{-2}}{-2} \right]_0^{1-x-y} dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \int_0^{1-x} \left[(x+y+(1-x-y)+1)^{-2} - (x+y+1)^{-2} \right] dy\,dx$$

$$= \frac{-1}{2}\int_0^1\int_0^{1-x}\left[(2)^{-2} - (x+y+1)^{-2}\right] dy\,dx$$

$$= \frac{-1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{4} - (x+y+1)^{-2}\right] dy\,dx$$

$$= \frac{-1}{2}\int_0^1\left[\frac{1}{4}y - \frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x} dx$$

$$= \frac{-1}{2}\int_0^1\left[\frac{1}{4}(1-x) + (x+1+1-x)^{-1}\right]dx - (x+1)^{-1}\,dx$$

$$= \frac{-1}{2}\int_0^1\left[\frac{1-x}{4} + (2)^{-1} - \frac{1}{x+1}\right]dx$$

$$= \frac{-1}{2}\int_0^1\left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1}\right]dx \qquad \int\frac{dx}{x+1}$$

$$= \frac{-1}{2}\int_0^1\left[\frac{1}{4} - \frac{x}{4} + \frac{1}{2} - \frac{1}{x+1}\right]dx$$

$$= \frac{-1}{2}\int_0^1\left[\frac{3}{4} - \frac{x}{4} - \frac{1}{x+1}\right) dx$$

$$= \frac{-1}{2}\left[\frac{3}{4}x - \frac{1}{4}\frac{x^2}{2} - \log(1+x)\right]_0^1$$

$$= \frac{-1}{2}\left[\frac{3}{4} - \frac{1}{8} - \log 2\right) = \frac{-1}{2}\left[\frac{6-1}{8} - \log 2\right]$$

$$= -\tfrac{1}{2}\left[\tfrac{5}{8} - \log 2\right] = \frac{-5}{16} + \frac{1}{2}\log 2$$

H.W

1) E

2) E

b

Area

Form

i)

Probl

1)

H.W

1) Evaluate $\int_0^1 \int_0^x dx\,dy$ (Ans : $1/2$)

2) Evaluate $\iint xy(x+y)\,dy\,dx$ over the area between $y = x^2$ & $y = x$.

Area as a Double Integral

Formulae :

i) Area of a region R in cartesian form is $\iint_R dx\,dy$

ii) Area of a region R in polar form is $\iint_R r\,dr\,d\theta$.

Problems : U.B.

1) Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ using double integration.

Soln : Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

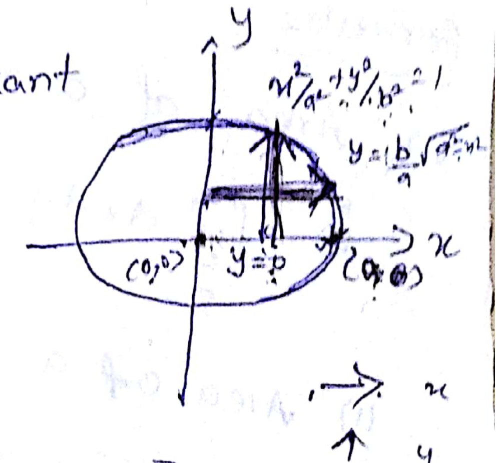
$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow \frac{y^2}{b^2} = \frac{a^2-x^2}{a^2} \Rightarrow \frac{b^2}{a^2}(a^2-x^2)$$

$$\therefore y = \frac{b}{a}\sqrt{a^2-x^2}.$$

y varies from $y=0$ to $y = \frac{b}{a}\sqrt{a^2-x^2}$

x varies from $x=0$ to $x=a$



$A = 4 \times$ Area in the 1st quadrant

$$= 4\int_0^a \int_0^{b/a\sqrt{a^2-x^2}} dy\, dx.$$

$$= 4\int_0^a [y]_0^{b/a\sqrt{a^2-x^2}} dx = 4\int_0^a \left[\frac{b}{a}\sqrt{a^2-x^2}\right] dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\, dx.$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1} x/a\right]_0^a$$

$$= \frac{4b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}(1)\right] = \frac{4b}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2}$$

$$= \pi ab \text{ sq. units}$$

(2

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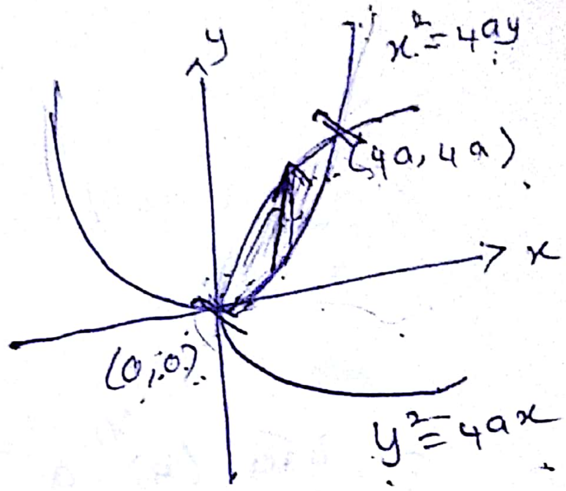
Find the area between the two parabolas $y^2 = 4ax$ & $x^2 = 4ay$ using double integration. U.Q.

Soln.,

Gn., $y^2 = 4ax$ — ①

$x^2 = 4ay$ — ②

$\Rightarrow y = x^2/4a.$



$$① \Rightarrow \left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\Rightarrow \frac{x^4}{16a^2} = 4ax$$

$$\Rightarrow x^4 = 64a^3x$$

$$x^4 - 64a^3x = 0$$

$$x(x^3 - 64a^3) = 0$$

$x = 0$ $\quad x^3 = 64a^3$

$x = 4a.$ $\quad$ |||ly $y = 4a.$ $\therefore (4a, 4a)$

Hence x varies from : $x = 0 \rightarrow 4a.$

y varies from : $y : x^2/4a$ to $2\sqrt{ax}.$

$$\text{Area} = \iint_R dx\,dy.$$

$$= \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx = \int_0^{4a} [y]_{x^2/4a}^{2\sqrt{ax}}\,dx.$$

$$= \int_0^{4a} \left(2\sqrt{ax} - x^2/4a\right) dx$$

$$= \int_0^{4a} \left(2\sqrt{a}(x)^{1/2} - \frac{x^2}{4a}\right) dx$$

$$= \left[2\sqrt{a}\,\frac{x^{3/2}}{3/2} - \frac{x^3}{12a}\right]_0^{4a}$$

$$= \frac{2\times 2\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^3}{12a} - (0)$$

$$= \frac{4\sqrt{a}}{3}(4)^{3/2}\cdot a^{3/2} - \frac{16a^2}{3}$$

$(4)^{3/2} = (2^2)^{3/2}$

$= (2)^3$

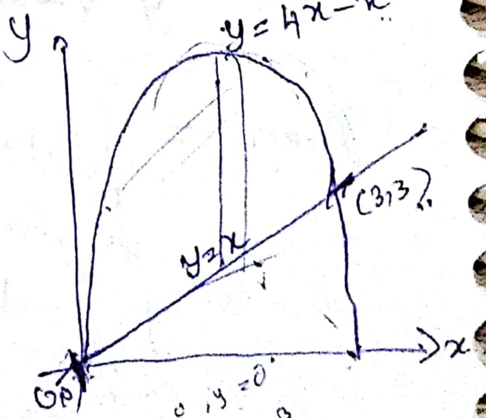
$a^{3/2}\cdot a^{1/2} = a^{4/2}$

$= a^2$

$$= \frac{4a^2 (2)^3}{3} - \frac{16a^2}{3}$$

$$= \frac{32a^2 - 16a^2}{3} \Rightarrow \frac{16a^2}{3} \text{ sq. units.}$$

3) Find by double integration the area lying between the parabola $y = 4x - x^2$ & the line $y = x$.



$x=0, y=0$

$x=1, y=3$

$2, y=4$

Soln

$y = 4x - x^2$, $y = x$

$x = 4x - x^2$

$\cancel{x^2}$ $x^2 - 3x = 0$

$x(x-3) = 0$

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$\therefore x = 0 ; x = 3$

$y = x , y = 4x - x^2$

The required area $= \iint_R dx\,dy$

$$= \int_0^3 \int_x^{4x-x^2} dy\, dx = \int_0^3 [y]_x^{4x-x^2} dx$$

$$= \int_0^3 [(4x - x^2) - (x)] dx$$

$$= \int_0^3 (3x - x^2) dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \frac{3}{2} \cdot 9 - \frac{27}{3} \Rightarrow \frac{27}{2} - \frac{27}{3}$$

$$= 27\left(\frac{3-2}{6}\right)$$

$$= 27 \times 1/6 = 9/2 \text{ sq.units.}$$

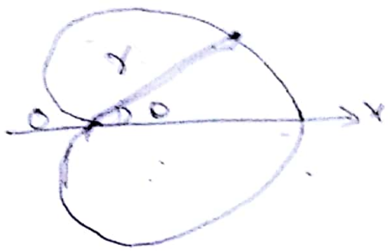
4) Find the area of the Cardiod $r = a(1 + \cos\theta)$ by using double integration

Soln Gn., $r = a(1 + \cos\theta)$

$\therefore$ Area of the cardiod $= 2 \times$ area above the initial line.

when $\theta = 0,\ r = 2a$

$\theta = \pi,\ r = 0$



$\therefore\ \theta : 0$ to π

$r : 0$ to $a(1+\cos\theta)$

($\sin n\pi = 0$, $\cos n\pi = \pm 1$)

$$\therefore \text{Area} = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta.$$

$$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta.$$

$$= a^2\int_0^{\pi}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= a^2\int_0^{\pi}\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{2}\,\frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= a^2\left[\left(\frac{3\pi}{2}+0\right)-0\right] = \frac{3\pi a^2}{2}\ //.$$

5) Find

the

Soln,

Gn

Sub:

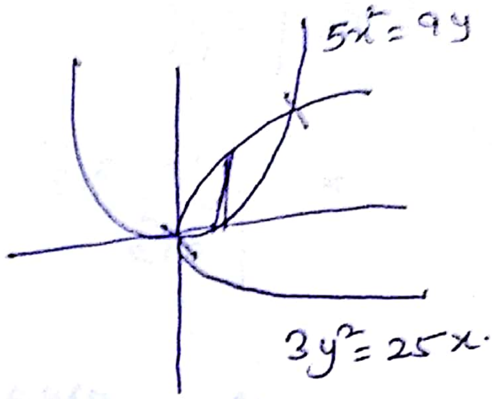
3(

5) Find by double integration the area bet^n, the two parabolas $3y^2 = 25x$ & $5x^2 = 9y$

Soln,

Gn, $3y^2 = 25x$ — (1) , $5x^2 = 9y$

$y = \frac{5x^2}{9}$



Subst. in (1)

$3\left(\frac{5x^2}{9}\right)^2 = 25x.$

$\frac{75x^4}{81} = 25x$

$\frac{75x^4}{25} = 81x$

$3x^4 - 81x = 0$

$x(3x^3 - 81) = 0$

$x = 0 \ , \ 3x^3 = 81$

$x^3 = 27$

$x = 3$

x varies from 0 to 3.

y " " $\frac{5x^2}{9}$ to $\frac{5\sqrt{x}}{\sqrt{3}}$

$$\text{Area} = \int_0^3 \int_{\frac{5x^2}{9}}^{\frac{5\sqrt{x}}{\sqrt{3}}} dy\, dx.$$

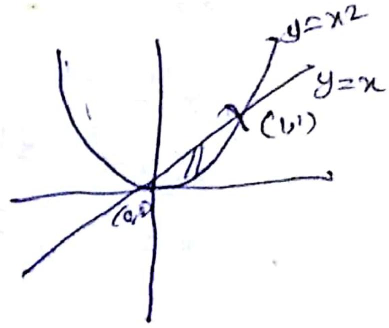
$$= \int_0^3 [y]_{\frac{5x^2}{9}}^{\frac{5\sqrt{x}}{\sqrt{3}}} dx = \int_0^3 \left[\frac{5\sqrt{x}}{\sqrt{3}} - \frac{5}{9}x^2\right] dx$$

$$\Rightarrow \left[\frac{5}{\sqrt{3}} \frac{x^{3/2}}{3/2} - \frac{5}{9}\frac{x^3}{3}\right]_0^3$$

$$\Rightarrow \left[\frac{10}{3\sqrt{3}} 3^{3/2} - \frac{5}{27} 3^3\right]$$

$$= \left[\frac{10}{3\sqrt{3}} 3\sqrt{3} - \frac{5}{27}(27)\right] = 5 \text{ sq. units.}$$

6) Find the area enclosed by the curve $y = x$ and $y = x^2$ in the 1st quadrant



Soln $y = x$, $y = x^2$

$x = x^2 \Rightarrow x^2 - x = 0$

$x(x-1) = 0$

$x = 0,\ x = 1$

x: varies from 0 to 1

y: " " x^2 to x.

$$\text{Area} = \iint_R dx\, dy = \int_0^1 \int_{x^2}^{x} dy\, dx$$

$$= \int_0^1 [y]_{x^2}^{x} dx = \int_0^1 [x - x^2] dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Volume as Triple Integral :-

1) Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by triple integrals.

Formula:

If V is the volume enclosed by the region D, then volume

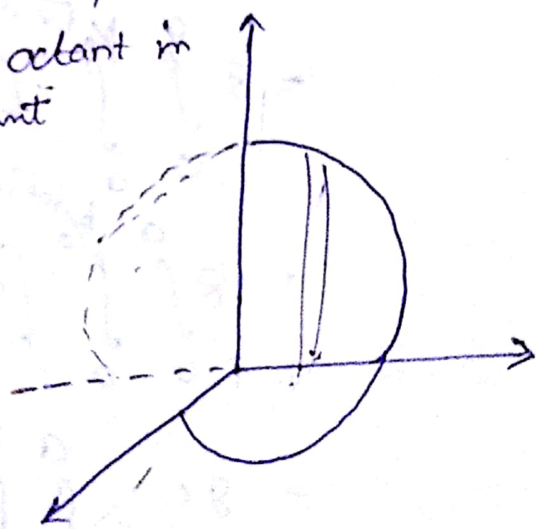
$$V=\iiint_D dx\,dy\,dz.$$

Soln:

On surface in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

The ellipse in 3D is an ellipsoid.

Volume = 8 x volume of octant in the 1st octant



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

$$z^2=c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

$$z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

z : 0 to $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$z=0 \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$y = \frac{b}{a}\sqrt{a^2-x^2}$

$y : 0$ to $\frac{b}{a}\sqrt{a^2-x^2}$ (or) $b\sqrt{1-\frac{x^2}{a^2}}$

$z=0 \; y=0 \Rightarrow x^2=a^2$

$x=a.$

Volume $= 8 \times \iiint_V dz\,dy\,dx$

$$= 8\int_0^a \int_0^{b/a\sqrt{a^2-x^2}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= 8\int_0^a \int_0^{b/a\sqrt{a^2-x^2}} [z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\,dy\,dx.$$

$$= 8\int_0^a \int_0^{b/a\sqrt{a^2-x^2}} \left[c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right] dy\,dx.$$

$$= \frac{8c}{b}\int_0^a \int_0^{b/a\sqrt{a^2-x^2}} \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\;dy\,dx.$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2} + \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\left(\frac{y}{b\sqrt{1-\frac{x^2}{a^2}}}\right)\right]_0^{b\sqrt{1-\frac{x^2}{a^2}}}$$

$= \frac{4\,8c}{ab}\int_0^a [$

$= \frac{4cb^2}{b}\int_0^a$

$= 4cb$

$= 4c$

2) Find

bounde

$\frac{x}{a}+\frac{y}{b}$

Soln

$= \int_0^a$

$$\left(\because \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right)$$

$$= \frac{4\cancel{8}c}{2b}\int_0^a \left[0 + b^2\left(1-\frac{x^2}{a^2}\right)\sin^{-1}(1) - (0) \right] dx.$$

$$= \frac{4cb^{\cancel{2}}}{\cancel{b}}\int_0^a \left(1-\frac{x^2}{a^2}\right)\pi/2 \; dx.$$

$$= 4cb\frac{\pi}{2}\left[x - \frac{x^3}{3a^2} \right]_0^a$$

$$= \frac{4cb\pi}{2}\left[a - \frac{a^3}{3a^2} \right] = 2\pi bc\left[a - \frac{a}{3} \right]$$

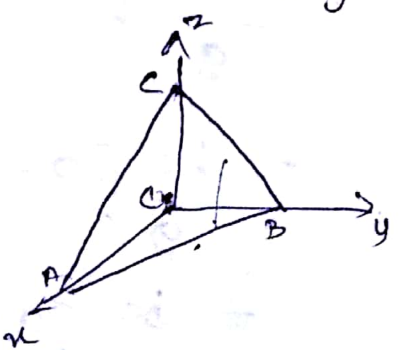
$$\text{Volume} = \frac{4\pi abc}{3} \;//.$$

2) Find the volume of the tetrahedron bounded by the co-ordinate plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$,

Soln $V = \iiint dz\,dy\,dx$

($\because$ Co-ordinate plane laws are, $x=y=z=0$)

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz\,dy\,dx.$$



$$= \int_0^a \int_0^{b(1-x/a)} [z]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} \, dy \, dx.$$

$$= \int_0^a \int_0^{b(1-x/a)} \left[c\left(1 - \frac{x}{a} - \frac{y}{b}\right)\right] dy \, dx$$

$$= c \int_0^a \int_0^{b(1-x/a)} \left[\left(1 - \frac{x}{a}\right) - \frac{y}{b}\right] dy\, dx$$

$$= c \int_0^a \left[\left(1 - x/a\right) y - \frac{y^2}{2b}\right]_0^{b(1-x/a)} dx$$

$$= c \int_0^a b\left(1 - \frac{x}{a}\right)^2 - \frac{b^2 (1-x/a)^2}{2b} \, dx$$

$$= c \int_0^a (1-x/a)^2 \left[b - b/2\right] dx.$$

$$= \frac{bc}{2} \int_0^a \left(1 - \frac{2x}{a} + \frac{x^2}{a^2}\right) dx$$

$$= \frac{bc}{2} \left[x - \frac{2x^2}{2a} + \frac{x^3}{3a^2}\right]_0^a$$

$$= \frac{bc}{2} \left[a - \frac{a^2}{a} + \frac{a^3}{3a^2}\right]$$

$$= \frac{bc}{2} \left[a - a + \frac{a}{3}\right] = \frac{abc}{6}$$

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3) Find the volume of the sphere $x^2+y^2+z^2=a^2$ using triple integrals. U.B.

Soln

Volume = $8\times$ Volume in the I octant

$8\iiint dz\,dy\,dx$

$x^2+y^2+z^2=a^2$

$z=\sqrt{a^2-x^2-y^2}$ $\quad\therefore$ z varies 0 to $\sqrt{a^2-x^2-y^2}$

y varies from 0 to $\sqrt{a^2-x^2}$

x " " 0 to a.

$$V = 8\times\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

$$= 8\times\int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}}\,dy\,dx.$$

$$= 8\times\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{(a^2-x^2)-y^2}\,dy\,dx.$$

$$= 8\times\int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2}+\frac{(a^2-x^2)}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}}$$

$$= 8\int_0^a \left[0+\frac{(a^2-x^2)}{2}\sin^{-1}(1)\right]dx.$$

$$= \frac{{}^2\cancel{8}\pi}{\cancel{4}}\int_0^a (a^2-x^2)\,dx = 2\pi\left[a^2x-\frac{x^3}{3}\right]_0^a$$

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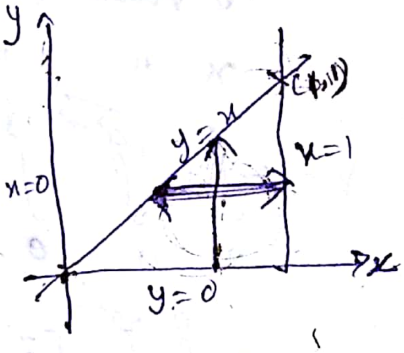
$$2\pi\left[a^3 - a^3/3\right] = \frac{4}{3}\pi a^3$$ //

Change the order of Integration

(Changing $dy\,dx$ to $dx\,dy$ is the main concept)

1) Change the order of integration and evaluate $\int_0^1\int_0^x dy\,dx$



Soln Gn., $x=0$ $x=1$

$y=0$ $y=x$

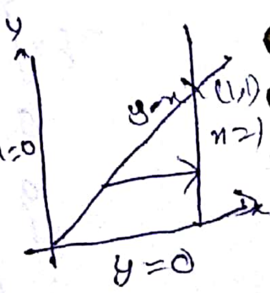
After changing the order of integration, the limits are x : y to 1

y : 0 to 1

$\rightarrow$ 'x'

$\uparrow$ y

$$\therefore \int_0^1\int_0^x dy\,dx = \int_0^1\int_y^1 dx\,dy$$



$$= \int_0^1 [x]_y^1\,dy$$

$$= \int_0^1 (1-y)\,dy$$

27 Change ... evalu...

Soln

After ... of inte... we h... x=

y

$\therefore$

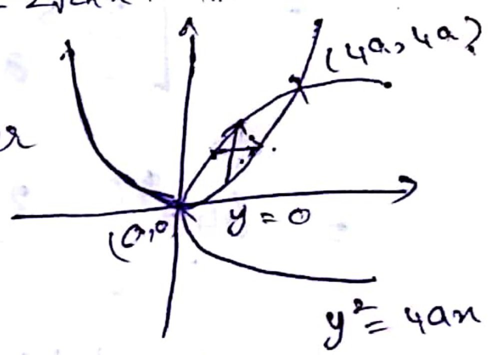
$$= \left[y - \frac{y^2}{2}\right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$ //.

27) Change the order of integration & hence evaluate $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy \, dy \, dx$.

Soln

Gn, $x = 0$ $x = 4a$

$y = \frac{x^2}{4a}$ $y = 2\sqrt{ax}$, $x^2 = 4ay$.



After changing the order of integration we have,

$x = \frac{y^2}{4a}$, $x = 2\sqrt{ay}$

$y = 0$ to $4a$.

$$\therefore \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy \, dy \, dx = \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} xy \, dx \, dy$$

$$= \int_0^{4a} \left[\frac{x^2 y}{2}\right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a} \left[\frac{4ay \cdot y}{2} - \frac{y}{2} \cdot \frac{y^4}{16a^2}\right]$$

$$= \int_0^{4a}\left[\frac{4a}{2}(y^2) - \frac{1}{32a^2}(y^5)\right]dy$$

$$= \left[\frac{4a}{2}\times\frac{y^3}{3} - \frac{1}{32a^2}\,\frac{y^6}{6}\right]_0^{4a}$$

$$= \left[\frac{{}^{2}\cancel{4}a}{\cancel{2}}\cdot\frac{64a^3}{3} - \frac{1}{32a^2}\cdot\frac{(4)^6a^6}{6} - 0\right]$$

$$= \left[\frac{2}{3}\,\frac{64a^4}{\cancel{a}} - \frac{1}{\cancel{32}\cancel{a}}\,\frac{(\cancel{64})^{2}(64)a^4}{\cancel{6}\,3}\right]$$

$$= 64a^4\left[\frac{2}{3} - \frac{1}{3}\right]$$

$$= 64a^4\left(\tfrac{1}{3}\right) = \frac{64a^4}{3} \;//.$$

3) Change the order of integration in the integral $\int_0^a\int_{x^2/a}^{2a-x} xy\,dy\,dx$ & hence evaluate

Soln. Gn., $\int_0^a\int_{x^2/a}^{2a-x} xy\,dy\,dx$.

The region of integration is bounded by $x=0$, $x=a$, $y=x^2/a$, $y=2a+x$

After changing
of integration,

I) $x=0$ to $x=$

$y=a$ to $2a$

II).

$x=0$

$y=$

$\therefore \int_0^a\int_{x^2/a}^{2a-x} x$

$= \int_a^{2a}[$

$= \frac{1}{2}\int_a^{2a}[$

$= \frac{1}{2}[$

$= \frac{1}{2}[$

$= \frac{1}{2}$

$= 1/2$

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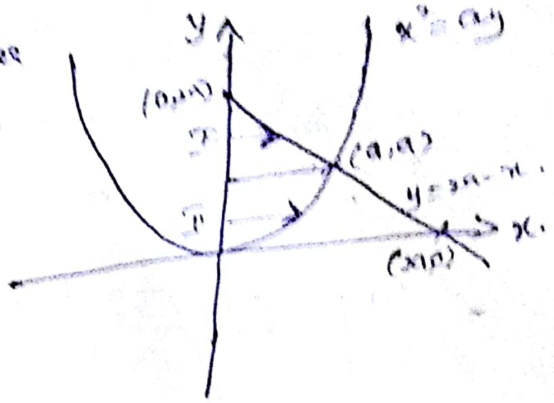
After changing the order of integration,

I) $x = 0$ to $x = 2a - y$

$y = a$ to $2a$.



II).

$x = 0$ to $\sqrt{ay}$

$y = 0$ to a.

$$\therefore \int_0^a \int_{x^2/a}^{2a-x} xy\, dy\, dx = \iint \mathrm{I} + \iint \mathrm{II}.$$

$$= \int_a^{2a}\int_0^{2a-y} xy\, dx\, dy + \int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy.$$

$$= \int_a^{2a} \left[\frac{x^2}{2} \cdot y\right]_0^{2a-y} dy + \int_0^a \left[\frac{x^2}{2} y\right]_0^{\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_a^{2a} \left[y(2a-y)^2 dy + \frac{1}{2}\int_0^a (ay).y\, dy.\right.$$

$$= \frac{1}{2}\left[\int_a^{2a} y(4a^2 - 4ay + y^2)dy + \int_0^a ay^2\, dy.\right]$$

$$= \frac{1}{2}\left[\int_a^{2a} (4a^2y - 4ay^2 + y^3)\, dy + \int_0^a ay^2\, dy\right]$$

$$= \frac{1}{2}\left[\left[4a^2.\frac{y^2}{2} - 4a.\frac{y^3}{3} + \frac{y^4}{4}\right]_a^{2a} + \left[\frac{ay^3}{3}\right]_0^a\right]$$

$$= \frac{1}{2}\left[4a^2.\frac{(4a^2)}{2} - 4a\frac{(8a^3)}{3} + \frac{16a^4}{4} - \left[4a^2.\frac{a^2}{2} - 4a.\frac{a^3}{3} + \frac{a^4}{4}\right]\right] + \left[\frac{a^4}{3}\right]\Big)$$

$$= \frac{1}{2}\left[8a^4 - \frac{32a^4}{3} + 4a^4 + \frac{a^4}{3}\right]$$

$$= \frac{a^4}{2}\left[\frac{24 - 32 + 12 + 1}{3}\right]$$

$$= \frac{a^4}{2}\left[5/3\right] = \frac{5a^4}{6}$$

$$= \frac{1}{2}\left[8a^4 - \frac{32a^4}{3} + 4a^4 - 2a^4 + \frac{4a^4}{3} - \frac{a^4}{4} + \frac{a^4}{3}\right]$$

$$= \frac{1}{2}\left[10a^4 - \frac{27a^4}{3} - \frac{a^4}{4}\right]$$

$$= \frac{a^4}{2}\left[\frac{120 - 108 - 3}{12}\right] \qquad \begin{array}{r}27\times4\\ \hline 108\end{array}$$

$$= \frac{a^4}{2} \cdot \frac{\cancel{9}^{3}}{\cancel{12}_{4}} = \frac{3a^4}{8} \;//.$$

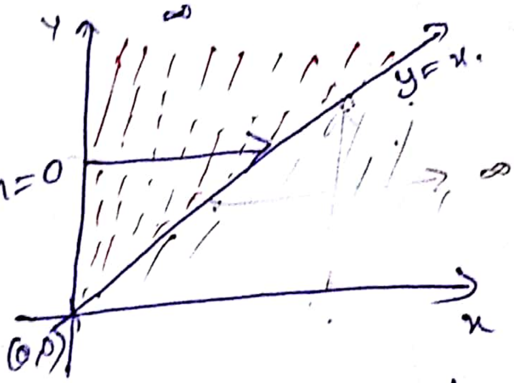
4) Evaluate : $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx$ by changing the order of integration.

Soln

Gn.. $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx.$

The region of Integration bounded by.

$x = 0 \quad x = \infty \quad y = x \quad y = \infty$



y to ∞

After changing the order of integration,

$x = 0 \quad x = y$

$y = 0 \quad y = \infty.$

$$\therefore \int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx = \int_0^{\infty}\int_0^{y} \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_0^{\infty} \left[\frac{e^{-y}}{y}\right] [x]_0^y \,dy.$$

$$= \int_0^{\infty} \left[\frac{e^{-y}}{y}\cdot y\right] dy = \left[\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= -\left(e^{-\infty} - e^{-0}\right) = -(0-1) = 1.$$

$\frac{1}{}\left(\frac{a^3}{3}\right)$

H.W

5) change the order of integration in $\int_0^a \int_y^a \frac{x \, dx \, dy}{x^2+y^2}$ and hence evaluate.

[Ans : $\frac{\pi a}{4}$]

6) change the order of integration in $\int_0^1 \int_y^{2-y} xy \, dx \, dy$ & hence evaluate.

7) Change the order of integration & evaluate

$\int_0^a \int_x^a (x^2+y^2) \, dy \, dx$.

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Multiple Integrals

Double Integration:

1) Evaluate : $\int_0^3 \int_0^2 xy(x+y)\, dx\, dy$

Soln:

$$\int_0^3 \int_0^2 (x^2y + xy^2)\, dx\, dy$$

$$\int_0^3 \left[\frac{x^3 y}{3} + \frac{x^2 y^2}{2} \right]_0^2 dy = \int_0^3 \left[\frac{8y}{3} + \frac{4}{2} y^2 \right] dy$$

$$= \left[\frac{8y^2}{6} + \frac{4y^3}{6} \right]_0^3 = \left[\left[\frac{8(9)}{6} + \frac{4(27)}{6} \right] - (0) \right]$$

$$= 12 + 18 = 30$$

2) Evaluate $\int_0^1 \int_1^2 x(x+y)\, dy\, dx$.

Soln

$$\int_0^1 \int_1^2 (x^2 + xy)\, dy\, dx.$$

$$\int_0^1 \left[x^2 y + \frac{xy^2}{2} \right]_1^2 dx = \int_0^1 \left[\left(2x^2 + \frac{4x}{2}\right) - \left(x^2 + \frac{x}{2}\right) \right] dx$$

$$= \int_0^1 \left(x^2 + \frac{3x}{2}\right) dx = \left[\frac{x^3}{3} + \frac{3x^2}{4} \right]_0^1$$

$$= \left[\frac{1}{3} + \frac{3}{4} - (0) \right]$$

$$= 4 + 9 = 13$$

3. S.T $\int_0^a \int_0^b (x+y)\,dx\,dy = \int_0^b \int_0^a (x+y)\,dy\,dx$

Soln: L.H.S $\int_0^a \left[\frac{x^2}{2} + yx \right]_a^b dy$ $\qquad \int y\,dx$

$$= \int_0^a \left[\frac{b^2}{2} + yb \right] dy$$

$$= \left[\frac{b^2 y}{2} + \frac{by^2}{2} \right]_0^a$$

$$= \frac{b^2 a}{2} + \frac{ba^2}{2} = \frac{ab}{2} [b+2]$$

R.H.S $\int_0^b \int_0^a (x+y)\,dy\,dx$

$$= \int_0^b \left[xy + \frac{y^2}{2} \right]_0^a dx = \int_0^b \left[xa + \frac{a^2}{2} \right] dx$$

$$= \left[\frac{x^2 a}{2} + \frac{a^2}{2} x \right]_0^b = \frac{b^2 a}{2} + \frac{a^2 b}{2}$$

$$= \frac{ab}{2} [b+a]$$

$\therefore$ L.H.S = R.H.S

4) $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dy\,dx}{\sqrt{a^2-x^2}}$

Soln: $\displaystyle\int_0^a \left[\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx = \int_0^a \left(\left[\frac{\sqrt{\cancel{a^2-x^2}}}{\sqrt{\cancel{a^2-x^2}}}\right] - 0\right] dx$

$\cancel{\displaystyle\int_0^a \left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^{\sqrt{a^2-x^2}} dy}$

$= [x]_0^a = a$

Triple Integration:-

1) S.T $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}} = \pi^2/8$

Soln

Let $a^2 = 1-x^2-y^2$, then $a = \sqrt{1-x^2-y^2}$

$\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-z^2}}$

$\cancel{\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \sin^{-1}(z/a)\,dz}$

$\Rightarrow \displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx.$

$\left(\because \displaystyle\int \frac{dz}{\sqrt{a^2-z^2}} = \sin^{-1}(z/a)\right)$

$\Rightarrow \displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \sin^{-1}\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}}\right) - \sin^{-1}(0)\Big]\,dy\,dx$

$\Rightarrow \displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} [\sin^{-1}(1) - \sin^{-1}(0)]\,dy\,dx$

$$\int_0^1\int_0^{\sqrt{1-x^2}} \pi/2 \, dy\,dx = \pi/2 \int_0^1 [y]_0^{\sqrt{1-x^2}} dx$$

$$\Rightarrow \pi/2 \int_0^1 \sqrt{1-x^2}\, dx.$$

$\left(\because \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right)$

$$\Rightarrow \pi/2 \left[\frac{x}{2}\sqrt{1-x^2} + 1/2 \sin^{-1}\left(\frac{x}{1}\right)\right]_0^1$$

$$= \pi/2 \left[1/2\,(0) + 1/2 \sin^{-1}(1) - 0\right]$$

$$= \pi/2 \left[1/2 \cdot \pi/2\right] = \pi^2/8.$$

2) S.T $\int_0^2\int_1^3\int_1^2 xy^2z \, dz\,dy\,dx$

Sol:

$$\int_0^2\int_1^3 xy^2\left[\frac{z^2}{2}\right]_1^2 dy\,dx$$

$$\Rightarrow \int_0^2\int_1^3 \frac{xy^2}{2}[4-1] = \int_0^2\int_1^3 \frac{3}{2}xy^2\,dy\,dx$$

$$\Rightarrow \int_0^2 \frac{3}{2}x\left[\frac{y^3}{3}\right]_1^3 dx = \frac{3}{2}\int_0^2 \frac{x}{3}(27-1)\,dx$$

$$\Rightarrow \frac{26}{2}\int_0^2 x\,dx = \frac{26}{2}\left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{26}{\cancel{2}}\left[\frac{4}{\cancel{2}}\right] = 26.$$

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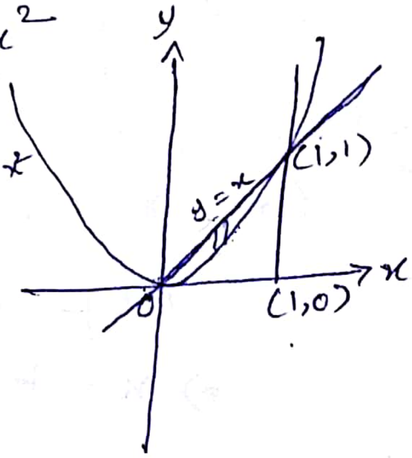
3) Evaluate $\iint (x+y)\,dy\,dx$ over the area between $y = x$ & $y = x^2$

Soln,

Gn., $y = x$ and $y = x^2$



$x^2 = x$

$\Rightarrow x(x-1) = 0$

$x = 0 \quad x = 1$

$\therefore y = 0,\ y = 1$

$\therefore$ point of intersection are $O(0,0)$ & $A(1,1)$

$$\therefore \iint (x-y)\,d\cancel{x}\,dy = \int_0^1 \int_{x^2}^{x} (x-y)\,dy\,dx$$

$$= \int_0^1 \left[xy - y^2/2\right]_{x^2}^{x} dx$$

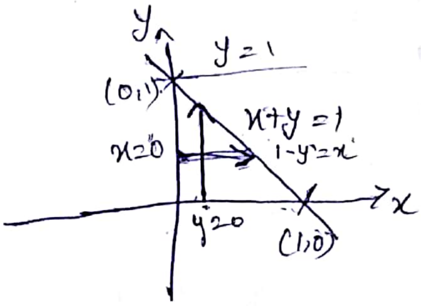
$$= \int_0^1 (x^2 - x^2/2) - (x^3 - \frac{x^4}{2})\,dx$$

$$= \int_0^1 (x^2/2 - x^3 + x^4/2)\,dx$$

$$= \left[\frac{x^3}{6} - \frac{x^4}{4} + \frac{x^5}{10}\right]_0^1$$

$$= 1/6 - 1/4 + 1/10 - 0 = \frac{10-15+6}{60} = 1/60$$

4) Evaluate $\iint xy\,dx\,dy$ over the region in the +ve quadrant for which $x+y=1$



Soln

Gn., $x=0$,

$x+y=1$

$\Rightarrow x = 1-y$

$x+y=1$

$x=0 \Rightarrow y=1$

$y=0 \Rightarrow x=1$

$\therefore x \to 0$ to $1-y$

$y \to 0$ to 1

$$\iint xy\,dx\,dy = \int_0^1 \int_0^{1-y} (xy\,dx)\,dy$$

$$= \int_0^1 \left[\frac{yx^2}{2}\right]_0^{1-y} dy \Rightarrow \int_0^1 \frac{y(1-y)^2}{2}\,dy$$

$$= \frac{1}{2}\int_0^1 y(1-2y+y^2)\,dy$$

$$= \frac{1}{2}\int_0^1 (y-2y^2+y^3)\,dy = \frac{1}{2}\left[\frac{y^2}{2}-\frac{2y^3}{3}+\frac{y^4}{4}\right]_0^1$$

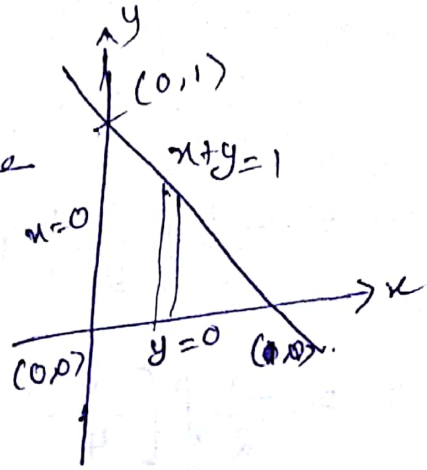
$$= \frac{1}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right] = \frac{1}{2}\left[\frac{6-8+3}{12}\right]$$

$$= \frac{1}{24}[9-8] = \frac{1}{24}$$

5) Evaluate $\iiint \frac{dz\,dy\,dx}{(x+y+z+1)^3}$ over the region bounded by the planes $x=0, y=0, z=0,$ $x+y+z=1$

<u>Soln:</u>

Gn., region in the xy plane is a triangle bounded by the lines $x=0, y=0$ & $x+y=1$



Here z varies from $z=0$ to $z=1-x-y$

y varies from $y=0$ to $y=1-x$

x varies from $x=0$ to $x=1$

$\left[\int x^n dx = \frac{x^{n+1}}{n+1}\right.$

$$\iiint \frac{dz\,dy\,dx}{(x+y+z+1)^3} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dz\,dy\,dx}{(x+y+z+1)^3}$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \int_0^{1-x} \left[(x+y+(1-x-y)+1)^{-2} - (x+y+1)^{-2}\right] dy\,dx$$

$$= \frac{-1}{2}\int_0^1\int_0^{1-x}\left[(2)^{-2} - (x+y+1)^{-2}\right] dy\, dx$$

$$= \frac{-1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{4} - (x+y+1)^{-2}\right] dy\, dx$$

$$= \frac{-1}{2}\int_0^1\left[\frac{1}{4}y - \frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x} dx$$

$$= \frac{-1}{2}\int_0^1\left[\frac{1}{4}(1-x) + (x+1+1-x)^{-1}\right] dx - (x+1)^{-1}\Big] dx$$

$$= \frac{-1}{2}\int_0^1\left[\frac{1-x}{4} + (2)^{-1} - \frac{1}{x+1}\right] dx$$

$$= \frac{-1}{2}\int_0^1\left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1}\right] dx \qquad \int \frac{dx}{x+1}$$

$$= \frac{-1}{2}\int_0^1\left[\frac{1}{4} - \frac{x}{4} + \frac{1}{2} - \frac{1}{x+1}\right] dx$$

$$= \frac{-1}{2}\int_0^1\left[\frac{3}{4} - \frac{x}{4} - \frac{1}{x+1}\right) dx$$

$$= \frac{-1}{2}\left[\frac{3}{4}x - \frac{1}{4}\frac{x^2}{2} - \log(1+x)\right]_0^1$$

$$= \frac{-1}{2}\left[\frac{3}{4} - \frac{1}{8} - \log 2\right) = \frac{-1}{2}\left[\frac{6-1}{8} - \log 2\right]$$

$$= -\tfrac{1}{2}\left[\tfrac{5}{8} - \log 2\right] = \frac{-5}{16} + \frac{1}{2}\log 2$$

H.W

1) Evaluate $\int_0^1 \int_0^x dx\, dy$ (Ans : $1/2$)

2) Evaluate $\iint xy(x+y)\,dy\,dx$ over the area between $y = x^2$ & $y = x$.

Area as a Double Integral

Formulae:

i) Area of a region R in cartesian form is $\iint_R dx\,dy$

ii) Area of a region R in polar form is $\iint_R r\,dr\,d\theta$.

Problems: U.B.

1) Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ using double integration:

Soln: Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

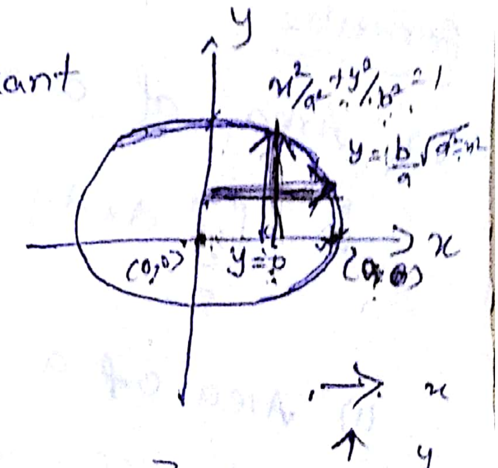
$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow \frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2} \Rightarrow \frac{b^2}{a^2}(a^2 - x^2)$$

$$\therefore y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

y varies from $y = 0$ to $y = \frac{b}{a}\sqrt{a^2 - x^2}$

x varies from $x = 0$ to $x = a$



$A = 4 \times$ Area in the 1st quadrant

$$= 4\int_0^a \int_0^{b/a\sqrt{a^2-x^2}} dy\, dx.$$

$$= 4\int_0^a [y]_0^{b/a\sqrt{a^2-x^2}} dx = 4\int_0^a \left[\frac{b}{a}\sqrt{a^2-x^2}\right] dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\, dx.$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1} x/a\right]_0^a$$

$$= \frac{4b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}(1)\right] = \frac{4b}{a}\,\frac{a^2}{2}\cdot\frac{\pi}{2}$$

$$= \pi ab \text{ sq. units}$$

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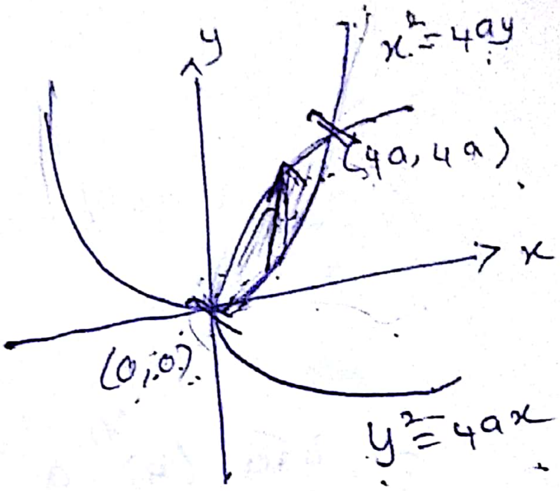
Find the area between the two parabolas $y^2=4ax$ & $x^2=4ay$ using double integration. U.Q.

Soln.,



Gn., $y^2=4ax$ — ①

$x^2=4ay$ — ②

$\Rightarrow y = x^2/4a$.

$$① \Rightarrow \left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\Rightarrow \frac{x^4}{16a^2} = 4ax$$

$$\Rightarrow x^4 = 64a^3x$$

$$x^4-64a^3x=0$$

$$x(x^3-64a^3)=0$$

$x=0$ $\quad x^3=64a^3$

$x=4a$. $\quad$ |||ly $y=4a$. $\therefore (4a,4a)$

Hence x varies from: $x:0 \to 4a$.

y varies from: $y: x^2/4a$ to $2\sqrt{ax}$.

$$\text{Area} = \iint_R dx\,dy.$$

$$= \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx = \int_0^{4a} [y]_{x^2/4a}^{2\sqrt{ax}} dx.$$

$$= \int_0^{4a} \left(2\sqrt{ax} - x^2/4a\right) dx$$

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$$= \int_0^{4a} \left(2\sqrt{a}(x)^{1/2} - \frac{x^2}{4a}\right) dx$$

$$= \left[2\sqrt{a}\,\frac{x^{3/2}}{3/2} - \frac{x^3}{12a}\right]_0^{4a}$$

$$= \frac{2\times 2\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^3}{12a} - (0)$$

$$= \frac{4\sqrt{a}}{3}(4)^{3/2}\cdot a^{3/2} - \frac{16a^2}{3}$$

$(4)^{3/2} = ((2)^2)^{3/2}$

$= (2)^3$

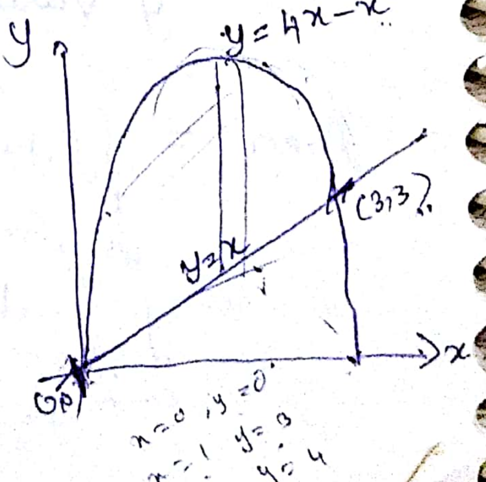
$a^{3/2}\cdot a^{1/2} = a^{4/2}$

$= a^2$

$$= \frac{4a^2(2)^3}{3} - \frac{16a^2}{3}$$

$$= \frac{32a^2 - 16a^2}{3} \Rightarrow \frac{16a^2}{3} \text{ sq. units.}$$

3) Find by double integration the area lying between the parabola $y = 4x - x^2$ & the line $y = x$.



Soln

$y = 4x - x^2$, $y = x$

$x = 4x - x^2$

$\cancel{x^2}\ \ x^2 - 3x = 0$

$x(x-3) = 0$

Scanned by CamScanner

$\therefore x=0\ ;\ x=3$

$y=x\ ,\ y=4x-x^2$

The required area $= \iint_R dx\,dy$

$$= \int_0^3 \int_x^{4x-x^2} dy\,dx = \int_0^3 [y]_x^{4x-x^2}\,dx$$

$$= \int_0^3 [(4x-x^2)-(x)]\,dx$$

$$= \int_0^3 (3x-x^2)\,dx \Rightarrow \left[\frac{3x^2}{2}-\frac{x^3}{3}\right]_0^3$$

$$= \frac{3}{2}\cdot 9 - \frac{27}{3} \Rightarrow \frac{27}{2}-\frac{27}{3}$$

$$= 27\left(\frac{3-2}{6}\right)$$

$$= 27 \times 1/6 = 9/2 \text{ sq.units}$$

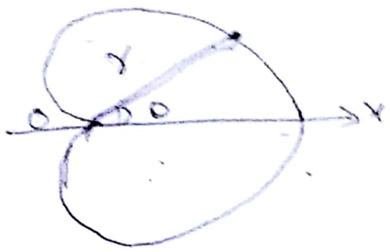
4) Find the area of the cardiod $r=a(1+\cos\theta)$ by using double integration

Soln Gn., $r=a(1+\cos\theta)$

$\therefore$ Area of the cardiod $= 2\times$ area above the initial line.

when $\theta = 0$, $r = 2a$

$\theta = \pi$, $r = 0$



$\therefore\ \theta : 0$ to π

$r : 0$ to $a(1+\cos\theta)$

(sin π = 0
cos π = −1)

$$\therefore \text{Area} = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta.$$

$$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \cancel{2}\int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta.$$

$$= a^2\int_0^{\pi}\left(1+2\cos\theta+\frac{1+\cos2\theta}{2}\right)d\theta$$

$$= a^2\int_0^{\pi}\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos2\theta\right)d\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{2}\frac{\sin2\theta}{2}\right]_0^{\pi}$$

$$= a^2\left[\left(\frac{3\pi}{2}+0\right)-0\right] = \frac{3\pi a^2}{2}\ /\!/.$$

8) Find
the
Soln.,
Gn
Sub:
3(

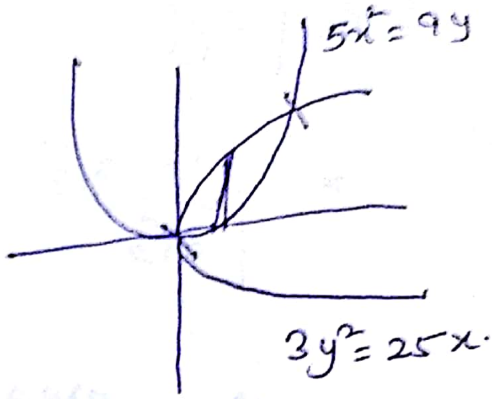
5) Find by double integration the area bet^n, the two parabolas $3y^2 = 25x$ & $5x^2 = 9y$

Soln.,

Gn., $3y^2 = 25x$ — ① , $5x^2 = 9y$

$y = \frac{5x^2}{9}$

Subst. in ①



$3\left(\frac{5x^2}{9}\right)^2 = 25x$.

$\frac{75x^4}{81} = 25x$

$\frac{75x^4}{25} = 81x$

$3x^4 - 81x = 0$

$x(3x^3 - 81) = 0$

$x = 0$, $3x^3 = 81$

$x^3 = 27$

$x = 3$

x varies from 0 to 3.

y " " $\frac{5x^2}{9}$ to $\frac{5\sqrt{x}}{\sqrt{3}}$

$$\text{Area} = \int_0^3 \int_{\frac{5x^2}{9}}^{\frac{5\sqrt{x}}{\sqrt{3}}} dy\, dx.$$

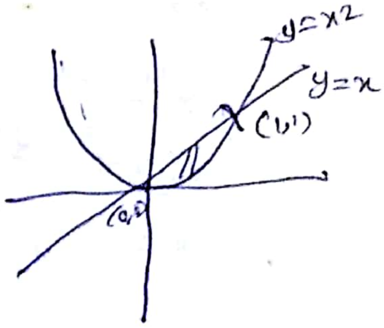
$$= \int_0^3 [y]_{\frac{5x^2}{9}}^{\frac{5\sqrt{x}}{\sqrt{3}}} dx = \int_0^3 \left[\frac{5\sqrt{x}}{\sqrt{3}} - \frac{5}{9}x^2\right] dx$$

$$\Rightarrow \left[\frac{5}{\sqrt{3}} \frac{x^{3/2}}{3/2} - \frac{5}{9}\frac{x^3}{3}\right]_0^3$$

$$\Rightarrow \left[\frac{10}{3\sqrt{3}} 3^{3/2} - \frac{5}{27} 3^3\right]$$

$$= \left[\frac{10}{3\sqrt{3}} 3\sqrt{3} - \frac{5}{27}(27)\right] = 5 \text{ sq. units.}$$

6). Find the area enclosed by the curve $y = x$ and $y = x^2$ in the 1st quadrant



Soln $y = x$, $y = x^2$

$x = x^2 \Rightarrow x^2 - x = 0$

$x(x-1) = 0$

$x = 0, x = 1$

x: varies from 0 to 1

y: " " x^2 to x.

$$\text{Area} = \iint_R dx\,dy = \int_0^1 \int_{x^2}^{x} dy\,dx$$

$$= \int_0^1 [y]_{x^2}^{x} dx = \int_0^1 [x - x^2] dx$$

$$= \left[\frac{x^2}{2} - x^3/3\right]_0^1 = 1/2 - 1/3 = 1/6$$

Volume as Triple Integral :-

1) Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by triple integrals.

Formula:

If V is the volume enclosed by the region D, then volume

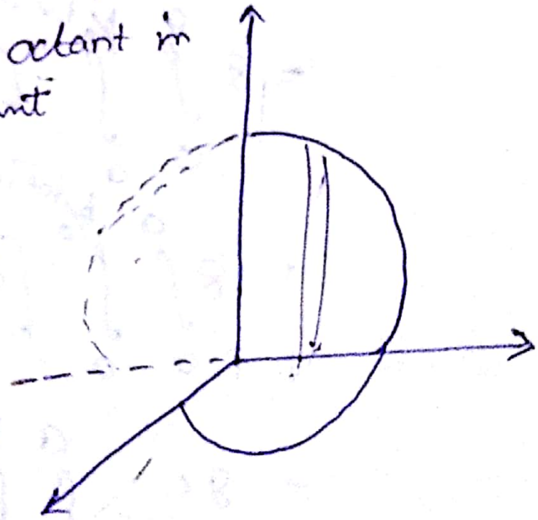
$$V = \iiint_D dx\,dy\,dz.$$

Soln:

In surface in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

The ellipse in 3D is an ellipsoid.

Volume $= 8 \times$ Volume of octant in the 1st octant



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

$$z^2=c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

$$z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$$z: 0 \text{ to } c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$z=0 \Rightarrow \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

$y=\dfrac{b}{a}\sqrt{a^2-x^2}$

$y : 0$ to $\dfrac{b}{a}\sqrt{a^2-x^2}$ (or) $b\sqrt{1-\dfrac{x^2}{a^2}}$

$z=0\ y=0 \Rightarrow x^2=a^2$

$x=a.$

Volume $= 8 \times \iiint\limits_V dz\,dy\,dx$

$= 8\displaystyle\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$

$= 8\displaystyle\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} [z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy\,dx.$

$= 8\displaystyle\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \left[c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right] dy\,dx.$

$= \dfrac{8c}{b}\displaystyle\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\ dy\,dx.$

$= \dfrac{8c}{b}\displaystyle\int_0^a \left[\frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2} + \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\left(\frac{y}{b\sqrt{1-\frac{x^2}{a^2}}}\right)\right]_0^{b\sqrt{1-\frac{x^2}{a^2}}}$

$= \dfrac{4 8c}{a b}\displaystyle\int_0^a [$

$= \dfrac{4cb^2}{b}\displaystyle\int_0^a$

$= 4cb$

$= 4c$

2) Find

bound

$\dfrac{x}{a}+\dfrac{y}{b}$

Soln

$= \displaystyle\int_0^a$

$$\left(\because \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right)$$

$$= \frac{4 \cancel{8} c}{2b} \int_0^a \left[0 + b^2\left(1-\frac{x^2}{a^2}\right)\sin^{-1}(1) - (0) \right] dx.$$

$$= \frac{4cb^{\cancel{2}}}{\cancel{b}} \int_0^a \left(1-\frac{x^2}{a^2}\right) \pi/2 \, dx.$$

$$= 4cb\frac{\pi}{2}\left[x - \frac{x^3}{3a^2}\right]_0^a$$

$$= \frac{4cb\pi}{2}\left[a - \frac{a^3}{3a^2}\right] = 2\pi bc\left[a - \frac{a}{3}\right]$$

$$\text{Volume} = \frac{4\pi abc}{3} //.$$

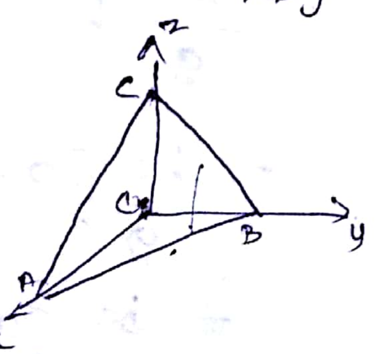
2) Find the volume of the tetrahedron bounded by the co-ordinate plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

Soln. $V = \iiint dz\,dy\,dx$

($\because$ Co-ordinate plane lines is, $x = y = z = 0$)

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz\,dy\,dx.$$



$$= \int_0^a \int_0^{b(1-x/a)} [z]_0^{c(1-\frac{x}{a}-\frac{y}{b})} \, dy \, dx.$$

$$= \int_0^a \int_0^{b(1-x/a)} \left[c\left(1-\frac{x}{a}-\frac{y}{b}\right)\right] dy \, dx$$

$$= c \int_0^a \int_0^{b(1-x/a)} \left[\left(1-\frac{x}{a}\right)-\frac{y}{b}\right] dy dx$$

$$= c \int_0^a \left[(1-x/a)y - \frac{y^2}{2b}\right]_0^{b(1-x/a)} dx$$

$$= c \int_0^a b\left(1-\frac{x}{a}\right)^2 - \frac{b^2(1-x/a)^2}{2b} \, dx$$

$$= c \int_0^a (1-x/a)^2 \left[b - b/2\right] dx.$$

$$= \frac{bc}{2} \int_0^a \left(1 - \frac{2x}{a} + \frac{x^2}{a^2}\right) dx$$

$$= \frac{bc}{2} \left[x - \frac{2x^2}{2a} + \frac{x^3}{3a^2}\right]_0^a$$

$$= \frac{bc}{2} \left[a - \frac{a^2}{a} + \frac{a^3}{3a^2}\right]$$

$$= \frac{bc}{2} \left[a - a + \frac{a}{3}\right] = \frac{abc}{6}$$

3) Find the volume of the sphere $x^2+y^2+z^2=a^2$ using triple integrals. U.B.

Soln

Volume = 8× Volume in the I octant

$8 \iiint dz\,dy\,dx$

$x^2+y^2+z^2=a^2$

$z = \sqrt{a^2-x^2-y^2}$ $\therefore$ z varies 0 to $\sqrt{a^2-x^2-y^2}$

y varies from 0 to $\sqrt{a^2-x^2}$

x " " 0 to a.

$$V = 8 \times \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$\because \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

$$= 8 \times \int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx.$$

$$= 8 \times \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{(a^2-x^2)-y^2}\,dy\,dx.$$

$$= 8 \times \int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{(a^2-x^2)}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}} \right]_0^{\sqrt{a^2-x^2}}$$

$$= 8\int_0^a \left[0 + \frac{(a^2-x^2)}{2}\sin^{-1}(1) \right] dx.$$

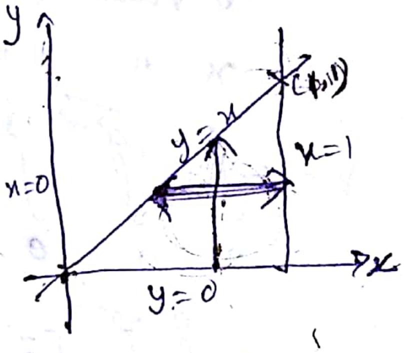
$$= \frac{{}^{2}\cancel{8}\pi}{\cancel{4}} \int_0^a (a^2-x^2)\,dx = 2\pi \left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$2\pi\left[a^3 - a^3/3\right] = \frac{4\pi a^3}{3}$$ //.

Change the order of Integration

[Changing $dy\,dx$ to $dx\,dy$ is the main concept]

1) Change the order of integration and evaluate $\int_0^1\int_0^x dy\,dx$



Soln

Gn., $x=0$ $\quad x=1$

$y=0$ $\quad y=x$

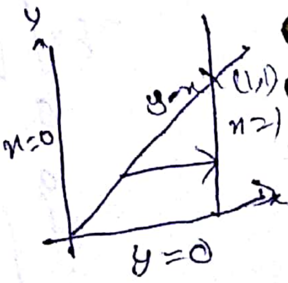
After changing the order of integration, the limits are x : y to 1

y : 0 to 1

$\rightarrow$ 'x'

$\uparrow$ y

$$\therefore \int_0^1\int_0^x dy\,dx = \int_0^1\int_y^1 dx\,dy$$



$$= \int_0^1 [x]_y^1\,dy$$

$$= \int_0^1 (1-y)\,dy$$

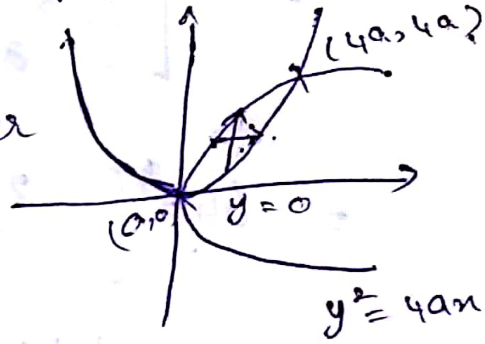
$$= \left[y - \frac{y^2}{2}\right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$ //.

27) Change the order of integration & hence evaluate $\displaystyle\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx$.

Soln

Gn, $x = 0$ $\quad x = 4a$

$y = \dfrac{x^2}{4a}$ $\quad y = 2\sqrt{ax}$, $x^2 = 4ay$.



After changing the order of integration we have,

$x = \dfrac{y^2}{4a}$, $x = 2\sqrt{ay}$

$y = 0$ to $4a$.

$$\therefore \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} xy\,dx\,dy$$

$$= \int_0^{4a}\left[\frac{x^2y}{2}\right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a}\left[\frac{4ay.y}{2} - \frac{y}{2}\cdot\frac{y^4}{16a^2}\right]$$

$$= \int_0^{4a} \left[\frac{4a}{2}(y^2) - \frac{1}{32a^2}(y^5)\right] dy$$

$$= \left[\frac{4a}{2} \times \frac{y^3}{3} - \frac{1}{32a^2}\frac{y^6}{6}\right]_0^{4a}$$

$$= \left[\frac{{}^2\cancel{4}a}{\cancel{2}} \cdot \frac{64a^3}{3} - \frac{1}{32a^2} \cdot \frac{(4)^6 a^6}{6} - 0\right]$$

$$= \left[\frac{2}{3}\frac{64a^4}{\cancel{a}} - \frac{1}{\cancel{32a}}\frac{(\cancel{64})(64)a^4}{\cancel{6}3}\right]$$

$$= 64a^4\left[\frac{2}{3} - \frac{1}{3}\right]$$

$$= 64a^4\left(\tfrac{1}{3}\right) = \frac{64a^4}{3}$$ //.

3) Change the order of integration in the integral $\int_0^a \int_{x^2/a}^{2a-x} xy \, dy\, dx$ & hence evaluate

Soln.

Gn., $\int_0^a \int_{x^2/a}^{2a-x} xy \, dy\, dx$.

The region of integration is bounded by $x=0$, $x=a$, $y = x^2/a$, $y = 2a+x$

After changing of integration,

I) $x = 0$ to $x =$

$y = a$ to $2a$

II)

$x = 0$

$y =$

$\therefore \iint_0^a \int_{x^2/a}^{2a-x} x$

$= \int_a^{2a} [$

$= \frac{1}{2}\int_a^{2a} [$

$= \frac{1}{2}[$

$= \frac{1}{2}[$

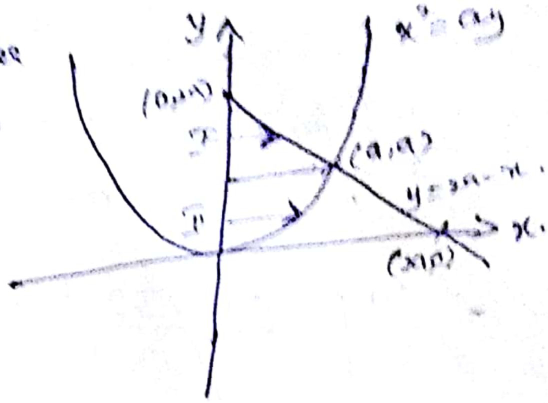
$= \frac{1}{2}$

$= \frac{1}{2}$

After changing the order of integration,

I) $x = 0$ to $x = 2a - y$

$y = a$ to $2a$.



II).

$x = 0$ to $\sqrt{ay}$

$y = 0$ to a.

$$\therefore \int_0^a \int_{x^2/a}^{2a-x} xy\, dy\, dx = \iint \mathrm{I} + \iint \mathrm{II}.$$

$$= \int_a^{2a} \int_0^{2a-y} xy\, dx\, dy + \int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy.$$

$$= \int_a^{2a} \left[\frac{x^2}{2} \cdot y \right]_0^{2a-y} dy + \int_0^a \left[\frac{x^2}{2} y \right]_0^{\sqrt{ay}} dy$$

$$= \frac{1}{2} \int_a^{2a} \left[y(2a-y)^2 dy + \frac{1}{2} \int_0^a (ay) \cdot y\, dy \right.$$

$$= \frac{1}{2} \left[\int_a^{2a} y(4a^2 - 4ay + y^2) dy + \int_0^a ay^2\, dy \right]$$

$$= \frac{1}{2} \left[\int_a^{2a} (4a^2 y - 4ay^2 + y^3) dy + \int_0^a ay^2\, dy \right]$$

$$= \frac{1}{2} \left[\left[4a^2 \cdot \frac{y^2}{2} - 4a \cdot \frac{y^3}{3} + \frac{y^4}{4} \right]_a^{2a} + \left[\frac{ay^3}{3} \right]_0^a \right]$$

$$= \frac{1}{2} \left[4a^2 \cdot \frac{(4a^2)}{2} - 4a \frac{(8a^3)}{3} + \frac{16a^4}{4} - \left[4a^2 \cdot \frac{a^2}{2} - 4a \cdot a^3/3 + \frac{a^4}{4} \right] \right] + \left[\frac{a^4}{3} \right] \Big)$$

$$= \frac{1}{2}\left[8a^4 - \frac{32a^4}{3} + 4a^4 + \frac{a^4}{3}\right]$$

$$= \frac{a^4}{2}\left[\frac{24 - 32 + 12 + 1}{3}\right]$$

$$= \frac{a^4}{2}\left[5/3\right] = \frac{5a^4}{6}.$$

$$= \frac{1}{2}\left[8a^4 - \frac{32a^4}{3} + 4a^4 - 2a^4 + \frac{4a^4}{3} - \frac{a^4}{4} + \frac{a^4}{3}\right]$$

$$= \frac{1}{2}\left[10a^4 - \frac{27a^4}{3} - \frac{a^4}{4}\right]$$

$$= \frac{a^4}{2}\left[\frac{120 - 108 - 3}{12}\right]$$

$$\frac{27\times 4}{108}$$

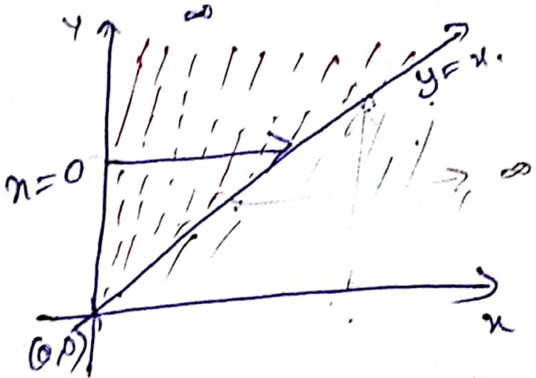
$$= \frac{a^4}{2}\cdot\frac{\cancel{9}^{3}}{\cancel{12}_{4}} = \frac{3a^4}{8}\ //.$$

4) Evaluate : $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx$ by changing the order of integration.

Soln Gn.. $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx$.

The region of Integration bounded by.

$x = 0$ $x = \infty$ $y = x$ $y = \infty$



$y \to \infty$

After changing the order of integration,

$x = 0$ $x = y$

$y = 0$ $y = \infty$.

$$\therefore \int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx = \int_0^{\infty}\int_0^{y} \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_0^{\infty} \left[\frac{e^{-y}}{y}\right] [x]_0^y \, dy.$$

$$= \int_0^{\infty} \left[\frac{e^{-y}}{y} \cdot y\right] dy = \left[\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= -(e^{-\infty} - e^{-0}) = -(0-1) = 1,$$

H.W

5) Change the order of integration in $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ and hence evaluate.

[Ans : $\frac{\pi a}{4}$]

6) Change the order of integration in $\int_0^1 \int_y^{2-y} xy\,dx\,dy$ & hence evaluate.

7) Change the order of integration & evaluate $\int_0^a \int_x^a (x^2+y^2)\,dy\,dx$.

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Basic Integration Formulae

The following results are a direct consequence of the definition of an integral.

(i) $\dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}\right) = \dfrac{1}{n+1}(n+1)x^n = x^n,\ n \neq -1 \Rightarrow \int x^n dx = \dfrac{x^{n+1}}{n+1} + c$

(ii) If $x > 0$, $\dfrac{d}{dx}(\log|x|) = \dfrac{d}{dx}(\log x) = \dfrac{1}{x}$

If $x < 0$, $\dfrac{d}{dx}(\log|x|) = \dfrac{d}{dx}[\log(-x)] = \dfrac{1}{-x}(-1) = \dfrac{1}{x}$

$\therefore \quad \dfrac{d}{dx}(\log|x|) = \dfrac{1}{x},\ x \neq 0 \Rightarrow \int \dfrac{1}{x} dx = \log|x| + c,\ x \neq 0$

(iii) $\dfrac{d}{dx}(-\cos x) = \sin x \Rightarrow \int \sin x dx = -\cos x + c$

(iv) $\dfrac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x dx = \sin x + c$

(v) $\dfrac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x\, dx = \tan x + c$

(vi) $\dfrac{d}{dx}(-\cot x) = \text{cosec}^2 x \Rightarrow \int \text{cosec}^2 c\, dx = -\cot x + c$

(vii) $\dfrac{d}{dx}(\sec x) = \sec x \tan x \Rightarrow \int \sec x \tan x\, dx = \sec x + c$

(viii) $\dfrac{d}{dx}(-\text{cosec}\, x) = \text{cosec}\, x \cot x \Rightarrow \int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + c$

(ix) $\dfrac{d}{dx}(e^x) = e^x \Rightarrow \int e^x dx = e^x + c$

(x) $\dfrac{d}{dx}\left(\dfrac{a^x}{\log a}\right) = \dfrac{a^x \log a}{\log a} = a^x \Rightarrow \int a^x dx = \dfrac{a^x}{\log a} + c$

(xi)
$$\left.\begin{array}{l}\dfrac{d}{dx}(\sin^{-1}x) = \dfrac{1}{\sqrt{1-x^2}} \\[2ex] \dfrac{d}{dx}(\cos^{-1}x) = -\dfrac{1}{\sqrt{1-x^2}}\end{array}\right\} \Rightarrow \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$$

$$= -\cos^{-1}x + c$$

(xii)
$$\left.\begin{array}{l}\dfrac{d}{dx}(\tan^{-1}x) = \dfrac{1}{1+x^2} \\[2ex] \dfrac{d}{dx}(\cot^{-1}x) = -\dfrac{1}{1+x^2}\end{array}\right\} \Rightarrow \int \frac{1}{1+x^2}\,dx = \tan^{-1} + c$$

$$= -\cot^{-1}x + c$$

(xiii)
$$\left.\begin{array}{l}\dfrac{d}{dx}(\sec^{-1}x) = \dfrac{1}{x\sqrt{x^2-1}} \\[2ex] \dfrac{d}{dx}(\operatorname{cosec}^{-1}x) = -\dfrac{1}{x\sqrt{x^2-1}}\end{array}\right\} \Rightarrow \int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + c$$

$$= -\operatorname{cosec}^{-1}x + c$$

(xiv) $$\frac{d}{dx}(x) = 1 \Rightarrow \int 1\,dx = x + c$$

(xv) $$\frac{d}{dx}(c) = 0 \Rightarrow \int 0\,dx = c \qquad \text{(constant)}$$

Double Integration :- It is used to find the Surface Integral.

① Evaluate $\int_0^3 \int_0^2 xy(x+y)\,dx\,dy$

$= \int_0^3 \int_0^2 (x^2y + xy^2)\,dx\,dy$

$= \int_0^3 \left[\frac{x^3}{3}y + \frac{x^2}{2}y^2\right]_0^2 dy = \int_0^3 \left[\frac{8}{3}y + \frac{4}{2}y^2\right] dy$

$= \left[\frac{8}{3}\frac{y^2}{2} + 2\frac{y^3}{3}\right]_0^3 = \left[\frac{\cancel{8}^{4\,2}}{\cancel{3}} \times \frac{9}{\cancel{2}} + \frac{2}{3}(27)\right]$

$= \frac{2}{3}\left[18+27\right] = \frac{2}{3}\left[45\right] = 30$

② Evaluate $\int_0^1 \int_1^2 x(x+y)\,dy\,dx$

$\int_0^1 \int_1^2 (x^2+xy)\,dy\,dx$

$\int_0^1 \left[x^2y + x\frac{y^2}{2}\right]_1^2 dx = \int_0^1 \left[x^2(2) + x\frac{4}{2}\right] - \left[x^2 + \frac{x}{2}\right] dx$

$= \int_0^1 x\left(2x + 2 - x - \frac{1}{2}\right)dx$

$= \int_0^1 x\left(x + \frac{3}{2}\right)dx$

$$= \left[\frac{x^3}{3} + \frac{3}{2}\frac{x^2}{2}\right]_0^1 = \frac{1}{3} + \frac{3}{4} = \frac{13}{12} //$$

(3) Show that $\int_0^a \int_0^b (x+y)\,dx\,dy = \int_0^b \int_0^a (x+y)\,dy\,dx$

<u>L.H.S.</u>

$$\int_0^a \left[\frac{x^2}{2} + xy\right]_0^b dy = \int_0^a \left(\frac{b^2}{2} + by\right) dy$$

$$= \left[\frac{b^2}{2}y + \frac{by^2}{2}\right]_0^a$$

$$= \frac{b}{2}\left[ab + a^2\right]$$

$$= \frac{ab}{2}\left[b + a\right]$$

<u>R.H.S.</u>

$$\int_0^b \int_0^a (x+y)\,dy\,dx = \int_0^b \left[xy + \frac{y^2}{2}\right]_0^a dx$$

$$= \int_0^b \left(ax + \frac{a^2}{2}\right) dx$$

$$= \left[a\frac{x^2}{2} + \frac{a^2}{2}x\right]_0^b$$

$$= \frac{a}{2}\left[b^2 + ab\right] = \frac{ab}{2}\left[b+a\right]$$

$\therefore$ L.H.S. = R.H.S.

(4) $$\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dy\,dx}{\sqrt{a^2-x^2}} = \int_0^a \left[\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{\cancel{\sqrt{a^2-x^2}}}{\cancel{\sqrt{a^2-x^2}}}\, dx = [x]_0^a = a \,//$$

⑤ $$\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$$

$$= \int_0^5 \int_0^{x^2} (x^3 + xy^2)\,dx\,dy \Rightarrow \int_0^5 \int_0^{x^2} (x^3+xy^2)\,dy\,dx$$

$$= \cancel{\int_0^5 \left[\frac{x^4}{4} + y^2\frac{x^2}{2}\right]_0^{x^2}} = \int_0^5 \left[x^3 y + x\frac{y^3}{3}\right]_0^{x^2} dx$$

$$= \int_0^5 \left[x^3(x^2) + \frac{x}{3}(x^6)\right] dx$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right) dx = \left[\frac{x^6}{6} + \frac{1}{3}\frac{x^8}{8}\right]_0^5$$

$$= \frac{1}{6}\left[5^6\right] + \frac{1}{3\times 8}\left[5^8\right] = 5^6\left[\frac{1}{6} + \frac{5^2}{24}\right]$$

$$= 5^6\left[\frac{1}{6} + \frac{25}{24}\right] = 5^6\left[\frac{4+25}{24}\right] = 5^6\left[\frac{29}{24}\right]$$

Triple Integration :-

1) S.T. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}} = \pi^2/8$

Solu :- Let $a^2 = 1 - x^2 - y^2$ then $a = \sqrt{1-x^2-y^2}$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-z^2}} \quad \because \int \frac{dz}{\sqrt{a^2-z^2}} = \sin^{-1}\left(\frac{z}{a}\right)$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}} \right]_0^{\sqrt{1-x^2-y^2}}$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \sin^{-1}\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}}\right) - \sin^{-1}(0) \, dy \, dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \sin^{-1}(1) - \sin^{-1}(0) \, dy \, dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \pi/2 \, dy \, dx \quad = \pi/2 \int_0^1 [y]_0^{\sqrt{1-x^2}} \, dx$$

$$= \pi/2 \int_0^1 \sqrt{1-x^2} \, dx \qquad \because \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$= \pi/2 \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_0^1$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\sqrt{(0)} + \frac{1}{2}\sin^{-1}(1)\right] = \frac{\pi}{2}\left[\frac{1}{2}\,\pi/2\right] = \frac{\pi^2}{8}$$

2) S.T. $\int_0^2 \int_1^3 \int_1^2 xy^2z \, dz \, dy \, dx$

Soln:- $\int_0^2 \int_1^3 xy^2 \left[\frac{z^2}{2}\right]_1^2 dy \, dx$

$= \int_0^2 \int_1^3 \frac{xy^2}{2} [4-1] \, dy \, dx = \int_0^2 \int_1^3 \frac{3}{2} xy^2 \, dy \, dx$

$= \int_0^2 \frac{3}{2} x \left[\frac{y^3}{3}\right]_1^3 dx = \frac{\cancel{3}}{2} \int_0^2 \frac{x}{\cancel{3}} (27-1) \, dx$

13

$\frac{\cancel{26}}{2}$

$\frac{13}{2}[x^2]_0$

$\frac{13 \times 4}{2}$

$\frac{\cancel{52}}{\cancel{2}}$ 26

$= \frac{26}{2} \int_0^2 x \, dx = \frac{26}{2} \left[\frac{x^2}{2}\right]_0^2 = \frac{26}{\cancel{4}} \left[\frac{\cancel{4}}{\cancel{2}}\right] = 26$

(1) $$\int_0^1 \int_0^2 \int_0^3 xyz \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^2 xy \left[\frac{z^2}{2}\right]_0^3 dy \, dx$$

$$= \int_0^1 \int_0^2 xy \left[\frac{9}{2}\right] dy \, dx$$

$$= \frac{9}{2} \int_0^1 x \left[\frac{y^2}{2}\right]_0^2 dx$$

$$= \frac{9}{\cancel{4}} \int_0^1 x \left[4\right] dx$$

$$= 9\left[x^2/2\right]_0^1 = 9/2$$

Note:- If the region is given first draw the region. If the order of integration is first w.r.to y and then w.r.to x draw a strip parallel to y-axis. The limits of y are the values of y in terms of x at the lower end of the strip and at the upper end of the strip. The limits of x are the values of x at the left extreme and at the right extreme of the region. {The outer intergral must be a constant value (ie) like $0 \to 1$ etc}

If the order of integration is first w.r.to x and then w.r.to y draw a strip parallel to x-axis. The limits of x are the values of x in terms of y at the extremities of the strip. Limits of y are the values of y at the lower end and at the upper end of the region.]

In this problem the order of integration is first w.r.to y & then w.r.to x.

$\therefore$ Take a strip parallel to y-axis

Its lower end is P on the parabola $y=x^2$ and upper end Q is on $y=x$

$\therefore$ limits of y are $y=x^2$ & $y=x$

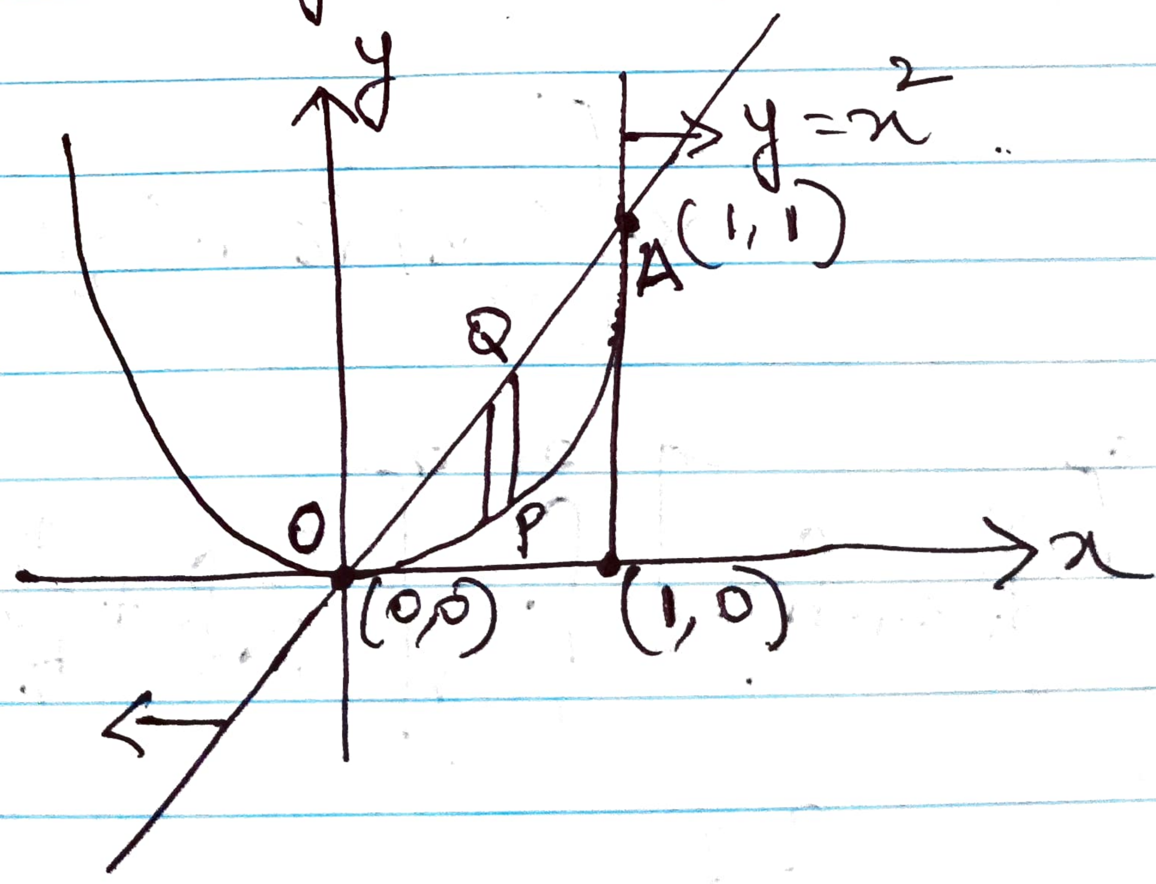
Then we move the strip PQ to cover the region

$\therefore$ x varies from 0 to 1.

Double Integral in Cartesian Co-ordinates

4) Using double Integration find the area bounded by [or]

Evaluate $\iint xy(x+y)\,dy\,dx$ over the area between $y=x^2$ and $y=x$



<u>Solu:-</u> Given that the region of integration is bounded by $y = x$ and $y = x^2$

To find the point of intersection

Solve $\underbrace{y = x^2}_{①}$ & $y = x$ — ②

Sub ② in ① $\Rightarrow x = x^2 \Rightarrow x^2 - x = 0$

$\therefore x(x-1) = 0$

$x = 0 \quad x = 1$

Sub in $y = x^2$ or $y = x$ we get

$y = 0, y = 1$

$\therefore$ The points are $O(0,0)$ & $A(1,1)$

$$\therefore \iint_R xy(x+y)\,dx\,dy = \int_0^1 \int_{x^2}^{x} xy\,(x+y)\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} x^2y + xy^2\,dy\,dx$$

$$= \int_0^1 \left[x^2\frac{y^2}{2} + x\frac{y^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left\{\left[x^2\frac{x^2}{2} + x\frac{x^3}{3}\right] - \left[x^2\frac{x^4}{2} + x\frac{x^6}{3}\right]\right\} dx$$

$$= \int_0^1 \left(\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right) dx$$

$$= \int_0^1 \left(\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right) dx = \left[\frac{\cancel{5}}{6}\frac{x^5}{\cancel{5}} - \frac{1}{2}\frac{x^7}{7} - \frac{1}{3}\frac{x^8}{8}\right]_0^1$$

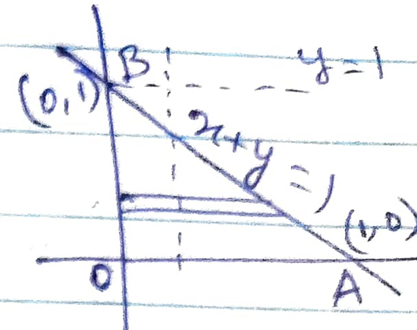
$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{28-12-7}{168} = \frac{9}{168} = \frac{3}{56}$$

3) Evaluate $\iint xy\,dx\,dy$ over the region in the positive quadrant for which $x+y=1$

Soln:-

In this problem the order of integr is w.r.to x then w.r.to y

So the left extreme is y axis (ie)



$x=0$ & $x+y=1$ $\therefore x=1-y$

$\therefore x \to 0 \to 1-y$ $\quad$ x varies:- $0 \to 1-y$

y=1

x-axis y=0

& $y=0$ & $y=1$ $\therefore y \to 0 \to 1$ $\quad$ y varies $0 \to 1$

$x+y=1$
when $x=0$ $y=1$ $(0,1)$
when $y=0$ $x=1$ $(1,0)$

$$= \int_0^1 \int_0^{1-y} xy\,dx\,dy$$

$$= \int_0^1 \left[y\frac{x^2}{2} \right]_0^{1-y} dy = \int_0^1 y\frac{(1-y)^2}{2}\,dy$$

$$= \frac{1}{2}\int_0^1 y(1-2y+y^2)\,dy = \frac{1}{2}\int_0^1 (y-2y^2+y^3)\,dy$$

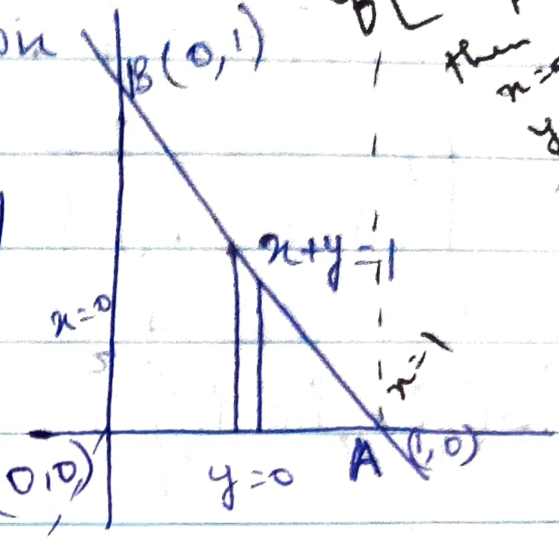
$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4} \right]_0^1 = \frac{1}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right]$$

$$= \frac{1}{2}\left[\frac{6-8+3}{12} \right]$$

$$= \frac{1}{24}\left[9-8 \right] = 1/24$$

5) Evaluate $\iiint \frac{dz\,dy\,dx}{(x+y+z+1)^3}$ over the region bounded by the planes $x=0, y=0, z=0$, $x+y+z=1$ (or) co-ordinate planes

Solu:- The given region is tetrahedron. The projection of the given region in the xy is a triangle bounded by the lines $x=0, y=0, \& x+y=1$

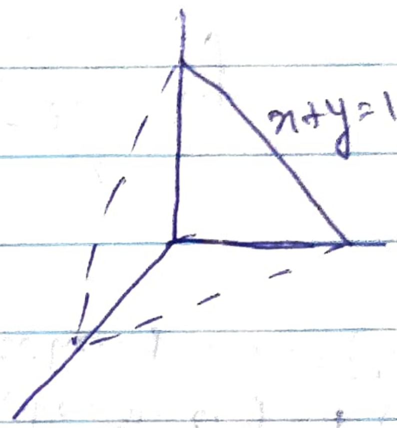


then $x=0$, $y=0$, $z=0$

Here z varies from $z=0$ to $z=1-x-y$

y varies from $y=0$ to $y=1-x$

x varies from $x=0$ to $x=1$



$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dz\,dy\,dx}{(x+y+z+1)^3}$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[(x+y+\cancel{1}+1-x-y)^{-2}-(x+y+1)^{-2}\right]dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[(2)^{-2}-(x+y+1)^{-2}\right]dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{4}-(x+y+1)^{-2}\right]dy\,dx$$

$$= -\frac{1}{2}\int_0^1\left[\frac{1}{4}y-\frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x}dx$$

$$= -\frac{1}{2}\int_0^1\left\{\left[\frac{1}{4}(1-x)+(x+1+1-x)^{-1}\right]\cancel{dx}-\left[x+1\right]^{-1}\right\}dx$$

$$= -\frac{1}{2}\int_0^1\left[\frac{1-x}{4}+(2)^{-1}-\frac{1}{x+1}\right]dx$$

$$= -\frac{1}{2}\int_0^1\left\{\frac{1-x}{4}+\frac{1}{2}-\frac{1}{x+1}\right\}dx$$

$$= -\frac{1}{2}\int_0^1\left[\frac{1}{4}-\frac{x}{4}+\frac{1}{2}-\frac{1}{x+1}\right]dx$$

$$= -\frac{1}{2}\int_0^1\left(\frac{3}{4}-\frac{x}{4}-\frac{1}{x+1}\right)dx$$

$$= -\frac{1}{2}\left[\frac{3}{4}x - \frac{1}{4}\frac{x^2}{2} - \log(1+x)\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{3}{4} - \frac{1}{8} - \log 2\right] = -\frac{1}{2}\left[\frac{6-1}{8} - \log 2\right]$$

$$= -\frac{1}{2}\left[\frac{5}{8} - \log 2\right] = \frac{-5}{16} + \frac{1}{2}\log 2 \;//$$

6) Evaluate $\int_0^1 \int_0^x dx\, dy$

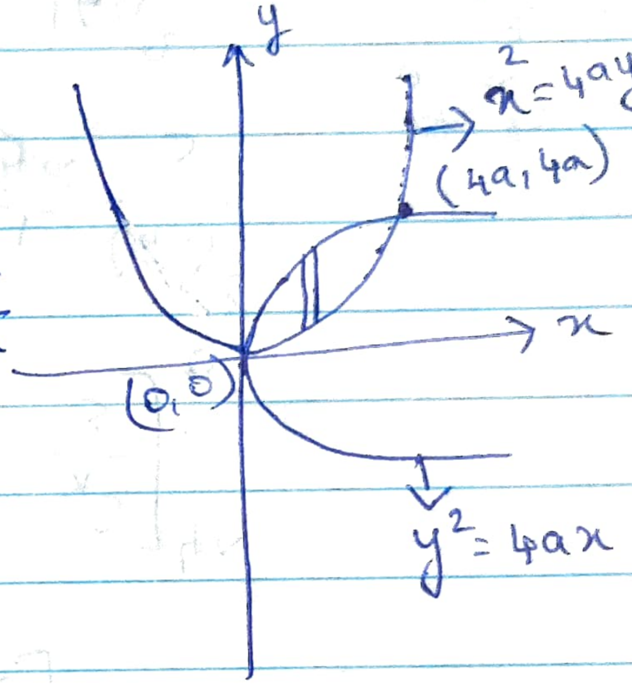
[Since the limits of the inner integral is a function of x then the integration should be w.r.to y first]

$$= \int_0^1 \int_0^x dy\, dx = \int_0^1 [y]_0^x dx = \int_0^1 x\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 = \underline{\underline{1/2}}$$

Area As a Double Integral :-

1. Find the area between the two parabolas $y^2 = 4ax$ and $x^2 = 4ay$ using double integration



Solu:- To find the limits

Given $y^2 = 4ax$ & $x^2 = 4ay$
① $\therefore y = \frac{x^2}{4a}$

Sub y in ①

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\frac{x^4}{16a^2} = 4ax$$

$$\therefore x^4 = 16a^2 \times 4ax$$

$$x^4 = 64a^3x \Rightarrow x^4 - 64a^3x = 0$$

$$x(x^3 - 64a^3) = 0$$

$$\therefore x = 0 \; ; \; x^3 = 64a^3$$

$$x = 4a$$

When sub $x = 4a$ in ①

$$y^2 = 4a \cdot 4a = 16a^2$$

$$y = 4a$$

$(4a, 4a)$

Hence x varies from $x : 0 \to 4a$

$\because y^2 = 4ax$ $\quad$ $x^2 = 4ay$

$\therefore y = 2\sqrt{ax}$ $\quad$ $\therefore y = \frac{x^2}{4a}$

Hence y varies from $y : \frac{x^2}{4a} \to 2\sqrt{ax}$

$$\text{Area} = \iint_R dx\, dy$$

$$= \int_{x=0}^{4a} \int_{y=x^2/4a}^{2\sqrt{ax}} dy\, dx = \int_0^{4a} \left[y\right]_{x^2/4a}^{2\sqrt{ax}} dx$$

$$= \int_0^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a}\right] dx$$

$$= \int_0^{4a} \left[2\sqrt{a}\, x^{1/2} - \frac{x^2}{4a}\right] dx$$

$$= \left[2\sqrt{a}\, \frac{x^{3/2}}{3/2} - \frac{1}{4a}\, \frac{x^3}{3}\right]_0^{4a}$$

$$= 2\sqrt{a}\, \frac{(4a)^{3/2}}{3/2} - \frac{1}{\cancel{4a}}\, \frac{(4a)^{\cancel{3}}}{3}$$

$$= \frac{4\sqrt{a}}{3} (4)^{3/2} (a)^{3/2} - \frac{16a^2}{3}$$

$a^{3/2}\ a^{1/2}$

$a^{4/2} = a^2$

$(4)^{3/2} = (4^{1/2})^3$

$= (\sqrt{4})^3$

$= (2)^3$

$$= \frac{4a^2(2)^3}{3} - \frac{16a^2}{3} = \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3} \text{ sq. units } //$$

Cartesian Co-or.

2) Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

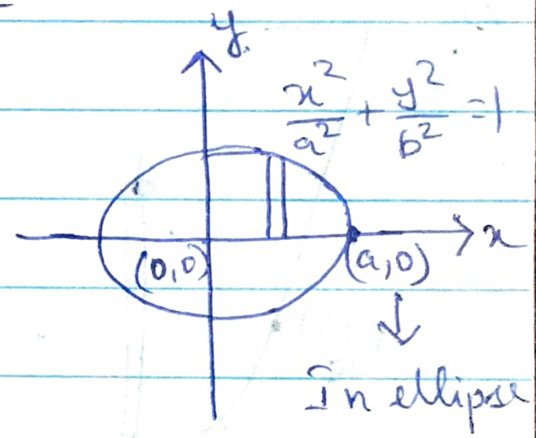
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow \frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2} \Rightarrow y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

y varies from $y = 0$ to $y = \frac{b}{a}\sqrt{a^2 - x^2}$

x varies from $x = 0$ to $x = a$



$$\text{Area} = 4 \iint_R dx\, dy$$

$$= 4 \int_0^a \int_0^{b/a\sqrt{a^2-x^2}} dy\, dx$$

$$= 4 \int_0^a [y]_0^{b/a\sqrt{a^2-x^2}} dx = 4 \int_0^a \left[\frac{b}{a}\sqrt{a^2-x^2}\right] dx$$

$$= \frac{4b}{a} \int_0^a \sqrt{a^2-x^2}\, dx = \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

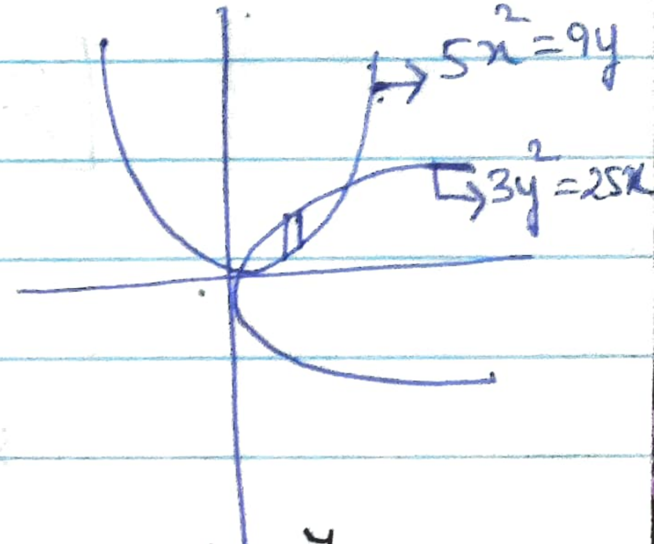
Cartesian Co-ordinate

4. Find by double integration the area between the two parabolas $3y^2 = 25x$ & $5x^2 = 9y$

Soln:-

Given $3y^2 = 25x$; $5x^2 = 9y$

① $\quad y = \frac{5x^2}{9}$



Sub y in ①

$$3\left(\frac{5x^2}{9}\right)^2 = 25x$$

$$25x = \frac{75x^4}{81}$$

$$\frac{75x^4}{81} = 25x$$

$$81x = \frac{75x^4}{81} \times \frac{1}{25}$$

$$\frac{\cancel{3}}{9 \times \cancel{3}\, 3}$$

$$\frac{75x^4}{25} = 81x \Rightarrow 3x^4 - 81x = 0$$

$$\frac{25}{27}$$

$$x(3x^3 - 81) = 0$$

$x = 0 \quad 3x^3 = 81$

$\therefore x$ varies from $x = 0$ to $x = 3$

$$x^3 = 81/\cancel{3}^{27}$$

$$x = 3$$

y varies from $y = \frac{5x^2}{9}$ to $y = \frac{5\sqrt{x}}{\sqrt{3}}$

$$\text{Area} = \int_0^3 \int_{\frac{5x^2}{9}}^{\frac{5\sqrt{x}}{\sqrt{3}}} dy\, dx \qquad \text{where } A = \iint_R dx\,dy$$

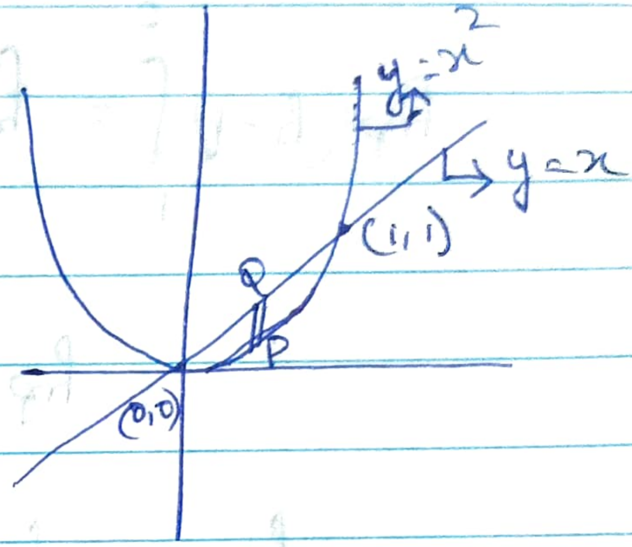
$$= \int_0^3 [y]_{\frac{5x^2}{9}}^{\frac{5\sqrt{x}}{\sqrt{3}}} dx = \int_0^3 \left[\frac{5\sqrt{x}}{\sqrt{3}} - \frac{5}{9}x^2\right] dx$$

$$= \left[\frac{5}{\sqrt{3}} \frac{x^{3/2}}{3/2} - \frac{5}{9}\frac{x^3}{3}\right]_0^3$$

$$= \left[\frac{5}{\sqrt{3}} \times \frac{2}{3} (3)^{3/2} - \frac{5}{27}(3)^3\right]$$

$$= \frac{10}{3\cancel{\sqrt{3}}}\, 3\cancel{\sqrt{3}} - \frac{5}{\cancel{27}} \times \cancel{27} = 5 \text{ sq. units}$$

5) Find the area enclosed by the curve $y = x$ and $y = x^2$ in the 1^{st} quadrant.



Soln:- $y = x \quad y = x^2$

$x^2 = x \Rightarrow x^2 - x = 0$

$x(x-1) = 0$

$x = 0 \quad x = 1$

x varies from $0 \to 1$

y varies from $x^2 \to x$

$$\text{Area} = \iint_R dx\,dy$$

$$= \int_0^1 \int_{x^2}^{x} dx\,dy = \int_0^1 \int_{x^2}^{x} dy\,dx = \int_0^1 [y]_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3}$$

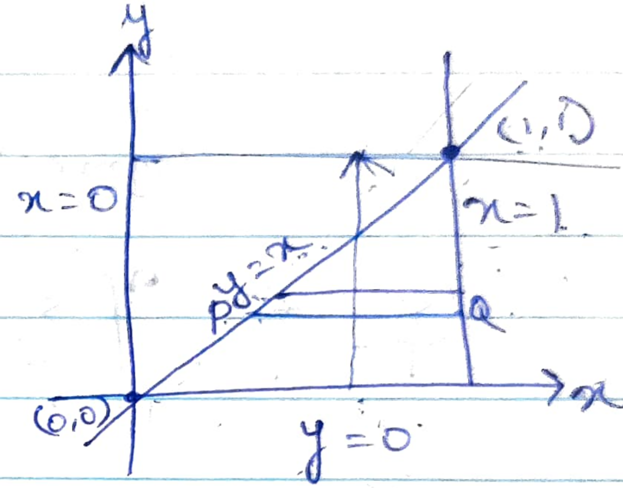
$$= \frac{3-2}{6} = 1/6 \text{ sq. units}$$

Change of Order of Integration :-

[Changing dy dx to dx dy is the main concept]

1. Change the order of integration and evaluate

$$\int_{x=0}^{1}\int_{y=0}^{x} dy\, dx$$



Solu :- The region of integration is bounded by $y=0$ to $y=x$ & $x=0$ to $x=1$

After changing the order of integration the limits are (ie) draw a strip ||el to x axis

$\therefore$ x varies from $x=y \to x=1$ $\quad x : y \to 1$

y varies from $y=0 \to y=1$ $\quad y : 0 \to 1$

$$\therefore \int_{x=0}^{1}\int_{y=0}^{x} dy\, dx = \int_{y=0}^{y=1}\int_{x=y}^{x=1} dx\, dy$$

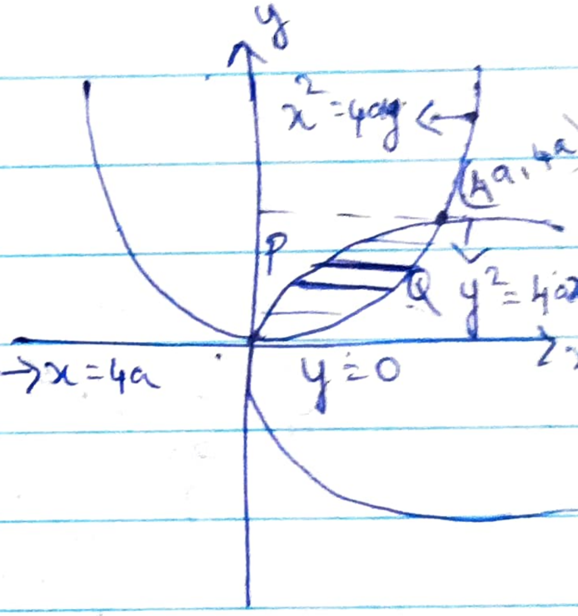
$$= \int_0^1 [x]_y^1\, dy = \int_0^1 (1-y)\, dy$$

$$= \left[y - \frac{y^2}{2}\right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

V.

2) Change the order of integration and hence evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\, dy\, dx$.

Soln:-



The region of integration is bounded by $y = \frac{x^2}{4a}$ & $y = 2\sqrt{ax}$ & $x = 0 \to x = 4a$

The region of integration is enclosed between two parabolas $x^2 = 4ay$ and $y^2 = 4ax$ as shown in fig.

After changing the region of integration the pt P lies on the parabola $y^2 = 4ax \therefore x = y^2/4a$ & pt Q lies on the parabola $x^2 = 4ay \therefore x = 2\sqrt{ay}$

$\therefore x :- y^2/4a \to 2\sqrt{ay}$

Then y varies from $y = 0 \to y = 4a$

$\therefore y :- 0 \to 4a$

Hence $$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\, dy\, dx = \int_{y=0}^{y=4a}\int_{x=\frac{y^2}{4a}}^{x=2\sqrt{ay}} xy\, dx\, dy$$

$$= \int_0^{4a} y \left[\frac{x^2}{2}\right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a} \left[y\,\frac{4ay}{2} - \frac{y^4\,y}{32a^2}\right] dy$$

$$= \int_0^{4a} \left(2ay^2 - \frac{y^5}{32a^2}\right) dy$$

$$= \left[2a\,\frac{y^3}{3} - \frac{1}{32a^2}\,\frac{y^6}{6}\right]_0^{4a}$$

$$= 2a\,\frac{(4a)^3}{3} - \frac{1}{32a^2}\,\frac{(4a)^6}{6}$$

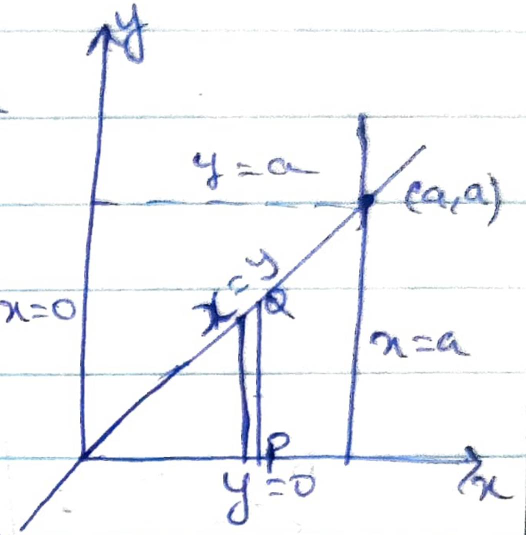
$$= \frac{2a}{3}\,64a^3 - \frac{1}{\cancel{32}\cancel{a^2}_{2}}\,\frac{\cancel{16} \times \cancel{256}^{\cancel{128}\,64}}{\cancel{6}_{3}}\,a^{\cancel{6}}$$

$$= \frac{128}{3}a^4 - \frac{64}{3}a^4$$

$$= \frac{64}{3}a^4$$

4) Change the order of integration in

$$\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$$ and hence evaluate it.



Soln:- The given limits show that the region of integration is bounded by the curves $x=y$; $x=a$; $y=0$; $y=a$

when we change the order of integration we should first integrate w.r.to y first then w.r.to x

The new limits are $x=0$ to $x=a$ $x:-0\to a$

& $y=0$ to $y=x$ $y:-0\to x$

$$\int_0^a\int_y^a \frac{x\,dx\,dy}{x^2+y^2} = \int_0^a\int_0^x \frac{x\,dy\,dx}{x^2+y^2} = \int_0^a x\left[\frac{1}{\cancel{x}}\tan^{-1}\frac{y}{x}\right]_0^x dx$$

$$\therefore \int \frac{dy}{x^2+y^2} = \frac{1}{x}\tan^{-1}\frac{y}{x}$$

$$= \int_0^a \pi/4\,dx = \pi/4\,[x]_0^a = \frac{\pi}{4}a$$

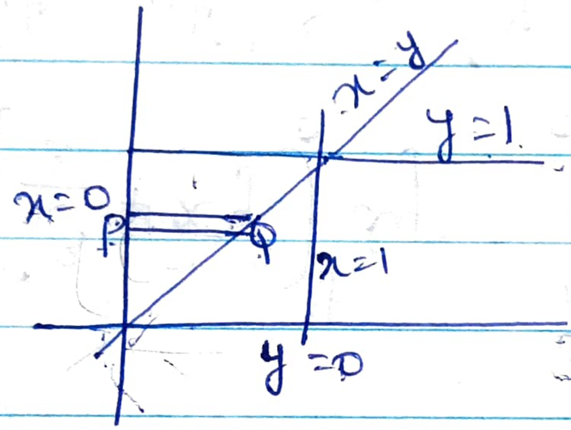
6. Change the order of integration in $\int_0^1 \int_0^y f(x,y)\,dx\,dy$

7. Solu:-

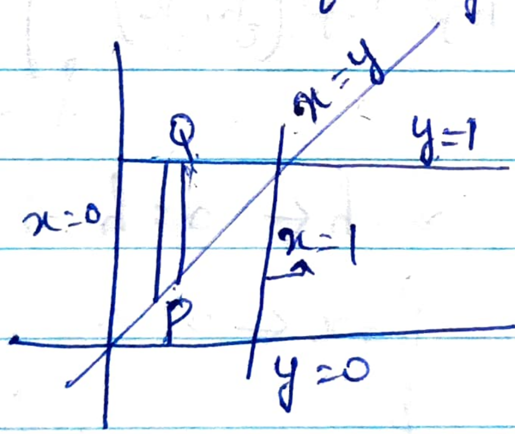
2 marks

Given $\int_0^1 \int_0^y f(x,y)\,dx\,dy$

$y=0 \quad y=1 \quad x=0 \quad x=y$



Change the order of integration $\quad \int\int dy\,dx$

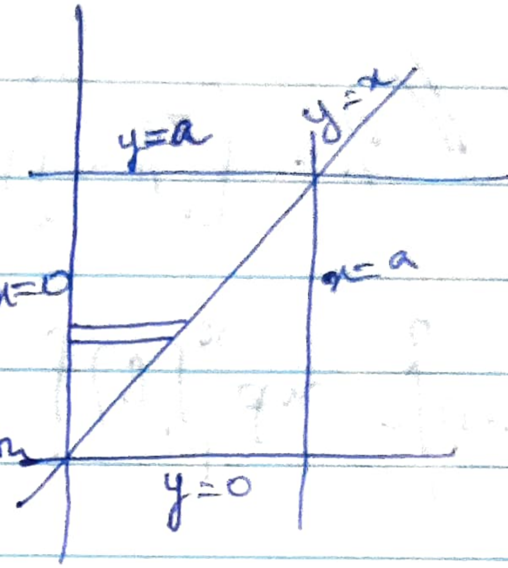


$$\int_{x=0}^{x=1}\int_{y=x}^{y=1} f(x,y)\,dy\,dx = \int_0^1 [y]_x^1\,dx = \int_0^1 (1-x)\,dx$$

$$= \left[x - \frac{x^2}{2}\right]_0^1 = \left[1 - \frac{1}{2}\right] = \frac{1}{2}$$

7) Change the order of integration and evaluate

$$\int_0^a \int_x^a (x^2+y^2)\,dy\,dx$$



(ie) $x=0 \quad x=a \quad y=x \quad y=a$

Changing the order of integration

x :- $x=0 \rightarrow x=y$

y :- $y=0 \rightarrow y=a$

$$= \int_{y=0}^{y=a} \int_{x=0}^{y} (x^2+y^2)\,dx\,dy$$

$$\int_0^a \left[\frac{x^3}{3} + xy^2\right]_0^y dy = \int_0^a \left[\frac{y^3}{3} + y^3\right] dy$$

$$= \left[\frac{y^4}{12} + \frac{y^4}{4}\right]_0^a = \left[\frac{a^4}{12} + \frac{a^4}{4}\right] = \frac{4a^4}{12} = \frac{a^4}{3}$$

(2) Change the order of integration

$$\int_0^1 \int_0^1 \frac{x}{x^2+y^2}\, dy\, dx$$

(3) Sketch roughly the region of integration for $\int_0^a \int_y^a f(x,y)\, dx\, dy$

(4) Change the order of integration in

$$\int_0^1 \int_x^1 \frac{x}{x^2+y^2}\, dy\, dx$$

(5) $\int_0^1 \int_0^x dy\, dx$

(6) $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\, dy\, dx$ → Problem worked in front

Volume as Triple Integral :-

1) Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by triple integrals.

Soln:- The ellipse in 3 D is an ellipsoid.

Volume = 8 x Volume of octant in the first octant

$\because$ First octant the lower limits $x=y=z=0$

$$V = 8\iiint dz\,dy\,dx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \left[z\right]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy\,dx$$

$$= 8c\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy\,dx$$

Put $\boxed{\frac{r^2}{b^2} = 1-\frac{x^2}{a^2}}$

$$= \frac{8c}{b}\int_0^a \int_0^r \sqrt{r^2-y^2}\, dy\,dx$$

$r^2 = \left(1-\frac{x^2}{a^2}\right)b^2$

$r = b\sqrt{1-\frac{x^2}{a^2}}$

$$= \frac{8c}{b} \int_0^a \int_0^r \sqrt{r^2 - y^2}\, dy\, dx$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{8c}{b} \int_0^a \left[\frac{y}{2}\sqrt{r^2 - y^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{y}{r}\right)\right]_0^r dx$$

$$= \frac{8c}{b} \int_0^a \frac{r^2}{2} \sin^{-1}(1)\, dx \qquad \sin^{-1}(1) = \frac{\pi}{2}$$

$$= \frac{8c}{b} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \int_0^a r^2\, dx$$

$$\frac{r^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$r^2 = b^2\left[1 - \frac{x^2}{a^2}\right]$$

$$= \frac{2c\pi}{b} \int_0^a b^2\left(1 - \frac{x^2}{a^2}\right) dx$$

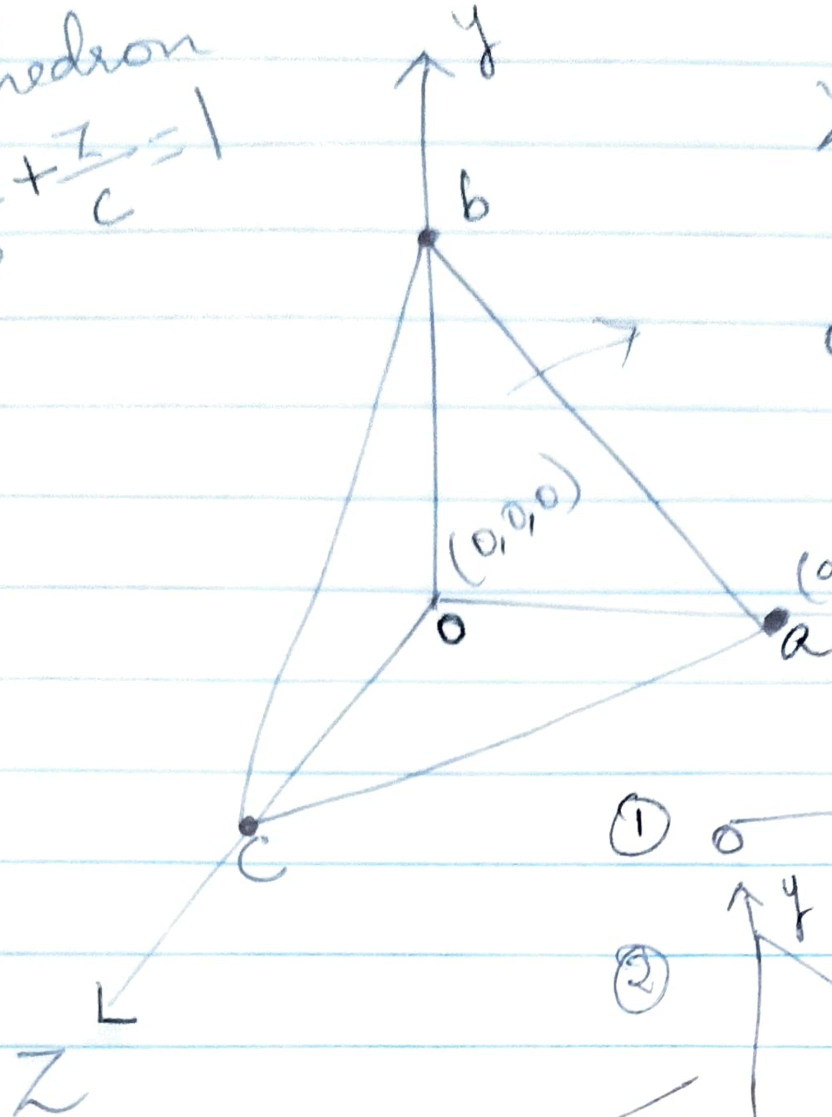
$$= \frac{2c\pi}{b} b^2 \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = 2c\pi b\left[x - \frac{1}{a^2}\frac{x^3}{3}\right]_0^a$$

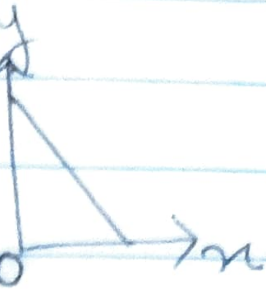
$$= 2cb\pi\left[a - \frac{a^3}{3a^2}\right] = 2cb\pi\left[\frac{3a^3 - a^3}{3a^2}\right]$$

$$2cb\pi\left[\frac{2a^3}{3a^2}\right] = \frac{4}{3}\pi abc$$

Tetrahedron

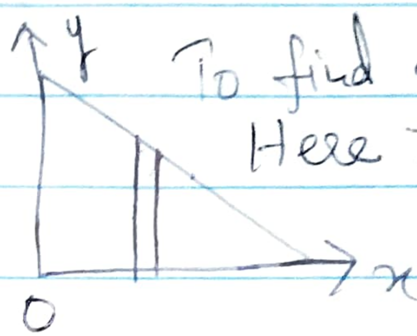
$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$





① 

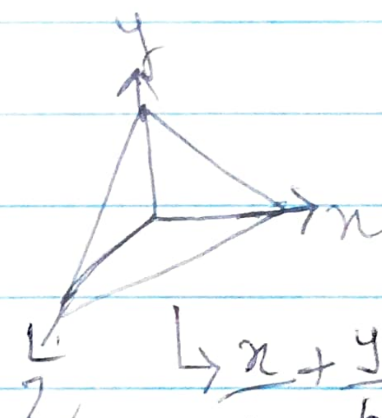
$x=0$ to $x=a$

② 

To find area
Here $z=0$

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\therefore y=b\left(1-x/a\right)$$

③ 

$$\hookrightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

$$z=c\left[1-\frac{x}{a}-\frac{y}{b}\right]$$

2) Find the volume of the tetrahedron bounded by the co-ordinate planes & $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Solu:- $V = \iiint dz\,dy\,dx$ $\quad$ ∵ co-ordinate planes lower limits = 0

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz\,dy\,dx$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} [z]_0^{c(1-\frac{x}{a}-y/b)}\,dy\,dx$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} \left(c\left(1-\frac{x}{a}-\frac{y}{b}\right)\right]dy\,dx$$

$$= c\int_0^a \int_0^{b(1-\frac{x}{a})} \left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right)y - \frac{1}{b}\frac{y^2}{2}\right]_0^{b(1-\frac{x}{a})} dx$$

$$= c\int_0^a \left[b\left(1-\frac{x}{a}\right)^2 - \frac{1}{2\cancel{b}}\, b^{\cancel{2}}\left(1-\frac{x}{a}\right)^2\right]dx$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right)^2\left(b-\frac{b}{2}\right)\right]dx$$

$$= \frac{cb}{2} \int_0^a \left(1 - \frac{2x}{a} + \frac{x^2}{a^2}\right) dx$$

$$= \frac{cb}{2} \left[x - \frac{\cancel{2}}{a}\frac{x^2}{\cancel{2}} + \frac{x^3}{3a^2} \right]_0^a$$

$$= \frac{cb}{2} \left[a - \frac{a^{\cancel{2}}}{\cancel{a}} + \frac{a^{\cancel{3}}}{3\cancel{a^2}} \right]$$

$$= \frac{cb}{2} \left[a - a + \frac{a}{3} \right] = \frac{abc}{6} \,//$$

3) Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ & $z=0$

Solu:- z varies from $z=0$ to $z=4-y$

x & y varies over all the points of the circle

$y^2 = 4-x^2 \Rightarrow$ ~~$y = \pm(2-x)$~~ $y = \pm\sqrt{4-x^2}$

& $x^2 = 4$ $\quad x = \pm 2$

$$I = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{4-y} dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[z\right]_{0}^{4-y} dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\, dy\, dx$$

$$= \int_{-2}^{2} \left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 4dy - \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\, dy \right] dx$$

$$= \int_{-2}^{2} \left[2 \int_{0}^{\sqrt{4-x^2}} 4\, dy - \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\, dy \right] dx \qquad \int_{-a}^{+a} dy = 2\int_{0}^{a}$$

$$= \int_{-2}^{2} \left[8 \left[y\right]_{0}^{\sqrt{4-x^2}} - 0 \right] dx \qquad \frac{y^2}{2}$$

$$= 8 \int_{-2}^{2} \sqrt{4-x^2}\, dx$$

$$= 8\times 2\int_{0}^{2} \sqrt{4-x^2}\,dx$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$= 16\int_{0}^{2} \sqrt{4-x^2}\,dx = 16\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= 16\left[0 + 2\sin^{-1}(1)\right]$$

$$= 16 \times 2\,\frac{\pi}{2} = 16\pi$$

4) Find the volume of the sphere $x^2+y^2+z^2=a^2$ without transformation.

Soln:- Volume $= 8 \times$ Volume in an octant

z varies from $0 \to \sqrt{a^2-x^2-y^2}$

y varies from $0 \to \sqrt{a^2-x^2}$

x varies from $0 \to a$

$$V = 8\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}}\int_{0}^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} \, dy \, dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \left(\sqrt{a^2-x^2-y^2}\right) dy \, dx$$

$$u^2 = a^2 - x^2$$

$$= 8\int_0^a \left[\frac{a^2-x^2}{2} \sin^{-1} \frac{y}{\sqrt{a^2-x^2}} + \frac{y}{2} \cancel{\sin^{-1}} \sqrt{a^2-x^2-y^2} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \frac{a^2-x^2}{2} \sin^{-1}(1) \, dx$$

$$= \frac{8}{2} \cdot \pi/2 \int_0^a \left(a^2-x^2\right) dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left[a^3 - \frac{a^3}{3} \right]$$

$$= \frac{4}{3} \pi a^3$$

Singularities

Singularities

Behavior of following functions f at 0:

- $f(z) = \dfrac{1}{z^9}$
- $f(z) = \dfrac{\sin z}{z}$
- $f(z) = \dfrac{e^z - 1}{z}$
- $f(z) = \dfrac{1}{\sin(\frac{1}{z})}$
- $f(z) = \operatorname{Log} z$
- $f(z) = e^{\frac{1}{z}}$

In the above we observe that all the functions are not analytic at 0, however in every neighborhood of 0 there is a point at which f is analytic.

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In the above we observe that all the functions are not analytic at 0, however in every neighborhood of 0 there is a point at which f is analytic.

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Singularities

Definition: The point z_0 is called a **singular point** or **singularity of f** if f is not analytic at z_0 but every neighborhood of z_0 contains at least one point at which f is analytic.

- Here $\dfrac{e^z - 1}{z}, \dfrac{1}{z^2}, \sin\frac{1}{z}, \text{Log } z$ etc. has singularity at $z = 0$.
- $\bar{z}, |z|^2, \text{Re } z, \text{Im } z, z\text{Re } z$ are nowhere analytic. That does not mean that every point of $\mathbb{C}$ is a singularity.
- A singularities are classified into TWO types:
  1. A singular point z_0 is said to be an **isolated singularity or isolated singular point** of f if f is analytic in $B(z_0, r) \setminus \{z_0\}$ for some $r > 0$.
  2. A singular point z_0 is said to be an **non-isolated singularity** if z_0 is not an isolated singular point.
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Singularities

If f has an isolated singularity at z_0, then f is analytic in $B(z_0, r) \setminus \{z_0\}$ for some $r > 0$. In this case f has the following Laurent series expansion:

$$f(z) = \cdots \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - a) + a_2(z - z_0)^2 + \cdots .$$

- If all $a_{-n} = 0$ for all $n \in \mathbb{N}$, then the point $z = z_0$ is a **removal singularity**.
- The point $z = z_0$ is called a **pole** if all but a finite number of a_{-n}'s are non-zero. If m is the highest integer such that $a_{-m} \neq 0$, then z_0 is a Pole of order m.
- If $a_{-n} \neq 0$ for infinitely many n's, then the point $z = z_0$ is a **essential singularity**.
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Removable singularities

- The following statements are equivalent:
  1. f has a removable singularity at z_0.
  2. If all $a_{-n} = 0$ for all $n \in \mathbb{N}$.
  3. $\lim_{z \to z_0} f(z)$ exists and finite.
  4. $\lim_{z \to z_0} (z - z_0) f(z) = 0$.
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Pole

The following statements are equivalent:

- f has a pole of order m at z_0.
- $f(z) = \dfrac{g(z)}{(z - z_0)^m}$, g is analytic at z_0 and $g(z_0) \neq 0$.
- $\dfrac{1}{f}$ has a zero of order m.
- $\lim\limits_{z \to z_0} |f(z)| = \infty$.
- $\lim\limits_{z \to z_0} (z - z_0)^{m+1} f(z) = 0$
- $\lim\limits_{z \to z_0} (z - z_0)^m f(z)$ has removal singularity at z_0.

Essential singularity

The following statements are equivalent:

- f has a essential singularity at z_0.
- The point z_0 is neither a pole nor removable singularity.
- $\lim_{z \to z_0} f(z)$ does not exists.
- Infinitely many terms in the principal part of Laurent series expansion around the point z_0.

Limit point of zeros is isolated essential singularity. For example:

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Singularities at ∞

Let f be a complex valued function. Define another function g by

$$g(z) = f\left(\frac{1}{z}\right).$$

Then the nature of singularity of f at $z = \infty$ is defined to be the the nature of singularity of g at $z = 0$.

- $f(z) = z^3$ has a pole of order 3 at ∞.
- e^z has an essential singularity at ∞.
- An entire function f has a removal singularity at ∞ if and only if f is constant.(Prove This!)
- An entire function f has a pole of order m at ∞ if and only if f is a polynomial of degree m.(Prove This!)

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