CHAPTER 1

MATHEMATICAL MODELS OF CONTROL SYSTEM

1.1 CONTROL SYSTEM

Control system theory evolved as an engineering discipline and due to universality of the principles involved, it is extended to various fields like economy, sociology, biology, medicine, etc. Control theory has played a vital role in the advance of engineering and science. The automatic control has become an integral part of modern manufacturing and industrial processes. For example, numerical control of machine tools in manufacturing industries, controlling pressure, temperature, humidity, viscosity and flow in process industry.

When a number of elements or components are connected in a sequence to perform a specific function, the group thus formed is called a ***system.*** In a system when the output quantity is controlled by varying the input quantity, the system is called ***control system.*** The output quantity is called controlled variable or response and input quantity is called command signal or excitation.

OPEN LOOP SYSTEM

Any physical system which does not automatically correct the variation in its output, is called an ***open loop system***, or control system in which the output quantity has no effect upon the input quantity are called open-loop control system. This means that the output is not fedback to the input for correction.

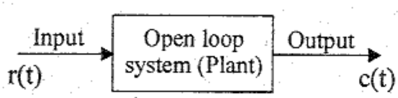


Fig 1.1 : *Open loop system.*

In open loop system the output can be varied by varying the input. But due to external disturbances the system output may change. When the output changes due to disturbances, it is not followed by changes in input to correct the output. In open loop systems the changes in output are corrected by changing the input manually.

CLOSED LOOP SYSTEM

Control systems in which the output has an effect upon the input quantity in order to maintain the desired output value are called ***closed loop systems.***

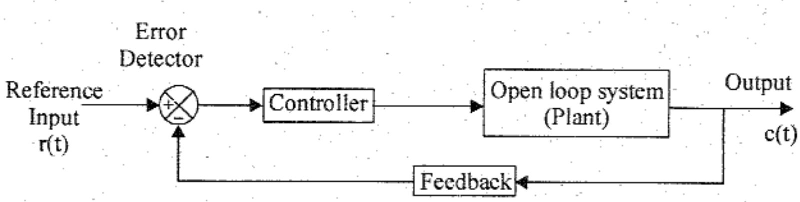


Fig 1.2 : *Closed loop system.*

The open loop system can be modified as closed loop system by providing a feedback. The provision of feedback automatically corrects the changes in output due to disturbances. Hence the closed loop system is also called ***automatic control system.*** The general block diagram of an automatic control system is shown in fig 1.2. It consists of an error detector, a controller, plant (open loop system) and feedback path elements.

The reference signal (or input signal) corresponds to desired output. The feedback path elements samples the output and converts it to a signal of same type as that of reference signal. The feedback signal is proportional to output signal and it is fed to the error detector. The error signal generated by the error detector is the difference between reference signal and feedback signal. The controller modifies and amplifies the error signal to produce better control action. The modified error signal is fed to the plant to correct its output.

Advantages of open loop systems

1. The open loop systems are simple and economical.
2. The open loop systems are easier to construct.
3. Generally the open loop systems are stable.

Disadvantages of open loop systems

1. The open loop systems are inaccurate and unreliable.
2. The changes in the output due to external disturbances are not corrected automatically.

Advantages of closed loop systems

1. The closed loop systems are accurate.
2. The closed loop systems are accurate even in the presence of non-linearities.
3. The sensitivity of the systems may be made small to make the system more stable.
4. The closed loop systems are less affected by noise.

Disadvantages of closed loop systems

1. The closed loop systems are complex and costly.
2. The feedback in closed loop system may lead to oscillatory response.
3. The feedback reduces the overall gain of the system.
4. Stability is a major problem in closed loop system and more care is needed to design a stable closed loop system.

1.2 EXAMPLES OF CONTROL SYSTEMS

EXAMPLE 1 : TEMPERATURE CONTROL SYSTEM

OPEN LOOP SYSTEM

The electric furnace shown in fig 1.3. is an open loop system. The output in the system is the desired temperature. The temperature of the system is raised by heat generated by the heating element. The output temperature depends on the time during which the supply to heater remains ON.

The ON and OFF of the supply is governed by the time setting of the relay. The temperature is measured by a sensor, which gives an analog voltage corresponding to the temperature of the furnace. The analog signal is converted to digital signal by an Analog - to - Digital converter (A/D converter).

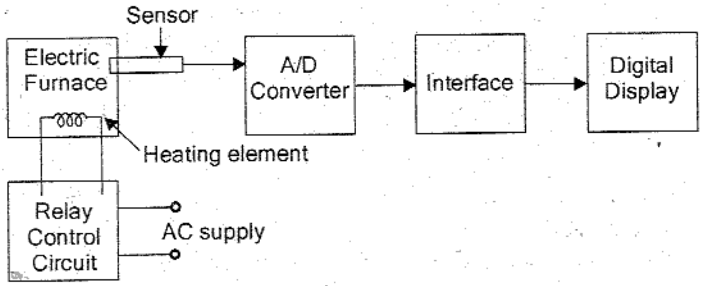


Fig 1.3 : Open loop temperature control system.

The digital signal is given to the digital display device to display the temperature. In this system if there is any change in output temperature then the time setting of the relay is not altered automatically.

CLOSED LOOP SYSTEM

The electric furnace shown in fig 1.4 is a closed loop system. The output of the system is the desired temperature and it depends on the time during which the supply to heater remains ON.

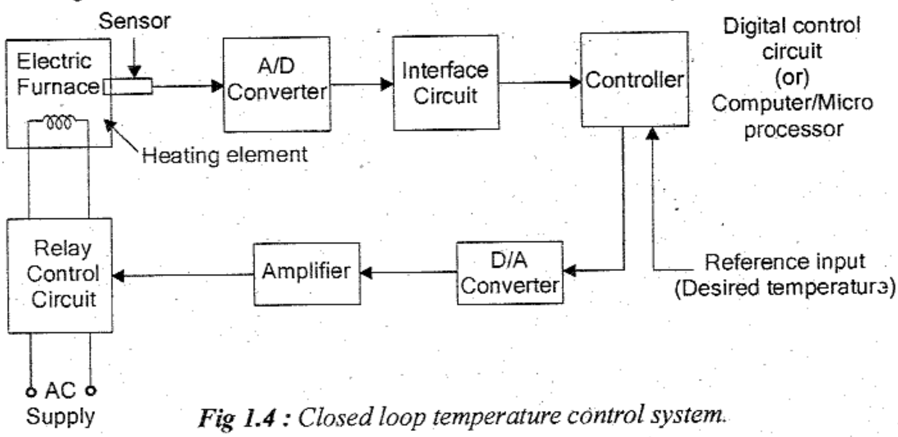


Fig 1.4 : Closed loop temperature control system.

The switching ON and OFF of the relay is controlled by a controller which is a digital system or computer. The desired temperature is input to the system through keyboard or as a signal corresponding to desired temperature via ports. The actual temperature is sensed by sensor and converted to digital signal by the A/D converter. The computer reads the actual temperature and compares with desired temperature. If it finds any difference then it sends signal to switch ON or OFF the relay through D/A converter and amplifier. Thus the system automatically corrects any changes in output. Hence it is a closed loop system.

EXAMPLE 2 : TRAFFIC CONTROL SYSTEM

OPEN LOOP SYSTEM

Traffic control by means of traffic signals operated on a time basis constitutes an open-loop control system. The sequence of control signals are based on a time slot given for each signal. The time slots are decided based on a traffic study. The system will not measure the density of the traffic before giving the signals. Since the time slot does not changes according to traffic density, the system is open loop system.

CLOSED LOOP SYSTEM

Traffic control system can be made as a closed loop system if the time slots of the signals are decided based on the density of traffic. In closed loop traffic control system, the density of the traffic is measured on all the sides and the information is fed to a computer. The timings of the control signals are decided by the computer based on the density of traffic. Since the closed loop system dynamically changes the timings, the flow of vehicles will be better than open loop system.

EXAMPLE 3 : NUMERICAL CONTROL SYSTEM

OPEN LOOP SYSTEM

Numerical control is a method of controlling the motion of machine components using numbers. Here, the position of work head tool is controlled by the binary information contained in a disk.

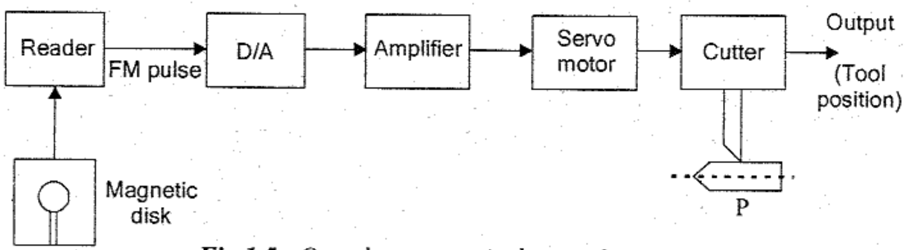


Fig 1.5 : Open loop numerical control system.

A magnetic disk is prepared in binary form representing the desired part P (P is the metal part to be machined). The tool will operate on the desired part P. To start the system, the disk is fed through the reader to the D/A converter. The D/A converter converts the FM(frequency modulated) output of the reader to a analog signal. It is amplified and fed to servometer which positions the cutter on the desired part P. The position of the cutter head is controlled by the angular motion of the servometer. This is an open loop system since no feedback path exists between the output and input. The system positions the tool for a given input command. Any deviation in the desired position is not checked and corrected automatically.

CLOSED LOOP SYSTEM

A magnetic disk is prepared in binary form representing the desired part P (P is the metal part to be machined). To start the system, the disk is loaded in the reader. The controller compares the frequency modulated input pulse signal with the feedback pulse signal. The controller is a computer or microprocessor system. The controller carries out mathematical operations on the difference in the pulse signals and generates an error signal. The D/A converter converts the controller output pulse (error signal) into an analog signal . The amplified analog signal rotates the servomotor to position the tool on the job. The position of the cutterhead is controlled according to the input of the servomotor.

The transducer attached to the cutterhead converts the motion into an electrical signal. The analog electrical signal is converted to the digital pulse signal by the A/D converter. Then this signal is compared with the input pulse signal. If there is any difference between these two, the controller sends a signal to the servomotor to reduce it. Thus the system automatically corrects any deviation in the desired output tool position. An advantage of numerical control is that complex parts can be produced with uniform tolerances at the maximum milling speed.

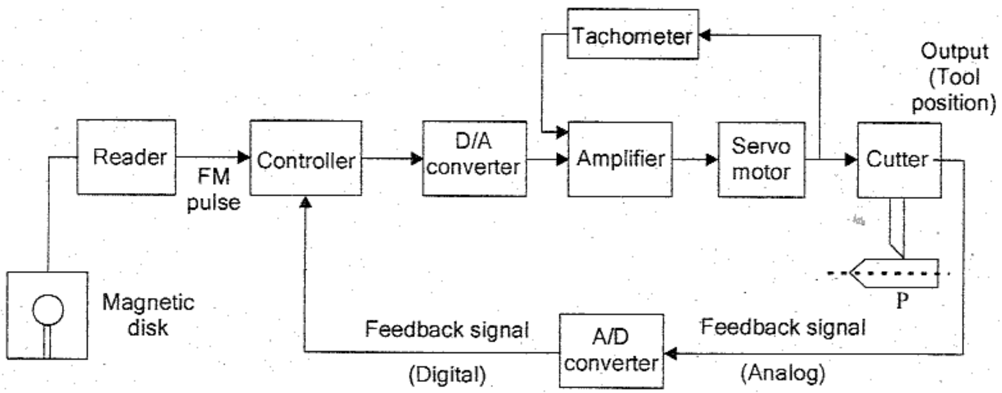


Fig 1.6 : Closed loop numerical control system.

EXAMPLE 4 : POSITION CONTROL SYSTEM USING SERVOMOTOR

The position control system shown in fig 1.7 is a closed loop system. The system consists of a servomotor powered by a generator. The load whose position has to be controlled is connected to motor shaft through gear wheels. Potentiometers are used to convert the mechanical motion to electrical signals. The desired load position (θ_R) is set on the input potentiometer and the actual load position (θ_c) is fed to feedback potentiometer. The difference between the two angular positions generates an error signal, which is amplified and fed to generator field circuit. The induced emf of the generator drives the motor. The rotation of the motor stops when the error signal is zero, i.e. when the desired load position is reached.

This type of control systems are called servomechanisms . The *servo* or *servomechanisms* are feedback control systems in which the output is mechanical position (or time derivatives of position e.g. velocity and acceleration).

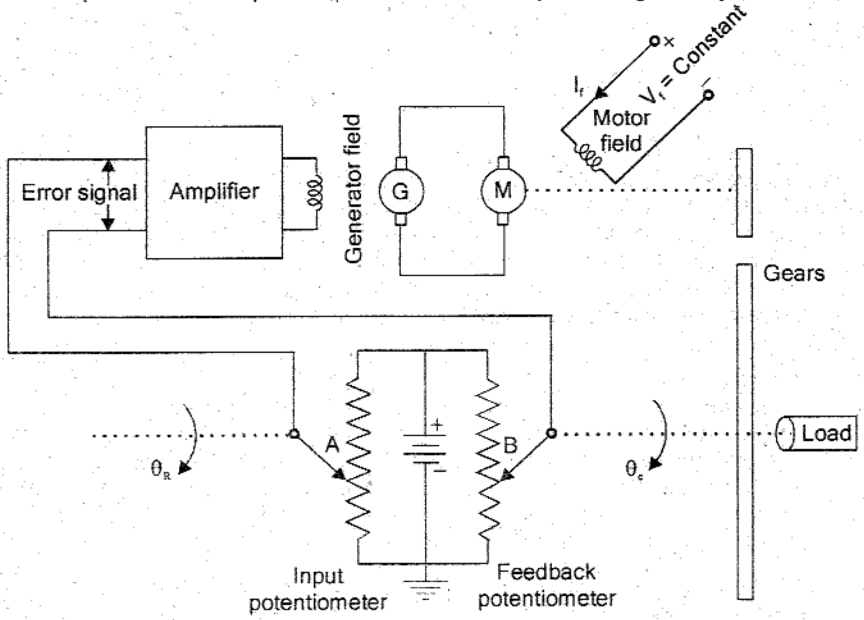


Fig 1.7 : *A position control system (servomechanism).*

1.3 MATHEMATICAL MODELS OF CONTROL SYSTEMS

A ***control system*** is a collection of physical objects (components) connected together to serve an objective. The input output relations of various physical components of a system are governed by ***differential equations.*** The mathematical model of a control system constitutes a set of differential equations. The response or output of the system can be studied by solving the differential equations for various input conditions.

The mathematical model of a system is linear if it obeys the principle of superposition and homogenity. This principle implies that if a system model has responses $y_1(t)$ and $y_2(t)$ to any inputs $x_1(t)$ and $x_2(t)$ respectively, then the system response to the linear combination of these inputs $a_1x_1(t) + a_2x_2(t)$ is given by linear combination of the individual outputs $a_1y_1(t)+a_2y_2(t)$, where a_1 and a_2 are constants.

The principle of superposition can be explained diagrammatically as shown in fig. 1.8.

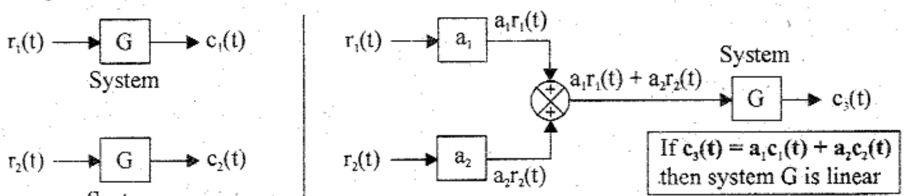


Fig 1.8 : *Principle of linearity and superposition.*

A mathematical model will be linear if the differential equations describing the system has constant coefficients (or the coefficients may be functions of independent variables). If the coefficients of the differential equation describing the system are constants then the model is ***linear time invariant.*** If the coefficients of differential equations governing the system are functions of time then the model is ***linear time varying.***

The differential equations of a linear time invariant system can be reshaped into different form for the convenience of analysis. One such model for single input and single output system analysis is transfer function of the system. The ***transfer function*** of a system is defined as the ratio of Laplace transform of output to the Laplace transform of input with zero initial conditions.

$$\boxed{\text{Transfer function} = \left.\frac{\text{Laplace Transform of output}}{\text{Laplace Transform of input}}\right|_{\text{with zero initial conditions}}} \qquad .....(1.1)$$

The transfer function can be obtained by taking Laplace transform of the differential equations governing the system with zero initial conditions and rearranging the resulting algebraic equations to get the ratio of output to input.

1.4 MECHANICAL TRANSLATIONAL SYSTEMS

The model of mechanical translational systems can be obtained by using three basic elements ***mass, spring and dash-pot.*** These three elements represents three essential phenomena which occur in various ways in mechanical systems.

The weight of the mechanical system is represented by the element ***mass*** and it is assumed to be concentrated at the center of the body. The elastic deformation of the body can be represented by a ***spring.*** The friction existing in rotating mechanical system can be represented by the ***dash-pot.*** The dash-pot is a piston moving inside a cylinder filled with viscous fluid.

When a force is applied to a translational mechanical system, it is opposed by opposing forces due to mass, friction and elasticity of the system. The force acting on a mechanical body are governed by ***Newton's second law of motion.*** For translational systems it states that the sum of forces acting on a body is zero. (or Newton's second law states that the sum of applied forces is equal to the sum of opposing forces on a body).

LIST OF SYMBOLS USED IN MECHANICAL TRANSLATIONAL SYSTEM

x = Displacement, m

$v = \frac{dx}{dt}$ = Velocity, m/sec

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ = Acceleration, m/sec^2

f = Applied force, N (Newtons)

f_m = Opposing force offered by mass of the body, N

f_k = Opposing force offered by the elasticity of the body (spring), N

f_b = Opposing force offered by the friction of the body (dash - pot), N

M = Mass, kg

K = Stiffness of spring, N/m

B = Viscous friction co-efficient, N-sec/m

Note : Lower case letters are functions of time

FORCE BALANCE EQUATIONS OF IDEALIZED ELEMENTS

Consider an ideal mass element shown in fig 1.9 which has negligible friction and elasticity. Let a force be applied on it. The mass will offer an opposing force which is proportional to acceleration of the body.

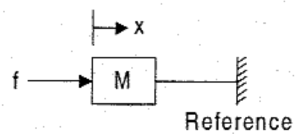


Fig 1.9 : Ideal mass element.

Let, f = Applied force

f_m = Opposing force due to mass

Here, $f_m \propto \dfrac{d^2x}{dt^2}$ or $f_m = M\dfrac{d^2x}{dt^2}$

By Newton's second law, $\boxed{f = f_m = M\dfrac{d^2x}{dt^2}}$ .....(1.2)

Consider an ideal frictional element dashpot shown in fig 1.10 which has negligible mass and elasticity. Let a force be applied on it. The dash-pot will offer an opposing force which is proportional to velocity of the body.

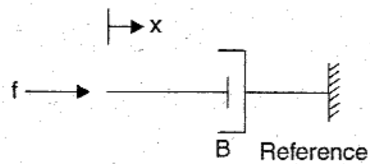


Fig 1.10 : Ideal dashpot with one end fixed to reference.

Let, f = Applied force

f_b = Opposing force due to friction

Here, $f_b \propto \dfrac{dx}{dt}$ or $f_b = B\dfrac{dx}{dt}$

By Newton's second law, $\boxed{f = f_b = B\dfrac{dx}{dt}}$ .....(1.3)

When the dashpot has displacement at both ends as shown in fig 1.11, the opposing force is proportional to differential velocity.

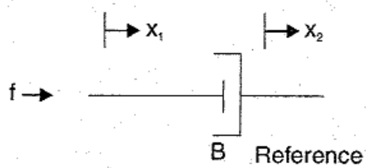


Fig 1.11 : Ideal dashpot with displacement at both ends.

$f_b \propto \dfrac{d}{dt}(x_1 - x_2)$ or $f_b = B\dfrac{d}{dt}(x_1 - x_2)$

$\therefore \boxed{f = f_b = B\dfrac{d}{dt}(x_1 - x_2)}$ .....(1.4)

Consider an ideal elastic element spring shown in fig 1.12, which has negligible mass and friction. Let a force be applied on it. The spring will offer an opposing force which is proportional to displacement of the body.

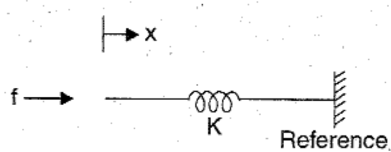


Fig 1.12 : Ideal spring with one end fixed to reference.

Let, f = Applied force

f_k = Opposing force due to elasticity

Here $f_k \propto x$ or $f_k = K\,x$

By Newton's second law, $\boxed{f = f_k = Kx}$ .....(1.5)

When the spring has displacement at both ends as shown in g 1.13 the opposing force is proportional to differential displacement.

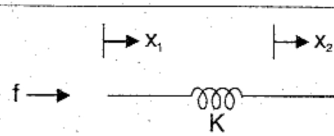


Fig 1.13 : Ideal spring with displacement at both ends.

$f_k \propto (x_1 - x_2)$ or $f_k = K(x_1 - x_2)$

$\therefore \boxed{f = f_k = K(x_1 - x_2)}$ .....(1.6)

Guidelines to determine the Transfer Function of Mechanical Translational System

1. In mechanical translational system, the differential equations governing the system are obtained by writing force balance equations at nodes in the system. The nodes are meeting point of elements. Generally the nodes are mass elements in the system. In some cases the nodes may be without mass element.

2. The linear displacement of the masses (nodes) are assumed as x_1, x_2, x_3, etc., and assign a displacement to each mass(node) . The first derivative of the displacement is velocity and the second derivative of the displacement is acceleration.

3. Draw the free body diagrams of the system . The free body diagram is obtained by drawing each mass separately and then marking all the forces acting on that mass (node). Always the opposing force acts in a direction opposite to applied force. The mass has to move in the direction of the applied force. Hence the displacement, velocity and acceleration of the mass will be in the direction of the applied force. If there is no applied force then the displacement, velocity and acceleration of the mass will be in a direction opposite to that of opposing force.

4. For each free body diagram, write one differential equation by equating the sum of applied forces to the sum of opposing forces.

5. Take Laplace transform of differential equations to convert them to algebraic equations. Then rearrange the s-domain equations to eliminate the unwanted variables and obtain the ratio between output variable and input variable. This ratio is the transfer function of the system.

> ***Note :*** *Laplace transform of* $x(t) = \mathcal{L}\{x(t)\} = X(s)$
>
> *Laplace transform of* $\dfrac{dx(t)}{dt} = \mathcal{L}\left\{\dfrac{d}{dt}x(t)\right\} = s\,X(s)$ *(with zero initial conditions)*
>
> *Laplace transform of* $\dfrac{d^2x(t)}{dt^2} = \mathcal{L}\left\{\dfrac{d^2}{dt^2}x(t)\right\} = s^2\,X(s)$ *(with zero initial conditions)*

EXAMPLE 1.1

Write the differential equations governing the mechanical system shown in fig 1. and determine the transfer function.

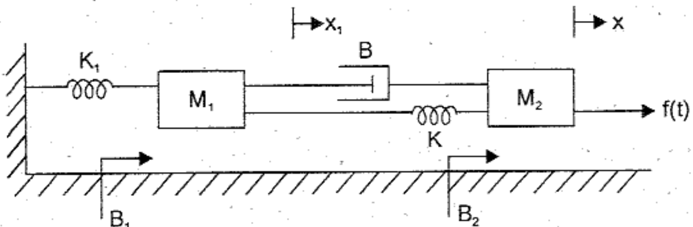


Fig 1.

SOLUTION

In the given system, applied force 'f(t)' is the input and displacement 'x' is the output.

Let, Laplace transform of f(t) $= \mathcal{L}\{f(t)\} = F(s)$

Laplace transform of $x = \mathcal{L}\{x\} = X(s)$

Laplace transform of $x_1 = \mathcal{L}\{x_1\} = X_1(s)$

Hence the required transfer function is $\frac{X(s)}{F(s)}$

The system has two nodes and they are mass M_1 and M_2. The differential equations governing the system are given by force balance equations at these nodes.

Let the displacement of mass M_1 be x_1. The free body diagram of mass M_1 is shown in fig 2. The opposing forces acting on mass M_1 are marked as f_{m1}, f_{b1}, f_b, f_{k1} and f_k.

$$f_{m1} = M_1 \frac{d^2x_1}{dt^2}\ ; \qquad f_{b1} = B_1 \frac{dx_1}{dt}\ ; \qquad f_{k1} = K_1x_1;$$

$$f_b = B\frac{d}{dt}(x_1 - x)\ ; \quad f_k = K(x_1 - x)$$

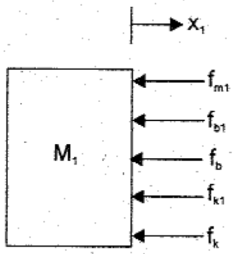


Fig 2 : Free body diagram of mass M_1 (node 1).

By Newton's second law,

$$f_{m1} + f_{b1} + f_b + f_{k1} + f_k = 0$$

$$\therefore M_1\frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + B\frac{d}{dt}(x_1 - x) + K_1x_1 + K(x_1 - x) = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_1s^2X_1(s) + B_1sX_1(s) + Bs\ [X_1(s) - X(s)] + K_1X_1(s) + K\ [X_1(s) - X(s)] = 0$$

$$X_1(s)\ [M_1s^2 + (B_1 + B)s + (K_1 + K)] - X(s)\ [Bs + K] = 0$$

$$X_1(s)\ [M_1s^2 + (B_1 + B)s + (K_1 + K)] = X(s)\ [Bs + K]$$

$$\therefore X_1(s) = X(s)\ \frac{Bs + K}{M_1s^2 + (B_1 + B)\ s + (K_1 + K)} \qquad \text{......(1)}$$

The free body diagram of mass M_2 is shown in fig 3. The opposing forces acting on M_2 are marked as f_{m2}, f_{b2}, f_b and f_k.

$$f_{m2} = M_2\ \frac{d^2x}{dt^2} \quad ; \qquad f_{b2} = B_2\ \frac{dx}{dt}$$

$$f_b = B\frac{d}{dt}\ (x - x_1) \quad ; \qquad f_k = K(x - x_1)$$

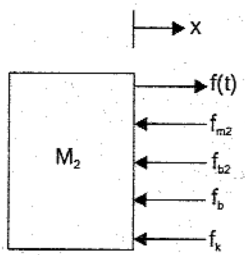


Fig 3 : Free body diagram of mass M_2 (node 2).

By Newton's second law,

$$f_{m2} + f_{b2} + f_b + f_k = f(t)$$

$$M_2\ \frac{d^2x}{dt^2} + B_2\ \frac{dx}{dt} + B\frac{d}{dt}\ (x - x_1) + K(x - x_1) = f(t)$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_2s^2X(s) + B_2sX(s) + Bs[X(s) - X_1(s)] + K[X(s) - X_1(s)] = F(s)$$

$$X(s)\ [M_2s^2 + (B_2 + B)s + K] - X_1(s)[Bs + K] = F(s) \qquad \text{.....(2)}$$

Substituting for $X_1(s)$ from equation (1) in equation (2) we get,

$$X(s)\ [M_2s^2 + (B_2 + B)s + K] - X(s)\ \frac{(Bs + K)^2}{M_1s^2 + (B_1 + B)s + (K_1 + K)} = F(s)$$

$$X(s)\left[\frac{[M_2s^2+(B_2+B)s+K]\,[M_1s^2+(B_1+B)s+(K_1+K)]-(Bs+K)^2}{M_1s^2+(B_1+B)\,s+(K_1+K)}\right]=F(s)$$

$$\therefore \frac{X(s)}{F(s)}=\frac{M_1s^2+(B_1+B)s+(K_1+K)}{[M_1s^2+(B_1+B)s+(K_1+K)]\,[M_2s^2+(B_2+B)s+K]-(Bs+K)^2}$$

RESULT

The differential equations governing the system are,

1. $M_1\frac{d^2x_1}{dt^2}+B_1\frac{dx_1}{dt}+B\frac{d}{dt}(x_1-x)+K_1x_1+K(x_1-x)=0$

2. $M_2\frac{d^2x}{dt^2}+B_2\frac{dx}{dt}+B\frac{d}{dt}(x-x_1)+K(x-x_1)=f(t)$

The transfer function of the system is,

$$\frac{X(s)}{F(s)}=\frac{M_1s^2+(B_1+B)\,s+(K_1+K)}{\left[M_1s^2+(B_1+B)\,s+(K_1+K)\right]\left[M_2s^2+(B_2+B)\,s+K\right]-(Bs+K)^2}$$

EXAMPLE 1.2

Determine the transfer function $\frac{Y_2(s)}{F(s)}$ of the system shown in fig 1.

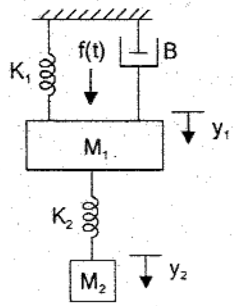


Fig 1.

SOLUTION

Let, Laplace transform of $f(t)=\mathcal{L}\{f(t)\}=F(s)$

Laplace transform of $y_1=\mathcal{L}\{y_1\}=Y_1(s)$

Laplace transform of $y_2=\mathcal{L}\{y_2\}=Y_2(s)$

The system has two nodes and they are mass M_1 and M_2. The differential equations governing the system are the force balance equations at these nodes.

The free body diagram of mass M_1 is shown in fig 2.

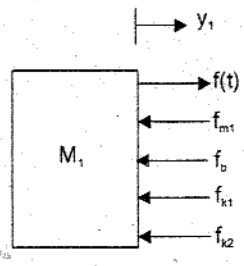


Fig 2.

The opposing forces are marked as f_{m1}, f_b, f_{k1} and f_{k2}

$$f_{m1}=M_1\frac{d^2y_1}{dt^2}\;;\quad f_b=B\frac{dy_1}{dt}\;;\quad f_{k1}=K_1y_1\;;\quad f_{k2}=K_2(y_1-y_2)$$

By Newton's second law, $f_{m1}+f_b+f_{k1}+f_{k2}=f(t)$

$$\therefore M_1\frac{d^2y_1}{dt^2}+B\frac{dy_1}{dt}+K_1y_1+K_2(y_1-y_2)=f(t) \quad .....(1)$$

On taking Laplace transform of equation (1) with zero initial condition we get,

$$M_1s^2Y_1(s)+BsY_1(s)+K_1Y_1(s)+K_2[Y_1(s)-Y_2(s)]=F(s)$$
$$Y_1(s)[M_1s^2+Bs+(K_1+K_2)]-Y_2(s)K_2=F(s) \quad .....(2)$$

The free body diagram of mass M_2 is shown in fig 3. The opposing forces acting on M_2 are f_{m2} and f_{k2}.

$$f_{m2}=M_2\frac{d^2y_2}{dt^2}\;;\qquad f_{k2}=K_2(y_2-y_1)$$

By Newton's second law, $f_{m2}+f_{k2}=0$

$$\therefore M_2 \frac{d^2y_2}{dt^2} + K_2(y_2 - y_1) = 0$$

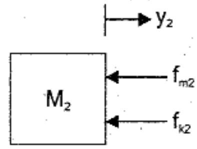


Fig 3.

On taking Laplace transform of above equation we get,

$$M_2 s^2 Y_2(s) + K_2[Y_2(s) - Y_1(s)] = 0$$

$$Y_2(s)\ [M_2 s^2 + K_2] - Y_1(s)\ K_2 = 0$$

$$\therefore Y_1(s) = Y_2(s)\ \frac{M_2 s^2 + K_2}{K_2} \qquad .....(3)$$

Substituting for $Y_1(s)$ from equation (3) in equation (2) we get,

$$Y_2(s)\left[\frac{M_2 s^2 + K_2}{K_2}\right]\left[M_1 s^2 + Bs + (K_1 + K_2)\right] - Y_2(s)\ K_2 = F(s)$$

$$Y_2(s)\left[\frac{(M_2 s^2 + K_2)\ [M_1 s^2 + Bs + (K_1 + K_2)] - K_2^2}{K_2}\right] = F(s)$$

$$\therefore \frac{Y_2(s)}{F(s)} = \frac{K_2}{\left[M_1 s^2 + Bs + (K_1 + K_2)\right]\left[M_2 s^2 + K_2\right] - K_2^2}$$

RESULT

The differential equations governing the system are,

1. $M_1 \frac{d^2y_1}{dt^2} + B\frac{dy_1}{dt} + K_1 y_1 + K_2(y_1 - y_2) = f(t)$

2. $M_2 \frac{d^2y_2}{dt^2} + K_2(y_2 - y_1) = 0$

The transfer function of the system is,

$$\frac{Y_2(s)}{F(s)} = \frac{K_2}{\left[M_1 s^2 + Bs + (K_1 + K_2)\right]\left[M_2 s^2 + K_2\right] - K_2^2}$$

EXAMPLE 1.3

Determine the transfer function, $\frac{X_1(s)}{F(s)}$ and $\frac{X_2(s)}{F(s)}$ for the system shown in fig 1.

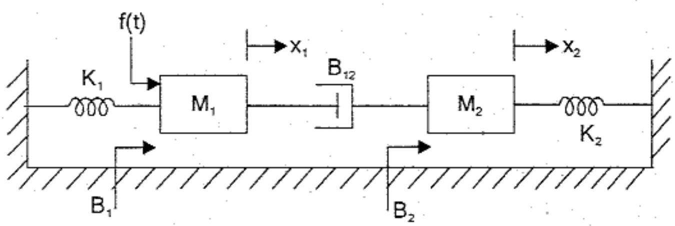


Fig 1.

SOLUTION

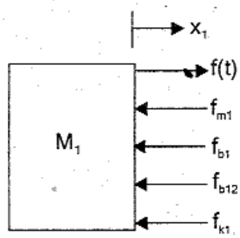


Fig 2.

Let, Laplace transform of f(t) = $\mathcal{L}\{f(t)\}$ = F(s)

Laplace transform of x_1 = $\mathcal{L}\{x_1\}$ = $X_1(s)$

Laplace transform of x_2 = $\mathcal{L}\{x_2\}$ = $X_2(s)$

The system has two nodes and they are mass M_1 and M_2. The differential equations governing the system are the force balance equations at these nodes. The free body diagram of mass M_1 is shown in fig 2. The opposing forces are marked as f_{m1}, f_{b1}, f_{b12} and f_{k1}.

$$f_{m1} = M_1\frac{d^2x_1}{dt^2} \;;\; f_{b1} = B_1\frac{dx_1}{dt} \;;\; f_{b12} = B_{12}\frac{d}{dt}(x_1 - x_2) \;;\; f_{k1} = K_1x_1$$

By Newton's second law, $f_{m1} + f_{b1} + f_{b12} + f_{k1} = f(t)$

$$M_1\frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + B_{12}\frac{d(x_1 - x_2)}{dt} + K_1x_1 = f(t)$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_1s^2X_1(s) + B_1s\,X_1(s) + B_{12}s\,[X_1(s) - X_2(s)] + K_1X_1(s) = F(s)$$

$$X_1(s)\,[M_1s^2 + (B_1 + B_{12})\,s + K_1] - B_{12}sX_2(s) = F(s) \quad .....(1)$$

The free body diagram of mass M_2 is shown in fig 3. The opposing forces are marked as f_{m2}, f_{b2}, f_{b12} and f_{k2}.

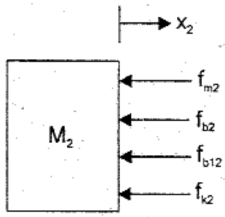


Fig 3.

$$f_{m2} = M_2\frac{d^2x_2}{dt^2} \quad ; \quad f_{b2} = B_2\frac{dx_2}{dt}$$

$$f_{b12} = B_{12}\frac{d}{dt}(x_2 - x_1)\;; \quad f_{k2} = K_2x_2$$

By Newton's second law, $f_{m2} + f_{b2} + f_{b12} + f_{k2} = 0$

$$M_2\frac{d^2x_2}{dt^2} + B_2\frac{dx_2}{dt} + B_{12}\frac{d(x_2 - x_1)}{dt} + K_2x_2 = 0 \quad .....(2)$$

On taking Laplace transform of equation (2) with zero initial conditions we get,

$$M_2s^2X_2(s) + B_2sX_2(s) + B_{12}s\,[X_2(s) - X_1(s)] + K_2X_2(s) = 0$$

$$X_2(s)\,[M_2s^2 + (B_2 + B_{12})\,s + K_2] - B_{12}\,s\,X_1(s) = 0$$

$$X_2(s)\,[M_2s^2 + (B_2 + B_{12})\,s + K_2] = B_{12}\,s\,X_1(s)$$

$$X_2(s) = \frac{B_{12}s\,X_1(s)}{[M_2s^2 + (B_2 + B_{12})\,s + K_2]} \quad .....(3)$$

Substituting for $X_2(s)$ from equation (3) in equation (1) we get,

$$X_1(s)\,[M_1s^2 + (B_1 + B_{12})\,s + K_1] - \frac{(B_{12}s)^2\,X_1(s)}{M_2s^2 + (B_2 + B_{12})\,s + K_2} = F(s)$$

$$\frac{X_1(s)\left[[M_1s^2 + (B_1 + B_{12})\,s + K_1]\,[M_2s^2 + (B_2 + B_{12})\,s + K_2] - (B_{12}s)^2\right]}{M_2s^2 + (B_2 + B_{12})\,s + K_2} = F(s)$$

$$\therefore \frac{X_1(s)}{F(s)} = \frac{M_2s^2 + (B_2 + B_{12})\,s + K_2}{[M_1s^2 + (B_1 + B_{12})\,s + K_1]\,[M_2s^2 + (B_2 + B_{12})\,s + K_2] - (B_{12}s)^2}$$

From equation (3) we get,

$$X_1(s) = \frac{[M_2s^2 + (B_2 + B_{12})\,s + K_2]\,X_2(s)}{B_{12}s} \quad .....(4)$$

Substituting for $X_1(s)$ from equation (4) in equation (1) we get,

$$\frac{X_2(s)\,[M_2s^2 + (B_2 + B_{12})\,s + K_2]}{B_{12}s}\,[M_1s^2 + (B_1 + B_{12})\,s + K_1] - B_{12}s\,X_2(s) = F(s)$$

$$X_2(s)\left[\frac{[M_2s^2+(B_2+B_{12})\,s+K_2]\,[M_1s^2+(B_1+B_{12})\,s+K_1]-(B_{12}s)^2}{B_{12}s}\right]=F(s)$$

$$\therefore \frac{X_2(s)}{F(s)}=\frac{B_{12}s}{[M_2s^2+(B_2+B_{12})\,s+K_2]\,[M_1s^2+(B_1+B_{12})\,s+K_1]-(B_{12}s)^2}$$

RESULT

The differential equations governing the system are,

1. $M_1\dfrac{d^2x_1}{dt^2}+B_1\dfrac{dx_1}{dt}+B_{12}\dfrac{d(x_1-x_2)}{dt}+K_1x_1=f(t)$
2. $M_2\dfrac{d^2x_2}{dt^2}+B_2\dfrac{dx_2}{dt}+B_{12}\dfrac{d(x_2-x_1)}{dt}+K_2x_2=0$

The transfer functions of the system are,

1. $\dfrac{X_1(s)}{F(s)}=\dfrac{M_2s^2+(B_2+B_{12})\,s+K_2}{[M_1s^2+(B_1+B_{12})\,s+K_1]\,[M_2s^2+(B_2+B_{12})\,s+K_2]-(B_{12}s)^2}$
2. $\dfrac{X_2(s)}{F(s)}=\dfrac{B_{12}s}{[M_2s^2+(B_2+B_{12})\,s+K_2]\,[M_1s^2+(B_1+B_{12})\,s+K_1]-(B_{12}s)^2}$

EXAMPLE 1.4

Write the equations of motion in s-domain for the system shown in fig 1. Determine the transfer function of the system.

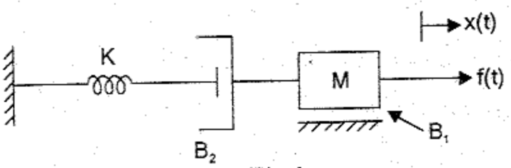


Fig 1.

SOLUTION

Let, Laplace transform of $x(t)=\mathcal{L}\{x(t)\}=X(s)$

Laplace transform of $f(t)=\mathcal{L}\{f(t)\}=F(s)$

Let x_1 be the displacement at the meeting point of spring and dashpot. Laplace transform of x_1 is $X_1(s)$.

The system has two nodes and they are mass M and the meeting point of spring and dashpot. The differential equations governing the system are the force balance equations at these nodes. The equations of motion in the s-domain are obtained by taking Laplace transform of the differential equations.

The free body diagram of mass M is shown in fig 2. The opposing forces are marked as f_m, f_{b1} and f_{b2}.

$$f_m=M\frac{d^2x}{dt^2}\;;\;f_{b1}=B_1\frac{dx}{dt}\;;\;f_{b2}=B_2\frac{d}{dt}(x-x_1)$$

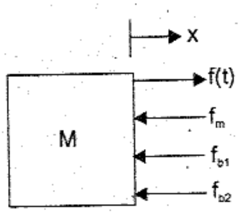


Fig 2.

By Newton's second law the force balance equation is,

$$f_m+f_{b1}+f_{b2}=f(t)$$

$$\therefore M\frac{d^2x}{dt^2}+B_1\frac{dx}{dt}+B_2\frac{d}{dt}(x-x_1)=f(t)$$

On taking Laplace transform of the above equation we get,

$$Ms^2\,X(s)+B_1\,s\,X(s)+B_2\,s\,[X(s)-X_1(s)]=F(s)$$

$$\left[Ms^2+(B_1+B_2)\,s\right]X(s)-B_2\,s\,X_1(s)=F(s) \qquad .....(1)$$

The free body diagram at the meeting point of spring and dashpot is shown in fig 3. The opposing forces are marked as f_k and f_{b2}.

$$f_{b2} = B_2 \frac{d}{dt}(x_1 - x) \; ; \quad f_k = K\, x_1$$

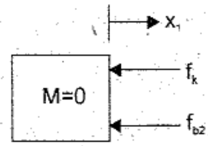


Fig 3.

By Newton's second law, $f_{b2} + f_k = 0$

$$\therefore B_2 \frac{d}{dt}(x_1 - x) + K\, x_1 = 0$$

On taking Laplace transform of the above equation we get,

$$B_2\, s\, [X_1(s) - X(s)] + K\, X_1(s) = 0$$

$$(B_2\, s + K)\, X_1(s) - B_2\, s\, X(s) = 0$$

$$\therefore X_1(s) = \frac{B_2\, s}{B_2\, s + K}\, X(s) \qquad .....(2)$$

Substituting for $X_1(s)$ from equation (2) in equation (1) we get,

$$\left[M\, s^2 + (B_1 + B_2)\, s\right] X(s) - B_2\, s \left[\frac{B_2\, s}{B_2\, s + K}\right] X(s) = F(s)$$

$$X(s)\, \frac{\left[[M\, s^2 + (B_1 + B_2)\, s]\,(B_2\, s + K) - (B_2\, s)^2\right]}{B_2\, s + K} = F(s)$$

$$\therefore \frac{X(s)}{F(s)} = \frac{B_2\, s + K}{[M\, s^2 + (B_1 + B_2)\, s]\,(B_2\, s + K) - (B_2\, s)^2}$$

RESULT

The differential equations governing the system are,

1. $M \dfrac{d^2x}{dt^2} + B_1 \dfrac{dx}{dt} + B_2 \dfrac{d}{dt}(x - x_1) = f(t)$

2. $B_2 \dfrac{d}{dt}(x_1 - x) + K\, x_1 = 0$

The equations of motion in s-domain are,

1. $[M\, s^2 + (B_1 + B_2)\, s]\, X(s) - B_2\, s\, X_1(s) = F(s)$
2. $(B_2\, s + K)\, X_1(s) - B_2\, s\, X(s) = 0$

The transfer function of the system is,

$$\frac{X(s)}{F(s)} = \frac{B_2\, s + K}{[M\, s^2 + (B_1 + B_2)\, s]\,(B_2\, s + K) - (B_2\, s)^2}$$

1.5 MECHANICAL ROTATIONAL SYSTEMS

The model of rotational mechanical systems can be obtained by using three elements, ***moment of inertia*** [J] of mass, ***dash-pot*** with rotational frictional coefficient [B] and ***torsional spring*** with stiffness [K].

The weight of the rotational mechanical system is represented by the moment of inertia of the mass. The moment of inertia of the system or body is considered to be concentrated at the centre of gravity of the body. The elastic deformation of the body can be represented by a spring (torsional spring). The friction existing in rotational mechanical system can be represented by the dash-pot. The dash-pot is a piston rotating inside a cylinder filled with viscous fluid.

When a torque is applied to a rotational mechanical system, it is opposed by opposing torques due to moment of inertia, friction and elasticity of the system. The torques acting on a rotational mechanical body are governed by ***Newton's second law of motion*** for rotational systems. It states that the sum of torques acting on a body is zero (or Newton's law states that the sum of applied torques is equal to the sum of opposing torques on a body).

LIST OF SYMBOLS USED IN MECHANICAL ROTATIONAL SYSTEM

θ = Angular displacement, rad

$\frac{d\theta}{dt}$ = Angular velocity, rad/sec

$\frac{d^2\theta}{dt^2}$ = Angular acceleration, rad/sec^2

T = Applied torque, N-m

J = Moment of inertia, Kg-m^2/rad

B = Rotational frictional coefficient, N-m/(rad/sec)

K = Stiffness of the spring, N-m/rad

TORQUE BALANCE EQUATIONS OF IDEALISED ELEMENTS

Consider an ideal mass element shown in fig 1.14 which has negligible friction and elasticity. The opposing torque due to moment of inertia is proportional to the angular acceleration.

Let, T = Applied torque.

T_j = Opposing torque due to moment of inertia of the body.

Here $T_j \propto \frac{d^2\theta}{dt^2}$ or $T_j = J\frac{d^2\theta}{dt^2}$

By Newton's second law,

$$\boxed{T = T_j = J\frac{d^2\theta}{dt^2}} \quad .....(1.7)$$

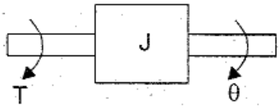


Fig 1.14 : Ideal rotational mass element.

Consider an ideal frictional element dash pot shown in fig 1.15 which has negligible moment of inertia and elasticity. Let a torque be applied on it. The dash pot will offer an opposing torque which is proportional to the angular velocity of the body.

Let, T = Applied torque.

T_b = Opposing torque due to friction.

$T_b \propto \frac{d\theta}{dt}$ or $T_b = B\frac{d\theta}{dt}$

By Newton's second law, $\boxed{T = T_b = B\frac{d\theta}{dt}}$ .....(1.8)

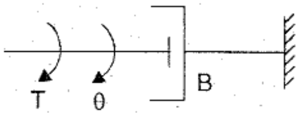


Fig 1.15 : Ideal rotational dash-pot with one end fixed to reference.

When the dash pot has angular displacement at both ends as shown in fig 1.16, the opposing torque is proportional to the differential angular velocity.

$T_b \propto \frac{d}{dt}(\theta_1 - \theta_2)$ or $T_b = B\frac{d}{dt}(\theta_1 - \theta_2)$

$$\boxed{\therefore T = T_b = B\frac{d}{dt}(\theta_1 - \theta_2)} \quad .....(1.9)$$

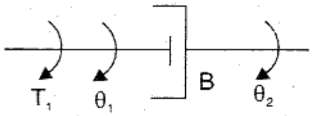


Fig 1.16 : Ideal dash-pot with angular displacement at both ends.

Consider an ideal elastic element, torsional spring as shown in fig 1.17, which has negligible moment of inertia and friction. Let a torque be applied on it. The torsional spring will offer an opposing torque which is proportional to angular displacement of the body.

Let, T = Applied torque.

T_k = Opposing torque due to elasticity.

$T_k \propto \theta$ or $T_k = K\theta$

By Newton's second law, $\boxed{T = T_k = K\theta}$ .....(1.10)

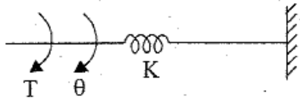


Fig 1.17 : Ideal spring with one end fixed to reference.

When the spring has angular displacement at both ends as shown in fig 1.18 the opposing torque is proportional to differential angular displacement.

$T_k \propto (\theta_1 - \theta_2)$ or $T_k = K(\theta_1 - \theta_2)$

$\therefore \boxed{T = T_k = K(\theta_1 - \theta_2)}$ .....(1.11)

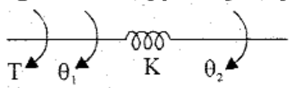


Fig 1.18 : Ideal spring with angular displacement at both ends.

Guidelines to determine the Transfer Function of Mechanical Rotational System

1. In mechanical rotational system, the differential equations governing the system are obtained by writing torque balance equations at nodes in the system. The nodes are meeting point of elements. Generally the nodes are mass elements with moment of inertia in the system. In some cases the nodes may be without mass element.
2. The angular displacement of the moment of inertia of the masses (nodes) are assumed as θ_1, θ_2, θ_3, etc., and assign a displacement to each mass (node). The first derivative of angular displacement is angular velocity and the second derivative of the angular displacement is angular acceleration.
3. Draw the free body diagrams of the system. The free body diagram is obtained by drawing each moment of inertia of mass separately and then marking all the torques acting on that body. Always the opposing torques acts in a direction opposite to applied torque.
4. The mass has to rotate in the direction of the applied torque. Hence the angular displacement, velocity and acceleration of the mass will be in the direction of the applied torque. If there is no applied torque then the angular displacement, velocity and acceleration of the mass is in a direction opposite to that of opposing torque.
5. For each free body diagram write one differential equation by equating the sum of applied torques to the sum of opposing torques.
6. Take Laplace transform of differential equation to convert them to algebraic equations. Then rearrange the s-domain equations to eliminate the unwanted variables and obtain the relation between output variable and input variable. This ratio is the transfer function of the system.

Note :

Laplace transform of $\theta = \mathcal{L}\{\theta\} = \theta(s)$

Laplace transform of $\dfrac{d\theta}{dt} = \mathcal{L}\left\{\dfrac{d\theta}{dt}\right\} = s\theta(s)$ *(with zero initial conditions)*

Laplace transform of $\dfrac{d^2\theta}{dt^2} = \mathcal{L}\left\{\dfrac{d^2\theta}{dt^2}\right\} = s^2\,\theta(s)$ *(with zero initial conditions)*

EXAMPLE 1.5

Write the differential equations governing the mechanical rotational system shown in fig 1. Obtain the transfer function of the system.

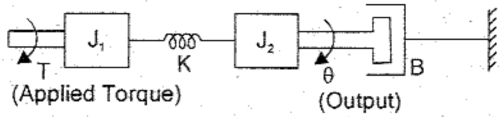


Fig 1.

SOLUTION

In the given system, applied torque T is the input and angular displacement θ is the output.

Let, Laplace transform of T $= \mathcal{L}\{T\} = T(s)$

Laplace transform of θ $= \mathcal{L}\{\theta\} = \theta(s)$

Laplace transform of $\theta_1 = \mathcal{L}\{\theta_1\} = \theta_1(s)$

Hence the required transfer function is $\frac{\theta(s)}{T(s)}$

The system has two nodes and they are masses with moment of inertia J_1 and J_2. The differential equations governing the system are given by torque balance equations at these nodes.

Let the angular displacement of mass with moment of inertia J_1 be θ_1. The free body diagram of J_1 is shown in fig 2. The opposing torques acting on J_1 are marked as T_{j1} and T_k.

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \quad ; \quad T_k = K(\theta_1 - \theta)$$

By Newton's second law, $T_{j1} + T_k = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T \qquad .....(1)$$

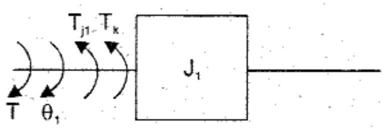


Fig 2 : Free body diagram of mass with moment of inertia J_1.

On taking Laplace transform of equation (1) with zero initial conditions we get,

$$J_1 s^2 \theta_1(s) + K\theta_1(s) - K\theta(s) = T(s)$$

$$(J_1 s^2 + K)\, \theta_1(s) - K\,\theta(s) = T(s) \qquad .....(2)$$

The free body diagram of mass with moment of inertia J_2 is shown in fig 3. The opposing torques acting on J_2 are marked as T_{j2}, T_b and T_k.

$$T_{j2} = J_2 \frac{d^2\theta}{dt^2} \quad ; \quad T_b = B\frac{d\theta}{dt} \quad ; \quad T_k = K(\theta - \theta_1)$$

By Newton's second law, $T_{j2} + T_b + T_k = 0$

$$\therefore J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

$$J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$$

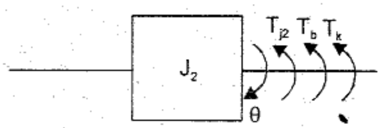


Fig 3 : Free body diagram of mass with moment of inertia J_2.

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_2 s^2\theta(s) + B\,s\,\theta(s) + K\theta(s) - K\theta_1(s) = 0$$

$$(J_2 s^2 + Bs + K)\,\theta(s) - K\theta_1(s) = 0$$

$$\theta_1(s) = \frac{(J_2 s^2 + Bs + K)}{K}\,\theta(s) \qquad .....(3)$$

Substituting for $\theta_1(s)$ from equation (3) in equation (2) we get,

$$(J_1 s^2 + K)\,\frac{(J_2 s^2 + Bs + K)}{K}\,\theta(s) - K\theta(s) = T(s)$$

$$\left[\frac{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}{K}\right]\theta(s) = T(s)$$

$$\therefore \frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}$$

RESULT

The differential equations governing the system are,

1. $J_1 \frac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T$

2. $J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$

The transfer function of the system is,

$$\frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}$$

EXAMPLE 1.6

Write the differential equations governing the mechanical rotational system shown in fig 1. and determine the transfer function $\theta(s)/T(s)$.

SOLUTION

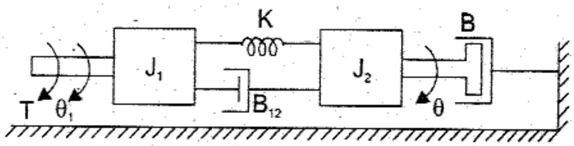


Fig 1.

In the given system, the torque T is the input and the angular displacement θ is the output.

Let, Laplace transform of $T = \mathcal{L}\{T\} = T(s)$

Laplace transform of $\theta = \mathcal{L}\{\theta\} = \theta(s)$

Laplace transform of $\theta_1 = \mathcal{L}\{\theta_1\} = \theta_1(s)$

Hence the required transfer function is $\frac{\theta(s)}{T(s)}$

The system has two nodes and they are masses with moment of inertia J_1 and J_2. The differential equations governing the system are given by torque balance equations at these nodes.

Let the angular displacement of mass with moment of inertia J_1 be θ_1. The free body diagram of J_1 is shown in fig 2. The opposing torques acting on J_1 are marked as T_{j1}, T_{b12} and T_k.

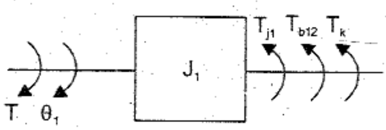


Fig 2 : Free body diagram of mass with moment of inertia J_1.

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \quad ; \quad T_{b12} = B_{12}\frac{d}{dt}(\theta_1 - \theta) \quad ; \quad T_k = K(\theta_1 - \theta)$$

By Newton's second law, $T_{j1} + T_{b12} + T_k = T$

$$J_1\frac{d^2\theta_1}{dt^2}+B_{12}\frac{d}{dt}(\theta_1-\theta)+K(\theta_1-\theta)=T$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_1s^2\theta_1(s)+s\,B_{12}\,[\theta_1(s)-\theta(s)]+K\theta_1(s)-K\theta(s)=T(s)$$

$$\theta_1(s)\,[J_1s^2+sB_{12}+K]-\theta(s)\,[sB_{12}+K]=T(s) \qquad .....(1)$$

The free body diagram of mass with moment of inertia J_2 is shown in fig 3. The opposing torques are marked as T_{j2}, T_{b12}, T_b and T_k.

$$T_{j2}=J_2\frac{d^2\theta}{dt^2} \quad ; \quad T_{b12}=B_{12}\frac{d}{dt}(\theta-\theta_1)$$

$$T_b=B\frac{d\theta}{dt} \quad ; \quad T_k=K(\theta-\theta_1)$$

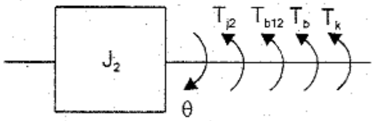


Fig 3 : Free body diagram of mass with moment of inertia J_2.

By Newton's second law, $T_{j2}+T_{b12}+T_b+T_k=0$

$$J_2\frac{d^2\theta}{dt^2}+B_{12}\frac{d}{dt}(\theta-\theta_1)+B\frac{d\theta}{dt}+K(\theta-\theta_1)=0$$

$$J_2\frac{d^2\theta}{dt^2}-B_{12}\frac{d\theta_1}{dt}+\frac{d\theta}{dt}(B_{12}+B)+K\theta-K\theta_1=0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_2s^2\theta(s)-B_{12}s\theta_1(s)+s\theta(s)\,[B_{12}+B]+K\theta(s)-K\theta_1(s)=0$$

$$\theta(s)\,[s^2J_2+s(B_{12}+B)+K]-\theta_1(s)\,[sB_{12}+K]=0$$

$$\theta_1(s)=\frac{[s^2J_2+s(B_{12}+B)+K]}{[sB_{12}+K]}\,\theta(s) \qquad .....(2)$$

Substituting for $\theta_1(s)$ from equation (2) in equation (1) we get,

$$[J_1s^2+sB_{12}+K]\,\frac{[J_2s^2+s(B_{12}+B)+K]\,\theta(s)}{(sB_{12}+K)}-(sB_{12}+K)\,\theta(s)=T(s)$$

$$\left[\frac{(J_1s^2+sB_{12}+K)\,[J_2s^2+s(B_{12}+B)+K]-(sB_{12}+K)^2}{(sB_{12}+K)}\right]\theta(s)=T(s)$$

$$\therefore \frac{\theta(s)}{T(s)}=\frac{(sB_{12}+K)}{(J_1s^2+sB_{12}+K)\,[J_2s^2+s(B_{12}+B)+K]-(sB_{12}+K)^2}$$

RESULT

The differential equations governing the system are,

1. $J_1\frac{d^2\theta_1}{dt^2}+B_{12}\frac{d}{dt}(\theta_1-\theta)+K(\theta_1-\theta)=T$

2. $J_2\frac{d^2\theta}{dt^2}-B_{12}\frac{d\theta_1}{dt}+\frac{d\theta}{dt}(B_{12}+B)+K(\theta-\theta_1)=0$

The transfer function of the system is,

$$\frac{\theta(s)}{T(s)}=\frac{(sB_{12}+K)}{(J_1s^2+sB_{12}+K)\,[J_2s^2+s(B_{12}+B)+K]-(sB_{12}+K)^2}$$

1.6 ELECTRICAL SYSTEMS

The models of electrical systems can be obtained by using resistor, capacitor and inductor. The current-voltage relation of resistor, inductor and capacitor are given in table-1. For modelling electrical systems, the electrical network or equivalent circuit is formed by using R, L and C and voltage or current source.

The differential equations governing the electrical systems can be formed by writing Kirchoff 's current law equations by choosing various nodes in the network or Kirchoff 's voltage law equations by choosing various closed paths in the network. The transfer function can be obtained by taking Laplace transform of the differential equations and rearranging them as a ratio of output to input.

TABLE-1.1 : Current-Voltage Relation of R, L and C

Element	Voltage across the element	Current through the element
i(t) R, + v(t) −	$v(t) = Ri(t)$	$i(t) = \dfrac{v(t)}{R}$
i(t) L, + v(t) −	$v(t) = L\dfrac{d}{dt}i(t)$	$i(t) = \dfrac{1}{L}\int v(t)\,dt$
i(t) C, + v(t) −	$v(t) = \dfrac{1}{C}\int i(t)\,dt$	$i(t) = C\dfrac{dv(t)}{dt}$

EXAMPLE 1.7

Obtain the transfer function of the electrical network shown in fig 1.

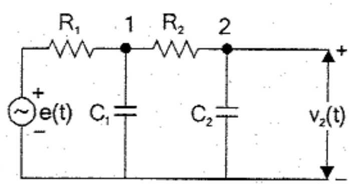


Fig 1.

SOLUTION

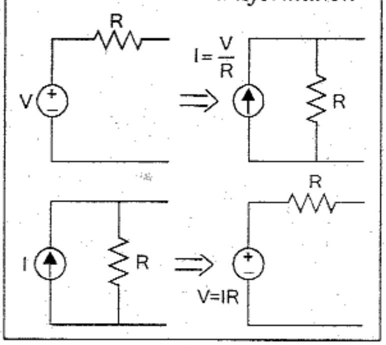
In the given network, input is e(t) and output is $v_2(t)$.

Let, Laplace transform of $e(t) = \mathcal{L}\{e(t)\} = E(s)$

Laplace transform of $v_2(t) = \mathcal{L}\{v_2(t)\} = V_2(s)$

The transfer function of the network is $\dfrac{V_2(s)}{E(s)}$

Note : Source transformation



Transform the voltage source in series with resistance R_1 into equivalent current source as shown in figure 2. The network has two nodes. Let the node voltages be v_1 and v_2. The Laplace transform of node voltages v_1 and v_2 are $V_1(s)$ and $V_2(s)$ respectively. The differential equations governing the network are given by the Kirchoff 's current law equations at these nodes.

At node-1, by Kirchoff's current law (refer fig 3)

$$\frac{v_1}{R_1} + C_1\frac{dv_1}{dt} + \frac{v_1 - v_2}{R_2} = \frac{e}{R_1}$$

On taking Laplace transform of above equation with zero initial conditions we get,

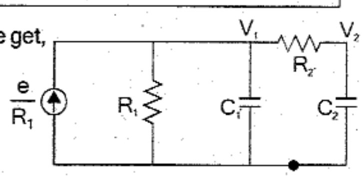


Fig 2.

$$\frac{V_1(s)}{R_1} + C_1 sV_1(s) + \frac{V_1(s)}{R_2} - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

$$V_1(s)\left[\frac{1}{R_1} + sC_1 + \frac{1}{R_2}\right] - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1} \qquad .....(1)$$

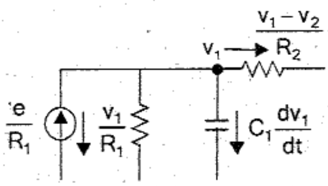


Fig 3.

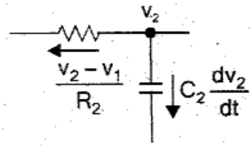


Fig 4.

At node-2, by Kirchoff's current law (refer fig 4)

$$\frac{v_2 - v_1}{R_2} + C_2\frac{dv_2}{dt} = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$\frac{V_2(s)}{R_2} - \frac{V_1(s)}{R_2} + C_2 s\, V_2(s) = 0$$

$$\frac{V_1(s)}{R_2} = \frac{V_2(s)}{R_2} + C_2 s\, V_2(s) = \left[\frac{1}{R_2} + sC_2\right] V_2(s)$$

$$\therefore V_1(s) = [1 + sC_2R_2]\, V_2(s) \quad \text{.....(2}$$

Substituting for $V_1(s)$ from equation (2) in equation (1) we get,

$$(1 + sR_2C_2)\, V_2(s)\left[\frac{1}{R_1} + sC_1 + \frac{1}{R_2}\right] - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

$$\left[\frac{(1 + sR_2C_2)\,(R_2 + R_1 + sC_1R_1R_2) - R_1}{R_1R_2}\right] V_2(s) = \frac{E(s)}{R_1}$$

$$\therefore \frac{V_2(s)}{E(s)} = \frac{R_2}{[(1 + sR_2C_2)\,(R_1 + R_2 + sC_1R_1R_2) - R_1]}$$

RESULT

The (node basis) differential equations governing the electrical network are,

1. $\frac{v_1}{R_1} + C_1\frac{dv_1}{dt} + \frac{v_1 - v_2}{R_2} = \frac{e}{R_1}$

2. $\frac{v_2 - v_1}{R_2} + C_2\frac{dv_2}{dt} = 0$

The transfer function of the electrical network is,

$$\frac{V_2(s)}{E(s)} = \frac{R_2}{[(1 + sR_2C_2)\,(R_1 + R_2 + sC_1R_1R_2) - R_1]}$$

1.7 TRANSFER FUNCTION OF ARMATURE CONTROLLED DC MOTOR

The speed of DC motor is directly proportional to armature voltage and inversely proportional flux in field winding. In armature controlled DC motor the desired speed is obtained by varying t armature voltage. This speed control system is an electro-mechanical control system. The electric system consists of the armature and the field circuit but for analysis purpose, only the armature circuit considered because the field is excited by a constant voltage. The mechanical system consists of t rotating part of the motor and load connected to the shaft of the motor. The armature controlled I motor speed control system is shown in fig 1.19.

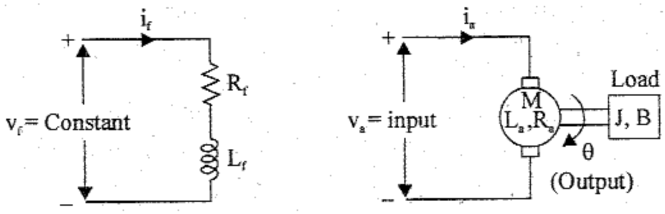


Fig 1.19 : Armature controlled DC motor.

Let, R_a = Armature resistance, Ω

L_a = Armature inductance, H

i_a = Armature current, A

v_a = Armature voltage, V

e_b = Back emf, V

K_t = Torque constant, N-m/A

T = Torque developed by motor, N-m

θ = Angular displacement of shaft, rad

J = Moment of inertia of motor and load, Kg-m²/rad

B = Frictional coefficient of motor and load, N-m/(rad/sec)

K_b = Back emf constant, V/(rad/sec)

The equivalent circuit of armature is shown in fig 1.20.

By Kirchoff's voltage law, we can write,

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = v_a \qquad .....(1.12)$$

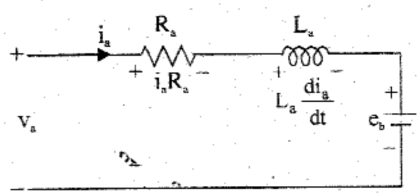


Fig 1.20 : Equivalent circuit of armature.

Torque of DC motor is proportional to the product of flux and current. Since flux is constant in this system, the torque is proportional to i_a alone.

$$T \propto i_a$$

$$\therefore \text{ Torque, } T = K_t\, i_a \qquad .....(1.13)$$

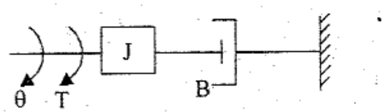


Fig 1.21.

The mechanical system of the motor is shown in fig 1.21.

The differential equation governing the mechanical system of motor is given by,

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \qquad .....(1.14)$$

The back emf of DC machine is proportional to speed (angular velocity) of shaft.

$$\therefore e_b \propto \frac{d\theta}{dt} \quad \text{or} \quad \text{Back emf, } e_b = K_b \frac{d\theta}{dt} \qquad .....(1.15)$$

The Laplace transform of various time domain signals involved in this system are shown below.

$$\mathcal{L}\{v_a\} = V_a(s); \quad \mathcal{L}\{e_b\} = E_b(s); \quad \mathcal{L}\{T\} = T(s); \quad \mathcal{L}\{i_a\} = I_a(s); \quad \mathcal{L}\{\theta\} = \theta(s)$$

The differential equations governing the armature controlled DC motor speed control system are,

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = v_a \quad ; \quad T = K_t i_a \quad ; \quad J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \quad ; \quad e_b = K_b \frac{d\theta}{dt}$$

Taking Laplace transform of the above equations with zero initial conditions we get,

$$I_a(s)\,R_a + L_a s I_a(s) + E_b(s) = V_a(s) \quad .....(1.16)$$

$$T(s) = K_t I_a(s) \quad .....(1.17)$$

$$Js^2\theta(s) + B\,s\,\theta(s) = T(s) \quad .....(1.18)$$

$$E_b(s) = K_b s\,\theta(s) \quad .....(1.19)$$

On equating equations (1.17) and (1.18) we get,

$$K_t I_a(s) = (Js^2 + Bs)\,\theta(s)$$

$$I_a(s) = \frac{(Js^2 + Bs)}{K_t}\,\theta(s) \quad .....(1.20)$$

Equation (1.16) can be written as,

$$(R_a + sL_a)\,I_a(s) + E_b(s) = V_a(s) \quad .....(1.21)$$

Substituting for $E_b(s)$ and $I_a(s)$ from equation (1.19) and (1.20) respectively in equation (1.21),

$$(R_a + sL_a)\,\frac{(Js^2 + Bs)}{K_t}\theta(s) + K_b s\,\theta(s) = V_a(s)$$

$$\left[\frac{(R_a + sL_a)\,(Js^2 + Bs) + K_b K_t s}{K_t}\right]\theta(s) = V_a(s)$$

The required transfer function is $\dfrac{\theta(s)}{V_a(s)}$

$$\therefore \frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a + sL_a)\,(Js^2 + Bs) + K_b K_t s} \quad .....(1.22)$$

$$= \frac{K_t}{R_a Js^2 + R_a Bs + L_a Js^3 + L_a Bs^2 + K_b K_t s}$$

$$= \frac{K_t}{s\left[JL_a s^2 + (JR_a + BL_a)\,s + (BR_a + K_b K_t)\right]}$$

$$= \frac{K_t / JL_a}{s\left[s^2 + \left(\dfrac{JR_a + BL_a}{JL_a}\right)s + \left(\dfrac{BR_a + K_b K_t}{JL_a}\right)\right]} \quad .....(1.23)$$

The transfer function of armature controlled dc motor can be expressed in another standard form as shown below. From equation (1.22) we get,

$$\frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a + sL_a)\,(Js^2 + Bs) + K_b K_t s} = \frac{K_t}{R_a\left(\dfrac{sL_a}{R_a} + 1\right)Bs\left(1 + \dfrac{Js^2}{Bs}\right) + K_b K_t\,s}$$

$$= \frac{K_t / R_a B}{s\left[(1 + sT_a)\,(1 + sT_m) + \dfrac{K_b K_t}{R_a B}\right]} \quad .....(1.24)$$

where, $\frac{L_a}{R_a} = T_a$ = Electrical time constant

$\frac{J}{B} = T_m$ = Mechanical time constant

1.8 TRANSFER FUNCTION OF FIELD CONTROLLED DC MOTOR

The speed of a DC motor is directly proportional to armature voltage and inversely proportional to flux. In field controlled DC motor the armature voltage is kept constant and the speed is varied by varying the flux of the machine. Since flux is directly proportional to field current, the flux is varied by varying field current. The speed control system is an electromechanical control system. The electrical system consists of armature and field circuit but for analysis purpose, only field circuit is considered because the armature is excited by a constant voltage. The mechanical system consists of the rotating part of the motor and the load connected to the shaft of the motor. The field controlled DC motor speed control system is shown in fig 1.22.

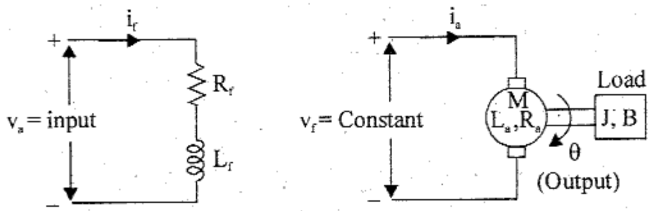


Fig 1.22 : Field controlled DC motor.

Let, R_f = Field resistance, Ω

L_f = Field inductance, H

i_f = Field current, A

v_f = Field voltage, V

T = Torque developed by motor, N-m

K_{tf} = Torque constant, N-m/A

J = Moment of inertia of rotor and load, Kg-m²/rad

B = Frictional coefficient of rotor and load, N-m/(rad/sec)

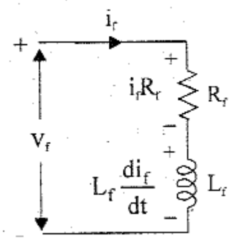


Fig 1.23 : Equivalent circuit of field.

The equivalent circuit of field is shown in fig 1.23.

By Kirchoff's voltage law, we can write

$$R_f i_f + L_f \frac{di_f}{dt} = v_f \qquad .....(1.25)$$

The torque of DC motor is proportional to product of flux and armature current. Since armature current is constant in this system, the torque is proportional to flux alone, but flux is proportional to field current.

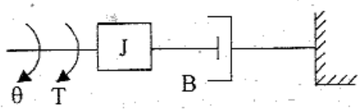


Fig 1.24.

$$T \propto i_f, \quad \therefore \text{Torque, } T = K_{tf}\, i_f \qquad .....(1.26)$$

The mechanical system of the motor is shown in fig 1.24. The differential equation governing the mechanical system of the motor is given by,

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \qquad .....(1.27)$$

The Laplace transform of various time domain signals involved in this system are shown below.

$\mathcal{L}\{i_f\} = I_f(s)$; $\mathcal{L}\{T\} = T(s)$; $\mathcal{L}\{v_f\} = V_f(s)$; $\mathcal{L}\{\theta\} = \theta(s)$

The differential equations governing the field controlled DC motor are,

$$K_f i_f + L_f \frac{di_f}{dt} = v_f \quad ; \quad T = K_{tf} i_f \quad ; \quad J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T$$

On taking Laplace transform of the above equations with zero initial condition we get,

$$R_f I_f(s) + L_f s\, I_f(s) = V_f(s) \qquad .....(1.28)$$

$$T(s) = K_{tf}\, I_f(s) \qquad .....(1.29)$$

$$Js^2\theta(s) + Bs\theta(s) = T(s) \qquad .....(1.30)$$

Equating equations (1.29) and (1.30) we get,

$$K_{tf}\, I_f(s) = Js^2\theta(s) + Bs\theta(s)$$

$$I_f(s) = s\,\frac{(Js+B)}{K_{tf}}\theta(s) \qquad .....(1.31)$$

The equation (1.28) can be written as,

$$(R_f + sL_f)\, I_f(s) = V_f(s) \qquad .....(1.32)$$

On substituting for $I_f(s)$ from equation (1.31) in equation (1.32) we get,

$$(R_f + sL_f)s\,\frac{(Js+B)}{K_{tf}}\,\theta(s) = V_f(s)$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_{tf}}{s\,(R_f + sL_f)\,(B + sJ)}$$

$$= \frac{K_{tf}}{sR_f\left(1+\frac{sL_f}{R_f}\right)B\left(1+\frac{sJ}{B}\right)} = \frac{K_m}{s\,(1+sT_f)\,(1+sT_m)} \qquad .....(1.33)$$

where, $K_m = \frac{K_{tf}}{R_f B}$ = Motor gain constant

$T_f = \frac{L_f}{R_f}$ = Field time constant

$T_m = \frac{J}{B}$ = Mechanical time constant

1.9 ELECTRICAL ANALOGOUS OF MECHANICAL TRANSLATIONAL SYSTEMS

Systems remain ***analogous*** as long as the differential equations governing the systems or transfer functions are in identical form. The electric analogue of any other kind of system is of greater importance since it is easier to construct electrical models and analyse them.

The three basic elements mass, dash-pot and spring that are used in modelling mechanical translational systems are analogous to resistance, inductance and capacitance of electrical systems.

The input force in mechanical system is analogous to either voltage source or current source in electrical systems. The output velocity (first derivative of displacement) in mechanical system is analogous to either current or voltage in an element in electrical system.

Since the electrical systems has two types of inputs either voltage or current source, there are two types of analogies : ***force-voltage analogy*** and ***force-current analogy***.

FORCE-VOLTAGE ANALOGY

The force balance equations of mechanical elements and their analogous electrical elements in force-voltage analogy are shown in table-1.2.The table-1.3 shows the list of analogous quantities in force-voltage analogy.

The following points serve as guidelines to obtain electrical analogous of mechanical systems based on force-voltage analogy.

1. In electrical systems the elements in series will have same current, likewise in mechanical systems, the elements having same velocity are said to be in series.
2. The elements having same velocity in mechanical system should have the same analogous current in electrical analogous system.
3. Each node (meeting point of elements) in the mechanical system corresponds to a closed loop in electrical system. A mass is considered as a node.
4. The number of meshes in electrical analogous is same as that of the number of nodes (masses) in mechanical system. Hence the number of mesh currents and system equations will be same as that of the number of velocities of nodes (masses) in mechanical system.

TABLE- 1.2 : Analogous Elements in Force-Voltage Analogy

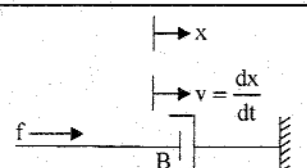
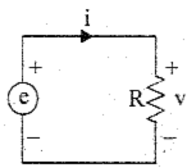
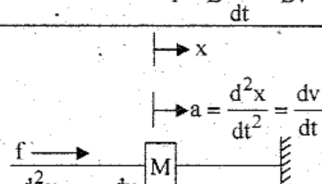
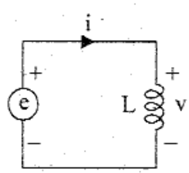
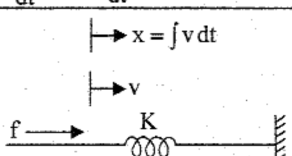
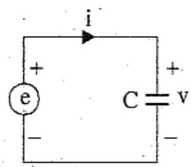
Mechanical system	Electrical system
Input : Force Output : Velocity	Input : Voltage source Output : Current through the element
 $f = B\frac{dx}{dt} = Bv$	 $e = v$ and $v = Ri$ $\therefore e = Ri$
 $f = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$	 $e = v$ and $v = L\frac{di}{dt}$ $\therefore e = L\frac{di}{dt}$
 $f = Kx = K\int v\,dt$	 $e = v$ and $v = \frac{1}{C}\int i\,dt$ $\therefore e = \frac{1}{C}\int i\,dt$

TABLE -1.3 : Analogous Quantities in Force-Voltage Analogy

Item	Mechanical system	Electrical system (mesh basis system)
Independent variable (input)	Force, f	Voltage, e, v
Dependent variable (output)	Velocity, v	Current, i
	Displacement, x	Charge, q
Dissipative element	Frictional coefficient of dashpot, B	Resistance, R
Storage element	Mass, M	Inductance, L
	Stiffness of spring, K	Inverse of capacitance, 1/C
Physical law	Newton's second law $\sum f = 0$	Kirchoff's voltage law $\sum v = 0$
Changing the level of independent variable	Lever $\frac{f_1}{f_2} = \frac{l_1}{l_2}$	Transformer $\frac{e_1}{e_2} = \frac{N_1}{N_2}$

TABLE-1.4 : Analogous Elements in Force-Current Analogy

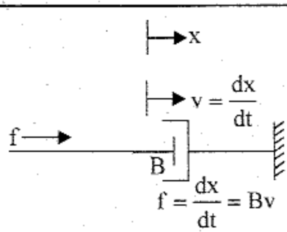
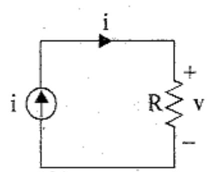
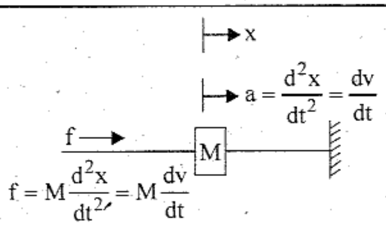
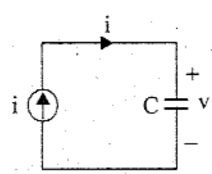
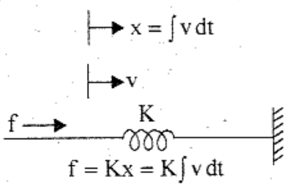
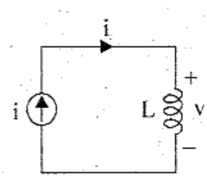
Mechanical system	Electrical system
Input : Force Output : Velocity	Input : Current source Output : Voltage across the element
 $v = \frac{dx}{dt}$, $f = B\frac{dx}{dt} = Bv$	 $i = \frac{1}{R}v$
 $a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$, $f = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$	 $i = C\frac{dv}{dt}$
 $x = \int v\,dt$, $f = Kx = K\int v\,dt$	 $i = \frac{1}{L}\int v\,dt$

TABLE-1.5 : Analogous Quantities in Force-Current Analogy

Item	Mechanical system	Electrical system (node basis system)
Independent variable (input)	Force, f	Current, i
Dependent variable (output)	Velocity, v	Voltage, v
	Displacement, x	Flux, ϕ
Dissipative element	Frictional coefficient of dashpot, B	Conductance G=1/R
Storage element	Mass, M	Capacitance, C
	Stiffness of spring, K	Inverse of inductance, 1/L
Physical law	Newton's second law $\Sigma f = 0$	Kirchoff's current law $\Sigma i = 0$
Changing the level of independent variable	Lever $\frac{f_1}{f_2} = \frac{l_1}{l_2}$	Transformer $\frac{i_1}{i_2} = \frac{N_2}{N_1}$

5. The mechanical driving sources (force) and passive elements connected to the node (mass) in mechanical system should be represented by analogous elements in a closed loop in analogous electrical system.
6. The element connected between two (nodes) masses in mechanical system is represented as a common element between two meshes in electrical analogous system.

FORCE-CURRENT ANALOGY

The force balance equations of mechanical elements and their analogous electrical elements in force-current analogy are shown in table-1.4. The table-1.5 shows the list of analogous quantities in force-current analogy.

The following points serve as guidelines to obtain electrical analogous of mechanical systems based on force-current analogy.

1. In electrical systems elements in parallel will have same voltage, likewise in mechanical systems, the elements having same force are said to be in parallel.
2. The elements having same velocity in mechanical system should have the same analogous voltage in electrical analogous system.
3. Each node (meeting point of elements) in the mechanical system corresponds to a node in electrical system. A mass is considered as a node.
4. The number of nodes in electrical analogous is same as that of the number of nodes (masses) in mechanical system. Hence the number of node voltages and system equations will be same as that of the number of velocities of (nodes) masses in mechanical system.
5. The mechanical driving sources (forces) and passive elements connected to the node (mass) in mechanical system should be represented by analogous elements connected to a node in electrical system.
6. The element connected between two nodes (masses) in mechanical system is represented as a common element between two nodes in electrical analogous system.

EXAMPLE 1.8

Write the differential equations governing the mechanical system shown in fig 1. Draw the force-voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

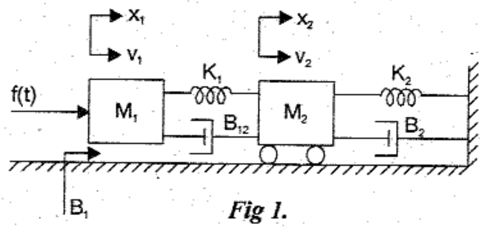


Fig 1.

SOLUTION

The given mechanical system has two nodes (masses). The differential equations governing the mechanical system are given by force balance equations at these nodes. Let the displacements of masses M_1 and M_2 be x_1 and x_2 respectively. The corresponding velocities be v_1 and v_2.

The free body diagram of M_1 is shown in fig 2. The opposing forces are marked as f_{m1}, f_{b1}, f_{b12} and f_{k1}.

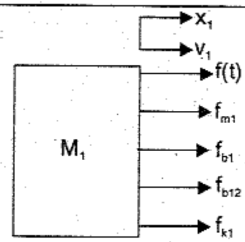


Fig 2.

$$f_{m1} = M_1 \frac{d^2x_1}{dt^2} \quad ; \quad f_{b1} = B_1 \frac{dx_1}{dt}$$

$$f_{b12} = B_{12} \frac{d}{dt}(x_1 - x_2) \quad ; \quad f_{k1} = K_1(x_1 - x_2)$$

By Newton's second law, $f_{m1} + f_{b1} + f_{b12} + f_{k1} = f(t)$

$$\therefore M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d}{dt}(x_1 - x_2) + K_1(x_1 - x_2) = f(t) \quad \text{.....(1)}$$

The free body diagram of M_2 is shown in fig 3.The opposing forces are marked as f_{m2}, f_{b2}, f_{b12}, f_{k1} and f_{k2}.

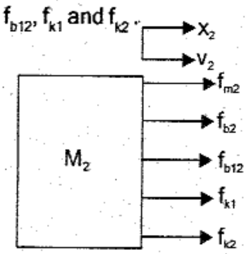


Fig 3.

$$f_{m2} = M_2 \frac{d^2x_2}{dt^2} \quad ; \quad f_{b2} = B_2 \frac{dx_2}{dt} \quad ; \quad f_{b12} = B_{12} \frac{d}{dt}(x_2 - x_1)$$

$$f_{k1} = K_1(x_2 - x_1) \quad ; \quad f_{k2} = K_2 x_2$$

By Newton's second law, $f_{m2} + f_{b2} + f_{k2} + f_{b12} + f_{k1} = 0$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 + B_{12} \frac{d}{dt}(x_2 - x_1) + K_1(x_2 - x_1) = 0 \quad \text{.....(2)}$$

On replacing the displacements by velocity in the differential equations (1) and (2) of the mechanical system we get,

$$\left(\text{i.e, } \frac{d^2x}{dt^2} = \frac{dv}{dt} \quad ; \quad \frac{dx}{dt} = v \text{ and } x = \int v\,dt \right)$$

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + B_{12}(v_1 - v_2) + K_1 \int (v_1 - v_2)\, dt = f(t) \quad \text{.....(3)}$$

$$M_2 \frac{dv_2}{dt} + B_2 v_2 + K_2 \int v_2\, dt + B_{12}(v_2 - v_1) + K_1 \int (v_2 - v_1)\, dt = 0 \quad \text{.....(4)}$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force-voltage analogous electrical circuit will have two meshes.

The force applied to mass, M_1 is represented by a voltage source in first mesh. The elements M_1, B_1, K_1 and B_{12} are connected to first node. Hence they are represented by analogous element in mesh-1 forming a closed path. The elements K_1, B_{12}, M_2, K_2, and B_2 are connected to second node. Hence they are represented by analogous element in mesh-2 forming a closed path.

The elements K_1 and B_{12} are common between node-1 and 2 and so they are represented by analogous element as common elements between two meshes. The force-voltage electrical analogous circuit is shown in fig 4.

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow e(t)$	$M_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$v_1 \rightarrow i_1$	$M_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_2 \rightarrow 1/C_2$
$v_2 \rightarrow i_2$		$B_{12} \rightarrow R_{12}$	

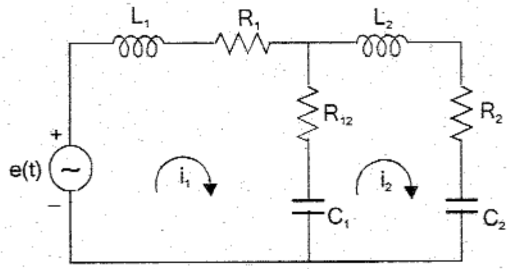


Fig 4 : Force-voltage electrical analogous circuit.

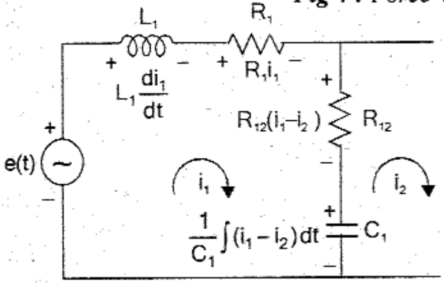


Fig 5.

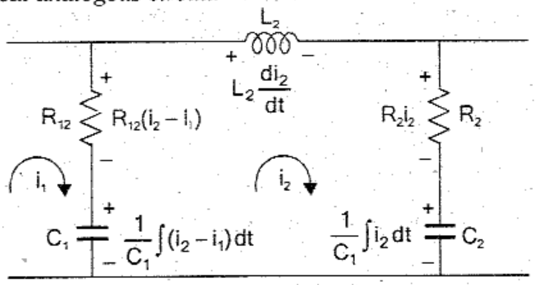


Fig 6.

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 4 are given below (Refer fig 5 and 6).

$$L_1\frac{di_1}{dt} + R_1i_1 + R_{12}(i_1 - i_2) + \frac{1}{C_1}\int(i_1 - i_2)dt = e(t) \quad .....(5)$$

$$L_2\frac{di_2}{dt} + R_2i_2 + \frac{1}{C_2}\int i_2\,dt + R_{12}(i_2 - i_1) + \frac{1}{C_1}\int(i_2 - i_1)dt = 0 \quad .....(6)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing the mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force-current analogous electrical circuit will have two nodes.

The force applied to mass M_1 is represented as a current source connected to node-1 in analogous electrical circuit. The elements M_1, B_1, K_1 and B_{12} are connected to first node. Hence they are represented by analogous elements connected to node-1 in analogous electrical circuit. The elements K_1, B_{12}, M_2, K_2, and B_2 are connected to second node. Hence they are represented by analogous elements as elements connected to node-2 in analogous electrical circuit.

The elements K_1 and B_{12} are common between node-1 and 2 and so they are represented by analogous elements as common element between two nodes in analogous circuit. The force-current electrical analogous circuit is shown in fig 7.

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow i(t)$	$M_1 \rightarrow C_1$	$B_1 \rightarrow 1/R_1$	$K_1 \rightarrow 1/L_1$
$v_1 \rightarrow v_1$	$M_2 \rightarrow C_2$	$B_2 \rightarrow 1/R_2$	$K_2 \rightarrow 1/L_2$
$v_2 \rightarrow v_2$	$B_{12} \rightarrow 1/R_{12}$		

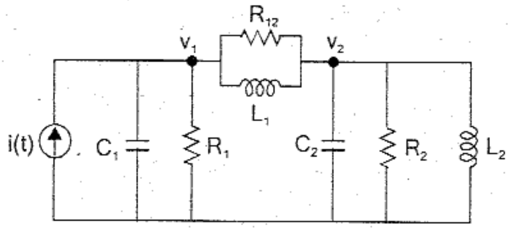


Fig 7 : Force-voltage electrical analogous circuit.

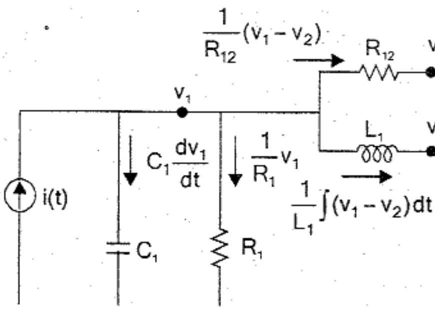


Fig 8.

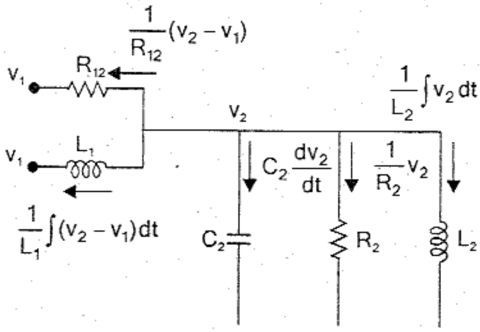


Fig 9.

The node basis equations using Kirchoff's current law for the circuit shown in fig 7 are given below (Refer fig 8 and 9).

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{R_{12}}(v_1-v_2)+\frac{1}{L_1}\int(v_1-v_2)dt=i(t) \qquad .....(7)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}v_2+\frac{1}{L_2}\int v_2dt+\frac{1}{R_{12}}(v_2-v_1)+\frac{1}{L_1}\int(v_2-v_1)dt=0 \qquad .....(8)$$

It is observed that the node basis equations (7) and (8) are similar to the differential equations (3) and (4) governing the mechanical system.

EXAMPLE 1.9

Write the differential equations governing the mechanical system shown in fig 1. Draw the force -voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

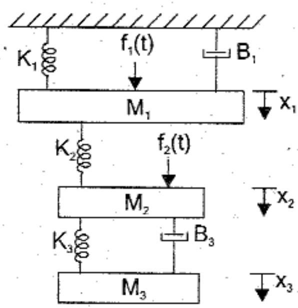


Fig 1.

SOLUTION

The given mechanical system has three nodes masses. The differential equations governing the mechanical system are given by force balance equations at these nodes. Let the displacements of masses M_1, M_2 and M_3 be x_1, x_2 and x_3 respectively. The corresponding velocities be v_1, v_2 and v_3.

The free body diagram of M_1 is shown in fig 2. The opposing forces are marked as f_{m1}, f_{b1}, f_{k2} and f_{k1}.

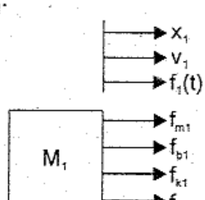


Fig 2.

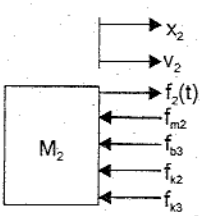


Fig 3.

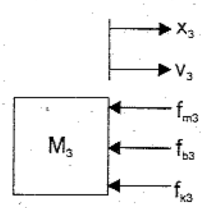


Fig 4.

$$f_{m1} = \frac{M_1 d^2x_1}{dt^2} \quad ; \quad f_{b1} = B_1\frac{dx_1}{dt} \quad ; \quad f_{k2} = K_2(x_1 - x_2) \quad ; \quad f_{k1} = K_1x_1$$

By Newton's second law, $f_{m1} + f_{b1} + f_{k2} + f_{k1} = f_1(t)$

$$M_1\frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + K_2(x_1 - x_2) + K_1x_1 = f_1(t) \qquad .....(1)$$

Free body diagram of M_2 is shown in fig 3. The opposing forces are marked as f_{m2}, f_{b3}, f_{k2} & f_{k3}.

$$f_{m2} = M_2\frac{d^2x_2}{dt^2} \quad ; \quad f_{b3} = B_3\frac{d}{dt}(x_2 - x_3) \quad ; \quad f_{k2} = K_2(x_2 - x_1) \quad ; \quad f_{k3} = K_3(x_2 - x_3)$$

By Newton's second law, $f_{m2} + f_{b3} + f_{k2} + f_{k3} = f_2(t)$

$$M_2\frac{d^2x_2}{dt^2} + B_3\frac{d}{dt}(x_2 - x_3) + K_2(x_2 - x_1) + K_3(x_2 - x_3) = f_2(t) \qquad .....(2)$$

The free body diagram of M_3 is shown in fig 4. The opposing forces are marked as f_{m3}, f_{b3} and f_{k3}.

$$f_{m3} = M_3\frac{d^2x_3}{dt^2} \quad ; \quad f_{b3} = B_3\frac{d}{dt}(x_3 - x_2) \quad ; \quad f_{k3} = K_3(x_3 - x_2)$$

By Newton's second law, $f_{m3} + f_{b3} + f_{k3} = 0$

$$M_3\frac{d^2x_3}{dt^2} + B_3\frac{d}{dt}(x_3 - x_2) + K_3(x_3 - x_2) = 0 \qquad .....(3)$$

On replacing the displacements by velocity in the differential equations (1), (2) and (3) governing the mechanical system we get,

$$\left(\text{i.e., } \frac{d^2x}{dt^2} = \frac{dv}{dt} \;;\; \frac{dx}{dt} = v \text{ and } x = \int v\,dt\right)$$

$$M_1\frac{dv_1}{dt} + B_1v_1 + K_1\int v_1dt + K_2\int(v_1 - v_2)dt = f_1(t) \qquad .....(4)$$

$$M_2\frac{dv_2}{dt} + B_3(v_2 - v_3) + K_2\int(v_2 - v_1)dt + K_3\int(v_2 - v_3)dt = f_2(t) \qquad .....(5)$$

$$M_3\frac{dv_3}{dt} + B_3(v_3 - v_2) + K_3\int(v_3 - v_2)dt = 0 \qquad .....(6)$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-voltage analogous electrical circuit will have three meshes. The force applied to mass, M_1 is represented by a voltage source in first mesh and the force applied to mass, M_2 is represented by a voltage source in second mesh.

The elements M_1, B_1, K_1 and K_2 are connected to first node. Hence they are represented by analogous element in mesh-1 forming a closed path. The elements M_2, B_3, K_2 and K_3 are connected to second node. Hence they are represented by analogous element in mesh-2 forming a closed path. The elements M_3, K_3 and B_3 are connected to third node. Hence they are represented by analogous element in mesh-3 forming a closed path.

The element K_2 is common between node-1 and 2 and so it is represented by analogous element as common element between mesh 1 and 2. The elements K_3 and B_3 are common between node-2 and 3 and so they are represented by analogous elements as common elements between mesh-2 and 3. The force-voltage electrical analogous circuit is shown in fig 5.

The electrical analogous elements for the elements of mechanical system are given below.

$f_1(t) \rightarrow e_1(t)$	$v_1 \rightarrow i_1$	$M_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$f_2(t) \rightarrow e_2(t)$	$v_2 \rightarrow i_2$	$M_2 \rightarrow L_2$	$B_3 \rightarrow R_3$	$K_2 \rightarrow 1/C_2$
	$v_3 \rightarrow i_3$	$M_3 \rightarrow L_3$		$K_3 \rightarrow 1/C_3$

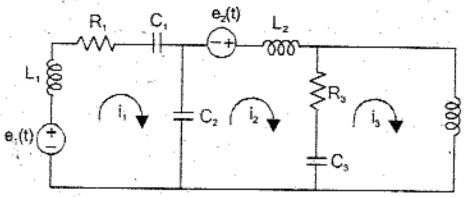


Fig 5 : Force-voltage electrical analogous circuit.

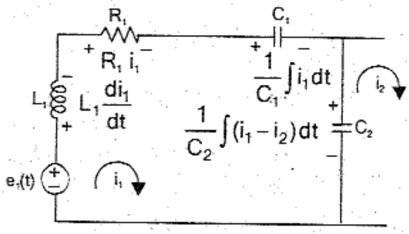


Fig 6.

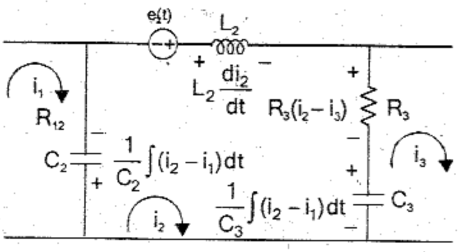


Fig 7.

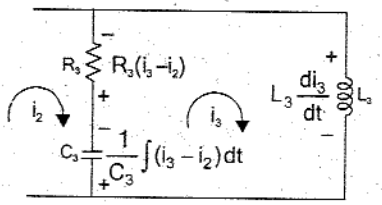


Fig 8.

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 5 are given below (Refer fig 6, 7, 8).

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1 dt + \frac{1}{C_2}\int (i_1 - i_2) dt = e_1(t) \quad .....(7)$$

$$L_2 \frac{di_2}{dt} + R_3(i_2 - i_3) + \frac{1}{C_3}\int (i_2 - i_3) dt + \frac{1}{C_2}\int (i_2 - i_1) dt = e_2(t) \quad .....(8)$$

$$L_3 \frac{di_3}{dt} + R_3(i_3 - i_2) + \frac{1}{C_3}\int (i_3 - i_2) dt = 0 \quad .....(9)$$

It is observed that the mesh equations (7), (8) and (9) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-current analogous electrical circuit will have three nodes.

The force applied to mass M_1 is represented as a current source connected to node-1 in analogous electrical circuit. The force applied to mass M_2 is represented as a current source connected to node-2 in analogous electrical circuit.

The elements M_1, B_1, K_1 and K_2 are connected to first node. Hence they are represented by analogous elements as elements connected to node-1 in analogous electrical circuit. The elements M_2, B_3, K_2 and K_3 are connected to second node. Hence they are represented by analogous elements as elements connected to node-2 in analogous electrical circuit. The elements M_3, B_3 and K_3 are connected to third node. Hence they are represented by analogous elements as elements connected to node-3 in analogous electrical circuit.

The element K_2 is common between node-1 and 2 and so it is represented by analogous element as common element between node-1 and 2 in analogous circuit. The elements B_3 and K_3 are common between node-2 and 3 and so they are represented by analogous elements as common elements between node-2 and 3. The force-current electrical analogous circuit is shown in fig 9.

The electrical analogous elements for the elements of mechanical system are given below.

$f_1(t) \rightarrow i_1(t)$	$v_1 \rightarrow v_1$	$M_1 \rightarrow C_1$	$B_1 \rightarrow 1/R_1$	$K_1 \rightarrow 1/L_1$
$f_2(t) \rightarrow i_2(t)$	$v_2 \rightarrow v_2$	$M_2 \rightarrow C_2$	$B_3 \rightarrow 1/R_3$	$K_2 \rightarrow 1/L_2$
	$v_3 \rightarrow v_3$	$M_3 \rightarrow C_3$		$K_3 \rightarrow 1/L_3$

The node basis equations using Kirchoff's current law for the circuit shown in fig 9. are given below. (Refer fig 10, 11,12).

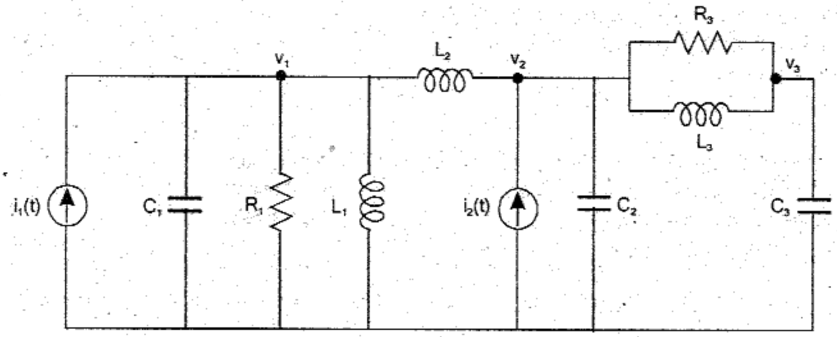


Fig 9 : Force-current electrical analogous circuit.

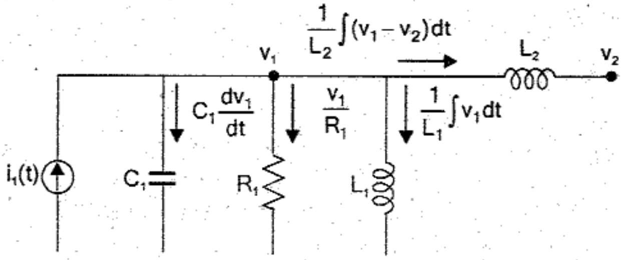


Fig 10.

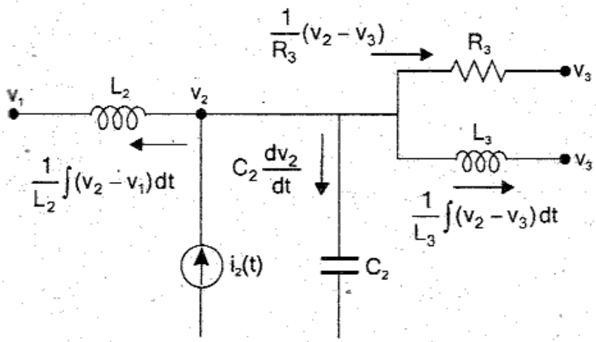


Fig 11.

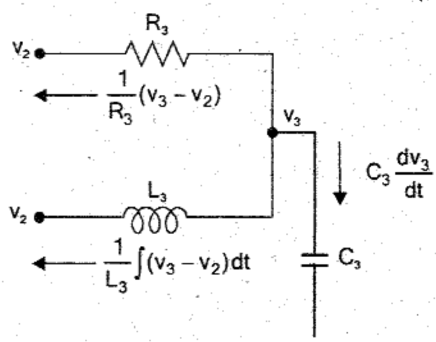


Fig 12.

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{L_1}\int v_1 dt+\frac{1}{L_2}\int(v_1-v_2)dt=i_1(t) \quad .....(10)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_3}(v_2-v_3)+\frac{1}{L_3}\int(v_2-v_3)dt+\frac{1}{L_2}\int(v_2-v_1)dt=i_2(t) \quad .....(11)$$

$$C_3\frac{dv_3}{dt}+\frac{1}{R_3}(v_3-v_2)+\frac{1}{L_3}\int(v_3-v_2)dt=0 \quad .....(12)$$

It is observed that node basis equations (10), (11) and (12) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

EXAMPLE 1-10

Write the differential equations governing the mechanical system shown in fig 1.Draw force-voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

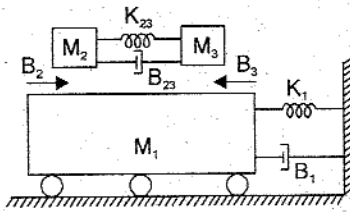


Fig 1.

SOLUTION

The given mechanical system has three nodes (masses). The differential equations governing the mechanical system are given by force balance equations at these nodes. Let the displacements of masses M_1, M_2 and M_3 be x_1, x_2 and x_3 respectively. The corresponding velocities be v_1, v_2 and v_3.

The free body diagram of M_1 is shown in fig 2. The opposing forces are marked as f_{b1}, f_{k1}, f_{b2}, f_{b3}, and f_{m1}.

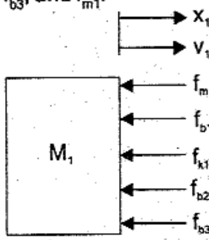


Fig 2.

$$f_{m1} = M_1 \frac{d^2x_1}{dt^2} \quad ; \quad f_{b1} = B_1 \frac{dx_1}{dt} \quad ; \quad f_{k1} = K_1 x_1$$

$$f_{b2} = B_2 \frac{d}{dt}(x_1 - x_2) \; ; \quad f_{b3} = B_3 \frac{d}{dt}(x_1 - x_3)$$

By Newton's second law, $f_{m1} + f_{b1} + f_{k1} + f_{b2} + f_{b3} = 0$

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1 x_1 + B_2 \frac{d}{dt}(x_1 - x_2) + B_3 \frac{d}{dt}(x_1 - x_3) = 0 \qquad .....(1)$$

The free body diagram of M_2 is shown in fig 3. The opposing forces are marked as f_{m2}, f_{b2}, f_{b23} and f_{k23}.

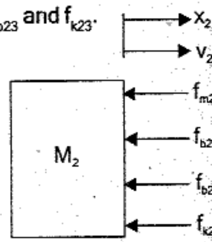


Fig 3.

$$f_{m2} = M_2 \frac{d^2x_2}{dt^2} \quad ; \quad f_{b2} = B_2 \frac{d}{dt}(x_2 - x_1)$$

$$f_{b23} = B_{23} \frac{d}{dt}(x_2 - x_3) \; ; \quad f_{k23} = K_{23}(x_2 - x_3)$$

By Newton's second law, $f_{m2} + f_{b2} + f_{b23} + f_{k23} = 0$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{d}{dt}(x_2 - x_1) + B_{23} \frac{d}{dt}(x_2 - x_3) + K_{23}(x_2 - x_3) = 0 \qquad ......(2)$$

The free body diagram of M_3 is shown in fig 4. The opposing forces are marked as f_{m3}, f_{b3}, f_{b23}, and f_{k23}.

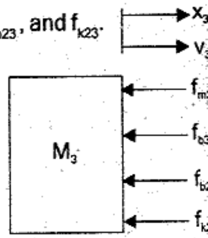


Fig 4.

$$f_{m3} = M_3 \frac{d^2x_3}{dt^2} \quad ; \quad f_{b3} = B_3 \frac{d}{dt}(x_3 - x_1)$$

$$f_{b23} = B_{23} \frac{d}{dt}(x_3 - x_2) \; ; \quad f_{k23} = K_{23}(x_3 - x_2)$$

By Newton's second law, $f_{m3} + f_{b3} + f_{b23} + f_{k23} = 0$

$$M_3 \frac{d^2x_3}{dt^2} + B_3 \frac{d}{dt}(x_3 - x_1) + B_{23} \frac{d}{dt}(x_3 - x_2) + K_{23}(x_3 - x_2) = 0 \qquad .....(3)$$

On replacing the displacements by velocity in the differential equations (1), (2) and (3) governing the mechanical system we get,

$$\left(\text{i.e., } \frac{d^2x}{dt^2} = \frac{dv}{dt}, \quad \frac{dx}{dt} = v \text{ and } x = \int v\,dt \right)$$

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + K_1 \int v_1 dt + B_2(v_1 - v_2) + B_3(v_1 - v_3) = 0 \qquad .....(4)$$

$$M_2 \frac{dv_2}{dt} + B_2(v_2 - v_1) + B_{23}(v_2 - v_3) + K_{23} \int (v_2 - v_3) dt = 0 \qquad .....(5)$$

$$M_3 \frac{dv_3}{dt} + B_3(v_3 - v_1) + B_{23}(v_3 - v_2) + K_{23} \int (v_3 - v_2) dt = 0 \qquad .....(6)$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-voltage analogous electrical circuit will have three meshes.

The elements M_1, K_1, B_1, B_3 and B_2 are connected to first node. Hence they are represented by analogous elements in mesh-1 forming a closed path. The elements M_2, K_{23}, B_{23} and B_2 are connected to second node. Hence they are represented by analogous elements in mesh-2 forming a closed path. The elements M_3, K_{23}, B_{23} and B_3 are connected to third node. Hence they are represented by analogous elements in mesh-3 forming a closed path.

The elements K_{23} and B_{23} are common between node-2 and 3 and so they are represented by analogous element as common elements between mesh-2 and 3. The element B_2 is common between node-1 and 2 and so it is represented by analogous element as common element between mesh-1 and 2. The element B_3 is common between node-1 and 3 and so it is represented by analogous element between mesh-1 and 3. The force-voltage electrical analogous circuit is shown in fig 5.

The electrical analogous elements for the elements of mechanical system are given below.

$v_1 \rightarrow i_1$	$M_1 \rightarrow L_1$	$K_1 \rightarrow 1/C_1$ $B_2 \rightarrow R_2$
$v_2 \rightarrow i_2$	$M_2 \rightarrow L_2$	$K_{23} \rightarrow 1/C_{23}$ $B_3 \rightarrow R_3$
$v_3 \rightarrow i_3$	$M_3 \rightarrow L_3$	$B_1 \rightarrow R_1$ $B_{23} \rightarrow R_{23}$

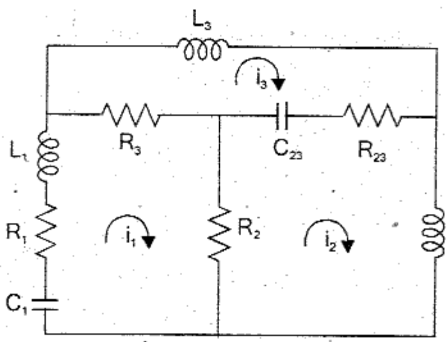


Fig 5 : Force-voltage electrical analogous circuit.

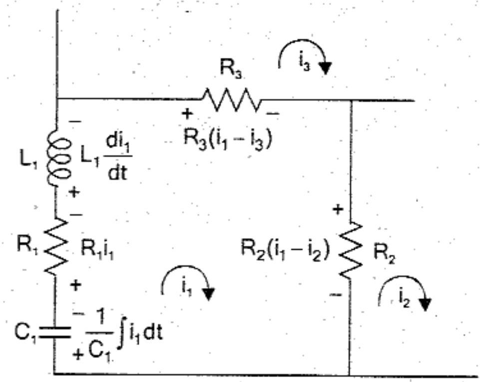


Fig 6.

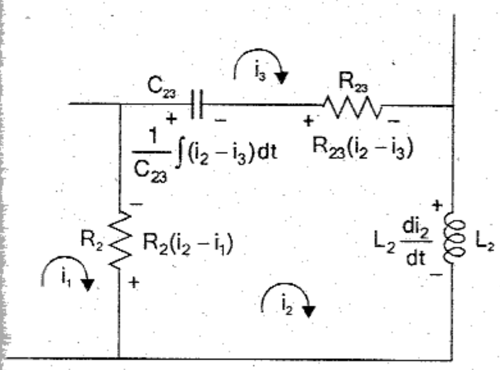


Fig 7.

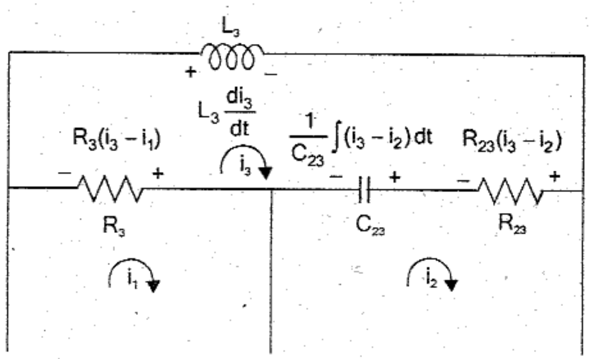


Fig 8.

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 5 are given below. (Refer fig 6, 7 & 8).

$$L_1\frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1 dt + R_2(i_1 - i_2) + R_3(i_1 - i_3) = 0 \qquad .....(7)$$

$$L_2\frac{di_2}{dt} + R_2(i_2 - i_1) + \frac{1}{C_{23}}\int (i_2 - i_3)dt + R_{23}(i_2 - i_3) = 0 \qquad .....(8)$$

$$L_3\frac{di_3}{dt}+R_3(i_3-i_1)+\frac{1}{C_{23}}\int(i_3-i_2)dt+R_{23}(i_3-i_2)=0 \qquad \ldots$$

It is observed that the mesh basis equations (7), (8) and (9) are similar to the differential equations (4), (5) and governing the mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-current analogous electrical circuit will ha three nodes.

The elements M_1, K_1, B_1, B_2 and B_3 are connected to first node. Hence they are represented by analogous elements elements connected to node-1 in analogous electrical circuit. The elements M_2, K_{23}, B_{23} and B_2 are connected to second no Hence they are represented by analogous elements as elements connected to node-2 in analogous electrical circuit. T elements M_3, K_{23}, B_{23} and B_3 are connected to third node. Hence they are represented by analogous elements as eleme connected to node-3 in analogous electrical circuit.

The elements K_{23} and B_{23} are common between node-2 and 3 and so they are represented by analogous element common elements between node-2 and 3 in electrical analogous circuit. The element B_2 is common between node-1 and 2 a so it is represented by analogous element as common element between node-1 and 2 in electrical analogous circuit. T element B_3 is common between node-1 and 3 and so it is represented by analogous element as common element between no 1 and 3 in electrical analogous circuit. The force-current electrical analogous circuit is shown in fig 9.

The electrical analogous elements for the elements of mechanical system are given below.

$v_1 \rightarrow v_1$	$M_1 \rightarrow C_1$	$K_1 \rightarrow 1/L_1$	$B_2 \rightarrow 1/R_2$
$v_2 \rightarrow v_2$	$M_2 \rightarrow C_2$	$K_{23} \rightarrow 1/L_{23}$	$B_3 \rightarrow 1/R_3$
$v_3 \rightarrow v_3$	$M_3 \rightarrow C_3$	$B_1 \rightarrow 1/R_1$	$B_{23} \rightarrow 1/R_{23}$

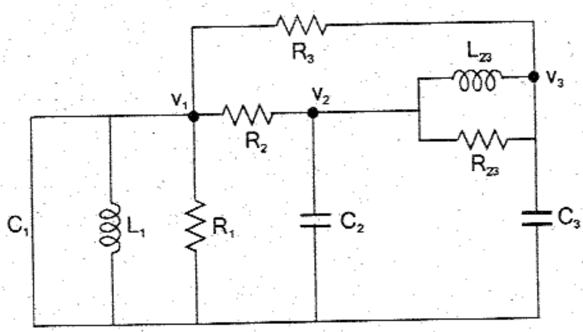


Fig 9 : Force-current electrical analogous circuit.

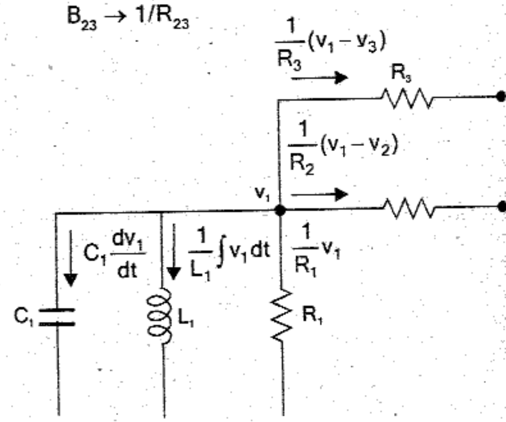


Fig 10.

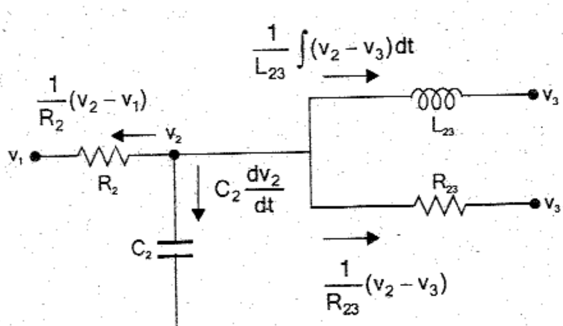


Fig 11.

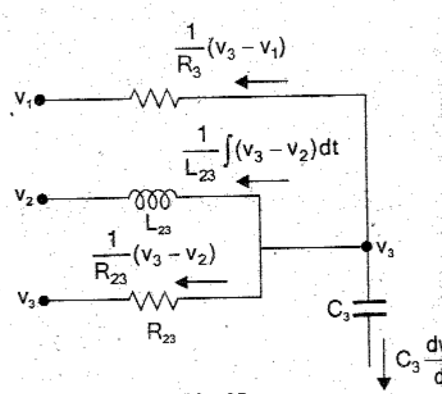


Fig 12.

The node basis equations using Kirchoff's current law for the circuit shown in fig 9 are given below. (Refer fig 10, 11 and 12).

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{L_1}\int v_1dt+\frac{1}{R_2}(v_1-v_2)+\frac{1}{R_3}(v_1-v_3)=0 \quad .....(10)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}(v_2-v_1)+\frac{1}{L_{23}}\int(v_2-v_3)dt+\frac{1}{R_{23}}(v_2-v_3)=0 \quad .....(11)$$

$$C_3\frac{dv_3}{dt}+\frac{1}{R_3}(v_3-v_1)+\frac{1}{L_{23}}\int(v_3-v_2)dt+\frac{1}{R_{23}}(v_3-v_2)=0 \quad .....(12)$$

It is observed that the node basis equations (10), (11) and (12) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

EXAMPLE 1.11

Write the differential equations governing the mechanical system shown in fig 1. Draw the force-voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

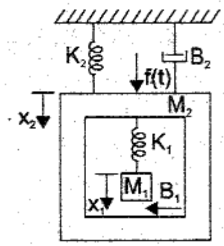


Fig 1.

SOLUTION

The given mechanical system has two nodes (masses). The differential equations governing the mechanical system are given by force balance equations at these nodes. Let the displacement of masses M_1 and M_2 be x_1 and x_2 respectively. The corresponding velocities be v_1 and v_2.

The free body diagram of M_1 is shown in fig 2. The opposing forces are marked as f_{m1}, f_{b1} and f_{k1}.

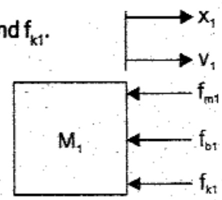


Fig 2.

$$f_{m1}=M_1\frac{d^2x_1}{dt^2}\ ;\ f_{b1}=B_1\frac{d(x_1-x_2)}{dt}\ ;\ f_{k1}=K_1(x_1-x_2)$$

By Newton's second law, $f_{m1}+f_{b1}+f_{k1}=0$

$$M_1\frac{d^2x_1}{dt^2}+B_1\frac{d(x_1-x_2)}{dt}+K_1(x_1-x_2)=0 \quad .....(1)$$

The free body diagram of M_2 is shown in fig 3. The opposing forces are marked as f_{m2}, f_{b2}, f_{b1}, f_{k2} and f_{k1}.

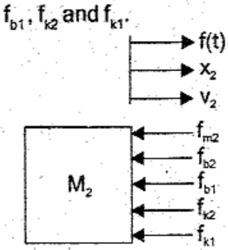


Fig 3.

$$f_{m2}=M_2\frac{d^2x_2}{dt^2}\ ;\ f_{b2}=B_2\frac{dx_2}{dt}\ ;\ f_{b1}=B_1\frac{d}{dt}(x_2-x_1)$$

$$f_{k2}=K_2x_2\ ;\ f_{k1}=K_1(x_2-x_1)$$

By Newton's second law, $f_{m2}+f_{b2}+f_{k2}+f_{b1}+f_{k1}=f(t)$

$$M_2\frac{d^2x_2}{dt^2}+B_2\frac{dx_2}{dt}+K_2x_2+B_1\frac{d}{dt}(x_2-x_1)+K_1(x_2-x_1)=f(t) \quad .....(2)$$

On replacing the displacements by velocity in the differential equations (1) and (2) governing the mechanical system we get,

$$\left(\text{i.e., } \frac{d^2x}{dt^2}=\frac{dv}{dt},\quad \frac{dx}{dt}=v \text{ and } x=\int v dt\right)$$

$$M_1\frac{dv_1}{dt}+B_1(v_1-v_2)+K_1\int(v_1-v_2)dt=0 \quad .....(3)$$

$$M_2\frac{dv_2}{dt}+B_2v_2+K_2\int v_2dt+B_1(v_2-v_1)+K_1\int(v_2-v_1)dt=f(t) \quad .....(4)$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force voltage analogous electrical circuit will ha two meshes. The force applied to mass, M_2 is represented by a voltage source in second mesh.

The elements M_1, K_1 and B_1 are connected to first node. Hence they are represented by analogous element in mesh forming a closed path. The elements M_2, K_2, B_2, B_1 and K_1 are connected to second node. Hence they are represented analogous element in mesh 2 forming a closed path.

The elements B_1 and K_1 are common between node 1 and 2 and so they are represented as common elements betwe mesh 1 and 2. The force-voltage electrical analogous circuit is shown in fig 4.

The electrical analogous elements for the elements of mechanical system are given below.

$$f(t) \rightarrow e(t) \quad v_1 \rightarrow i_1 \quad M_1 \rightarrow L_1 \quad K_1 \rightarrow 1/C_1 \quad B_1 \rightarrow R_1$$

$$v_2 \rightarrow i_2 \quad M_2 \rightarrow L_2 \quad K_2 \rightarrow 1/C_2 \quad B_2 \rightarrow R_2$$

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 4. are given below, (refer fig 5 and 6

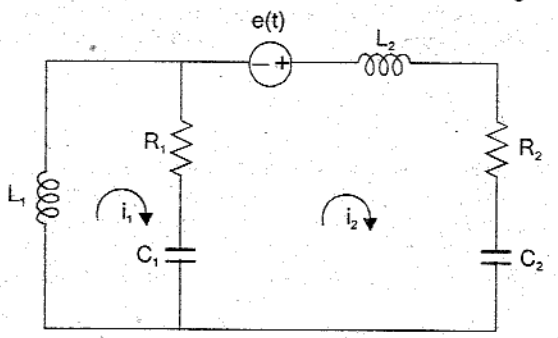


Fig 4 : Force-voltage electrical analogous circuit.

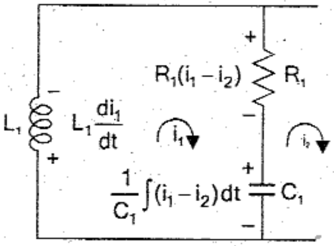


Fig 5.

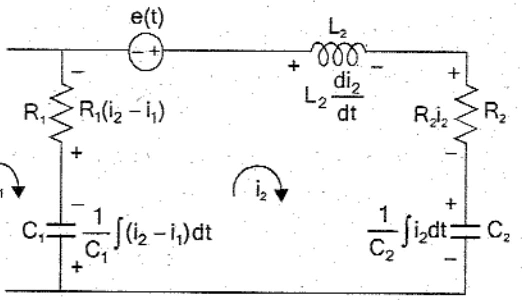


Fig 6.

$$L_1\frac{di_1}{dt} + R_1(i_1 - i_2) + \frac{1}{C_1}\int (i_1 - i_2)\,dt = 0 \qquad .....(5)$$

$$L_2\frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt + \frac{1}{C_1}\int (i_2 - i_1)\,dt + R_1(i_2 - i_1) = e(t) \qquad .....(6)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing th mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force-current analogous electrical circuit will ha two nodes. The force applied to mass M_2 is represented as a current source connected to node-2 in analogous electrical circu

The elements M_1, K_1 and B_1 are connected to first node. Hence they are represented by analogous elements a elements connected to node-1 in analogous electrical circuit. The elements M_2, K_2, B_2, B_1 and K_1 are connected to secon node. Hence they are represented by analogous elements as elements connected to node-1 in analogous electrical circuit.

The elements K_1 and B_1 is common to node-1 and 2 and so they are represented by analogous element as common elements between two nodes in analogous circuit. The force-current electrical analogous circuit is shown in fig 7.

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow i(t)$ $\quad v_1 \rightarrow v_1$ $\quad M_1 \rightarrow C_1$ $\quad B_1 \rightarrow 1/R_1$ $\quad K_1 \rightarrow 1/L_1$

$v_2 \rightarrow v_2$ $\quad M_2 \rightarrow C_2$ $\quad B_2 \rightarrow 1/R_2$ $\quad K_2 \rightarrow 1/L_2$

The node basis equations using Kirchoff's current law for the circuit shown in fig.7, are given below, (Refer fig 8 and 9).

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}(v_1-v_2)+\frac{1}{L_1}\int(v_1-v_2)\,dt=0 \qquad .....(7)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}v_2+\frac{1}{L_2}\int v_2dt+\frac{1}{R_1}(v_2-v_1)+\frac{1}{L_1}\int(v_2-v_1)dt=i(t) \qquad .....(8)$$

It is observed that the node basis equations (7) and (8) are similar to the differential equations (3) and (4) governing the mechanical system.

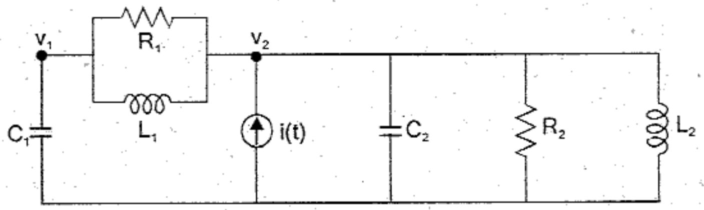


Fig 7 : Force-current electrical analogous circuit.

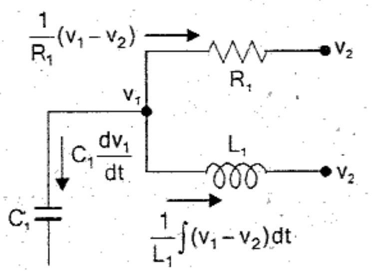


Fig 8.

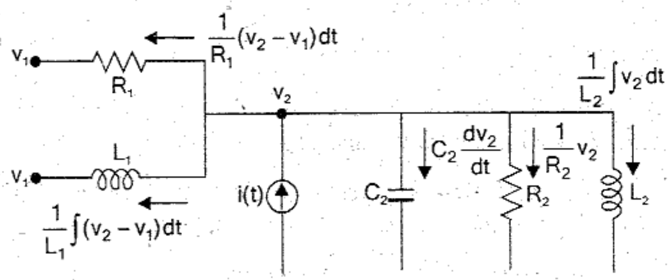


Fig 9.

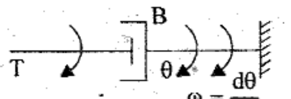
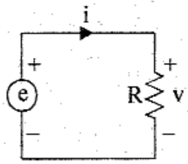
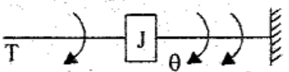
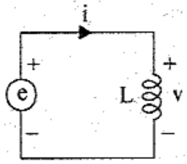
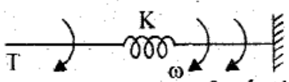
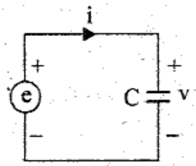
1.10 ELECTRICAL ANALOGOUS OF MECHANICAL ROTATIONAL SYSTEMS

The three basic elements moment of inertia, rotational dashpot and torsional spring that are used in modelling mechanical rotational systems are analogous to resistance, inductance and capacitance of electrical systems. The input torque in mechanical system is analogous to either voltage source or current source in electrical systems. The output angular velocity (first derivative of angular displacement) in mechanical rotational system is analogous to either current or voltage in an element in electrical system. Since the electrical systems has two types of inputs either voltage source or current source, there are two types of analogies: ***torque-voltage analogy and torque-current analogy***.

TORQUE-VOLTAGE ANALOGY

The torque balance equations of mechanical rotational elements and their analogous electrical elements in torque-voltage analogy are shown in table-1.6. The table-1.7 shows the list of analogous quantities in torque-voltage analogy.

TABLE-1.6 : Analogous Element of Torque-Voltage Analogy

Mechanical rotational system	Electrical system
Input : Torque Output : Angular velocity	Input : Voltage source Output : Current through the element
 $\omega = \frac{d\theta}{dt}$ $T = B\frac{d\theta}{dt} = B\omega$	 $e = v \; ; \; v = Ri$ $\therefore e = Ri$
 $T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$ $\quad \propto = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$	 $e = v \; ; \; v = L\frac{di}{dt}$ $\therefore e = L\frac{di}{dt}$
 $\theta = \int \omega\, dt$ $T = K\theta = K\int \omega\, dt$	 $e = v \; ; \; v = \frac{1}{C}\int i\, dt$ $\therefore e = \frac{1}{C}\int i\, dt$

The following points serve as guidelines to obtain electrical analogous of mechanical rotational systems based on torque-voltage analogy.

1. In electrical systems the elements in series will have same current, likewise in mechanical systems, the elements having same angular velocity are said to be in series.
2. The elements having same angular velocity in mechanical system should have analogous same current in electrical analogous system.
3. Each node (meeting point of elements) in the mechanical system corresponds to a closed loop in electrical system. The moment of inertia of mass is considered as a node.
4. The number of meshes in electrical analogous is same as that of the number of nodes (moment of inertia of mass) in mechanical system. Hence the number of mesh currents and system equations will be same as that of the number of angular velocities of nodes (moment of inertia of mass) in mechanical system.
5. The mechanical driving sources (Torque) and passive elements connected to the node (moment of inertia of mass) in mechanical system should be represented by analogous element in a closed loop in analogous electrical system.
6. The element connected between two nodes (moment of inertia) in mechanical system is represented as a common element between two meshes in electrical analogous system.

TABLE-1.7 : Analogous Quantities in Torque-Voltage Analogy

Item	Mechanical rotational system	Electrical system (mesh basis system)
Independent variable (input)	Torque, T	Voltage, e, v
Dependent variable (output)	Angular Velocity, ω	Current, i
	Angular displacement, θ	Charge, q
Dissipative element	Rotational coefficient of dashpot, B	Resistance, R
Storage element	Moment of inertia, J	Inductance, L
	Stiffness of spring, K	Inverse of capacitance, 1/C
Physical law	Newton's second law $\sum T = 0$	Kirchoff's voltage law $\sum v = 0$
Changing the level of independent variable	Gear $\frac{T_1}{T_2} = \frac{n_1}{n_2}$	Transformer $\frac{e_1}{e_2} = \frac{N_1}{N_2}$

TORQUE-CURRENT ANALOGY

The torque balance equations of mechanical elements and their analogous electrical elements in torque-current analogy are shown in table-1.8. The table-1.9 shows the list of analogous quantities in torque-current analogy.

The following points serve as guidelines to obtain electrical analogous of mechanical rotational systems based on Torque-current analogy.

1. In electrical systems the elements in parallel will have same voltage, likewise in mechanical systems, the elements having same torque are said to be in parallel.
2. The elements having same angular velocity in mechanical system should have analogous same voltage in electrical analogous system.
3. Each node (meeting point of elements) in the mechanical system corresponds to a node in electrical system. The moment of inertia of mass is considered as a node.
4. The number of nodes in electrical analogous is same as that of the number of nodes (moment of inertia of mass) in mechanical system. Hence the number of node voltages and system equations will be same as that of the number of angular velocities of nodes (moment of inertia of mass) in mechanical system.
5. The mechanical driving sources (Torque) and passive elements connected to the node in mechanical system should be represented by analogous element connected to a node in analogous electrical system.
6. The element connected between two nodes (moment of inertia of mass) in mechanical system is represented as a common element between two nodes in electrical analogous system.

TABLE-1.8 : Analogous Elements in Torque-Current Analogy

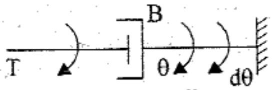
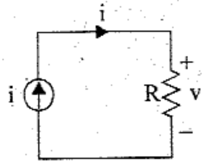
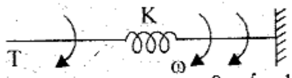
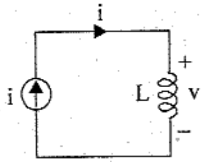
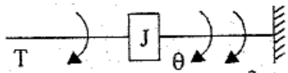
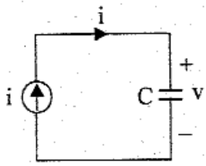
Mechanical rotational system	Electrical system
Input : Torque Output : Angular velocity	Input : Current source Output : Voltage across the element
 $\omega = \frac{d\theta}{dt}$ $T = B\frac{d\theta}{dt} = B\omega$	 $i = \frac{1}{R}v$
 $\theta = \int \omega\,dt$ $T = K\theta = K\int \omega\,dt$	 $i = \frac{1}{L}\int v\,dt$
 $\propto = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$ $T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$	 $i = C\frac{dv}{dt}$

TABLE-1.9 : Analogous Quantities in Torque-Current Analogy

Item	Mechanical rotational system	Electrical system (node basis system)
Independent variable (input)	Torque, T	Current, i
Dependent variable (output)	Angular Velocity, ω	Voltage, v
	Angular displacement, θ	Flux, ϕ
Dissipative element	Rotational frictional coefficient of dashpot, B	Conductance, G = 1/R
Storage element	Moment of inertia, J	Capacitance, C
	Stiffness of spring, K	Inverse of inductance, 1/L
Physical law	Newton's second law $\Sigma T = 0$	Kirchoff 's current law $\Sigma i = 0$
Changing the level of independent variable	Gear $\frac{T_1}{T_2} = \frac{n_1}{n_2}$	Transformer $\frac{i_1}{i_2} = \frac{N_2}{N_1}$

EXAMPLE 1.12

Write the differential equations governing the mechanical rotational system shown in fig 1. Draw the torque-voltage and torque-current electrical analogous circuits and verify by writing mesh and node equations.

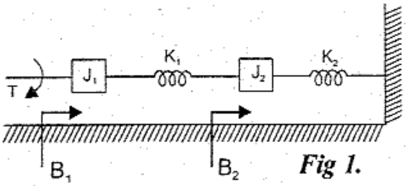


Fig 1.

SOLUTION

The given mechanical rotational system has two nodes (moment of inertia of masses). The differential equations governing the mechanical rotational system are given by torque balance equations at these nodes.

Let the angular displacements of J_1 and J_2 be θ_1 and θ_2 respectively. The corresponding angular velocities be ω_1 and ω_2.

The free body diagram of J_1 is shown in fig 2. The opposing torques are marked as T_{j1}, T_{b1} and T_{k1}.

$$T_{j1}=J_1\frac{d^2\theta_1}{dt^2}\quad;\quad T_{b1}=B_1\frac{d\theta_1}{dt}\quad;\quad T_{k1}=K_1(\theta_1-\theta_2)$$

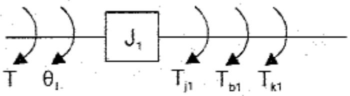


Fig 2.

By Newton's second law, $T_{j1}+T_{b1}+T_{k1}=T$

$$J_1\frac{d^2\theta_1}{dt^2}+B_1\frac{d\theta_1}{dt}+K_1(\theta_1-\theta_2)=T \qquad .....(1)$$

The free body diagram of J_2 is shown in fig 3. The opposing torques are marked as T_{j2}, T_{b2}, T_{k2} and T_{k1}.

$$T_{j2}=J_2\frac{d^2\theta_2}{dt^2}\quad;\quad T_{b2}=B_2\frac{d\theta_2}{dt}$$

$$T_{k2}=K_2\theta_2\quad;\quad T_{k1}=K_1(\theta_2-\theta_1)$$

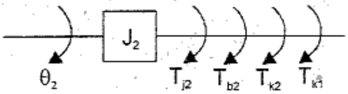


Fig 3.

By Newton's second law, $T_{j2}+T_{b2}+T_{k2}+T_{k1}=0$

$$J_2\frac{d^2\theta_2}{dt^2}+B_2\frac{d\theta_2}{dt}+K_2\theta_2)+K_1(\theta_2-\theta_1)=0 \qquad .....(2)$$

On replacing the angular displacements by angular velocity in the differential equations (1) and (2) governing the mechanical rotational system we get,

$$\left(\text{i.e., } \frac{d^2\theta}{dt^2}=\frac{d\omega}{dt};\quad \frac{d\theta}{dt}=\omega \text{ and } \theta=\int\omega\,dt\right)$$

$$J_1\frac{d\omega_1}{dt}+B_1\omega_1+K_1\int(\omega_1-\omega_2)dt=T \qquad .....(3)$$

$$J_2\frac{d\omega_2}{dt}+B_2\omega_2+K_2\int\omega_2 dt+K_1\int(\omega_2-\omega_1)dt=0 \qquad .....(4)$$

TORQUE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has two nodes (J_1 and J_2). Hence the torque-voltage analogous electrical circuit will have two meshes. The torque applied to J_1 is represented by a voltage source in first mesh. The elements J_1, B_1 and K_1 are connected to first node. Hence they are represented by analogous element in mesh-1 forming a closed path. The elements J_2, B_2, K_2 and K_1 are connected to second node. Hence they are represented by analogous elements in mesh-2 forming a closed path.

The element K_1 is common between node-1 and 2 and so it is represented by analogous element as common element between two meshes. The torque-voltage electrical analogous circuit is shown in fig 4.

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow e(t)$	$J_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$\omega_1 \rightarrow i_1$	$J_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_2 \rightarrow 1/C_2$
$\omega_2 \rightarrow i_2$			

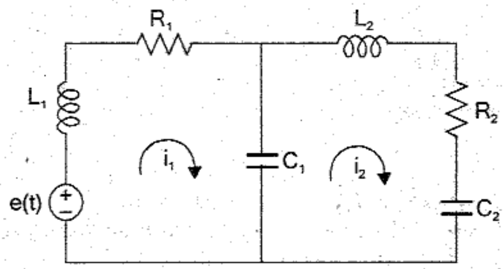


Fig 4 : Torque-voltage electrical analogous circuit.

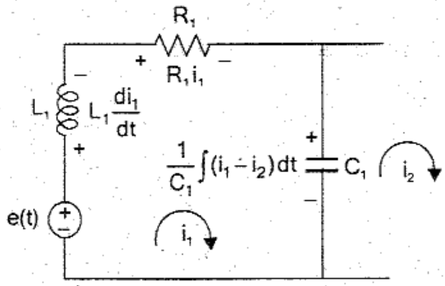


Fig 5.

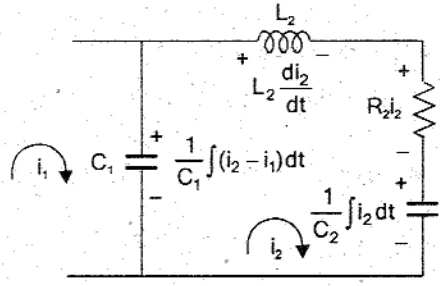


Fig 6.

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 4 are given below (Refer fig 5 and 6).

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int(i_1 - i_2) = e(t) \qquad .....(5)$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2 dt + \frac{1}{C_1}\int(i_2 - i_1)dt = 0 \qquad .....(6)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing the mechanical system.

TORQUE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has two nodes (J_1 and J_2). Hence the torque-current analogous electrical circuit will have two nodes. The torque applied to J_1 is represented as a current source connected to node-1 in analogous electrical circuit.

The elements J_1, B_1 and K_1 are connected to first node. Hence they are represented by analogous elements as elements connected to node-1 in analogous electrical circuit. The elements J_2, B_2, K_2 and K_1 are connected to second node. Hence they are represented by analogous elements as elements connected to node-2 in analogous electrical circuit.

The element K_1 is common between node-1 and 2. So it is represented by analogous element as common element between node-1 and 2. The torque-current electrical analogous circuit is shown in fig 7.

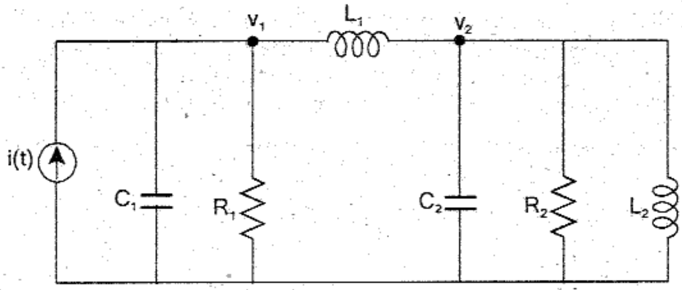


Fig 7 : Torque-current electrical analogous circuit.

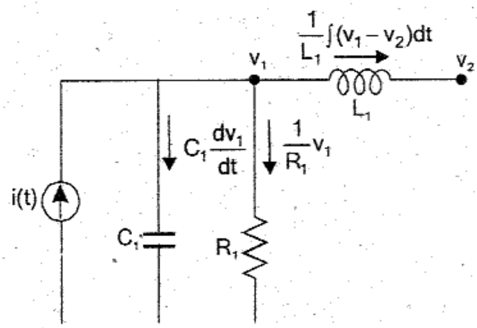


Fig 8.

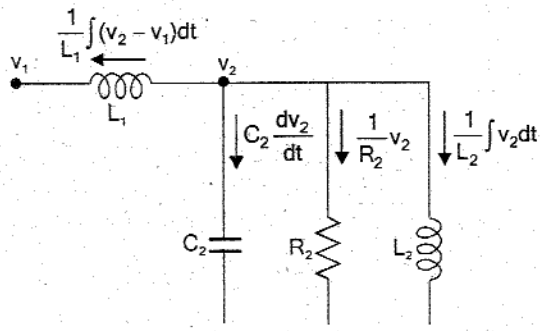


Fig 9.

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow i(t)$	$B_1 \rightarrow 1/R_1$	$\omega_1 \rightarrow v_1$	$J_1 \rightarrow C_1$	$K_1 \rightarrow 1/L_1$
	$B_2 \rightarrow 1/R_2$	$\omega_2 \rightarrow v_2$	$J_2 \rightarrow C_2$	$K_2 \rightarrow 1/L_2$

The node basis equations using Kirchoff's current law for the circuit shown in fig 7 are given below (Refer fig 8 and 9).

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{L_1}\int(v_1-v_2)dt=i(t) \qquad .....(7)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}v_2+\frac{1}{L_2}\int v_2dt+\frac{1}{L_1}\int(v_2-v_1)dt=0 \qquad .....(8)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing the mechanical system.

EXAMPLE 1.13

Write the differential equations governing the mechanical rotational system shown in fig 1. Draw the torque-voltage and torque-current electrical analogous circuits and verify by writing mesh and node equations.

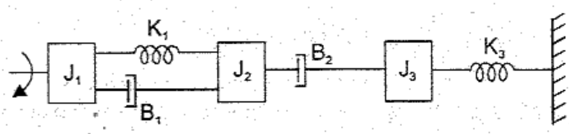


Fig 1.

SOLUTION

The given mechanical rotational system has three nodes (moment of inertia of masses). The differential equations governing the mechanical rotational system are given by torque balance equations at these nodes.

Let the angular displacements of J_1, J_2 and J_3 be θ_1, θ_2 and θ_3 respectively. The corresponding angular velocities be ω_1, ω_2 and ω_3.

The free body diagram of J_1 is shown in fig 2. The opposing torques are marked as T_{j1}, T_{b1} and T_{k1}.

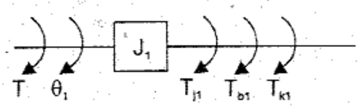


Fig 2.

$$T_{j1}=J_1\frac{d^2\theta_1}{dt^2}\ ;\ T_{b1}=B_1\frac{d(\theta_1-\theta_2)}{dt}$$

$$T_{k1}=K_1(\theta_1-\theta_2)$$

By Newton's second law, $T_{j1}+T_{b1}+T_{k1}=T$

$$J_1\frac{d^2\theta_1}{dt^2}+B_1\frac{d(\theta_1-\theta_2)}{dt}+K_1(\theta_1-\theta_2)=T \qquad .....(1)$$

The free body diagram of J_2 is shown in fig 3. The opposing torques are marked as T_{j2}, T_{b2}, T_{b1} and T_{k1}.

$$T_{j2}=J_2\frac{d^2\theta_2}{dt^2}\ ; \qquad T_{b2}=B_2\frac{d(\theta_2-\theta_3)}{dt}$$

$$T_{k1} = K_1(\theta_2 - \theta_1)\ ;\ \ T_{b1} = B_1 \frac{d(\theta_2 - \theta_1)}{dt}$$

By Newton's second law, $T_{j2} + T_{b2} + T_{b1} + T_{k1} = 0$

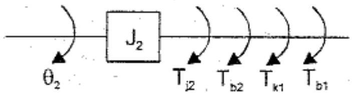


Fig 3.

$$J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d(\theta_2 - \theta_3)}{dt} + B_1 \frac{d(\theta_2 - \theta_1)}{dt} + K_1(\theta_2 - \theta_1) = 0 \qquad .....(2)$$

The free body diagram of J_3 is shown in fig 4. The opposing torques are marked as T_{j3}, T_{b2}, and T_{k3}.

$$T_{j3} = J_3 \frac{d^2\theta_3}{dt^2}\ ;\ \ T_{b2} = B_2 \frac{d(\theta_3 - \theta_2)}{dt}\ ;\ \ T_{k3} = K_3\theta_3$$

By Newton's second law, $T_{j3} + T_{b2} + T_{k3} = 0$

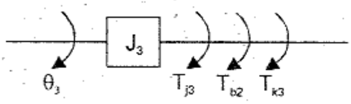


Fig 4.

$$\therefore J_3 \frac{d^2\theta_3}{dt^2} + B_2 \frac{d(\theta_3 - \theta_2)}{dt} + K_3\theta_3 = 0 \qquad .....(3)$$

On replacing the angular displacements by angular velocity in the differential equations (1) and (2) governing the mechanical rotational system we get,

$$\left(\text{i.e., } \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}\ ;\ \ \frac{d\theta}{dt} = \omega \ \text{ and } \ \theta = \int \omega \, dt \right)$$

$$J_1 \frac{d\omega_1}{dt} + B_1(\omega_1 - \omega_2) + K_1 \int (\omega_1 - \omega_2)\, dt = T \qquad .....(4)$$

$$J_2 \frac{d\omega_2}{dt} + B_1(\omega_2 - \omega_1) + B_2(\omega_2 - \omega_3) + K_1 \int (\omega_2 - \omega_1)\, dt = 0 \qquad .....(5)$$

$$J_3 \frac{d\omega_3}{dt} + B_2(\omega_3 - \omega_2) + K_3 \int \omega_3 dt = 0 \qquad .....(6)$$

TORQUE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has three nodes (J_1, J_2 and J_3). Hence the torque-voltage analogous electrical circuit will have three meshes. The torque applied to J_1 is represented by a voltage source in first mesh.

The elements J_1, K_1 and B_1 are connected to first node. Hence they are represented by analogous element in mesh-1 forming a closed path. The elements J_2, B_2, B_1 and K_1 are connected to second node. Hence they are represented by analogous element in mesh-2 forming a closed path. The element J_3, B_2 and K_3 are connected to third node. Hence they are represented by analogous element in mesh-3 forming a closed path.

The elements K_1 and B_1 are common between the nodes-1 and 2 and so they are represented by analogous element as common between mesh-1 and 2. The element B_2 is common between the nodes-2 and 3 and so it is represented by analogous element as common element between the mesh-2 and 3. The torque-voltage electrical analogous circuit is shown in fig 5.

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow e(t)$	$\omega_1 \rightarrow i_1$	$J_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
	$\omega_2 \rightarrow i_2$	$J_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_3 \rightarrow 1/C_3$
	$\omega_3 \rightarrow i_3$	$J_3 \rightarrow L_3$		

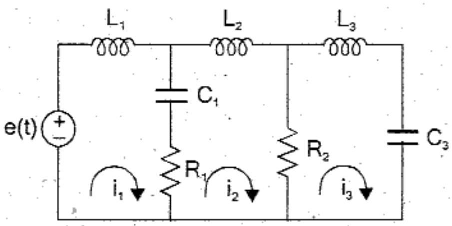


Fig 5 : Torque-voltage electrical analogous circuit.

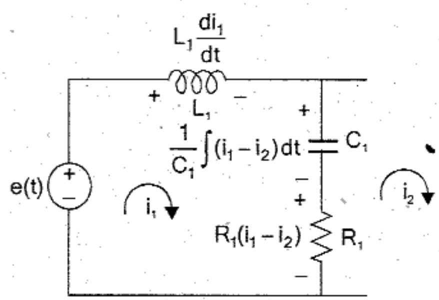


Fig 6.

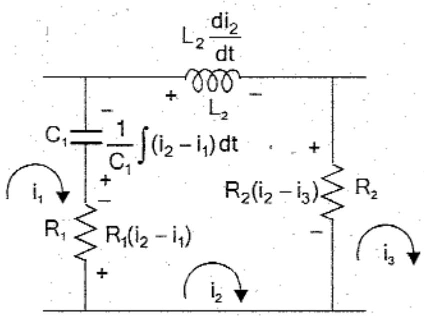


Fig 7.

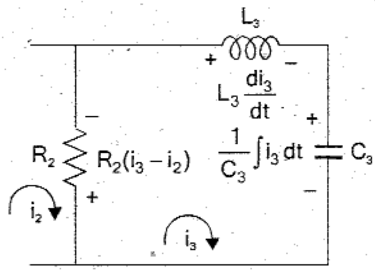


Fig 8.

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 5 are given below (Refer fig 6, 7 and 8).

$$L_1\frac{di_1}{dt}+R_1(i_1-i_2)+\frac{1}{C_1}\int(i_1-i_2)dt=e(t) \qquad .....(7)$$

$$L_2\frac{di_2}{dt}+R_1(i_2-i_1)+R_2(i_2-i_3)+\frac{1}{C_1}\int(i_2-i_1)dt=0 \qquad .....(8)$$

$$L_3\frac{di_3}{dt}+R_2(i_3-i_2)+\frac{1}{C_3}\int i_3dt=0 \qquad .....(9)$$

It is observed that the mesh basis equations (7), (8) and (9) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

TORQUE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has three nodes (J_1, J_2 and J_3). Hence the torque-current analogous electrical circuit will have three nodes. The torque applied to J_1 is represented as a current source connected to node-1 in analogous electrical circuit.

The elements K_1, J_1 and B_1 are connected to first node. Hence they are represented by analogous elements as elements connected to node-1 in analogous electrical circuit. The elements J_2, B_2, B_1 and K_1 are connected to second node. Hence they are represented by analogous elements as elements connected to node-2 in analogous electrical circuit. The elements J_3, B_2, and K_3 are connected to third node. Hence they are represented by analogous elements as elements connected to node-3 in analogous electrical circuit.

The elements K_1 and B_1 are common between node-1 and 2 and so they are represented by analogous element as common elements between node-1 and 2. The element B_2 is common between node-2 and 3 and so it is represented as common element between node-2 and 3 in analogous circuit. The torque-current electrical analogous circuit is shown in fig 9.

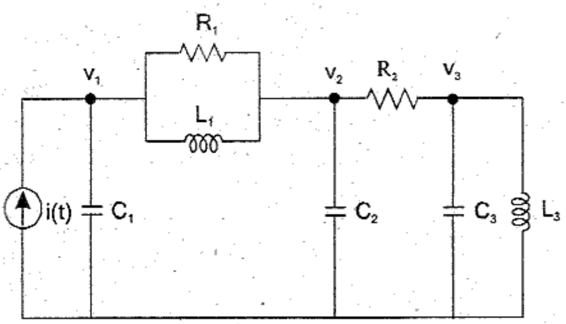


Fig 9 : Torque-current electrical analogous circuit.

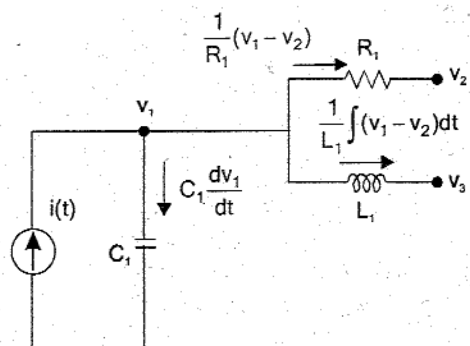


Fig 10.

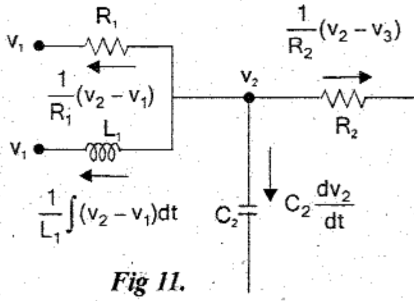


Fig 11.

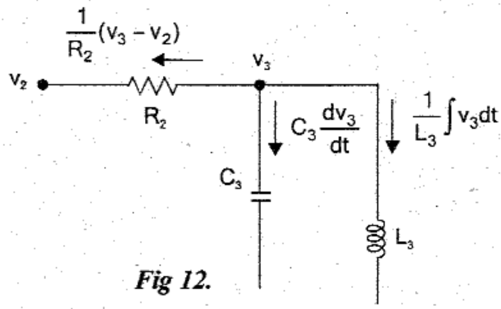


Fig 12.

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow i(t)$	$\omega_1 \rightarrow v_1$	$J_1 \rightarrow C_1$	$B_1 \rightarrow 1/R_1$	$K_1 \rightarrow 1/L_1$
	$\omega_2 \rightarrow v_2$	$J_2 \rightarrow C_2$	$B_2 \rightarrow 1/R_2$	$K_3 \rightarrow 1/L_3$
	$\omega_3 \rightarrow v_3$	$J_3 \rightarrow C_3$		

The node basis equations using Kirchoff's current law for the circuit shown in fig 9 are given below (Refer fig 10, 11 and 12).

$$C_1 \frac{dv_1}{dt} + \frac{1}{R_1}(v_1 - v_2) + \frac{1}{L_1}\int (v_1 - v_2)dt = i(t) \quad .....(10)$$

$$C_2 \frac{dv_2}{dt} + \frac{1}{R_1}(v_2 - v_1) + \frac{1}{R_2}(v_2 - v_3) + \frac{1}{L_1}\int (v_2 - v_1)\,dt = 0 \quad .....(11)$$

$$C_3 \frac{dv_3}{dt} + \frac{1}{R_2}(v_3 - v_2) + \frac{1}{L_3}\int v_3\,dt = 0 \quad .....(12)$$

It is observed that the node basis equations (10), (11) and (12) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

1.11 BLOCK DIAGRAMS

A control system may consist of a number of components. In control engineering to show the functions performed by each component, we commonly use a diagram called the block diagram. A ***block diagram*** of a system is a pictorial representation of the functions performed by each component and of the flow of signals. Such a diagram depicts the interrelationships that exist among the various components. The elements of a block diagram are ***block***, ***branch point*** and ***summing point***.

BLOCK

In a block diagram all system variables are linked to each other through functional blocks. The ***functional block*** or simply ***block*** is a symbol for the mathematical operation on the input signal to the block that produces the output. The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals. Figure 1.25 shows the block diagram of functional block.

The arrowhead pointing towards the block indicates the input, and the arrowhead leading away from the block represents the output. Such arrows are referred to as signals. The output signal from the block is given by the product of input signal and transfer function in the block.

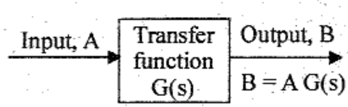


***Fig 1.25 :** Functional block.*

SUMMING POINT

Summing points are used to add two or more signals in the system. Referring to figure 1.26, a circle with a cross is the symbol that indicates a summing operation.

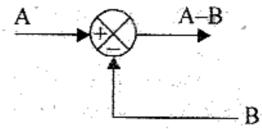


Fig 1.26 : Summing point.

The plus or minus sign at each arrowhead indicates whether the signal is to be added or subtracted. It is important that the quantities being added or subtracted have the same dimensions and the same units.

BRANCH POINT

A ***branch point*** is a point from which the signal from a block goes concurrently to other blocks or summing points.

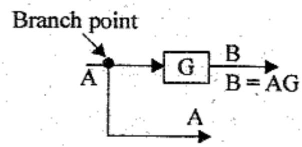


Fig 1.27 : Branch point.

CONSTRUCTING BLOCK DIAGRAM FOR CONTROL SYSTEMS

A control system can be represented diagramatically by block diagram. The differential equations governing the system are used to construct the block diagram. By taking Laplace transform the differential equations are converted to algebraic equations. The equations will have variables and constants. From the working knowledge of the system the input and output variables are identified and the block diagram for each equation can be drawn. Each equation gives one section of block diagram. The output of one section will be input for another section. The various sections are interconnected to obtain the overall block diagram of the system.

EXAMPLE 1.14

Construct the block diagram of armature controlled dc motor.

SOLUTION

The differential equations governing the armature controlled dc motor are (refer section 1.7),

$$V_a = i_a R_a + L_a \frac{di_a}{dt} + e_b \quad .....(1)$$

$$T = K_t i_a \quad .....(2)$$

$$T = J\,\frac{d\omega}{dt} + B\omega \quad .....(3)$$

$$e_b = K_b\,\omega \quad .....(4)$$

$$\omega = \frac{d\theta}{dt} \quad .....(5)$$

On taking Laplace transform of equation (1) we get,

$$V_a(s) = I_a(s)\,R_a + L_a\,s\,I_a(s) + E_b(s) \quad .....(6)$$

In equation (6), $V_a(s)$ and $E_b(s)$ are inputs and $I_a(s)$ is the output. Hence the equation (6) is rearranged and the block diagram for this equation is shown in fig 1.

$$V_a(s) - E_b(s) = I_a(s)\,[R_a + s\,L_a]$$

$$\therefore I_a(s) = \frac{1}{R_a + s\,L_a}\,[V_a(s) - E_b(s)]$$

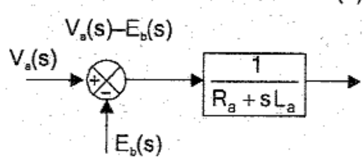


Fig 1.

On taking Laplace transform of equation (2) we get,

$$T(s) = K_t\,I_a(s) \quad .....(7)$$

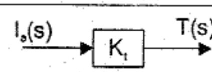


Fig 2.

In equation (7), $I_a(s)$ is the input and T(s) is the output. The block diagram for this equation is shown in fig 2.

On taking Laplace transform of equation (3) we get,

$$T(s) = Js\,\omega(s) + B\,\omega(s) \qquad .....(8)$$

In equation (8), T(s) is the input and $\omega(s)$ is the output. Hence the equation (8) is rearranged and the block diagram fo this equation is shown in fig (3).

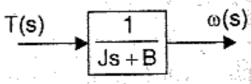


Fig 3.

$$T(s) = (Js + B)\,\omega(s)$$

$$\therefore \omega(s) = \frac{1}{Js + B}\,T(s)$$

On taking Laplace transform of equation (4) we get,

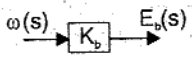


Fig 4.

$$E_b(s) = K_b\,\omega(s) \qquad .....(9)$$

In equation (9), $\omega(s)$ is the input and $E_b(s)$ is the output. The block diagram for this equation is shown in fig 4.

On taking Laplace transform of equation (5) we get,

$$\omega(s) = s\,\theta(s) \qquad .....(10)$$

In equation (10), $\omega(s)$ is the input and $\theta(s)$ is the output. Hence equation (10) is rearranged and the block diagram for th equation is shown in fig 5.

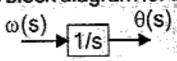


Fig 5.

$$\theta(s) = \frac{1}{s}\,\omega(s)$$

The overall block diagram of armature controlled dc motor is obtained by connecting the various sections shown in fig 1 fig 5. The overall block diagram is shown in fig 6.

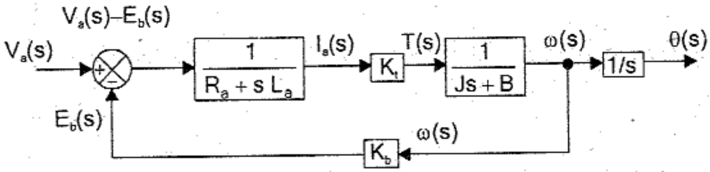


Fig 6 : Block diagram of armature controlled dc motor.

EXAMPLE 1.15

Construct the block diagram of field controlled dc motor.

SOLUTION

The differential equations governing the field controlled dc motor are (refer section 1.8),

$$v_f = R_f\,i_f + L_f\,\frac{di_f}{dt} \qquad .....$$

$$T = K_{tf}\,i_f \qquad .....$$

$$T = J\,\frac{d^2\theta}{dt^2} + B\,\frac{d\theta}{dt} \qquad .....$$

On taking Laplace transform of equation (1) we get,

$$V_f(s) = R_f\,I_f(s) + L_f\,s\,I_f(s) \qquad .....$$

In equation (4), $V_f(s)$ is the input and $I_f(s)$ is the output. Hence the equation (4) is rearranged and the block diagram for is equation is shown in fig 1.

$$V_f(s) = I_f(s)\left[R_f + sL_f\right]$$

$$\therefore I_f(s) = \frac{1}{R_f + s\,L_f}\,V_f(s)$$

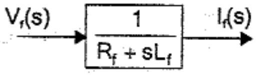


Fig 1.

On taking Laplace transform of equation (2) we get,

$$T(s) = K_{tf}\, I_f(s) \qquad .....(5)$$

In equation (5), $I_f(s)$ is the input and T(s) is the output. The block diagram for this uation is shown in fig 2.

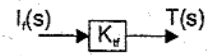


Fig 2.

On taking Laplace transform of equation (3) we get,

$$T(s) = J\,s^2\theta(s) + B\,s\,\theta(s) \qquad .....(6)$$

In equation (6), T(s) is input and $\theta(s)$ is the output. Hence equation (6) is rearranged the block diagram for this equation is shown in fig 3.

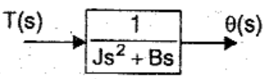


Fig 3.

$$T(s) = (J\,s^2 + Bs)\;\theta(s)$$

$$\therefore \theta(s) = \frac{1}{Js^2 + Bs}\;T(s)$$

The overall block diagram of field controlled dc motor is obtained by connecting the various section shown in fig 1 to The overall block diagram is shown in fig 4.

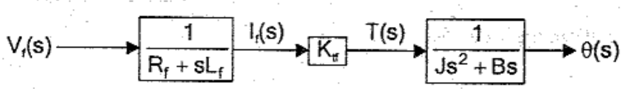


Fig 4 : Block diagram of field controlled dc motor.

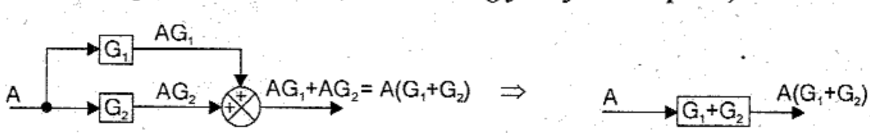
CK DIAGRAM REDUCTION

The block diagram can be reduced to find the overall transfer function of the system. The following rules can be used ock diagram reduction. The rules are framed such that any modification made on the diagram does not alter the input-output ation.

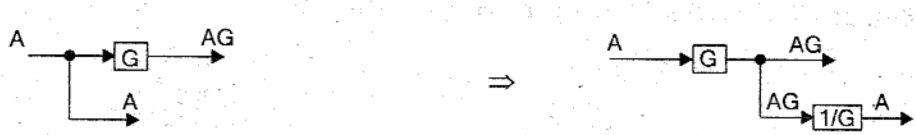
RULES OF BLOCK DIAGRAM ALGEBRA

***Rule-1 :* Combining the blocks in cascade**

***Rule-2 :* Combining Parallel blocks (or combining feed forward paths)**

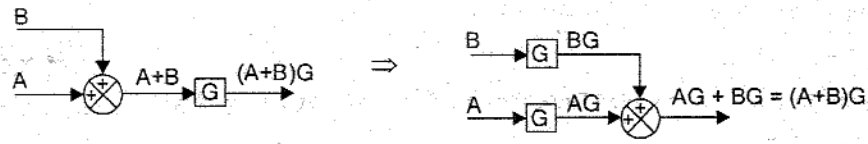


***Rule-3 :* Moving the branch point ahead of the block**

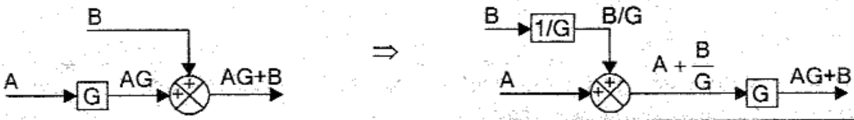


Rule-4 : ***Moving the branch point before the block***

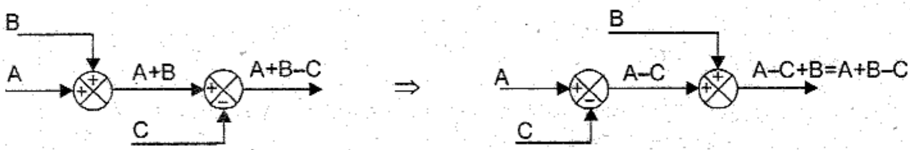
Rule-5 : ***Moving the summing point ahead of the block***



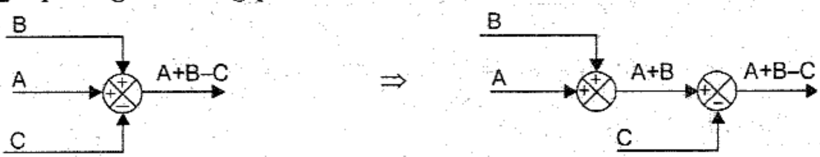
Rule-6 : ***Moving the summing point before the block***



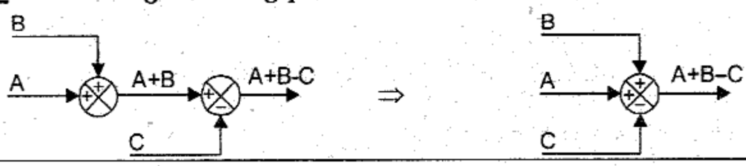
Rule-7 : ***Interchanging summing point***



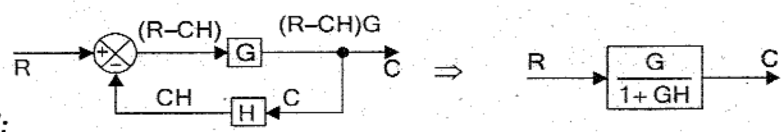
Rule-8 : ***Splitting summing points***



Rule-9 : ***Combining summing points***



Rule-10 : ***Elimination of (negative) feedback loop***

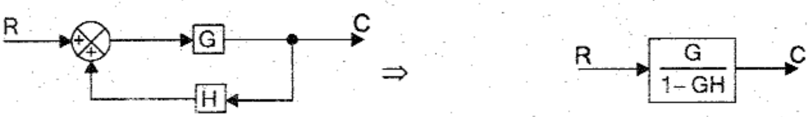


Proof:

$$C = (R - CH)G \quad \Rightarrow \quad C = RG - CHG \quad \Rightarrow \quad C + CHG = RG$$

$$\therefore C(1 + HG) = RG \quad \Rightarrow \quad \frac{C}{R} = \frac{G}{1 + GH}$$

Rule-11 : ***Elimination of (positive) feedback loop***



EXAMPLE 1.16

Reduce the block diagram shown in fig 1 and find C/R.

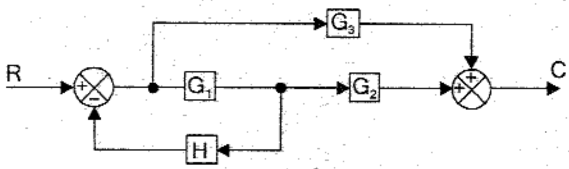
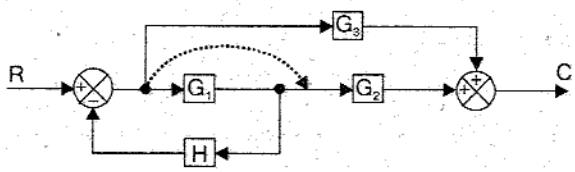


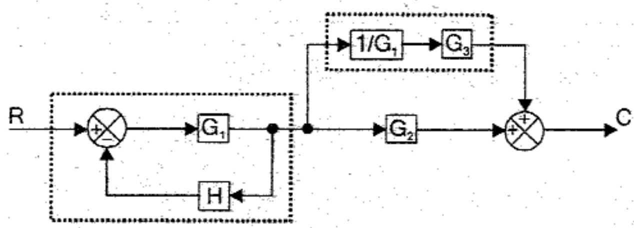
Fig 1.

SOLUTION

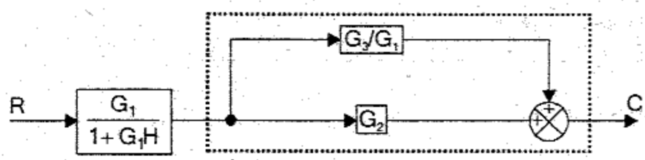
Step 1: Move the branch point after the block.



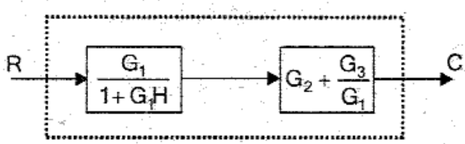
Step 2: Eliminate the feedback path and combining blocks in cascade.



Step 3: Combining parallel blocks



Step 4: Combining blocks in cascade



$$\frac{C}{R}=\left(\frac{G_1}{1+G_1H}\right)\left(G_2+\frac{G_3}{G_1}\right)=\left(\frac{G_1}{1+G_1H}\right)\left(\frac{G_1G_2+G_3}{G_1}\right)=\frac{G_1G_2+G_3}{1+G_1H}$$

RESULT

The overall transfer function of the system, $\frac{C}{R}=\frac{G_1G_2+G_3}{1+G_1H}$

EXAMPLE 1.17

Using block diagram reduction technique find closed loop transfer function of the system whose block diagram is shown in fig 1.

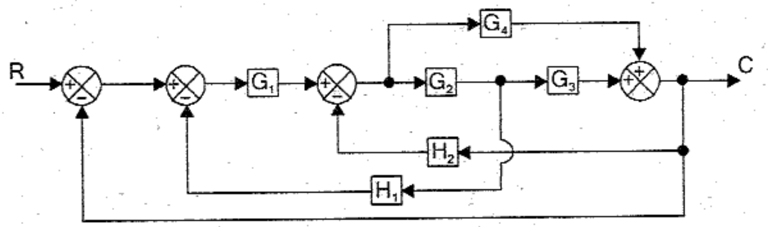
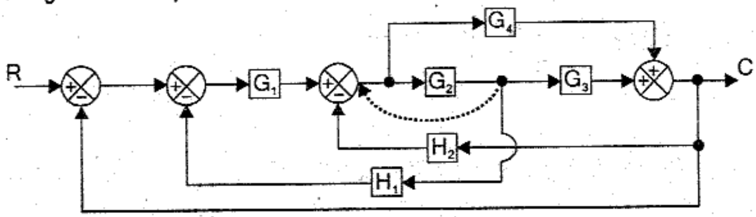


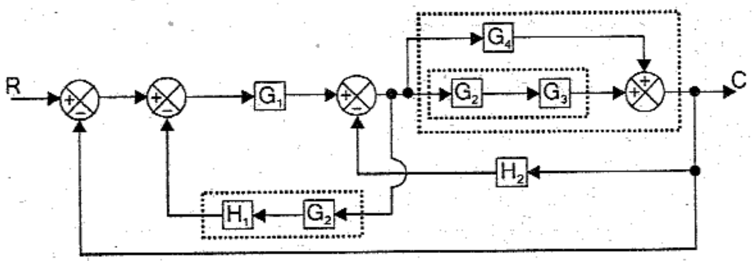
Fig 1.

SOLUTION

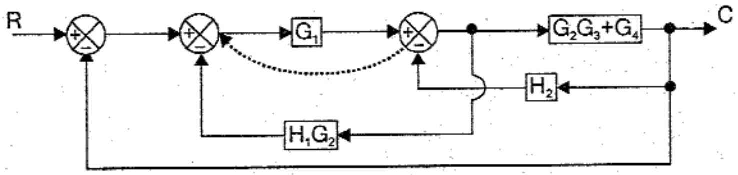
Step 1: Moving the branch point before the block



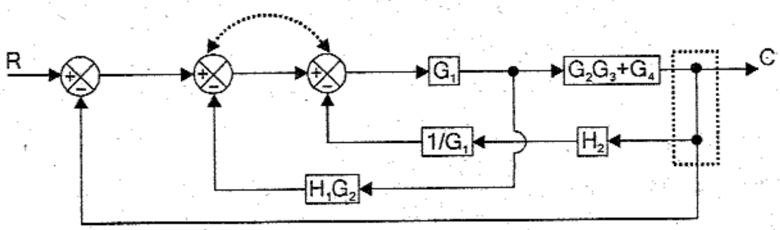
Step 2: Combining the blocks in cascade and eliminating parallel blocks



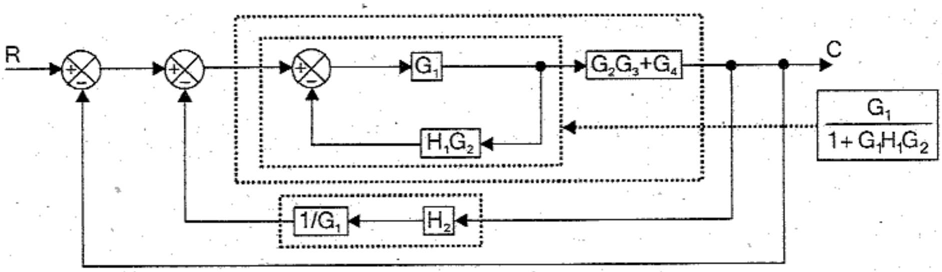
Step 3: Moving summing point before the block.



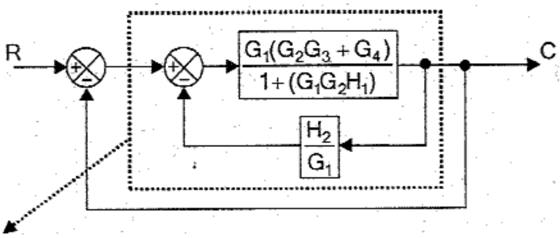
Step 4: Interchanging summing points and modifying branch points.



Step 5: Eliminating the feedback path and combining blocks in cascade

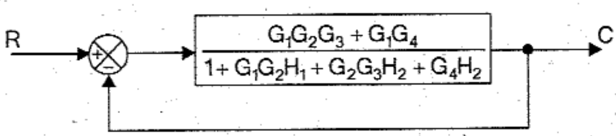


Step 6: Eliminating the feedback path



$$\frac{\dfrac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}}{1+\dfrac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}\,\dfrac{H_2}{G_1}} \Rightarrow \frac{\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1}}{\dfrac{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}{1+G_1G_2H_1}} \Rightarrow \frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}$$

Step 7: Eliminating the feedback path



$$\frac{C}{R}=\frac{\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}{1+\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}=\frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2+G_1G_2G_3+G_1G_4}$$

RESULT

The overall transfer function is given by,

$$\frac{C}{R}=\frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2+G_1G_2G_3+G_1G_4}$$

EXAMPLE 1.18

Determine the overall transfer function $\frac{C(s)}{R(s)}$ for the system shown in fig 1.

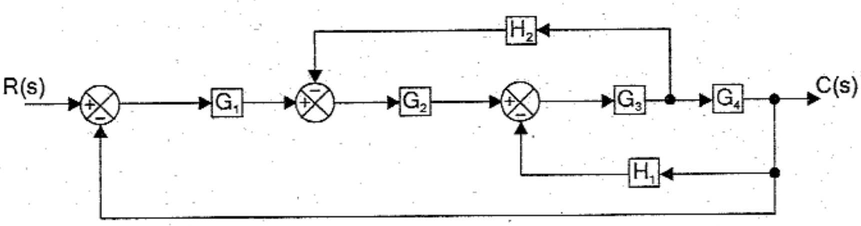
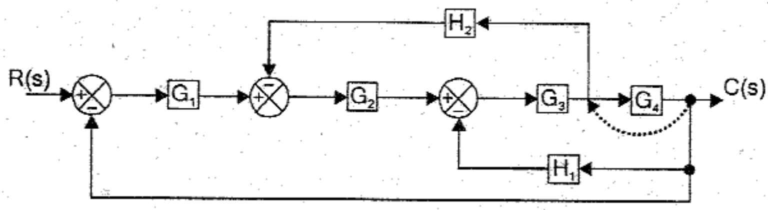


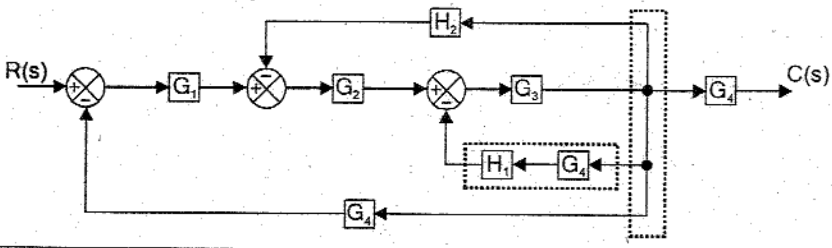
Fig 1.

SOLUTION

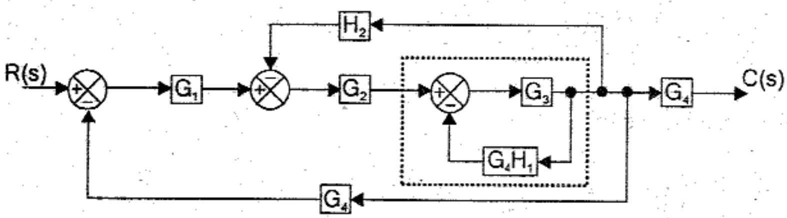
Step 1: Moving the branch point before the block



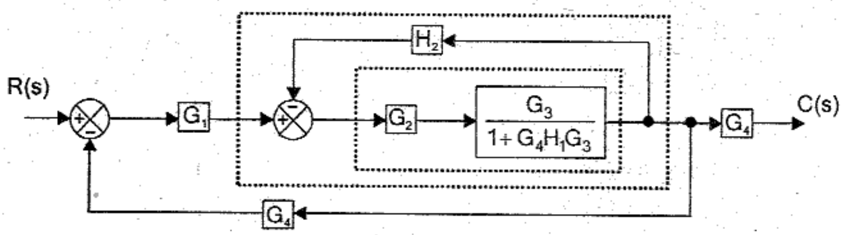
Step 2: Combining the blocks in cascade and rearranging the branch points



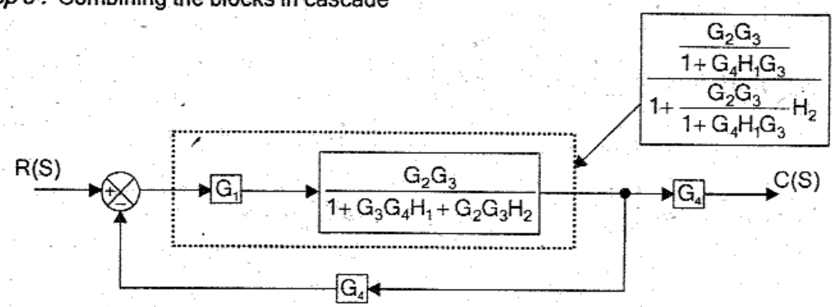
Step 3: Eliminating the feedback path



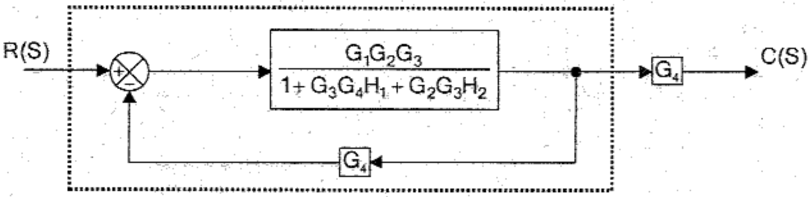
Step 4: Combining the blocks in cascade and eliminating feedback path



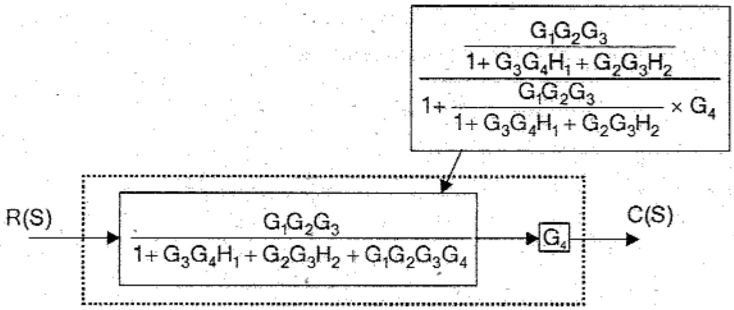
Step 5: Combining the blocks in cascade



Step 6 : Eliminating the feedback path



Step 7 : Combining the blocks in cascade



$$\frac{C(s)}{R(s)}=\frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4}$$

RESULT

The overall transfer function of the system is given by,

$$\frac{C(s)}{R(s)}=\frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4}$$

EXAMPLE 1.19

For the system represented by the block diagram shown in fig 1. Evaluate the closed loop transfer function when the input R is (i) at station-I (ii) at station-II.

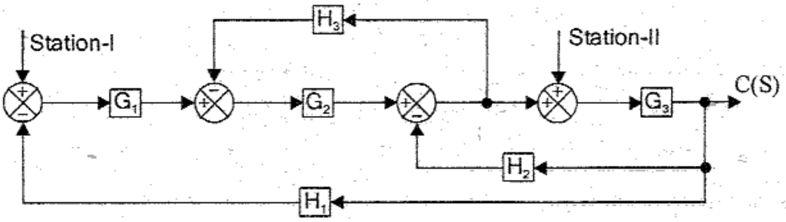


Fig 1.

SOLUTION

(i) Consider the input R is at station-I and so the input at station-II is made zero. Let the output be C_1. Since there is no input at station-II that summing point can be removed and resulting block diagram is shown in fig 2.

Step 1 : Shift the take off point of feedback H_3 beyond G_3 and rearrange the branch points

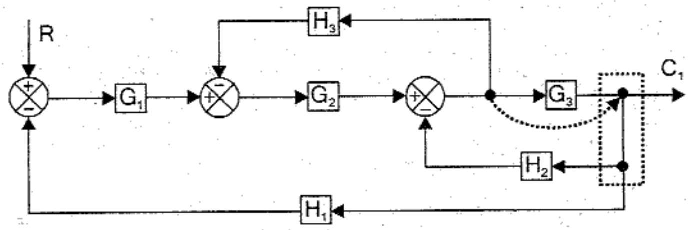
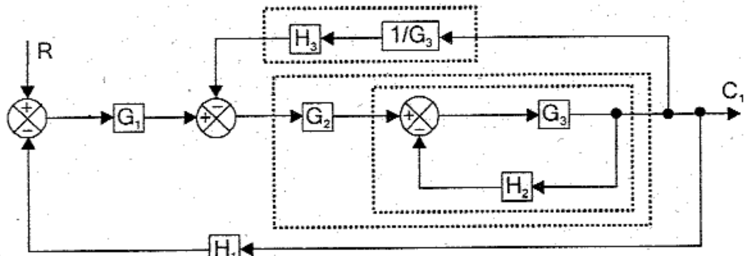
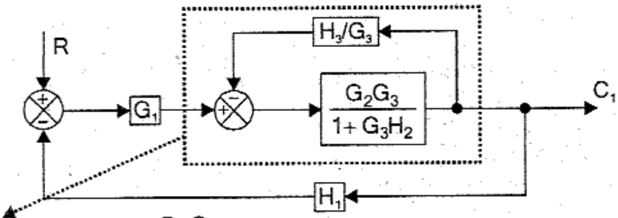


Fig 2.

Step 2 : Eliminating the feedback H_2 and combining blocks in cascade

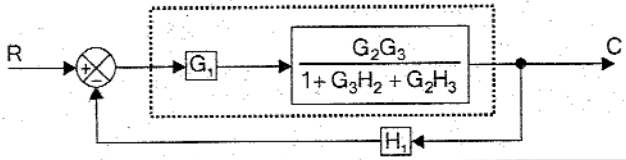


Step 3: Eliminating the feedback path

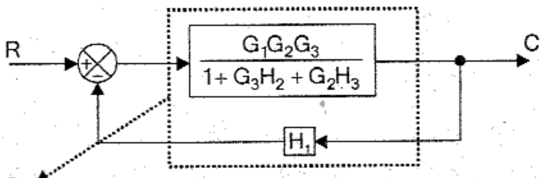


$$\frac{\dfrac{G_2G_3}{1+G_3H_2}}{1+\dfrac{G_2G_3}{1+G_3H_2}\times\dfrac{H_3}{G_3}} = \frac{\dfrac{G_2G_3}{1+G_3H_2}}{1+\dfrac{G_2H_3}{1+G_3H_2}} = \frac{G_2G_3}{1+G_3H_2+G_2H_3}$$

Step 4: Combining the blocks in cascade



Step 5: Eliminating feedback path H_1



$$\frac{\dfrac{G_1G_2G_3}{1+G_3H_2+G_2H_3}}{1+\dfrac{G_1G_2G_3}{1+G_3H_2+G_2H_3}\times H_1} = \frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}$$

$$\therefore \quad \frac{C_1(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}$$

(ii) Consider the input R at station-II, the input at station-I is made zero. Let output be C_2. Since there is no input in station-I that corresponding summing point can be removed and a negative sign can be attached to the feedback path gain H_1. The resulting block diagram is shown in fig 3.

Step 1: Combining the blocks in cascade, shifting the summing point of H_2 before G_2 and rearranging the branch points.

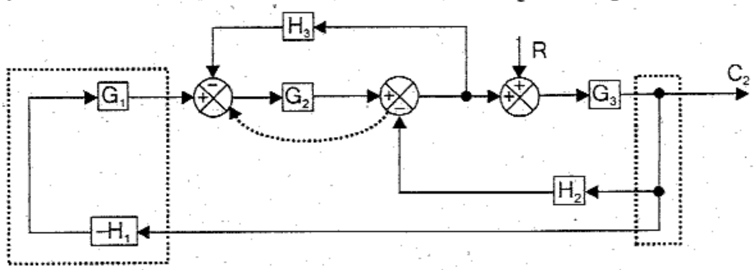
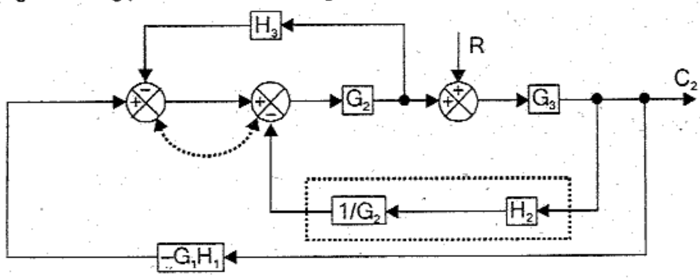
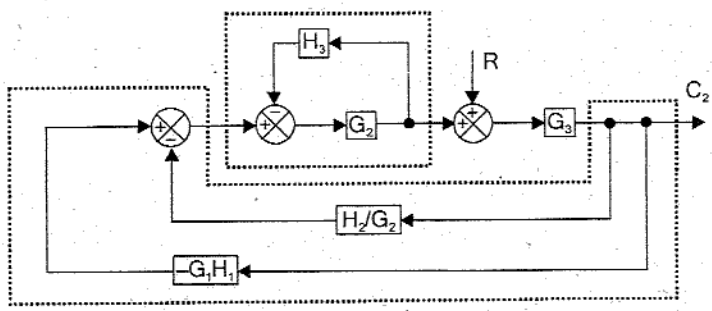


Fig 3.

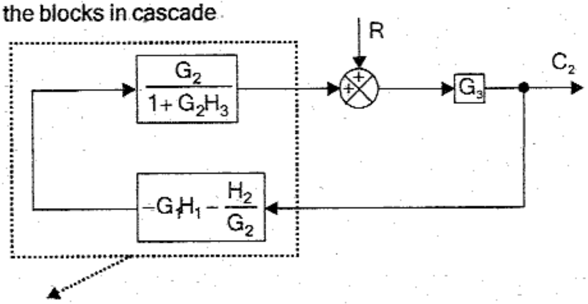
Step 2: Interchanging summing points and combining the blocks in cascade.



Step 3: Combining parallel blocks and eliminating feedback path

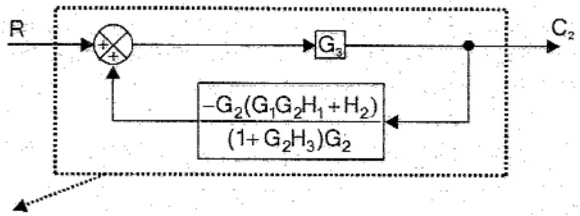


Step 4: Combining the blocks in cascade



$$\left(\frac{G_2}{1+G_2H_3}\right) \times \left(-G_1H_1 - \frac{H_2}{G_2}\right) = \left(\frac{G_2}{1+G_2H_3}\right) \times \left(\frac{-G_1H_1G_2 - H_2}{G_2}\right) = \frac{-G_2(G_1G_2H_1 + H_2)}{(1+G_2H_3)G_2}$$

Step 5: Eliminating the feedback path



$$\frac{G_3}{1-\left(\frac{-(G_1G_2H_1+H_2)}{1+G_2H_3}\right)G_3} = \frac{G_3}{\frac{1+G_2H_3+G_3(G_1G_2H_1+H_2)}{1+G_2H_3}} = \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_3(G_1G_2H_1+H_2)}$$

$$\therefore \; \frac{C_2}{R} = \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_3(G_1G_2H_1+H_2)}$$

RESULT

The transfer function of the system with input at station-I is,

$$\frac{C_1}{R} = \frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}$$

The transfer function of the system with input at station-II is,

$$\frac{C_2}{R} = \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_3(G_1G_2H_1+H_2)}$$

EXAMPLE 1.20

For the system represented by the block diagram shown in the fig 1, determine C_1/R_1 and C_2/R_1.

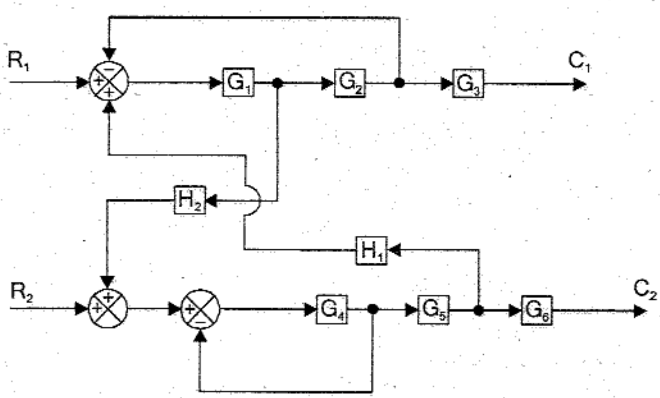


Fig 1

SOLUTION

Case (i) To find $\frac{C_1}{R_1}$

In this case set $R_2 = 0$ and consider only one output C_1. Hence we can remove the summing point which adds R_2 and need not consider G_6, since G_6 is on the open path. The resulting block diagram is shown in fig 2.

Step 1: Eliminating the feedback path

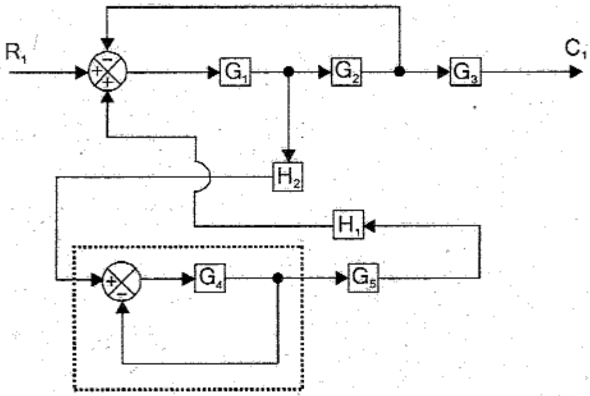
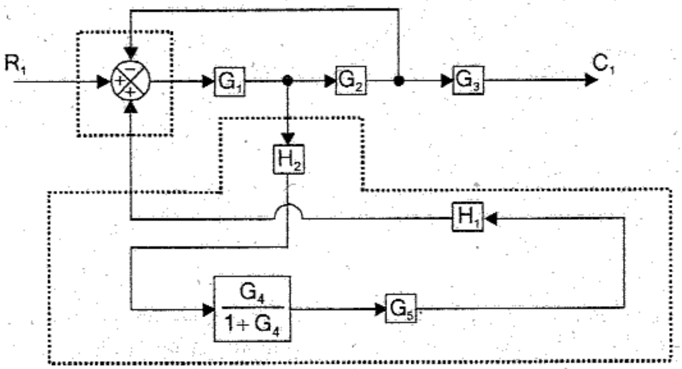
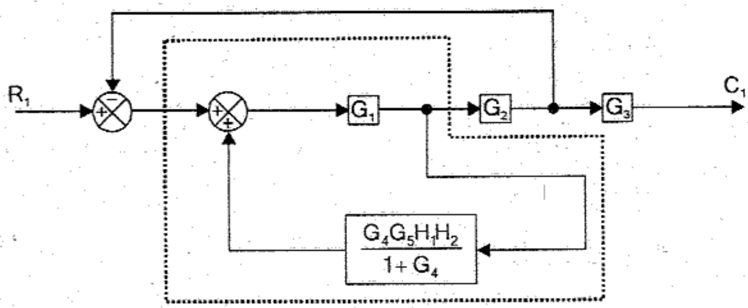


Fig 2.

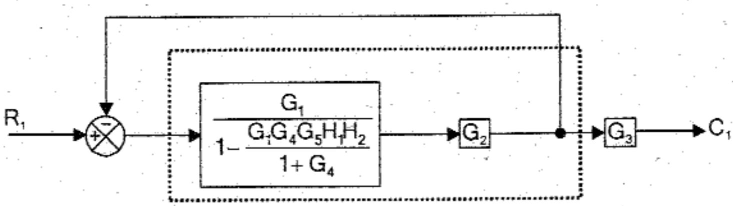
Step 2: Combining the blocks in cascade and splitting the summing point



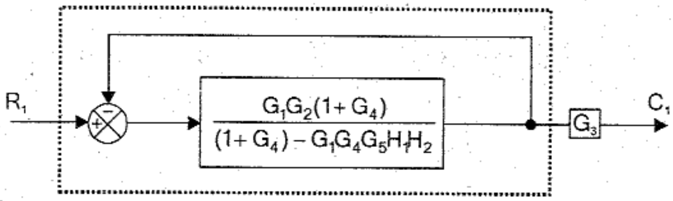
Step 3: Eliminating the feedback path

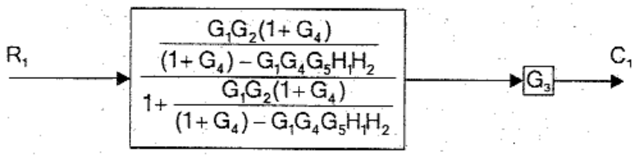


Step 4: Combining the blocks in cascade

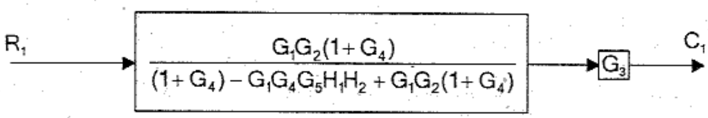


Step 5: Eliminating the feedback path





Step 6: Combining the blocks in cascade



$$\frac{C_1}{R_1}=\frac{G_1G_2G_3(1+G_4)}{(1+G_1G_2)\,(1+G_4)-G_1G_4G_5H_1H_2}$$

Case 2 : To find $\frac{C_2}{R_1}$

In this case set $R_2 = 0$ and consider only one output C_2. Hence we can remove the summing point which adds R_2 and need not consider G_3, since G_3 is on the open path. The resulting block diagram is shown in fig 3.

Step 1: Eliminate the feedback path.

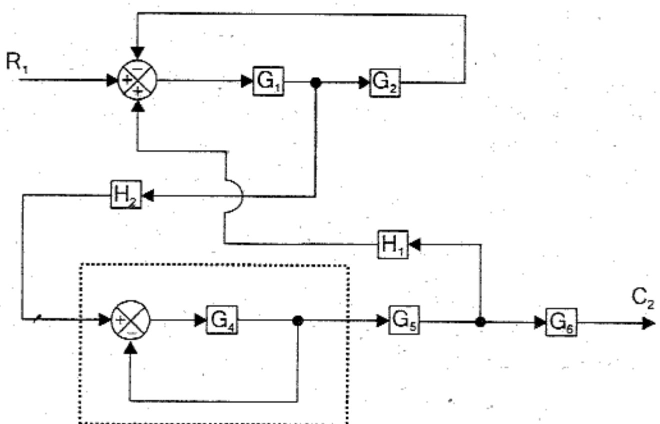
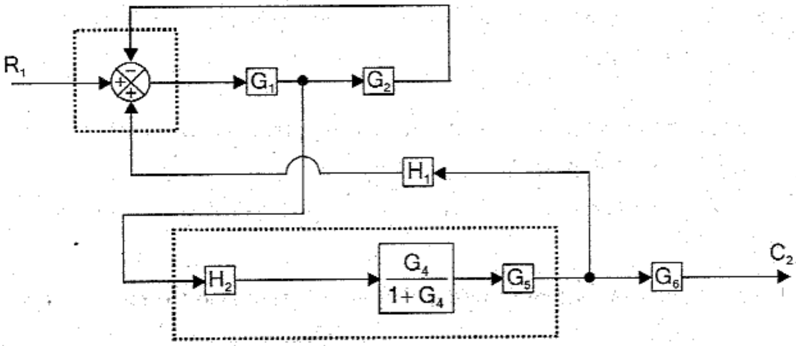
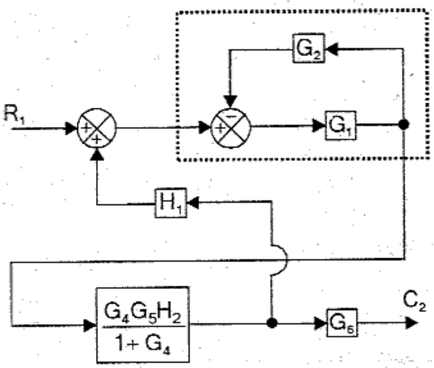


Fig 3.

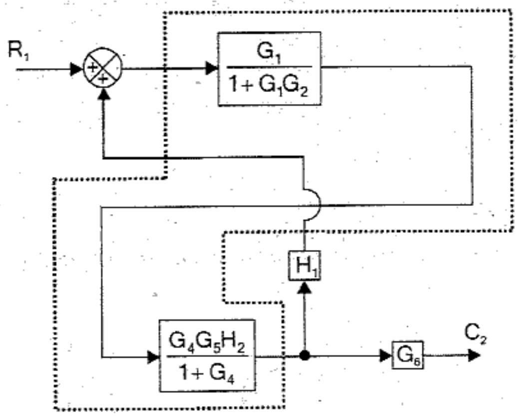
Step 2: Combining blocks in cascade and splitting the summing point



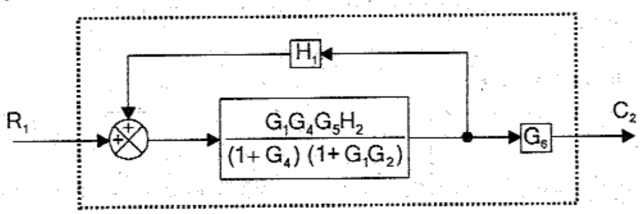
Step 3: Eliminating the feedback path

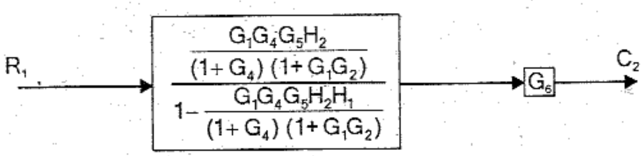


Step 4 : Combining the blocks in cascade

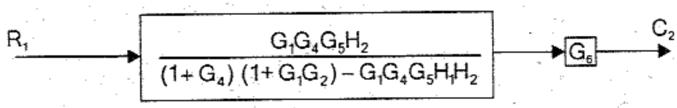


Step 5: Eliminating the feedback path





Step 6: Combining the blocks in cascade



$$\frac{C_2}{R_1} = \frac{G_1G_4G_5G_6H_2}{(1+G_4)\,(1+G_1G_2) - G_1G_4G_5H_1H_2}$$

RESULT

The transfer function of the system when the input and output are R_1 and C_1 is given by,

$$\frac{C_1}{R_1} = \frac{G_1G_2G_3\,(1+G_4)}{(1+G_1G_2)\,(1+G_4) - G_1G_4G_5H_1H_2}$$

The transfer function of the system when the input and output are R_1 and C_2 is given by,

$$\frac{C_2}{R_1} = \frac{G_1G_4G_5G_6\,H_2}{(1+G_4)\,(1+G_1G_2) - G_1G_4G_5H_1H_2}$$

EXAMPLE 1.21

Obtain the closed loop transfer function C(s)/R(s) of the system whose block diagram is shown in fig 1.

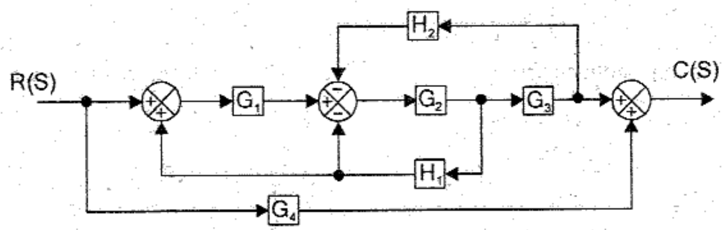
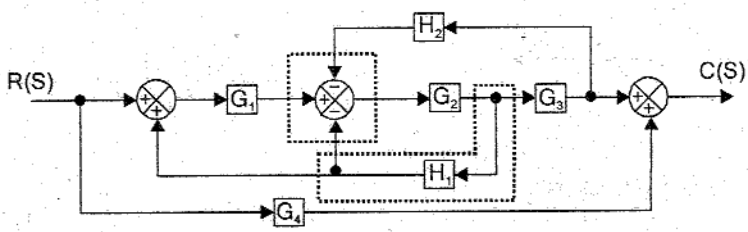


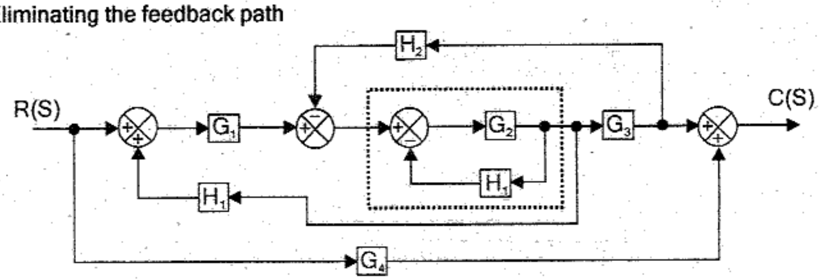
Fig 1.

SOLUTION

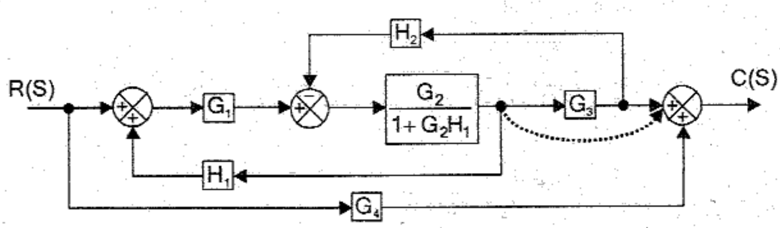
Step 1 : Splitting the summing point and rearranging the branch points



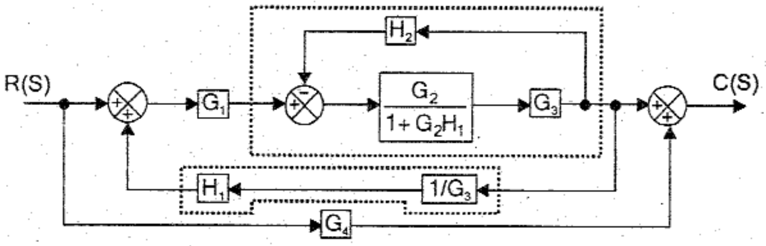
Step 2 : Eliminating the feedback path



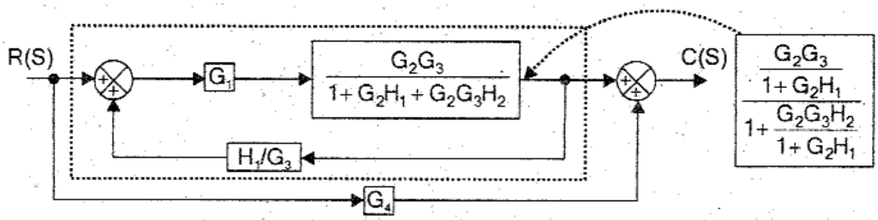
Step 3 : Shifting the branch point after the block.



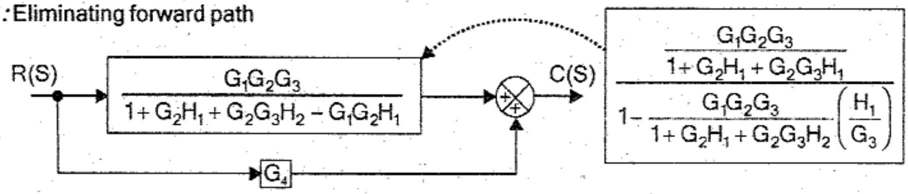
Step 4 : Combining the blocks in cascade and eliminating feedback path



Step 5 : Combining the blocks in cascade and eliminating feedback path



Step 6 : Eliminating forward path



$$\therefore \frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1} + G_4$$

RESULT

The transfer function of the system is $\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1} + G_4$

EXAMPLE 1.22

The block diagram of a closed loop system is shown in fig 1. Using the block diagram reduction technique determine the closed loop transfer function C(s)/R(s).

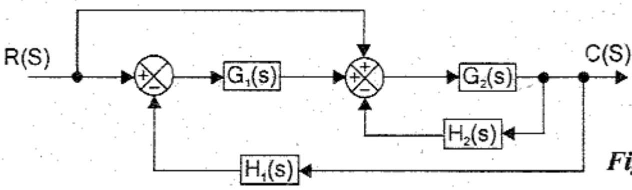
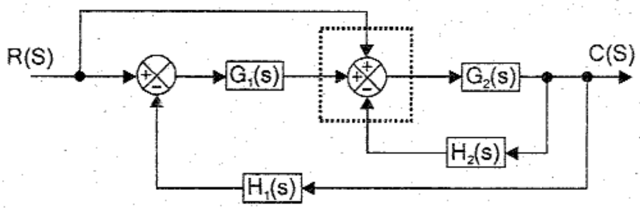


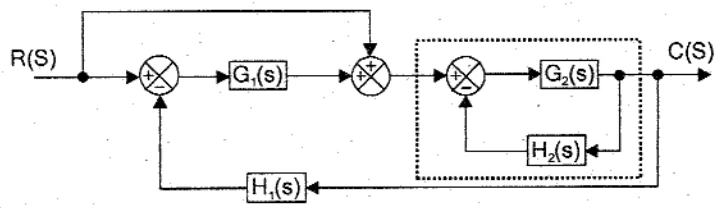
Fig 1.

SOLUTION

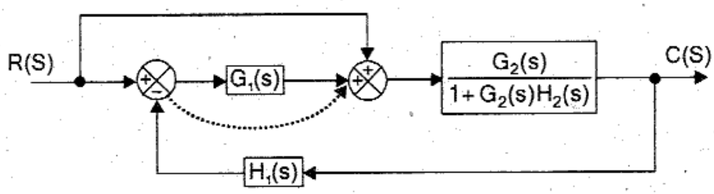
Step 1 : Splitting the summing point.



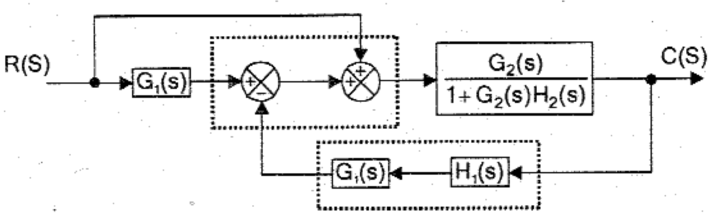
Step 2 : Eliminating the feedback path.



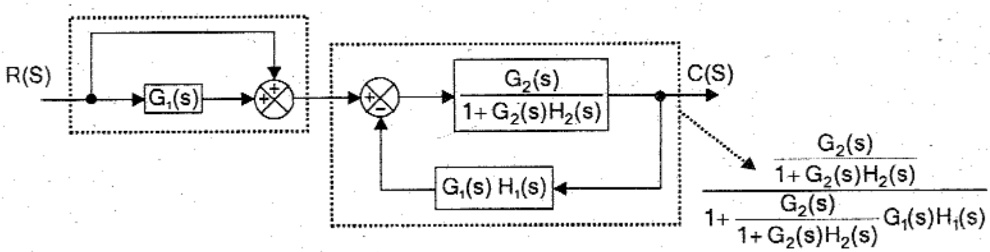
Step 3 : Moving the summing point after the block.



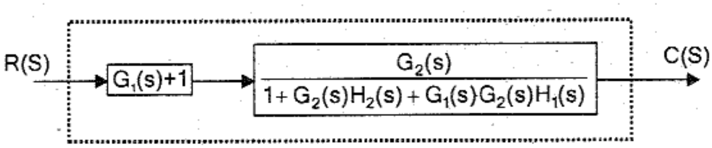
Step 4 : Interchanging the summing points and combining the blocks in cascade



Step 5 : Eliminating the feedback path and feed forward path



Step 6 : Combining the blocks in cascade



$$\therefore \frac{C(s)}{R(s)} = \frac{G_2(s)\left[G_1(s)+1\right]}{1+G_2(s)\,H_2(s)+G_1(s)\,G_2(s)\,H_1(s)}$$

RESULT

The transfer function of the system is,

$$\frac{C(s)}{R(s)} = \frac{G_2(s)\left[G_1(s)+1\right]}{1+G_2(s)\,H_2(s)+G_1(s)\,G_2(s)\,H_1(s)}$$

EXAMPLE 1.23

Using block diagram reduction technique find the transfer function C(s)/R(s) for the system shown in fig 1.

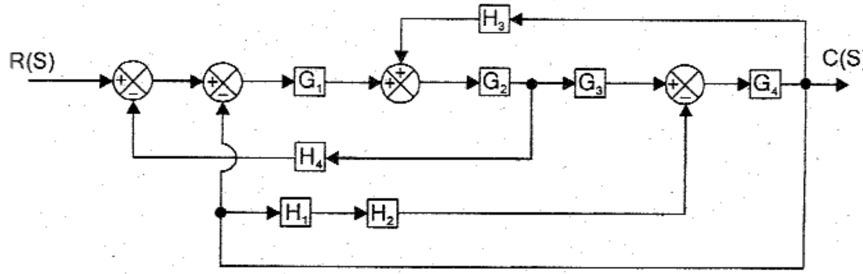
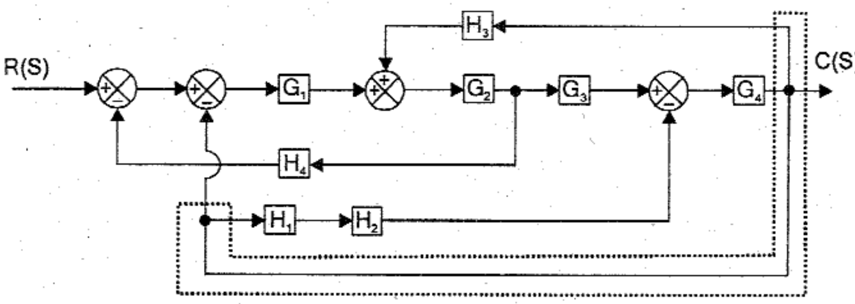


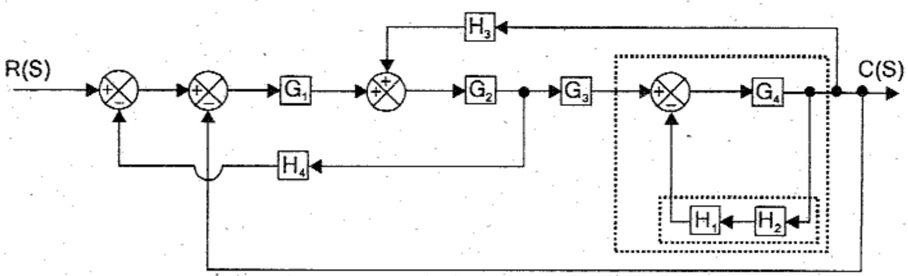
Fig 1.

SOLUTION

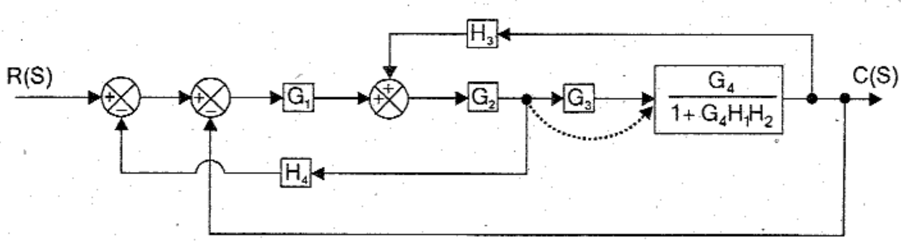
Step 1 : Rearranging the branch points



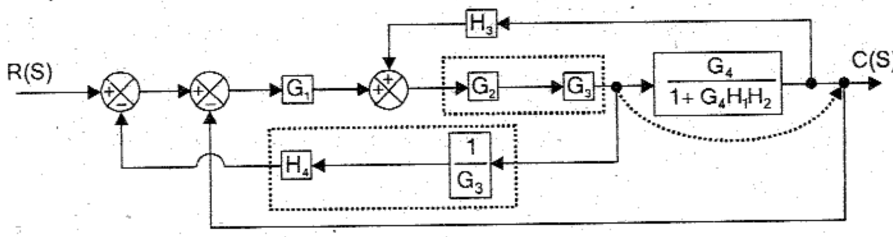
Step 2 : Combining the blocks in cascade and eliminating the feedback path.



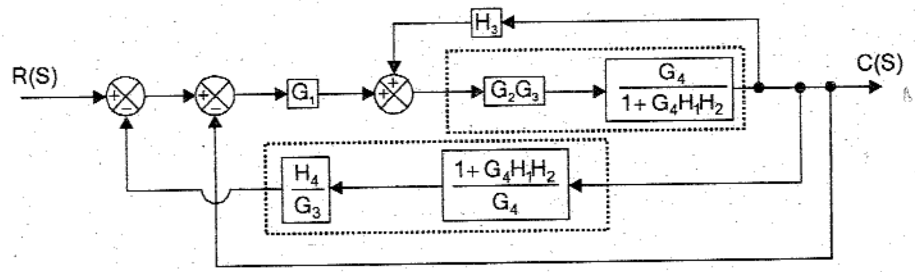
Step 3 : Moving the branch point after the block.



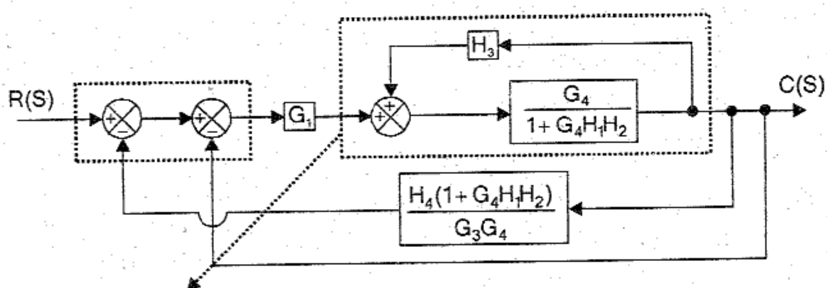
Step 4 : Moving the branch point and combining the blocks in cascade.



Step 5 : Combining the blocks in cascade

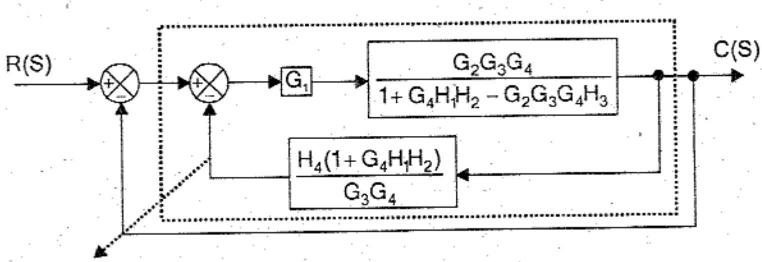


Step 6 : Eliminating feedback path and interchanging the summing points.



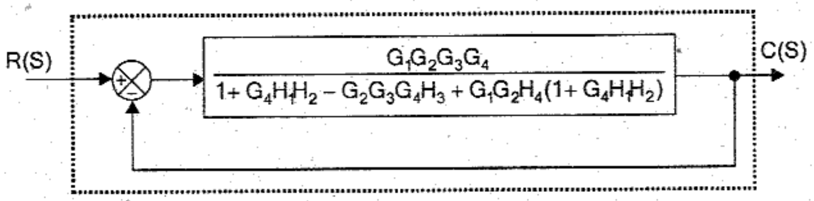
$$\frac{\dfrac{G_2G_3G_4}{1+G_4H_1H_2}}{1-\dfrac{G_2G_3G_4H_3}{1+G_4H_1H_2}} = \frac{G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}$$

Step 7 : Combining the blocks in cascade and eliminating the feedback path



$$\frac{\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}}{1+\left(\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}\right)\left(\dfrac{H_4(1+G_4H_1H_2)}{G_3G_4}\right)} = \frac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}$$

Step 8 : Eliminating the unity feedback path.



$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}{1+\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}$$

$$= \frac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)+G_1G_2G_3G_4}$$

$$= \frac{G_1G_2G_3G_4}{1+H_1H_2(G_4+G_1G_2G_4H_4)+G_1G_2(H_4+G_3G_4)-G_2G_3G_4H_3}$$

RESULT

The transfer function of the system is,

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1+H_1H_2(G_4+G_1G_2G_4H_4)+G_1G_2(H_4+G_3G_4)-G_2G_3G_4H_3}$$

1.12 BLOCK DIAGRAM REDUCTION USING MATLAB

TRANSFER FUNCTION OF A SYSTEM

Let, G(s) be the transfer function of a system. When the transfer function is a rational function of "s", then using MATLAB the transfer function can be obtained from the coefficients of the numerator and denominator polynomials as shown below. Let, the general form of G(s) be as shown below.

$$G(s) = \frac{b_0s^M + b_1s^{M-1} + b_2s^{M-2} + \dots\dots\dots + b_{M-1}s + b_M}{a_0s^N + a_1s^{N-1} + a_2s^{N-2} + \dots\dots\dots + a_{N-1}s + a_N}$$

First, the coefficients of the numerator and denominator polynomials are declared as two arrays as shown below.

```
num_cof  =  [b0  b1  b2  ........  bM];
den_cof  =  [a0  a1  a2  ........  aN];
```

Next, the transfer can be obtained using the following commands of MATLAB.

```
G  =  tf('s');
G  =  ([num_cof],  [den_cof])
```

TRANSFER FUNCTION OF CASCADE / PARALLEL / FEEDBACK SYSTEM

Consider two systems with transfer functions $G_1(s)$ and $G_2(s)$. Let the two transfer functions be rational function of "s" as shown below.

$$G_1(s) = \frac{b_0s^M + b_1s^{M-1} + b_2s^{M-2} + \dots\dots\dots + b_{M-1}s + b_M}{a_0s^N + a_1s^{N-1} + a_2s^{N-2} + \dots\dots\dots + a_{N-1}s + a_N}$$

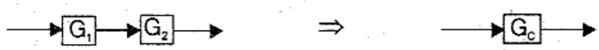
$$G_2(s) = \frac{d_0 s^M + d_1 s^{M-1} + d_2 s^{M-2} + \dots\dots\dots + d_{M-1} s + d_M}{c_0 s^N + c_1 s^{N-1} + c_2 s^{N-2} + \dots\dots\dots + c_{N-1} s + c_N}$$

When the two systems are connected as cascade / parallel / feedback system, then the over transfer function of cascaded system / parallel system / feedback system can be obtained using MATLA

In order to obtain the overall transfer function, first the coefficients of the numerator and denomina polynomials of $G_1(s)$ and $G_2(s)$ are declared as arrays as shown below.

```
num_cof1 = [b0 b1 b2 ........ bM];
den_cof1 = [a0 a1 a2 ........ aN];
num_cof2 = [d0 d1 d2 ........ dM];
den_cof2 = [c0 c1 c2 ........ cN];
```

When the two systems are connected in cascade as shown below, then the overall transfer functio $G_C(s)$ of the cascaded system can be obtained using the following commands of MATLAB.

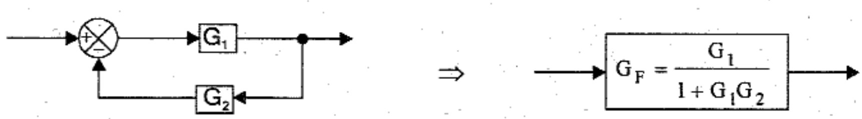



```
GC = tf('s');
[num_cofC, den_cofC] = series(num_cof1, den_cof1, num_cof2, den_cof2);
GC = ([num_cofC], [den_cofC])
```

When the two systems are connected in parallel as shown below, then the overall transfer functio $G_P(s)$ of parallel system can be obtained using the following commands of MATLAB.


```
GP = tf('s');
[num_cofP, den_cofP]=parallel(num_cof1, den_cof1, num_cof2, den_cof2);
GP = ([num_cofP], [den_cofP])
```

When the two systems are connected in feedback as shown below, then the overall trans function $G_F(s)$ of feedback system can be obtained using the following commands of MATLAB.




```
GF = tf('s');
[num_cofF, den_cofF] = feedback(num_cof1, den_cof1, num_cof2, den_cof2)
GF = ([num_cofF], [den_cofF])
```

PROGRAM 1.1

Consider the transfer functions of the two systems given below.

$G_1(s)=8/(s^2+2s+9)$ and $G_2(s)=4/(s+6)$

Write a MATLAB program to find the overall transfer function if the two systems are connected as cascade system, parallel system and feedback system.

```
clc
clear all
G1=tf('s'); G2=tf('s'); GC=tf('s');GP=tf('s');GF=tf('s');
num_cof1=[0 0 8];
den_cof1=[1 2 9];
disp('System1');
G1=tf([num_cof1], [den_cof1])
num_cof2=[0 4];
den_cof2=[1 6];
disp('System2');
G2=tf([num_cof2], [den_cof2])
[num_cofC,den_cofC]=series(num_cof1,den_cof1,num_cof2,den_cof2);
disp('Cascade system');
GC=tf([num_cofC], [den_cofC])
[num_cofP,den_cofP]=parallel(num_cof1,den_cof1,num_cof2,den_cof2);
disp('Parallel system');
GP=tf([num_cofP], [den_cofP])
[num_cofF,den_cofF]=feedback(num_cof1,den_cof1,num_cof2,den_cof2);
disp('Feedback system');
GF=tf([num_cofF], [den_cofF])
```

OUTPUT

```
System1
Transfer function:
          8
    -------------
    s^2 + 2 s + 9

System2
Transfer function:
      4
    -----
    s + 6

Cascade system
Transfer function:
                 32
    ----------------------
    s^3 + 8 s^2 + 21 s + 54

Parallel system
Transfer function:
         4 s^2 + 16 s + 84
    ----------------------
    s^3 + 8 s^2 + 21 s + 54

Feedback system
Transfer function:
              8 s + 48
    ----------------------
    s^3 + 8 s^2 + 21 s + 86
```

1.13 SIGNAL FLOW GRAPH

The signal flow graph is used to represent the control system graphically and it was developed by **S.J. Mason.**

A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. By taking Laplace transform, the time domain differential equations governing a control system can be transferred to a set of algebraic equations in s-domain. The signal flow graph of the system can be constructed using these equations.

It should be noted that the signal flow graph approach and the block diagram approach yield the same information. The advantage in signal flow graph method is that, using Mason's gain formula the overall gain of the system can be computed easily. This method is simpler than the tedious block diagram reduction techniques.

The signal flow graph depicts the flow of signals from one point of a system to another and gives the relationships among the signals. A signal flow graph consists of a network in which nodes are connected by directed branches. Each node represents a system variable and each branch connected between two nodes acts as a signal multiplier. Each branch has a gain or transmittance. When the signal pass through a branch, it gets multiplied by the gain of the branch.

In a signal flow graph, the signal flows in only one direction. The direction of signal flow is indicated by an arrow placed on the branch and the gain (multiplication factor) is indicated along the branch.

EXPLANATION OF TERMS USED IN SIGNAL FLOW GRAPH

Node	:	A node is a point representing a variable or signal.
Branch	:	A branch is directed line segment joining two nodes. The arrow on the branch indicates the direction of signal flow and the gain of a branch is the transmittance.
Transmittance	:	The gain acquired by the signal when it travels from one node to another is called transmittance. The transmittance can be real or complex.
Input node (Source)	:	It is a node that has only outgoing branches.
Output node (Sink)	:	It is a node that has only incoming branches.
Mixed node	:	It is a node that has both incoming and outgoing branches.
Path	:	A path is a traversal of connected branches in the direction of the branch arrows. The path should not cross a node more than once.
Open path	:	A open path starts at a node and ends at another node.
Closed path	:	Closed path starts and ends at same node.
Forward path	:	It is a path from an input node to an output node that does not cross any node more than once.
***Forward path* gain**	:	It is the product of the branch transmittances (gains) of a forward path.
Individual loop	:	It is a closed path starting from a node and after passing through a certain part of a graph arrives at same node without crossing any node more than once.
Loop gain	:	It is the product of the branch transmittances (gains) of a loop.
Non-touching Loops	:	If the loops does not have a common node then they are said to be non-touching loops.

PROPERTIES OF SIGNAL FLOW GRAPH

The basic properties of signal flow graph are the following :

(i) The algebraic equations which are used to construct signal flow graph must be in the form of cause and effect relationship.

(ii) Signal flow graph is applicable to linear systems only.

(iii) A node in the signal flow graph represents the variable or signal.

(iv) A node adds the signals of all incoming branches and transmits the sum to all outgoing branches.

(v) A mixed node which has both incoming and outgoing signals can be treated as an output node by adding an outgoing branch of unity transmittance.

(vi) A branch indicates functional dependence of one signal on the other.

(vii) The signals travel along branches only in the marked direction and when it travels it gets multiplied by the gain or transmittance of the branch.

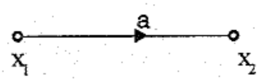
(viii) The signal flow graph of system is not unique. By rearranging the system equations different types of signal flow graphs can be drawn for a given system.

SIGNAL FLOW GRAPH ALGEBRA

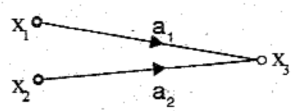
Signal flow graph for a system can be reduced to obtain the transfer function of the system using the following rules. The guideline in developing the rules for signal flow graph algebra is that the signal at a node is given by sum of all incoming signals.

Rule 1 **:** Incoming signal to a node through a branch is given by the product of a signal at previous node and the gain of the branch.

Example:



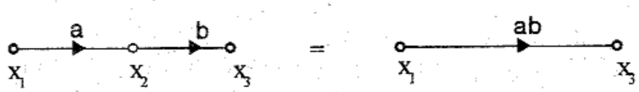
$x_2 = ax_1$



$x_3 = a_1x_1 + a_2x_2$

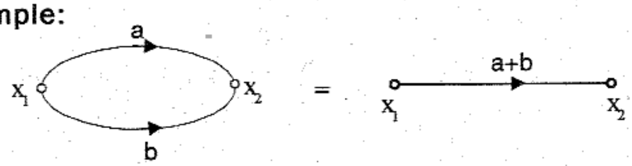
Rule 2 **:** Cascaded branches can be combined to give a single branch whose transmittance is equal to the product of individual branch transmittance.

Example:



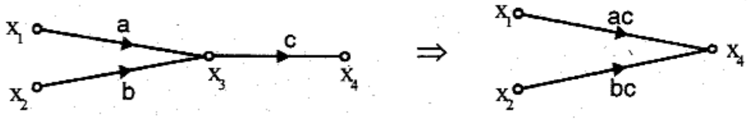
Rule 3 **:** Parallel branches may be represented by single branch whose transmittance is the sum of individual branch transmittances.

Example:



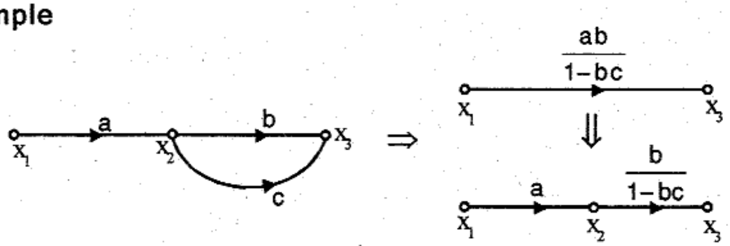
Rule 4 **:** A mixed node can be eliminated by multiplying the transmittance of outgoing branch (from the mixed node) to the transmittance of all incoming branches to the mixed node.

Example



Rule 5 **:** A loop may be eliminated by writing equations at the input and output node and rearranging the equations to find the ratio of output to input. This ratio gives the gain of resultant branch.

Example



***Proof*:**

$x_2 = ax_1 + cx_3 \ ; \quad x_3 = bx_2$

Put, $x_2 = ax_1 + cx_3$ in the equation for x_3.

$\therefore x_3 = b\,(ax_1 + cx_3) \ \Rightarrow \ x_3 = abx_1 + bcx_3 \ \Rightarrow \ x_3 - bc\,x_3 = ab\,x_1 \ \Rightarrow x_3(1-bc) = ab\,x_1$

$$\therefore \frac{x_3}{x_1} = \frac{ab}{1-bc}$$

SIGNAL FLOW GRAPH REDUCTION

The signal flow graph of a system can be reduced either by using the rules of a signal flow graph algebra or by using Mason's gain formula.

For signal flow graph reduction using the rules of signal flow graph, write equations at every node and then rearrange these equations to get the ratio of output and input (transfer function).

The signal flow graph reduction by above method will be time consuming and tedious. **S.J.Mason** has developed a simple procedure to determine the transfer function of the system represented as a signal flow graph. He has developed a formula called by his name **Mason's gain formula** which can be directly used to find the transfer function of the system.

MASON'S GAIN FORMULA

The Mason's gain formula is used to determine the transfer function of the system from the signal flow graph of the system.

Let, $R(s)$ = Input to the system

$C(s)$ = Output of the system

Now, Transfer function of the system, $T(s) = \dfrac{C(s)}{R(s)}$ .....(1.34)

Mason's gain formula states the overall gain of the system [transfer function] as follows,

Overall gain, $T = \dfrac{1}{\Delta} \sum\limits_K P_K \Delta_K$ .....(1.35) where, T = $T(s)$ = Transfer function of the system

P_K = Forward path gain of K^{th} forward path

K = Number of forward paths in the signal flow graph

$$\Delta = 1 - (\text{Sum of individual loop gains}) + \begin{pmatrix}\text{Sum of gain products of all possible} \\ \text{combinations of two non - touching loops}\end{pmatrix} - \begin{pmatrix}\text{Sum of gain products of all possible} \\ \text{combinations of three non - touching loops}\end{pmatrix} + \ldots\ldots$$

Δ_K = Δ for that part of the graph which is not touching K^{th} forward path

CONSTRUCTING SIGNAL FLOW GRAPH FOR CONTROL SYSTEMS

A control system can be represented diagrammatically by signal flow graph. The differential equations governing the system are used to construct the signal flow graph.The following procedure can be used to construct the signal flow graph of a system.

1. Take Laplace transform of the differential equations governing the system in order to convert them to algebraic equations in s-domain.
2. The constants and variables of the s-domain equations are identified.
3. From the working knowledge of the system, the variables are identified as input, output and intermediate variables.
4. For each variable a node is assigned in signal flow graph and constants are assigned as the gain or transmittance of the branches connecting the nodes.
5. For each equation a signal flow graph is drawn and then they are interconnected to give overall signal flow graph of the system.

PROCEDURE FOR CONVERTING BLOCK DIAGRAM TO SIGNAL FLOW GRAPH

The signal flow graph and block diagram of a system provides the same information but there is no standard procedure for reducing the block diagram to find the transfer function of the system. Also the block diagram reduction technique will be tedious and it is difficult to choose the rule to be applied for simplification. Hence it will be easier if the block diagram is converted to signal flow graph and **Mason's gain formula** is applied to find the transfer function. The following procedure can be used to convert block diagram to signal flow graph.

1. Assume nodes at input, output, at every summing point, at every branch point and in between cascaded blocks.
2. Draw the nodes separately as small circles and number the circles in the order 1, 2, 3, 4, ... etc.
3. From the block diagram find the gain between each node in the main forward path and connect all the corresponding circles by straight line and mark the gain between the nodes.
4. Draw the feed forward paths between various nodes and mark the gain of feed forward path along with sign.
5. Draw the feedback paths between various nodes and mark the gain of feedback paths along with sign.

EXAMPLE 1.24

Construct a signal flow graph for armature controlled dc motor.

SOLUTION

The differential equations governing the armature controlled dc motor are (refer section 1.7).

$$v_a = i_a R_a + L_a \frac{di_a}{dt} + e_b \; ; \qquad T = K_t i_a \; ; \qquad T = J\frac{d\omega}{dt} + B\omega \; ; \qquad e_b = K_b\,\omega \; ; \qquad \omega = d\theta / dt$$

On taking Laplace transform of above equations we get,

$$V_a(s) = I_a(s)\,R_a + L_a\,s\,I_a(s) + E_b(s) \qquad .....(1)$$

$$T(s) = K_t I_a(s) \qquad .....(2)$$

$$T(s) = J\,s\,\omega(s) + B\,\omega(s) \qquad .....(3)$$

$$E_b(s) = K_b\,\omega(s) \qquad .....(4)$$

$$\omega(s) = s\,\theta(s) \qquad .....(5)$$

The input and output variables of armature controlled dc motor are armature voltage $V_a(s)$ and angular displacement $\theta(s)$ respectively. The variables $I_a(s)$, $T(s)$, $E_b(s)$ and $\omega(s)$ are intermediate variables.

The equations (1) to (5) are rearranged & individual signal flow graph are shown in fig 1 to fig 5.

$$V_a(s) - E_b(s) = I_a(s)\,[R_a + s\,L_a]$$

$$\therefore I_a(s) = \frac{1}{R_a + s\,L_a}\,[V_a(s) - E_b(s)]$$

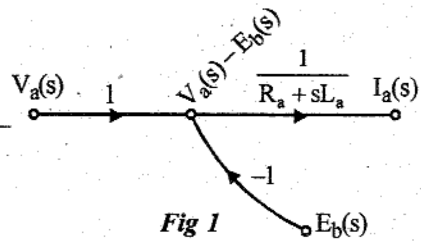


Fig 1

$$T(s) = K_t I_a(s)$$

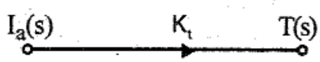


Fig 1

$$T(s) = \omega(s)\,[Js + B]$$

$$\therefore \omega(s) = \frac{1}{Js + B}\,T(s)$$

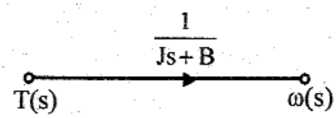


Fig 3

$$E_b(s) = K_b\,\omega(s)$$

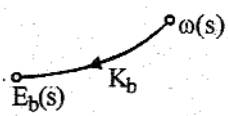


Fig 4

$$\omega(s) = s\theta(s)$$

$$\therefore \theta(s) = \frac{1}{s}\omega(s)$$

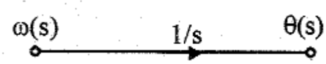


Fig 5

The overall signal flow graph of armature controlled dc motor is obtained by interconnecting the individual signal flow graphs shown in fig 1 to fig 5. The overall signal flow graph is shown in fig 6.

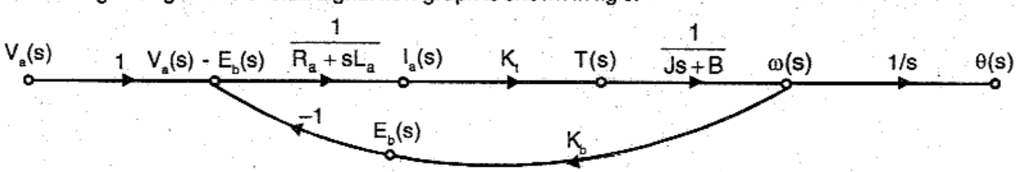


Fig 6 : Signal flow graph of armature controlled dc motor.

EXAMPLE 1.25

Find the overall transfer function of the system whose signal flow graph is shown in fig 1.

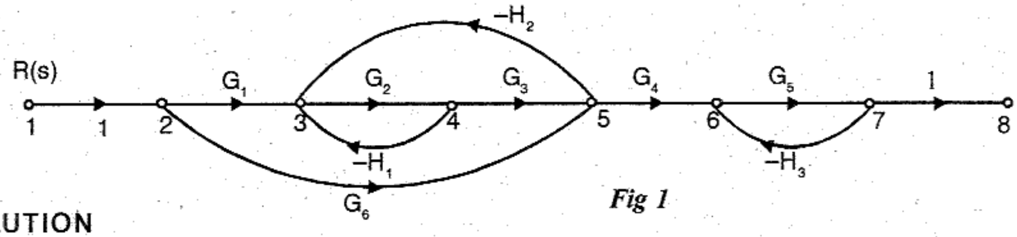


Fig 1

SOLUTION

Forward Path Gains

There are two forward paths. $\therefore K = 2$

Let forward path gains be P_1 and P_2.

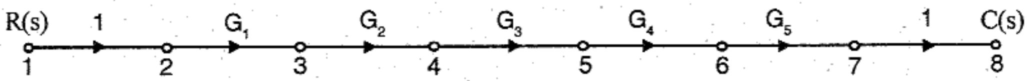


***Fig 2 :** Forward path-1.*

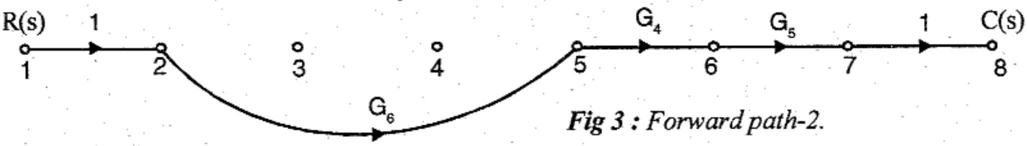


***Fig 3 :** Forward path-2.*

Gain of forward path-1, $P_1 = G_1\,G_2\,G_3\,G_4\,G_5$

Gain of forward path-2, $P_2 = G_4 G_5 G_6$

Individual Loop Gain

There are three individual loops. Let individual loop gains be P_{11}, P_{21} and P_{31}.

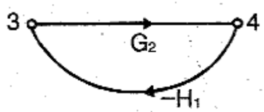


***Fig 4 :** Loop-1.*

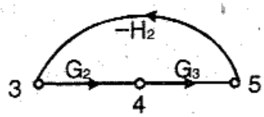


***Fig 5 :** Loop-2.*

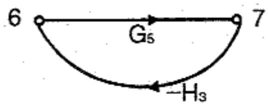


***Fig 6 :** Loop-3.*

Loop gain of individual loop-1, $P_{11} = -G_2 H_1$

Loop gain of individual loop-2, $P_{21} = -G_2 G_3 H_2$

Loop gain of individual loop-3, $P_{31} = -G_5 H_3$

Gain Products of Two Non-touching Loops

There are two combinations of two non-touching loops. Let the gain products of two non touching loops be P_{12} and P_{22}.

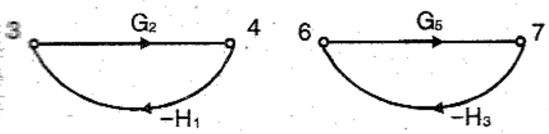


***Fig 7 :** First combination of 2 non-touching loops.*

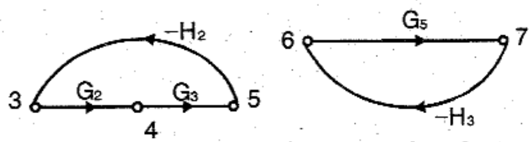


***Fig 8 :** Second combination of 2 non-touching loops.*

Gain product of first combination of two non touching loops $\Big\}$ $P_{12} = P_{11}P_{31} = (-G_2H_1)\,(-G_5H_3) = G_2G_5H_1H_3$

Gain product of second combination of two non touching loops $\Big\}$ $P_{22} = P_{21}P_{31} = (-G_2G_3H_2)\,(-G_5H_3) = G_2G_3G_5H_2H_3$

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + (P_{12} + P_{22})$$

$$= 1 - (-G_2H_1 - G_2G_3H_2 - G_5H_3) + (G_2G_5H_1H_3 + G_2G_3G_5H_2H_3)$$

$$= 1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3$$

$\Delta_1 = 1$, Since there is no part of graph which is not touching with first forward path.

The part of the graph which is non touching with second forward path is shown in fig 9.

$$\Delta_2 = 1 - P_{11} = 1 - (-G_2H_1) = 1 + G_2H_1$$

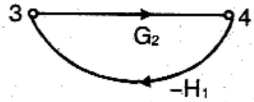


Fig 9

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2) \quad \text{(Number of forward paths is 2 and so K = 2)}$$

$$= \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6(1+G_2H_1)}{1+G_2H_1+G_2G_3H_2+G_5H_3+G_2G_5H_1H_3+G_2G_3G_5H_2H_3}$$

$$= \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6 + G_2G_4G_5G_6H_1}{1+G_2H_1+G_2G_3H_2+G_5H_3+G_2G_5H_1H_3+G_2G_3G_5H_2H_3}$$

$$= \frac{G_2G_4G_5\left[G_1G_3 + G_6/G_2 + G_6H_1\right]}{1+G_2H_1+G_2G_3H_2+G_5H_3+G_2G_5H_1H_3+G_2G_3G_5H_2H_3}$$

EXAMPLE 1.26

Find the overall gain of the system whose signal flow graph is shown in fig 1.

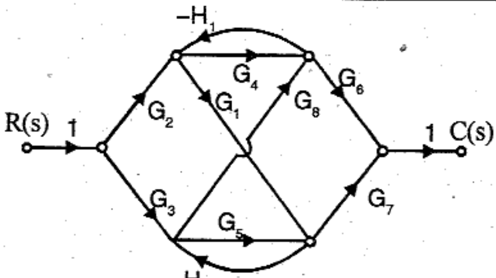


Fig 1

SOLUTION

Let us number the nodes as shown in fig 2.

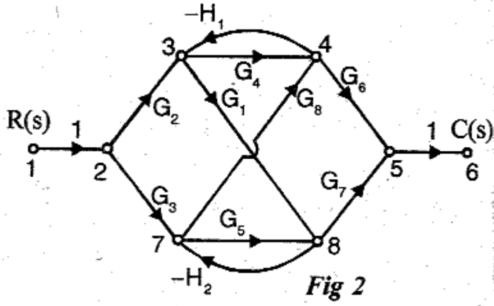


Fig 2

I. Forward Path Gains

There are six forward paths. $\therefore K = 6$

Let the forward path gains be P_1, P_2, P_3, P_4, P_5 and P_6.

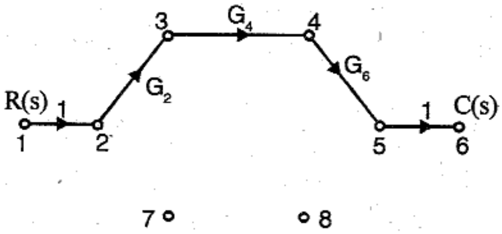


Fig 3 : Forward path-1.

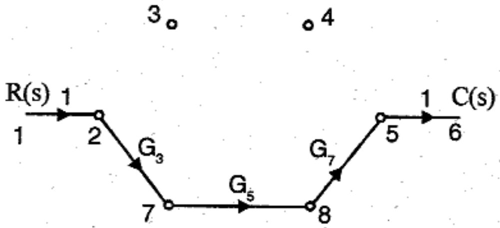


Fig 4 : Forward path-2.

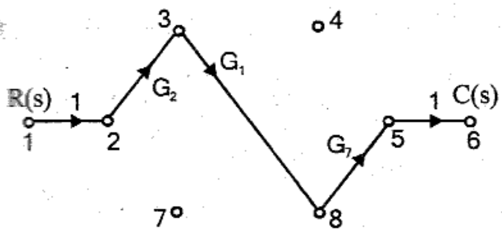


Fig 5 : Forward path-3

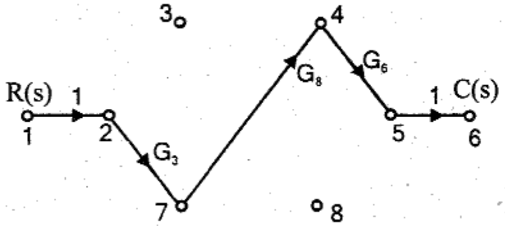


Fig 6 : Forward path-4

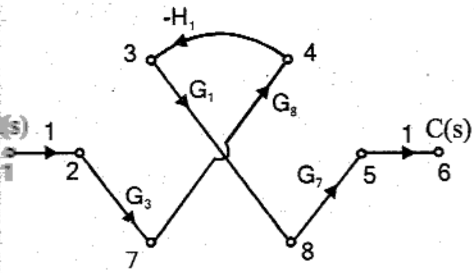


Fig 7 : Forward path-5

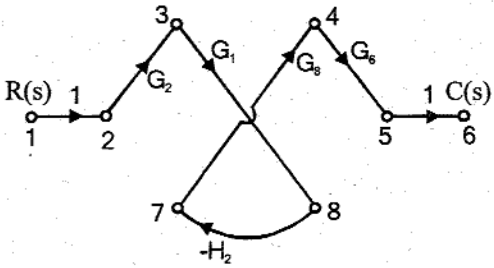


Fig 8 : Forward path-6

Gain of forward path-1, $P_1 = G_2 G_4 G_6$

Gain of forward path-2, $P_2 = G_3 G_5 G_7$

Gain of forward path-3, $P_3 = G_1 G_2 G_7$

Gain of forward path-4, $P_4 = G_3 G_8 G_6$

Gain of forward path-5, $P_5 = -G_1G_3G_7G_8H_1$

Gain of forward path-6, $P_6 = -G_1G_2G_6G_8H_2$

Individual Loop Gain

There are three individual loops.

Let individual loop gains be P_{11}, P_{21} and P_{31}.

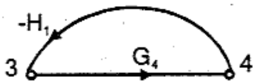


Fig 9 : Loop-1

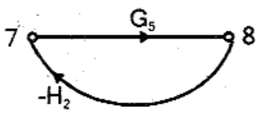


Fig 10 : Loop-2

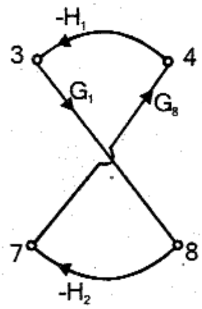


Fig 11 : Loop-3

Loop gain of individual loop-1, $P_{11} = -G_4H_1$

Loop gain of individual loop-2, $P_{21} = -G_5H_2$

Loop gain of individual loop-3, $P_{31} = G_1G_8H_1H_2$

Gain Products of Two Non-touching Loops

There is only one combination of two non-touching loops. Let gain product of two non-touching loops be P_{12}.

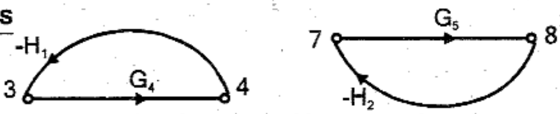


Fig 12 : Combination of 2 non-touching loops

$$\left.\begin{array}{l}\text{Gain product of first combination}\\ \text{of two non-touching loops}\end{array}\right\} P_{12} = P_{11}P_{21} = (-G_4H_1)\,(-G_5H_2) = G_4G_5H_1H_2$$

Calculation of Δ and Δ_K

$$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + P_{12} = 1 - (-G_4H_1 - G_5H_2 + G_1G_8H_1H_2) + G_4G_5H_1H_2$$

$$= 1 + G_4H_1 + G_5H_2 - G_1G_8H_1H_2 + G_4G_5H_1H_2$$

The part of the graph non-touching forward path - 1 is shown in fig 13.

$$\therefore \Delta_1 = 1 - (-G_5H_2) = 1 + G_5H_2$$

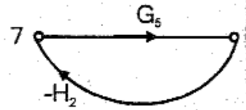


Fig 13

The part of the graph non-touching forward path -2 is shown in fig 14.

$$\therefore \Delta_2 = 1 - (-G_4H_1) = 1 + G_4H_1$$

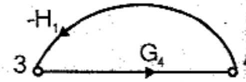


Fig 14

There is no part of the graph which is non-touching with forward paths 3, 4, 5 and 6.

$$\therefore \Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 1$$

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\left(\sum_K P_K \Delta_K\right) \qquad \text{(Number of forward paths is six and so K = 6)}$$

$$= \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4 + P_5\Delta_5 + P_6\Delta_6)$$

$$= \frac{G_2G_4G_6(1+G_5H_2) + G_3G_5G_7(1+G_4H_1) + G_1G_2G_7 + G_3G_6G_8 - G_1G_3G_7G_8H_1 - G_1G_2G_6G_8H_2}{1+G_4H_1+G_5H_2-G_1G_8H_1H_2+G_4G_5H_1H_2}$$

EXAMPLE 1.27

Find the overall gain C(s)/R(s) for the signal flow graph shown in fig 1.

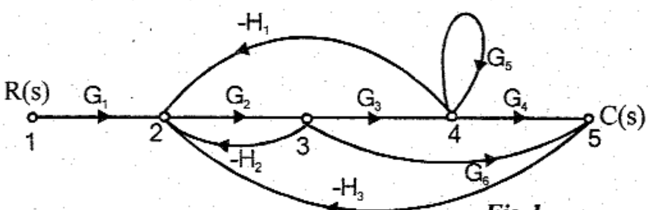


Fig 1

SOLUTION

I. Forward Path Gains

There are two forward paths. $\therefore K = 2$. Let the forward path gains be P_1 and P_2.

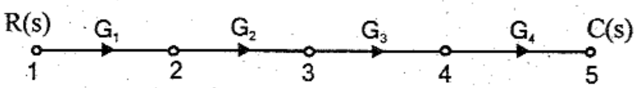


Fig 2 : Forward path-1

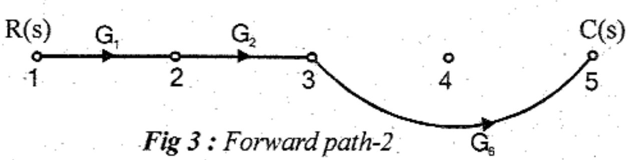


Fig 3 : Forward path-2

Gain of forward path-1, $P_1 = G_1G_2G_3G_4$

Gain of forward path-2, $P_2 = G_1G_2G_6$

Individual Loop Gain

There are five individual loops. Let the individual loop gains be p_{11}, p_{21}, p_{31}, p_{41} and p_{51}.

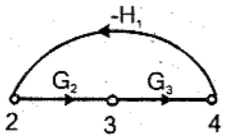


Fig 4 : loop-1

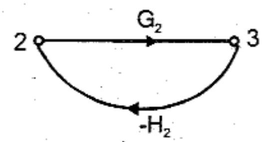


Fig 5 : loop-2

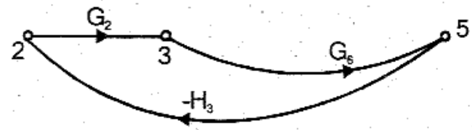


Fig 6 : loop-3

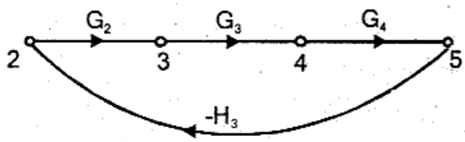


Fig 7 : loop-4

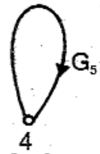


Fig 8 : loop-5

Loop gain of individual loop-1, $P_{11} = -G_2G_3H_1$

Loop gain of individual loop-2, $P_{21} = -H_2G_2$

Loop gain of individual loop-3, $P_{31} = -G_2G_6H_3$

Loop gain of individual loop-4, $P_{41} = -G_2G_3G_4H_3$

Loop gain of individual loop-5, $P_{51} = G_5$

Gain Products of Two Non-touching Loops

There are two combinations of two non-touching loops.

Let the gain products of two non-touching loops be P_{12} and P_{22}.

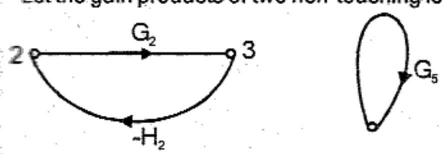


Fig 9 : First combination of two non-touching loops

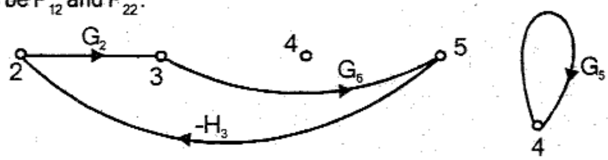


Fig 10 : Second combination of two non-touching loops

Gain product of first combination of two non touching loops $\Big\}$ $P_{12} = P_{21}P_{51} = (-G_2H_2)(G_5) = G_2G_5H_2$

Gain product of second combination of two non touching loops $\Big\}$ $P_{22} = P_{31}P_{51} = (-G_2G_6H_3)(G_5) = -G_2G_5G_6H_3$

Calculation of Δ and Δ_K

$$\Delta = 1-(P_{11}+P_{21}+P_{31}+P_{41}+P_{51})+(P_{12}+P_{22})$$
$$= 1-(-G_2G_3H_1 - H_2G_2 - G_2G_3G_4H_3 + G_5 - G_2G_6H_3)$$
$$+(-G_2H_2G_5 - G_2G_5G_6H_3)$$

Since there is no part of graph which is not touching forward path-1, $\Delta_1 = 1$.

The part of graph which is not touching forward path-2 is shown in fig 11.

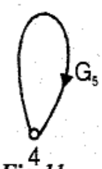


Fig 11

$\therefore \Delta_2 = 1 - G_5$

Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K \text{ (Number of forward path is 2 and so K = 2)}$$

$$= \frac{1}{\Delta}\left[P_1\Delta_1 + P_2\Delta_2\right] = \frac{1}{\Delta}\left[G_1G_2G_3G_4 \times 1 + G_1G_2G_6(1-G_5)\right]$$

$$= \frac{G_1G_2G_3G_4 + G_1G_2G_6 - G_1G_2G_5G_6}{1 + G_2G_3H_1 + H_2G_2 + G_2G_3G_4H_3 - G_5 + G_2G_6H_3 - G_2H_2G_5 - G_2G_5G_6H_3}$$

EXAMPLE 1.28

Find the overall gain C(s)/R(s) for the signal flow graph shown in fig 1.

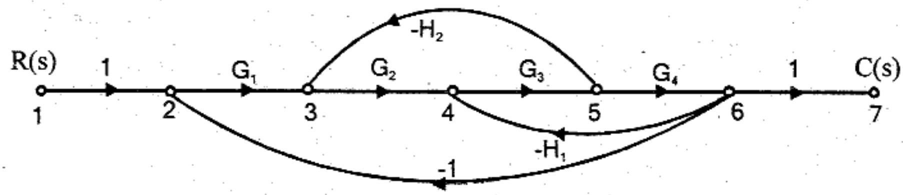


Fig 1

SOLUTION

I. Forward Path Gains

There is only one forward path. $\therefore K = 1$.

Let the forward path gain be P_1.

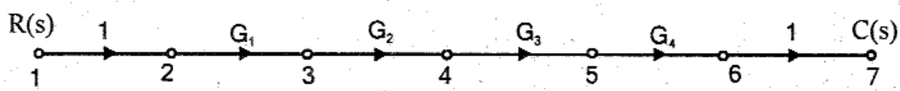


Fig 1 : Forward path-1

Gain of forward path-1, $P_1 = G_1G_2G_3G_4$

II. Individual Loop Gain

There are three individual loops. Let the loop gains be P_{11}, P_{21}, P_{31}.

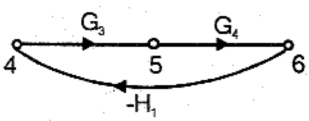


Fig 3 : loop-1

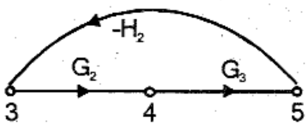


Fig 4 : loop-2

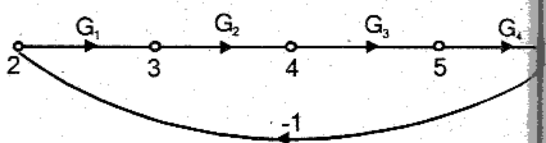


Fig 5 : loop-3

Loop gain of individual loop-1, $P_{11} = -G_3G_4H_1$

Loop gain of individual loop-2, $P_{21} = -G_2G_3H_2$

Loop gain of individual loop-3, $P_{31} = -G_1G_2G_3G_4$

III. Gain Products of Two Non-touching Loops

There are no possible combinations of two non-touching loops, three non-touching loops, etc.

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - (P_{11} + P_{21} + P_{31})$$
$$= 1 - (-G_3G_4H_1 - G_2G_3H_2 - G_1G_2G_3G_4)$$
$$= 1 + G_3G_4H_1 + G_2G_3H_2 + G_1G_2G_3G_4$$

Since no part of the graph is non-touching with forward path-1, $\Delta_1 = 1$.

Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{C(s)}{R(s)} = \frac{1}{\Delta}\sum_K P_K \Delta_K = \frac{1}{\Delta} P_1 \Delta_1 \text{ (Number of forward path is 1 and so K = 1)}$$

$$= \frac{G_1G_2G_3G_4}{1 + G_3G_4H_1 + G_2G_3H_2 + G_1G_2G_3G_4}$$

EXAMPLE 1.29

The signal flow graph for a feedback control system is shown in fig 1. Determine the closed loop transfer function C(s)/R(s).

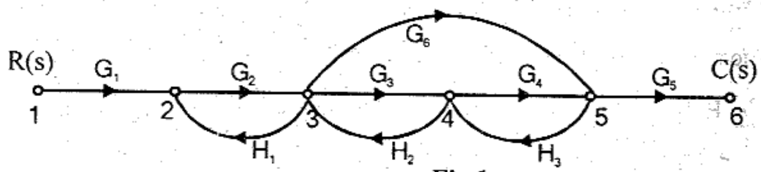


Fig 1

SOLUTION

Forward Path Gains

There are two forward paths. $\therefore$ K = 2.

Let forward path gains be P_1 and P_2.

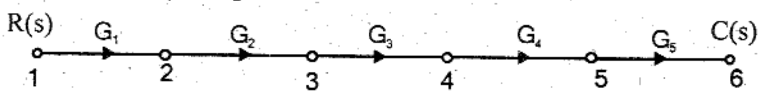


***Fig 2 :** Forward path-1*

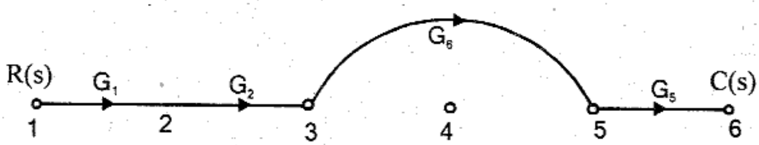


***Fig 3 :** Forward path-2*

Gain of forward path-1, $P_1 = G_1G_2G_3G_4G_5$

Gain of forward path-2, $P_2 = G_1G_2G_6G_5$

Individual Loop Gain

There are four individual loops. Let individual loop gains be P_{11}, P_{21}, P_{31} and P_{41}.

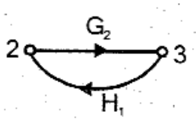


***Fig 4 :** loop-1*

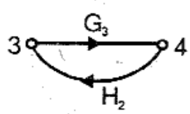


***Fig 5 :** loop-2*

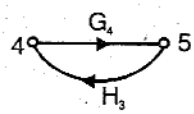


***Fig 6 :** loop-3*

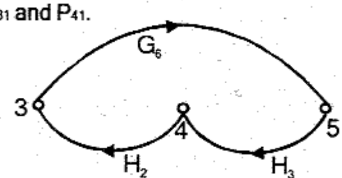


***Fig 7 :** loop-4*

Loop gain of individual loop-1, $P_{11} = G_2H_1$

Loop gain of individual loop-2, $P_{21} = G_3H_2$

Loop gain of individual loop-3, $P_{31} = G_4H_3$

Loop gain of individual loop-4, $P_{41} = G_6H_2H_3$

Gain Products of Two Non-touching Loops

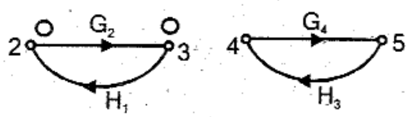


***Fig 8 :** First combination of two non touching loops*

There is only one combination of two non-touching loops. Let the gain products of two non-touching loops be P_{12}.

Gain product of first combination of two non-touching loops $\left.\right\}$ $P_{12} = (G_2H_1)(G_4H_3)$

$$= G_2G_4H_1H_3$$

IV. Calculation of Δ and Δ_K

$$\Delta = 1-(P_{11}+P_{21}+P_{31}+P_{41})+P_{12}$$
$$=1-(G_2H_1+G_3H_2+G_4H_3+G_6H_2H_3)+G_2G_4H_1H_3$$
$$=1-G_2H_1-G_3H_2-G_4H_3-G_6H_2H_3+G_2G_4H_1H_3$$

Since there is no part of graph which is non-touching with forward path-1 and 2, $\Delta_1 = \Delta_2 = 1$

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2) \quad \text{(Number of forward paths is two and so K = 2)}$$

$$= \frac{G_1G_2G_3G_4G_5 + G_1G_2G_5G_6}{1-G_2H_1-G_3H_2-G_4H_3-G_6H_2H_3+G_2G_4H_1H_3}$$

EXAMPLE 1.30

Convert the given block diagram to signal flow graph and determine C(s)/R(s).

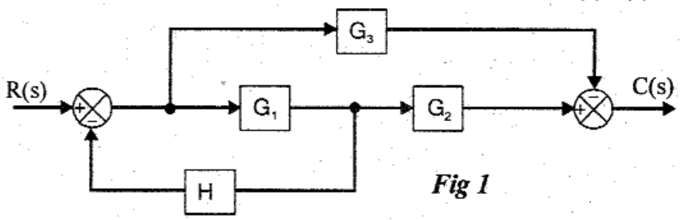


Fig 1

SOLUTION

The nodes are assigned at input, output, at every summing point & branch point as shown in fig 2.

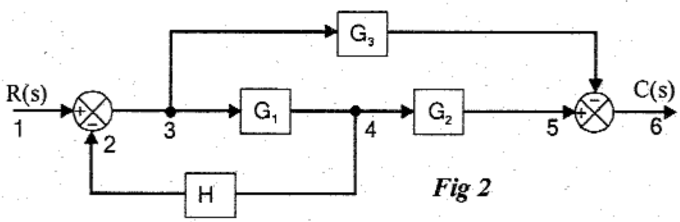


Fig 2

The signal flow graph of the above system is shown in fig 3.

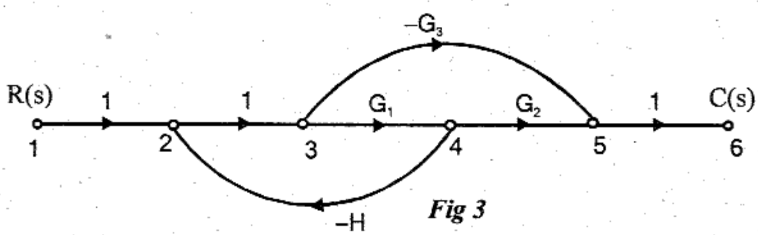


Fig 3

I. Forward Path Gains

There are two forward paths. $\therefore$ K=2

Let the forward path gains be P_1 and P_2.

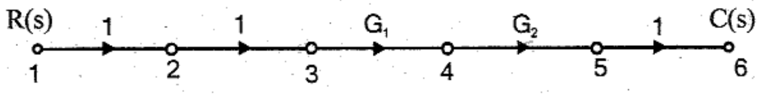


Fig 4 : Forward path-1

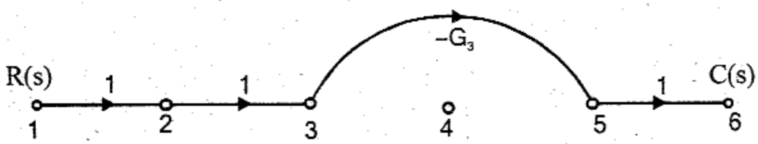


Fig 5 : Forward path-2

Gain of forward path-1, $P_1 = G_1G_2$

Gain of forward path-2, $P_2 = -G_3$

II. Individual Loop Gain

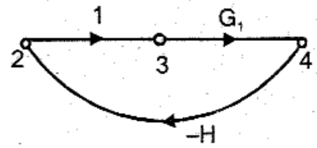


Fig 3 : loop-1

There is only one individual loop. Let the individual loop gain be P_{11}.

Loop gain of individual loop-I, $P_{11} = -G_1H$.

III. Gain Products of Two Non-touching Loops

There are no combinations of non-touching Loops.

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - [P_{11}] = 1 + G_1H$$

Since there are no part of the graph which is non-touching with forward path-1 and 2,

$$\Delta_1 = \Delta_2 = 1$$

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2] = \frac{G_1G_2 - G_3}{1 + G_1H}$$

EXAMPLE 1.31

Convert the block diagram to signal flow graph and determine the transfer function using Mason's gain formula.

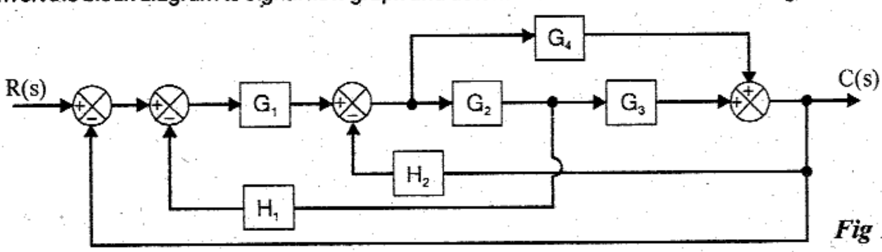


Fig 1

SOLUTION

The nodes are assigned at input, ouput, at every summing point & branch point as shown in fig 2.

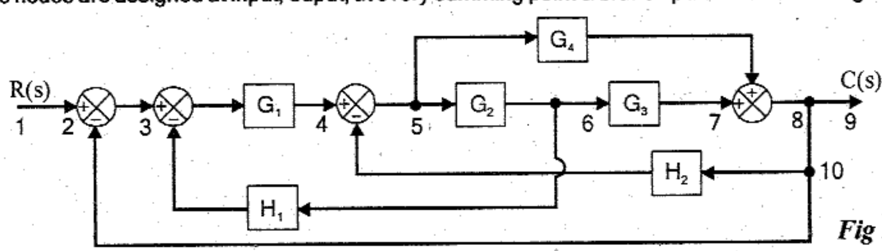


Fig 2

The signal flow graph for the above block diagram is shown in fig 3.

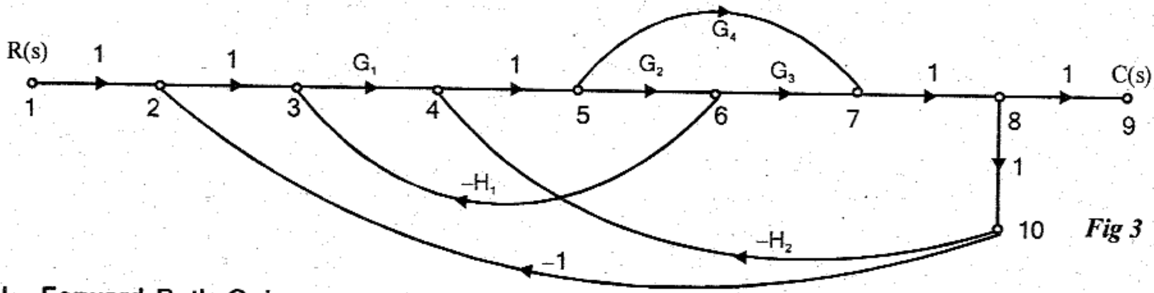


Fig 3

I. Forward Path Gains

There are two forward paths. $\therefore$ K=2.

Let the gain of the forward paths be P_1 and P_2.

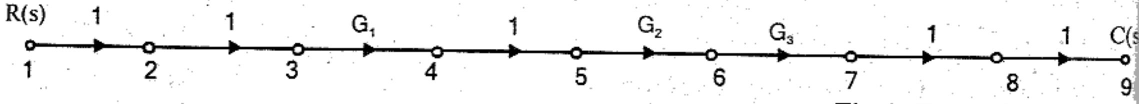


Fig 4 : Forward path-1

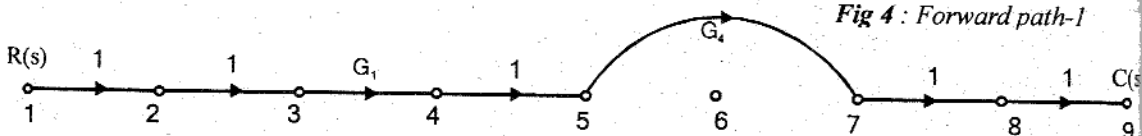


Fig 5 : Forward path-2

Gain of forward path-1, $P_1 = G_1G_2G_3$

Gain of forward path-2, $P_2 = G_1G_4$

II. Individual Loop Gain

There are five individual loops. Let the individual loop gain be P_{11}, P_{21}, P_{31}, P_{41} and P_{51}.

Loop gain of individual loop-1, $P_{11} = -G_1G_2G_3$

Loop gain of individual loop-2, $P_{21} = -G_2G_1H_1$

Loop gain of individual loop-3, $P_{31} = -G_2G_3H_2$

Loop gain of individual loop-4, $P_{41} = -G_1G_4$

Loop gain of individual loop-5, $P_{51} = -G_4H_2$

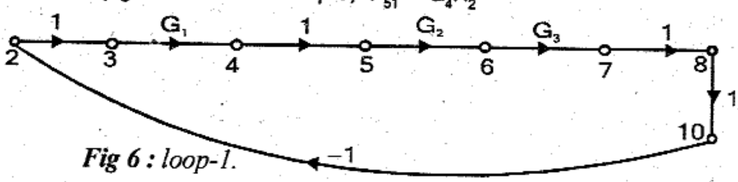


Fig 6 : loop-1.

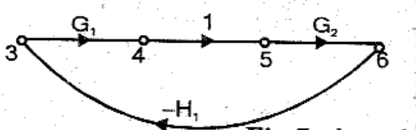


Fig 7 : loop-2.

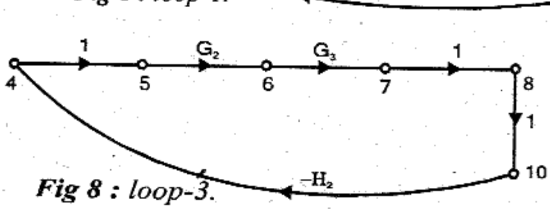


Fig 8 : loop-3.

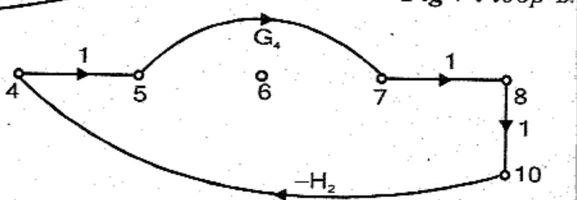


Fig 10 : loop-5.

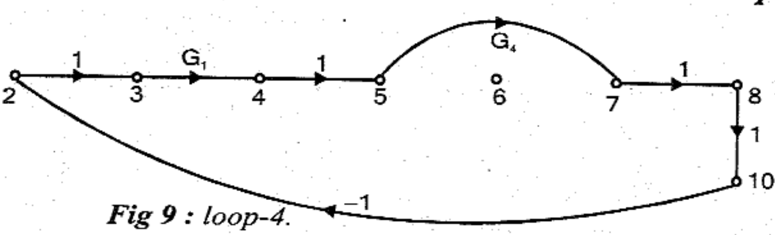


Fig 9 : loop-4.

Gain Products of Two Non-touching Loops

There are no possible combinations of two non-touching loops, three non-touching loops, etc.,.

Calculation of Δ and Δ_K

$$\Delta = 1 - [P_{11}+P_{21}+P_{31}+P_{41}+P_{51}] = 1+G_1G_2G_3+G_1G_2H_1+G_2G_3H_2+G_1G_4+G_4H_2$$

Since no part of graph is non touching with forward paths-I and 2, $\Delta_1 = \Delta_2 = 1$.

Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_k P_k\Delta_k = \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2]$$

$$= \frac{G_1G_2G_3 + G_1G_4}{1+G_1G_2G_3+G_1G_2H_1+G_2G_3H_2+G_1G_4+G_4H_2}$$

EXAMPLE 1.32

Convert the block diagram to signal flow graph and determine the transfer function using Mason's gain formula.

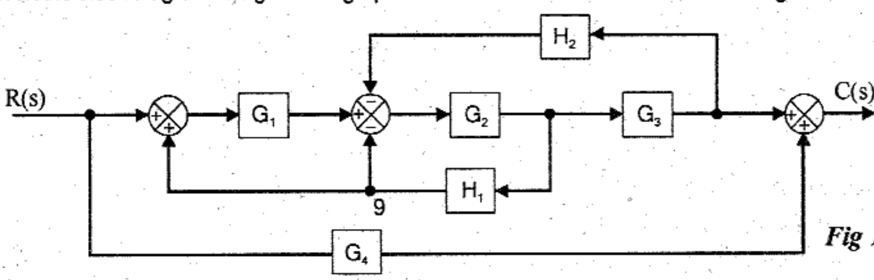


Fig 1

SOLUTION

The nodes are assigned at input, output, at every summing point & branch point as shown in fig 2.

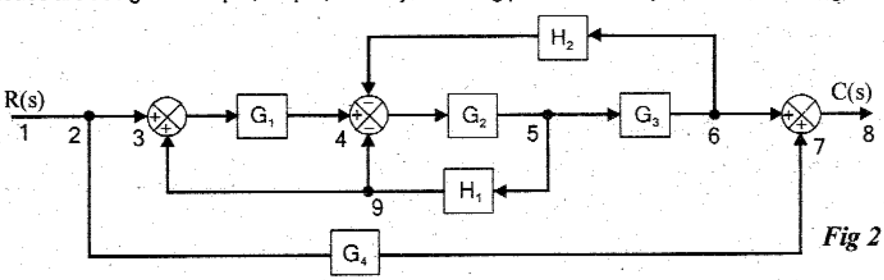


Fig 2

The signal flow graph for the above block diagram is shown in fig 3.

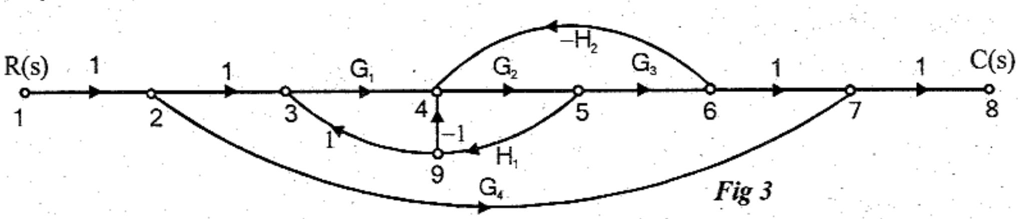


Fig 3

Forward Path Gains

There are two forward path, $\therefore$ K=2.

Let the forward path gains be P_1 and P_2.

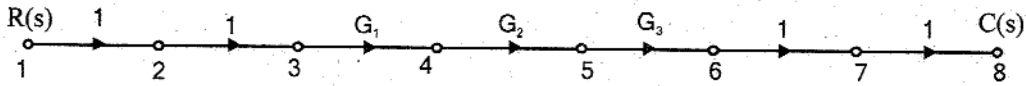


Fig 4 : Forward path-1.

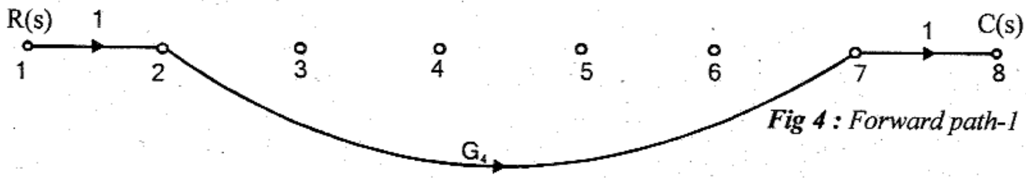


Fig 4 : Forward path-1

Gain of forward path-1, $P_1 = G_1G_2G_3$

Gain of forward path-2, $P_2 = G_4$

II. Individual Loop Gain

There are three individual loops with gains P_{11}, P_{21} and P_{31}.

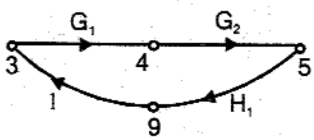


Fig 6 : loop-1.

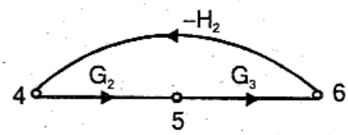


Fig 7 : loop-2.

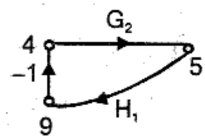


Fig 8 : loop-3.

Gain of individual loop-1, $P_{11} = G_1G_2H_1$

Gain of individual loop-2, $P_{21} = -G_2G_3H_2$

Gain of individual loop-3, $P_{31} = -G_2H_1$

III. Gain Products of Two Non-touching Loops

There are no possible combinations of two-non touching loops, three non-touching loops, etc.,.

IV. Calculation of Δ and Δ_K

$$\Delta = 1-[P_{11}+P_{21}+P_{31}] = 1 - G_1G_2H_1 + G_2G_3H_2 + G_2H_1$$

Since no part of graph touches forward path-1, $\Delta_1 = 1$.

The part of graph non touching forward path-2 is shown in fig 9.

$$\therefore \Delta_2 = 1-[G_1G_2H_1 - G_2G_3H_2 - G_2H_1]$$

$$= 1 - G_1G_2H_1 + G_2G_3H_2 + G_2H_1$$

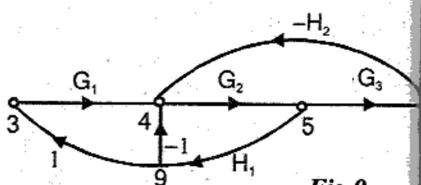


Fig 9

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_k P_k\Delta_k = \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2] \text{ (Number of forward paths is 2 and so K = 2)}$$

$$= \frac{1}{\Delta}[G_1G_2G_3 + G_4(1 - G_1G_2H_1 + G_2G_3H_2 + G_2H_1)]$$

$$= \frac{1}{\Delta}[G_1G_2G_3 + G_4 - G_1G_2G_4H_1 + G_2G_3G_4H_2 + G_2G_4H_1]$$

$$= \frac{G_1G_2G_3 + G_4 - G_1G_2G_4H_1 + G_2G_3G_4H_2 + G_2G_4H_1}{1 - G_1G_2H_1 + G_2G_3H_2 + G_2H_1}$$

EXAMPLE 1.33

Draw a signal flow graph and evaluate the closed loop transfer function of a system whose block diagram is shown in fig 1.

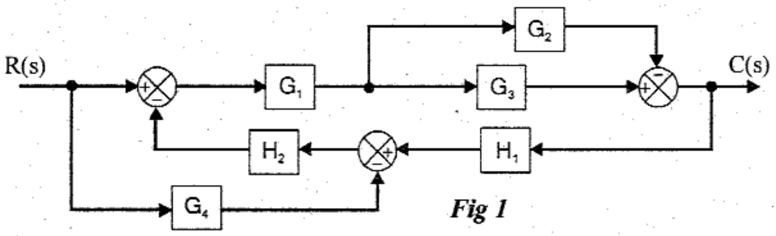


Fig 1

SOLUTION

The nodes are assigned at input, output, at every summing point & branch point as shown in fig 2.

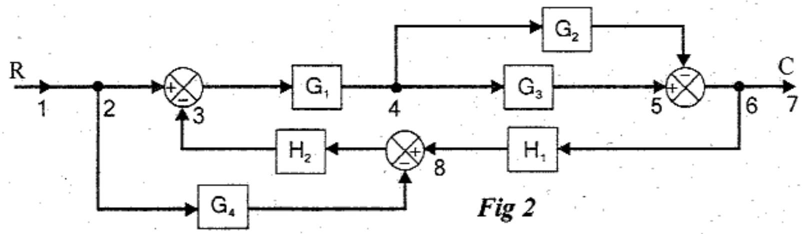


Fig 2

The signal flow graph for the block diagram of fig 2, is shown in fig 3.

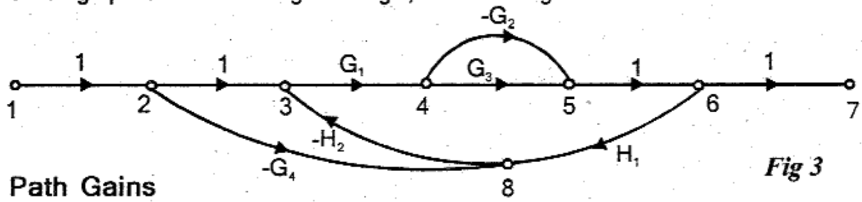


Fig 3

Forward Path Gains

There are four forward paths, $\therefore K = 4$

Let the forward path gains be P_1, P_2, P_3 and P_4.

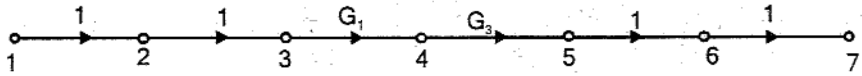


Fig 4 : Forward path-1.

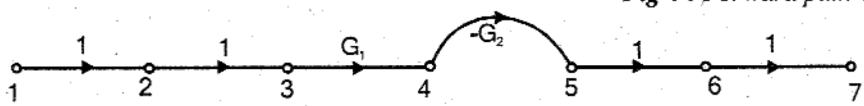


Fig 5 : Forward path-2.

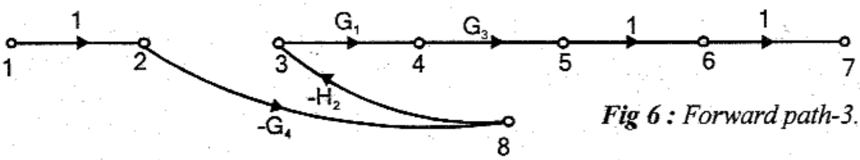


Fig 6 : Forward path-3.

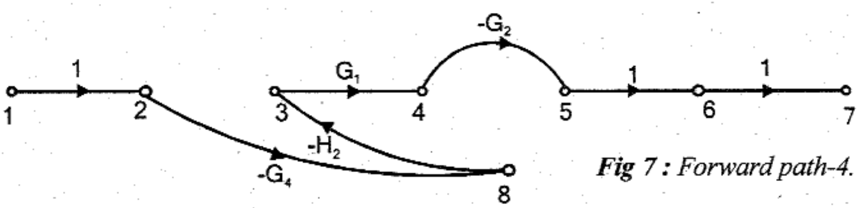


Fig 7 : Forward path-4.

Gain of forward path-1, $P_1 = G_1G_3$

Gain of forward path-2, $P_2 = -G_1G_2$

Gain of forward path-3, $P_3 = G_1G_3G_4H_2$

Gain of forward path-4, $P_4 = -G_1G_2G_4H_2$

II. Individual Loop Gain

There are two individual loops, let individual loop gains be P_{11} and P_{21}.

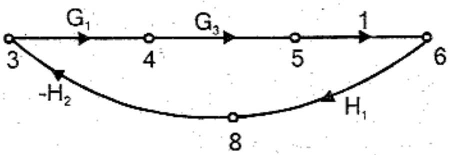


Fig 7 : loop-1

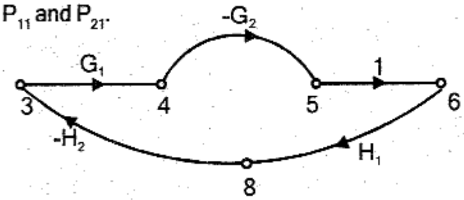


Fig 7 : loop-2

Loop gain of individual loop-1, $P_{11} = -G_1G_3H_1H_2$

Loop gain of individual loop-2, $P_{21} = G_1G_2H_1H_2$

III. Gain Products of Two Non-touching Loops

There are no possible combinations of two non-touching loops, three non-touching loops, etc.,.

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - [\text{sum of individual loop gain}] = 1 - (P_{11} + P_{21})$$

$$= 1 - [-G_1G_3H_1H_2 + G_1G_2H_1H_2] = 1 + G_1G_3H_1H_2 - G_1G_2H_1H_2$$

Since no part of graph is non touching with the forward paths, $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1$.

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{P_1 + P_2 + P_3 + P_4}{\Delta} \text{ (Number of forward paths is 4 and so K = 4)}$$

$$= \frac{G_1G_3 - G_1G_2 + G_1G_3G_4H_2 - G_1G_2G_4H_2}{1 + G_1G_3H_1H_2 - G_1G_2H_1H_2}$$

$$= \frac{G_1(G_3 - G_2) + G_1G_4H_2(G_3 - G_2)}{1 + G_1H_1H_2(G_3 - G_2)} = \frac{G_1(G_3 - G_2)(1 + G_4H_2)}{1 + G_1H_1H_2(G_3 - G_2)}$$

1.14 SHORT QUESTIONS AND ANSWERS

Q1.1 What is system?

When a number of elements or components are connected in a sequence to perform a specific function, the group thus formed is called a system.

Q1.2 What is control system?

A system consists of a number of components connected together to perform a specific function. In a system when the output quantity is controlled by varying the input quantity, then the system is called control system. The output quantity is called controlled variable or response and input quantity is called command signal or excitation.

Q1.3 What are the two major type of control systems?

The two major type of control systems are open loop and closed loop systems.

Q1.4 Define open loop system.

The control system in which the output quantity has no effect upon the input quantity are called open loop control system. This means that the output is not fedback to the input for correction.

Q.1.5 ***Define closed loop system.***

The control systems in which the output has an effect upon the input quantity in order to maintain the desired output value are called closed loop control systems.

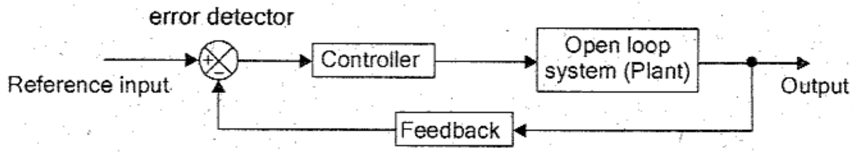
Q.1.6 ***What is feedback? What type of feedback is employed in control system?***

The feedback is a control action in which the output is sampled and a proportional signal is given to input for automatic correction of any changes in desired output.

Negative feedback is employed in control system.

Q.1.7 ***What are the components of feedback control system?***

The components of feedback control system are plant, feedback path elements, error detector and controller.



Q.1.8 ***Why negative feedback is invariably preferred in a closed loop system?***

The negative feedback results in better stability in steady state and rejects any disturbance signals. It also has low sensitivity to parameter variations. Hence negative feedback is preferred in closed loop systems.

Q.1.9 ***What are the characteristics of negative feedback?***

The characteristics of negative feedback are as follows :

(i) accuracy in tracking steady state value.

(ii) rejection of disturbance signals.

(iii) low sensitivity to parameter variations.

(iv) reduction in gain at the expense of better stability.

Q.1.10. ***What is the effect of positive feedback on stability?***

The positive feedback increases the error signal and drives the output to instability. But sometimes the positive feedback is used in minor loops in control systems to amplify certain internal signals or parameters.

Q.1.11. ***Distinguish between open loop and closed loop system.***

Open loop	Closed loop
1. Inaccurate & unreliable.	1. Accurate & reliable.
2. Simple and economical.	2. Complex and costly.
3. Changes in output due to external disturbances are not corrected automatically.	3. Changes in output due to external disturbances are corrected automatically.
4. They are generally stable.	4. Great efforts are needed to design a stable system.

Q.1.12 ***What is servomechanism?***

The servomechanism is a feedback control system in which the output is mechanical position (or time derivatives of position e.g. velocity and acceleration).

Q.1.13 ***State the principle of homogenity (or) State the principle of superposition.***

The principle of superposition and homogenity states that if the system has responses $c_1(t)$ and $c_2(t)$ for the inputs $r_1(t)$ and $r_2(t)$ respectively then the system response to the linear combination of these input $a_1r_1(t) + a_2r_2(t)$ is given by linear combination of the individual outputs $a_1c_1(t) + a_2c_2(t)$, where a_1 and a_2 are constants.

Q1.14 ***Define linear system.***

A system is said to be linear, if it obeys the principle of superposition and homogenity, which states that the response of a system to a weighed sum of signals is equal to the corresponding weighed sum of the responses of the system to each of the individual input signals. The concept of linear system is diagrammatically shown below.

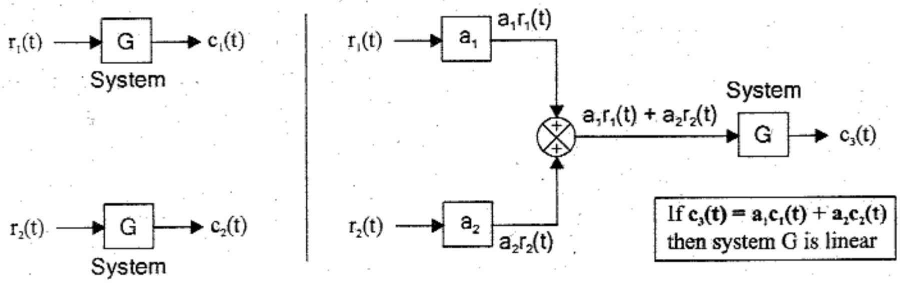


***Fig Q1.14 :** Principle of linearity and superposition.*

Q1.15 ***What is time invariant system?***

A system is said to be time invariant if its input-output characteristics do not change with time. A linear time invariant system can be represented by constant coefficient differential equations. (In linear time varying systems the coefficients of the differential equation governing the system are function of time).

Q1.16 ***Define transfer function.***

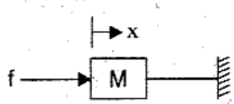
The transfer function of a system is defined as the ratio of Laplace transform of output to Laplace transform of input with zero initial conditions. (It is also defined as the Laplace transform of the impulse response of system with zero initial conditions).

Q1.17 ***What are the basic elements used for modelling mechanical translational system?***

The model of mechanical translational system can be obtained by using three basic elements mass, spring and dashpot.

Q1.18 ***Write the force balance equation of ideal mass element.***

Let a force f be applied to an ideal mass M. The mass will offer an opposing force, f_m which is proportional to acceleration.

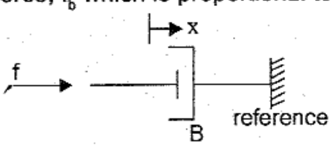


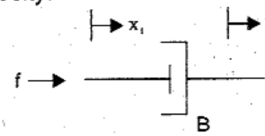
$$\therefore\ f = f_m = M\frac{d^2x}{dt^2}$$

Q1.19 ***Write the force balance equation of ideal dashpot.***

Let a force f be applied to an ideal dashpot, with viscous frictional coefficient B. The dashpot will offer an opposing force, f_b which is proportional to velocity.

$$f = f_b = B\frac{dx}{dt}$$

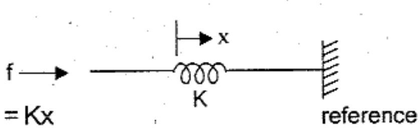




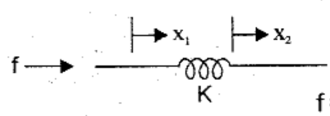
$$f = f_b = B\frac{d}{dt}(x_1 - x_2)$$

Q1.20 ***Write the force balance equation of ideal spring.***

Let a force f be applied to an ideal spring with spring constant K. The spring will offer an opposing force, f_k which is proportional to displacement.



$$f = f_k = Kx$$



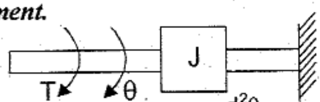
$$f = f_k = K(x_1 - x_2)$$

1.21 What are the basic elements used for modelling mechanical rotational system?

The model of mechanical rotational system can be obtained using three basic elements mass with moment of inertia, J, dash-pot with rotational frictional coefficient, B and torsional spring with stiffness, K.

1.22 Write the torque balance equation of an ideal rotational mass element.

Let a torque T be applied to an ideal mass with moment of inertia, J. The mass will offer an opposing torque T_j which is proportional to angular acceleration.

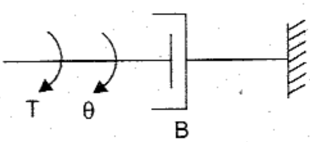


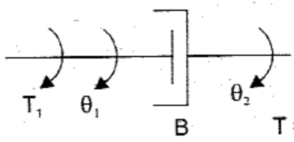
$$T = T_j = J\frac{d^2\theta}{dt^2}$$

1.23 Write the torque balance equation of an ideal rotational dash-pot.

Let a torque T be applied to a rotational dash-pot with frictional coefficient B. The dashpot will offer an opposing torque which is proportional to angular velocity.

$$T = T_b = B\frac{d\theta}{dt}$$



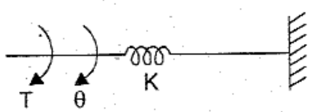


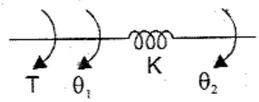
$$T = T_b = B\frac{d}{dt}(\theta_1 - \theta_2)$$

1.24 Write the torque balance equation of ideal rotational spring.

Let a torque T be applied to an ideal rotational spring with spring constant K. The spring will offer an opposing torque T_k which is proportional to angular displacement.

$$T_k = K\theta$$





$$T = T_k = K(\theta_1 - \theta_2)$$

1.25 Name the two types of electrical analogous for mechanical system.

The two types of analogies for the mechanical system are force-voltage and force-current analogy.

1.26 Write the analogous electrical elements in force-voltage analogy for the elements of mechanical translational system.

Force, f	→ Voltage, e	Frictional coefficient , B	→ Resistance, R
Velocity, v	→ Current, i	Stiffness, K	→ Inverse of capacitance, 1/C
Displacement, x	→ Charge, q	Newton's second law, $\Sigma f = 0$	→ Kirchoff 's voltage law, $\Sigma v = 0$
Mass, M	→ Inductance, L		

1.27 Write the analogous electrical elements in force-current analogy for the elements of mechanical translational system.

Force, f	→ Current, i	Frictional coefficient, B	→ Conductance, G =1/ R
Velocity, v	→ Voltage, v	Stiffness, K	→ Inverse of Inductance, 1/L
Displacement, x	→ Flux, ϕ	Newton's second law, $\Sigma f = 0$	→ Kirchoff's current law, $\Sigma i = 0$
Mass, M	→ Capacitance, C		

1.28 Write the analogous electrical elements in torque-voltage analogy for the elements of mechanical rotational system.

Torque, T	→Voltage, e	Stiffness of spring, K	→ Inverse of capacitance, 1/C
Angular velocity, ω	→Current, i	Frictional coefficient, B	→ Resistance, R
Moment of inertia, J	→Inductance, L	Newton's second law, $\Sigma T = 0$	→ Kirchoff 's voltage law, $\Sigma v = 0$.
Angular displacement, θ	→ Charge, q		

Q1.29 ***Write the analogous electrical elements in torque-current analogy for the elements of mechanica rotational system.***

Torque, T	→ Current, i	Frictional coefficient , B	→ Conductance, G = 1/R
Angular velocity, ω	→ Voltage, v	Stiffness of spring, K	→ Inverse of inductance, 1/L
Angular displacement, θ	→ Flux, ϕ	Newton's second law, $\Sigma T = 0$	→ Kirchoff 's current law, $\Sigma i =$
Moment of inertia, J	→ Capacitance, C		

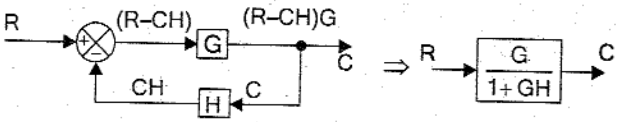
Q1.30 ***What is block diagram? What are the basic components of block diagram?***

A block diagram of a system is a pictorial representation of the functions performed by each componen of the system and shows the flow of signals. The basic elements of block diagram are block, branch point and summing point.

Q1.31 ***What is the basis for framing the rules of block diagram reduction technique?***

The rules for block diagram reduction technique are framed such that any modification made on the diagram does not alter the input output relation.

Q1.32 ***Write the rule for eliminating negative feedback loop.***



Proof

$$C = (R - CH)\,G$$

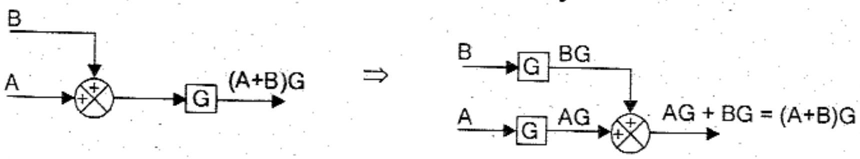
$$C = RG - CHG$$

$$C + CHG = RG$$

$$C(1 + HG) = RG$$

$$\frac{C}{R} = \frac{G}{1+GH}$$

Q1.33 ***Write the rule for moving the summing point ahead of a block.***



Q1.34 ***What is a signal flow graph?***

A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. By taking Laplace transform, the time domain differential equations governing a control system can be transferred to a set of algebraic equations in s-domain. The signal flow graph of the system can be constructed using these equations.

Q1.35 ***What is transmittance?***

The transmittance is the gain acquired by the signal when it travels from one node to another node in signal flow graph.

Q1.36 ***What is sink and source?***

Source is the input node in the signal flow graph and it has only outgoing branches. Sink is a output node in the signal flow graph and it has only incoming branches.

Q1.37 ***Define non-touching loop.***

The loops are said to be non-touching if they do not have common nodes.

Q1.38 ***What are the basic properties of signal flow graph?***

The basic properties of signal flow graph are,

(i) Signal flow graph is applicable to linear systems.

(ii) It consists of nodes and branches. A node is a point representing a variable or signal. A branch indicates functional dependence of one signal on the other.

(iii) A node adds the signals of all incoming branches and transmits this sum to all outgoing branches.

(iv) Signals travel along branches only in the marked direction and when it travels it gets multiplied by the gain or transmittance of the branch.

(v) The algebraic equations must be in the form of cause and effect relationship.

Q1.39 ***Write the Mason's gain formula.***

Mason's gain formula states that the overall gain of the system [transfer function] as follows,

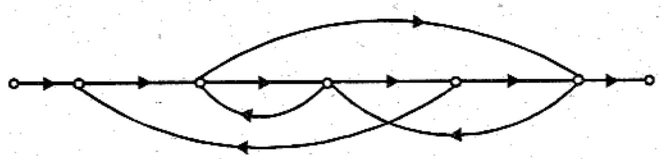
Overall gain, $T = \frac{1}{\Delta} \sum_K P_K \Delta_K$

$T = T(s)$ = Transfer function of the system
K = Number of forward paths in the signal flow graph
P_K = Forward path gain of K^{th} forward path

$$\Delta = 1 - \begin{bmatrix} \text{sum of individual} \\ \text{loop gains} \end{bmatrix} + \begin{bmatrix} \text{sum of gain products of all possible} \\ \text{combinations of two non-touching loops} \end{bmatrix} - \begin{bmatrix} \text{sum of gain products of all possible} \\ \text{combinations of three non-touching loops} \end{bmatrix} + \ldots$$

$\Delta_K = \Delta$ for that part of the graph which is not touching K^{th} forward path

Q1.40 ***For the given signal flow graph, identify the number of forward path and number of individual loop.***



Number of forward paths = 2
Number of individual loops = 4

1.15 EXERCISES

E1.1 For the mechanical system shown in fig E1.1derive the transfer function. Also draw the force-voltage and force-current analogous circuits.

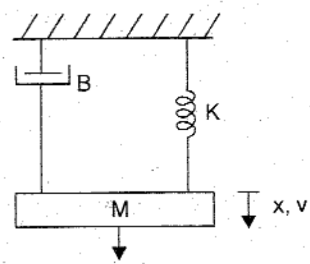


Fig E1.1

E1.2 For the mechanical system shown in fig E1.2 draw the force-voltage and force-current analogous circuits.

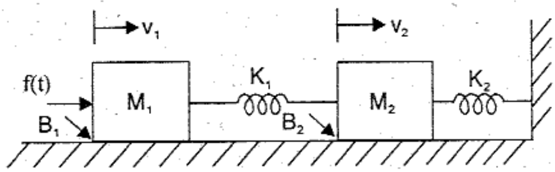


Fig E1.2

E1.3 Write the differential equations governing the mechanical system shown in fig E1.3(a) & (b). Also draw the force-voltage and force-current analogous circuit.

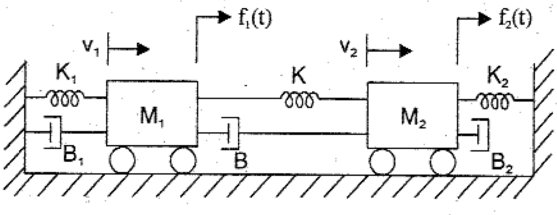


Fig E1.3(a)

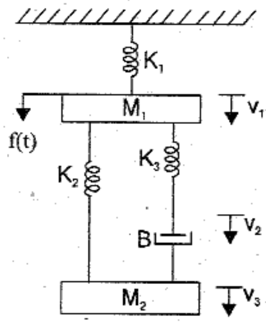


Fig E1.3(b)

E1.4 Consider the mechanical translational system shown in fig E1.4, Draw(a) force-voltage and (b) force-current analogous circuits.

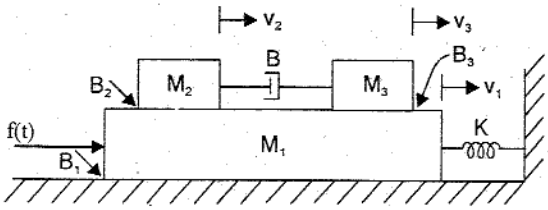


Fig E1.4

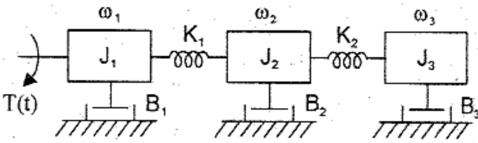


Fig E1.5

E1.5 Write the differential equations governing the rotational mechanical system shown in fig E1.5. Also draw the torque-voltage and torque-current analogous circuits.

E1.6 In an electrical circuit the elements resistance, capacitance and inductance are connected in parallel across the voltage source E as shown in fig E1.6, Draw(a) Translation mechnical analogous system (b) Rotational mechanical analogous system.

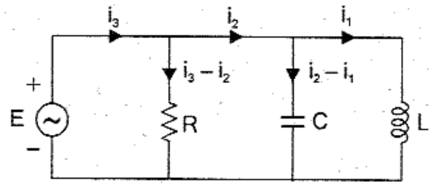


Fig E1.6

E1.7 Consider the block diagram shown in fig E.1.7(a), (b) (c) & (d). Using the block diagram reduction technique, find C/R.

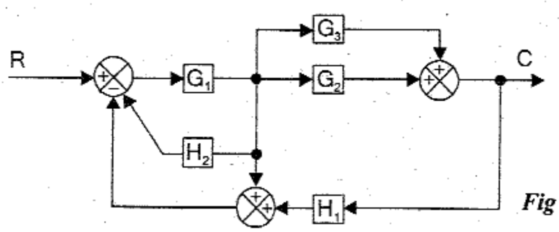


Fig E1.7(a)

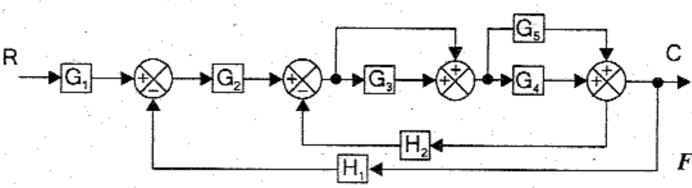


Fig E1.7(b)

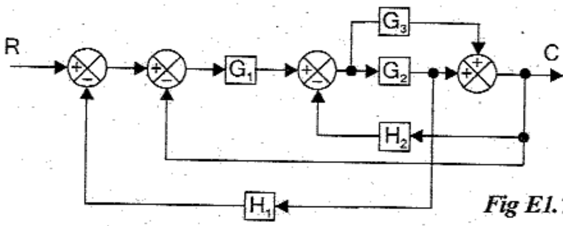


Fig E1.7(c)

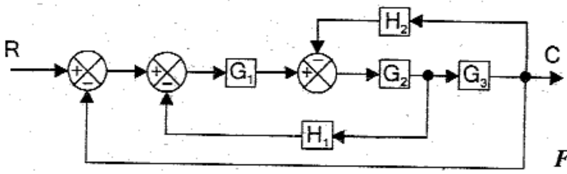


Fig E1.7(d)

E 8 Convert the block diagram shown in fig E1.8 to signal flow graph and find the transfer function of the system.

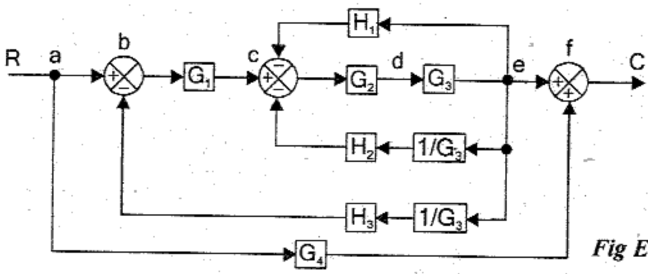


Fig E1.8

E 9 Consider the system shown in fig E1.9(a), (b), (c) & (d). obtain the transfer function using Mason's gain formula.

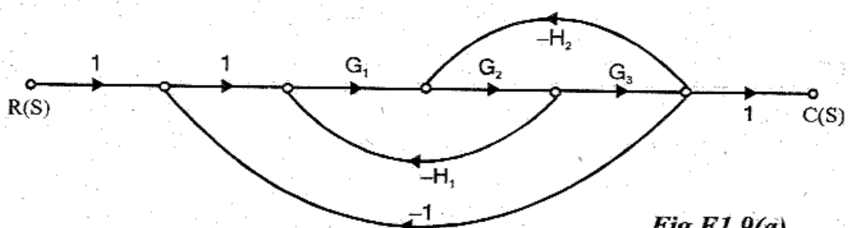


Fig E1.9(a)

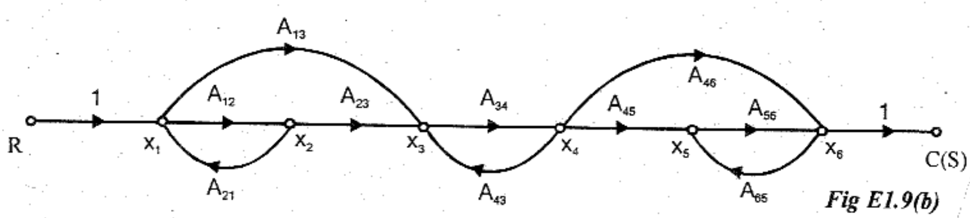


Fig E1.9(b)

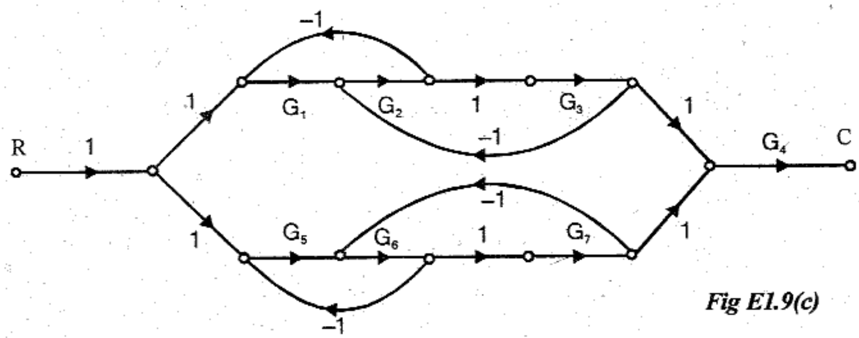


Fig E1.9(c)

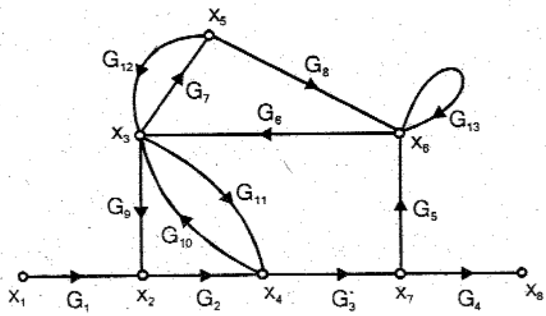


Fig E1.9(d)

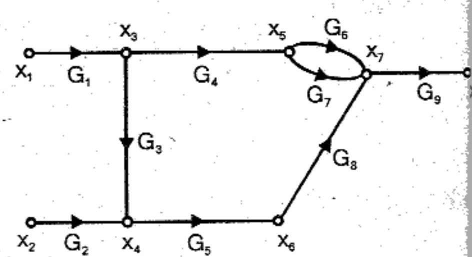


Fig E1.10

E1.10 Consider the signal flow graph shown in fig E.1.10 obtain $\dfrac{X_8}{X_1}$ and $\dfrac{X_8}{X_2}$

E1.11 Find the transfer functions of the networks shown in fig E1.11(a), (b), (c) & (d).

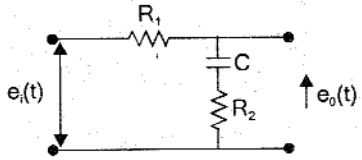


Fig E1.11(a)

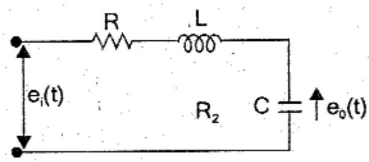


Fig E1.11(b)

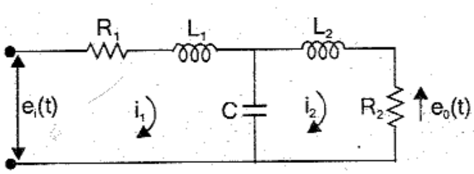


Fig E1.11(c)

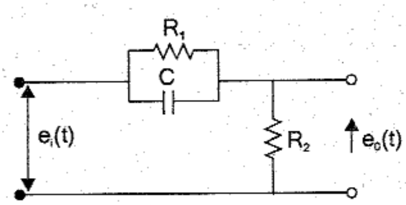


Fig E1.11(d)

E1.12 Find the transfer function of the circuit shown in fig E1.12.

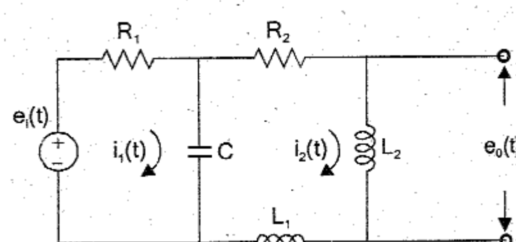
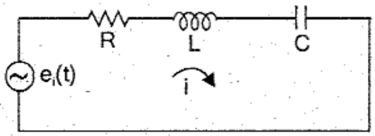


Fig E1.12

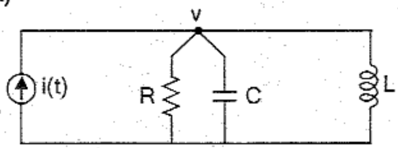
ANSWER FOR EXERCISE PROBLEMS

1. The transfer function is $\dfrac{X(s)}{F(s)} = \dfrac{1}{(Ms^2 + Bs + K)}$



$f(t) \to e(t)$ $M \to L$ $K \to 1/C$
$v \to i$ $B \to R$

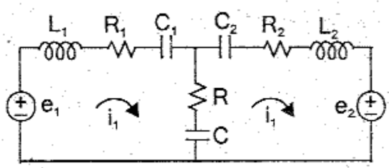
Force-voltage analogous circuit



$f(t) \to i(t)$ $M \to C$ $K \to 1/L$
$v \to v$ $B \to 1/R$

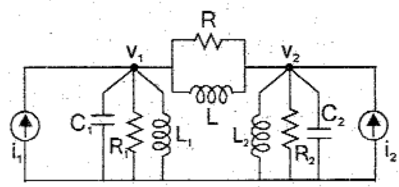
Force-current analogous circuit

2



$f_1 \to e_1$ $M_1 \to L_1$ $B \to R$
$f_2 \to e_2$ $M_2 \to L_2$ $K_1 \to 1/C_1$
$v_1 \to i_1$ $B_1 \to R_1$ $K_2 \to 1/C_2$
$v_2 \to i_2$ $B_2 \to R_2$ $K \to 1/C$

Force-voltage analogous circuit

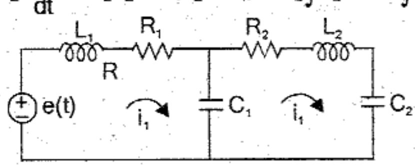


$f_1 \to i_1$ $M_1 \to C_1$ $B \to 1/R$
$f_2 \to i_2$ $M_2 \to C_2$ $K_1 \to 1/L_1$
$v_1 \to v_1$ $B_1 \to 1/R_1$ $K_2 \to 1/L_2$
$v_2 \to v_2$ $B_2 \to 1/R_2$ $K \to 1/L$

Force-current analogous circuit

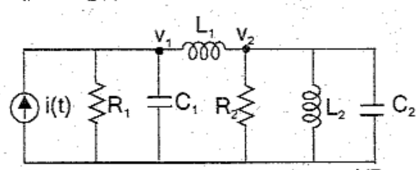
3 a) $M_1 \dfrac{dv_1}{dt} + B_1 v_1 + B(v_1 - v_2) + K_1 \int v_1 dt + K \int (v_1 - v_2) dt = f_1(t)$

$M_2 \dfrac{dv_2}{dt} + B_2 v_2 + B(v_2 - v_1) + K_2 \int v_2 dt + K \int (v_2 - v_1) dt = f_2(t)$



$f(t) \to e(t)$ $M_1 \to L_1$ $B_2 \to R_2$
$v_1 \to i_1$ $M_2 \to L_2$ $K_1 \to 1/C_1$
$v_2 \to i_2$ $B_1 \to R_1$ $K_2 \to 1/C_2$

Force-voltage analogous circuit

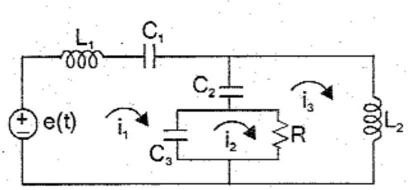


$f(t) \to i(t)$ $M_1 \to C_1$ $B_2 \to 1/R_2$
$v_1 \to v_1$ $M_2 \to C_2$ $K_1 \to 1/L_1$
$v_2 \to v_2$ $B_1 \to 1/R_1$ $K_2 \to 1/L_2$

Force-current analogous circuit

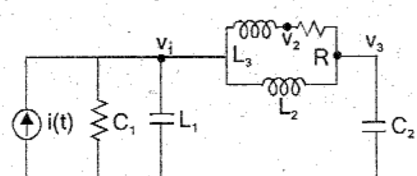
3 b) $M_1 \dfrac{dv_1}{dt} + K_1 \int v_1 dt + K_2 \int (v_1 - v_3) dt + K_3 \int (v_1 - v_2) dt = f(t)$

$K_3 \int (v_2 - v_1) dt + B(v_2 - v_3) = 0;$ $M_2 \dfrac{dv_3}{dt} + B(v_3 - v_2) + K_2 \int (v_3 - v_1) dt = 0$



$f(t) \to e(t)$ $M_1 \to L_1$ $K_1 \to 1/C_1$
$v_1 \to i_1$ $M_2 \to L_2$ $K_2 \to 1/C_2$
$v_2 \to i_2$ $B \to R$ $K_3 \to 1/C_3$
$v_3 \to i_3$

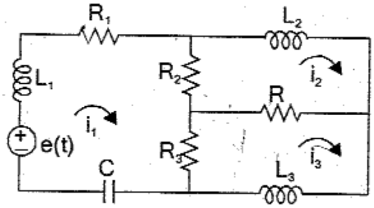
Force-voltage analogous circuit



$f(t) \to i(t)$ $M_1 \to C_1$ $K_1 \to 1/L_1$
$v_1 \to v_1$ $M_2 \to C_2$ $K_2 \to 1/L_2$
$v_2 \to v_2$ $B \to 1/R$ $K_3 \to 1/L_3$
$v_3 \to v_3$

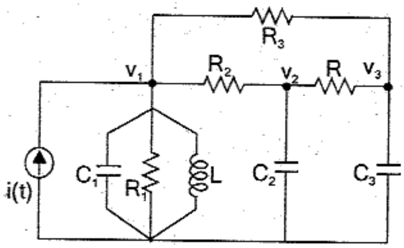
Force-current analogous circuit

E1.4



$f(t) \to e(t)$ $\quad M_1 \to L_1$ $\quad B_1 \to R_1$
$v_1 \to i_1$ $\quad M_2 \to L_2$ $\quad B_2 \to R_2$
$v_2 \to i_2$ $\quad M_3 \to L_3$ $\quad B_3 \to R_3$
$v_3 \to i_3$ $\quad B \to R$ $\quad K \to 1/C$

Force voltage-analogous circuit

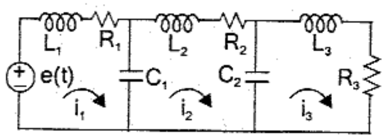


$f(t) \to i(t)$ $\quad M_1 \to C_1$ $\quad B_1 \to 1/R_1$
$v_1 \to v_1$ $\quad M_2 \to C_2$ $\quad B_2 \to 1/R_2$
$v_2 \to v_2$ $\quad M_3 \to C_3$ $\quad B_3 \to 1/R_3$
$v_3 \to v_3$ $\quad B \to 1/R$ $\quad K \to 1/L$

Force-current analogous circuit

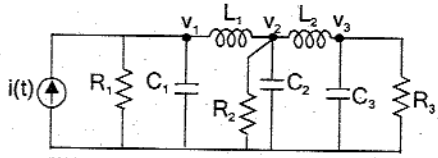
E1.5 $J_1\dfrac{d\omega_1}{dt} + B_1\omega_1 + K_1\int(\omega_1 - \omega_2)dt = T(t);\quad J_2\dfrac{d\omega_2}{dt} + B_2\omega_2 + K_1\int(\omega_2 - \omega_1)dt + K_2\int(\omega_2 - \omega_3)dt = 0$

$$J_3\frac{d\omega_3}{dt} + B_3\omega_3 + K_2\int(\omega_3 - \omega_2)dt = 0$$



$T(t) \to e(t)$ $\quad J_1 \to L_1$ $\quad B_1 \to R_1$ $\quad K_1 \to 1/C_1$
$\omega_1 \to i_1$ $\quad J_2 \to L_2$ $\quad B_2 \to R_2$ $\quad K_2 \to 1/C_2$
$\omega_2 \to i_2$ $\quad J_3 \to L_3$ $\quad B_3 \to R_3$ $\quad \omega_3 \to i$

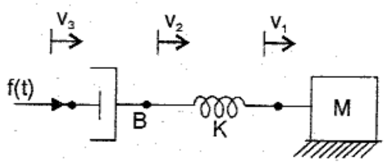
Torque-voltage analogous circuit



$T(t) \to i(t)$ $\quad J_1 \to C_1$ $\quad B_1 \to 1/R_1$ $\quad K_1 \to 1/L_1$
$\omega_1 \to v_1$ $\quad J_2 \to C_2$ $\quad B_2 \to 1/R_2$ $\quad K_2 \to 1/L_2$
$\omega_2 \to v_2$ $\quad J_3 \to C_3$ $\quad B_3 \to 1/R_3$ $\quad \omega_3 \to v_3$

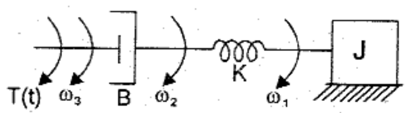
Torque-current analogous circuit

E1.6



$e(t) \to f(t)$ $\quad i_1 \to v_1$ $\quad i_3 \to v_3$ $\quad R \to B$
$i_2 \to v_2$ $\quad L \to M$ $\quad 1/C \to K$

Analogous mechanical translational system



$e(t) \to T(t)$ $\quad i_1 \to \omega_1$ $\quad i_3 \to \omega_3$ $\quad R \to B$
$i_2 \to \omega_2$ $\quad L \to J$ $\quad 1/C \to K$

Analogous mechanical rotational system

E1.7 **(a)** $\dfrac{C}{R} = \dfrac{G_1G_2 + G_1G_3}{1 + G_1H_2 + G_1 + G_1G_2H_1 + G_1G_3H_1}$

(b) $\dfrac{C}{R} = \dfrac{G_1G_2(1+G_3)\,(G_4+G_5)}{1 + (1+G_3)\,(G_4+G_5)\,H_2 + (1+G_3)\,(G_4+G_5)\,G_2H_1}$

(c) $\dfrac{C}{R} = \dfrac{G_1G_2 + G_1G_3}{1 + G_1G_2H_1 + G_1G_2 + G_1G_3 + G_2H_2 + G_3H_2}$

(d) $\dfrac{C}{R} = \dfrac{G_1G_2G_3}{1 + G_2G_3H_2 + G_1G_2H_1 + G_1G_2G_3}$

$$\frac{G_1G_2G_3 + G_4 + G_2G_3G_4H_1 + G_2G_4H_2 + G_1G_2G_4H_3}{1 + G_2G_3H_1 + G_2H_2 + G_1G_2H_3}$$

(a) $\dfrac{C(s)}{R(s)} = \dfrac{G_1G_2G_3}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3}$

(b) $\dfrac{C}{R} = \dfrac{A_{12}A_{23}A_{34}A_{45}A_{56} + A_{13}A_{34}A_{45}A_{56} + A_{12}A_{23}A_{34}A_{46} + A_{13}A_{34}A_{46}}{1 - (A_{12}A_{21} + A_{34}A_{43} + A_{56}A_{65}) + (A_{12}A_{21}A_{34}A_{43} + A_{12}A_{21}A_{56}A_{65} + A_{34}A_{43}A_{56}A_{65}) - (A_{12}A_{21}A_{34}A_{43}A_{56}A_{65})}$

(c) $\dfrac{C}{R} = \dfrac{G_1G_2G_3G_4\,(1 + G_5G_6 + G_6G_7) + G_4G_5G_6G_7(1 + G_1G_2 + G_2G_3)}{1 + G_1G_2 + G_2G_3 + G_5G_6 + G_6G_7 + G_1G_2G_5G_6 + G_5G_6G_2G_3 + G_1G_2G_6G_7 + G_2G_3G_6G_7}$

(d) $\dfrac{x_8}{x_1} = \dfrac{[G_1G_2G_3G_4]\,[1 - (G_7G_{12} + G_6G_7G_8 + G_{13}) + G_7G_{12}G_{13}]}{1 - [G_2G_9G_{10} + G_{10}G_{11} + G_2G_3G_5G_6G_9 + G_3G_5G_6G_{11} + G_7G_{12} + G_6G_7G_8 + G_{13}] + G_2G_9G_{10}G_{13} + G_{10}G_{11}G_{13} + G_7G_{12}G_{13}}$

$\dfrac{x_8}{x_1} = G_1G_4G_6G_9 + G_1G_4G_7G_9 + G_1G_3G_5G_8G_9 \quad ; \quad \dfrac{x_8}{x_2} = G_2G_5G_8G_9$

(a) $\dfrac{E_o(s)}{E_i(s)} = \dfrac{1 + s\,R_2\,C}{1 + s\,(R_1 + R_2)\,C}$

(b) $\dfrac{E_o(s)}{E_i(s)} = \dfrac{1}{s^2LC + s\,R\,C + 1}$

(c) $\dfrac{E_o(s)}{E_i(s)} = \dfrac{s\,R_2\,C}{(s^2L_1C + s\,R_1\,C + 1)\,(s^2L_2C + s\,R_2\,C + 1) - 1}$

(d) $\dfrac{E_o(s)}{E_i(s)} = \dfrac{sR_1R_2C + R_2}{sR_1R_2C + (R_1 + R_2)}$

$$\frac{C(s)}{E(s)} = \frac{s^2\,L_2\,C}{[s\,R_1\,C + 1]\,[s^2(L_1 + L_2)\,C + s\,R_2\,C + 1] - 1}$$

CHAPTER 2

TIME RESPONSE ANALYSIS

2.1 TIME RESPONSE

The time response of the system is the output of the closed loop system as a function of time. It is denoted by c(t). The time response can be obtained by solving the differential equation governing the system. Alternatively, the response c(t) can be obtained from the transfer function of the system and the input to the system.

$$\text{The closed loop transfer function, } \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)} = M(s) \qquad .....(2.1)$$

The Output or Response in s-domain, C(s) is given by the product of the transfer function and the input, R(s). On taking inverse Laplace transform of this product the time domain response, c(t) can be obtained.

$$\text{Response in s-domain, } C(s) = R(s)\,M(s) \qquad .....(2.2)$$

$$\text{Response in time domain, } c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\{R(s) \times M(s)\} \qquad .....(2.3)$$

$$\text{where, } M(s) = \frac{G(s)}{1+G(s)H(s)}$$

The time response of a control system consists of two parts : *the transient and the steady state response.* The transient response is the response of the system when the input changes from one state to another. The steady state response is the response as time, **t** approaches infinity.

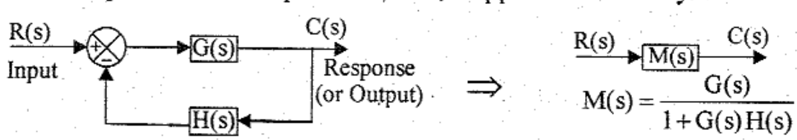


Fig 2.1 : Closed loop system.

2.2 TEST SIGNALS

The knowledge of input signal is required to predict the response of a system. In most of the systems the input signals are not known ahead of time and also it is difficult to express the input signals mathematically by simple equations. The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity and a constant acceleration. Hence test signals which resembles these characteristics are used as input signals to predict the performance of the system. The commonly used test input signals are impulse, step, ramp, acceleration and sinusoidal signals.

The standard test signals are,

1. a) Step signal
   b) Unit step signal
2. a) Ramp signal
   b) Unit ramp signal
3. a) Parabolic signal
   b) Unit parabolic signal
4. Impulse signal
5. Sinusoidal signal.

Since the test signals are simple functions for time, they can be easily generated in laboratories. The mathematical and experimental analysis of control systems using these signals can be carried out easily. The use of the test signals can be justified because of a correlation existing between the response characteristics of a system to a test input signal and capability of the system to cope with actual input signals.

STEP SIGNAL

The step signal is a signal whose value changes from zero to A at t = 0 and remains constant at A for t > 0. The step signal resembles an actual steady input to a system. A special case of step signal is unit step in which **A** is unity.

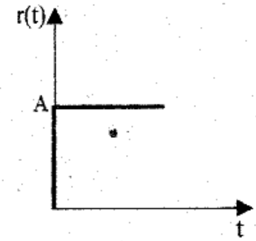


Fig 2.2 : Step signal.

The mathematical representation of the step signal is,

$$r(t) = 1 \; ; \; t \geq 0$$
$$= 0 \; ; \; t < 0 \qquad .....(2.4)$$

RAMP SIGNAL

The ramp signal is a signal whose value increases linearly with time from an initial value of zero at t = 0. The ramp signal resembles a constant velocity input to the system. A special case of ramp signal is unit ramp signal in which the value of **A** is unity.

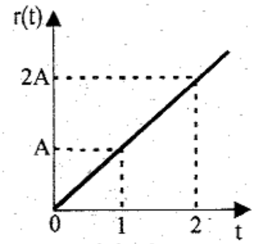


Fig 2.3 : Ramp signal.

The mathematical representation of the ramp signal is,

$$r(t) = A\,t \; ; \; t \geq 0$$
$$= 0 \; ; \; t < 0 \qquad .....(2.5)$$

PARABOLIC SIGNAL

In parabolic signal, the instantaneous value varies as square of the time from an initial value of zero at t = 0. The sketch of the signal with respect to time resembles a parabola. The parabolic signal resembles a constant acceleration input to the system. A special case of parabolic signal is unit parabolic signal in which A is unity.

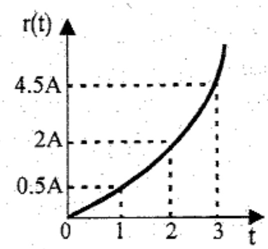


Fig 2.4 : Parabolic signal.

The mathematical representation of the parabolic signal is,

$$r(t) = \frac{At^2}{2} \; ; \; t \geq 0$$
$$= 0 \; ; \; t < 0 \qquad .....(2.6)$$

***Note :** Integral of step signal is ramp signal. Integral of ramp signal is parabolic signal.*

IMPULSE SIGNAL

A signal of very large magnitude which is available for very short duration is called ***impulse signal.*** Ideal impulse signal is a signal with infinite magnitude and zero duration but with an area of A. The unit impulse signal is a special case, in which A is unity.

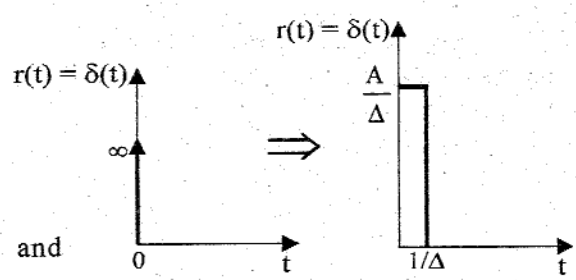


Fig 2.5 : Impulse signal.

The impulse signal is denoted by $\delta(t)$ and mathematically it is expressed as,

$$\delta(t) = \infty ; \; t = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t)dt = A$$
$$= 0 ; \; t \neq 0 \qquad .....(2.7)$$

Since a perfect impulse cannot be achieved in practice it is usually approximated by a pulse of small width but with area, A. Mathematically an impulse signal is the derivative of a step signal. Laplace transform of the impulse function is unity.

TABLE 2-1 : Standard Test Signals

Name of the signal	Time domain equation of signal, r(t)	Laplace transform of the signal, R(s)
Step	A	$\frac{A}{s}$
Unit step	1	$\frac{1}{s}$
Ramp	At	$\frac{A}{s^2}$
Unit ramp	t	$\frac{1}{s^2}$
Parabolic	$\frac{At^2}{2}$	$\frac{A}{s^3}$
Unit parabolic	$\frac{t^2}{2}$	$\frac{1}{s^3}$
Impulse	$\delta(t)$	1

2.3 IMPULSE RESPONSE

The response of the system, with input as impulse signal is called ***weighing function*** or ***impulse response*** of the system. It is also given by the inverse Laplace transform of the system transfer function, and denoted by m(t).

$$\text{Impulse response, } m(t) = \mathcal{L}^{-1}\{R(s)\,M(s)\} = \mathcal{L}^{-1}\{M(s)\} \quad .....(2.8)$$

$$\text{where, } M(s) = \frac{G(s)}{1+G(s)H(s)} \qquad \boxed{R(s) = 1, \text{ for impulse}}$$

Since impulse response (or weighing function) is obtained from the transfer function of the system, it shows the characteristics of the system. Also the response for any input can be obtained by convolution of input with impulse response.

2.4 ORDER OF A SYSTEM

The input and output relationship of a control system can be expressed by n^{th} order differential equation shown in equation (2.9).

$$a_0\frac{d^n}{dt^n}p(t) + a_1\frac{d^{n-1}}{dt^{n-1}}p(t) + a_2\frac{d^{n-2}}{dt^{n-2}}p(t) + ...+ a_{n-1}\frac{d}{dt}p(t) + a_n\,p(t) = b_0\frac{d^m}{dt^m}q(t)$$
$$+ b_1\frac{d^{m-1}}{dt^{m-1}}q(t) + b_2\frac{d^{m-2}}{dt^{m-2}}q(t) + ...+ b_{m-1}\frac{d}{dt}q(t) + b_m\,q(t) \quad .....(2.9)$$

where, $p(t) = $ Output / Response ; $q(t) = $ Input / Excitation.

The order of the system is given by the order of the differential equation governing the system. If the system is governed by n^{th} order differential equation, then the system is called ***n^{th} order system.***

Alternatively, the order can be determined from the transfer function of the system. The transfer function of the system can be obtained by taking Laplace transform of the differential equation governing the system and rearranging them as a ratio of two polynomials in **s**, as shown in equation (2.10).

$$\text{Transfer function, } T(s)=\frac{P(s)}{Q(s)}=\frac{b_0 s^m+b_1 s^{m-1}+b_2 s^{m-2}+........+b_{m-1}s+b_m}{a_0 s^n+a_1 s^{n-1}+a_2 s^{n-2}+........+a_{n-1}s+a_n} \qquad .....(2.10)$$

where, $P(s)$ = Numerator polynomial

$Q(s)$ = Denominator polynomial

The order of the system is given by the maximum power of **s** in the denominator polynomial, $Q(s)$.

Here, $Q(s)=a_0\, s^n+a_1\, s^{n-1}+a_2\, s^{n-2}+ .... +a_{n-1}s+a_n$.

Now, **n** is the order of the system

When n = 0, the system is zero order system.

When n = 1, the system is first order system.

When n = 2, the system is second order system and so on.

***Note :** The order can be specified for both open loop system and closed loop system.*

The numerator and denominator polynomial of equation (2.10) can be expressed in the factorized form as shown in equation (2.11).

$$T(s)=\frac{P(s)}{Q(s)}=\frac{(s+z_1)(s+z_2).......(s+z_m)}{(s+p_1)(s+p_2).......(s+p_n)} \qquad .....(2.11)$$

where, $z_1, z_2, ...z_m$ are zeros of the system.

$p_1, p_2, ...p_n$ are poles of the system.

Now, the value of **n** gives the number of poles in the transfer function. Hence the order is also given by the number of poles of the transfer function.

***Note :** The zeros and poles are critical value, of s, at which the function T(s) attains extreme values 0 or ∞. When s takes the value of a zero, the function T(s) will be zero. When s takes the value of a pole, the function T(s) will be infinite.*

2.5 REVIEW OF PARTIAL FRACTION EXPANSION

The time response of the system is obtained by taking the inverse Laplace transform of the product of input signal and transfer function of the system. Taking inverse Laplace transform requires the knowledge of partial fraction expansion. In control systems three different types of transfer function are encountered. They are,

Case 1 : Functions with separate poles.

Case 2 : Functions with multiple poles.

Case 3 : Functions with complex conjugate poles.

The partial fraction of all the three cases are explained with an example.

Case 1 : When the transfer function has distinct poles

$$\text{Let, } T(s)=\frac{K}{s(s+p_1)(s+p_2)}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s)=\frac{K}{s(s+p_1)(s+p_2)}=\frac{A}{s}+\frac{B}{s+p_1}+\frac{C}{s+p_2}$$

The residues A, B and C are given by,

$$A=T(s)\times s\Big|_{s=0}\qquad B=T(s)\times(s+p_1)\Big|_{s=-p_1}\qquad C=T(s)\times(s+p_2)\Big|_{s=-p_2}$$

Example

$$\text{Let, } T(s)=\frac{2}{s(s+1)(s+2)}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s)=\frac{2}{s(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

A is obtained by multiplying T(s) by s and letting s = 0.

$$A=T(s)\times s\Big|_{s=0}=\frac{2}{s(s+1)(s+2)}\times s\Big|_{s=0}=\frac{2}{(s+1)(s+2)}\Bigg|_{s=0}=\frac{2}{1\times 2}=1$$

B is obtained by multiplying T(s) by (s +1) and letting s = –1.

$$B=T(s)\times(s+1)\Big|_{s=-1}=\frac{2}{s(s+1)(s+2)}\times(s+1)\Big|_{s=-1}=\frac{2}{s(s+2)}\Bigg|_{s=-1}=\frac{2}{-1(-1+2)}=-2$$

C is obtained by multiplying T(s) by (s +2) and letting s = –2.

$$C=T(s)\times(s+2)\Big|_{s=-2}=\frac{2}{s(s+1)(s+2)}\times(s+2)\Big|_{s=-2}=\frac{2}{s(s+1)}\Bigg|_{s=-2}=\frac{2}{-2(-2+1)}=+1$$

$$\therefore T(s)=\frac{2}{s(s+1)(s+2)}=\frac{1}{s}-\frac{2}{s+1}+\frac{1}{s+2}$$

Case 2 : When the transfer function has multiple poles

$$\text{Let, } T(s)=\frac{K}{s(s+p_1)(s+p_2)^2}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s)=\frac{K}{s(s+p_1)(s+p_2)^2}=\frac{A}{s}+\frac{B}{s+p_1}+\frac{C}{(s+p_2)^2}+\frac{D}{(s+p_2)}$$

The residues A, B, C and D are given by,

$$A=T(s)\times s\Big|_{s=0}\qquad B=T(s)\times(s+p_1)\Big|_{s=-p_1}$$

$$C=T(s)\times(s+p_2)^2\Big|_{s=-p_2}\qquad D=\frac{d}{ds}\left[T(s)\times(s+p_2)^2\right]\Big|_{s=-p_2}$$

Example

$$\text{Let, } T(s)=\frac{2}{s(s+1)(s+2)^2}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s)=\frac{K}{s(s+1)(s+2)^2}=\frac{A}{s}+\frac{B}{(s+1)}+\frac{C}{(s+2)^2}+\frac{D}{(s+2)}$$

A is obtained by multiplying T(s) by s and letting s = 0.

$$A = T(s) \times s\Big|_{s=0} = \frac{2}{s(s+1)(s+2)^2} \times s\Big|_{s=0} = \frac{2}{(s+1)(s+2)^2}\Big|_{s=0} = \frac{2}{1 \times 2^2} = 0.5$$

B is obtained by multiplying T(s) by (s +1) and letting s = –1.

$$B = T(s) \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+1)(s+2)^2} \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+2)^2}\Big|_{s=-1} = \frac{2}{-1(-1+2)^2} = -2$$

C is obtained by multiplying T(s) by $(s+2)^2$ and letting s = –2.

$$C = T(s) \times (s+2)^2\Big|_{s=-2} = \frac{2}{s(s+1)(s+2)^2} \times (s+2)^2\Big|_{s=-2} = \frac{2}{s(s+1)}\Big|_{s=-2} = \frac{2}{-2(-2+1)} = 1$$

D is obtained by differentiating the product T(s) $(s+2)^2$ with respect to s and then letting s = –2.

$$D = \frac{d}{ds}\left[T(s) \times (s+2)^2\right]\Big|_{s=-2} = \frac{d}{ds}\left[\frac{2}{s(s+1)}\right]\Big|_{s=-2} = \frac{-2(2s+1)}{s^2(s+1)^2}\Big|_{s=-2} = \frac{-2(2(-2)+1)}{(-2)^2(-2+1)^2} = +1.5$$

$$\therefore\ T(s) = \frac{2}{s(s+1)(s+2)^2} = \frac{0.5}{s} - \frac{2}{s+1} + \frac{1}{(s+2)^2} + \frac{1.5}{s+2}$$

Case 3 : When the transfer function has complex conjugate poles

$$\text{Let, } T(s) = \frac{K}{(s+p_1)(s^2+bs+c)}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s) = \frac{K}{(s+p_1)(s^2+bs+c)} = \frac{A}{s+p_1} + \frac{Bs+C}{s^2+bs+c} \qquad .....(2.12)$$

The residue A is given by, $A = T(s) \times (s+p_1)\big|_{s=-p_1}$

The residues B and C are solved by cross multiplying the equation (2.12) and then equating the coefficient of like power of **s.**

Finally express T(s) as shown below,

$$T(s) = \frac{A}{s+p_1} + \frac{Bs+C}{s^2+bs+c}$$

$$\boxed{(x+y)^2 = x^2+2xy+y^2}$$

Let us express, $s^2 + bs$, in the form of $(x + y)^2$. This will require addition and subtraction of an extra term $(b/2)^2$.

$$\therefore\ T(s) = \frac{A}{s+p_1} + \frac{Bs+C}{s^2 + 2 \times \frac{b}{2}s + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2} = \frac{A}{s+p_1} + \frac{Bs+C}{\left(s+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)}$$

$$= \frac{A}{s+p_1} + \frac{Bs}{\left(s+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)} + \frac{C}{\left(s+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)}$$

Example

Let, $T(s) = \dfrac{1}{(s+2)\,(s^2+s+1)}$

By partial fraction expansion,

$$T(s) = \frac{1}{(s+2)\,(s^2+s+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+s+1}$$

A is obtained by multiplying T(s) by (s +2) and letting s = –2.

$$\therefore A = T(s) \times (s+2)\Big|_{s=-2} = \frac{1}{(s+2)\,(s^2+s+1)} \times (s+2)\Bigg|_{s=-2} = \frac{1}{(-2)^2-2+1} = \frac{1}{3}$$

To solve B and C, cross multiply the following equation and substitute the value of A. Then equate the like power of s.

$$\frac{1}{(s+2)\,(s^2+s+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+s+1}$$

$$1 = A(s^2+s+1) + (Bs+C)\,(s+2)$$

$$1 = \frac{1}{3}(s^2+s+1) + Bs^2 + 2Bs + Cs + 2C$$

$$1 = \frac{s^2}{3} + \frac{s}{3} + \frac{1}{3} + Bs^2 + 2Bs + Cs + 2C$$

$$\boxed{\begin{aligned} &s^2+s+1\\ &\quad = s^2 + 2\times\frac{s}{2} + \left(\frac{1}{2}\right)^2 + 1 - \left(\frac{1}{2}\right)^2\\ &\quad = \left(s+\frac{1}{2}\right)^2 + \left(1-\frac{1}{4}\right)\\ &\quad = (s+0.5)^2 + 0.75 \end{aligned}}$$

On equating the coefficient of s^2 terms, $0 = \dfrac{1}{3} + B$; $\therefore B = -\dfrac{1}{3}$

On equating the coefficient of s terms, $0 = \dfrac{1}{3} + 2B + C$; $\therefore C = -\dfrac{1}{3} - 2B = -\dfrac{1}{3} + \dfrac{2}{3} = \dfrac{1}{3}$

$$T(s) = \frac{\frac{1}{3}}{s} + \frac{\frac{-1}{3}s + \frac{1}{3}}{s^2+s+1} = \frac{1}{3s} - \frac{1}{3}\frac{s}{(s^2+s+1)} + \frac{1}{3}\frac{1}{(s^2+s+1)}$$

$$= \frac{1}{3s} - \frac{1}{3}\frac{s}{(s+0.5)^2+0.75} + \frac{1}{3}\frac{1}{(s+0.5)^2+0.75}$$

2.6 RESPONSE OF FIRST ORDER SYSTEM FOR UNIT STEP INPUT

The closed loop order system with unity feedback is shown in fig 2.6.

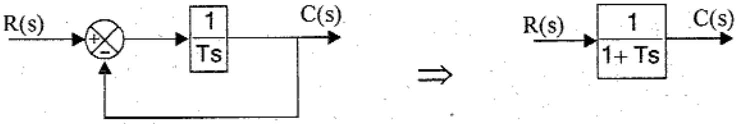


Fig 2.6 : *Closed loop for first order system.*

The closed loop transfer function of first order system, $\dfrac{C(s)}{R(s)} = \dfrac{1}{1+Ts}$

If the input is unit step then, $r(t) = 1$ and $R(s) = \dfrac{1}{s}$.

$$\therefore \text{The response in s-domain, } C(s) = R(s)\frac{1}{(1+Ts)} = \frac{1}{s}\frac{1}{(1+Ts)} = \frac{1}{sT\left(\frac{1}{T}+s\right)} = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)}$$

By partial fraction expansion,

$$C(s)=\frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)}=\frac{A}{s}+\frac{B}{\left(s+\frac{1}{T}\right)}$$

A is obtained by multiplying C(s) by s and letting s = 0.

$$A=C(s)\times s\Big|_{s=0}=\left.\frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)}\times s\right|_{s=0}=\left.\frac{\frac{1}{T}}{s+\frac{1}{T}}\right|_{s=0}=\frac{\frac{1}{T}}{\frac{1}{T}}=1$$

B is obtained by multiplying C(s) by (s+1/T) and letting s = –1/T.

$$B=C(s)\times\left(s+\frac{1}{T}\right)\Big|_{s=-\frac{1}{T}}=\left.\frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)}\times\left(s+\frac{1}{T}\right)\right|_{s=-\frac{1}{T}}=\left.\frac{\frac{1}{T}}{s}\right|_{s=-\frac{1}{T}}=\frac{\frac{1}{T}}{\frac{-1}{T}}=-1$$

$$\therefore C(s)=\frac{1}{s}-\frac{1}{s+\frac{1}{T}}$$

$$\boxed{\mathcal{L}\left\{e^{-at}\right\}=\frac{1}{s+a}}$$

The response in time domain is given by,

$$c(t)=\mathcal{L}^{-1}\{C(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s+\frac{1}{T}}\right\}=1-e^{-\frac{t}{T}} \qquad .....(2.13)$$

The equation (2.13) is the response of the closed loop first order system for unit step input. For step input of step value, A, the equation (2.13) is multiplied by A.

$$\boxed{\begin{aligned}\therefore \text{For closed loop first order system, Unit step response } &= 1-e^{-\frac{t}{T}}\\ \text{Step response } &= A\left(1-e^{-\frac{t}{T}}\right)\end{aligned}}$$

When, $t = 0$, $c(t) = 1 - e^{0} = 0$

When, $t = 1T$, $c(t) = 1 - e^{-1} = 0.632$

When, $t = 2T$, $c(t) = 1 - e^{-2} = 0.865$

When, $t = 3T$, $c(t) = 1 - e^{-3} = 0.95$

When, $t = 4T$, $c(t) = 1 - e^{-4} = 0.9817$

When, $t = 5T$, $c(t) = 1 - e^{-5} = 0.993$

When, $t = \infty$, $c(t) = 1 - e^{-\infty} = 1$

Here T is called Time constant of the system. In a time of 5T, the system is assumed to have attained steady state. The input and output signal of the first order system is shown in fig 2.7.

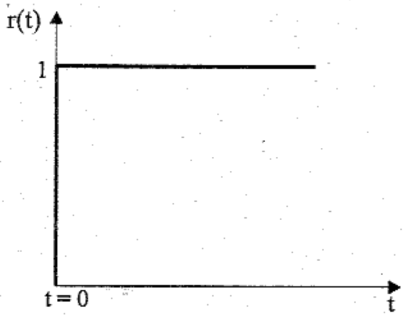


Fig 2.7a : Unit step input.

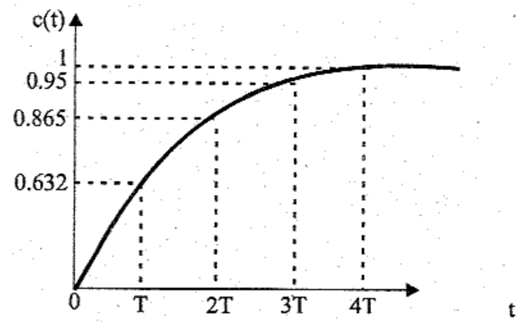


Fig 2.7b : Response for Unit step input.

Fig 2.7 : Response of first order system to Unit step input.

2.7 SECOND ORDER SYSTEM

The closed loop second order system is shown in fig 2.8

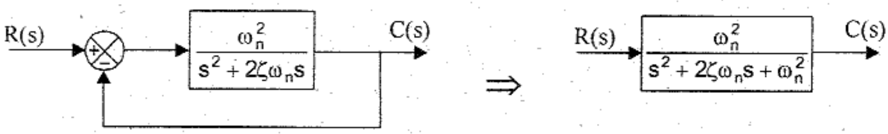


Fig 2.8 : Closed loop for second order system.

The standard form of closed loop transfer function of second order system is given by,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad .....(2.14)$$

where, ω_n = Undamped natural frequency, rad/sec.

ζ = Damping ratio.

*The **damping ratio*** is defined as the ratio of the actual damping to the critical damping. The response c(t) of second order system depends on the value of damping ratio. Depending on the value of ζ, the system can be classified into the following four cases,

Case 1 : Undamped system, $\zeta = 0$

Case 2 : Under damped system, $0 < \zeta < 1$

Case 3 : Critically damped system, $\zeta = 1$

Case 4 : Over damped system, $\zeta > 1$

The characteristics equation of the second order system is,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad .....(2.15)$$

It is a quadratic equation and the roots of this equation is given by,

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2 - 1)}}{2}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \qquad .....(2.16)$$

When $\zeta = 0$, $s_1, s_2 = \pm j\omega_n$; $\begin{cases}\text{roots are purely imaginary} \\ \text{and the system is undamped}\end{cases}$ .....(2.17)

When $\zeta = 1$, $s_1, s_2 = -\omega_n$; $\begin{cases}\text{roots are real and equal and} \\ \text{the system is critically damped}\end{cases}$ .....(2.18)

When $\zeta > 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$; $\begin{cases}\text{roots are real and unequal and} \\ \text{the system is overdamped}\end{cases}$ .....(2.19)

When $0 < \zeta < 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)}$

$$= -\zeta\omega_n \pm \omega_n\sqrt{-1}\,\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$= -\zeta\omega_n \pm j\omega_d \,; \begin{cases}\text{roots are complex conjugate} \\ \text{the system is underdamped}\end{cases} \quad .....(2.20)$$

$$\text{where, } \omega_d = \omega_n\sqrt{1-\zeta^2} \quad .....(2.21)$$

Here ω_d is called damped frequency of oscillation of the system and its unit is rad/sec.

2.7.1 RESPONSE OF UNDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For undamped system, z = 0.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2} \quad .....(2.22)$$

When the input is unit step, $r(t) = 1$ and $R(s) = \frac{1}{s}$.

$$\therefore \text{The response in s-domain, } C(s) = R(s)\,\frac{\omega_n^2}{s^2 + \omega_n^2} = \frac{1}{s}\,\frac{\omega_n^2}{s^2 + \omega_n^2} \quad .....(2.23)$$

By partial fraction expansion,

$$C(s) = \frac{\omega_n^2}{s\,(s^2 + \omega_n^2)} = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2}$$

A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times s\Big|_{s=0} = \frac{\omega_n^2}{s\,(s^2 + \omega_n^2)} \times s\Bigg|_{s=0} = \frac{\omega_n^2}{s^2 + \omega_n^2}\Bigg|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

B is obtained by multiplying C(s) by $(s^2 + \omega_n^2)$ and letting $s^2 = -\omega_n^2$ or $s = j\omega_n$.

$$B = C(s) \times (s^2 + \omega_n^2)\Big|_{s=j\omega} = \frac{\omega_n^2}{s\,(s^2 + \omega_n^2)} \times (s^2 + \omega_n^2)\Bigg|_{s=j\omega} = \frac{\omega_n^2}{s}\Bigg|_{s=j\omega_n} = \frac{\omega_n^2}{j\omega_n} = -j\omega_n = -s$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2} = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

$\mathcal{L}\{1\} = \frac{1}{s}$	$\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}$

$$\text{Time domain response, } c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2 + \omega_n^2}\right\} = 1 - \cos\omega_n t \quad .....(2.24)$$

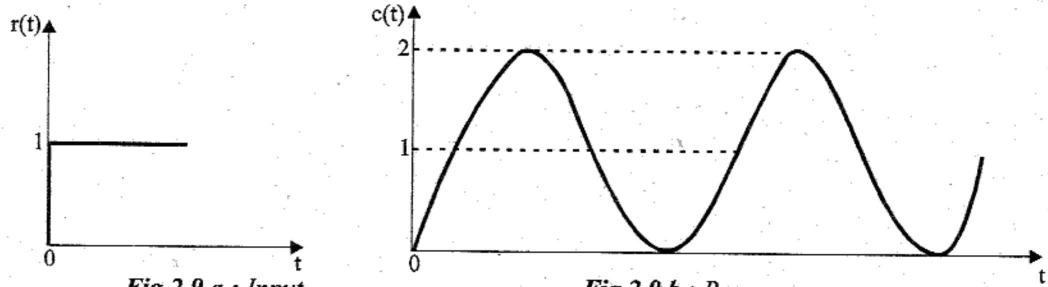


Fig 2.9.a : Input. *Fig 2.9.b : Response.*

Fig 2.9 : Response of undamped second order system for unit step input.

Using equation (2.24), the response of undamped second order system for unit step input is sketched in fig 2.9, and observed that the response is completely oscillatory.

> ***Note :*** *Every practical system has some amount of damping. Hence undamped system does not exist in practice.*

The equation (2.24) is the response of undamped closed loop second order system for unit step input. For step input of step value A, the equation (2.24) should be multiplied by A.

$\therefore$ For closed loop undamped second order system,

Unit step response $= 1 - \cos \omega_n t$

Step response $= A(1 - \cos \omega_n t)$

2.7.2 RESPONSE OF UNDERDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For underdamped system, $0 < \zeta < 1$ and roots of the denominator (characteristic equation) are complex conjugate.

The roots of the denominator are, $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

Since $\zeta < 1$, ζ^2 is also less then 1, and so $1 - \zeta^2$ is always positive.

$$\therefore\ s = -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore s = -\zeta\omega_n \pm j\omega_d$$

The response in s-domain, $C(s) = R(s)\,\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

For unit step input, $r(t) = 1$ and $R(s) = 1/s$.

$$\therefore C(s) = \frac{\omega_n^2}{s\,(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

By partial fraction expansion, $$C(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad .....(2.25)$$

A is obtained by multiplying C(s) by s and letting s = 0.

$$\therefore\ A = s \times C(s)\Big|_{s=0} = s \times \left.\frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}\right|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

To solve for B and C, cross multiply equation (2.25) and equate like power of s.

On cross multiplication equation (2.25) after substituting A = 1, we get,

$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + (Bs + C)s$$

$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + Bs^2 + Cs$$

Equating coefficients of s^2 we get, $0 = 1 + B$ $\quad\therefore B = -1$

Equating coefficient of s we get, $0 = 2\zeta\omega_n + C$ $\quad\therefore C = -2\zeta\omega_n$

$$\therefore\ C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad .....(2.26)$$

Let us add and subtract $\zeta^2\omega_n^2$ to the denominator of second term in the equation (2.26).

$$\therefore\ C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2 + \zeta^2\omega_n^2 - \zeta^2\omega_n^2} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2) + (\omega_n^2 - \zeta^2\omega_n^2)}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \qquad \boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \qquad .....(2.27)$$

Let us multiply and divide by ω_d in the third term of the equation (2.27).

$$\therefore\ C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\,\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}}$$

$$\boxed{\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2 + \omega^2}}$$

$$\boxed{\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2 + \omega^2}}$$

The response in time domain is given by,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\,\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right\}$$

$$= 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}\,e^{-\zeta\omega_n t}\sin\omega_d t = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t\right) \qquad \boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\ \cos\omega_d t + \zeta\ \sin\omega_d t\right) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\omega_d t \times \zeta + \cos\omega_d t \times \sqrt{1-\zeta^2}\right)$$

Let us express c(t) in a standard form as shown below.

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\sin\omega_d t \times \cos\theta + \cos\omega_d t \times \sin\theta)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta) \qquad .....(2.28)$$

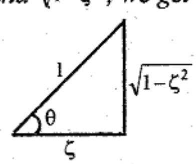
$$\text{where, } \left(\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

***Note :** On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get*

$\sin\theta = \sqrt{1-\zeta^2}$

$\cos\theta = \zeta$

$\tan\theta = \dfrac{\sqrt{1-\zeta^2}}{\zeta}$



The equation (2.28) is the response of under damped closed loop second order system for unit step input. For step input of step value, A, the equation (2.28) should be multiplied by A.

∴ For closed loop under damped second order system,

$$\text{Unit step response } = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)\ ;\ \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\text{Step response } = A\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)\ ;\ \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$$

Using equation (2.28) the response of underdamped second order system for unit step input is sketched and observed that the response oscillates before settling to a final value. The oscillations depends on the value of damping ratio.

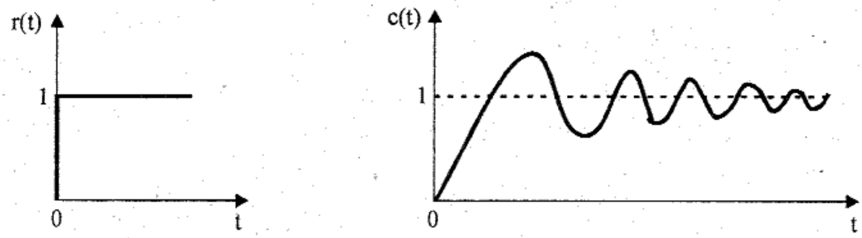


Fig 2.10.a : Input. *Fig 2.10.b : Response.*

Fig 2.10 : Response of under damped second order system for unit step input.

2.7.3 RESPONSE OF CRITICALLY DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For critical damping $\zeta = 1$.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2} \qquad .....(2.29)$$

When input is unit step, $r(t) = 1$ and $R(s) = 1/s$.

∴ The response in s-domain,

$$C(s) = R(s)\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{1}{s}\,\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{\omega_n^2}{s\,(s+\omega_n)^2} \qquad .....(2.30)$$

By partial fraction expansion, we can write,

$$C(s) = \frac{\omega_n^2}{s\,(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n}$$

$$A = s \times C(s)\Big|_{s=0} = \left.\frac{\omega_n^2}{(s+\omega_n)^2}\right|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+\omega_n)^2 \times C(s)\Big|_{s=-\omega_n} = \left.\frac{\omega_n^2}{s}\right|_{s=-\omega_n} = -\omega_n$$

$$C = \frac{d}{ds}\left[(s+\omega_n)^2 \times C(s)\right]\Big|_{s=-\omega_n} = \frac{d}{ds}\left(\frac{\omega_n^2}{s}\right)\Bigg|_{s=-\omega_n} = \left.\frac{-\omega_n^2}{s^2}\right|_{s=-\omega_n} = -1$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n} = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

$\mathcal{L}\{1\} = \frac{1}{s}$
$\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$
$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$

The response in time domain,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right\}$$

$$c(t) = 1 - \omega_n t\, e^{-\omega_n t} - e^{-\omega_n t}$$

$$c(t) = 1 - e^{-\omega_n t}(1+\omega_n t) \qquad .....(2.31)$$

The equation (2.31) is the response of critically damped closed loop second order system for unit step input. For step input of step value, A, the equation (2.31) should be multiplied by A.

$\therefore$ For closed loop critically damped second order system,

Unit step response $= 1 - e^{-\omega_n t}(1+\omega_n t)$

Step response $= A\left[1 - e^{-\omega_n t}(1+\omega_n t)\right]$

Using equation (2.31), the response of critically damped second order system is sketched as shown in fig 2.11 and observed that the response has no oscillations.

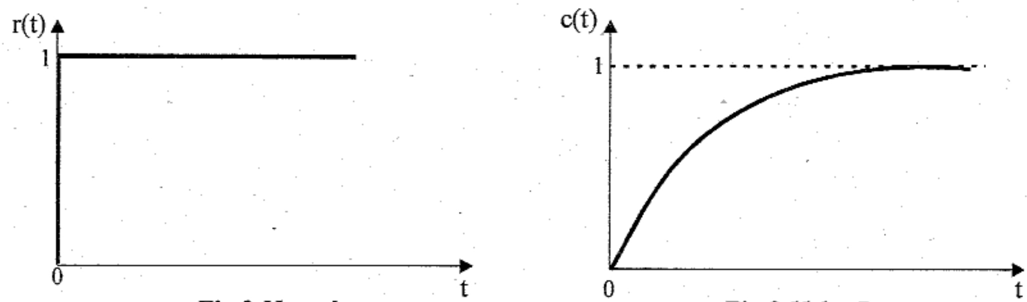


Fig 2.11.a : Input. *Fig 2.11.b : Response.*

Fig 2.11 : Response of critically damped second order system for unit step input.

2.7.4 RESPONSE OF OVER DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For overdamped system $\zeta > 1$. The roots of the denominator of transfer function are real and distinct. Let the roots of the denominator be s_a, s_b.

$$s_a, s_b = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\left[\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}\right] \qquad .....(2.32)$$

Let $s_1 = -s_a$ and $s_2 = -s_b$ $\qquad \therefore s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$ $\qquad .....(2.33)$

$$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1} \qquad .....(2.34)$$

The closed loop transfer function can be written in terms of s_1 and s_2 as shown below.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+s_1)\,(s+s_2)} \qquad .....(2.35)$$

For unit step input r(t) = 1 and R(s) = 1/s.

$$\therefore C(s) = R(s)\frac{\omega_n^2}{(s+s_1)\,(s+s_2)} = \frac{\omega_n^2}{s\,(s+s_1)\,(s+s_2)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{\omega_n^2}{s\,(s+s_1)\,(s+s_2)} = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2}$$

$$A = s \times C(s)\big|_{s=0} = s \times \left.\frac{\omega_n^2}{s\,(s+s_1)\,(s+s_2)}\right|_{s=0} = \frac{\omega_n^2}{s_1 s_2}$$

$$= \frac{\omega_n^2}{\left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]\left[\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]} = \frac{\omega_n^2}{\zeta^2\omega_n^2 - \omega_n^2\left(\zeta^2-1\right)} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+s_1) \times C(s)\big|_{s=-s_1} = \left.\frac{\omega_n^2}{s(s+s_2)}\right|_{s=-s_1} = \frac{\omega_n^2}{-s_1(-s_1+s_2)}$$

$$= \frac{-\omega_n^2}{s_1\left[-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]} = \frac{-\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_1} = \frac{-\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}$$

$$C = C(s) \times (s+s_2)\big|_{s=-s_2} = \left.\frac{\omega_n^2}{s\,(s+s_1)}\right|_{s=-s_2} = \frac{\omega_n^2}{-s_2(-s_2+s_1)}$$

$$= \frac{\omega_n^2}{-s_2\left[-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]} = \frac{\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_2} = \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_2}$$

The response in time domain, c(t) is given by,

$$c(t) = \mathcal{L}^{-1}\left\{= \frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}\,\frac{1}{(s+s_1)} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_2}\,\frac{1}{(s+s_2)}\right\}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}\,e^{-s_1 t} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_2}\,e^{-s_2 t}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right) \qquad .....(2.36$$

where, $s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$

The equation (2.36) is the response of overdamped closed loop system for unit step input. For step input of value, A, the equation (2.36) is multiplied by A.

∴ For closed loop over damped second order system,

Unit step response $= 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right)$ where, $s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

Step response $= A\left[1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right)\right]$ $s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$

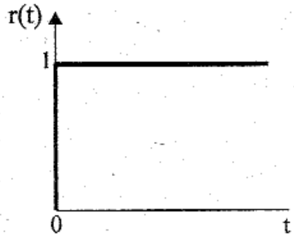


Fig 2.12.a : Input.

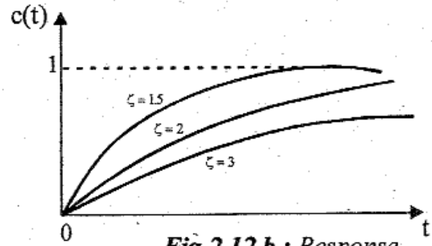


Fig 2.12.b : Response.

Fig 2.12 : Response of over damped second order system for unit step input.

Using equation (2.36), the response of overdamped second order system is sketched as shown in fig 2.12 and observed that the response has no oscillations but it takes longer time for the response to reach the final steady value.

2.8 TIME DOMAIN SPECIFICATIONS

The desired performance characteristics of control systems are specified in terms of time domain specifications. Systems with energy storage elements cannot respond instantaneously and will exhibit transient responses, whenever they are subjected to inputs or disturbances.

The desired performance characteristics of a system of any order may be specified in terms of the transient response to a unit step input signal. The response of a second order system for unit-step input with various values of damping ratio is shown in fig 2.13.

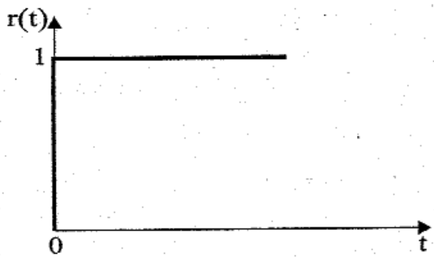


Fig 2.13.a : Input.

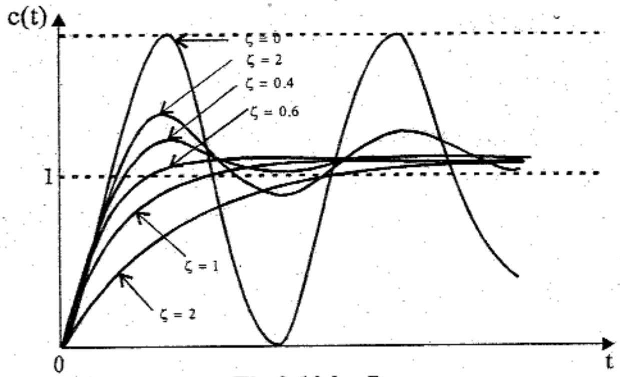


Fig 2.13.b : Response.

Fig 2.13 : Unit step response of second order system.

The transient response of a system to a unit step input depends on the initial conditions. Therefore to compare the time response of various systems it is necessary to start with standard initial conditions. The most practical standard is to start with the system at rest and so output and all time derivatives before $t = 0$ will be zero. The transient response of a practical control system often exhibits damped oscillation before reaching steady state. A typical damped oscillatory response of a system is shown in fig 2.14.

The transient response characteristics of a control system to a unit step input is specified in terms of the following time domain specifications.

1. Delay time, t_d
2. Rise time, t_r
3. Peak time, t_p
4. Maximum overshoot, M_p
5. Settling time, t_s

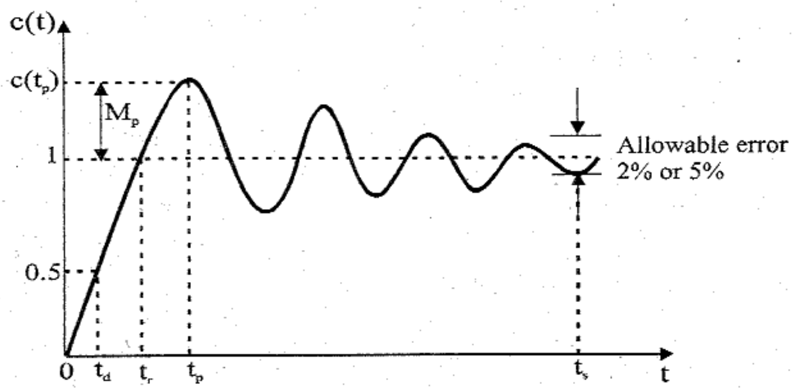


Fig 2.14 : Damped oscillatory response of second order system for unit step input.

The time domain specifications are defined as follows.

1. DELAY TIME (t_d) : It is the time taken for response to reach 50% of the final value, for the very first time.

2. RISE TIME (t_r) : It is the time taken for response to raise from 0 to 100% for the very first time. For underdamped system, the rise time is calculated from 0 to 100%. But for overdamped system it is the time taken by the response to raise from 10% to 90%. For critically damped system, it is the time taken for response to raise from 5% to 95%.

3. PEAK TIME (t_p) : It is the time taken for the response to reach the peak value the very first time. (or) It is the time taken for the response to reach the peak overshoot, M_p.

4. PEAK OVERSHOOT (M_p) : It is defined as the ratio of the maximum peak value to the final value, where the maximum peak value is measured from final value.

Let, $c(\infty)$ = Final value of c(t).

$c(t_p)$ = Maximum value of c(t).

Now, Peak overshoot, $$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \quad .....(2.37)$$

% Peak overshoot, $$\%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100 \quad .....(2.38)$$

5. SETTLING TIME (t_s) : It is defined as the time taken by the response to reach and stay within a specified error. It is usually expressed as % of final value. The usual tolerable error is 2 % or 5% of the final value.

EXPRESSIONS FOR TIME DOMAIN SPECIFICATIONS

Rise time (t_r)

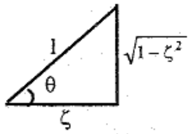
The unit step response of second order system for underdamped case is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

At $t = t_r$, $c(t) = c(t_r) = 1$ **(Refer fig 2.14).**

$$\therefore c(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 1$$

$$\therefore \frac{-e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 0$$

> ***Note :** On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get*
>
> $\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$
>
> 
>

Since $-e^{-\zeta\omega_n t_r} \neq 0$, the term, $\sin(\omega_d t_r + \theta) = 0$

When, $\phi = 0, \pi, 2\pi, 3\pi \ldots$, $\sin\phi = 0$

$$\therefore \omega_d t_r + \theta = \pi$$

$$\omega_d t_r = \pi - \theta$$

$$\boxed{\therefore \text{ Rise Time, } t_r = \frac{\pi - \theta}{\omega_d}} \quad .....(2.39)$$

Here, $\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$; Damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ ***(refer note)***

$$\boxed{\therefore \text{Rise time, } t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}} \text{ in sec}} \quad .....(2.40)$$

> ***Note :** θ or $\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ should be measured in radians.*

Peak time (t_p)

To find the expression for peak time, t_p, differentiate c(t) with respect to t and equate to 0.

$$\text{i.e., } \frac{d}{dt}c(t)\Big|_{t=t_p} = 0$$

The unit step response of under damped second order system is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

Differentiating c(t) with respect to **t**.

$$\frac{d}{dt}c(t) = \frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(-\zeta\omega_n)\sin(\omega_d t + \theta) + \left(\frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\cos(\omega_d t + \theta)\omega_d$$

Put, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore \frac{d}{dt}c(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\zeta\omega_n)\sin(\omega_d t + \theta) - \frac{\omega_n\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_d t + \theta)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\zeta\sin(\omega_d t + \theta) - \sqrt{1-\zeta^2}\cos(\omega_d t + \theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\cos\theta\sin(\omega_d t + \theta) - \sin\theta\cos(\omega_d t + \theta)\right] \quad \textit{(refer note)}$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\sin(\omega_d t + \theta)\cos\theta - \cos(\omega_d t + \theta)\sin\theta\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\left[\sin((\omega_d t+\theta)-\theta)\right] = \frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\sin(\omega_d t)$$

at $t = t_p$, $\frac{d}{dt}c(t) = 0$

$$\therefore \frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t_p}\sin(\omega_d t_p) = 0$$

Since, $e^{-\zeta\omega_n t_p} \neq 0$, the term, $\sin(\omega_d t_p) = 0$

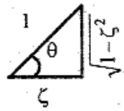
When $\phi = 0, \pi, 2\pi, 3\pi$, $\sin\phi = 0$

$\therefore \; \omega_d t_p = \pi$

Note : On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get

$\sin\theta = \sqrt{1-\zeta^2}$

$\cos\theta = \zeta$



$$\boxed{\therefore \text{Peak time, } t_p = \frac{\pi}{\omega_d}} \qquad .....(2.41)$$

The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\boxed{\therefore \text{Peak time, } t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} \qquad .....(2.42)$$

Peak overshoot (M_p)

$$\boxed{\%\text{Peak overshoot, } \%M_p = \frac{c(t_p)-c(\infty)}{c(\infty)}\times 100} \qquad .....(2.43)$$

where, $c(t_p)$ = Peak response at $t = t_p$.

$c(\infty)$ = Final steady state value.

The unit step response of second order system is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)$$

$$\text{At } t=\infty, \quad c(t) = c(\infty) = 1 - \frac{e^{-\infty}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta) = 1-0 = 1$$

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

$$\text{At } t=t_p, \quad c(t) = c(t_p) = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_p+\theta)$$

$$= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d\frac{\pi}{\omega_d}+\theta\right)$$

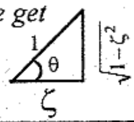
$$\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$$

$$\boxed{\sin(\pi+\theta) = -\sin\theta}$$

$$= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}\sin(\pi+\theta)}{\sqrt{1-\zeta^2}}$$

Note : On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get

$\sin\theta = \sqrt{1-\zeta^2}$



$$= 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sin\theta$$

$$= 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sqrt{1-\zeta^2} = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \qquad .....(2.44)$$

$$\text{Percentage Peak Overshoot, } \%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100 = \frac{1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} - 1}{1} \times 100$$

$$= e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

$$\boxed{\therefore \text{ Percentage Peak Overshoot, } \%M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100} \quad .....(2.45)$$

Settling time (t_s)

The response of second order system has two components. They are,

1. Decaying exponential component, $\dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$.
2. Sinusoidal component, $\sin(\omega_d t + \theta)$.

In this the decaying exponential term dampens (or) reduces the oscillations produced by sinusoidal component. Hence the settling time is decided by the exponential component. The settling time can be found out by equating exponential component to percentage tolerance errors.

For 2 % tolerance error band, at $t = t_s$, $\dfrac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$

For least values of ζ , $e^{-\zeta\omega_n t_s} = 0.02$.

On taking natural logarithm we get,

$$-\zeta\omega_n t_s = ln(0.02) \quad \Rightarrow \quad -\zeta\omega_n t_s = -4 \quad \Rightarrow \quad t_s = \frac{4}{\zeta\omega_n}$$

For the second order system, the time constant, $T = \dfrac{1}{\zeta\omega_n}$

$$\boxed{\therefore \text{ Settling time, } t_s = \frac{1}{\zeta\omega_n} = 4T \qquad \text{(for 2\% error)}} \quad .....(2.46)$$

For 5% error, $e^{-\zeta\omega_n t_s} = 0.05$

On taking natural logarithm we get,

$$-\zeta\omega_n t_s = ln(0.05) \quad \Rightarrow \quad -\zeta\omega_n t_s = -3 \quad \Rightarrow \quad t_s = \frac{3}{\zeta\omega_n}$$

$$\boxed{\therefore \text{ Settling time, } t_s = \frac{3}{\zeta\omega_n} = 3T \qquad \text{(for 5\% error)}} \quad .....(2.47)$$

In general for a specified percentage error, Settling time can be evaluted using equation (2.48).

$$\boxed{\therefore \text{ Settling time, } t_s = \frac{ln(\%\text{ error})}{\zeta\omega_n} = \frac{ln(\%\text{ error})}{T}} \quad .....(2.48)$$

EXAMPLE 2.1

Obtain the response of unity feedback system whose open loop transfer function is $G(s) = \frac{4}{s(s+5)}$ and when the input is unit step.

SOLUTION

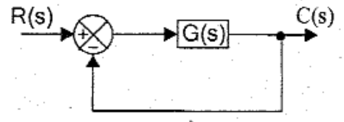


Fig 1 : Closed loop system.

The closed loop system is shown in fig 1.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{4}{s\,(s+5)}}{1+\frac{4}{s\,(s+5)}} = \frac{\frac{4}{s\,(s+5)}}{\frac{s\,(s+5)+4}{s\,(s+5)}} = \frac{4}{s\,(s+5)+4} = \frac{4}{s^2+5s+4} = \frac{4}{(s+4)\,(s+1)}$$

The response in s-domain, $C(s) = R(s)\ \frac{4}{(s+1)\,(s+4)}$

Since the input is unit step, $R(s) = \frac{1}{s}$; $\quad \therefore C(s) = \frac{4}{s\,(s+1)\,(s+4)}$

By partial fraction expansion, we can write,

$$C(s) = \frac{4}{s\,(s+1)\,(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = C(s)\ \times\ s\big|_{s=0} = \left.\frac{4}{(s+1)\,(s+4)}\right|_{s=0} = \frac{4}{1\times 4} = 1$$

$$B = C(s)\ \times\ (s+1)\big|_{s=-1} = \left.\frac{4}{s\,(s+4)}\right|_{s=-1} = \frac{4}{-1(-1+4)} = \frac{-4}{3}$$

$$C = C(s)\ \times\ (s+4)\big|_{s=-4} = \left.\frac{4}{s\,(s+1)}\right|_{s=-4} = \frac{4}{-4(-4+1)} = \frac{1}{3}$$

The time domain response c(t) is obtained by taking inverse Laplace transform of C(s).

Response in time domain, $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{4}{3}\,\frac{1}{s+1} + \frac{1}{3}\,\frac{1}{s+4}\right\}$

$$= 1 - \frac{4}{3}\,e^{-t} + \frac{1}{3}\,e^{-4t} = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right]$$

RESULT

Response of unity feedback system, $c(t) = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right]$

EXAMPLE 2.2

A positional control system with velocity feedback is shown in fig 1. What is the response of the system for unit step input.

SOLUTION

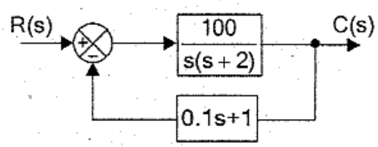


Fig 1 : Positional control system.

The closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$$

Given that, $G(s) = \frac{100}{s(s+2)}$ and $H(s) = 0.1s+1$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{100}{s\,(s+2)}}{1+\left(\frac{100}{s\,(s+2)}\right)(0.1s+1)} = \frac{\frac{100}{s\,(s+2)}}{\frac{s(s+2)+100\,(0.1s+1)}{s\,(s+2)}} = \frac{100}{s^2+2s+10s+100} = \frac{100}{s^2+12s+100}$$

Here $(s^2 + 12s + 100)$ is characteristic polynomial. The roots of the characteristic polynomial are,

$$s_1, s_2 = \frac{-12 \pm \sqrt{144-400}}{2} = \frac{-12 \pm j16}{2} = -6 \pm j8$$

The roots are complex conjugate. The system is underdamped and so the response of the system will have damped oscillations.

$$\text{The response in s-domain, } C(s) = R(s)\,\frac{100}{s^2+12s+100}$$

$$\text{Since input is unit step, } R(s) = \frac{1}{s}$$

$$\therefore C(s) = \frac{1}{s}\;\frac{100}{s^2+12s+100} = \frac{100}{s(s^2+12s+100)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{100}{s\,(s^2+12s+100)} = \frac{A}{s} + \frac{Bs+C}{s^2+12s+100}$$

The residue A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times s\big|_{s=0} = \left.\frac{100}{s^2+12s+100}\right|_{s=0} = \frac{100}{100} = 1$$

The residue B and C are evaluated by cross multiplying the following equation and equating the coefficients of like power of s.

$$\frac{100}{s\,(s^2+12s+100)} = \frac{A}{s} + \frac{Bs+C}{s^2+12s+100}$$

$$100 = A(s^2+12s+100) + (Bs+C)\,s$$

$$100 = As^2 + 12As + 100A + Bs^2 + Cs$$

On equating the coefficients of s^2 we get, $0 = A + B$ $\quad\therefore B = -A = -1$

On equating coefficients of s we get, $0 = 12A + C$ $\quad\therefore C = -12A = -12$

$$\therefore C(s) = \frac{1}{s} + \frac{-s-12}{s^2+12s+100} = \frac{1}{s} - \frac{s+12}{s^2+12s+36+64} = \frac{1}{s} - \frac{s+6+6}{(s+6)^2+8^2}$$

$$= \frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{(s+6)^2+8^2} = \frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{8}\,\frac{8}{(s+6)^2+8^2}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

$$\text{Time response, } c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{8}\,\frac{8}{(s+6)^2+8^2}\right\}$$

$$= 1 - e^{-6t}\cos 8t - \frac{6}{8}\,e^{-6t}\sin 8t = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right]$$

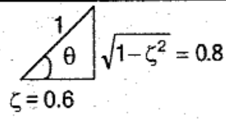
The result can be converted to another standard form by constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$. The damping ratio ζ is evaluated by comparing the closed loop transfer function of the system with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} = \frac{100}{s^2+12s+100}$$

On comparing we get, $\omega_n^2 = 100$ $\quad\therefore \omega_n = 10$

$2\zeta\omega_n = 12$ $\quad\therefore \zeta = \dfrac{12}{2\omega_n} = \dfrac{12}{2\times 10} = 0.6$

Constructing right angled triangle with ζ and $\sqrt{1-\zeta^2}$ we get,



$\cos\theta = 0.6$

$\sin\theta = 0.8$

$$\sin\theta = 0.8 \; ; \; \cos\theta = 0.6 \; ; \; \tan\theta = \frac{0.8}{0.6}$$

$$\therefore \theta = \tan^{-1}\frac{0.8}{0.6} = 53° = 53° \times \frac{\pi}{180°}\text{rad} = 0.925 \text{ rad.}$$

$$\therefore \text{ Time response, } c(t) = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right] = 1 - e^{-6t}\,\frac{10}{8}\left[\frac{6}{10}\sin 8t + \frac{8}{10}\cos 8t\right]$$

$$= 1 - \frac{10}{8}e^{-6t}\left[\sin 8t \times 0.6 + \cos 8t \times 0.8\right] = 1 - 1.25\, e^{-6t}\left[\sin 8t \cos\theta + \cos 8t \sin\theta\right]$$

$$= 1 - 1.25\, e^{-6t}\left[\sin\,(8t+\theta)\right] = 1 - 125\, e^{-6t}\,\sin(8t + 0.925)$$

Note : θ is expressed in radians

RESULT

The response in time domain,

$$c(t) = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right] \quad \text{or} \quad c(t) = 1 - 1.25\, e^{-6t}\sin(8t + 0.925)$$

EXAMPLE 2.3

The response of a servomechanism is, $c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$ when subject to a unit step input. Obtain an expression for closed loop transfer function. Determine the undamped natural frequency and damping ratio.

SOLUTION

Given that, $c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$

On taking Laplace transform of c(t) we get,

$$C(s) = \frac{1}{s} + 0.2\,\frac{1}{(s+60)} - 12\,\frac{1}{(s+10)} = \frac{(s+60)\,(s+10) + 0.2\,s\,(s+10) - 1.2\,s\,(s+60)}{s\,(s+60)\,(s+10)}$$

$$= \frac{s^2 + 70s + 600 + 0.2\,s^2 + 2s - 12s^2 - 72s}{s\,(s+60)\,(s+10)} = \frac{600}{s\,(s+60)\,(s+10)} = \frac{1}{s}\,\frac{600}{(s+60)\,(s+10)}$$

Since input is unit step, R(s) = 1/s.

$$\therefore C(s) = R(s)\,\frac{60}{(s+60)\,(s+10)} = R(s)\,\frac{600}{s^2 + 70s + 600}$$

$$\therefore \text{ The closed loop transfer function of the system, } \frac{C(s)}{R(s)} = \frac{600}{s^2 + 70s + 600}$$

The damping ratio and natural frequency of oscillation can be estimated by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{600}{s^2 + 70s + 600}$$

On comparing we get,

$$\omega_n^2 = 600$$

$$\therefore \omega_n = \sqrt{600} = 24.49 \text{ rad / sec}$$

$$2\zeta\omega_n = 70$$

$$\therefore \zeta = \frac{70}{2\omega_n} = \frac{70}{2 \times 24.49} = 143$$

RESULT

The closed loop transfer function of the system, $\dfrac{C(s)}{R(s)} = \dfrac{600}{s^2 + 70s + 600}$

Natural frequency of oscillation, $\omega_n = 24.49$ rad/sec

Damping ratio, $\zeta = 1.43$

EXAMPLE 2.4

The unity feedback system is characterized by an open loop transfer function G(s) =K/s (s +10). Determine the gain K, so that the system will have a damping ratio of 0.5 for this value of K. Determine peak overshoot and time at peak overshoot for a unit step input.

SOLUTION

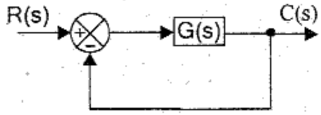


Fig 1 : Unity feedback system.

The unity feedback system is shown in fig 1.

The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

Given that, G(s) = K/s (s +10)

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}} = \frac{K}{s\,(s+10)+K} = \frac{K}{s^2+10s+K}$$

The value of K can be evaluated by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{K}{s^2+10s+K}$$

On comparing we get,

$\omega_n^2 = K$	$2\zeta\omega_n = 10$	$K = 100$
$\therefore \omega_n = \sqrt{K}$	Put $\zeta = 0.5$ and $\omega_n = \sqrt{K}$	$\omega_n = 10$ rad / sec
	$\therefore 2 \times 0.5 \times \sqrt{K} = 10$	
	$\sqrt{K} = 10$	

The value of gain, K=100.

$$\text{Percentage peak overshoot, } \%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

$$= e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100 = 0.163 \times 100 = 16.3\%$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{10\sqrt{1-0.5^2}} = 0.363 \text{ sec}$$

RESULT

The value of gain,	K	=	100
Percentage peak overshoot,	$\%M_p$	=	16.3%
Peak time,	t_p	=	0.363 sec.

EXAMPLE 2.5

The open loop transfer function of a unity feedback system is given by G(s) = K/s (sT + 1), where K and T are positive constant. By what factor should the amplifier gain K be reduced, so that the peak overshoot of unit step response of the system is reduced from 75% to 25%.

SOLUTION

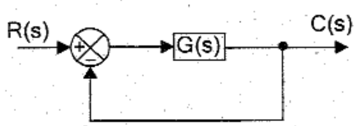


Fig 1 : Unity feedback system.

The unity feedback system is shown in fig 1.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

Given that, $G(s) = K/s\,(sT+1)$

$$\therefore \frac{C(s)}{R(s)} = \frac{K/s\,(sT+1)}{1+K/s(sT+1)} = \frac{K}{s\,(sT+1)+K} = \frac{K}{s^2T+s+K} = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}$$

Expression for ζ and ω_n can be obtained by comparing the transfer function with the standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}$$

On comparing we get,

$$\omega_n^2 = K/T$$

$$\therefore \omega_n = \sqrt{K/T}$$

$$2\zeta\omega_n = 1/T$$

$$\zeta = \frac{1}{2\omega_n T} = \frac{1}{2\sqrt{\frac{K}{T}}\;T} = \frac{1}{2\sqrt{KT}}$$

The peak overshoot, M_p is reduced by increasing the damping ratio ζ. The damping ratio ζ is increased by reducing the gain K.

When $M_p = 0.75$, Let $\zeta = \zeta_1$ and $K = K_1$

When $M_p = 0.25$, Let $\zeta = \zeta_2$ and $K = K_2$

Peak overshoot, $M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

Taking natural logarithm on both sides, $ln\, M_p = \dfrac{-\zeta\pi}{\sqrt{1-\zeta^2}}$

On squaring we get, $(ln\, M_p)^2 = \dfrac{\zeta^2\pi^2}{1-\zeta^2}$

On crossing multiplication we get,

$$(1-\zeta^2)\,(ln\, M_p)^2 = \zeta^2\pi^2$$

$$(ln\, M_p)^2 - \zeta^2(ln\, M_p)^2 = \zeta^2\pi^2$$

$$(ln\, M_p)^2 = \zeta^2\pi^2 + \zeta^2(ln\, M_p)^2$$

$$(ln\, M_p)^2 = \zeta^2\left[\pi^2+(ln\, M_p)^2\right]$$

$$\therefore \zeta^2 = \frac{(ln\, M_p)^2}{\pi^2+(ln\, M_p)^2} \qquad ....(1)$$

$$\text{But } \zeta = \frac{1}{2\sqrt{KT}}, \quad \therefore \zeta^2 = \frac{1}{4KT} \qquad ....(2)$$

On equating, equation (1) & (2) we get,

$$\frac{1}{4KT} = \frac{(ln\, M_p)^2}{\pi^2+(ln\, M_p)^2}$$

$$\frac{1}{K} = \frac{4T\,(ln\, M_p)^2}{\pi^2+(ln\, M_p)^2}$$

$$K = \frac{\pi^2+(ln\, M_p)^2}{4T\,(ln\, M_p)^2}$$

When, $K = K_1$, $M_p = 0.75$, $\therefore K_1 = \dfrac{\pi^2+(ln 0.75)^2}{4T\,(ln 0.75)^2} = \dfrac{9.952}{0.331T} = \dfrac{30.06}{T}$

When, $K = K_2$, $M_p = 0.25$, $\therefore K_2 = \dfrac{\pi^2+(ln 0.25)^2}{4T\,(ln 0.25)^2} = \dfrac{11.79}{7.68T} = \dfrac{1.53}{T}$

$$\therefore \frac{K_1}{K_2} = \frac{(1/T)\,30.06}{(1/T)\,1.53} = 19.6$$

$$K_1 = 19.6\,K_2 \quad \text{(or)} \quad K_2 = \frac{1}{19.6}K_1$$

To reduce peak overshoot from 0.75 to 0.25, K should be reduced by 19.6 times (approximately 20 times).

RESULT

The value of gain, K should be reduced approximately 20 times to reduce peak overshoot from 0.75 to 0.25.

EXAMPLE 2.6

A positional control system with velocity feedback is shown in fig 1. What is the response c(t) to the unit step input. Given that $\zeta = 0.5$. Also calculate rise time, peak time, maximum overshoot and settling time.

SOLUTION

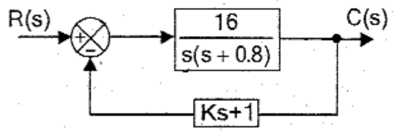


Fig 1

The closed loop transfer function, $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)\,H(s)}$

Given that G(s) = 16/s(s + 0.8) and H(s) = Ks +1

$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{16}{s\,(s+0.8)}}{1+\dfrac{16}{s\,(s+0.8)}\,(Ks+1)} = \frac{16}{s\,(s+0.8)+16(Ks+1)}$$

$$= \frac{16}{s^2+0.8s+16Ks+16} = \frac{16}{s^2+(0.8+16K)\,s+16}$$

The values of K and ω_n are obtained by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{16}{s^2+(0.8+16K)\,s+16}$$

On comparing we get.

$$\omega_n^2 = 16 \qquad\qquad 0.8+16K = 2\zeta\omega_n$$

$$\therefore \omega_n = 4 \text{ rad/sec} \qquad\qquad \therefore K = \frac{2\zeta\omega_n - 0.8}{16} = \frac{2\times 0.5\times 4 - 0.8}{16} = 0.2$$

$$\therefore \frac{C(s)}{R(s)} = \frac{16}{s^2+(0.8+16\times 0.2)\,s+16} = \frac{16}{s^2+4s+16}$$

Given that the damping ratio, $\zeta = 0.5$. Hence the system is underdamped and so the response of the system will have damped oscillations. The roots of characteristic polynomial will be complex conjugate.

The response in s - domain, $C(s) = R(s)\,\dfrac{16}{s^2+4s+16}$

For unit step input, R(s) = 1/s.

$$\therefore C(s) = \frac{1}{s}\,\frac{16}{s^2+4s+16} = \frac{16}{s\,(s^2+4s+16)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{16}{s\,(s^2+4s+16)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+16}$$

The residue A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s)\times s\big|_{s=0} = \left.\frac{16}{s^2+4s+16}\right|_{s=0} = \frac{16}{16} = 1$$

The residues B and C are evaluated by cross multiplying the following equation and equating the coefficients of like powers of s.

$$\frac{16}{s\,(s^2+4s+16)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+16}$$

On cross multiplication we get, $16 = A\,(s^2+4s+16) + (Bs+C)\,s$

$$16 = As^2 + 4As + 16A + Bs^2 + Cs$$

On equating the coefficients of s^2 we get, $0 = A + B \quad \therefore\ B = -A = -1$

On equating the coefficients of s we get, $0 = 4A + C \ \therefore\ C = -4A = -4$

$$\therefore\ C(s) = \frac{1}{s} + \frac{-s-4}{s^2+4s+16} = \frac{1}{s} - \frac{s+4}{s^2+4s+4+12}$$

$$= \frac{1}{s} - \frac{s+2+2}{(s+2)^2+12} = \frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}}\ \frac{\sqrt{12}}{(s+2)^2+12}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

The response in time domain,

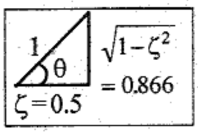
$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}}\ \frac{\sqrt{12}}{(s+2)^2+12}\right\}$$

$$= 1 - e^{-2t}\cos\sqrt{12}\,t - \frac{2}{2\sqrt{3}}\ e^{-2t}\sin\sqrt{12}\,t$$

$$= 1 - e^{-2t}\left[\frac{1}{\sqrt{3}}\sin(\sqrt{12}\,t) + \cos(\sqrt{12}\,t)\right]$$

The result can be converted to another standard form by constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$.

On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$ we get,



$$\sin\theta = 0.866 = \sqrt{3}/2\ ; \quad \cos\theta = 0.5 \quad = 1/2\ ; \ \tan\theta = 1732$$

$$\therefore\ \theta = \tan^{-1}1732 = 60° = 1047 \text{ rad}$$

$\therefore$ The response in time domain,

$$c(t) = 1 - e^{-2t}\left[\frac{1}{\sqrt{3}} \times 2 \times \sin\sqrt{12}\,t \times \frac{1}{2} + \frac{2}{\sqrt{3}} \times \cos\sqrt{12}\,t \times \frac{\sqrt{3}}{2}\right]$$

$$= 1 - e^{-2t}\frac{2}{\sqrt{3}}\left[\sin\sqrt{12}\,t\cos\theta + \cos\sqrt{12}\,t\sin\theta\right]$$

Note : θ is expressed in radians.

$$= 1 - \frac{2}{\sqrt{3}}\ e^{-2t}\left[\sin(\sqrt{12}\,t + \theta)\right] = 1 - \frac{2}{\sqrt{3}}e^{-2t}\left[\sin(\sqrt{12}\,t + 1047)\right]$$

$$\left.\begin{array}{l}\text{Damped frequency}\\ \text{of oscillation}\end{array}\right\}\omega_d = \omega_n\sqrt{1-\zeta^2} = 4\sqrt{1-0.5^2} = 3.464 \text{ rad / sec}$$

$$\therefore\ \text{Rise time, } t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - 1047}{3.464} = 0.6046 \text{ sec}$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3.464} = 0.907 \text{ sec}$$

$$\left.\begin{array}{r}\%\text{ Maximum}\\ \text{overshoot}\end{array}\right\}\%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.5\times\pi}{\sqrt{1-0.5^2}}} \times 100 = 0.163 \times 100 = 16.3\%$$

$$\text{Time constant, } T = \frac{1}{\zeta\omega_n} = \frac{1}{0.5 \times 4} = 0.5 \text{ sec}$$

For 5% error, Settling time, $t_s = 3T = 3 \times 0.5 = 1.5$ sec

For 2% error, Settling time, $t_s = 4T = 4 \times 0.5 = 2$ sec

RESULT

The time domain response, $c(t) = 1 - e^{-2t}\left[\frac{1}{\sqrt{3}}\sin(\sqrt{12}\,t) + \cos(\sqrt{12}\,t)\right]$

$$\text{(or)} \quad c(t) = 1 - \frac{2}{\sqrt{3}}e^{-2t}\left[\sin(\sqrt{12}\,t + 1047)\right]$$

Rise time, $t_r = 0.6046$ sec

Peak time, $t_p = 0.907$ sec

% Maximum overshoot, $\%M_p = 16.3\%$

Settling time, $t_s = 1.5$ sec, for 5% error

$= 2$ sec, for 2% error

EXAMPLE 2.7

A unity feedback control system is characterized by the following open loop transfer function G(s) = (0.4 s +1)/s(s +0.6). Determine its transient response for unit step input and sketch the response. Evaluate the maximum overshoot and the corresponding peak time.

SOLUTION

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$

Given that, $G(s) = (0.4\,s+1)/s(s+0.6)$

For unity feedback system, $H(s) = 1$.

$$\therefore \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{0.4\,s+1}{s(s+0.6)}}{1+\frac{0.4\,s+1}{s(s+0.6)}} = \frac{0.4\,s+1}{s\,(s+0.6)+0.4\,s+1}$$

$$= \frac{0.4\,s+1}{s^2+0.6s+0.4s+1} = \frac{0.4\,s+1}{s^2+s+1}$$

The s - domain response, $C(s) = R(s) \times \frac{0.4\,s+1}{s^2+s+1}$

For step input, $R(s) = 1/s$.

$$\therefore C(s) = \frac{1}{s}\,\frac{0.4\,s+1}{s^2+s+1} = \frac{0.4\,s+1}{s(s^2+s+1)}$$

By partial fraction expansion C(s) can be expressed as,

$$C(s) = \frac{0.4\,s+1}{s\,(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

The residue A is solved by multiplying C(s) by s and letting s = 0.

$$\therefore A = C(s) \times s\Big|_{s=0} = \left.\frac{0.4\,s+1}{s^2+s+1}\right|_{s=0} = 1$$

The residues B and C are solved by cross multiplying the following equation and equating the coefficients of like powers of s.

$$\frac{0.4\,s+1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

On cross multiplication we get,

$$0.4\,s+1 = A(s^2+s+1) + (Bs+C)\,s$$

$$0.4\,s+1 = As^2 + As + A + Bs^2 + Cs$$

On equating coefficients of s^2 we get, $0 = A + B$ $\quad\therefore B = -A = -1$

On equating coefficients of s we get, $0.4 = A + C$ $\quad\therefore C = 0.4 - A = -0.6$

$$\therefore C(s) = \frac{1}{s} + \frac{-s-0.6}{s^2+s+1} = \frac{1}{s} - \frac{s+0.6}{s^2+s+0.25+0.75} = \frac{1}{s} - \frac{s+0.6}{\left(s^2+2\times 0.5s+0.5^2\right)+0.75}$$

$$= \frac{1}{s} - \frac{s+0.5+0.1}{(s+0.5)^2+0.75} = \frac{1}{s} - \frac{s+0.5}{(s+0.5)^2+0.75} - \frac{0.1}{\sqrt{0.75}}\;\frac{\sqrt{0.75}}{(s+0.5)^2+0.75}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

∴ The response in time domain,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+0.5}{(s+0.5)^2+0.75} - \frac{0.1}{\sqrt{0.75}}\frac{\sqrt{0.75}}{(s+0.5)^2+0.75}\right\}$$

$$= 1 - e^{-0.5t}\cos\sqrt{0.75}\,t - \frac{0.1}{\sqrt{0.75}}\,e^{-0.5t}\sin\sqrt{0.75}\,t$$

$$= 1 - e^{-0.5t}\left[0.1155\,\sin(\sqrt{0.75}\,t) + \cos(\sqrt{0.75}\,t)\right]$$

The transient response is the part of the output which vanishes as t tends to infinity. Here as t tends to infinity the exponential component $e^{-0.5t}$ tends to zero. Hence the transient response is given by the damped sinusoidal component.

The transient response of $c(t) = e^{-0.5t}\left[0.1155\,\sin(\sqrt{0.75}\,t) + \cos(\sqrt{0.75}\,t)\right]$

The value of ζ and ω_n can be estimated by comparing the characteristic equation of the system with standard form of second order characteristic equation.

$$\therefore\ s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + s + 1$$

On comparing we get,

$$\omega_n^2 = 1 \qquad\qquad 2\zeta\omega_n = 1$$

$$\therefore\ \omega_n = 1\ \text{rad/sec} \qquad\qquad \therefore\ \zeta = \frac{1}{2\omega_n} = \frac{1}{2} = 0.5$$

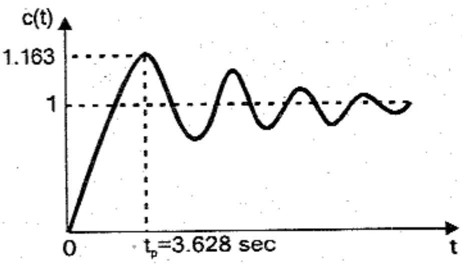


Fig 1 : Response of under damped system.

Maximum overshoot, $M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}} = 0.163$

% Maximum overshoot, $\%M_p = M_p \times 100 = 0.163 \times 100 = 16.3\%$

Peak time, $t_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{\pi}{1\times\sqrt{1-0.5^2}} = 3.628$ sec

The response of the system is underdamped and it is shown in fig 1.

RESULT

Transient response of the system, $c(t) = e^{-0.5t}\left[0.1155\sin(\sqrt{0.75}\,t) + \cos(\sqrt{0.75}\,t)\right]$

% Maximum peak overshoot, $\%M_p$ = 16.3 %

Peak time, t_p = 3.628 sec

EXAMPLE 2.8

A unity feedback control system has an amplifier with gain K_A = 10 and gain ratio, G(s) = 1/s(s+2) in the feed forward path. A derivative feedback, H(s) = sK_o is introduced as a minor loop around G(s). Determine the derivative feedback constant, K_o so that the system damping factor is 0.6.

SOLUTION

The given system can be represented by the block diagram shown in fig 1.

Here, $K_A = 10$; $G(s) = \dfrac{1}{s(s+2)}$ and $H(s) = sK_o$

The closed loop transfer function of the system can be obtained by block diagram reduction techniques.

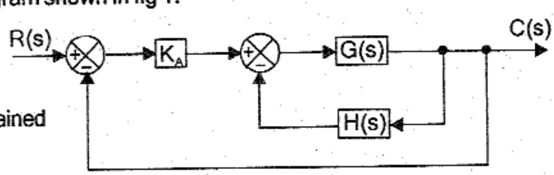
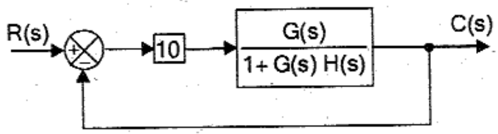
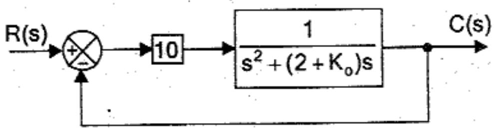


Fig 1.

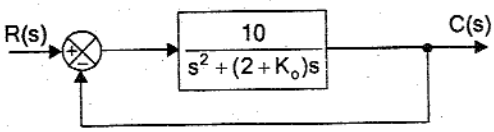
Step 1: Reducing the inner feedback loop.



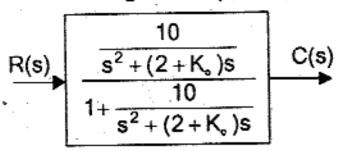
$$\frac{G(s)}{1+G(s)\,H(s)}=\frac{\dfrac{1}{s\,(s+2)}}{1+\dfrac{1}{s\,(s+2)}\,sK_o}=\frac{1}{s\,(s+2)+sK_o}=\frac{1}{s^2+2s+sK_o}=\frac{1}{s^2+(2+K_o)\,s}$$

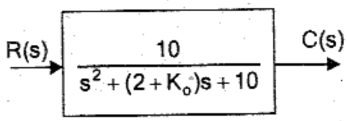


Step 2 : Combining blocks in cascade



Step 3 : Reducing the unity feedback path





The closed loop transfer function, $\dfrac{C(s)}{R(s)}=\dfrac{10}{s^2+(2+K_o)\,s+10}$ .....(1)

The given system is a second order system. The value of K_o can be determined by comparing the system transfer function with standard form of second order transfer function given below.

$$\left.\begin{array}{l}\text{Standard form of}\\ \text{Second order transfer function}\end{array}\right\}\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \quad .....(2)$$

On comparing equation (1) & (2) we get,

$$\omega_n^2=10$$
$$\therefore\ \omega_n=\sqrt{10}=3.162\ \text{rad / sec}$$

$$2+K_o=2\zeta\omega_n$$
$$\therefore\ K_o=2\zeta\omega_n-2$$
$$=2\times0.6\times3.162-2=1.7944$$

RESULT

The value of constant, $K_o=1.7944$

EXAMPLE 2.9

A unity feedback control system has an open loop transfer function, G(s) = 10/s(s+2). Find the rise time, percentage overshoot, peak time and settling time for a step input of 12 units.

SOLUTION

Note : The formulae for rise time, percentage overshoot and peak time remains same for unit step and step input.

The unity feedback system is shown in fig 1.

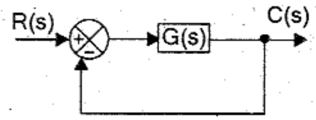


Fig 1 : Unity feedback system.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

The closed loop transfer function,

Given that, G(s) = 10/s (s+2)

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{10}{s(s+2)}}{1+\frac{10}{s(s+2)}} = \frac{10}{s(s+2)+10} = \frac{10}{s^2+2s+10} \qquad .....(1)$$

The values of damping ratio ζ and natural frequency of oscillation ω_n are obtained by comparing the system transfer function with standard form of second order transfer function.

$$\left.\begin{array}{l}\text{Standard form of}\\ \text{Second order transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad .....(2)$$

On comparing equation (1) & (2) we get,

$$\omega_n^2 = 10 \qquad\qquad 2\zeta\omega_n = 2$$

$$\therefore \omega_n = \sqrt{10} = 3.162 \text{ rad/sec} \qquad \therefore \zeta = \frac{2}{2\omega_n} = \frac{1}{3.162} = 0.316$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-0.316^2}}{0.316} = 1.249 \text{ rad}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.162\sqrt{1-0.316^2} = 3 \text{ rad/sec}$$

$$\text{Rise time, } t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-1.249}{3} = 0.63 \text{ sec}$$

$$\text{Percentage overshoot, } \%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.316\pi}{\sqrt{1-0.316^2}}} \times 100$$

$$= 0.3512 \times 100 = 35.12\%$$

$$\text{Peak overshoot} = \frac{35.12}{100} \times 12 \text{ units} = 4.2144 \text{ units}$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3} = 1.047 \text{ sec}$$

$$\text{Time constant, } T = \frac{1}{\zeta\omega_n} = \frac{1}{0.316 \times 3.162} = 1 \text{ sec}$$

$\therefore$ For 5% error, Settling time, $t_s = 3T = 3$ sec

For 2% error, Settling time, $t_s = 4T = 4$ sec

RESULT

Rise time, t_r	=	0.63 sec
Percentage overshoot, $\%M_p$	=	35.12%
Peak overshoot	=	4.2144 units, (for a input of 12 units)
Peak time, t_p	=	1.047 sec
Settling time, t_s	=	3 sec for 5% error
	=	4 sec for 2% error

EXAMPLE 2.10

A closed loop servo is represented by the differential equation $\frac{d^2c}{dt^2}+8\frac{dc}{dt}=64\,e$

Where c is the displacement of the output shaft, r is the displacement of the input shaft and e = r – c. Determine undamped natural frequency, damping ratio and percentage maximum overshoot for unit step input.

SOLUTION

The mathematical equations governing the system are,

$$\frac{d^2c}{dt^2}+8\frac{dc}{dt}=64\,e \qquad .....(1)$$

$$e=r-c \qquad .....(2)$$

Put $e = r - c$ in equation (1),

$$\therefore\ \frac{d^2c}{dt^2}+8\frac{dc}{dt}=64(r-c) \qquad .....(3)$$

Let $\mathcal{L}\{c\} = C(s)$ and $\mathcal{L}\{r\} = R(s)$

On taking Laplace transform of equation (3) we get,

$$s^2\,C(s)+8s\,C(s)=64\,[R(s)-C(s)]$$

$$\therefore\ s^2\,C(s)+8s\,C(s)+64\,C(s)=64\,R(s)$$

$$(s^2+8s+64)\,C(s)=64\,R(s)$$

$$\therefore\ \frac{C(s)}{R(s)}=\frac{64}{s^2+8s+64} \qquad .....(4)$$

The ratio C(s)/R(s) is the closed loop transfer function of the system. On comparing the system transfer function with standard form of second order transfer function, we can estimate the values of ζ and ω_n.

$$\left.\begin{array}{l}\text{Standard form of}\\ \text{Second order transfer function}\end{array}\right\}\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad .....(5)$$

On comparing equation (1) & (2) we get,

$$\omega_n^2=64 \qquad\qquad 2\zeta\omega_n=8$$

$$\therefore\ \omega_n=8\ \text{rad/sec} \qquad\qquad \zeta=\frac{8}{2\omega_n}=\frac{8}{2\times 8}=0.5$$

Percentage peak overshoot, $\%M_p=e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\times 100 = e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}}\times 100 = 16.3\%$

RESULT

Undamped natural frequency of oscillation, ω_n = 8 rad/sec

Damping ratio, ζ = 0.5

Percentage peak overshoot, $\%M_p$ = 16.3%

2.9 TYPE NUMBER OF CONTROL SYSTEMS

The type number is specified for loop transfer function G(s) H(s). The number of poles of the loop transfer function lying at the origin decides the type number of the system. In general, if **N** is the number of poles at the origin then the type number is **N.**

The loop transfer function can be expressed as a ratio of two polynomials in **s.**

$$G(s)\,H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\,\ldots\ldots\ldots\ldots}{s^N\,(s+p_1)\,(s+p_2)\,(s+p_3)\,\ldots\ldots} \quad \ldots\ldots(2.49)$$

where, $z_1, z_2, z_3, \ldots\ldots\ldots\ldots$ are zeros of transfer function

$p_1, p_2, p_3, \ldots\ldots\ldots\ldots$ are poles of transfer function

K = Constant

N = Number of poles at the origin

The value of **N** in the denominator polynomial of loop transfer function shown in equation (2.49) decides the type number of the system.

If N = 0, then the system is type – 0 system

If N = 1, then the system is type – 1 system

If N = 2, then the system is type – 2 system

If N = 3, then the system is type – 3 system and so on.

2.10 STEADY STATE ERROR

The steady state error is the value of error signal e(t), when **t** tends to infinity. The steady state error is a measure of system accuracy. These errors arise from the nature of inputs, type of system and from non linearity of system components. The steady state performance of a stable control system is generally judged by its steady state error to step, ramp and parabolic inputs.

Consider a closed loop system shown in fig 2.15.

Let, R(s) = Input signal

E(s) = Error signal

C(s) H(s) = Feedback signal

C(s) = Output signal or response

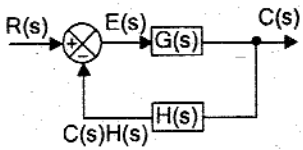


Fig 2.15.

The error signal, E(s) = R(s) – C(s) H(s) .....(2.50)

The output signal, C(s) = E(s) G(s) .....(2.51)

On substituting for C(s) from equation (2.51) in equation (2.50) we get,

$$E(s) = R(s) - [E(s)\,G(s)]\,H(s)$$

$$E(s) + E(s)\,G(s)\,H(s) = R(s)$$

$$E(s)\,[1 + G(s)\,H(s)] = R(s)$$

$$\therefore E(s) = \frac{R(s)}{1+G(s)\,H(s)} \quad \ldots\ldots(2.52)$$

Let, e(t) = error signal in time domain.

$$\therefore e(t) = \mathcal{L}^{-1}\{E(s)\} = \mathcal{L}^{-1}\left\{\frac{R(s)}{1+G(s)\,H(s)}\right\} \qquad .....(2.53)$$

Let, e_{ss} = steady state error.

The steady state error is defined as the value of e(t) when **t** tends to infinity.

$$\therefore e_{ss} = \underset{t\to\infty}{\mathrm{Lt}}\; e(t) \qquad .....(2.54)$$

The final value theorem of Laplace transform states that,

$$\textbf{If,}\quad \mathbf{F(s) = \mathcal{L}\{f(t)\}}\ \textbf{then,}\quad \underset{t\to\infty}{\mathbf{Lt}}\ \mathbf{f(t)} = \underset{s\to0}{\mathbf{Lt}}\ \mathbf{s\,F(s)} \qquad .....(2.55)$$

Using final value theorem,

$$\text{The steady state error, } e_{ss} = \underset{t\to\infty}{\mathrm{Lt}}\; e(t) = \underset{s\to0}{\mathrm{Lt}}\; sE(s) = \underset{s\to0}{\mathrm{Lt}}\; \frac{sR(s)}{1+G(s)\,H(s)} \qquad .....(2.56)$$

2.11 STATIC ERROR CONSTANTS

When a control system is excited with standard input signal, the steady state error may be zero, constant or infinity. The value of steady state error depends on the type number and the input signal. Type-0 system will have a constant steady state error when the input is step signal. Type-1 system will have a constant steady state error when the input is ramp signal or velocity signal. Type-2 system will have a constant steady state error when the input is parabolic signal or acceleration signal. For the three cases mentioned above the steady state error is associated with one of the constants defined as follows,

$$\text{Positional error constant,}\quad K_p = \underset{s\to0}{\mathrm{Lt}}\; G(s)\,H(s) \qquad .....(2.57)$$

$$\text{Velocity error constant,}\quad K_v = \underset{s\to0}{\mathrm{Lt}}\; s\,G(s)\,H(s) \qquad .....(2.58)$$

$$\text{Acceleration error constant,}\quad K_a = \underset{s\to0}{\mathrm{Lt}}\; s^2G(s)\,H(s) \qquad .....(2.59)$$

The K_p, K_v and K_a are in general called static error constants.

2.12 STEADY STATE ERROR WHEN THE INPUT IS UNIT STEP SIGNAL

$$\text{Steady state error, } e_{ss} = \underset{s\to0}{\mathrm{Lt}}\; \frac{sR(s)}{1+G(s)\,H(s)}$$

When the input is unit step, $R(s) = 1/s$

$$\therefore e_{ss} = \underset{s\to0}{\mathrm{Lt}}\; \frac{s\frac{1}{s}}{1+G(s)\,H(s)} = \underset{s\to0}{\mathrm{Lt}}\; \frac{1}{1+G(s)\,H(s)} = \frac{1}{1+\underset{s\to0}{\mathrm{Lt}}\; G(s)\,H(s)} = \frac{1}{1+K_p} \qquad .....(2.60)$$

$$\text{where, } K_p = \underset{s\to0}{\mathrm{Lt}}\; G(s)\,H(s)$$

The constant K_p is called ***positional error constant.***

Type-0 system

$$K_p = \underset{s\to0}{\mathrm{Lt}}\; G(s)\,H(s) = \underset{s\to0}{\mathrm{Lt}}\; K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3......} = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \text{constant}$$

Hence in type-0 systems when the input is unit step there will be a constant steady state error.

Type-1 system

$$K_p = \underset{s\to 0}{Lt}\, G(s)\,H(s) = \underset{s\to 0}{Lt}\, K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

In systems with type number 1 and above, for unit step input the value of K_p is infinity and so the steady state error is zero.

2.13 STEADY STATE ERROR WHEN THE INPUT IS UNIT RAMP SIGNAL

Steady state error, $e_{ss} = \underset{s\to 0}{Lt}\ \dfrac{sR(s)}{1+G(s)\,H(s)}$

When the input is unit ramp, $R(s) = \dfrac{1}{s^2}$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\ \frac{s\frac{1}{s^2}}{1+G(s)\,H(s)} = \underset{s\to 0}{Lt}\ \frac{1}{s+sG(s)\,H(s)} = \frac{1}{\underset{s\to 0}{Lt}\ sG(s)\,H(s)} = \frac{1}{K_v} \qquad .....(2.61)$$

where, $K_v = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s)$

The constant K_v is called ***velocity error constant.***

Type-0 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\ sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = 1/K_v = 1/0 = \infty$$

Hence in type-0 systems when the input is unit ramp, the steady state error is infinity.

Type-1 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\ sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3......} = \text{constant}$$

$$\therefore e_{ss} = 1/K_v = \text{constant}$$

Hence in type-1 systems when the input is unit ramp there will be a constant steady state error.

Type-2 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\ sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^2\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = 1/K_v = 1/\infty = 0$$

In systems with type number 2 and above, for unit ramp input, the value of K_v is infinity so the steady state error is zero.

2.14 STEADY STATE ERROR WHEN THE INPUT IS UNIT PARABOLIC SIGNAL

Steady state error, $e_{ss} = \underset{s\to 0}{Lt}\ \dfrac{sR(s)}{1+G(s)\,H(s)}$

When the input is unit parabola, $R(s) = \dfrac{1}{s^3}$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\ \frac{s\frac{1}{s^3}}{1+G(s)\,H(s)} = \underset{s\to 0}{Lt}\ \frac{1}{s^2+s^2\,G(s)\,H(s)} = \frac{1}{\underset{s\to 0}{Lt}\ s^2G(s)\,H(s)} = \frac{1}{K_a} \qquad .....(2.62)$$

where, $K_a = \underset{s\to 0}{Lt}\ s^2G(s)\,H(s)$

The constant K_a is called ***acceleration error constant.***

Type-0 system

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)\,H(s) = \underset{s\to 0}{Lt}\ s^2K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-0 systems for unit parabolic input, the steady state error is infinity.

Type-1 system

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)\,H(s) = \underset{s\to 0}{Lt}\ s^2K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-1 systems for unit parabolic input, the steady state error is infinity.

Type-2 system

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)\,H(s) = \underset{s\to 0}{Lt}\ s^2K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^2(s+p_1)\,(s+p_2)\,(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3.....} = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{K_a} = \text{constant}$$

Hence in type-2 system when the input is unit parabolic signal there will be a constant steady state error.

Type-3 system

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)\,H(s) = \underset{s\to 0}{Lt}\ s^2K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^3\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0$$

In systems with type number 3 and above for unit parabolic input the value of K_a is infinity and so the steady state error is zero.

TABLE-2.2 : Static Error Constant for Various Type Number of Systems

Error Constant	Type number of system			
	0	1	2	3
K_p	constant	∞	∞	∞
K_v	0	constant	∞	∞
K_a	0	0	constant	∞

TABLE-2.3 : Steady State Error for Various Types of Inputs

Input Signal	Type number of system			
	0	1	2	3
Unit Step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0

2.15 GENERALIZED ERROR COEFFICIENT

The drawback in static error coefficients is that it does not show the variation of error with time and input should be a standard input. The generalized error coefficients gives the steady state error as a function of time. Also using the generalized error coefficients, the steady state error can be found for any type of input.

The error signal in s-domain, E(s) can be expressed as a product of two s-domain functions.

$$E(s) = \frac{R(s)}{1+G(s)\,H(s)} = \frac{1}{1+G(s)\,H(s)}R(s) = F(s)\,R(s) \qquad .....(2.63)$$

$$\text{where,}\quad F(s) = \frac{1}{1+G(s)\,H(s)}$$

Let, $e(t) = \mathcal{L}^{-1}\{E(s)\}$ (error signal in time domain)

$f(t) = \mathcal{L}^{-1}\{F(s)\}$

$r(t) = \mathcal{L}^{-1}\{R(s)\}$ (input signal in time domain)

The convolution theorem of Laplace transform states that the Laplace transform of the convolution of two time domain signals is equal to the product of their individual Laplace transform.

i.e., $\mathcal{L}\{f(t) * r(t)\} = F(s)\,R(s)$

where $*$ is the symbol for convolution operation

$$\therefore \mathcal{L}^{-1}\{F(s)\,R(s)\} = f(t) * r(t) \qquad .....(2.64)$$

From equation (2.63) & (2.64) we can write,

$e(t) = f(t) * r(t)$

Mathematically the convolution of f(t) and r(t) is defined as,

$$f(t) * r(t) = \int_{-\infty}^{+\infty} f(T)\,r(t-T)\,dT \;; \quad \text{where T is a dummy variable}$$

$$\therefore\; e(t) = \int_{-\infty}^{+\infty} f(T)\,r(t-T)\,dT$$

It is assumed that the input signal starts only at t = 0 and does not exist before t = 0. Also we are interested in finding error signal at any time t after t = 0 (i.e., for t > 0). Hence in the above equation the limit of integral can be changed as 0 to t.

$$\therefore e(t) = \int_{0}^{t} f(T)\,r(t-T)\,dT$$

Using Taylor's series expansion the signal r(t–T) can be expressed as,

$$r(t-T) = r(t) - T\,\dot{r}(t) + \frac{T^2}{2!}\ddot{r}(t) - \frac{T^3}{3!}\dddot{r}(t) + \ldots\ldots + (-1)^n \frac{T^n}{n!}\overset{n.}{r}(t)\ldots$$

where, $\dot{r}(t) = 1^{st}$ derivative of r(t)

$\ddot{r}(t) = 2^{nd}$ derivative of r(t)

$\vdots$

$\overset{n.}{r}(t) = n^{th}$ derivative of r(t)

On substituting the Taylor's series expansion of r(t – T), the error e(t) can be written as,

$$e(t) = \int_0^t f(T)\left[r(t) - T\,\dot{r}(t) + \frac{T^2}{2!}\ddot{r}(t) - \frac{T^3}{3!}\dddot{r}(t) + \ldots + (-1)^n \frac{T^n}{n!}\overset{n.}{r}(t)\ldots\right] dT$$

$$e(t) = \int_0^t f(T)\,r(t)\,dT - \int_0^t f(T)\,T\,\dot{r}(t)\,dT + \int_0^t f(T)\frac{T^2}{2!}\ddot{r}(t)\,dT$$

$$-\int_0^t f(T)\frac{T^3}{3!}\dddot{r}(t) + \ldots + \int_0^t f(T)\,(-1)^n\frac{T^n}{n!}\overset{n.}{r}(t)\,dT\ldots\infty$$

Since $r(t), \dot{r}(t), \ddot{r}(t), \ldots \overset{n.}{r}(t)$ are constants when the integration is done with respect to T, the error signal can be written as,

$$e(t) = r(t)\int_0^t f(T)\,dT - \dot{r}(t)\int_0^t Tf(T)\,dt + \frac{\ddot{r}(t)}{2!}\int_0^t T^2 f(T)\,dt$$

$$-\frac{\dddot{r}(t)}{3!}\int_0^t T^3 f(T)\,dt + \ldots + (-1)^n\frac{\overset{n.}{r}(t)}{n!}\int_0^t T^n f(T)\,dt\ldots$$

Let, $C_0 = +\int_0^t f(T)\,dT$ $\quad C_3 = -\int_0^t T^3 f(T)\,dT$

$C_1 = -\int_0^t Tf(T)\,dT$ $\quad\vdots$

$C_2 = +\int_0^t T^2 f(T)\,dT$ $\quad C_n = (-1)^n\int_0^t T^n f(T)\,dT$

$$e(t) = r(t)\,C_0 + \dot{r}(t)\,C_1 + \ddot{r}(t)\frac{C_2}{2!} + \dddot{r}(t)\frac{C_3}{3!} + \ldots + \overset{n.}{r}(t)\frac{C_n}{n!} + \ldots$$

$$= C_0\,r(t) + C_1\,\dot{r}(t) + \frac{C_2}{2!}\ddot{r}(t) + \frac{C_3}{3!}\dddot{r}(t) + \ldots + \frac{C_n}{n!}\overset{n.}{r}(t)\ldots\ldots \qquad \ldots\ldots(2.65)$$

The equation (2.65) is the general equation for error signal, e(t).

The coefficients $C_0, C_1, C_2, \ldots\ldots C_n$ are called the generalized error coefficients or dynamic error coefficients.

The steady state error e_{ss} is obtained by taking limit $t \to \infty$ on e(t).

$\therefore$ Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\left[r(t)\,C_0 + \dot{r}(t)\,C_1 + \ddot{r}(t)\frac{C_2}{2!} + \dddot{r}(t)\frac{C_3}{3!} + \ldots + \overset{n.}{r}(t)\frac{C_n}{n!} + \ldots\right]$

$$= C_0\,r(t) + C_1\,\dot{r}(t) + \frac{C_2}{2!}\ddot{r}(t) + \frac{C_3}{3!}\dddot{r}(t) + \ldots + \frac{C_n}{n!}\overset{n.}{r}(t)\ldots\ldots \qquad \ldots\ldots(2.66)$$

2.16 EVALUATION OF GENERALIZED ERROR COEFFICIENTS

The generalized error coefficient is given by,

$$C_n = (-1)^n \int_0^t T^n f(T)\, dT \,; \qquad \text{where } F(s) = \frac{1}{1+G(s)\,H(s)}$$

We know that $\mathcal{L}\{f(T)\} = F(s)$, hence by the definition of Laplace transform,

$$F(s) = \int_0^t f(T)\, e^{-sT} dT \qquad .....(2.67)$$

On taking $\underset{s\to 0}{\text{Lt}}$ on both sides of equation (2.67) we get,

$$\underset{s\to 0}{\text{Lt}}\, F(s) = \underset{s\to 0}{\text{Lt}} \int_0^t f(T)\, e^{-sT} dT$$

$$= \int_0^t f(T) \underset{s\to 0}{\text{Lt}}\, e^{-sT} dT = \int_0^t f(T)\, dT = C_0$$

$$\therefore \boxed{\mathbf{C_0 = \underset{s\to 0}{Lt}\, F(s)}} \qquad .....(2.68)$$

On differentiating equation (2.68) with respect to s we get,

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^t f(T)\, e^{-sT}\, dT$$

$$= \int_0^t f(T) \frac{d}{ds} (e^{-sT}) dT = \int_0^t f(T)\, (-T)\, e^{-sT} dT$$

$$= -\int_0^t Tf(T)\, e^{-sT} dT \qquad .....(2.69)$$

On taking $\underset{s\to 0}{\text{Lt}}$ on both sides of equation (2.69) we get,

$$\underset{s\to 0}{\text{Lt}} \frac{d}{ds} F(s) = \underset{s\to 0}{\text{Lt}} \; -\int_0^t Tf(T)\, e^{-sT} dT$$

$$= -\int_0^t Tf(T) \underset{s\to 0}{\text{Lt}}\, e^{-sT} dT = -\int_0^t Tf(T)\, dT = C_1$$

$$\therefore \boxed{\mathbf{C_1 = \underset{s\to 0}{Lt} \frac{d}{ds} F(s)}} \qquad .....(2.70)$$

On differentiating equation (2.68) on both sides with respect to s we get,

$$\frac{d}{ds}\left[\frac{d}{ds}(F(s))\right] = \frac{d}{ds}\left[-\int_0^t Tf(T)\, e^{-sT}\, dT\right]$$

$$\frac{d^2}{ds^2} F(s) = \left[-\int_0^t Tf(T) \frac{d}{ds} (e^{-sT}) dT\right] = -\int_0^t Tf(T)\, (-T)\, e^{-sT} dT$$

$$\frac{d^2\,(F(s))}{ds^2} = \int_0^t T^2 f(T)\, e^{-sT} dT \qquad .....(2.71)$$

Applying the limit $s \to 0$ on both sides of the equation (2.71) we get,

$$\underset{s\to 0}{Lt}\frac{d^2}{ds^2}F(s) = \underset{s\to 0}{Lt}\int_0^t T^2\, f(T)\, e^{-st}\, dT$$

$$= \int_0^t T^2\, f(T) \underset{s\to 0}{Lt}\, e^{-st}\, dT = \int_0^t T^2\, f(T)\, dT = C_2$$

$$\therefore \boxed{C_2 = \underset{s\to 0}{Lt}\frac{d^2}{ds^2}F(s)} \quad .....(2.72)$$

Similarly it can be shown that,

$$\boxed{C_n = \underset{s\to 0}{Lt}\frac{d^n}{ds^n}F(s)} \quad .....(2.73)$$

2.17 CORRELATION BETWEEN STATIC AND DYNAMIC ERROR COEFFICIENTS

The values of dynamic error coefficients can be used to calculate static error coefficients. The following expressions shows the relationship between them.

$$C_0 = \frac{1}{1+K_p} \quad .....(2.74)$$

$$C_1 = \frac{1}{K_v} \quad .....(2.75)$$

$$C_2 = \frac{1}{K_a} \quad .....(2.76)$$

Proof

$$C_0 = \underset{s\to 0}{Lt}\, F(s) = \underset{s\to 0}{Lt}\frac{1}{1+G(s)\,H(s)} = \frac{1}{1+\underset{s\to 0}{Lt}\, G(s)\,H(s)} = \frac{1}{1+K_p}$$

2.18 ALTERNATE METHOD FOR GENERALIZED ERROR COEFFICIENTS

The error signal in s-domain, $E(s) = \dfrac{R(s)}{1+G(s)\,H(s)}$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)\,H(s)} \quad .....(2.77)$$

The equation (2.77) can be expressed as a power series of s as shown in equation (2.78).

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)\,H(s)} = C_0 + C_1 s + \frac{C_2}{2!}s^2 + \frac{C_3}{3!}s^3 + ... \quad .....(2.78)$$

$$\therefore E(s) = C_0\, R(s) + C_1 s R(s) + \frac{C_2}{2!}s^2\, R(s) + \frac{C_3}{3!}s^3\, R(s) + ... \quad .....(2.79)$$

On taking inverse Laplace transform of equation (2.79) we get,

$$e(t) = C_0\, r(t) + C_1 s r(t) + \frac{C_2}{2!}s^2\, r(t) + \frac{C_3}{3!}s^3\, r(t) + ... \quad .....(2.80)$$

The equation (2.80) is same as that of equation (2.65) in section 2.14.This method will be useful to find the generalized error coefficients without using differentiation, but using laplace transform.

EXAMPLE 2.11

For a unity feedback control system the open loop transfer function, $G(s) = \frac{10(s+2)}{s^2(s+1)}$. Find

a) the position, velocity and acceleration error constants,

b) the steady state error when the input is R(s), where $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

SOLUTION

a) To find static error constants

For a unity feedback system, H(s)=1

$$\text{Position error constant, } K_p = \underset{s\to 0}{\text{Lt}}\, G(s)H(s) = \underset{s\to 0}{\text{Lt}}\, G(s) = \underset{s\to 0}{\text{Lt}}\, \frac{10(s+2)}{s^2(s+1)} = \infty$$

$$\text{Velocity error constant, } K_v = \underset{s\to 0}{\text{Lt}}\, s\,G(s)H(s) = \underset{s\to 0}{\text{Lt}}\, s\,G(s) = \underset{s\to 0}{\text{Lt}}\, s\frac{10(s+2)}{s^2(s+1)} = \infty$$

$$\text{Acceleration error constant, } K_a = \underset{s\to 0}{\text{Lt}}\, s^2G(s)H(s) = \underset{s\to 0}{\text{Lt}}\, s^2G(s)$$

$$= \underset{s\to 0}{\text{Lt}}\, s^2\frac{10(s+2)}{s^2(s+1)} = \frac{10\times 2}{1} = 20$$

b) To find steady state error

Method-I

Steady state error for non-standard input is obtained using generalized error series, given below.

$$\text{The error signal, } e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!} + \ldots\ldots + \overset{n}{r}(t)\frac{C_n}{n!} + \ldots.$$

$$\text{Given that, } R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$$

$$\text{Input signal in time domain, } r(t) = \mathcal{L}^{-1}\{R(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}\right\}$$

$$= 3 - 2t + \frac{1}{3}\frac{t^2}{2!} = 3 - 2t + \frac{t^2}{6}$$

$$\therefore \dot{r}(t) = \frac{d}{dt}r(t) = -2 + \frac{1}{6}2t = -2 + \frac{t}{3}$$

$$\ddot{r}(t) = \frac{d^2}{dt^2}r(t) = \frac{d}{dt}\dot{r}(t) = \frac{1}{3}$$

$$\dddot{r}(t) = \frac{d^3}{dt^3}r(t) = \frac{d}{dt}\ddot{r}(t) = 0$$

The derivatives of r(t) is zero after second derivative. Hence we have to evaluate only three constants C_0, C_1 and C_2. The generalized error constants are given by,

$$C_0 = \underset{s\to 0}{\text{Lt}}\, F(s)\,; \qquad C_1 = \underset{s\to 0}{\text{Lt}}\, \frac{d}{ds}F(s)\,; \qquad C_2 = \underset{s\to 0}{\text{Lt}}\, \frac{d^2}{ds^2}F(s)$$

$$F(s) = \frac{1}{1+G(s)H(s)} = \frac{1}{1+G(s)} = \frac{1}{1+\dfrac{10(s+2)}{s^2(s+1)}} = \frac{s^2(s+1)}{s^2(s+1)+10(s+2)} = \frac{s^3+s^2}{s^3+s^2+10s+20}$$

$$C_0 = \underset{s\to 0}{\text{Lt}}\, F(s) = \underset{s\to 0}{\text{Lt}}\left[\frac{s^3+s^2}{s^3+s^2+10s+20}\right] = 0$$

$$C_1 = \underset{s\to 0}{Lt}\frac{d}{ds}F(s) = \underset{s\to 0}{Lt}\frac{d}{ds}\left[\frac{s^3+s^2}{s^3+s^2+10s+20}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{(s^3+s^2+10s+20)(3s^2+2s)-(s^3+s^2)(3s^2+2s+10)}{(s^3+s^2+10s+20)^2}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{3s^5+2s^4+3s^4+2s^3+30s^3+20s^2+60s^2+40s-3s^5-2s^4-10s^3-3s^4-2s^3-10s^2}{(s^3+s^2+10s+20)^2}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{20s^3+70s^2+40s}{(s^3+s^2+10s+20)^2}\right] = 0$$

$$C_2 = \underset{s\to 0}{Lt}\frac{d^2}{ds^2}F(s) = \underset{s\to 0}{Lt}\frac{d}{ds}\left[\frac{d}{ds}F(s)\right] = \underset{s\to 0}{Lt}\frac{d}{ds}\left[\frac{20s^3+70s^2+40s}{(s^3+s^2+10s+20)^2}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{(s^3+s^2+10s+20)^2(60s^2+140s+40) - (20s^3+70s^2+40s)\,2\times(s^3+s^2+10s+20)\,(3s^2+2s+10)}{(s^3+s^2+10s+20)^4}\right] = \frac{20^2\times 40}{20^4} = \frac{1}{10}$$

Error signal, $e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!} = \left(3-2t+\frac{t^2}{6}\right)\times 0 + \left(-2+\frac{t}{3}\right)\times 0 + \frac{1}{3}\times\frac{1}{10}\times\frac{1}{2!} = \frac{1}{60}$

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{t\to\infty}{Lt}\frac{1}{60} = \frac{1}{60}$

Method - II

The error signal in s - domain, $E(s) = \frac{R(s)}{1+G(s)H(s)}$

Given that, $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$; $G(s) = \frac{10(s+2)}{s^2(s+1)}$; $H(s) = 1$

$$\therefore E(s) = \frac{\frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}}{1+\frac{10(s+2)}{s^2(s+1)}} = \frac{\frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}}{\frac{s^2(s+1)+10(s+2)}{s^2(s+1)}}$$

$$= \frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to 0}{Lt}\, s\,E(s)$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\; s\left\{\frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{3s^2(s+1)}{s^2(s+1)+10(s+2)} - \frac{2s(s+1)}{s^2(s+1)+10(s+2)} + \frac{(s+1)}{3s^2(s+1)+30\,(s+2)}\right\} = 0 - 0 + \frac{1}{60}$$

$$= \frac{1}{60}$$

Method - III

$$\text{Error signal in s-domain, } E(s) = \frac{R(s)}{1+G(s)H(s)}$$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

$$\text{Given that, } G(s) = \frac{10(s+2)}{s^2(s+1)}; \qquad H(s) = 1$$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+\dfrac{10(s+2)}{s^2(s+1)}} = \frac{s^2(s+1)}{s^2(s+1)+10(s+2)}$$

$$= \frac{s^3+s^2}{s^3+s^2+10s+20} = \frac{s^2+s^3}{20+10s+s^2+s^3} = \frac{s^2}{20}+\frac{s^3}{40}+....$$

$$\begin{array}{r|l} & \dfrac{s^2}{20}+\dfrac{s^3}{40}\;\cdots\cdots \\ \hline 20+10s+s^2+s^3 & s^2+s^3 \\ & \underset{(-)}{s^2}\underset{(-)}{+}\dfrac{s^3}{2}\underset{(-)}{+}\dfrac{s^4}{20}\underset{(-)}{+}\dfrac{s^5}{20} \\ \hline & \dfrac{s^3}{2}-\dfrac{s^4}{20}-\dfrac{s^5}{20} \\ & \underset{(-)}{\dfrac{s^3}{2}}\underset{(-)}{+}\dfrac{s^4}{4}\underset{(-)}{+}\dfrac{s^5}{40}\underset{(-)}{+}\dfrac{s^6}{40} \\ \hline & -\dfrac{3s^4}{10}-\dfrac{3s^5}{40}-\dfrac{s^6}{40} \\ & \vdots \end{array}$$

Dividing numerator polynomial by denominator polynomial.

$$E(s) = R(s)\left[\frac{s^2}{20}+\frac{s^3}{40}+...\right] = \frac{1}{20}s^2R(s)+\frac{1}{40}s^3R(s)+...$$

On taking inverse Laplace transform of the above equation we get,

$$e(t) = \frac{1}{20}\ddot{r}(t)+\frac{1}{40}\dddot{r}(t)+...$$

$$\text{Given that, } R(s) = \frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}$$

$$\therefore r(t) = \mathcal{L}^{-1}\{R(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}\right\} = 3-2t+\frac{1}{3}\frac{t^2}{2!} = 3-2t+\frac{t^2}{6}$$

$$\dot{r}(t) = \frac{d}{dt}r(t) = -2+\frac{1}{6}2t = -2+\frac{t}{3}$$

$$\ddot{r}(t) = \frac{d^2}{dt^2}r(t) = \frac{d}{dt}\dot{r}(t) = \frac{1}{3}$$

$$\dddot{r}(t) = \frac{d^3}{dt^3}r(t) = \frac{d}{dt}\ddot{r}(t) = 0$$

$$\therefore \text{Error signal in time domain, } e(t) = \frac{1}{20}\ddot{r}(t) = \frac{1}{20}\left(\frac{1}{3}\right) = \frac{1}{60}$$

$$\text{Steady state error, } e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{t\to\infty}{Lt}\,\frac{1}{60} = \frac{1}{60}$$

RESULT

(a) Position error constant, $K_p = \infty$
Velocity error constant, $K_v = \infty$
Acceleration error constant, $K_a = 20$

(b) When, $R(s) = \dfrac{3}{s}-\dfrac{2}{s^2}+\dfrac{1}{3s^3}$, Steady state error, $e_{ss} = \dfrac{1}{60}$

EXAMPLE 2.12

For servomechanisms with open loop transfer function given below explain what type of input signal give rise to a constant steady state error and calculate their values.

a) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$; b) $G(s) = \dfrac{10}{(s+2)(s+3)}$; c) $G(s) = \dfrac{10}{s^2(s+1)(s+2)}$

SOLUTION

a) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$

Let us assume unity feedback system, $\therefore H(s)=1$

The open loop system has a pole at origin. Hence it is a type-1 system. In systems with type number-1, the velocity (ramp) input will give a constant steady state error.

The steady state error with unit velocity input, $e_{ss} = \dfrac{1}{K_v}$

Velocity error constant, $K_v = \underset{s \to 0}{Lt}\ s\,G(s)\,H(s) = \underset{s \to 0}{Lt}\ s\,G(s)$

$$= \underset{s \to 0}{Lt}\ s\ \frac{20(s+2)}{s(s+1)(s+3)} = \frac{20 \times 2}{1 \times 3} = \frac{40}{3}$$

Steady state error, $e_{ss} = \dfrac{1}{K_v} = \dfrac{3}{40} = 0.075$

---

b) $G(s) = \dfrac{10}{(s+2)(s+3)}$

Let us assume unity feedback system, $\therefore H(s)=1$.

The open loop system has no pole at origin. Hence it is a type-0 system. In systems with type number-0, the step input will give a constant steady state error.

The steady state error with unit step input, $e_{ss} = \dfrac{1}{1+K_p}$

Position error constant, $K_p = \underset{s \to 0}{Lt}\ G(s)H(s) = \underset{s \to 0}{Lt}\ G(s) = \underset{s \to 0}{Lt}\ \dfrac{10}{(s+2)(s+3)} = \dfrac{10}{2 \times 3} = \dfrac{5}{3}$

Steady state error, $e_{ss} = \dfrac{1}{1+K_p} = \dfrac{1}{1+\frac{5}{3}} = \dfrac{3}{3+5} = \dfrac{3}{8} = 0.375$

---

c) $G(s) = \dfrac{10}{s^2(s+1)(s+2)}$

Let us assume unity feedback system, $\therefore H(s)=1$.

The open loop system has two poles at origin. Hence it is a type-2 system. In systems with type number-2, the acceleration (parabolic) input will give a constant steady state error.

The steady state error with unit acceleration input, $e_{ss} = \dfrac{1}{K_a}$

Acceleration error constant, $K_a = \underset{s \to 0}{Lt}\ s^2\,G(s)H(s) = \underset{s \to 0}{Lt}\ s^2\,G(s) = \underset{s \to 0}{Lt}\ s^2\ \dfrac{10}{s^2(s+1)(s+2)} = \dfrac{10}{1 \times 2} = 5$

Steady state error, $e_{ss} = \dfrac{1}{K_a} = \dfrac{1}{5} = 0.2$

RESULT

1. In system (a) with unit velocity input, Steady state error $= 0.075$
2. In system (b) with unit step input, Steady state error $= 0.375$
3. In system (c) with unit acceleration input, Steady state error $= 0.2$

EXAMPLE 2.13

The open loop transfer function of a servo system with unity feedback is G(s) = 10/s(0.1s+1). Evaluate the static error constants of the system. Obtain the steady state error of the system, when subjected to an input given by the polynominal, $r(t) = a_0 + a_1 t + \frac{a_2}{2}t^2$.

SOLUTION

To find static error constant

For unity feedback system, H(s) = 1.

$\therefore$ Loop transfer function, G(s) H(s) = G(s)

The static error constants are K_p, K_v and K_a.

$$\text{Position error constant, } K_p = \underset{s\to 0}{\text{Lt}}\ G(s) = \underset{s\to 0}{\text{Lt}} \frac{10}{s(0.1s+1)} = \infty$$

$$\text{Velocity error constant, } K_v = \underset{s\to 0}{\text{Lt}}\ sG(s) = \underset{s\to 0}{\text{Lt}}\ s\frac{10}{s(0.1s+1)} = 10$$

$$\text{Acceleration error constant, } K_a = \underset{s\to 0}{\text{Lt}}\ s^2G(s) = \underset{s\to 0}{\text{Lt}}\ s^2\frac{10}{s(0.1s+1)} = 0$$

To find steady state error

Method - I

Steady state error for non-standard input is obtained using generalized error series, given below.

$$\text{The error signal, } e(t) = r(t)C_o + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!} + ... + \overset{n}{r}(t)\frac{C_n}{n!} + ...$$

$$\text{Given that, } r(t) = a_o + a_1 t + \frac{a_2}{2}t^2$$

$$\therefore\ \dot{r}(t) = \frac{d}{dt}r(t) = \frac{d}{dt}\left(a_o + a_1 t + \frac{a_2}{2}t^2\right) = a_1 + a_2 t$$

$$\ddot{r}(t) = \frac{d^2}{dt^2}r(t) = \frac{d}{dt}\left(\frac{d}{dt}r(t)\right) = \frac{d}{dt}(a_1 + a_2 t) = a_2$$

$$\dddot{r}(t) = \frac{d^3}{dt^3}r(t) = \frac{d}{dt}\left(\frac{d^2}{dt^2}r(t)\right) = \frac{d}{dt}(a_2) = 0$$

Derivatives of r(t) is zero after 2nd derivative. Hence, let us evaluate three constants C_0, C_1 & C_2.

The generalized error constants are given by,

$$C_o = \underset{s\to 0}{\text{Lt}}\ F(s)\ ; \qquad C_1 = \underset{s\to 0}{\text{Lt}} \frac{d}{ds} F(s)\ ; \qquad C_2 = \underset{s\to 0}{\text{Lt}} \frac{d^2}{ds^2} F(s)$$

$$F(s) = \frac{1}{1+G(s)H(s)} = \frac{1}{1+G(s)} = \frac{1}{1+\frac{10}{s(0.1s+1)}} = \frac{s(0.1s+1)}{s(0.1s+1)+10} = \frac{0.1\,s^2+s}{0.1s^2+s+10}$$

$$C_o = \underset{s\to 0}{\text{Lt}}\ F(s) = \underset{s\to 0}{\text{Lt}} \frac{0.1s^2+s}{0.1s^2+s+10} = 0$$

$$C_1 = \underset{s\to 0}{\text{Lt}} \frac{d}{ds}F(s) = \underset{s\to 0}{\text{Lt}} \frac{d}{ds}\left[\frac{0.1s^2+s}{0.1s^2+s+10}\right]$$

$$= \underset{s\to 0}{\text{Lt}}\left[\frac{(0.1s^2+s+10)(0.2s+1) - (0.1s^2+s)(0.2s+1)}{(0.1s^2+s+10)^2}\right] = \underset{s\to 0}{\text{Lt}} \frac{2s+10}{(0.1s^2+s+10)^2} = \frac{10}{10^2} = 0.1$$

$$C_2 = \underset{s\to 0}{Lt}\ \frac{d^2}{ds^2}F(s) = \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{d}{ds}F(s)\right] = \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{2s+10}{(0.1s^2+s+10)^2}\right]$$

$$= \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{(0.1s^2+s+10)^2\times 2-(2s+10)\times 2\,(0.1s^2+s+10)\,(0.2s+1)}{(0.1s^2+s+10)^4}\right]$$

$$\therefore C_2 = \frac{10^2\times 2-10\times 2\times 10\times 1}{10^4} = 0$$

Error signal, $e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!} = \dot{r}(t)C_1 + 0 + 0 = (a_1 + a_2 t)\ 0.1$

$\therefore$ Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{t\to\infty}{Lt}\ \left[\ (a_1 + a_2 t)\ 0.1\ \right] = \infty$

Method - II

The error signal in s-domain, $E(s) = \frac{R(s)}{1+G(s)H(s)}$

Given that, $r(t) = a_o + a_1 t + \frac{a_2}{2}t^2$; $\quad G(s) = \frac{10}{s(0.1s+1)}$; $\quad H(s) = 1$

On taking Laplace transform of r(t) we get R(s),

$$\therefore\ R(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{2}\frac{2!}{s^3} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}$$

$$\therefore\ E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{\dfrac{a_o}{s} + \dfrac{a_1}{s^2} + \dfrac{a_2}{s^3}}{1+\dfrac{10}{s\,(0.1s+1)}} = \frac{\dfrac{a_o}{s} + \dfrac{a_1}{s^2} + \dfrac{a_2}{s^3}}{\dfrac{s\,(0.1s+1)+10}{s\,(0.1s+1)}}$$

$$= \frac{a_0}{s}\left[\frac{s\,(0.1s+1)}{s\,(0.1s+1)+10}\right] + \frac{a_1}{s^2}\left[\frac{s\,(0.1s+1)}{s\,(0.1s+1)+10}\right] + \frac{a_2}{s^3}\left[\frac{s\,(0.1s+1)}{s\,(0.1s+1)+10}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ s\,E(s)$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\ s\left\{\frac{a_o}{s}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_1}{s^2}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_2}{s^3}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{a_o s(0.1s+1)}{s(0.1s+1)+10} + \frac{a_1(0.1s+1)}{s(0.1s+1)+10} + \frac{a_2(0.1s+1)}{s[s(0.1s+1)+10]}\right\} = 0 + \frac{a_1}{10} + \infty = \infty$$

Method - III

Error signal in s-domain, $E(s) = \frac{R(s)}{1+G(s)H(s)}$; $\quad \therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$

Given that, $G(s) = \frac{10}{s(0.1s+1)}$ and $H(s) = 1$.

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+\dfrac{10}{s(0.1s+1)}} = \frac{s(0.1s+1)}{s(0.1s+1)+10} = \frac{0.1s^2+s}{0.1s^2+s+10} = \frac{s+0.1s^2}{10+s+0.1s^2} = \frac{s}{10} - \frac{s^3}{1000} + \ldots$$

Dividing numerator polynomial by denominator polynomial.

$$\therefore E(s) = \frac{s}{10}R(s) - \frac{s^3}{1000}R(s) + \ldots$$

On taking inverse Laplace transform,

$$e(t)=\frac{1}{10}\dot{r}-\frac{1}{1000}\dddot{r}(t)+.....$$

$$\begin{array}{r|l} & \frac{s}{10}-\frac{s^3}{1000} \\ \hline 10+s+0.1s^2 & s+0.1s^2 \\ & \underset{(-)}{s}\underset{(-)}{+}\frac{s^2}{10}\underset{(-)}{+}\frac{s^3}{100} \\ \cline{2-2} & -\frac{s^3}{100} \\ & \underset{(+)}{-}\frac{s^3}{100}\underset{(+)}{-}\frac{s^4}{1000}\underset{(+)}{-}\frac{s^5}{10000} \\ \cline{2-2} & \frac{s^4}{1000}+\frac{s^5}{10000} \end{array}$$

Given that, $r(t)=a_0+a_1t+\frac{a_2}{2}t^2$

$$\therefore\ \dot{r}=\frac{d}{dt}r(t)=a_1+a_2t$$

$$\ddot{r}(t)=\frac{d}{dt}\dot{r}(t)=a_2$$

$$\dddot{r}(t)=\frac{d}{dt}\ddot{r}(t)=0$$

$\therefore$ Error signal in time domain, $e(t)=\frac{1}{10}\dot{r}(t)=\frac{1}{10}(a_1+a_2t)$

Steady state error, $e_{ss}=\underset{t\to\infty}{Lt}\ e(t)=\underset{t\to\infty}{Lt}\ \frac{1}{10}(a_1+a_2t)=\infty$

RESULT

(a) Position error constant, $K_p=\infty$

(b) Velocity error constant, $K_v=10$

(c) Acceleration error constant, $K_a=0$

(d) When input, $r(t)=a_o+a_1t+\frac{a_2t^2}{2}$, Steady state error, $e_{ss}=\infty$

EXAMPLE 2.14

Consider a unity feedback system with a closed loop transfer function $\frac{C(s)}{R(s)}=\frac{Ks+b}{s^2+as+b}$. Determine open loop transfer function G(s). Show that steady state error with unit ramp input is given by $\frac{(a-K)}{b}$.

SOLUTION

For unity feedback system, H(s)=1

The closed loop transfer function, $M(s)=\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{G(s)}{1+G(s)}$

$$\therefore \frac{G(s)}{1+G(s)}=M(s)$$

On cross multiplication of the above equation we get,

$$G(s)=M(s)[1+G(s)]=M(s)+M(s)\,G(s)$$

$$\therefore\ G(s)-M(s)\,G(s)=M(s)\quad\Rightarrow\quad G(s)[1-M(s)]=M(s)\quad\Rightarrow\quad M(s)=\frac{Ks+b}{s^2+as+b}$$

$\therefore$ Open loop transfer function,

$$G(s)=\frac{M(s)}{1-M(s)}=\frac{\frac{Ks+b}{s^2+as+b}}{1-\frac{Ks+b}{s^2+as+b}}=\frac{Ks+b}{\left(s^2+as+b\right)-(Ks+b)}$$

$$=\frac{Ks+b}{s^2+as+b-Ks-b}=\frac{Ks+b}{s^2+(a-k)s}=\frac{Ks+b}{s[s+(a.K)]}$$

Velocity error constant, $K_V = \underset{s\to 0}{Lt}\ s\,G(s)H(s) = \underset{s\to 0}{Lt}\ s\,G(s) = \underset{s\to 0}{Lt}\ s\dfrac{Ks+b}{s[s+(a-K)]} = \dfrac{b}{a-K}$

With velocity input, Steady state error, $e_{ss} = \dfrac{1}{K_V} = \dfrac{a-K}{b}$

RESULT

Open loop transfer function, $G(s) = \dfrac{Ks+b}{s[s+(a-K)]}$

With velocity input, Steady state error, $e_{ss} = \dfrac{a-K}{b}$

EXAMPLE 2.15

A unity feedback system has the forward transfer function $G(s) = \dfrac{K_1(2s+1)}{s(5s+1)(1+s)^2}$. When the input r(t) = 1+6t, determine the minimum value of K_1 so that the steady error is less than 0.1.

SOLUTION

Given that, input $r(t) = 1 + 6t$

On taking laplace transform of r(t) we get R(s).

$$\therefore R(s) = \mathcal{L}\{r(t)\} = \mathcal{L}\{1+6t\} = \frac{1}{s}+\frac{6}{s^2}$$

The error signal in s-domain E(s) is given by,

$$\therefore E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{\frac{1}{s}+\frac{6}{s^2}}{1+\frac{K_1(2s+1)}{s(5s+1)(1+s)^2}} = \frac{\frac{1}{s}+\frac{6}{s^2}}{\frac{s(5s+1)(1+s)^2+K_1(2s+1)}{s(5s+1)(1+s)^2}}$$

$\boxed{\text{Here } H(s)=1}$

$$= \frac{1}{s}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right]+\frac{6}{s^2}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

$$e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ s\,E(s)$$

$$= \underset{s\to 0}{Lt}\ s\left\{\frac{1}{s}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right]+\frac{6}{s^2}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}+\frac{6(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right\} = 0+\frac{6}{K_1} = \frac{6}{K_1}$$

Given that, $e_{ss} < 0.1$, $\quad \therefore 0.1 = \dfrac{6}{K_1}$ $\quad$ or $\quad K_1 = \dfrac{6}{0.1} = 60$

RESULT

For steady state error, $e_{ss} < 0.1$, the value of K_1 should be greater than 60.

2.19 COMPONENTS OF AUTOMATIC CONTROL SYSTEM

The basic components of an automatic control system are Error detector, Amplifier and Controller, Actuator (Power actuator), Plant and Sensor or Feedback system. The block diagram of an automatic control system is shown in fig 2.16.

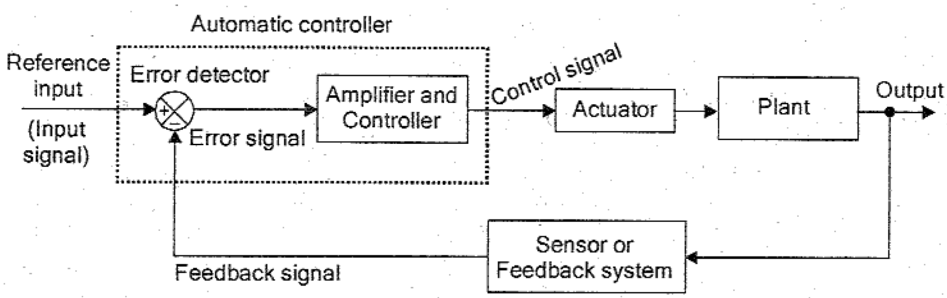


Fig 2.16: Block diagram of automatic control system.

The plant is the open loop system whose output is automatically controlled by closed loop system. The combined unit of error detector, amplifier and controller is called ***automatic controller***, because without this unit the system becomes open loop system.

In automatic control systems the reference input will be an input signal proportional to desired output. The feedback signal is a signal proportional to current output of the system. The error detector compares the reference input and feedback signal and if there is a difference it produces an error signal. An amplifier can be used to amplify the error signal and the controller modifies the error signal for better control action.

The actuator amplifies the controller output and converts to the required form of energy that is acceptable for the plant. Depending on the input to the plant, the output will change. This process continues as long as there is a difference between reference input and feedback signal. If the difference is zero, then there is no error signal and the output settles at the desired value.

Generally, the error signal will be a weak signal and so it has to be amplified and then modified for better control action. In most of the system the controller itself amplifies the error signal and integrates or differentiates to produce a control signal (i.e., modified error signal). The different types of controllers are P, PI, PD and PID controllers.

2.20 CONTROLLERS

A controller is a device introduced in the system to modify the error signal and to produce a control signal. The manner in which the controller produces the control signal is called the ***control action***. The controller modifies the transient response of the system. The electronic controllers using operational amplifiers are presented in this section.

The following six basic control actions are very common among industrial analog controllers.

1. Two-position or ON-OFF control action.
2. Proportional control action.
3. Integral control action.
4. Proportional- plus- integral control action.
5. Proportional-plus-derivative control action.
6. Proportional-plus-integral-plus-derivative control action.

Depending on the control actions provided the controllers can be classified as follows.

1. Two position or ON-OFF controllers.
2. Proportional controllers.
3. Integral controllers.
4. Proportional-plus-integral controllers.
5. Proportional-plus-derivative controllers.
6. Proportional-plus-integral-plus-derivative controllers.

ON-OFF (OR) TWO POSITION CONTROLLER

The ON-OFF or two position controller has only two fixed positions. They are either on or off. The on-off control system is very simple in construction and hence less expensive. For this reason, it is very widely used in both industrial and domestic control systems.

The ON-OFF control action may be provided by a relay. There are different types of relay. The most popular one is electromagnetic relay. It is a device which has NO (Normally Open) and NC (Normally Closed) contacts, whose opening and closing are controlled by the relay coil. When the relay coil is excited, the relay operates and the contacts change their positions (i.e., NO $\rightarrow$ NC and NC $\rightarrow$ NO).

Let the output signal from the controller be u(t) and the actuating error signal be e(t). In this controller, u(t) remains at either a maximum or minimum value.

$$u(t) = u_1 \; ; \quad \text{for } e(t) < 0$$
$$\quad\;\; = u_2 \; ; \quad \text{for } e(t) > 0$$

$$\boxed{E(s) = \mathcal{L}\{e(t)\} \; ; \quad U(s) = \mathcal{L}\{u(t)\}}$$

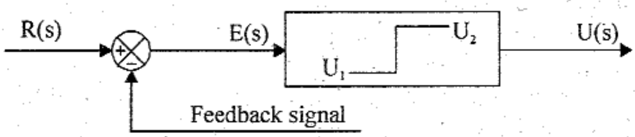


***Fig 2.17 :** Block diagram of on-off controller.*

PROPORTIONAL CONTROLLER (P - CONTROLLER)

The proportional controller is a device that produces a control signal, u(t) proportional to the input error signal, e(t).

In P-controller, $u(t) \propto e(t)$

$$\therefore \quad u(t) = K_p \, e(t) \qquad .....(2.81)$$

where, K_p = Proportional gain or constant

On taking Laplace transform of equation (2.81) we get,

$$U(s) = K_p E(s) \qquad .....(2.82)$$

$$\boxed{\therefore \; \text{Transfer function of P - controller, } \frac{U(s)}{E(s)} = K_p} \qquad .....(2.83)$$

The equation (2.82) gives the output of the P-controller for the input E(s) and equation (2.83) is the transfer function of the P-controller. The block diagram of the P-controller is shown in fig 2.18.

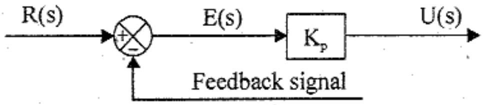


***Fig 2.18 :** Block diagram of proportional controller.*

From the equation (2.82), we can conclude that the proportional controller amplifies the error signal by an amount K_p. Also the introduction of the controller on the system increases the loop gain by an amount K_p. The increase in loop gain improves the steady state tracking accuracy, disturbance signal rejection and the relative stability and also makes the system less sensitive to parameter variations. But increasing the gain to very large values may lead to instability of the system. The drawback in P-controller is that it leads to a constant steady state error.

EXAMPLE OF ELECTRONIC P-CONTROLLER

The proportional controller can be realized by an amplifier with adjustable gain. Either the non-inverting operational amplifier or the inverting operational amplifier followed by sign changer will work as a proportional controller. The op-amp proportional controller is shown in fig 2.19 and 2.20.

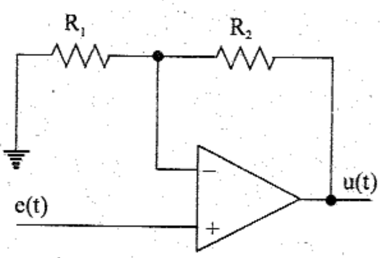


Fig 2.19 : Op-amp P-controller using non-inverting amplifier.

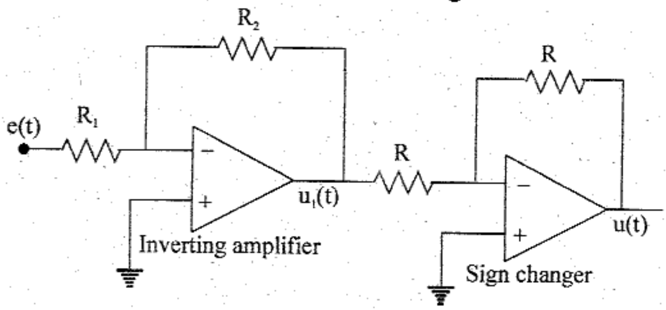


Fig 2.20 : Op-amp P-controller using inverting amplifier.

By deriving the transfer function of the controllers shown in fig 2.11 and 2.12 and comparing with the transfer function of P-controller defined by equation (2.83), it can be shown that they work as P-controllers.

ANALYSIS OF P-CONTROLLER SHOWN IN FIG 2.19

In fig 2.19, the input e(t) is applied to positive input. By symmetry of op-amp the voltage of negative input is also e(t). Also we assume an ideal op-amp so that input current is zero. Based on the above assumptions the equivalent circuit of the controller is shown in fig 2.21.

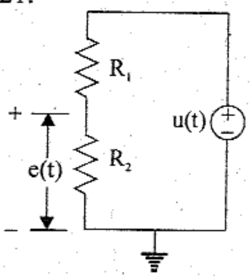


Fig 2.21 : Equivalent circuit of P-controller shown in fig 2.19.

By voltage division rule,

$$e(t) = \frac{R_1}{R_1 + R_2}\, u(t) \;; \qquad \therefore u(t) = \frac{R_1 + R_2}{R_1}\, e(t) \qquad .....(2.84)$$

On taking Laplace transform of equation (2.84) we get,

$$U(s) = \frac{R_1 + R_2}{R_1}\, E(s) \qquad .....(2.85)$$

$$\therefore \frac{U(s)}{E(s)} = \frac{R_1 + R_2}{R_1} \qquad .....(2.86)$$

The equation (2.86) is the transfer function of op-amp P-controller. On comparing equation (2.86) with equation (2.83) we get,

$$\text{Proportional gain, } K_P = \frac{R_1 + R_2}{R_1} \qquad .....(2.87)$$

Therefore by adjusting the values of R_1 and R_2 the value of gain, K_p can be varied.

ANALYSIS OF P-CONTROLLER SHOWN IN FIG 2.20

The assumption made in op-amp circuit analysis are,

1. The voltages at both inputs are equal
2. The input current is zero.

Based on the above assumptions, the equivalent circuit of op-amp amplifier and sign changer are shown in fig 2.22 and 2.23.

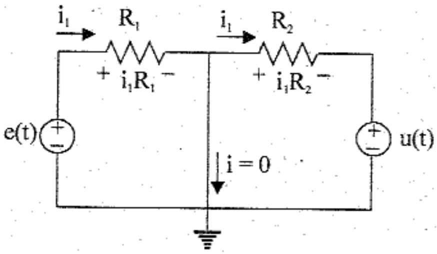


Fig 2.22 : Equivalent circuit of amplifier.

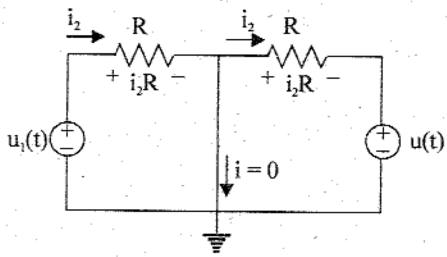


Fig 2.23 : Equivalent circuit of sign changer.

From fig 2.22, $e(t) = i_1R_1$; $\therefore\ i_1 = \dfrac{e(t)}{R_1}$ .....(2.88)

$u_1(t) = -i_1R_2$ .....(2.89)

Substitute for i_1 from equation (2.88) in equation (2.89).

$$\therefore\ u_1(t) = -\frac{e(t)}{R_1}R_2 \quad .....(2.90)$$

From fig 2.23, $u(t) = -i_2R$; $\therefore i_2 = -\dfrac{u(t)}{R}$ .....(2.91)

$u_1(t) = i_2R$ .....(2.92)

Substitute for i_2 from equation (2.91) in equation (2.92).

$$\therefore\ u_1(t) = -\frac{u(t)}{R}R = -u(t) \quad .....(2.93)$$

On equating the equations (2.90) and (2.93) we get,

$$-u(t) = -\frac{e(t)}{R_1}R_2\ ; \qquad u(t) = \frac{R_2}{R_1}e(t) \quad .....(2.94)$$

On taking Laplace transform of equation (2.94) we get,

$$U(s) = \frac{R_2}{R_1}E(s) \quad .....(2.95)$$

$$\therefore\ \frac{U(s)}{E(s)} = \frac{R_2}{R_1} \quad .....(2.96)$$

The equation (2.96) is the transfer function of op-amp P-controller. On comparing equation (2.96) with equation (2.83) we get,

$$\text{Proportional gain, } K_p = \frac{R_2}{R_1} \quad .....(2.97)$$

Therefore by adjusting the values of R_1 and R_2 the value of gain K_p can be varied.

INTEGRAL CONTROLLER (I-CONTROLLER)

The integral controller is a device that produces a control signal u(t) which is proportional to integral of the input error signal, e(t).

In I-controller, $u(t) \ \alpha \int e(t)\, dt$; $\therefore u(t) = K_i \int e(t)\, dt$ .....(2.98)

where, K_i = Integral gain or constant.

On taking Laplace transform of equation (2.98) with zero initial conditions we get,

$$U(s) = K_i \frac{E(s)}{s} \qquad .....(2.99)$$

$$\boxed{\therefore \text{ Transfer function of I-controller, } \frac{U(s)}{E(s)} = \frac{K_i}{s}} \qquad .....(2.100)$$

The equation (2.99) gives the output of the I-controller for the input E(s) and equation (2.101) is the transfer function of the I-controller. The block diagram of I-controller is shown in fig 2.24.

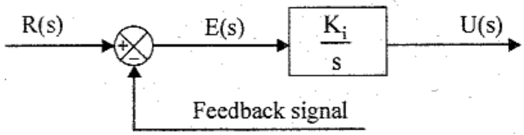


Fig 2.24 : *Block diagram of an integral controller.*

The integral controller removes or reduces the steady error without the need for manual reset. Hence the I-controller is sometimes called ***automatic reset***. The drawback in integral controller is that it may lead to oscillatory response of increasing or decreasing amplitude which is undesirable and the system may become unstable.

EXAMPLE OF ELECTRONIC I-CONTROLLER

The integral controller can be realized by an integrator using op-amp followed by a sign changer as shown in fig 2.25.

By deriving the transfer function of the controller shown in fig 2.25 and comparing with the transfer function of I-controller defined by equation(2.101), it can be shown that it work as I-controller.

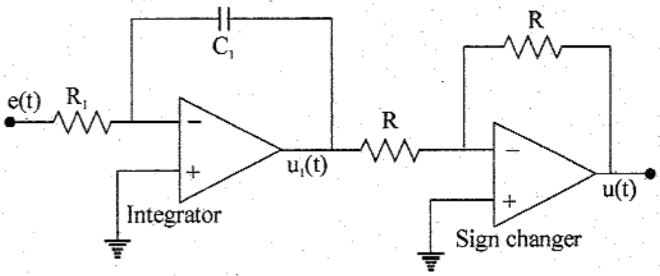


Fig 2.25 : *I-controller using op-amp.*

ANALYSIS OF I-CONTROLLER SHOWN IN FIG 2.25

The assumptions made in op-amp circuit analysis are,

1. The voltages of both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp integrator and sign changer are shown in fig 2.26 and 2.27.

From fig 2.26, $e(t) = i_1 R_1$; $\therefore i_1 = \frac{e(t)}{R_1}$ .....(2.101)

$$u_1(t) = -\frac{1}{C_1}\int i_1 dt \qquad .....(2.102)$$

Substitute for i_1 from equation (2.101) in equation (2.102).

$$\therefore\ u_1(t) = -\frac{1}{C_1}\int \frac{e(t)}{R_1}dt = -\frac{1}{R_1C_1}\int e(t)dt \qquad .....(2.103)$$

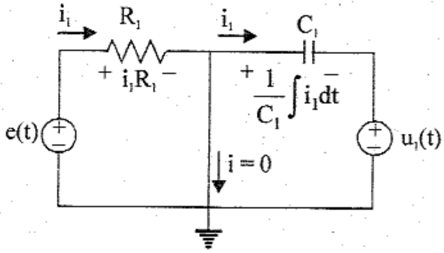


Fig 2.26 : Equivalent circuit of integrator.

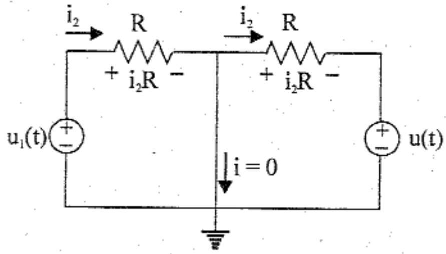


Fig 2.27 : Equivalent circuit of sign changer.

From fig 2.27, $u(t) = -i_2R$, $\therefore\ i_2 = \frac{-u(t)}{R}$ .....(2.104)

$$u_1(t) = i_2R \qquad .....(2.105)$$

Substitute for i_2 from equation (2.106) in equation (2.107),

$$\therefore\ u_1(t) = -\frac{u(t)}{R}R = -u(t) \qquad .....(2.106)$$

On equating the equations (2.103) and (2.106) we get,

$$-u(t) = -\frac{1}{R_1C_1}\int e(t)\,dt$$

$$\therefore\ u(t) = \frac{1}{R_1C_1}\int e(t)\,dt \qquad .....(2.107)$$

On taking Laplace transform of equation (2.107) with zero initial conditions we get,

$$U(s) = \frac{1}{R_1C_1}\frac{E(s)}{s} \qquad .....(2.108)$$

$$\therefore\ \frac{U(s)}{E(s)} = \frac{1}{R_1C_1}\frac{1}{s} \qquad .....(2.109)$$

The equation (2.109) is the transfer function of op-amp I-controller. On comparing equation (2.109) with equation (2.100) we get,

$$\text{Integral gain, } K_i = \frac{1}{R_1C_1} \qquad .....(2.110)$$

Therefore by adjusting the values of R_1 and C_1 the value of gain K_i can be varied.

PROPORTIONAL PLUS INTEGRAL CONTROLLER (PI-CONTROLLER)

The proportional plus integral controller (PI-controller) produces an output signal consisting of two terms : *one proportional to error signal and the other proportional to the integral of error signal.*

In PI - controller, $u(t)\ \alpha\ \left[e(t) + \int e(t)\,dt\right]$; $\quad \therefore\ u(t) = K_p\,e(t) + \frac{K_p}{T_i}\int e(t)dt$ ....(2.111)

where, K_p = Proportional gain

T_i = Integral time.

On taking Laplace transform of equation (2.111) with zero initial conditions we get,

$$U(s) = K_p\,E(s) + \frac{K_p}{T_i}\,\frac{E(s)}{s} \qquad .....(2.112)$$

$$\boxed{\therefore \text{ Transfer function of PI - controller, } \frac{U(s)}{E(s)} = K_p\left(1+\frac{1}{T_i s}\right)} \qquad .....(2.113)$$

The equation (2.112) gives the output of the PI-controller for the input E(s) and equation (2.113) is the transfer function of the PI controller. The block diagram of PI-controller is shown in fig 2.28.

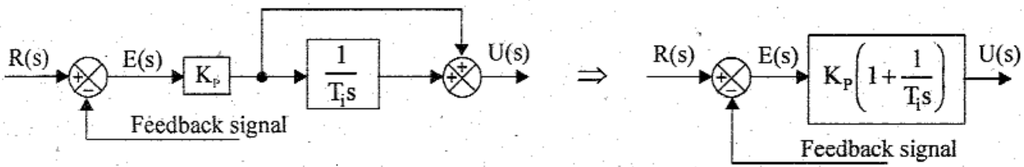


Fig 2.28 : Block diagram of PI-controller.

The advantages of both P-controller and I-controller are combined in PI-controller. The proportional action increases the loop gain and makes the system less sensitive to variations of system parameters. The integral action eliminates or reduces the steady state error.

The integral control action is adjusted by varying the integral time. The change in value of K_p affects both the proportional and integral parts of control action. The inverse of the integral time T_i is called the ***reset rate.***

EXAMPLE OF ELECTRONIC PI-CONTROLLER

The PI-controller can be realized by an op-amp integrator with gain followed by a sign changer as shown in fig 2.29.

By deriving the transfer function of the controller shown in fig (2.29) and comparing with the transfer function of PI-controller defined by equation (2.114), it can be proved that the circuit shown in fig 2.29, work as PI-controller.

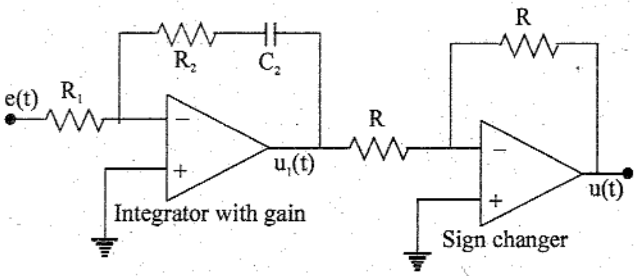


Fig 2.29 : PI-controller using op-amp.

ANALYSIS OF PI-CONTROLLER SHOWN IN FIG 2.29

The assumptions made in op-amp circuit analysis are,

1. The voltages at both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp integrator and sign changer are shown in fig 2.30. and 2.31.

From fig 2.30, $e(t) = i_1 R_1$; $\therefore\ i_1 = \frac{e(t)}{R_1}$ .....(2.114)

$$u_1(t) = -i_1 R_2 - \frac{1}{C_2}\int i_1 dt \qquad .....(2.115)$$

Substitute for i_1 from equation (2.114) in equation (2.115).

$$\therefore\ u_1(t) = -\frac{e(t)}{R_1}R_2 - \frac{1}{C_2}\int \frac{e(t)}{R_1}\,dt \qquad .....(2.116)$$

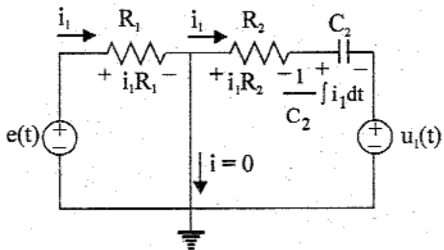


Fig 2.30 : Equivalent circuit of integrator.

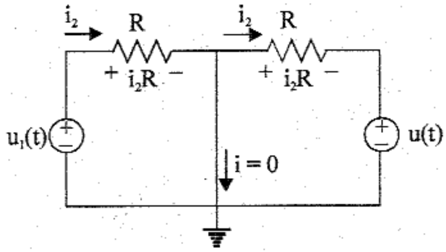


Fig 2.31 : Equivalent circuit of sign changer.

From fig 2.31, $u(t) = -i_2R$, $\therefore\ i_2 = \dfrac{-u(t)}{R}$ .....(2.117)

$u_1(t) = i_2R$ .....(2.118)

Substitute for i_2 from equation (2.117) in equation (2.118),

$$\therefore\ u_1(t) = -\frac{u(t)}{R}R = -u(t) \qquad .....(2.119)$$

On equating the equations (2.116) and (2.119) we get,

$$-u(t) = -\frac{e(t)}{R_1}R_2 - \frac{1}{C_2}\int \frac{e(t)}{R_1}\,dt$$

$$\therefore\ u(t) = \frac{R_2}{R_1}e(t) + \frac{1}{R_1C_2}\int e(t)\,dt \qquad .....(2.120)$$

On taking Laplace transform of equation (2.120) with zero initial conditions we get,

$$U(s) = \frac{R_2}{R_1}E(s) + \frac{1}{R_1C_2}\frac{E(s)}{s}$$

$$\therefore\ \frac{U(s)}{E(s)} = \frac{R_2}{R_1}\left(1 + \frac{1}{R_2C_2s}\right) \qquad .....(2.121)$$

The equation (2.121) is the transfer function of op-amp PI-controller. On comparing equation(2.121) with equation (2.113) we get,

Proportional gain, $K_p = \dfrac{R_2}{R_1}$; Integral time, $T_i = R_2C_2$

By varying the values of R_1 and R_2, the value of gain K_p and T_i can be adjusted.

PROPORTIONAL PLUS DERIVATIVE CONTROLLER (PD-CONTROLLER)

The proportional plus derivative controller produces an output signal consisting of two terms : *one proportional to error signal and the other proportional to the derivative of error signal.*

In PD - controller, $u(t)\ \alpha \left[e(t) + \frac{d}{dt}e(t)\right]$; $\therefore u(t) = K_p\,e(t) + K_pT_d\frac{d}{dt}e(t)$ .....(2.122)

where, K_p = Proportional gain

T_d = Derivative time

On taking Laplace transform of equation (2.123) with zero initial conditions we get,

$$U(s) = K_p\,E(s) + K_p\,T_d\,sE(s) \qquad .....(2.123)$$

$$\therefore \text{ Transfer function of PD-controller, } \frac{U(s)}{E(s)} = K_p(1+T_d s) \qquad .....(2.124)$$

The equation (2.123) gives the output of the PD-controller for the input E(s) and equation (2.124) is the transfer function of PD-controller.

The block diagram of PD-controller is shown in fig 2.32.

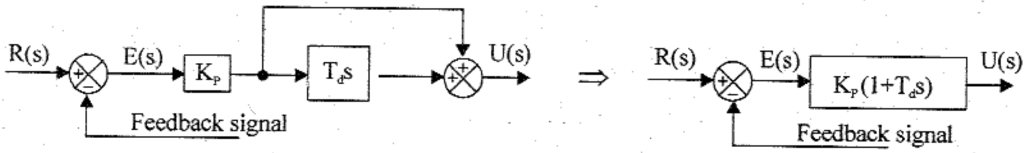


Fig 2.32 : Block diagram of PD- controller.

The derivative control acts on rate of change of error and not on the actual error signal. The derivative control action is effective only during transient periods and so it does not produce corrective measures for any constant error. Hence the derivative controller is never used alone, but it is employed in association with proportional and integral controllers. The derivative controller does not affect the steady-state error directly but anticipates the error, initiates an early corrective action and tends to increase the stability of the system. While derivative control action has an advantage of being anticipatory it has the disadvantage that it amplifies noise signals and may cause a saturation effect in the actuator.

The derivative control action is adjusted by varying the derivative time. The change in the value of K_p affects both the proportional and derivative parts of control action. The derivative control is also called ***rate control.***

EXAMPLE OF ELECTRONIC PD-CONTROLLER

The PD-controller can be realized by an op-amp differentiator with gain followed by a sign changer as shown in fig 2.33.

By deriving the transfer function of the controller shown in fig 2.33 and comparing with the transfer function of PD-controller defined by equation (2.124) it can be proved that the circuit shown in fig 2.33 will work as PD-controller.

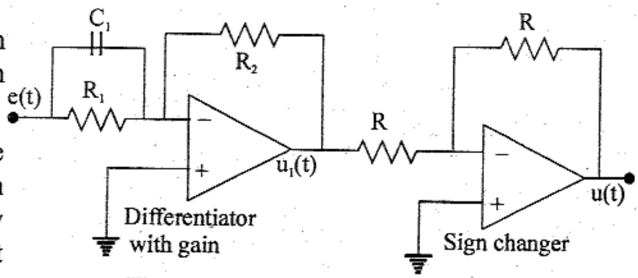


Fig 2.33 : PD controller using op-amp.

ANALYSIS OF PD-CONTROLLER SHOWN IN FIG 2.33

The assumptions made in op-amp circuit analysis are,

1. The voltages at both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp differentiator and sign changer are shown in fig 2.34 and 2.35.

From fig 2.34, $\therefore\ i_1 = \dfrac{e(t)}{R_1} + C_1\dfrac{de(t)}{dt}$ .....(2.125)

$$i_1R_2 = -u_1(t), \qquad \therefore\ i_1 = \frac{-u_1(t)}{R_2} \qquad .....(2.126)$$

On equating the equations (2.125) and (2.126) we get,

$$-\frac{u_1(t)}{R_2} = \frac{e(t)}{R_1} + C_1\frac{d}{dt}e(t)\,; \qquad \therefore\ u_1(t) = -\left(\frac{R_2}{R_1}e(t) + R_2C_1\frac{d}{dt}e(t)\right) \qquad .....(2.127)$$

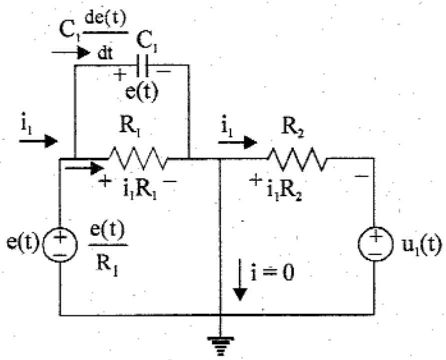


Fig 2.34 : Equivalent circuit of differentiator.

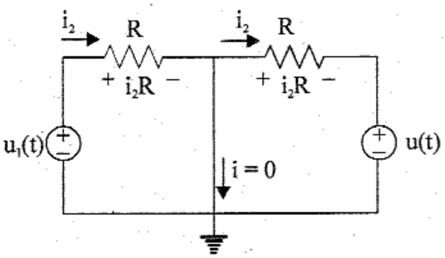


Fig 2.35 : Equivalent circuit of sign changer.

From fig 2.35, $u(t) = -i_2R$; $\therefore\ i_2 = \dfrac{-u(t)}{R}$ .....(2.128)

$$u_1(t) = i_2R \qquad .....(2.129)$$

Substitute for i_2 from equation (2.128) in equation (2.129).

$$\therefore\ u_1(t) = -\frac{u(t)}{R}R = -u(t) \qquad .....(2.130)$$

On equating the equations (2.127) and (2.130) we get,

$$-u(t) = -\left(\frac{R_2}{R_1}e(t) + R_2C_1\frac{d}{dt}e(t)\right)$$

$$\therefore\ u(t) = \frac{R_2}{R_1}e(t) + R_2C_1\frac{d}{dt}e(t) \qquad .....(2.131)$$

On taking Laplace transform of equation (2.131) with zero initial conditions we get,

$$U(s) = \frac{R_2}{R_1}\ E(s) + R_2C_1\ s\ E(s) \qquad .....(2.132)$$

$$\therefore\ \frac{U(s)}{E(s)} = \frac{R_2}{R_1}\ (1 + R_1C_1\ s) \qquad .....(2.133)$$

The equation (2.133) is the transfer function of op-amp PD-controller. On comparing equation (2.133) with equation (2.124) we get,

Proportional gain, $K_p = \dfrac{R_2}{R_1}$

Derivative time, $T_d = R_1C_1$

By varying the values of R_1 and R_2, the value of gain K_p and T_d can be adjusted.

PROPORTIONAL PLUS INTEGRAL PLUS DERIVATIVE CONTROLLER (PID-CONTROLLER)

The PID-controller produces an output signal consisting of three terms : *one proportional to error signal, another one proportional to integral of error signal and the third one proportional to derivative of error signal.*

In PID - controller, $u(t) \;\alpha\; \left[e(t)+\int e(t)\,dt+\frac{d}{dt}e(t)\right]$

$$\therefore u(t) = K_p\, e(t)+\frac{K_p}{T_i}\int e(t)dt+K_pT_d\frac{d}{dt}e(t) \quad .....(2.134)$$

where, K_p = Proportional gain

T_i = Integral time

T_d = Derivative time

On taking Laplace transform of equation (2.134) with zero initial conditions we get,

$$U(s) = K_pE(s)+\frac{K_p}{T_i}\,\frac{E(s)}{s}+K_p\,T_d\,s\,E(s) \quad .....(2.135)$$

$$\boxed{\therefore \text{ Transfer function of PID - controller, } \frac{U(s)}{E(s)} = K_p\left(1+\frac{1}{T_is}+T_ds\right)} \quad .....(2.136)$$

The equation (2.135) gives the output of the PID-controller for the input E(s) and equation (2.136) is the transfer function of the PID-controller. The block diagram of PID-contoller is shown in fig 2.36.

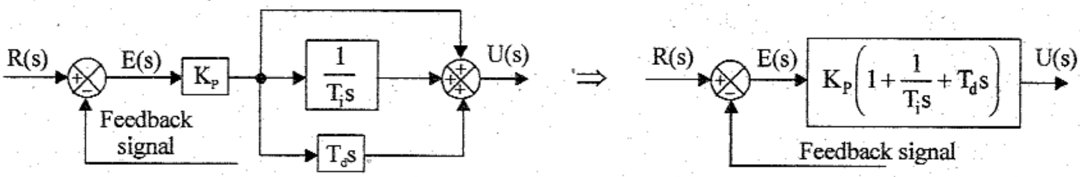


Fig 2.36: Block diagram of PID- controller.

The combination of proportional control action, integral control action and derivative control action is called PID-control action. This combined action has the advantages of the each of the three individual control actions.

The proportional controller stabilizes the gain but produces a steady state error. The integral controller reduces or eliminates the steady state error. The derivative controller reduces the rate of change of error.

EXAMPLE OF ELECTRONIC PID-CONTROLLER

The PID-controller can be realized by op-amp amplifier with integral and derivative action followed by sign changer as shown in fig 2.37.

By deriving the transfer function of the controller shown in fig (2.37) and comparing with the transfer function of PID-controller defined by equation (2.136) it can be proved that the circuit shown in fig 2.37 work as PID-controller.

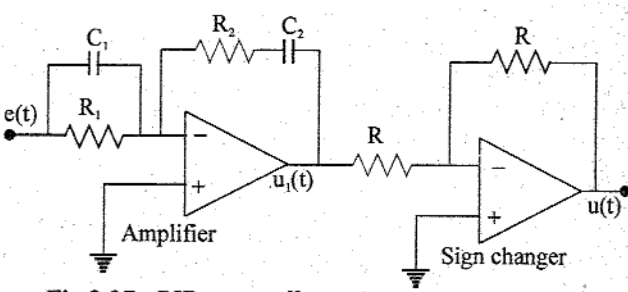


Fig 2.37 : PID- controller using op-amp.

ANALYSIS OF PID-CONTROLLER SHOWN IN FIG 2.37

The assumptions made in op-amp circuit analysis are.

1. The voltages of both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp amplifier and sign changer are shown in fig 2.38 and 2.39.

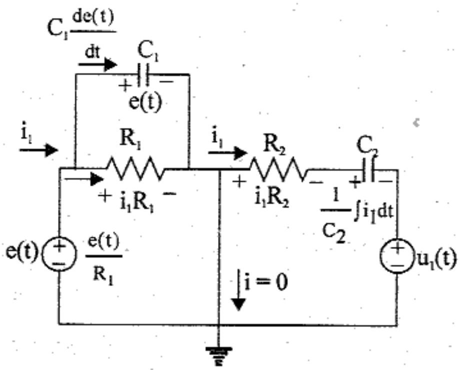


Fig 2.38 : Equivalent circuit of amplifier.

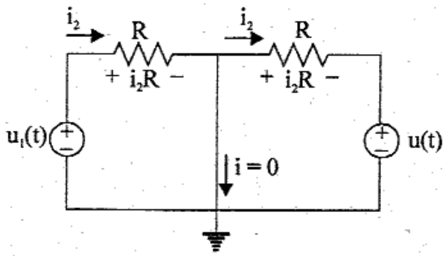


Fig 2.39 : Equivalent circuit of sign changer.

From fig 2.38, $i_1 = \frac{e(t)}{R_1} + C_1\frac{de(t)}{dt}$ .....(2.137)

On taking Laplace transform of equation (2.137) with zero initial conditions we get,

$$I_1(s) = \frac{1}{R_1} E(s) + C_1 s E(s)$$

$$I_1(s) = \left(\frac{1}{R_1} + C_1 s\right) E(s) \qquad .....(2.138)$$

From fig 2.38, $i_1R_2 + \frac{1}{C_2}\int i_1 dt = -u_1(t)$ .....(2.139)

On taking Laplace transform of equation (2.138) with zero initial conditions we get,

$$I_1(s)\, R_2 + \frac{1}{C_2}\frac{I_1(s)}{s} = -U_1(s)$$

$$\therefore I_1(s)\left(R_2 + \frac{1}{C_2 s}\right) = -U_1(s) \qquad .....(2.140)$$

Substitute for $I_1(s)$ from equation (2.138) in equation (2.140).

$$\therefore \left(\frac{1}{R_1} + C_1 s\right) E(s) \left(R_2 + \frac{1}{C_2 s}\right) = -U_1(s)$$

$$-\left(\frac{R_2}{R_1} + \frac{C_1}{C_2} + \frac{1}{R_1 C_2 s} + R_2 C_1 s\right) E(s) = U_1(s) \qquad .....(2.141)$$

From fig 2.39, $u(t) = -i_2R$; $\therefore i_2 = -\frac{u(t)}{R}$ .....(2.142)

$$u_1(t) = i_2 R \qquad .....(2.143)$$

Substitute for i_2 from equation (2.142) in equation (2.143).

$$\therefore u_1(t) = -\frac{u(t)}{R} R = -u(t) \qquad .....(2.144)$$

On taking Laplace transform of equation (2.144) we get,

$$U_1(s) = -U(s) \qquad .....(2.145)$$

From equations (2.142) and (2.146) we get,

$$U(s)=\left(\frac{R_2}{R_1}+\frac{C_1}{C_2}+\frac{1}{R_1C_2s}+R_2C_1s\right)E(s)$$

$$\therefore\ \frac{U(s)}{E(s)}=\left(\frac{R_2C_2+R_1C_1}{R_1C_2}+\frac{1}{R_1C_2s}+R_2C_1s\right)$$

$$=\frac{R_2}{R_1}\left(\frac{R_2C_2+R_1C_1}{R_2C_2}+\frac{1}{R_2C_2s}+R_1C_1s\right) \qquad .....(2.146)$$

The equation (2.146) is the transfer function of op-amp PID-controller. On comparing equation (2.146) with equation (2.136) we get,

Proportional gain, $K_p=\dfrac{R_2}{R_1}$

Derivative time, $T_d=R_1C_1$; Integral time, $T_i=R_2C_2$

Also, $\dfrac{R_1C_1+R_2C_2}{R_2C_2}=1$

By varying the values of R_1 and R_2 the values of K_p, T_d and T_i are adjusted.

2.21 RESPONSE WITH P, PI, PD AND PID CONTROLLERS

In feedback control systems a controller may be introduced to modify the error signal and to achieve better control action. The introduction of controllers will modify the transient response and the steady state error of the system. The effects due to introduction of P, PI, PD and PID controllers are discussed in this section.

EFFECT OF PROPORTIONAL CONTROLLER (P-CONTROLLER)

The proportional controller produces an output signal which is proportional to error signal. The transfer function of proportional controller is given below. (Refer equation 2.83).

Transfer function of P-controller, $\dfrac{U(s)}{E(s)} = K_p$

The term K_p in the transfer function of proportional controller is called the gain of the controller. Hence the proportional controller amplifies the error signal and increases the loop gain of the system. The following aspects of system behaviour are improved by increasing loop gain.

* Steady state tracking accuracy.
* Disturbance signal rejection.
* Relative stability.

In addition to increase in loop gain it decreases the sensitivity of the system to parameter variations. The drawback in proportional control action is that it produces a constant steady state error.

EFFECT OF PI-CONTROLLER

The proportional plus integral controller (PI-controller) produces an output signal consisting of two terms : *one proportional to error signal and the other proportional to the integral of error signal.*

Transfer function of PI-controller, $G_c(s) = K_p\left(1+\dfrac{1}{T_is}\right)=K_p\left(\dfrac{T_is+1}{T_is}\right)$ (Refer equation 2.113)

where, K_p is proportional gain and, T_i is integral time.

The block diagram of unity feedback system with PI-controller is shown in fig 2.40.

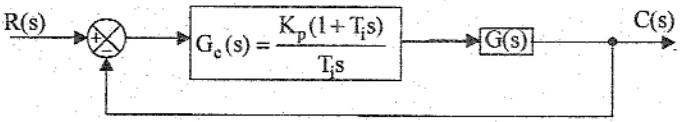


Fig 2.40 : Block diagram of feedback system with PI-controller.

Let the open loop transfer function G(s) be a second order system with transfer function, as shown in equation (2.148).

Open loop transfer function, $G(s) = \dfrac{\omega_n^2}{s\,(s+2\zeta\omega_n)}$ ......(2.147)

Now, loop transfer function $= G_c(s)\,G(s)\,H(s) = G_c(s)\,G(s)$ $\boxed{H(s)=1}$

$$= K_p\left(\frac{1+T_i s}{T_i s}\right) \times \frac{\omega_n^2}{s\,(s+2\zeta\omega_n)} = \frac{K_p\omega_n^2\,(1+T_i s)}{s^2 T_i (s+2\zeta\,\omega_n)} \quad .....(2.148)$$

Now the closed loop transfer function is given by,

$$\frac{C(s)}{R(s)} = \frac{G_c(s)G(s)}{1+G_c(s)G(s)} = \frac{\dfrac{K_p\omega_n^2(1+T_i s)}{s^2 T_i\,(s+2\zeta\omega_n)}}{1+\dfrac{K_p\omega_n^2(1+T_i s)}{s^2 T_i\,(s+2\zeta\omega_n)}} = \frac{K_p\omega_n^2\,(1+T_i s)}{s^2 T_i(s+2\zeta\,\omega_n)+K_p\omega_n^2\,(1+T_i s)}$$

$$= \frac{K_p\omega_n^2.(1+T_i s)}{T_i s^3 + 2\zeta\omega_n T_i s^2 + K_p\omega_n^2\,T_i\,s + K_p\omega_n^2}$$

$$= \frac{(K_p/T_i)\,\omega_n^2(1+T_i s)}{s^3 + 2\zeta\omega_n s^2 + K_p\omega_n^2\,s + \dfrac{K_p}{T_i}\omega_n^2} \qquad \boxed{K_i = \frac{K_p}{T_i}}$$

$$= \frac{K_i\omega_n^2\,(1+T_i s)}{s^3 + 2\zeta\omega_n s^2 + K_p\omega_n^2 s + K_i\omega_n^2} \quad .....(2.149)$$

From the closed loop transfer function (equation (3.149)) it is observed that the PI-controller introduces a zero in the system and increases the order by one. The increase in the order of the system results in a less stable system than the original one because higher order systems are less stable than lower order systems.

From the loop transfer function (equation (3.148)) it is observed that the PI-controller increase the type number by one. The increase in type number results in reducing the steady state error. For example if the steady state error of the original system is constant, then the integral controller will reduce the error to zero.

EFFECT OF PD-CONTROLLER

The proportional plus derivative controller produces an output signal consisting of two terms : *one proportional to error signal and the other proportional to the derivative of error signal.*

The transfer function of PD - controller, $G_c(s) = K_p\,(1+T_d\,s)$ (Refer equation 2.124)

where K_p is Proportional gain, T_d is Derivative time.

The block diagram of unity feedback system with PD-controller is shown in fig 2.41.

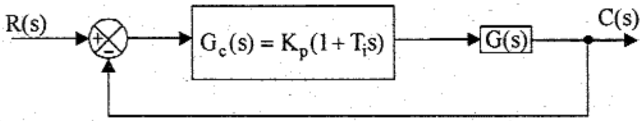


Fig 2.41 : Block diagram of feedback system with PD-controller.

Let the open loop transfer function G(s) be a second order system with transfer function as shown in equation (2.150).

Open loop transfer function, $G(s) = \dfrac{\omega_n^2}{s\,(s+2\zeta\omega_n)}$ .....(3.150)

Now, loop transfer function $= G_c(s)\,G(s)\,H(s) = G_c(s)\,G(s)$ $\boxed{H(s)=1}$

$$= K_p\big(1+T_d(s)\big) \times \frac{\omega_n^2}{s\,(s+2\zeta\omega_n)} = \frac{K_p\omega_n^2\,(1+T_d s)}{s(s+2\zeta\,\omega_n)} \qquad .....(2.151)$$

Now the closed loop transfer function is given by,

$$\frac{C(s)}{R(s)} = \frac{G_c(s)G(s)}{1+G(s)G_c(s)} = \frac{\dfrac{K_p\omega_n^2(1+T_d s)}{s\,(s+2\zeta\omega_n)}}{1+\dfrac{K_p\omega_n^2\,(1+T_d s)}{s\,(s+2\zeta\omega_n)}}$$

$$= \frac{K_p\omega_n^2\,(1+T_d s)}{s\,(s+2\zeta\omega_n)+K_p\omega_n^2\,(1+T_d s)}$$

$$= \frac{K_p\omega_n^2\,(1+T_d s)}{s^2+2\zeta\omega_n s+K_p\omega_n^2+K_p\omega_n^2 T_d s}$$

$$= \frac{K_p\omega_n^2\,(1+T_d s)}{s^2+(2\zeta\omega_n+K_p\omega_n^2 T_d)\,s+K_p\omega_n^2} \qquad \boxed{K_d = K_p T_d}$$

$$= \frac{\omega_n^2\,(K_p+K_d s)}{s^2+(2\zeta\omega_n+K_d\omega_n^2)\,s+K_p\omega_n^2} \qquad .....(2.152)$$

From the closed loop transfer function (equation (2.152)) it is observed that the PD-controller introduces a zero in the system and increases the damping ratio. The addition of the zero may increase the peak overshoot and reduce the rise time. But the effect of increased damping ultimately reduces the peak overshoot.

From the loop transfer function (equation (2.151)) it is observed that the PD-controller does not modify the type number of the system. Hence PD-controller will not act modify steady state error.

EFFECT OF PID-CONTROLLER

A suitable combination of the three basic modes : *proportional, integral and derivative* (PID) can improve all aspects of the system performance.

The proportional controller stabilizes the gain but produces a steady state error. The integral controller reduces or eliminates the steady state error. The derivative controller reduces the rate of change of error. The combined effect of all the three cannot be judged from the parameters K_p, K_i and K_d.

2.22 TIME RESPONSE ANALYSIS USING MATLAB

In general, the closed loop transfer function of a system is denoted as M(s).

Let, M(s) be a rational function of "s", as shown below.

$$M(s) = \frac{b_0 s^M + b_1 s^{M-1} + b_2 s^{M-2} + ... + b_{M-1} s + b_M}{a_0 s^N + a_1 s^{N-1} + a_2 s^{N-2} + ... + a_{N-1} s + a_N}$$

For time response analysis, the coefficients of the numerator and denominator polynomials are declared as two arrays as shown below.

```
num_cof = [b0 b1 b2 ........ bM];
den_cof = [a0 a1 a2 ........ aN];
```

UNIT STEP RESPONSE

To compute step response

The unit step response can be computed and displayed using following commands.

```
syms s complex;
R = 1/s;
M = (b0*s^M+b1*s^(M-1)+...+bM)/(a0*s^N+a1*s^(N-1)+...+aN);
S = R*M;
disp('Unit step response of the system is,');
step_res = ilaplace(S)
```

To plot step response

Method 1 :

The unit step response can be plotted using the following command.

```
step(num_cof, den_cof);
```

Method 2 :

The unit step response of the system can be plotted using the following commands.

```
t = t_start : t_step : t_end ;
c = step(num_cof, den_cof,t);
plot(t,c,'k');
     where, c is an array where the values of response are stored.
```

The unit step response can be computed "n" times by varying some parameter of the system (coefficient / damping ratio / natural frequency of oscillation) using the following commands.

```
t = t_start : t_step : t_end ;
for i = 1 : n
     .
     .
c(1:k, i) = step(num_cof, den_cof,t);
     .
     .
end
plot(t,c,'k');
     where, c is an array where the values of response are stored.
            k is the number of samples of response to be computed.
```

Method 3 :

The unit step response of the system can be plotted using the following commands.

```
s = tf('s');
M = (b0*s^M+b1*s^(M-1)+...+bM)/(a0*s^N+a1*s^(N-1)+...+aN);
t = t_start : t_step : t_end ;
sr = step(M,t);
plot(t,sr,'k');
```

IMPULSE RESPONSE

To compute impulse response

The impulse response can be computed and displayed using following commands.

```
syms s complex;
M = (b0*s^M+b1*s^(M-1)+...+bM)/(a0*s^N+a1*s^(N-1)+...+aN);
disp('Unit step response of the system is,');
imp_res = ilaplace(S)
```

To plot impulse response

Method 1 :

The impulse response can be plotted using the following command.

```
impulse(num_cof, den_cof);
```

Method 2 :

The impulse response of the system can be plotted using the following commands.

```
t = t_start : t_step : t_end ;
m = impulse(num_cof, den_cof,t);
plot(t,m,'k');
     where, m is an array where the values of impulse response are stored.
```

Method 3 :

The impulse response of the system can be plotted using the following commands.

```
s = tf('s');
M = (b0*s^M+b1*s^(M-1)+...+bM)/(a0*s^N+a1*s^(N-1)+...+aN);
t = t_start : t_step : t_end ;
imp = impulse(M,t);
plot(t,imp,'k');
```

RESPONSE FOR ARBITRARY INPUT

The response of a system for an arbitrary input, r(t) can be plotted using the following commands.

```
t = t_start : t_step : t_end ;
c = Lsim(num_cof, den_cof, r, t);
plot(t,c,'k');
     where, c is an array where the values of response are stored.
```

PROGRAM 2.1

Consider the standard closed loop transfer function of the second order system given below.

$$M(s)=\omega_n^2/(s^2+2\zeta\omega_n s+\omega_n^2)$$

Write a MATLAB program to find the unit step response for various values of damping ratio, ζ. Take, natural frequency of oscillation, ω_n=1 rad/sec.

```
%Unit step response for various values of damping ratio, zeta.
%The natural frequency of oscillation, wn=1.

clc
t=0:0.2:12;                          %specify a time vector
c=zeros(61,6);                       %initialize response array as zero
zeta=[0 0.2 0.4 0.6 0.8 1];        %store zeta as an array
    for n=1:6;                        %for loop to compute c(t) 6 times
        num_cof=[0 0 1];
        den_cof=[1 2*zeta(n) 1];
        c(1:61,n)=step(num_cof,den_cof,t);
    end

plot(t,c,'k'); grid
xlabel('time,t in sec'); ylabel('Unit step response,c(t)');

text(2.8,1.86,'\zeta=0')
text(2.8,1.58,'\zeta=0.2')
text(2.8,1.30,'\zeta=0.4')
text(2.8,1.12,'\zeta=0.6')
text(2.8,0.95,'\zeta=0.8')
text(2.8,0.72,'\zeta=1.0')
```

OUTPUT

The output waveforms are shown in fig p2.1.

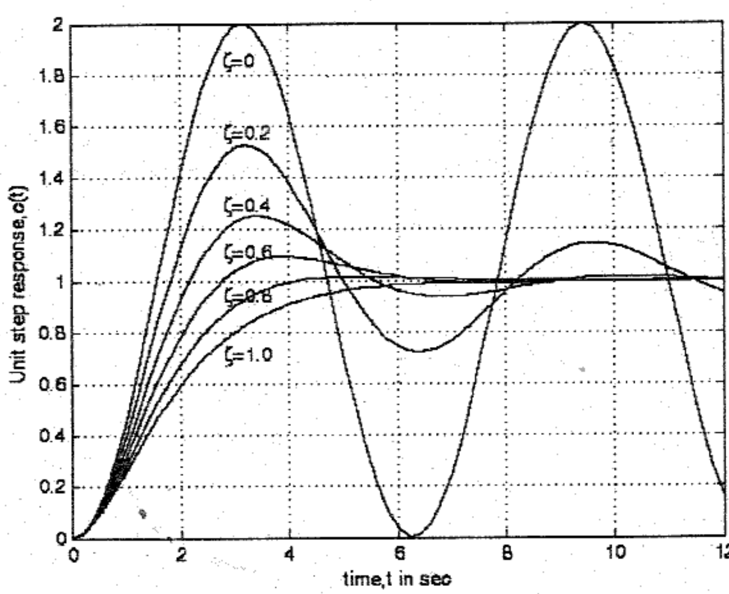


Fig P2.1 : Unit step response of second order system for various values of damping ratio.

PROGRAM 2.2

Consider the standard closed loop transfer function of the second order system given below.

$$M(s)=\omega_n^2/(s^2+2\zeta\omega_n s+\omega_n^2)$$

Write a MATLAB program to find the unit step response for various values of natural frequency of oscillation, ω_n. Take, damping ratio, ζ=0.4.

```
%Unit step response for various natural frequency of oscillation,wn.
%The damping ratio, zeta=0.4.
clc
t=0:0.1:8;                  %specify a time vector
wn=[1 2 4 6];               %store wn as an array
zeta=0.4;
c=zeros(81,4);              %initialize the response array as zeros

   for i=1:4;               %for loop to compute c(t) 4 times
   b2=wn(i)*wn(i);
   a1=2*zeta*wn(i);
   num_cof=[0 0 b2];
   den_cof=[1 a1 b2];
   c(1:81,i)=step(num_cof,den_cof,t);
   end

plot(t,c(:,1),'--k',t,c(:,2),'xk',t,c(:,3),'-k',t,c(:,4),'-.k');
grid; xlabel('time,t in sec'); ylabel('Unit step response,c(t)');
text(4.25,1.25,'wn=1')
text(1.5,1.30,'wn=2')
text(0.7,1.30,'wn=3')
text(0.1,1.25,'wn=4')
```

OUTPUT

The output waveforms are shown in fig p2.2.

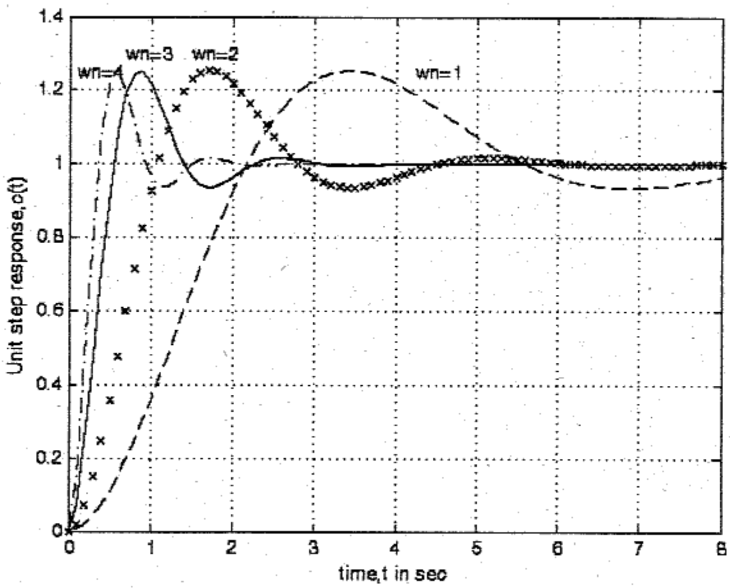


Fig P2.2 : Unit step response of second order system for various values of natural frequency of oscillation.

PROGRAM 2.3

Write a MATLAB program to find impulse response of the following systems.

a) $M_1(s)=(2s+1)/(s+1)^2$ b) $M_2(s)=s/(s+1)$ c) $M_3(s)=1/(s^2+1)$

```
%Program to find impulse response
clc
syms s complex;
M1=(2*s+1)/((s+1)^2);
disp('Impulse response of the system1 is,');
m1=ilaplace(M1)

M2=s/(s+1);
disp('Impulse response of the system2 is,');
m2=ilaplace(M2)

M3=1/(s^2+1);
disp('Impulse response of the system3 is,');
m3=ilaplace(M3)

s=tf('s');
M1=(2*s+1)/((s+1)^2);
M2=s/(s+1);
M3=1/(s^2+1);

t=0:.005:10;
m1=impulse(M1,t);
m2=impulse(M2,t);
m3=impulse(M3,t);

plot(t,m1,'--k',t,m2,'-.k',t,m3,'-k');grid
xlabel('time,t in sec');
ylabel('Impulse responses,m1(t),m2(t),m3(t)');
text(0.4,1.30,'m1(t)')
text(0.3,-0.30,'m2(t)')
text(2.2,0.90,'m3(t)')
```

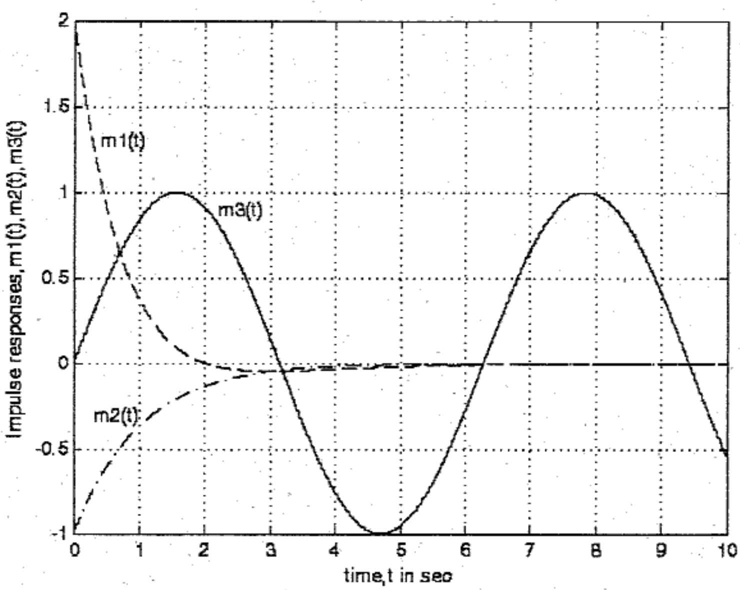


Fig P2.3 : Impulse response of systems given in program 2.3.

OUTPUT

```
Impulse response of the system1 is,
        m1 =
              (2-t)*exp(-t)

Impulse response of the system2 is,
        m2 =
               dirac(t)-exp(-t)

Impulse response of the system3 is,
        m3 =
              sin(t)

The output waveforms are shown in fig p2.3.
```

PROGRAM 2.4

Write a MATLAB program to find unit step response of the following systems.

a) $M_1(s)=4/(s^2+5s+4)$ b) $M_2(s)=100/(s^2+12s+100)$ c) $M_3(s)=600/(s^2+70s+600)$

```
%program to find unit step response
clc
syms s complex;
R=1/s;                          %Laplace of unit step input
M1=4/(s^2+5*s+4);
S1=R*M1;                        %s-domain unit step response of system1
disp('Unit step response of the system1 is,');
s1=ilaplace(S1)               %time domain unit step response of system1

M2=100/(s^2+12*s+100);
S2=R*M2;                        %s-domain unit step response of system2
disp('Unit step response of the system2 is,');
s2=ilaplace(S2)               %time domain unit step response of system2

M3=600/(s^2+70*s+600);
S3=R*M3;                        %s-domain unit step response of system3
disp('Unit step response of the system3 is,');
s3=ilaplace(S3)               %time domain unit step response of system3

s=tf('s');
M1=4/(s^2+5*s+4);
M2=100/(s^2+12*s+100);
M3=600/(s^2+70*s+600);

t=0:.005:10;
s1=step(M1,t);
s2=step(M2,t);
s3=step(M3,t);

plot(t,s1,'--k',t,s2,'-.k',t,s3,'-k');grid
xlabel('time,t in sec');
ylabel('Unit step responses,s1(t),s2(t),s3(t)');
text(2.2,0.85,'s1(t)')
text(0.2,1.15,'s2(t)')
text(0.5,0.95,'s3(t)')
```

OUTPUT

```
Unit step response of the system1 is,
        s1 =
                1/3*exp(-4*t)+1-4/3*exp(-t)

Unit step response of the system2 is,
        s2 =
                1-exp(-6*t)*cos(8*t)-3/4*exp(-6*t)*sin(8*t)

Unit step response of the system3 is,
        s3 =
                1+1/5*exp(-60*t)-6/5*exp(-10*t)

The output waveform is shown in fig p2.4.
```

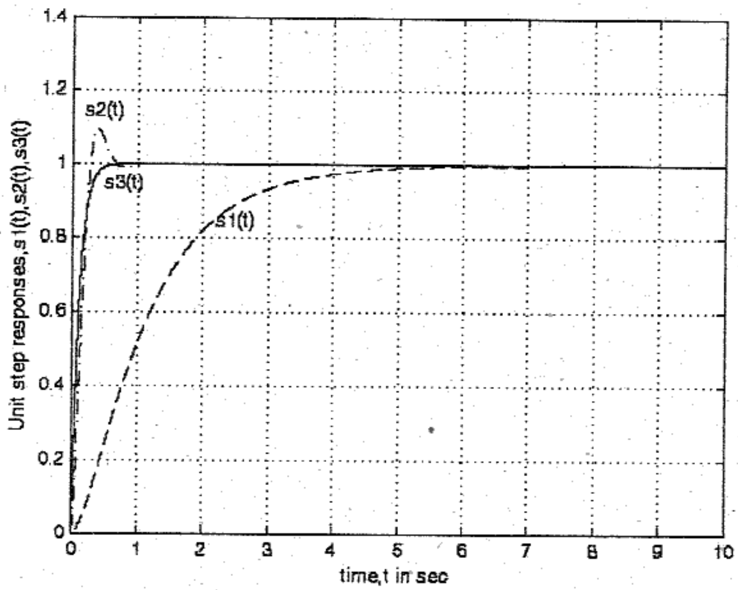


Fig P2.4 : Unit step response of systems given in program 2.4.

PROGRAM 2.5

Consider the closed loop transfer function of the following second order system,

$M(s)=16/(s^2+4s+16)$

Write a MATLAB program to find the rise time, peak time, maximum peak overshoot, and settling time from the unit step response of the system.

```
clc
t=0:0.005:5;                %set time vector

num_cof=[0 0 16];           %store the numerator coefficients as an array
den_cof=[1 4 16];           %store denominator coefficients as an array
[c,x,t]=step(num_cof,den_cof,t);

n=1;                         %initialize count as 1
while c(n)<1.0001;          %count the time index as along as c(t)<1
     n=n+1;
end;
```

```
rise_time=(n-1)*0.005   %rise time=(count-1)*time interval

[cmax,tp]=max(c);         %determine maximum value of c(t) &
                           %corresponding time
peak_time=(tp-1)*0.005 %peak time=(tp-1)*time interval

max_overshoot=cmax-1    %compute peak overshoot

n=1001;                     %initialize count as (5/.005)+1=1001
while c(n)>0.95&c(n)<1.05;
     n=n-1;                 %count time index between c(t)>0.95&c(t)<1.05
end;
settling_time_5per_err=(n-1)*0.005

n=1001;                     %initialize count as (5/.005)+1=1001
while c(n)>0.98 & c(n)<1.02;
     n=n-1;                 %count time index between c(t)>0.98&c(t)<1.02
end;
settling_time_2per_err=(n-1)*0.005
```

OUTPUT

```
rise_time =
             0.6050

peak_time =
             0.9050

max_overshoot =
             0.1630

settling_time_5per_err =
                       1.3200

settling_time_2per_err =
                       2.0150
```

PROGRAM 2.6

Consider the closed loop transfer function of the following second order system,

$M(s)=64/(s^2+8s+64)$

Write a MATLAB program to find the response for unit step, unit ramp and unit parabolic input signals.

```
%unit step/ramp/parabolic response

clc
num_cof=[0 0 64];
den_cof=[1 8 64];

t=0:0.005:2;

r1=t;                 %unit ramp input signal
r2=0.5*t.^2;       %unit parabolic input signal

c1=step(num_cof, den_cof,t);
c2=Lsim(num_cof, den_cof,r1,t);
c3=Lsim(num_cof, den_cof,r2,t);
```

```
plot(t,c1,'--k',t,c2,'-.k',t,c3,'-k'); grid
xlabel('time,t in sec');
ylabel('Responses,c1(t),c2(t),c3(t)');
text(0.25,1.15,'c1(t)')
text(1.45,1.5,'c2(t)')
text(1.35,0.7,'c3(t)')
```

OUTPUT

The output waveform is shown in fig p2.6.

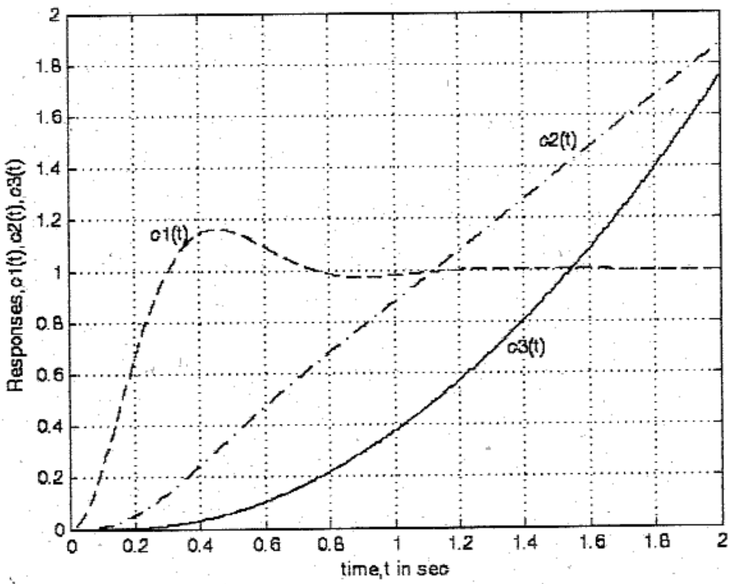


Fig P2.6 : *Step, ramp and parabolic response of system given in program 2.6.*

PROGRAM 2.7

Consider the closed loop transfer function of the following second order system,

$M(s)=5/(s^2+s+5)$

Write a MATLAB program to find the response for the input signal, $r(t)=2-2t+t^2$.

```
%program to find response for given input
clc
num_cof=[0 0 5];
den_cof=[1 1 5];
t=0:0.005:3;                          %specify a time vector

r=2-2*t+t.^2;                        %input signal

c=Lsim(num_cof,den_cof,r,t);    %compute response using Lsim function

plot(t,r,'--k',t,c,'-.k'); grid
xlabel('time,t in sec');
ylabel('Input,r(t) and output,c(t)');

text(0.25,1.65,'r(t)')
text(0.25,0.6,'c(t)')
```

OUTPUT

The output waveform is shown in fig p2.7.

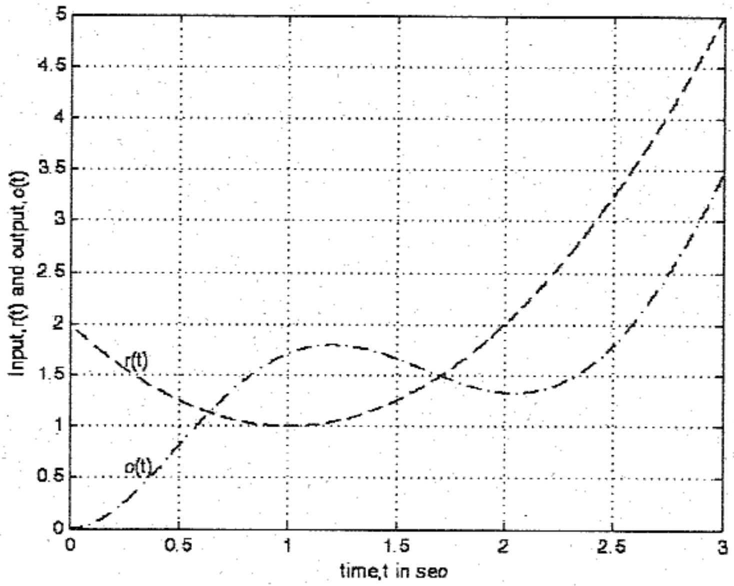


Fig P2.7 : Input and Output of the system given in program 2.7.

2.23 SHORT QUESTIONS AND ANSWERS

Q2.1 ***What is time response?***

The time response is the output of the closed loop system as a function of time. It is denoted by c(t). It is given by inverse Laplace of the product of input and transfer function of the system.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$

Response in s-domain, $C(s) = \frac{R(s)\,G(s)}{1+G(s)\,H(s)}$

Response in time domain, $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{R(s)\,G(s)}{1+G(s)\,H(s)}\right\}$

Q2.2 ***What is transient and steady state response?***

The transient response is the response of the system when the input changes from one state to another. The response of the system as $t \to \infty$ is called steady state response.

Q2.3 ***What is the importance of test signals?***

The test signals can be easily generated in test laboratories and the characteristics of test signals resembles, the characteristics of actual input signals. The test signals are used to predetermine the performance of the system. If the response of a system is satisfactory for a test signal, then the system will be suitable for practical applications.

Q2.4 ***Name the test signals used in control system.***

The commonly used test input signals in control system are Impulse, Step, Ramp, Acceleration and Sinusoidal signals.

Q2.5 ***Define step signal.***

The step signal is a signal whose value changes from 0 to A and remains constant at A for $t > 0$. The mathematical representation of step signal is,

$$r(t) = A, \; t \geq 0$$
$$= 0, \; t < 0$$

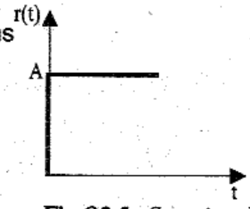


Fig Q2.5 : Step signal.

Q2.6 ***Define ramp signal.***

A ramp signal is a signal whose value increases linearly with time from an initial value of zero at $t = 0$. Mathematical representation of ramp signal is,

$$r(t) = At, \; t \geq 0$$
$$= 0, \; t < 0$$

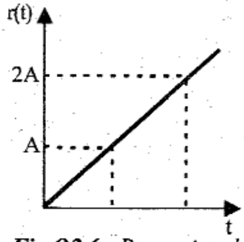


Fig Q2.6 : Ramp signal.

Q2.7 ***Define parabolic signal.***

It is a signal in which the instantaneous value varies as square of the time from an initial value of zero at $t = 0$. The mathematical representation of parabolic signal is,

$$r(t) = \frac{At^2}{2}, \; t \geq 0$$
$$= 0, \; t < 0$$

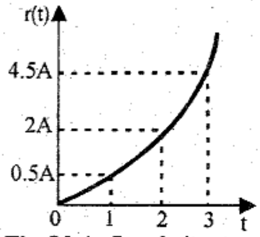


Fig Q2.4 : Parabolic signal.

Q2.8 ***What is weighing function?***

The impulse response of system is called weighing function. It is given by inverse Laplace transform of system transfer function.

Q2.9 ***What is an impulse signal?***

A signal which is available for very short duration is called impulse signal. Ideal impulse signal is a unit impulse signal which is defined as a signal having zero values at all time except at $t = 0$. At $t = 0$ the magnitude becomes infinite. It is denoted by $\delta(t)$ and mathematically expressed as,

$$\delta(t) = \infty \; ; \; t = 0 \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$
$$= 0 \; ; \; t = 0$$

Q2.10 ***Define pole.***

The pole of a function, F(s) is the value at which the function, F(s) becomes infinite, where F(s) is a function of complex variable s.

Q2.11 ***Define zero.***

The zero of a function, F(s) is the value at which the function, F(s) becomes zero, where F(s) is a function of complex variable s.

Q2.12 ***What is the order of a system?***

The order of the system is given by the order of the differential equation governing the system. It is also given by the maximum power of s in the denominator polynomial of transfer function. The maximum power of s also gives the number of poles of the system and so the order of the system is also given by number of poles of the transfer function.

Q2.13 ***Define damping ratio.***

The damping ratio is defined as the ratio of actual damping to critical damping.

Q2.14 ***Give the expression for damping ratio of mechanical and electrical system.***

The damping ratio of second order mechanical translational system, $\zeta = \dfrac{B}{2\sqrt{MK}}$

The damping ratio of second order mechanical rotational system, $\zeta = \dfrac{B}{2\sqrt{JK}}$

The damping ratio of second order electrical system, $\zeta = \frac{R}{2\sqrt{L/C}}$

Q2.15 ***How the system is classified depending on the value of damping?***

Depending on the value of damping, the system can be classified into the following four cases.

Case 1 : Undamped system, $\zeta = 0$

Case 2 : Underdamped system, $0 < \zeta < 1$

Case 3 : Critically damped system, $\zeta = 1$

Case 4 : Over damped system, $\zeta > 1$

Q2.16. ***Sketch the response of a second order under damped system.***

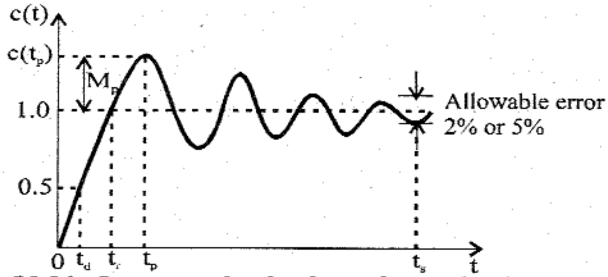


Fig Q2.16 : *Response of under damped second order system.*

Q2.17 ***What will be the nature of response of a second order system with different types of damping?***

For undamped system the response is oscillatory.
For underdamped system the response is damped oscillatory.
For critically damped system the response is exponentially rising.
For overdamped system the response is exponentially rising but the rise time will be very large.

Q2.18. ***What is damped frequency of oscillation?***

In underdamped system the response is damped oscillatory. The frequency of damped oscillation is given by, $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

Q2.19. ***Give the expression for natural frequency of oscillations of electrical and mechanical system.***

The natural frequency of oscillation of second order mechanical translational system $\Big\}$ $\omega_n = \sqrt{\frac{K}{M}}$

The natural frequency of oscillation of second order mechanical rotational system $\Big\}$ $\omega_n = \sqrt{\frac{K}{J}}$

The natural frequency of oscillation of second order electrical system $\Big\}$ $\omega_n = \frac{1}{\sqrt{LC}}$

Q2.20. ***The closed loop transfer function of second order system is $\frac{C(s)}{R(s)} = \frac{10}{s^2+6s+10}$. What is the type of damping in the system?.***

Let us compare the given transfer function with the standard form of second order transfer function

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{10}{s^2+6s+10}$$

$\omega_n^2 = 10$

$\therefore \omega_n = \sqrt{10} = 3.1622 \text{ rad/sec}$

$2\zeta\omega_n = 6$

$\therefore \zeta = \frac{6}{2\times\omega_n} = \frac{6}{2\times\sqrt{10}} = 0.95$

Since $\zeta < 1$, the system is underdamped.

Q2.21 ***The closed loop transfer function of a second order system is given by $\frac{200}{s^2+20s+200}$. Determine the damping ratio and natural frequency of oscillation.***

Let us compare the given transfer function with the standard form of second order transfer function

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{200}{s^2+20s+200}$$

$$\therefore \omega_n^2 = 200 \qquad\qquad 2\zeta\omega_n = 20$$

$$\omega_n = \sqrt{200} = 14.14 \text{ rad/sec} \qquad\qquad \zeta = \frac{20}{2\times\omega_n} = \frac{20}{2\times 14.14} = 0.707$$

Damping ratio, $\zeta = 0.707$

Natural frequency of oscillation, $\omega_n = 14.14$ rad/sec.

Q2.22 ***A second order system has a damping ratio of 0.6 and natural frequency of oscillation is 10 rad/sec. Determine the damped frequency of oscillation.***

Damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2} = 10\sqrt{1-(0.6)^2} = 10\times 0.8 = 8 \text{ rad/sec}$

Q2.23 ***The open loop transfer function of a unity feedback system is $G(s) = \frac{20}{s(s+10)}$. What is the nature of response of closed loop system for unit step input.***

The closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{20/s(s+10)}{1+\frac{20}{s(s+10)}} = \frac{20}{s(s+10)+20} = \frac{20}{s^2+10s+20}$$

The standard form of second order transfer function is, $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

On comparing system transfer function with standard form of second order transfer function we get,

$$\omega_n^2 = 20 \qquad\qquad 2\zeta\omega_n = 10$$

$$\therefore \omega_n = \sqrt{20} = 4.47 \text{ rad/sec} \qquad\qquad \zeta = \frac{10}{2\times\omega_n} = \frac{10}{2\times 4.47} = 1.12$$

Since damping ratio, $\zeta > 1$, the system is overdamped and the response will be exponentially rising.

Q2.24 ***List the time domain specifications.***

The time domain specifications are,

(i) Delay time (ii) Rise time (iii) Peak time
(iv) Maximum overshoot (v) Settling time.

Q2.25 ***Define delay time.***

It is the time taken for response to reach 50% of the final value, the very first time.

Q2.26 ***Define rise time.***

It is the time taken for response to raise from 0 to 100% , the very first time. For underdamped system, the rise time is calculated from 0 to 100%. But for overdamped system it is the time taken by the response to raise from 10% to 90%. For critically damped system, it is the time taken for response to raise from 5% to 95%.

Q2.27 ***Define peak time.***

It is the time taken for the response to reach the peak value, the very first time (or) It is the time taken for the response to reach peak overshoot, M_p.

Q2.28 ***Define peak overshoot.***

It is defined as the ratio of the maximum peak value to final value, where maximum peak value is measured from final value.

Let final value = $c(\infty)$, Maximum value = $c(t_p)$ $\therefore$ Peak overshoot, $M_p = \frac{c(t_p)-c(\infty)}{c(\infty)}$.

Q2.29 ***Define settling time.***

It is defined as the time taken by the response to reach and stay within a specified error and the error is usually specified as % of final value. The usual tolerable error is 2% or 5% of the final value.

Q2.30 ***The damping ratio of a system is 0.75 and the natural frequency of oscillation is 12 rad/sec. Determine the peak overshoot and the peak time.***

Peak overshoot, $M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{-\frac{0.75\times\pi}{\sqrt{1-(0.75)^2}}} = 0.028$; $\therefore \%M_p = 0.028 \times 100 = 2.8\%$

Damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2} = 12\sqrt{1-(0.75)^2} = 7.94 \text{ rad/sec}$

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{7.94} = 0.396 \text{ sec}$

Q2.31 ***The damping ratio of system is 0.6 and the natural frequency of oscillation is 8 rad/sec. Determine the rise time.***

Rise time, $t_r = \frac{\pi-\theta}{\omega_d}$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-(0.6)^2}}{0.6} = 53.13^\circ = \frac{53.13}{180}\times\pi \text{ rad} = 0.927 \text{ rad}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 8\sqrt{1-(0.6)^2} = 6.4 \text{ rad/sec}$$

$\therefore$ Rise time, $t_r = \frac{\pi-0.927}{6.4} = 0.34 \text{ sec}$

Q2.32 ***What is type number of a system? What is its significance?***

The type number is given by number of poles of loop transfer function at the origin. The type number of the system decides the steady state error.

Q2.33 ***Distinguish between type and order of a system.***

(i) Type number is specified for loop transfer function but order can be specified for any transfer function. (open loop or closed loop transfer function).

(ii) The type number is given by number of poles of loop transfer function lying at origin of s-plane but the order is given by the number of poles of transfer function.

Q2.34 ***For the system with following transfer function, determine type and order of the system.***

(i) $G(s)\,H(s) = \frac{K}{s(s+1)\,(s^2+6s+8)}$ (ii) $G(s)\,H(s) = \frac{20\,(s+2)}{s^2(s+3)\,(s+0.5)}$

(iii) $G(s)\,H(s) = \frac{(s+4)}{(s-2)\,(s+0.25)}$ (iv) $G(s)\,H(s) = \frac{10}{s^3(s^2+2s+1)}$

Ans: (i) Type - 1, order - 4 (ii) Type - 2, order - 4

(iii) Type - 0, order - 2 (iv) Type - 3, order - 5.

Q2.35 ***What is steady state error?***

The steady state error is the value of error signal e(t), when t tends to infinity. The steady state error is a measure of system accuracy. These errors arise from the nature of inputs, type of system and from non-linearity of system components.

Q2.36 ***What are static error constants?***

The K_p, K_v and K_a are called static error constants. These constants are associated with steady state error in a particular type of system and for a standard input.

Q2.37 ***Define positional error constant.***

The positional error constant $K_p = \underset{s\to 0}{\text{Lt}}\ G(s)\,H(s)$. The steady state error in type-0 system when the input is unit step is given by $1/1+K_p$.

Q2.38 ***Define velocity error constant.***

The velocity error constant $K_v = \underset{s \to 0}{Lt}\ sG(s)\,H(s)$. The steady state error in type-1 system for unit ramp input is given by $1/K_v$.

Q2.39 ***Define acceleration error constant.***

The acceleration error constant $K_a = \underset{s \to 0}{Lt}\ s^2\,G(s)\,H(s)$. The steady state error in type-2 system for unit parabolic input is given by $1/K_a$.

Q2.40 ***A unity feedback system has a open loop transfer function of $G(s) = \dfrac{10}{(s+1)\,(s+2)}$. Determine the steady state error for unit step input.***

The steady state error for unit step input, $e_{ss} = \dfrac{1}{1+K_p}$, where, $K_p = \underset{s \to 0}{Lt}\ G(s)\,H(s)$.

For unity feedback system H(s) = 1.

$$\therefore K_p = \underset{s \to 0}{Lt}\ G(s) = \underset{s \to 0}{Lt}\ \frac{10}{(s+1)\,(s+2)} = 5 \quad \text{and} \quad e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+5} = \frac{1}{6}$$

Q2.41 ***A unity feedback system has a open loop transfer function of $G(s) = \dfrac{25\,(s+4)}{s(s+0.5)\,(s+2)}$. Determine the steady state error for unit ramp input.***

The steady state error for unit ramp input is, $e_{ss} = \dfrac{1}{K_v}$, where, $K_v = \underset{s \to 0}{Lt}\ s\,G(s)\,H(s)$. For unity feedback system H(s) = 1.

$$\therefore K_v = \underset{s \to 0}{Lt}\ sG(s) = \underset{s \to 0}{Lt}\ s\left[\frac{25\,(s+4)}{s(s+0.5)\,(s+2)}\right] = \frac{25 \times 4}{0.5 \times 2} = 100 \quad \text{and} \quad e_{ss} = \frac{1}{K_v} = \frac{1}{100} = 0.01$$

Q2.42 ***A unity feedback system has a open loop transfer function of $G(s) = \dfrac{20\,(s+5)}{s(s+0.1)\,(s+3)}$. Determine the steady state error for parabolic input.***

The steady state error for unit ramp input is $e_{ss} = \dfrac{1}{K_a}$, where, $K_a = \underset{s \to 0}{Lt}\ s^2 G(s)\,H(s)$. For unity feedback system H(s) = 1.

$$\therefore K_a = \underset{s \to 0}{Lt}\ s^2\left[\frac{20\,(s+5)}{s^2(s+0.1)\,(s+3)}\right] = \frac{20 \times 5}{0.1 \times 3} = \frac{100}{0.3} = 333.33 \quad \text{and} \quad e_{ss} = \frac{1}{K_a} = \frac{1}{333.33} = 0.003$$

Q2.43 ***What are generalized error coefficients?***

They are the coefficients of generalized error series.The generalized error series is given by,

$$e(t) = C_0 r(t) + C_1 \dot{r}(t) + \frac{C_2}{2!}\ddot{r}(t) + \frac{C_3}{3!}\dddot{r}(t) + \ldots.. + \frac{C_n}{n!}\overset{n}{r}(t)\ldots.$$

The coefficients $C_0, C_1, C_2 \ldots\ldots C_n$ are called generalized error coefficients or dynamic error coefficients.

The n^{th} coefficient, $C_n = \underset{s \to 0}{Lt}\ \dfrac{d^n}{ds^n} F(s)$, where, $F(s) = \dfrac{1}{1+G(s)\,H(s)}$.

Q2.44 ***Give the relation between generalized and static error coefficients.***

The following expression shows the relation between generalized and static error coefficient.

$$C_0 = \frac{1}{1+K_p}\,; \qquad C_1 = \frac{1}{K_v}; \qquad C_2 = \frac{1}{K_a}$$

Q2.45 ***Mention two advantages of generalized error constants over static error constants.***

(i) Generalized error series gives error signal as a function of time.

(ii) Using generalized error constants the steady state error can be determined for any type of input but static error constants are used to determine steady state error when the input is anyone of the standard input.

Q2.46. What are the basic components of an automatic control system ?

The basic components of an automatic control system are,

1. Error detector
2. Amplifier and controller
3. Actuator (Power actuator)
4. Plant
5. Sensor or feedback system

Q2.47 What is automatic controller ?

The combined unit of error detector, amplifier and controller is called automatic controller.

Q2.48 What is the need for a controller?

The controller is provided to modify the error signal for better control action.

Q2.49 What are the different types of controllers?

The different types of controller used in control system are P, PI, PD and PID controllers.

Q2.50 What is Proportional controller and what are its advantages?

The Proportional controller is a device that produces a control signal which is proportional to the input error signal.

The advantages in the proportional controller are improvement in steady-state tracking accuracy disturbance signal rejection and the relative stability. It also makes a system less sensitive to paramete variations.

Q2.51 What is the drawback in P-controller?

The drawback in P-controller is that it develop a constant steady-state error.

Q2.52 What is integral control action?

In integral control action, the control signal is proportional to integral of error signal.

Q2.53 What is the advantage and disadvantage in integral controller?

The advantage in Integral controller is that it eliminates or reduces the steady-state error. The disadvantage is that it can make a system unstable.

Q2.54 Write the transfer function of P, PI, PD and PID controllers.

The transfer function of P-controller, $\frac{U(s)}{E(s)} = K_p$; where, K_p = Proportional gain.

The transfer function of PI-controller, $\frac{U(s)}{E(s)} = K_p\left(1+\frac{1}{T_i s}\right)$; where, T_i = Integral time constant.

The transfer function of PD-controller, $\frac{U(s)}{E(s)} = K_p(1+T_d s)$; where, T_d = Derivative time constant.

The transfer function of PID-controller, $\frac{U(s)}{E(s)} = K_p\left(1+\frac{1}{T_i s}+T_d s\right)$

Q2.55 What is Reset rate?

The Reset rate is the reciprocal of integral time or reset time. The reset rate is the number of times per minute that the proportional part of the control action is duplicated and it is measured in terms of repeats/minute.

Q2.56 Why derivative control is not employed in isolation?

A derivative control mode in isolation produces no corrective efforts for any constant errors. Because it acts only on rate of change of error.

Q2.57 What is PI-controller?

The PI-controller is a device which produces a control signal consisting of two terms : *one proportional to error signal and the other proportional to the integral of error signal.*

Q2.58 What is PD-controller?

The PD-controller is a device which produces a control signal consisting of two terms : *one proportional to error signal and the other proportional to the derivative of error signal.*

Q2.59 What is PID-Controller?

The PID-controller is a device which produces a control signal consisting of three terms : *one proportional to error signal, another one proportional to integral of error signal and the third one proportional to derivative of error signal.*

Q2.60 Give an example of electronic PID-controller

The electronic PID-controller can be realized by an op-amp amplifier with integral and derivative action followed by sign changer, as shown in figure Q2.60.

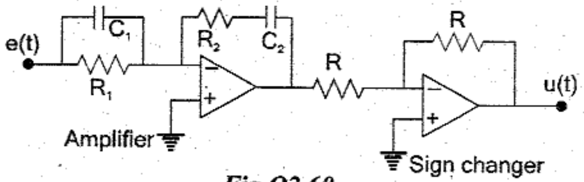


Fig Q2.60

Q2.61 Sketch the step response of a P and PI-controller ?

Let e(t) be the input signal to the controller and u(t) be the output signal to the controller. The input and output signals are shown in the figure Q2.61.

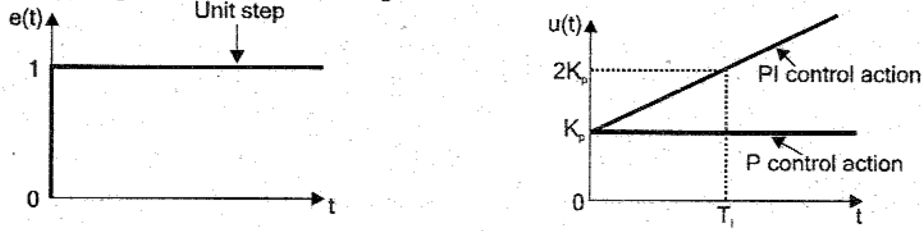


Fig Q2.61

Q2.62 Sketch the ramp response of P, PD and PID-controller?

Let e(t) be the input signal to the controller and u(t) be the output signal to the controller. The input and output signals are shown in the figure Q2.62.

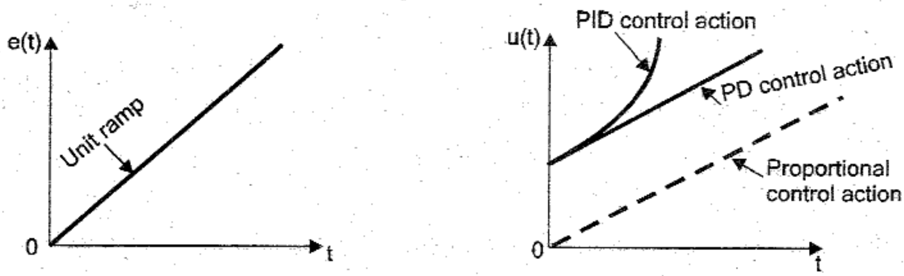


Fig Q2.62

Q2.63 What is the effect on system performance when a proportional controller is introduced in a system?

The proportional controller improves the steady-state tracking accuracy, disturbance signal rejection and relative stability of the system. It also increases the loop gain of the system which results in reducing the sensitivity of the system to parameter variations.

Q2.64 What is the disadvantage in proportional controller?

The disadvantage in proportional controller is that it produces a constant steady state error.

Q2.65 ***What is the effect of PI-controller on the system performance?***

The PI - controller increases the order of the system by one, which results in reducing, the steady state error. But the system becomes less stable than the original system.

Q2.66 ***What is the effect of PD-controller on the system performance?***

The effect of PD - controller is to increase the damping ratio of the system and so the peak overshoot is reduced.

Q2.67 ***Why derivative controller is not used in control systems?***

The derivative controller produces a control action based on rate of change of error signal and it does not produce corrective measures for any constant error. Hence derivative controller is not used in control systems.

Q2.68 ***Determine the impulse response of the feedback system governed by the closed loop transfer function,*** $M(s) = \dfrac{2s+1}{(s+1)^2}$.

By partial fraction expansion the given closed loop transfer function can be expressed as,

$$\therefore M(s) = \frac{2s+1}{(s+1)^2} = \frac{A}{(s+1)^2} + \frac{B}{s+1}$$

$$A = \left.\frac{2s+1}{(s+1)^2} \times (s+1)^2\right|_{s=-1} = 2s+1\Big|_{s=-1} = 2(-1)+1 = -1$$

$$B = \left.\frac{d}{ds}\left[\frac{2s+1}{(s+1)^2} \times (s+1)^2\right]\right|_{s=-1} = \left.\frac{d}{ds}[2s+1]\right|_{s=-1} = 2$$

$$\therefore M(s) = \frac{-1}{(s+1)^2} + \frac{2}{(s+1)}$$

The impulse response is given by inverse Laplace transform of closed loop transfer function.

$$\therefore \text{Impulse response, } m(t) = \mathcal{L}^{-1}\left\{\frac{-1}{(s+1)^2} + \frac{2}{s+1}\right\} = -t\,e^{-t} + 2\,e^{-t}$$

Q2.69 ***Determine the impulse response of the feedback systems governed by the following closed loop transfer functions,***

a) $M(s) = \dfrac{s}{s+1}$; b) $M(s) = \dfrac{1}{s^2+1}$.

a) The impulse response is given by inverse Laplace transform of closed loop transfer function.

$$\therefore \text{Impulse response, } m(t) = \mathcal{L}^{-1}\left\{\frac{s}{s+1}\right\} = \mathcal{L}^{-1}\left\{1 - \frac{1}{s+1}\right\} = \delta(t) - e^{-t}$$

b) The impulse response is given by inverse Laplace transform of closed loop transfer function.

$$\therefore \text{Impulse response, } m(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

Q2.70 ***Determine the impulse response of the feedback system governed by the closed loop transfer function,*** $M(s) = \dfrac{2(s+3)}{(s+3)^2+1}$.

The impulse response is given by inverse Laplace transform of closed loop transfer function.

$$\therefore \text{Impulse response, } m(t) = \mathcal{L}^{-1}\left\{\frac{2(s+3)}{(s+3)^2+1}\right\} = 2\,e^{-3t}\cos t$$

2.24 EXERCISES

E2.1 What is the unit-step response of the system shown in fig E2.1

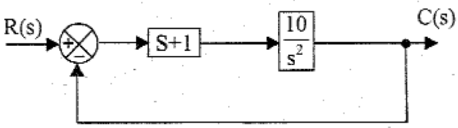


Fig E2.1.

E2.2 Obtain the unit-step response of a unity-feedback system

whose open-loop transfer function is $G(s)=\dfrac{5(s+20)}{s(s+4.59)\,(s^2+3.41s+16.35)}$.

E2.3 The open loop transfer function of an unity feedback control system is given by $G(s)=\dfrac{100}{s(s+2)\,(s+5)}$.

For unit step input,find the time response of the closed loop system and determine % over shoot and the rise time.

E2.4 A Servomechanism has its moment of inertia J = 10×10^{-6} Kg-m^2, retarding friction, B = 400×10^{-6} N-m/(rad/sec) and elasticity coefficient, K = 0.004 N-m/rad. Find the natural frequency and damping factor of the system.

E2.5 For a second order system whose open loop transfer function $G(s)=\dfrac{4}{s(s+2)}$, determine the maximum over shoot, the time to reach the maximum overshoot when a step displacement of 18° is given to the system. Find the rise time, time constant and the settling time for an error of 7% .

E2.6 Consider the unity feedback closed loop system where the forward transfer function is $G(s)=\dfrac{25}{s(s+5)}$.

Obtain the rise time, Peak time, Maximum overshoot and the settling time when the system is subjected to a unit-step input.

E2.7 Consider the system shown in fig E2.7, where $\zeta = 0.6$ and $\omega_n = 0.5$ rad/sec. Determine the rise time, peak time, maximum overshoot and settling time, when the system is subjected to a unit-step input.

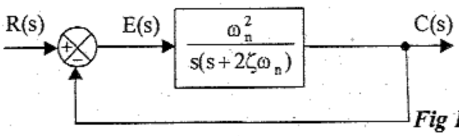


Fig E2.7.

E2.8 For the system shown in fig E2.8, determine the values of K and K_h so that the maximum overshoot in the unit step response is 0.2 and the peak time is 1 sec. With these values of K and K_h, obtain rise time and settling time.

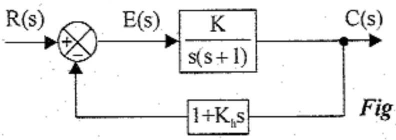


Fig E2.8.

E2.9 The system shown in fig E2.9 subjected to a unit-step input. Determine the values of K and T, where the Maximum overshoot of the system is 25.4% corresponding to $\zeta = 0.4$.

E2.10 Determine the values of K and T of the closed-loop system shown in Fig E2.10, so that the maximum overshoot in unit-step response is 25% and the peak time is 2 sec. Assume that J=1 Kg-m^2.

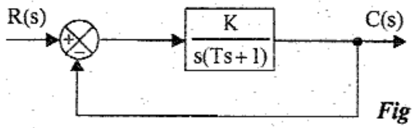


Fig E2.9.

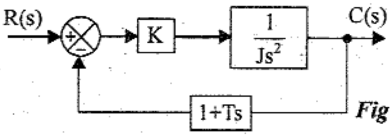


Fig E2.10.

E2.11 A unity-feedback system is characterized by the open-loop transfer function $G(s) = \dfrac{1}{s(0.5s+1)\,(0.2s+1)}$.

a) Determine the steady-state errors to unit-step, unit-ramp and unIt parabolic inputs.

b) Determine rise time, peak time, peak overshoot and settling time of the unit-step respone of the system.

E2.12 For a system whose $G(s) = \dfrac{10}{s(s+1)\,(s+2)}$,
find the steady state error when it is subjected to the input, $r(t) = 1+2t+1.5\,t^2$.

E2.13 A unity feedback system has $G(s) = \dfrac{1}{s(1+s)}$.
The input to the system is described by $r(t) = 4+6t+2t^3$. Find the generalized error coefficients and steady state error.

E2.14 A unity feedback system has the forward path transfer function $G(s) = \dfrac{10}{(s+1)}$.

Find the steady state error and the generalised error coefficient for $r(t) = t$.

E2.15 Find out the position, velocity and acceleration error coefficients for the following unity feedback systems having forward loop transfer function G(s) as,

(a) $\dfrac{100}{(1+0.5s)\,(1+2s)}$ (b) $\dfrac{K}{s(1+0.1s)\,(1+s)}$

(c) $\dfrac{K}{s^2(s^2+8s+100)}$ (d) $\dfrac{K(1+s)\,(1+2s)}{s^2(s^2+4s+20)}$

E2.16 The open loop transfer function of a unity feedback control system is $G(s) = 9/(s+1)$, using the generalized error series determine the error signal and steady state error of the system when the system is excited by,

(i) $r(t) = 2$ (ii) $r(t) = t$

(iii) $r(t) = 3t^2/2$ (iv) $r(t) = 1+2t+3t^2/2$

E2.17 For unity feedback system having open loop transfer function as $G(s) = \dfrac{K(s+2)}{s^2(s^2+7s+12)}$. Determine,

(i) type of system,

(ii) error constants K_p, K_v and K_a

(iii) steady state error for parabolic input.

ANSWER FOR EXERCISE PROBLEMS

E2.1 $c(t) = -1.1455\, e^{-8.87t} + 0.1455 e^{-1.13t} + 1$

E2.2 $c(t) = 1 + \frac{3}{8} e^{-t} \cos 3t - \frac{17}{24} e^{-t} \sin 3t - \frac{11}{8} e^{-3t} \cos t - \frac{13}{8} e^{-3t} \sin t$

E2.3 $c(t) = \left[1 - 0.186\, e^{-7.45t} - 0.88\, e^{0.225t} \cos(3.65t - 22°)\right]$

As t tends to infinity, c(t) tends to infinity and so the system is unstable. Therefore % over shoot and rise time are not defined.

E2.4 Natural frequency, $\omega_n = 20$ rad/sec, Damping factor, $\zeta = 1$.

E2.5 Maximum overshoot = 0.16, when input is 18%, $M_p = 2.88\%$
Peak time, $t_p = 1.81$sec. Rise time, $t_r = 1.21$ sec
Time constant, T=1 sec, Settling time for 7% error = 2.66 sec.

E2.6 Rise time, $t_r = 0.55$sec, %Peak overshoot, $M_p = 9.5\%$
Peak time, $t_p = 0.785$sec, Settling time, $t_s = 1.33$sec (for 2% error); $t_s = 1$ sec(for 5% error)

E2.7 Rise time, $t_r = 0.55$sec, Maximum overshoot, $M_p = 0.095$
Peak time, $t_p = 0.785$sec, Settling time, $t_s = 1$ sec (for 5% criterion)

E2.8 K=12.5, Rise time, $t_r = 0.65$sec
$K_h = 0.178$; Settling time, $t_s = 2.48$ sec (for 2% error) ; $t_s = 1.86$ sec (for 5% error)

E2.9 K=1.42, T=1.09

E2.10 K=2.95 N-m T=0.471 sec

E2.10 (a) $e_{ss}\big|_{\text{unit step}} = 0$

$e_{ss}\big|_{\text{unit ramp}} = 1$

$e_{ss}\big|_{\text{unit parabola}} = \infty$

(b) Rise time, $t_r = 1.91$ sec
Peak time, $t_p = 2.79$ sec
Peak overshoot, $M_p = 0.1265$
Settling time, $t_s = 5.4$ sec

E2.12 The total steady state error is ∞.

E2.13 $C_0 = 0$; $C_1 = 1$; $C_2 = 0$; $C_3 = -6$; $e_{ss} = \infty$

E2.14 $C_0 = 1/11$; $C_1 = 10/121$; $e_{ss} = \infty$.

E2.15

Question	K_p	K_v	K_a
(a)	100	0	0
(b)	∞	K	0
(c)	∞	∞	K/100
(d)	∞	∞	K/20

E2.16 (i) $e(t) = 0.2$; $e_{ss} = 0.2$
(ii) $e(t) = 0.1t + 0.09$; $e_{ss} = \infty$.
(iii) $e(t) = 0.15t^2 + 0.27\,t - 0.054$; $e_{ss} = \infty$.
(iv) $e(t) = 0.15t^2 + 0.77\,t + 0.226$; $e_{ss} = \infty$.

E2.17 (i) It is type-2 system
(ii) $K_p = \infty$; $K_v = \infty$; $K_a = K/6$
(iii) $e_{ss} = 6/K$