ORDINARY DIFFERENTIAL EQUATIONS

A differential equation of the type

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = X \quad - (1)$$

where $P_0, P_1, P_2, \ldots P_n$ and X are function of x or constants is called a linear differential equation of nth order.

If the coefficients $P_0, P_1, P_2, \ldots P_{n-1}, P_n$ of the derivatives are constants $a_0, a_1, a_2, \ldots a_n$ then (1) reduces to

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = X \quad - (2)$$

this is a linear equations with constant coefficients. The general solution of equation (2) is $y = C.F + P.I$.

Complementary Function (C.F)

To find the C.F, we consider the equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \quad - (I)$$

If we use the differential operator symbols,

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}, \quad \ldots, \quad D^n = \frac{d^n}{dx^n}.$$

Equation (2) becomes

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) y = 0 \quad - (II)$$

First we're to find the auxillary equation (A.E)

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which is obtained by simply replacing D by m in the operator polynomial and then by equating it to zero.

ie $a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0$ — (III)

which is a polynomial equation in m of degree n.

By solving this we get n roots say $m_1, m_2, \dots, m_n$.

The solution of equation (I) depends on the nature of the roots.

The nature of roots of the A.E are as follows:

Case (i): when the roots $m_1, m_2, \dots m_n$ are all real and distinct then the solution ie C.F is given by $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$.

Case (ii): If any 2 roots are equal ie if $m_1 = m_2$ then the C.F is

$$y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}.$$

If any 3 roots are equal ie if $m_1 = m_2 = m_3$ then C.F is $y = (c_1 x^2 + c_2 x + c_3) e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$.

Case (iii): If any 2 roots are complex, let $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the C.F is

$$y = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right] + c_3 e^{m_3 x} + \dots$$

Case (iv): Two pairs of complex roots are equal, say, $m_1 = m_3 = \alpha + i\beta$ and $m_2 = m_4 = \alpha - i\beta$, then

$$y = e^{\alpha x}\left[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x\right] + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}.$$

Problems:

2. Q.B.

1) Solve $(D^2-4D+3)y=0$.

Soln: A.E is $m^2-4m+3=0$

$(m-1)(m-3)=0 \Rightarrow m=1,3$.

$\therefore$ C.F $=C_1e^{x}+C_2e^{3x}$

2) Solve $2\frac{d^2y}{dx^2}+5\frac{dy}{dx}-12y=0$

Soln: The given Equation can be written in the form

$(2D^2+5D-12)y=0$.

A.E is $2m^2+5m-13=0$.

$(m+4)(m-3/2)=0 \Rightarrow m=3/2\,;-4$

$\therefore$ C.F $=C_1e^{3/2x}+C_2e^{-4x}$.

3) Solve $\frac{d^2y}{dx^2}+3\frac{dy}{dx}-40y=0$

Soln. $(D^2+3D-40)y=0$

A.E is $m^2+3m-40=0$

$\Rightarrow (m-5)(m+8)=0 \Rightarrow m=5,-8$

$\therefore$ C.F $=C_1e^{5x}+C_2e^{-8x}$

Q.B 4

4) Solve $(D^2-6D+9)y=0$.

Soln: A.E is $m^2-6m+9=0$.

$(m-3)(m-3)=0$.

$\Rightarrow m=3,3$.

$\therefore$ C.F $=(C_1x+C_2)e^{3x}$.

5.) Solve $(D^2+6D+9)y=0$.

Soln: A.E is $m^2+6m+9=0$.

$(m+3)(m+3)=0 \Rightarrow m=-3,-3$.

$\therefore$ C.F $y=(C_1x+C_2)e^{-3x}$.

Q.B. 6) $(D^2+8D+25)y=0$

$m^2+8m+25=0$

$m=\frac{-8\pm\sqrt{64-100}}{2}$

$=\frac{-8\pm 6i}{2}$

$=-4\pm 3i$.

$\therefore y=e^{-4x}[A\cos 3x+B\sin 3x]$

Q.B

5) $(D^2+4D+4)y=0$

$m^2+4m+4=0$

$(m+2)^2=0$

$m=-2,-2$

$\therefore y=(A+Bx)e^{-2x}$

Q.B

1.) $(D^2-5D+6)y=0$

A.E is $m^2-5m+6=0$

$(m-3)(m-2)=0$

$m=2,3$

$\therefore y=Ae^{2x}+Be^{3x}$

3.) $(D^2-4)y=0$

$m^2-4=0$

$(m-2)(m+2)=0$

$m=2,-2$

$\therefore y=Ae^{2x}+Be^{-2x}$

6) Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 0$.

Soln: $(D^2 + 2D + 3)y = 0$.

A.E is $m^2 + 2m + 3 = 0$.

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4-12}}{2} = \frac{-2 \pm \sqrt{-8}}{2}$.

$= \frac{-2 \pm 2\sqrt{2}\,i}{2} = -1 \pm i\sqrt{2}$.

$\therefore$ The roots are $-1+i\sqrt{2}$, $-1-i\sqrt{2}$.

$\therefore$ C.F is $e^{-x}\left[c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right]$.

7) Solve $(D^4 + 8D^2 + 16)y = 0$.

Soln: The eqn. is $(D^2+4)^2 = 0$.

The A.E is $(m^2+4)^2 = 0 \Rightarrow m^2 + 4 = 0$

$\Rightarrow m^2 = -4$

$\Rightarrow m = \pm 2i$ (Repeated twice)

$\therefore$ C.F $y = (c_1 x + c_2)\cos 2x + (c_3 x + c_4)\sin 2x$.

P.I Particular Integral:

When the RHS of the given equation is zero the general solution of the equation is only C.f.

When the RHS of the given differential equation is a function of x, the general solutions includes P.I also

There are different methods to find P.I.

Method of variation of parameters.

Consider the second order non-homogeneous linear Equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ — (1)

where P, q, r are functions of x or constants.

The homogeneous equation corresponding to Eqn ①

is $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + qy = 0$ — ②

ie $y'' + Py' + qy = 0$.

The general soln. of ② is

$y = c_1 y_1(x) + c_2 y_2(x)$.

Replacing the constants c_1, c_2 by variable functions.

$u(x)$ and $v(x)$ we get,

$y = uy_1 + vy_2$.

where $u = -\int \frac{r y_2}{w} dx + k_1$

$v = \int \frac{r y_1}{w} dx + k_2$.

Here $w = y_1 y_2' - y_2 y_1'$ is called Wronskian of y_1 and

y_2 $(w \neq 0)$.

1.) Solve $y'' + y = \sec x$ by method of variation of parameters.

Soln: $y'' + y = \sec x$ — ①

The homogeneous equation corresponding to Eqn. ① is

$y'' + y = 0$.

A.E is $m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i$

$\therefore$ C.F is $y = c_1 \cos x + c_2 \sin x$

$\Rightarrow y = uy_1 + vy_2$

Here $y_1 = \cos x$ $\quad y_2 = \sin x$

$y_1' = -\sin x$ $\quad y_2' = \cos x$.

$\therefore w = y_1 y_2' - y_2 y_1' = \cos^2 x + \sin^2 x = 1$

Now.

$$u = -\int \frac{r y_2}{\omega} dx + k_1$$

$$= -\int \sec x \sin x \, dx + k_1$$

$$= -\int \tan x \, dx + k_1$$

$$= -\log \sec x + k_1$$

$$u = \log \cos x + k_1$$

$$v = \int \frac{r y_1}{\omega} dx + k_2$$

$$= \int \sec x \cos x \, dx + k_2$$

$$= \int dx + k_2$$

$$= x + k_2.$$

$\therefore$ The general solution of Equation ① is

$$y = (\log \cos x + k_1)\cos x + (x + k_2)\sin x.$$

2) $y'' + y = \tan x$ — ①.

Soln: Homogeneous Eqn. of ① is $y'' + y = 0$.

A.E is $m^2 + 1 = 0 \Rightarrow m = \pm i$

C.F is $y = c_1 \cos x + c_2 \sin x$.

$y_1 = \cos x \quad y_1' = -\sin x, \quad y_2 = \sin x, \quad y_2' = \cos x$

$\therefore \omega = y_1 y_2' - y_2 y_1' = \cos^2 x + \sin^2 x = 1$

$$u = -\int \tan x \sin x \, dx + k_1$$

$$= -\int \frac{\sin^2 x}{\cos x} dx + k_1$$

$$= -\int \frac{1 - \cos^2 x}{\cos x} dx + k_1$$

$$= -\left[\int \sec x \, dx - \int \cos x \, dx + k_1\right]$$

$$= -\left[\log(\sec x + \tan x) - \sin x\right] + k_1$$

$$u = \sin x - \log(\sec x + \tan x) + k_1$$

$$v = \int \tan x \cos x \, dx + k_2$$

$$= \int \sin x \, dx + k_2$$

$$v = -\cos x + k_2$$

$\therefore$ The general solution is

$y = \cos x\left[\sin x - \log(\sec x + \tan x) + K_1\right] + \sin x(-\cos x + K_2$

3) Solve $(D^2 + 2D + 5)y = e^{-x}\tan x$ by method of variation of parameters.

Soln: Homogeneous Equation is $D^2 + 2D + 5 = 0$.

A.E is $m^2 + 2m + 5 = 0$.

$m = -1 \pm 2i, \quad \alpha = -1, \quad \beta = 2$.

C.F is $y = e^{-x}\left[C_1\cos 2x + C_2\sin 2x\right]$.

$y = C_1 e^{-x}\cos 2x + C_2 e^{-x}\sin 2x$.

P.I is $y = uy_1 + vy_2$

$y_1 = e^{-x}\cos 2x \qquad\qquad y_2 = e^{-x}\sin 2x$

$y_1' = -e^{-x}\cos 2x - 2e^{-x}\sin 2x \qquad y_2' = -e^{-x}\sin 2x + 2e^{-x}\cos 2x$

$W = y_1 y_2' - y_2 y_1'$

$= e^{-x}\cos 2x\left[2e^{-x}\cos 2x - e^{-x}\sin 2x\right]$

$- e^{-x}\sin 2x\left[-2e^{-x}\sin 2x - e^{-x}\cos 2x\right]$

$= 2e^{-2x}\cos^2 2x - e^{-2x}\cos 2x\sin 2x + 2e^{-2x}\sin^2 2x + e^{-2x}\sin 2x\cos 2x$.

$= 2e^{-2x}(\sin^2 2x + \cos^2 2x)$

$\Rightarrow W = 2e^{-2x}$

$u = -\int \frac{ry_2}{W}dx + K_1$

$= -\int \frac{e^{-x}\tan x\, e^{-x}\sin 2x}{2e^{-2x}}dx + K_1$

$= -\frac{1}{2}\int \tan x \sin 2x\, dx + K_1$

$$= -\frac{1}{2}\int \frac{\sin x}{\cos x} \cdot 2\sin x\cos x \, dx + K_1.$$

$$= -\int \frac{1-\cos 2x}{2} \, dx + K_1$$

$$= -\frac{1}{2}\left[\int dx - \int \cos 2x \, dx\right] + K_1$$

$$= -\frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + K_1$$

$$\Rightarrow u = \frac{\sin 2x}{4} - \frac{x}{2} + K_1.$$

$$v = \int \frac{r y_1}{W} \, dx + K_2.$$

$$= \int \frac{e^{-x}\tan x \, e^{-x}\cos 2x}{2e^{-2x}} \, dx + K_2.$$

$$= \frac{1}{2}\int \tan x \left(2\cos^2 x - 1\right) dx + K_2$$

$$= \frac{1}{2}\int \left(2\sin x\cos x - \tan x\right) dx + K_2$$

$$= \frac{1}{2}\int \left(\sin 2x - \tan x\right) dx + K_2$$

$$= -\frac{1}{2}\frac{\cos 2x}{2} - \frac{1}{2}\log\sec x + K_2$$

$$= -\frac{\cos 2x}{4} - \frac{1}{2}\log\sec x + K_2$$

$$= \frac{1}{2}\log\cos x - \frac{\cos 2x}{4} + K_2.$$

$\therefore$ The complete solution is

$$y = \left(\frac{\sin 2x}{4} - \frac{x}{2} + K_1\right)e^{-x}\cos 2x + \left(\frac{1}{2}\log\cos x - \frac{\cos 2x}{4} + K_2\right) e^{-x}\sin 2x$$

H/W.

4) $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

Hint: $m = \pm i$, $u = -\int \operatorname{cosec} x \sin x \, dx$ $= -\int dx + k_1 = -x + k_1$, $v = \int \operatorname{cosec} x \cos x \, dx$ $= \int \cot x \, dx$ $= \log \sin x + k_2$.

$y_1 = \cos x, y_2 = \sin x$,

$W = 1$, C.F $= c_1 \cos x + c_2 \sin x + (-x + k_1)\cos x + (\log \sin x + k_2)\sin x$.

5) $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$.

Hint: $m = \pm 2i$, $u = \sin 2x - \log(\sec 2x + \tan 2x)$, $v = -\cos 2x$.

C.F $= c_1 \cos 2x + c_2 \sin 2x$ Soln: $(c_1 \cos 2x + c_2 \sin 2x) - \cos 2x - \log(\sec 2x + \tan 2x) + \sin 2x$.

$W = 2$

To find Particular Integral

when the RHS of the given differential equation is zero, the general soln of the eqn. is only complementary function.

when the RHS of the given differential equation is a function of x namely $e^{\alpha x}$, $\sin \alpha x$, $\cos \alpha x$, algebraic function, the general solution includes particular Integral also.

Type I:

1) solve $(D^2 - 5D + 6)y = e^{-2x}$

Soln: A.E is $m^2 - 5m + 6 = 0$

$m = 2, 3$.

C.F is $c_1 e^{2x} + c_2 e^{3x}$

$$P.I = \frac{1}{D^2 - 5D + 6} \cdot e^{-2x}$$

Q.B

7) $(D^2 + 4D + 4)y = e^{-2x}$

$$P.I = \frac{1}{D^2 + 4D + 4} \cdot e^{-2x}$$

$$= \frac{1}{(-2)^2 + 4(-2) + 4} \cdot e^{-2x}$$

$$= \frac{1}{0} e^{-2x} = \frac{x e^{-2x}}{2D + 4}$$

$$= \frac{x}{0} e^{-2x} = \frac{x^2 e^{-2x}}{2}$$

$$= \frac{1}{(-2)^2 - 5(-2) + 6} \cdot e^{-2x} \quad [\text{Replacing } D \text{ by } -2].$$

$$= \frac{1}{20} \cdot e^{-2x}.$$

$\therefore$ The complete solution is $y = C.F + P.I$

$$\therefore y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{20} e^{-2x}$$

2) Solve $(D^2 - 6D + 9)y = e^{3x}$.

<u>Soln</u>: A.E is $m^2 - 6m + 9 = 0 \Rightarrow m = 3, 3$.

C.F is $(C_1 x + C_2) e^{3x}$.

$$P.I = \frac{1}{D^2 - 6D + 9} \cdot e^{3x} = \frac{1}{9 - 18 + 9} \cdot e^{3x}$$

$$= \frac{1}{0} e^{3x} \quad [\text{This is not possible}]$$

$$= \frac{x}{2D - 6} \cdot e^{3x} = \frac{x}{6 - 6} e^{3x} = \frac{x}{0} e^{3x} \quad [\quad " \quad]$$

$$= \frac{x^2}{2} \cdot e^{3x}$$

$\therefore$ The complete solution is $y = (C_1 x + C_2) e^{3x} + \frac{x^2}{2} e^{3x}$.

3) Solve $y'' + 3y' + 2y = 10e^{5x}$

<u>Soln</u>: A.E is $m^2 + 3m + 2 = 0$

$$(m+1)(m+2) = 0 \Rightarrow m = -1, -2.$$

$\therefore$ C.F is $C_1 e^{-x} + C_2 e^{-2x}$.

$$P.I = \frac{10}{D^2 + 3D + 2} \cdot e^{5x} = \frac{10}{25 + 15 + 2} \cdot e^{5x} = \frac{10}{42} e^{5x}$$

$$= \frac{5}{21} e^{5x}.$$

$\therefore$ The complete solution is

$$y = C_1 e^{-x} + C_2 e^{-2x} + \frac{5}{21} e^{5x}.$$

4) Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 16$. Q.B. 5

Soln: A.E is $m^2 + 4m + 4 = 0 \Rightarrow m = -2, -2$. $(D^2+4D+4)y = 0$.

$\therefore$ C.F is $(c_1 x + c_2)e^{-2x}$.

$$P.I = \frac{1}{D^2+4D+4} \cdot 16e^{0x} = \frac{16}{4}e^{0x} = 4.$$

$\therefore$ The complete solution is $y = (c_1 x + c_2)e^{-2x} + 4$.

5) Solve $(D^3 - 3D^2 + 4D - 2)y = e^x$.

Soln. A.E is $m^3 - 3m^2 + 4m - 2 = 0$.

$(m-1)(m^2 - 2m + 2) = 0$.

$\Rightarrow m = 1$, (or) $m^2 - 2m + 2 = 0$.

$$m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

C.F is $c_1 e^x + e^x[c_2 \cos x + c_3 \sin x]$.

$$P.I = \frac{1}{D^3 - 3D^2 + 4D - 2} \cdot e^x.$$

$$= \frac{1}{1-3+4-2} \cdot e^x.$$

$$= \frac{1}{0}e^x = \frac{x}{3D^2 - 6D + 4} \cdot e^x = \frac{x}{3-6+4}e^x = xe^x$$

$\therefore$ The complete solution is

$$y = c_1 e^x + e^x[c_2 \cos x + c_3 \sin x] + xe^x.$$

Q.B. 8) $(D^2+6D+5)y = e^{2x}$

6) $y'' + 3y' + 2y = e^{5x}$

Soln: $(c_1 e^{-2x} + c_2 e^{-x}) + \frac{1}{42}e^{5x}$

$$P.I = \frac{1}{D^2+6D+5} \cdot e^{2x}.$$

$$= \frac{1}{4+12+5} \cdot e^{2x}$$

$$= \frac{e^{2x}}{21}$$

7) $(D^2 + 2D + 1)y = e^{-x} + 3$.

TYPE - II

1) solve $(D^2-3D+2)y = 6e^{3x} + \sin 2x$.

Soln: A.E is $m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$.

C.F is $c_1 e^{x} + c_2 e^{2x}$.

$$P.I_1 = \frac{1}{D^2-3D+2} \cdot 6e^{3x} = \frac{6}{2} e^{3x} = 3e^{3x}.$$

$$P.I_2 = \frac{1}{D^2-3D+2} \cdot \sin 2x$$

$$= \frac{1}{-4-3D+2} \cdot \sin 2x \quad [\text{Replace } D^2 \text{ by } -2^2 = -4].$$

$$= \frac{1}{-2-3D} \cdot \sin 2x$$

$$= \frac{1}{-(2+3D)} \times \frac{(2-3D)}{(2-3D)} \cdot \sin 2x$$

$$= -\frac{[2-3D]}{4-9D^2} \cdot \sin 2x.$$

$$= -\frac{(2-3D)}{4+36} \cdot \sin 2x$$

$$= -\frac{1}{40}\left[2\sin 2x - 3D(\sin 2x)\right]$$

$$= -\frac{1}{40}\left[2\sin 2x - 6\cos 2x\right]$$

$$= -\frac{1}{20}\left[\sin 2x - 3\cos 2x\right].$$

$\therefore$ The complete solution is

$$y = c_1 e^{x} + c_2 e^{2x} + 3e^{3x} - \frac{1}{20}(\sin 2x - 3\cos 2x).$$

Q.B. 9) $(D^2+16)y = \cos 3x$

$$P.I = \frac{\cos 3x}{D^2+16} \quad D^2 = -a^2$$

$$= \frac{\cos 3x}{-9+16} = \frac{\cos 3x}{7}.$$

10) $(D^2+a^2)y = \sin ax$.

$$P.I = \frac{\sin ax}{D^2+a^2} = \frac{\sin ax}{-a^2+a^2}$$

$$= \frac{1}{0}\sin ax$$

$$= \frac{x \sin ax}{2D}$$

$$= \frac{x}{2} \cdot \int \sin ax \, dx$$

$$= \frac{x}{2} \cdot \frac{-\cos ax}{a} = \frac{-x\cos ax}{2a}.$$

2) Solve $(D^2-4D+3)y = \sin 3x \cos 2x$.

Soln: A.E is $m^2-4m+3=0 \Rightarrow m=1,3$.

$\therefore$ C.F is $c_1e^x + c_2e^{3x}$.

$\sin 3x \cos 2x = \frac{1}{2}[\sin 5x + \sin x] = \frac{1}{2}\sin 5x + \frac{1}{2}\sin x$.

$$P.I_1 = \frac{1}{D^2-4D+3}\cdot\left(\tfrac{1}{2}\sin 5x\right)$$

$$= \frac{1}{2}\left[\frac{1}{-25-4D+3}\cdot \sin 5x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-(22+4D)}\cdot \sin 5x\right]$$

$$= -\frac{1}{2}\left[\frac{(22-4D)}{484-16D^2}\cdot \sin 5x\right]$$

$$= -\frac{1}{2}\left[\frac{(22-4D)}{484-16(-25)}\cdot \sin 5x\right]$$

$$= -\frac{1}{2}\left[\frac{22\sin 5x - 4D(\sin 5x)}{484+400}\right]$$

$$= -\frac{1}{2}\left[\frac{22\sin 5x - 20\cos 5x}{884}\right]$$

$$= \frac{1}{884}\left[10\cos 5x - 11\sin 5x\right]$$

$$P.I_2 = \frac{1}{D^2-4D+3}\left(\tfrac{1}{2}\sin x\right)$$

$$= \frac{1}{2}\left[\frac{1}{-1-4D+3}\cdot \sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{2-4D}\cdot \sin x\right]$$

$$= \frac{1}{2}\left[\frac{(2+4D)}{4-16D^2}\cdot \sin x\right]$$

$$= \frac{1}{2}\left[\frac{(2+4D)}{4-16(-1)}\cdot \sin x\right]$$

$$= \frac{1}{2}\left[\frac{(2+4D)}{4+16}\cdot \sin x\right]$$

$$= \frac{1}{2}\left[\frac{2\sin x + 4D(\sin x)}{20}\right]$$

$$= \frac{1}{2}\left[\frac{2\sin x + 4\cos x}{20}\right]$$

$$= \frac{1}{20}\left[\sin x + 2\cos x\right]$$

$\therefore$ The complete solution is

$$y = c_1e^x + c_2e^{3x} + \frac{1}{884}\left[10\cos 5x - 11\sin 5x\right] + \frac{1}{20}\left[2\cos x + \sin x\right]$$

H/W

3) Solve $(D^2-3D+2)y = \cos 3x \cos 2x$.

C.F is $c_1e^{x} + c_2e^{2x}$

$$\cos 3x \cos 2x = \frac{1}{2}\left[\cos 5x + \cos x\right] = \frac{1}{2}\cos 5x + \frac{1}{2}\cos x.$$

$$P.I_1 = \frac{1}{(D^2-3D+2)}\cdot\frac{1}{2}\cos 5x$$

$$= \frac{1}{2}\left[\frac{1}{-25-3D+2}\cdot\cos 5x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-(23+3D)}\cdot\cos 5x\right]$$

$$= -\frac{1}{2}\left[\frac{(23-3D)}{529-9D^2}\cdot\cos 5x\right]$$

$$= -\frac{1}{2}\left[\frac{(23-3D)}{529-9(-25)}\cdot\cos 5x\right]$$

$$= -\frac{1}{2}\left[\frac{23\cos 5x-3D(\cos 5x)}{529+225}\right]$$

$$= -\frac{1}{2}\left[\frac{23\cos 5x+15\sin 5x}{754}\right]$$

$$= -\left[\frac{23\cos 5x+15\sin 5x}{1508}\right]$$

$$P.I_2 = \frac{1}{(D^2-3D+2)}\cdot\frac{1}{2}\cos x$$

$$= \frac{1}{2}\left[\frac{1}{-1-3D+2}\right]\cdot\cos x$$

$$= \frac{1}{2}\left[\frac{1}{1-3D}\right]\cdot\cos x$$

$$= \frac{1}{2}\left[\frac{1+3D}{1-9D^2}\right]\cdot\cos x$$

$$= \frac{1}{2}\left[\frac{(1+3D)}{1-9(-1)}\right]\cdot\cos x$$

$$= \frac{1}{2}\left[\frac{\cos x+3D(\cos x)}{1+9}\right]$$

$$= \frac{1}{2}\left[\frac{\cos x-3\sin x}{10}\right]$$

$$= \frac{\cos x-3\sin x}{20}$$

∴ The complete solution is

$$y = c_1e^{x} + c_2e^{2x} - \frac{1}{1508}\left[23\cos 5x+15\sin 5x\right] + \frac{1}{20}\left[\cos x-3\sin x\right]$$

H/W.

4) Solve $(D^2+6D+8)y = e^{-2x} + \cos^2 x$.

Soln A.E is $m^2+6m+8=0 \Rightarrow (m+4)(m+2)=0 \Rightarrow m=-2,-4$

C.F is $C_1e^{-4x}+C_2e^{-2x}$.

$\cos^2 x = \frac{1+\cos 2x}{2}$.

$P.I_1 = \frac{1}{D^2+6D+8} \cdot e^{-2x}$

$= \frac{1}{4-12+8} \cdot e^{-2x}$

$= \frac{1}{0} e^{-2x}$

$= \frac{x}{2D+6} \cdot e^{-2x}$

$= \frac{x}{-4+6} \cdot e^{-2x}$

$P.I_1 = \frac{x}{2} e^{-2x}$

$P.I_2 = \frac{1}{D^2+6D+8} \cdot \frac{1}{2} e^{0x}$

$= \frac{1}{2} \cdot \frac{1}{8} \cdot e^{0x}$

$= \frac{1}{16}$.

$P.I_3 = \frac{1}{D^2+6D+8} \cdot \frac{\cos 2x}{2}$

$= \frac{1}{2}\left[\frac{1}{D^2+6D+8} \cdot \cos 2x\right]$

$= \frac{1}{2}\left[\frac{1}{-4+6D+8} \cdot \cos 2x\right]$

$= \frac{1}{2}\left[\frac{1}{4+6D} \cdot \cos 2x\right]$

$= \frac{1}{2}\left[\frac{(4-6D)}{16-36D^2} \cos 2x\right]$

$= \frac{1}{2}\left[\frac{(4-6D)}{16-36(-4)} \cos 2x\right]$

$= \frac{1}{2}\left[\frac{4\cos 2x - 6D(\cos 2x)}{16+144}\right]$

$= \frac{1}{2}\left[\frac{4\cos 2x + 12\sin 2x}{160}\right]$

$= \frac{1}{80}\left[\cos 2x + 3\sin 2x\right]$

∴ The complete solution is

$y = C_1 e^{-4x} + C_2 e^{-2x} + \frac{x}{2} e^{-2x} + \frac{1}{16} + \frac{1}{80}\left[\cos 2x + 3\sin 2x\right]$

5) Solve $(D^2+4)y = \cos 3x$.

A.E is $m^2+4=0 \Rightarrow m = \pm 2i$.

C.F is $y = c_1 \cos 2x + c_2 \sin 2x$.

$$P.I = \frac{1}{D^2+4} \cdot \cos 3x = \frac{1}{-9+4} \cos 3x = \frac{1}{-5} \cos 3x$$

$\therefore$ The complete solution is $y = C.F + P.I$.

$$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{5} \cos 3x.$$

H/W

6) Solve $(D^2+D+1)y = \sin 2x$.

Soln: $y = e^{-x/2}\left[c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right] - \frac{1}{13}\left[2\cos 2x + 3\sin 2x\right]$

TYPE III:

If $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n$ which is a polynomial in x of degrees n or algebraic function then

$$P.I = \frac{1}{\phi(D)} a_0 x^n + a_1 x^{n-1} + \dots + a_n$$

$$= [\phi(D)]^{-1} \left[a_0 x^n + a_1 x^{n-1} + \dots + a_n\right]$$

Expand $[\phi(D)]^{-1}$ by using binomial theorem in ascending power of D.

1.) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

2) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

3) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

4) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

Note : $\frac{1}{D} f(x) = \int f(x)$ wrt x.

1) Solve $(D^2 - 4D + 3)y = x^2$

Soln: A.E is $m^2 - 4m + 3 = 0 \Rightarrow m = 1, 3$.

C.f $= C_1 e^{x} + C_2 e^{3x}$.

$$P.I = \frac{1}{D^2 - 4D + 3} \cdot x^2$$

$$= \frac{1}{3\left[1 + \left(\frac{D^2 - 4D}{3}\right)\right]} \cdot x^2$$

$$= \frac{1}{3}\left[1 + \left(\frac{D^2 - 4D}{3}\right)\right]^{-1} \cdot x^2.$$

$$= \frac{1}{3}\left[1 - \left(\frac{D^2 - 4D}{3}\right) + \left(\frac{D^2 - 4D}{3}\right)^2 - \cdots\right] \cdot x^2$$

$$= \frac{1}{3}\left[1 - \frac{D^2}{3} + \frac{4D}{3} + \frac{D^4 - 8D^3 + 16D^2}{9} - \cdots\right] \cdot x$$

$$= \frac{1}{3}\left[1 - \frac{D^2}{3} + \frac{4D}{3} + \frac{16D^2}{9}\right] \cdot x^2$$

$$= \frac{1}{3}\left[x^2 - \frac{2}{3} + \frac{8x}{3} + \frac{32}{9}\right]$$

$$= \frac{1}{3}\left[x^2 + \frac{8x}{3} + \frac{26}{9}\right].$$

$\therefore$ The complete solution is $y = Cf + PI$

$$\Rightarrow y = C_1 e^{x} + C_2 e^{3x} + \frac{1}{3}\left[x^2 + \frac{8x}{3} + \frac{26}{9}\right]$$

2) Solve $(D^3 + 8)y = x^4 + 2x + 1$.

Soln: A.E is $m^3 + 8 = 0$.

$(m+2)(m^2 - 2m + 4) = 0$.

$m = -2,\ m = \dfrac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3}.$

$C.F = e^{x}\left[C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right] + C_3 e^{-2x}.$

$$P.I = \frac{1}{D^3+8}\cdot(x^4+2x+1)$$

$$= \frac{1}{8\left(1+\frac{D^3}{8}\right)}\cdot(x^4+2x+1)$$

$$= \frac{1}{8}\left[1+\frac{D^3}{8}\right]^{-1}(x^4+2x+1)$$

$$= \frac{1}{8}\left[1-\frac{D^3}{8}+\frac{D^6}{8}-\cdots\right]\cdot(x^4+2x+1)$$

$$= \frac{1}{8}\left[x^4+2x+1-\frac{(\overset{3}{\cancel{24}}x)}{\cancel{8}}\right]$$

$$= \frac{1}{8}\left[x^4-x+1\right].$$

$\therefore$ The complete solution is $y = Cf + PI$.

$\Rightarrow y = e^{x}\left[C_1\cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right] + \frac{1}{8}\left[x^4 - x + 1\right] + C_3 e^{-2x}$

3) solve $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^{2x} + x^2 + x$

Soln: $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x.$

A.E is $m^3 + 2m^2 + m = 0.$

$m(m^2+2m+1) = 0.$

$m(m+1)(m+1) = 0.$

$\Rightarrow m = 0,\ m = -1,\ m = -1.$

$C.F = C_1 e^{0x} + (C_2 \cancel{e^{x}}x + C_3)e^{-x}.$

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$$P.I_1 = \frac{1}{(D^3+2D^2+D)} \cdot e^{2x} = \frac{1}{8+8+2} \cdot e^{2x} = \frac{1}{18} e^{2x}.$$

$$P.I_2 = \frac{1}{(D^3+2D^2+D)} \cdot (x^2+x)$$

$$= \frac{1}{D\left[1+\left(\frac{D^3+2D^2}{D}\right)\right]} \cdot (x^2+x)$$

$$= \frac{1}{D}\left[1+(D^2+2D)\right]^{-1} \cdot (x^2+x)$$

$$= \frac{1}{D}\left[1-(D^2+2D)+(D^2+2D)^2-\cdots\right] \cdot (x^2+x)$$

$$= \frac{1}{D}\left[1-D^2-2D+4D^2+4D^3+D^4 \ldots\right] \cdot (x^2+x)$$

$$= \frac{1}{D}\left[x^2+x-2(2x+1)-2+4(2)\right]$$

$$= \frac{1}{D}\left[x^2+x-4x-2-2+8\right]$$

$$= \frac{1}{D}\left[x^2-3x+4\right] = \frac{x^3}{3} - \frac{3x^2}{2} + 4x.$$

The complete solution is $y = C.f + P.I$.

$$\therefore\ y = C_1 + (C_2 x + C_3)e^{-x} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x.$$

H/w.

4) $(D^2-3D+2)y = 2x^2+1.$

<u>soln:</u> A.E is $m^2-3m+2=0.$

$$\Rightarrow (m-2)(m-1)=0.$$

$$\Rightarrow m=1,2.$$

$\therefore$ $C.f = C_1 e^{x} + C_2 e^{2x}.$

$$P.I = \frac{1}{D^2-3D+2} \cdot (2x^2+1)$$

$$= \frac{1}{2\left[1+\left(\frac{D^2-3D}{2}\right)\right]^{-1}} \cdot (2x^2+1)$$

$$= \frac{1}{2}\left[1+\left(\frac{D^2-3D}{2}\right)\right]^{-1} \cdot (2x^2+1)$$

$$= \frac{1}{2}\left[1-\left(\frac{D^2-3D}{2}\right)+\left(\frac{D^2-3D}{2}\right)^2 - \cdots\right] \cdot (2x^2+1)$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}+\frac{3D}{2}+\frac{D^4-6D^3+9D^2}{4} - \cdots\right] \cdot (2x^2+1)$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}+\frac{3D}{2}+\frac{9D^2}{4} - \cdots\right] \cdot (2x^2+1)$$

$$= \frac{1}{2}\left[2x^2+1-\frac{4}{2}+\frac{3}{2}\cancel{4}x+\frac{9}{4}\cdot 4\right]$$

$$= \frac{1}{2}\left[2x^2+1-2+6x+9\right]$$

$$= \frac{1}{2}\left[2x^2+6x+8\right].$$

The complete solution is $y = C.f + P.I$.

$\Rightarrow$ $y = c_1e^{2x} + c_2e^{x} + x^2 + 3x + 4$.

5) Solve $(D^2+5D+4)y = x^2+7x+9$.

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TYPE - IV

If $f(x) = e^{\alpha x} \cdot X$ where X is any function of x.

$$P.I = \frac{1}{\phi(D)} \cdot e^{\alpha x} \cdot X.$$

$$= e^{\alpha x} \cdot \frac{1}{\phi(D+\alpha)} \cdot X \quad \text{(Replace } D \text{ by } D+\alpha\text{)}.$$

1) Solve $(D^2 - 5D + 6)y = e^x \cos 2x$.

Soln: A.E is $m^2 - 5m + 6 = 0 \Rightarrow (m-2)(m-3) = 0 \Rightarrow m = 2, 3$

$\therefore$ $C.F = C_1 e^{2x} + C_2 e^{3x}$.

$$P.I = \frac{1}{D^2 - 5D + 6} \cdot e^x \cos 2x.$$

$$= e^x \cdot \frac{1}{(D+1)^2 - 5(D+1) + 6} \cdot \cos 2x$$

$$= e^x \cdot \frac{1}{D^2 + 2D + 1 - 5D - 5 + 6} \cdot \cos 2x$$

$$= e^x \cdot \frac{1}{D^2 - 3D + 2} \cdot \cos 2x$$

$$= e^x \cdot \frac{1}{-4 - 3D + 2} \cdot \cos 2x$$

$$= e^x \cdot \frac{1}{-(2+3D)} \times \frac{(2-3D)}{(2-3D)} \cdot \cos 2x$$

$$= -e^x \cdot \frac{(2-3D)}{4 - 9D^2} \cdot \cos 2x$$

$$= -e^{+x} \cdot \frac{(2-3D)}{4 - 9(-4)} \cdot \cos 2x$$

$= -e^{+x} \dfrac{(2-3D)}{40} \cdot \cos 2x \cdot$

$= \dfrac{-e^{x}}{40} \left[2\cos 2x - 3D(\cos 2x) \right]$

$= -\dfrac{e^{x}}{40} \left[2\cos 2x + 6\sin 2x \right].$

$\therefore$ The complete solution is $y = C.F + P.I.$

ie $y = C_1 e^{2x} + C_2 e^{3x} - \dfrac{e^{x}}{20} \left[\cos 2x + 3\sin 2x \right].$

2) Solve $(D^2 + 2D + 2)y = e^{-x}\sin x.$

Soln: A.E is $m^2 + 2m + 2 = 0.$

$m = \dfrac{-2 \pm 2i}{2} = -1 \pm i$

$C.F = e^{-x} \left[C_1 \cos x + C_2 \sin x \right].$

$P.I = \dfrac{1}{(D^2 + 2D + 2)} \cdot e^{-x}\sin x.$

$= e^{-x} \cdot \dfrac{1}{(D-1)^2 + 2(D-1) + 2} \cdot \sin x$

$= e^{-x} \cdot \dfrac{1}{D^2 - \cancel{2D} + 1 + \cancel{2D} - \cancel{2} + \cancel{2}} \cdot \sin x.$

$= e^{-x} \dfrac{1}{D^2 + 1} \cdot \sin x$

$= e^{-x} \dfrac{1}{-1+1} \cdot \sin x = e^{-x} \cdot \dfrac{1}{0} \sin x$

(Not possible)

$= e^{-x} \cdot \frac{1}{2D} \cdot \sin x$

$= \frac{x}{2} e^{-x} (-\cos x) = -\frac{x e^{-x}}{2} \cos x$

$\therefore$ The complete solution is $y = e^{-x}(c_1 \cos x + c_2 \sin x) - \frac{x e^{-x} \cos x}{2}$.

<u>H/W</u> 3.) $(D^4 - 1)y = \cos x \cosh x$.

<u>Soln.</u> A.E is $m^4 - 1 = 0$.

$(m^2)^2 - 1 = 0$.

$(m^2 - 1)(m^2 + 1) = 0$

$(m-1)(m+1)(m^2+1) = 0 \Rightarrow m = 1, -1, \pm i$

$\therefore$ C.F is $c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$.

$\cos x \cosh x = \cos x \left(\frac{e^{x} + e^{-x}}{2} \right) = \frac{e^{x}}{2} \cos x + \frac{e^{-x}}{2} \cos x$.

$P.I_1 = \frac{1}{D^4 - 1} \cdot \frac{e^{x}}{2} \cos x$.

$= \frac{e^{x}}{2} \cdot \frac{1}{(D+1)^4 - 1} \cdot \cos x$

$= \frac{e^{x}}{2} \cdot \frac{1}{D^4 + 4D^3 + 6D^2 + 4D + 1 - 1} \cdot \cos x$

$= \frac{e^{x}}{2} \cdot \frac{1}{(D^2)^2 + 4D^2 \cdot D + 6D^2 + 4D} \cdot \cos x$

$= \frac{e^{x}}{2} \cdot \frac{1}{1 - \cancel{4D} - 6 + \cancel{4D}} \cdot \cos x$

$= \frac{e^{x}}{2} \cdot \frac{1}{-5} \cdot \cos x = -\frac{e^{x}}{10} \cos x$

$$P.I_2 = \frac{1}{D^4-1} \cdot \frac{e^{-x}}{2} \cos x$$

$$= \frac{e^{-x}}{2} \cdot \frac{1}{(D-1)^4-1} \cdot \cos x$$

$$= \frac{e^{-x}}{2} \cdot \frac{1}{D^4-4D^3+6D^2-4D+1-1} \cdot \cos x.$$

$$= \frac{e^{-x}}{2} \cdot \frac{1}{(D^2)^2-4D^2 \cdot D+6D^2-4D} \cdot \cos x$$

$$= \frac{e^{-x}}{2} \cdot \frac{1}{1+\cancel{4D}-6-\cancel{4D}} \cdot \cos x$$

$$= \frac{e^{-x}}{2} \cdot \frac{1}{-5} \cdot \cos x$$

$$= -\frac{1}{10} e^{-x} \cos x$$

$\therefore$ $P.I = P.I_1 + P.I_2 = -\frac{1}{5}\left[\frac{e^{x}+e^{-x}}{2}\right]\cos x.$

$$= -\frac{1}{5} \cos x \cosh x.$$

$\therefore$ The complete solution is $y = C.F + P.I.$

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5} \cos x \cosh x.$$

H/W.

4.) Solve $\frac{d^4y}{dx^4} - y = e^{x} \cos x.$

Soln: $(D^4-1)y = e^{x} \cos x.$

A.E is $m^4-1 = 0 \Rightarrow (m^2)^2-1 = 0.$

$\Rightarrow (m^2-1)(m^2+1) = 0.$

$\Rightarrow (m-1)(m+1)(m^2+1) = 0$

$\Rightarrow m = 1, -1, \pm i$

C.F is $C_1 e^{x} + C_2 e^{-x} + C_3 \cos x + C_4 \sin x$.

$$P.I = \frac{1}{D^4 - 1} e^{x} \cos x.$$

$$= e^{x} \cdot \frac{1}{(D+1)^4 - 1} \cdot \cos x.$$

$$= e^{x} \cdot \frac{1}{(D+1)^2 (D+1)^2 - 1} \cdot \cos x$$

$$= e^{x} \cdot \frac{1}{(D^2 + 2D + 1)(D^2 + 2D + 1) - 1} \cdot \cos x.$$

$$= e^{x} \cdot \frac{1}{(-\cancel{1} + 2D + \cancel{1})(-\cancel{1} + 2D + \cancel{1}) - 1} \cdot \cos x$$

$$= e^{x} \cdot \frac{1}{4D^2 - 1} \cdot \cos x$$

$$= e^{x} \cdot \frac{1}{-4 - 1} \cdot \cos x$$

$$= e^{x} \cdot \frac{1}{-5} \cos x.$$

$\therefore$ The complete solution is

$$y = C_1 e^{x} + C_2 e^{-x} + C_3 \cos x + C_4 \sin x + \frac{e^{x}}{-5} \cos x.$$

5) Solve $(D^3 - D)y = e^{x} \cdot x$

<u>Soln</u>. A.E is $m^3 - m = 0 \Rightarrow m(m^2 - 1) = 0 \Rightarrow m = 0, m = \pm 1$

C.F is $C_1 e^{0x} + C_2 e^{x} + C_3 e^{-x}$

$$P.I = \frac{1}{D^3 - D} \cdot e^{x} \cdot x$$

EULERS HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS.

The Equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X \quad -(1)$$

where $a_0, a_1, \ldots a_n$ are constants and X is a function of x is called Euler's homogeneous linear differential equation. We can transform the Eqn (1) into an Equation with constant coefficients by using

$x = e^z$ ie $z = \log x$.

$$xD = x\frac{d}{dx} = D', \quad x^2 D^2 = x^2 \frac{d^2}{dx^2} = D'(D'-1).$$

$$x^3 D^3 = x^3 \frac{d^3}{dx^3} = D'(D'-1)(D'-2).$$

(1) Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x^2$

Soln: put $x = e^z \Rightarrow z = \log x$.

$$x\frac{d}{dx} = D', \quad x^2 \frac{d^2}{dx^2} = D'(D'-1)$$

Then the given equation becomes.

$$[D'(D'-1) - D' + 1]y = e^{2z}$$

$$[D'^2 - D' - D' + 1]y = e^{2z}$$

$$[D'^2 - 2D' + 1]y = e^{2z}$$

A.E is $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

$$C.F = (c_1 z + c_2)e^z.$$

$$P.I = \frac{1}{D'^2 - 2D' + 1} \cdot e^{2z}$$

$$= \frac{1}{4-4+1} \cdot e^{2z} = e^{2z}.$$

$\therefore$ The complete soln. is $y = (C_1 z + C_2)e^{z} + e^{2z}$.

ie $y = (C_1 \log x + C_2)x + x^2$.

2) Solve $y'' + \frac{1}{x} y' = \frac{12 \log x}{x^2}$

Soln: $x^2 y'' + x y' = 12 \log x$.

Put $x = e^{z} \Rightarrow z = \log x$

$(x^2 D^2 + x D)y = 12 \log x$.

$[D'(D'-1) + D']y = \cancel{12\log}\ \cancel{12\log}$ $12z$

$[D'^2 - D' + D']y = 12z$.

$(D')^2 y = 12z$.

A.E is $m^2 = 0 \Rightarrow m = 0, 0$.

$\therefore$ C.F $= (C_1 z + C_2)e^{0z}$

$\therefore$ C.F $= C_1 \log x + C_2$.

$$P.I = \frac{1}{D'^2} \cdot 12z = \frac{1}{D'} \cdot \int 12z\,dz = \frac{1}{D'} \cdot 12 \cdot \frac{z^2}{2}$$

$$= 6 \int z^2\,dz = 6 \cdot \frac{z^3}{3} = 2z^3.$$

$\therefore$ P.I $= 2(\log x)^3$.

$\therefore$ The complete solution is $y = C.F + P.I$.

$y = C_1 \log x + C_2 + 2(\log x)^3$.

H/w

3) Solve the equation $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = x^2 + \frac{1}{x^2}$

Soln: Put $x = e^z \Rightarrow z = \log x$.

$x\frac{d}{dx} = D'$, $x^2\frac{d^2}{dx^2} = D'(D'-1)$.

Then the given equation becomes.

$$[D'(D'-1) + 4D' + 2]y = e^{2z} + e^{-2z}.$$

$$[D'^2 + 3D' + 2]y = e^{2z} + e^{-2z}.$$

A.E is $m^2 + 3m + 2 = 0$

$\Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2$.

$\therefore$ C.F $= c_1 e^{-z} + c_2 e^{-2z}$.

$$P.I_1 = \frac{1}{D'^2 + 3D' + 2} \cdot e^{2z}$$

$$= \frac{1}{4+6+2} \cdot e^{2z}$$

$$= \frac{1}{12} e^{2z}$$

$$P.I_2 = \frac{1}{D'^2 + 3D' + 2} \cdot e^{-2z}$$

$$= \frac{1}{4-6+2} \cdot e^{-2z}$$

$$= \frac{1}{0} \cdot e^{-2z}$$

$$= \frac{z}{2D'+3} \cdot e^{-2z}$$

$$= \frac{z}{-4+3} \cdot e^{-2z}$$

$$= -ze^{-2z}.$$

$\therefore P.I = P.I_1 + P.I_2 = \frac{1}{12}e^{2z} - \frac{z}{e^{2z}}$.

$\therefore$ The complete solution is $y = C.f + P.I$

$$\Rightarrow y = c_1 e^{-z} + c_2 e^{-2z} + \frac{1}{12}e^{2z} - ze^{-2z}.$$

$$\Rightarrow y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{x^2}{12} - \frac{\log x}{x^2}.$$

H/W 4) Solve $(x^2D^2 - xD + 1)y = \left(\frac{\log x}{x}\right)^2$

Soln. put $x = e^z \Rightarrow z = \log x$.

$xD = D'$, $x^2D^2 = D'(D'-1)$.

Then the given Eqn. becomes

$$[D'(D'-1) - D' + 1]y = \left(\frac{z}{e^z}\right)^2$$

$$[D'^2 - 2D' + 1]y = e^{-2z}.z^2.$$

A.E is $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$

$\therefore$ C.f $= (c_1 z + c_2)e^z.$

$$P.I = \frac{1}{D^2 - 2D + 1} . e^{-2z}. z^2.$$

$$= \frac{1}{(D-1)^2} . e^{-2z}. z^2.$$

$$= e^{-2z}. \frac{1}{(D-3)^2} . z^2.$$

$$= e^{-2z}. \frac{1}{(-3)^2\left[1 - \frac{D}{3}\right]^2} . z^2.$$

$$= \frac{e^{-2z}}{9}\left[1 + \frac{2D}{3} + \frac{\cancel{3}D^2}{\cancel{9}3}\right]z^2$$

$$= \frac{e^{-2z}}{9}\left[z^2 + \frac{4z}{3} + \frac{2}{3}\right]$$

$$= \frac{e^{-2z}}{27}\left[3z^2 + 4z + 2\right].$$

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The Complete solution is

$$y = (c_1 z + c_2)e^{z} + \frac{e^{-2z}}{27}(3z^2+4z+2).$$

$$y = (c_1 \log x + c_2)x + \frac{1}{27x^2}\left[3(\log x)^2 + 4\log x + 2\right].$$

H/W

5) Solve $(x^2D^2 + xD + 1)y = \sin(2\log x)\sin(\log x)$

Soln: $c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{1}{16}\sin(3\log x)$
$- \frac{1}{4}\log x \cos(\log x)$

6) solve $(x^2D^2 - 2xD - 4)y = 32(\log x)^2$

Soln: $c_1 x^4 + \frac{c_2}{x} - 8(\log x)^2 + 12\log x - 13.$

7) Solve $(x^2D^2 - xD + 4)y = x^2\sin(\log x).$

Soln: $x\left[c_1\cos(\sqrt{3}\log x) + c_2\sin(\sqrt{3}\log x)\right] - \frac{1}{13}x^2\left[2\cos(\log x)\right.$
$\left. - 3\sin(\log x)\right]$

LEGENDRE'S LINEAR DIFFERENTIAL EQUATIONS:

An Equation of the form

$$a_0(a+bx)^n\frac{d^ny}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = X.$$

where $a_0, \ldots, a_n$ are constant and X is a fn of x ①

is called Legendre's Linear differential Equation

put $(a+bx) = e^z \Rightarrow x = \frac{e^z - a}{b}$

$$z = \log(a+bx)$$

$$(a+bx)\frac{dy}{dx} = bDy$$

$$(a+bx)^2\frac{d^2y}{dx^2} = b^2D(D-1)y.$$

1) Solve $(3x+2)^2 \dfrac{d^2y}{dx^2} + 5(3x+2)\dfrac{dy}{dx} - 3y = x^2+x+1$

Soln: put $3x+2 = e^z \Rightarrow z = \log(3x+2)$.

$\therefore \; x = \dfrac{e^z - 2}{3}$.

$(3x+2)\dfrac{d}{dx} = 3D, \; (3x+2)^2 \dfrac{d^2}{dx^2} = 3^2 D(D-1) = 9D(D-1)$

$\therefore$ The given equation becomes

$$[9D(D-1) + 5\cdot 3D - 3]y = \left(\frac{e^z - 2}{3}\right)^2 + \left(\frac{e^z-2}{3}\right) + 1$$

$$(9D^2 - 9D + 15D - 3)y = \frac{e^{2z} - 4e^z + 4}{9} + \frac{e^z}{3} - \frac{2}{3} + 1$$

$$(9D^2 + 6D - 3)y = \frac{e^{2z} - 4e^z + 4 + 3e^z - 6 + 9}{9}$$

$$(9D^2 + 6D - 3)y = \frac{e^{2z} - e^z + 7}{9}.$$

$$3(3D^2 + 2D - 1)y = \frac{e^{2z} - e^z + 7}{9}.$$

$$(3D^2 + 2D - 1)y = \frac{1}{27}\left[e^{2z} - e^z + 7\right].$$

A.E is $3m^2 + 2m - 1 = 0$.

$\Rightarrow m = -1, \; 1/3$.

C.f is $C_1 e^{-z} + C_2 e^{z/3}$.

$$P.I_1 = \frac{1}{3D^2 + 2D - 1} \cdot \frac{e^{2z}}{27}$$

$$= \frac{1}{27} \cdot \frac{1}{15} e^{2z}$$

$$= \frac{e^{2z}}{405}$$

$$P.I_2 = \frac{1}{3D^2+2D-1} \cdot \frac{-e^{+z}}{27}$$

$$= \frac{-1}{27} \cdot \frac{1}{4} e^z$$

$$= -\frac{1}{108} e^z$$

$$P.I_3 = \frac{1}{3D^2+2D-1}$$

$$= \frac{-7}{27} e^{0z}$$

$$= \frac{-7}{27}.$$

$\therefore P.I = P.I_1 + P.I_2 + P.I_3$.

$$= \frac{e^{2z}}{405} - \frac{e^z}{108} - \frac{7}{27}.$$

$\therefore$ The complete solution is $y = C.F + P.I$.

$$y = c_1 e^{-z} + c_2 e^{z/3} + \frac{e^{2z}}{405} - \frac{e^z}{108} - \frac{7}{27}.$$

$$= c_1 (3x+2)^{-1} + c_2 (3x+2)^{1/3} + \frac{(3x+2)^2}{405} - \frac{(3x+2)}{108} - \frac{7}{27}$$

1W

2) Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$.

Soln: put $(1+x) = e^z \Rightarrow z = \log(1+x)$

$\therefore x = e^z - 1$

$(1+x)\frac{d}{dx} = D$, $(1+x)^2 \frac{d^2}{dx^2} = D(D-1)$.

Then the given Eqn. becomes.

$[D(D-1) + D + 1]y = 2\sin z$.

$(D^2+1)y = 2\sin z$.

A.E is $m^2+1 = 0 \Rightarrow m = \pm i$

C.F is $C_1 \cos z + C_2 \sin z$.

$P.I = 2 \cdot \frac{1}{D^2+1} \sin z = 2 \cdot \frac{1}{-1+1} \sin z = 2 \cdot \frac{1}{0} \sin z$.

$$= \frac{2z}{2D} \sin z = z(-\cos z).$$

The complete soln. is

$y = C_1 \cos z + C_2 \sin z - z\cos z$.

$= C_1 \cos[\log(1+x)] + C_2 \sin[\log(1+x)] - \log(1+x) \cdot \cos[\log(1+x)]$

H/W

3) Solve $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Soln. put $(3x+2) \cancel{\frac{d}{dx}} = e^z \Rightarrow x = \frac{e^z - 2}{3}$.

$\log(3x+2) = z$.

$(3x+2)\frac{d}{dx} = 3D$, $(3x+2)^2\frac{d^2}{dx^2} = 3^2 D(D-1) = 9D(D-1)$.

Then the given Equation becomes,

$\left[9D(D-1) + 3(3D) - 36\right]y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1.$

$(9D^2 - 9D + 9D - 36)y = 3\left[\frac{e^{2z} - 4e^z + 4}{9}\right] + \frac{4e^z}{3} - \frac{8}{3} + 1$

$9(D^2 - 4)y = \frac{e^{2z}}{3} + \frac{4}{3} - \cancel{\frac{4e^z}{3}} + \cancel{\frac{4e^z}{3}} - \frac{8}{3} + 1.$

$9(D^2 - 4)y = \frac{e^{2z}}{3} - \frac{1}{3}.$

$\Rightarrow (D^2 - 4)y = \frac{1}{27}\left[e^{2z} - 1\right]$

A.E is $m^2 - 4 = 0 \Rightarrow m^2 = 4 \Rightarrow m = \pm 2.$

$C.F = C_1 e^{2z} + C_2 e^{-2z} = C_1(3x+2)^2 + C_2(3x+2)^{-2}.$

$P.I = \frac{1}{27} \cdot \frac{1}{D^2-4}(e^{2z} - 1).$

$= \frac{1}{27}\left[\frac{1}{D^2-4}e^{2z} - \frac{1}{D^2-4} \cdot e^{0z}\right].$

$= \frac{1}{27}\left[\frac{1}{0}e^{2z} - \frac{1}{-4} \cdot e^{0z}\right] = \frac{1}{27}\left[\frac{z}{2D}e^{2z} + \frac{1}{4}\right].$

$= \frac{1}{27}\left[\frac{z}{4} \cdot e^{2z} + \frac{1}{4}\right] = \frac{1}{108}\left[ze^{2z} + 1\right].$

$= \frac{1}{108}\left[\log(3x+2)(3x+2)^2 + 1\right].$

$Y = C_1(3x+2)^2 + C_2(3x+2)^{-2} + \frac{1}{108}\left[\log(3x+2)(3x+2)^2 + 1\right].$

H/W: 4) Solve $(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x$.

5) Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos[\log(1+x)]$.

SIMULTANEOUS DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS:

1) Solve $\frac{dx}{dt} + 2x - 3y = 5t$; $\frac{dy}{dt} - 3x + 2y = 2e^{2t}$.

Soln: The given Equations can be written as

$Dx + 2x - 3y = 5t \Rightarrow (D+2)x - 3y = 5t$ — (1)

$Dy - 3x + 2y = 2e^{t} \Rightarrow -3x + (D+2)y = 2e^{2t}$ — (2)

(1) $\times 3 \Rightarrow 3(D+2)x - 9y = 15t$.

(2) $\times (D+2) = -3(D+2)x + (D+2)^2 y = 2(D+2)e^{2t}$.

$$-9y + (D+2)^2 y = 2(D+2)e^{2t} + 15t.$$

$\Rightarrow -9y + (D^2+4D+4)y = 2(2e^{2t} + 2e^{2t}) + 15t$.

$\Rightarrow (D^2 + 4D - 5)y = 8e^{2t} + 15t$ — (I)

A.E is $m^2 + 4m - 5 = 0 \Rightarrow m = 1, -5$.

C.F is $c_1 e^{t} + c_2 e^{-5t}$.

$$P.I_1 = \frac{1}{D^2+4D-5} \cdot 15t$$

$$= 15 \cdot \frac{1}{-5\left[1 - \left(\frac{D^2+4D}{5}\right)\right]} \cdot t$$

$$= -\frac{15}{5}\left[1 - \left(\frac{D^2+4D}{5}\right)\right]^{-1} \cdot t$$

$$= -3\left[1 + \frac{D^2+4D}{5}\right] \cdot t$$

$$= -3\left[t + \frac{4}{5}\right] = -3t - \frac{12}{5}$$

$$P.I_2 = \frac{1}{D^2+4D-5} \cdot 8e^{2t}$$

$$= 8 \cdot \frac{1}{4+8-5} e^{2t}$$

$$= \frac{8}{7} e^{2t}.$$

$\therefore y = c_1e^{t} + c_2 e^{-5t} - 3t - \frac{12}{5} + \frac{8}{7} e^{2t}$.

Again,

$① \times (D+2) \Rightarrow (D+2)^2 x - 3\cancel{(D+2)y} = 5(D+2)t$.

$② \times 3 \Rightarrow \quad -9x + \cancel{3(D+2)y} = 6e^{2t}$.

$$(D+2)^2 x - 9x = 5(1+2t) + 6e^{2t}$$

$\Rightarrow (D^2 + 4D + 4 - 9)x = 5 + 10t + 6e^{2t}$.

$\Rightarrow (D^2 + 4D - 5)x = 5(1+2t) + 6e^{2t}$.

$A.E = m^2 + 4m - 5 = 0 \Rightarrow m = 1, -5$

$C.F = c_1e^{t} + c_2e^{-5t}$.

$P.I_1 = \frac{1}{D^2+4D-5} \cdot 5(1+2t)$

$= \frac{5}{-5}\left[1 - \left(\frac{D^2+4D}{5}\right)\right]^{-1} \cdot (1+2t)$

$= -1\left[1 + \left(\frac{D^2+4D}{5}\right)\right](1+2t)$

$= -\left[(1+2t) + \frac{8}{5}\right]$.

$= -\left[2t + \frac{13}{5}\right]$

$P.I_2 = \frac{1}{D^2+4D-5} \cdot 6e^{2t}$

$= 6 \cdot \frac{1}{4+8-5} e^{2t}$

$= \frac{6}{7} e^{2t}$.

$x = c_1e^{t} + c_2 e^{-5t} - \left(\frac{13}{5} + 2t\right) + \frac{6}{7} e^{2t}$.

2) Solve $\frac{d^2x}{dt^2} + y = \sin t \; ; \; \frac{d^2y}{dt^2} + x = \cos t$.

G.B. (14)

Soln. The given Equation can be written as.

$D^2x + y = \sin t \quad -①$

$D^2y + x = \cos t \quad -②$

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$① \times D^2 \Rightarrow D^4x + \cancel{D^2y} = D^2(\sin t)$.

$② \Rightarrow \underset{(-)}{(-)x} \; \underset{(-)}{+} \cancel{D^2y} = \underset{(\mp)}{\pm}\cos t$.

$D^2\left[D(\sin t)\right]$

$D\left[\cos t\right]$

$D^4x - x = D^2(\sin t) - \cos t$.

$(D^4 - 1)x = -\sin t - \cos t$ — (I)

A.E $= m^4 - 1 = 0 \Rightarrow (m^2)^2 - 1 = 0$

$\Rightarrow (m^2-1)(m^2+1) = 0 \Rightarrow m^2 = 1, m^2 = -1 \Rightarrow m = \pm 1, \pm i$

C.f is $c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t$.

$P.I_1 = \dfrac{1}{D^4-1}(-\sin t)$

$= \dfrac{-1}{+1-1}\sin t \cdot$ (NP)

$= \dfrac{-t}{4D^3}\sin t$.

$= \dfrac{-t}{-4D}\sin t$

$= \dfrac{t}{4}(-\cos t)$

$= -\dfrac{t\cos t}{4}$

$P.I_2 = \dfrac{1}{D^4-1}(-\cos t)$

$= \dfrac{-1}{1-1}\cos t$

$= \dfrac{-t}{4D^3}\cos t$

$= \dfrac{-t}{-4D}\cos t$

$= \dfrac{t}{4D}\cos t$

$= \dfrac{t}{4}\sin t$

$\therefore x = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t - \frac{t}{4}\cos t + \frac{t}{4}\sin t$.

Again $y = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t - \frac{t}{4}\cos t + \frac{t}{4}\sin t$.

$① \Rightarrow \cancel{D^2x} + y = \sin t$.

$② \times D^2 \Rightarrow \underset{-}{\cancel{D^2x}} \; \underset{-}{+} D^4y \; \underset{-}{=} D^2(\cos t)$

$(1 - D^4)y = \sin t - D^2(\cos t)$.

$\Rightarrow -(D^4-1)y = \sin t + \cos t$. (SAME AS THE ABOVE).

$\Rightarrow (D^4-1)y = -\sin t - \cos t$. — (II)

H/W.

3) Solve $\frac{d^2x}{dt^2} - 3x - 4y = 0$; $\frac{d^2y}{dt^2} + x + y = 0$.

$m = \pm 1, \pm 1$; $x = (c_1 t + c_2)e^t + (c_3 t + c_4)e^{-t}$. $|||^{ly}$ $y =$

4) Solve $\frac{dx}{dt} - \frac{dy}{dt} + 2y = \cos 2t$; $\frac{dx}{dt} + \frac{dy}{dt} - 2x = \sin 2t$

$m = 1 \pm i$; $x = e^t(c_1 \cos t + c_2 \sin t) - \frac{1}{2}\cos 2t$;

$y = e^t(c_1 \cos t + c_2 \sin t) - \frac{1}{2}\sin 2t$.

Formula :

1) $\int dx = x$.

2) $\int x^n dx = \frac{x^{n+1}}{n+1}$

3) $\int \sin mx \, dx = -\frac{\cos mx}{m}$

4) $\int \cos mx \, dx = \frac{\sin mx}{m}$

5) $\int \tan x \, dx = \log \sec x$.

$-\int \tan x \, dx = \log \cos x$.

6) $\int \cot x \, dx = \log \sin x$

7) $\int \sec x \, dx = \log(\sec x + \tan x)$

8) $\int \operatorname{cosec} x \, dx = -\log(\operatorname{cosec} x + \cot x)$.

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UNIT - I . ODE.

Part - B. 16 marks Questions.

(11) Solve $(D^2-4D+4)y = e^{-4x} + 5\cos 3x$.

Soln. A.E is $m^2 - 4m + 4 = 0$.

$$(m-2)^2 = 0 \Rightarrow m = 2, 2.$$

$\therefore$ C.f $= (C_1 + C_2 x)e^{2x}$.

$$P.I_1 = \frac{e^{-4x}}{D^2-4D+4} = \frac{e^{-4x}}{(-4)^2-4(4)+4)} = \frac{e^{-4x}}{36}.$$

$$P.I_2 = \frac{5\cos 3x}{D^2-4D+4} = \frac{5\cos 3x}{-9-4D+4} = \frac{5\cos 3x}{-5-4D}.$$

$$= \frac{-5\cos 3x(5-4D)}{(5+4D)(5-4D)} = \frac{-25\cos 3x + 20D(\cos 3x)}{25-16D^2}.$$

$$= \frac{-25\cos 3x + 20(-3\cdot\sin 3x)}{25-(16)(-9)}$$

$$= \frac{-25\cos 3x - 60\sin 3x}{25+144}$$

$$= \frac{-5}{169}\left[5\cos 3x + 12\sin 3x\right].$$

$\therefore y = C.f + P.I_1 + P.I_2$.

$$y = (C_1 + C_2 x)e^{2x} + \frac{e^{-4x}}{36} - \frac{5}{169}\left[5\cos 3x + 12\sin 3x\right].$$

(12) Solve by method of variation of parameters

$(D^2+4)y = \tan 2x$.

Soln: The A.E is $m^2+4=0$

$m^2=-4 \Rightarrow m = \pm 2i$.

$\therefore$ C.F $= c_1 \cos 2x + c_2 \sin 2x$.

C.F $= c_1 y_1 + c_2 y_2$.

$\Rightarrow y_1 = \cos 2x$, $y_2 = \sin 2x$, here $r = \tan 2x$.

$y_1' = -2\sin 2x$, $y_2' = 2\cos 2x$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x.$$

$$= 2(\sin^2 2x + \cos^2 2x)$$

$$= 2(1) = 2.$$

$$u = -\int \frac{r y_2}{W}\, dx + k_1.$$

$$= -\int \frac{\sin 2x \cdot \tan 2x}{2}\, dx + k_1.$$

$$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\, dx + k_1 = \frac{-1}{2}\int \left(\frac{1-\cos^2 2x}{\cos 2x}\right) dx + k_1$$

$$= -\frac{1}{2}\left[\int \frac{1}{\cos 2x}\, dx - \int \cos 2x\, dx\right] + k_1$$

$$= -\frac{1}{2}\left[\int \sec 2x\, dx - \int \cos 2x\, dx\right] + k_1$$

$$= -\frac{1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right] + K_1$$

$$u = -\frac{1}{4}\left[\log(\sec 2x + \tan 2x) - \sin 2x\right] + K_1$$

$$v = \int \frac{r y_1}{W}\, dx + K_2$$

$$= \int \frac{\cos 2x \cdot \tan 2x}{2}\, dx + K_2 = \int \frac{\sin 2x}{2}\, dx + K_2$$

$$= \frac{1}{2}\left(\frac{-\cos 2x}{2}\right) + K_2 = -\frac{1}{4}\cos 2x + K_2$$

$$\therefore v = -\frac{1}{4}\cos 2x + K_2$$

$$\therefore P.I = u y_1 + v y_2$$

$$= -\frac{1}{4}\left[\log(\sec 2x + \tan 2x) - \sin 2x\right] \cdot \cos 2x$$

$$- \frac{1}{4}\cos 2x \sin 2x .$$

$$= -\frac{1}{4}\log(\sec 2x + \tan 2x)^{\cos 2x} + \cancel{\frac{1}{4}\sin 2x \cos 2x}$$

$$- \cancel{\frac{1}{4}\sin 2x \cos 2x}$$

$$= -\frac{1}{4}\log(\sec 2x + \tan 2x)\cos 2x + K_1$$

(5) Solve the simultaneous equations:

$$\frac{dx}{dt} + 2x - 3y = 5e^{-t}, \quad \frac{dy}{dt} - 3x + 2y = 0,$$

Soln. Equation can be written as

$$(D+2)x - 3y = 5e^{-t} \quad - (1)$$

$$-3x + (D+2)y = 0 \quad - (2)$$

$$(1) \times (D+2): (D+2)^2 x - 3(D+2)y = 5(D+2)e^{-t}.$$

$$(2) \times 3: -9x + 3(D+2)y = 0.$$

$$[(D+2)^2 - 9]x = 5(D+2)e^{-t}.$$

$$(D^2+4D-5)x = 5[D(e^{-t}) + 2e^{-t}]$$

$$= 5(-e^{-t} + 2e^{-t}) = 5e^{-t}.$$

$$\Rightarrow (D^2+4D-5)x = 5e^{-t}.$$

A.E is $m^2 + 4m - 5 = 0$.

$(m+5)(m-1) = 0$.

$\Rightarrow m = 1, -5$

c.f $x = c_1 e^{t} + c_2 e^{-5t}$.

$$P.I = \frac{5e^{-t}}{D^2+4D-5} = \frac{5e^{-t}}{1-4-5} = \frac{-5}{8}e^{-t}.$$

$$\therefore x = c_1 e^{t} + c_2 e^{-5t} - \frac{5}{8}e^{-t}.$$

$$(1) \times 3: 3(D+2)x - 9y = 15e^{-t}.$$

$$(2) \times (D+2): -3(D+2)x + (D+2)^2 y = 0.$$

$$[(D+2)^2 - 9]y = 15e^{-t}.$$

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(13) Solve: $(x^2D^2 + xD + 1)y = 4\sin\log x$.

Soln. Put $x = e^z$, $\theta = \frac{d}{dz}$, $xD = \theta$

$\log x = z$. $\quad x^2D^2 = \theta(\theta-1)$.

∴ Eqn. becomes

$[\theta(\theta-1) + \theta + 1]y = 4\sin z$.

$(\theta^2 - \theta + \theta + 1)y = 4\sin z$.

$(\theta^2+1)y = 4\sin z$.

A.E is $m^2+1=0 \Rightarrow m^2=-1 \Rightarrow m = \pm i$

$C.f = c_1\cos z + c_2\sin z$

$P.I = \frac{4\sin z}{\theta^2+1} = \frac{4\sin z}{-1+1} = \frac{4}{0}\sin z$

$\therefore P.I = \frac{\cancel{4}^{2} z\sin z}{\cancel{2}\theta} = 2z \cdot \frac{1}{\theta}(\sin z)$

$= 2z\int \sin z\, dz$

$= 2z(-\cos z) + c.$

$\cancel{= 2\int z\sin z\, dz}$

$\cancel{= 2[z(-\cos z) - (1)(-\sin z)]}$

$\cancel{= 2[-z\cos z + \sin z]}$

$\cancel{= 2[\sin z - z\cos z]}$.

$\therefore y = P.I + C.f = -2\log x\cos(\log x) + c_1\cos(\log x) + c_2\sin(\log x)$

$\cancel{= c_1\cos(\log x) + c_2\sin(\log x) + 2\sin(\log x) - 2\log x\cos(\log x)}$.

SOLUTION OF SECOND AND HIGHER ORDER LINEAR ODE WITH CONSTANT COEFFICIENTS.

The general form of the linear differential equation of second order is

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

where P and Q are constants and R is a function of x or constant.

(ii) Differential operators

The symbol D stands for the operation of differential (i.e.,) $Dy = \frac{dy}{dx}, D^2 y = \frac{d^2 y}{dx^2}$

$\frac{1}{D}$ stands for the operation of integration.

$\frac{1}{D^2}$ stands for the operation of integration twice.

$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = R$ can be written in the operator form

$$D^2 y + P\,Dy + Qy = R \text{ (or) } (D^2 + PD + Q)\,y = R$$

(iii) Complete solution = Complementary function + Particular Integral

(iv) To find the Complementary functions

	Roots of A.E.	C.F.
1.	Roots are Real and different m_1, m_2 $(m_1 \neq m_2)$	$y = Ae^{m_1 x} + Be^{m_2 x}$
2.	Roots are Real and equal $m_1 = m_2 = m$ (say)	$y = (Ax + B)\,e^{mx}$ (or) $(A + Bx)\,e^{mx}$
3.	Roots are imaginary $\alpha \pm i\beta$	$y = e^{\alpha x}(A \cos\beta x + B \sin\beta x)$

(v) To find the particular integral :

$$\text{P.I.} = \frac{1}{f(D)} X$$

	X	P.I
1.	e^{ax}	$\text{P.I.} = \frac{1}{f(D)} e^{ax} = e^{ax} \frac{1}{f(a)}, \quad f(a) \neq 0$ $= x\,e^{ax} \frac{1}{f'(a)}, f(a) = 0, f'(a) \neq 0$ $= x^2 e^{ax} \frac{1}{f''(a)}, f'(a) = 0, f''(a) \neq 0$
2.	x^n	$\text{P.I.} = \frac{1}{f(D)} x^n$ $= [f(D)]^{-1} x^n$ Expand $[f(D)]^{-1}$ and then operate.

3.	$\sin ax$ (or) $\cos ax$	P.I. $= \dfrac{1}{f(D)} \cos ax$ (or) $\sin ax$ Replace D^2 by $-a^2$
4.	$e^{ax}\ \phi(x)$	P.I. $= \dfrac{1}{f(D)}\ e^{ax}\ \phi(x)$ $= e^{ax}\ \dfrac{1}{f(D+a)}\ \phi(x)$

I. PROBLEMS BASED ON R.H.S. OF THE GIVEN DIFFERENTIAL EQUATION IS ZERO.

Example 2.1.1. Solve $(D^2 - 5D + 6)\,y = 0$

Solution : Given : $(D^2 - 5D + 6)\,y = 0$

The auxiliary equation is $m^2 - 5m + 6 = 0$

$$\text{i.e.,} \quad (m-3)(m-2) = 0$$

$$\text{i.e.,} \quad m = 2,\ m = 3$$

$$\therefore \text{C.F.} = Ae^{2x} + Be^{3x}$$

$\therefore$ The general solution is given by

$$y = Ae^{2x} + Be^{3x}$$

Example 2.1.2. Solve $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 13y = 0.$

Solution : Given : $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 13y = 0$

$$\text{i.e.,} \quad (D^2 - 6D + 13)\,y = 0$$

The auxiliary equation is $m^2 - 6m + 13 = 0$

$$\text{i.e.,} \quad m = \frac{6 \pm \sqrt{36 - 52}}{2}$$

$$= \frac{6 \pm \sqrt{-16}}{2}$$

$$m = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

The roots are imaginary and occur in conjugate pairs.

Hence the solution is $y = e^{3x}$ (A cos $2x$ + B sin $2x$).

Example 2.1.3. Solve $(D^2 + 1)\, y = 0$ given $y(0) = 0$, $y''(0) = 1$

[AU, April 1996]

Solution : Given $(D^2 + 1)\, y = 0$

A.E is $m^2 + 1 = 0$

$$m = \pm i$$

$$y = A \cos x + B \sin x$$

i.e., $y(x) = A \cos x + B \sin x$

$$y(0) = A = 0$$

$$y'(x) = -A \sin x + B \cos x$$

$$y'(0) = B = 1$$

$$\therefore A = 0, B = 1$$

i.e., $y = (0)\, (\cos x) + \sin x$

$$y = \sin x$$

Type I

If $f(x) = e^{ax}$, then the particular integral is given by

$$P.I. = \frac{1}{\phi(D)} \cdot e^{ax}$$

$$= \frac{1}{\phi(a)} e^{ax}, \text{ provided } \phi(a) \neq 0$$

If $\phi(a) = 0$, *then* $P.I. = \frac{1}{\phi(D)} \cdot e^{ax}$

$$= x \cdot \frac{1}{\phi'(D)} e^{ax}$$

where $\phi'(D)$ *means derivative of* $\phi(D)$ *w.r.t. 'D'.*

$$= x \cdot \frac{1}{\phi'(a)} e^{ax}, \text{ provided } \phi'(a) \neq 0.$$

If $\phi'(a) = 0$, *then* $P.I. = x^2 \cdot \frac{1}{\phi''(D)} e^{ax}$

$$= x^2 \cdot \frac{1}{\phi''(a)} e^{ax}, \text{ provided } \phi''(a) \neq 0.$$

If $\phi''(a) = 0$, *then* $P.I. = x^3 \cdot \frac{1}{\phi'''(D)} e^{ax}$

$$= x^3 \cdot \frac{1}{\phi'''(a)} e^{ax}, \text{ provided } \phi'''(a) \neq 0$$

and so on.

EXAMPLE 1

Solve the equation $(D^2 - 4D + 13)\, y = e^{2x}$.

Solution

The given differential equation is $(D^2 - 4D + 13)y = e^{2x}$.

Auxiliary equation is $m^2 - 4m + 13 = 0$.

$$m = \frac{4 \pm \sqrt{16-52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

Complementary function $y_c = e^{2x}(\text{A}\cos 3x + \text{B}\sin 3x)$

Particular Integral $y_p = \dfrac{1}{\text{D}^2 - 4\text{D} + 13} e^{2x}$

$$= \frac{1}{4-8+13} e^{2x} \quad \text{(Replacing D by 2)}$$

$$= \frac{1}{9} e^{2x}$$

$\therefore$ The complete solution is $y = y_c + y_p$

$$= e^{2x}(\text{A}\cos 3x + \text{B}\sin 3x) + \frac{e^{2x}}{9}$$

■ EXAMPLE 2 ■

Solve the equation $(D^3 - 3D^2 + 4D - 2)\,y = e^x$.

● Solution

The given equation is

$$m^3 - 3m^2 + 4m - 2 = 0$$

$$(m-1)(m^2 - 2m + 2) = 0$$

$$m = 1 \text{ or } m = 1 \pm i$$

The Complementary function $y_c = \text{A}e^x + e^x(\text{B}\cos x + \text{C}\sin x)$

Particular Integral $y_p = \dfrac{1}{\text{D}^3 - 3\text{D}^2 + 4\text{D} - 2} e^x$

$$= \frac{1}{(1)^3 - 3(1)^2 + 4(1) - 2} e^x$$

[Replacing D by 1]

$$= \frac{1}{1-3+4-2} e^x = \frac{1}{0} e^x$$

$$= \frac{x}{3\text{D}^2 - 6\text{D} + 4} e^x$$

$$= \frac{1}{3-6+4} e^x \quad \text{[Replacing D by 1]}$$

$$= x e^x$$

The complete solution is $y = y_c + y_p$

$$= \text{A}e^x + e^x(\text{B}\cos x + \text{C}\sin x) + xe^x$$

3 . ***Solve the equation*** $(D^2 + 2D + 1)\, y = e^{-x} + 3.$ **[Apr. 90]**

• Solution

The given differential equation is

$$(D^2 + 2D + 1)\, y = e^{-x} + 3$$

Auxiliary equation is $m^2 + 2m + 1 = 0$

$$(m+1)(m+1) = 0$$

$$m = -1, -1$$

Complementary Function $y_c = (Ax + B)\, e^{-x}$

The Particular Integral $y_p = y_{p_1} + y_{p_2}$

$$y_{p_1} = \frac{1}{D^2 + 2D + 1}\, e^{-x}$$

$$= \frac{1}{(-1)^2 + 2(-1) + 1}\, e^{-x}$$

[Replacing D by –1]

$$= \frac{1}{1 - 2 + 1}\, e^{-x}$$

$$= \frac{x}{2D + 2}\, e^{-x} \qquad [\because \text{Dr. is } 0]$$

$$= \frac{x}{2(-1) + 2}\, e^{-x} \qquad [\text{Replacing D by } -1]$$

$$= \frac{x}{-2 + 2}\, e^{-x} = \frac{x^2}{2}\, e^{-x} \qquad [\because \text{Dr. is } 0]$$

$$y_{p_2} = \frac{1}{D^2 + 2D + 1} \cdot 3e^{0x} \qquad [\because e^{0x} = 1]$$

$$= \frac{1}{(0)^2 + 2(0) + 1}\, 3e^{0x} \; (\text{Replacing D by } 0)$$

$$= 3$$

The complete solution is $y = y_c + y_{p_1} + y_{p_2} = (Ax + B)e^{-x} + \frac{x^2}{2}\, e^{-x} + 3$

Type II

If $f(x) = \sin ax$ or $\cos ax$, then Particular Integral is given by

$$P.I. = \frac{1}{\phi(D)} \sin ax \text{ (or) } \cos ax$$

In $\phi(D)$ replace D^2 by $-a^2$, provided $\phi(D) \neq 0$.

If $\phi(D) = 0$, when we replace D^2 by $-a^2$ then

$$P.I. = x \cdot \frac{1}{\phi'(D)} \sin ax \text{ (or) } \cos ax.$$

Again replace D^2 by $-a^2$ in $\phi'(D)$ provided $\phi'(D) \neq 0$, then

$$P.I. = x^2 \cdot \frac{1}{\phi''(D)} \sin ax \text{ (or) } \cos ax$$

and this process may be repeated if $\phi''(D) = 0$ and so on.

NOTE : If $f(x) = \sin(ax + b)$ or $\cos(ax + b)$ then the method of finding particular integral is the same as explained earlier.

For example, $$P.I. = \frac{1}{D^2 + 4}\{\sin(3x + 2) \text{ (or) } \cos(3x + 2)\}$$

Replacing D^2 by -9 $$= \frac{1}{-9 + 4}\{\sin(3x + 2) \text{ (or) } \cos(3x + 2)\}$$

$$= \frac{-1}{5}\{\sin(3x + 2) \text{ (or) } \cos(3x + 2)\}$$

If $\phi(D) = 0$, then

$$P.I. = \frac{1}{D^2 + 4}\{\sin(2x + 3) \text{ (or) } \cos(2x + 3)\}$$

Replacing D^2 by -2^2,

$$= \frac{1}{-4 + 4}\{\sin(2x + 3) \text{ (or) } \cos(2x + 3)\}$$

$$= x\frac{1}{2D}\{\sin(2x + 3) \text{ (or) } \cos(2x + 3)\}$$

$$= x \cdot \frac{1}{2}\left[\frac{-\cos(2x + 3)}{2} \text{ or } \frac{\sin(2x + 3)}{2}\right]$$

■ **EXAMPLE 1** ■

Solve the equation $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \sin 3x.$

● **Solution**

The given differential equation can be written in operator form as $(D^2 + 3D + 2)\, y = \sin 3x.$

$$\text{Auxiliary equation is} \quad m^2 + 3m + 2 = 0$$

$$(m+1)(m+2) = 0$$

$$m = -1, -2$$

$$\text{Complementary function } y_c = Ae^{-x} + Be^{-2x}$$

$$\text{Particular integral } y_p = \frac{1}{D^2 + 3D + 2} \sin 3x$$

$$= \frac{1}{-(3)^2 + 3D + 2} \sin 3x \qquad [\text{Replacing } D^2 \text{ by } -3^2]$$

$$= \frac{1}{3D - 7} \sin 3x = \frac{3D+7}{(3D-7)(3D+7)} \sin 3x$$

$$= \frac{3D+7}{9D^2 - 49} \sin 3x \qquad [\text{Replacing } D^2 \text{ by } -3^2]$$

$$= \frac{3D+7}{-81-49} \sin 3x$$

$$= \frac{3D(\sin 3x) + 7\sin 3x}{-130} = \frac{9\cos 3x + 7\sin 3x}{-130}$$

Complete solution is $y = y_c + y_p$

$$= Ae^{-x} + Be^{-2x} - \frac{1}{130}(9\cos 3x + 7\sin 3x)$$

2 Solve the equation $\frac{d^2y}{dx^2} + 4y = \sin 2x.$

• Solution

The given differential equation can be written as $(D^2 + 4)y = \sin 2x.$

$$\text{Auxiliary equation is } m^2 + 4 = 0$$

$$m = \pm 2i$$

Complementary function $y_c = A\cos 2x + B\sin 2x$

$$\text{Particular integral } y_p = \frac{1}{D^2+4}\sin 2x$$

$$= \frac{1}{-2^2+4}\sin 2x = \frac{1}{0}\sin 2x$$

$$= \frac{x}{2D}\sin 2x \qquad [\because \text{ denominator is zero}]$$

$$= \frac{x}{2}\cdot\frac{1}{D}(\sin 2x) = \frac{x}{2}\,\frac{-\cos 2x}{2}$$

$$= -\frac{x}{4}\cos 2x$$

Complete solution is $y = y_c + y_p$

$$= A\cos 2x + B\sin 2x - \frac{x}{4}\cos 2x$$

■ EXAMPLE 5 ■

3. *Solve the equation $(D^2 + 16)y = \cos 4x$*

• Solution

The auxiliary equation is $m^2 + 16 = 0$ i.e., $m = \pm 4i$.

$\therefore$ Complementary function $y_c = A\cos 4x + B\sin 4x$.

$$\text{Particular integral } y_p = \frac{1}{D^2+16}\cos 4x$$

$$= \frac{1}{-4^2+16}\cos 4x \qquad \text{[Replace } D^2 \text{ by } -4^2]$$

$$= \frac{x}{2D}\cos 4x$$

$$= \frac{x}{2}\cdot\frac{1}{D}(\cos 4x)$$

$$= \frac{x}{2}\cdot\frac{\sin 4x}{4} = \frac{x}{8}\sin 4x$$

Complete solution is $y = y_c + y_p$

$$= A\cos 4x + B\sin 4x + \frac{x}{8}\sin 4x$$

■ EXAMPLE 6 ■

4. *Solve the equation $(D^2 + 6D + 8)y = e^{-2x} + \cos^2 x$*

• Solution

The given differential equation is

$(D^2 + 6D + 8)y = e^{-2x} + \cos^2 x$

Auxiliary equation is $m^2 + 6m + 8 = 0$

$$(m+4)(m+2) = 0$$

$$m = -4, -2$$

Complementary function is $y_c = Ae^{-4x} + Be^{-2x}$

Particular integral $y_p = y_{p_1} + y_{p_2}$

$$y_{p_1} = \frac{1}{D^2+6D+8}e^{-2x} = \frac{1}{(-2)^2+6(-2)+8}e^{-2x}$$

$$= \frac{1}{4-12+8}e^{-2x} \qquad [\because \text{Dr. is zero}]$$

$$= \frac{x}{2D+6}e^{-2x} = \frac{x}{2(-2)+6}e^{-2x} = \frac{x}{2}e^{-2x}$$

$$y_{p_2} = \frac{1}{D^2+6D+8}\cos^2 x = \frac{1}{D^2+6D+8}\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{D^2+6D+8}\left(\frac{1}{2}\right) + \frac{1}{D^2+6D+8}\left(\frac{\cos 2x}{2}\right)$$

$$= \frac{1}{D^2+6D+8}\left(\frac{1}{2}e^{0x}\right) + \frac{1}{2}\cdot\frac{1}{-2^2+6D+8}\cos 2x$$

$$= \frac{1}{2}\cdot\frac{1}{8} + \frac{1}{2}\cdot\frac{1}{6D+4}\cos 2x = \frac{1}{16} + \frac{1}{4}\,\frac{1}{3D+2}\cos 2x$$

$$= \frac{1}{16} + \frac{1}{4}\,\frac{1}{3D+2}\cos 2x = \frac{1}{16} + \frac{1}{4}\,\frac{3D-2}{(3D+2)(3D-2)}\cos 2x$$

$$= \frac{1}{16} + \frac{1}{4}\,\frac{3D-2}{9D^2-4}\cos 2x = \frac{1}{16} + \frac{1}{4}\,\frac{3D(\cos 2x) - 2\cos 2x}{9(-4)-4}$$

$$= \frac{1}{16} + \frac{1}{4}\,\frac{-6\sin 2x - 2\cos 2x}{-40} = \frac{1}{16} + \frac{1}{80}(3\sin 2x + \cos 2x)$$

Complete solution is $y = y_c + y_{p_1} + y_{p_2}$

$$= A\,e^{-4x} + B\,e^{-2x} + \frac{x}{2}\,e^{-2x} + \frac{1}{16} + \frac{1}{80}(3\sin 2x + \cos 2x)$$

Type III

If $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$

where $a_0x^n + a_1x^{n-1} + \ldots + a_n$ *is a pure algebraic function then*

$$P.I = \frac{1}{\phi(D)}(a_0x^n + a_1x^{n-1} + \ldots + a_n)$$

$$= [\phi(D)]^{-1}(a_0x^n + a_1x^{n-1} + \ldots + a_n)$$

Expand $[\phi(D)]^{-1}$ *by using Binomial theorem in ascending powers of D and then operate on* $a_0x^n + a_1x^{n-1} + \ldots + a_n$

NOTE 1 : The following formulas are very important :

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

NOTE 2 : $\frac{1}{D} f(x)$ *is equal to the integral of* $f(x)$ *one time w.r.t 'x'.*

$\frac{1}{D^2} f(x)$ *is equal to the integral of* $f(x)$ *two times w.r.t 'x' and so on.*

■ EXAMPLE 1 ■

1. *Solve the equation* $(D^2 + 5D + 4)y = x^2 + 7x + 9$

● Solution

The given equation is $(D^2 + 5D + 4)y = x^2 + 7x + 9$.

Auxiliary equation is $m^2 + 5m + 4 = 0$

i.e., $(m+4)(m+1) = 0$

i.e., $m = -4, -1$.

Complementary function is $y_c = Ae^{-4x} + Be^{-x}$.

Particular integral is $y_p = \frac{1}{D^2 + 5D + 4}(x^2 + 7x + 9)$

$$= \frac{1}{4\left[1 + \frac{D^2 + 5D}{4}\right]}(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 + \left(\frac{D^2 + 5D}{4}\right)\right]^{-1}(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 - \left(\frac{D^2 + 5D}{4}\right) + \left(\frac{D^2 + 5D}{4}\right)^2 - \ldots\right](x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 - \frac{D^2}{4} - \frac{5D}{4} + \frac{D^4}{16} + \frac{25D^2}{16} + \frac{10D^3}{16} - \ldots\right](x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[x^2+7x+9-\frac{1}{4}D^2(x^2+7x+9)-\frac{5}{4}D(x^2+7x+9)+\frac{25}{16}D^2(x^2+7x+9)\right]$$

[$\because$ Third, Fourth, ... etc derivatives are zero]

$$= \frac{1}{4}\left[x^2+7x+9-\frac{1}{2}-\frac{10x}{4}-\frac{35}{4}+\frac{50}{16}\right]$$

$$= \frac{1}{4}\left[x^2+\frac{9x}{2}+\frac{23}{8}\right]$$

$$= \frac{1}{32}(8x^2+36x+23) \text{ (simplifying)}$$

$\therefore$ complete solution of the given differential equation is $y = y_c + y_p$

$$= Ae^{-4x} + Be^{-x} + \frac{1}{32}(8x^2+36x+23)$$

2. **Solve the equation** $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2 + 3.$

• Solution

The given equation can be written as

$$(D^2 - 5D + 6)y = x^2 + 3$$

Auxiliary equation is $m^2 - 5m + 6 = 0$

$$(m-3)(m-2) = 0$$

$$m = 3, 2$$

Complementary function is $y_c = Ae^{3x} + Be^{2x}$

Particular integral is $y_p = \frac{1}{D^2 - 5D + 6}(x^2 + 3)$

$$= \frac{1}{6\left[1 + \left(\frac{D^2 - 5D}{6}\right)\right]}(x^2 + 3) = \frac{1}{6}\left[1 + \left(\frac{D^2 - 5D}{6}\right)\right]^{-1}(x^2 + 3)$$

$$= \frac{1}{6}\left[1 - \left(\frac{D^2 - 5D}{6}\right) + \left(\frac{D^2 - 5D}{6}\right)^2 - \ldots\right](x^2 + 3)$$

$$= \frac{1}{6}\left[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{D^4}{36} + \frac{25D^2}{36} - \frac{10D^3}{36} - \ldots\right](x^2 + 3)$$

$$= \frac{1}{6}\left[x^2 + 3 - \frac{2}{6} + \frac{10x}{6} + \frac{50}{36}\right]$$

[$\because D^3(x^2 + 3), D^4(x^2 + 3), \ldots$ etc are equal to zero]

$$= \frac{1}{6}\,\frac{36x^2 + 60x + 146}{36}$$

$\therefore$ Complete solution is $y = y_c + y_p$

$$y = A\,e^{3x} + Be^{2x} + \frac{1}{108}(18x^2 + 30x + 73)$$

3. ***Solve the equation $D^2 (D^2 + 4) y = 96x^2$.***

• Solution

The given equation is

$$D^2 (D^2 + 4) y = 96x^2$$

Auxiliary equation is $m^2 (m^2 + 4) = 0$

i.e., $m^2 = 0$ (or) $m^2 + 4 = 0$

i.e., $m = 0, 0$ (or) $m^2 = -4$ $(m = \pm\sqrt{-4} = \pm 2i)$

i.e., $m = 0, 0, 2i, -2i$

$\therefore$ Complementary function

$$y_c = (Ax + B)e^{0x} + e^{0x}(C\cos 2x + D\sin 2x)$$

$$= (Ax + B) + C\cos 2x + D\sin 2x$$

Particular Integral is $y_p = \dfrac{1}{D^2(D^2+4)} 96x^2$

$$= \frac{1}{4D^2\left[1 + \frac{D^2}{4}\right]} 96x^2$$

$$= \frac{96}{4D^2}\left(1 + \frac{D^2}{4}\right)^{-1} x^2$$

$$= \frac{24}{D^2}\left[1 - \frac{D^2}{4} + \frac{D^4}{16} - \ldots\right] x^2$$

$$= \frac{24}{D^2}\left[x^2 - \frac{D^2}{4}(x^2)\right]$$

[$\because$ Fourth, sixth, etc. derivatives are zero]

$$= \frac{24}{D^2}\left[x^2 - \frac{1}{2}\right]$$

$$= 24\left[\frac{1}{D^2}(x^2) - \frac{1}{2D^2}\right]$$

$$= 24 \cdot \frac{x^4}{12} - 12\frac{x^2}{2} \qquad \left[\because \frac{1}{D^2}(x^2) \text{ means integrate } x^2 \text{ twice}\right]$$

$$= 2x^4 - 6x^2$$

$\therefore$ The complete solution is $y = y_c + y_p$

$$= Ax + B + C\cos 2x + D\sin 2x + 2x^4 - 6x^2$$

Type IV : Given ODE : $[\phi(D)]y = f(x)$, with $f(x) = e^{ax} \cdot X(x)$.

If $f(x) = e^{ax} X$ where X is $\sin ax$ or $\cos ax$ or x^n, then

$$P.I. = \frac{1}{\phi(D)} \cdot e^{ax} X = e^{ax} \cdot \frac{1}{\phi(D+a)} \cdot X$$

Here $\dfrac{1}{\phi(D+a)} X$ can be evaluated by using any one of the first three types.

■ EXAMPLE 1 ■

Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-x}\sin 2x.$

● Solution

The given differential equation can be written as

$$(D^2 + 4D + 4)y = e^{-x}\sin 2x$$

The auxiliary equation is $m^2 + 4m + 4 = 0$

$$(m+2)(m+2) = 0$$

i.e., $m = -2, -2$

$$\therefore \text{ C.F } = y_c = (Ax + B)\, e^{-2x}$$

$$\text{P.I.} = y_p = \frac{1}{D^2 + 4D + 4} e^{-x}\sin 2x$$

$$= e^{-x}\frac{1}{(D-1)^2 + 4(D-1) + 4}\sin 2x \quad \text{[Replacing D by D} -1]$$

$$= e^{-x}\frac{1}{D^2 + 2D + 1}\sin 2x$$

$$= e^{-x}\frac{1}{-4 + 2D + 1}\sin 2x \qquad \text{[Replacing } D^2 \text{ by } -2^2]$$

$$= e^{-x}\frac{1}{2D-3}\sin 2x$$

$$= e^{-x}\frac{(2D+3)}{(2D-3)(2D+3)}\sin 2x \qquad \text{multiply and divide by } 2D+3$$

$$= e^{-x}\frac{(2D+3)\sin 2x}{(4D^2 - 9)}$$

$$= e^{-x}\frac{2D(\sin 2x) + 3\sin 2x}{4(-4) - 9} \qquad \text{[Replacing } D^2 \text{ by } -2^2]$$

$$= e^{-x}\frac{(4\cos 2x + 3\sin 2x)}{-25}$$

$\therefore$ The complete solution is $y = y_c + y_p$

$$= (Ax + B)\, e^{-2x} - \frac{1}{25} e^{-x}(4\cos 2x + 3\sin 2x)$$

■ EXAMPLE 2 ■

Solve $(D^2 + 9)y = (x^2 + 1)\, e^{3x}.$

● Solution

The auxiliary equation is $m^2 + 9 = 0$

$$m^2 = -9 \text{ (or) } m = \pm\sqrt{-9}$$

i.e., $m = \pm 3i.$

$\therefore$ The complementary function is $y_c = A\cos 3x + B\sin 3x.$

To find particular integral :

$$\text{Particular integral } y_p = \frac{1}{D^2+9}(x^2+1)\ e^{3x}$$

$$= e^{3x}\frac{1}{(D+3)^2+9}(x^2+1) \qquad \text{(Replacing D by D + 3)}$$

$$= e^{3x}\frac{1}{D^2+6D+18}(x^2+1)$$

$$= e^{3x}\frac{1}{18\left(1+\frac{D^2+6D}{18}\right)}(x^2+1)$$

$$= \frac{e^{3x}}{18}\left[1+\left(\frac{D^2+6D}{18}\right)\right]^{-1}(x^2+1)$$

$$= \frac{e^{3x}}{18}\left[1-\left(\frac{D^2+6D}{18}\right)+\left(\frac{D^2+6D}{18}\right)^2-\dots\right](x^2+1)$$

$$= \frac{e^{3x}}{18}\left[1-\frac{D^2}{18}-\frac{D}{3}+\frac{D^4}{324}+\frac{D^2}{9}+\frac{D^3}{27}-\dots\right](x^2+1)$$

$$= \frac{e^{3x}}{18}\left[x^2+1-\frac{2}{18}-\frac{2x}{3}+\frac{2}{9}\right]$$

[$\because$ Third, Fourth, ... etc derivatives of (x^2+1) is zero]

$$= \frac{e^{3x}}{18}\left[x^2-\frac{2x}{3}+\frac{10}{9}\right]$$

$\therefore$ The Complete solution is $y = y_c + y_p$

$$= A\cos 3x + B\sin 3x + \frac{e^{3x}}{18}\left(x^2-\frac{2x}{3}+\frac{10}{9}\right)$$

■ EXAMPLE 3 ■

Solve $(D^2+4D+3)y = e^x\cos 2x - \cos 3x$.

● Solution

The auxiliary equation is $m^2+4m+3=0$

$$m^2+4m+3 = 0$$

i.e., $(m+1)(m+3) = 0$

i.e., $m = -1$ or -3

$\therefore$ The complementary function is

$$y_c = A\,e^{-x} + Be^{-3x}.$$

$$\text{Particular integral} \quad y_p = \frac{1}{D^2+4D+3}(e^x\cos 2x - \cos 3x)$$

$$= \frac{1}{D^2+4D+3}e^x\cos 2x - \frac{1}{D^2+4D+3}\cos 3x$$

$$= y_{p_1} - y_{p_2}$$

$$\text{Now}\quad y_{p_1} = \frac{1}{D^2+4D+3}\, e^x \cos 2x$$

$$= e^x \frac{1}{(D+1)^2+4(D+1)+3} \cos 2x \qquad \text{[Replacing D by D + 1]}$$

$$= e^x \frac{1}{D^2+6D+8} \cos 2x$$

$$= e^x \frac{1}{-4+6D+8} \qquad \text{[Replacing } D^2 \text{ by } -2^2\text{]}$$

$$= e^x \frac{1}{6D+4} \cos 2x = \frac{e^x}{2} \frac{1}{(3D+2)} \cos 2x$$

$$= \frac{e^x}{2} \frac{(3D-2)}{(3D+2)(3D-2)} \cos 2x$$

$$= \frac{e^x}{2} \frac{3D-2}{9D^2-4} \cos 2x$$

$$= \frac{e^x}{2} \frac{3D-2}{9(-4)-4} \cos 2x \qquad \text{[Replacing } D^2 \text{ by } -4\text{]}$$

$$= \frac{e^x}{2} \frac{3D(\cos 2x) - 2\cos 2x}{-40}$$

$$= \frac{e^x}{-80}(-6\sin 2x - 2\cos 2x)$$

$$= \frac{e^x}{40}(3\sin 2x + \cos 2x)$$

$$y_{p_2} = \frac{1}{D^2+4D+3} \cos 3x$$

$$= \frac{1}{-(3)^2+4D+3} \cos 3x$$

$$= \frac{1}{4D-6} \cos 3x$$

$$= \frac{4D+6}{(4D-6)(4D+6)} \cos 3x$$

$$= \frac{4D+6}{16D^2-36} \cos 3x$$

$$= \frac{4D(\cos 3x) + 6\cos 3x}{16(-9)-36}$$

$$= \frac{-12\sin 3x + 6\cos 3x}{-144-36}$$

$$= \frac{1}{180}(12\sin 3x - 6\cos 3x)$$

$\therefore$ The complete solution is $y = y_c + y_p$

$$= y_c + y_{p_1} - y_{p_2}$$

$$= Ae^{-x} + Be^{-3x} + \frac{e^x}{40}(3\sin 2x + \cos 2x) - \frac{1}{30}(2\sin 3x - \cos 3x)$$

■ EXAMPLE 4 ■

Solve $(D^2 - 2D + 1)y = x^2 e^{3x}$.

Solution

Auxiliary equation is $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$\therefore \text{C.F.} = (c_1 + c_2 x)\, e^x$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1}\, x^2 e^{3x}$$

$$= e^{3x} \frac{1}{(D+3)^2 - 2(D+3) + 1}\, x^2$$

$$= e^{3x} \frac{1}{D^2 + 4D + 4}\, x^2$$

$$= e^{3x} \frac{1}{4\left[1 + \left(\frac{D^2 + 4D}{4}\right)\right]} \cdot x^2$$

$$= \frac{e^{3x}}{4}\left[1 + \left(\frac{D^2 + 4D}{4}\right)\right]^{-1} x^2$$

$$= \frac{e^{3x}}{4}\left[1 - \left(\frac{D^2 + 4D}{4}\right) + \left(\frac{D^2 + 4D}{4}\right)^2 \ldots\right] x^2$$

$$= \frac{e^{3x}}{4}\left[1 - \frac{D^2}{4} - D + D^2\right] x^2$$

$$= \frac{e^{3x}}{4}\left[x^2 + \frac{6}{4} - 2x\right]$$

Complete solution is $y = \text{C.F.} + \text{P.I.}$

$$= (c_1 + c_2 x)\, e^x + \frac{e^{3x}}{4}\left[x^2 - 2x + \frac{3}{2}\right]$$

■ **EXAMPLE** 5

Solve $[D^2 - 4D + 4]\,y = x^3 e^{2x},\ D \equiv \dfrac{d}{dx}$.

• Solution

$$\text{Auxiliary equation is} \quad m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$\therefore \text{ Complementary function is} \quad \text{C.F.} = (c_1 x + c_2)\, e^{2x}$$

$$\text{The particular integral is} \quad \text{P.I} = \frac{1}{D^2 - 4D + 4} \cdot x^3 e^{2x}$$

$$= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^3$$

$$= e^{2x} \frac{1 \cdot x^3}{D^2} = e^{2x} \cdot \frac{x^5}{20}$$

$$\therefore \text{ Complete solution is} \quad y = \text{C.F} + \text{P.I}$$

$$= (c_1 x + c_2) + \frac{e^{2x} \cdot x^5}{20}$$

Example 6: **Find the particular integral of $(D^2 + 1)\, y = xe^x$**

[AU, Nov. 2001]

Solution : P.I. $= \dfrac{1}{D^2+1} x e^x = e^x \dfrac{1}{(D+1)^2+1} x$

$$= e^x \frac{1}{[D^2+2D+1]+1} x = e^x \frac{1}{D^2+2D+2} x$$

$$= \frac{e^x}{2} \frac{1}{\left[1+\dfrac{D^2+2D}{2}\right]} x = \frac{e^x}{2}\left[1+\frac{D^2+2D}{2}\right]^{-1} x$$

$$= \frac{e^x}{2}\left[1-\left(\frac{D^2+2D}{2}\right)+\left[\frac{D^2+2D}{2}\right]^2-\ldots\right] x$$

$$= \frac{e^x}{2}\left[x-\left(\frac{0+2}{2}\right)\right] \quad [\text{since } D^n(x) = 0, \text{ when } n = 2, 3, \ldots]$$

$$= \frac{e^x}{2}[x-1]$$

Example 2.1.37. Solve $(D+2)^2 y = e^{-2x} \sin x$

[AU, April 1996]

Solution : Given $(D+2)^2 y = e^{-2x} \sin x$

A.E. is $(m+2)^2 = 0$

$m = -2, -2$

C.F. $= (Ax+B)\, e^{-2x}$

P.I. $= \dfrac{1}{(D+2)^2} e^{-2x} \sin x$

$= e^{-2x} \dfrac{1}{(D-2+2)^2} \sin x$ [Replace D by D – 2]

$= e^{-2x} \dfrac{1}{D^2} \sin x = e^{-2x} \dfrac{1}{-1} \sin x$ [Replace D^2 by -1^2]

$= -e^{-2x} \sin x$

y = C.F + P.I.

$= (Ax+B)\, e^{-2x} - e^{-2x} \sin x$

Example 2.1.38. Obtain the P.I. of $(D^2 - 2D + 1)\, y = e^x (3x^2 - 2)$

[AU, May 2001]

Solution : Given $(D^2 - 2D + 1)\, y = e^x (3x^2 - 2)$

P.I. $= \dfrac{1}{(D-1)^2} e^x [3x^2 - 2]$

$= e^x \dfrac{1}{[D+1-1]^2} [3x^2 - 2]$ [Replace D by D + 1]

$= e^x \dfrac{1}{D^2} [3x^2 - 2]$

$= e^x \dfrac{1}{D^2} [3x^2 - 2] = e^x \dfrac{1}{D} \left[\dfrac{3x^3}{3} - 2x\right] = e^x \dfrac{1}{D} [x^3 - 2x]$

$= e^x \left[\dfrac{x^4}{4} - \dfrac{2x^2}{2}\right] = e^x \left[\dfrac{x^4}{4} - x^2\right]$

Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$ **[AU, April 2002]**

Solution : Given : $[D^2 + 4D + 5]y = -2\cosh x$

The A.E is $m^2 + 4m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

$$\text{C.F.} = e^{-2x}[A\cos x + B\sin x]$$

$$\text{P.I} = \frac{1}{D^2 + 4D + 5} - 2\cosh x = -2\frac{1}{D^2 + 4D + 5}\left[\frac{e^x + e^{-x}}{2}\right]$$

$$= \frac{-1}{D^2 + 4D + 5}[e^x + e^{-x}]$$

$$= \frac{-1}{D^2 + 4D + 5}e^x + \frac{-1}{D^2 + 4D + 5}e^{-x}$$

$$= \left[\frac{-e^x}{1 + 4 + 5}\right] + \left[\frac{-e^{-x}}{1 - 4 + 5}\right] = \left[\frac{-e^x}{10}\right] + \left[\frac{-e^{-x}}{2}\right]$$

$$\therefore y = \text{C.F.} + \text{P.I.}$$

$$= e^{-2x}[A\cos x + B\sin x] - \frac{e^{x}}{10} - \frac{e^{-x}}{2}$$

❑ SIMULTANEOUS DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Consider the following equations

$$\left.\begin{aligned} a_1x + b_1y + c_1 &= 0 \\ a_2x + b_2y + c_2 &= 0 \end{aligned}\right\} \quad \ldots (1)$$

Equation (1) is known as simultaneous algebraic equations with two unknowns. Similarly we have simultaneous equations with three, four, *etc.* unknowns.

Now consider the following equations

$$\left.\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= a_1x + b_1y \end{aligned}\right\} \quad \ldots (2)$$

Solve $\frac{dx}{dt} + y = e^t, \ x - \frac{dy}{dt} = t.$

Solution

Put $\frac{d}{dt} = D$ in the given equations, we get,

$$Dx + y = e^t \quad \dots (1)$$
$$x - Dy = t \quad \dots (2)$$

Multiplying (1) by D, $D^2x + Dy = D(e^t) = e^t \quad \dots (3)$

$$x - Dy = t \quad \dots (4)$$

Adding (3) and (4), $(D^2 + 1)x = e^t + t \quad \dots (5)$

Equation (5) is a linear differential equation with constant coefficients.

$\therefore$ Solution of (5) is $x = \text{C.F.} + \text{P.I.}$

To find C.F.

Auxiliary equation is $m^2 + 1 = 0$

i.e., $m = \pm i.$

$\therefore \text{C.F.} = c_1 \cos t + c_2 \sin t$

To find P.I.

$$\text{P.I.} = \frac{1}{D^2+1}(e^t + t)$$
$$= \frac{1}{D^2+1} e^t + \frac{1}{D^2+1} t$$
$$= \frac{1}{2} e^t + (1 + D^2)^{-1} t \quad \textbf{(Replacing D by 1)}$$
$$= \frac{1}{2} e^t + (1 - D^2 + D^4 - \dots) t$$
$$= \frac{1}{2} e^t + t$$

$\therefore$ Solution of (5) is $x = c_1 \cos t + c_2 \sin t + \frac{e^t}{2} + t \quad \dots (6)$

Now to find y we have two methods.

Method *(i)* : From (1) we get

$$y = e^t - D(x)$$

$$y = e^t - \frac{d}{dt}(x) \qquad \dots (7)$$

Substituting (6) in (7) we get

$$y = e^t - \frac{d}{dt}\left[c_1 \cos t + c_2 \sin t + \frac{e^t}{2} + t\right]$$

$$y = e^t + c_1 \sin t - c_2 \cos t - \frac{e^t}{2} - 1$$

$$y = c_1 \sin t - c_2 \cos t + \frac{1}{2} e^t - 1 \qquad \dots (8)$$

$\therefore$ The solution of the given simultaneous differential equation is

$$x = c_1 \cos t + c_2 \sin t + \frac{e^t}{2} + t$$

$$y = \frac{e^t}{2} + c_1 \sin t - c_2 \cos t - 1.$$

❑ LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS (Cauchy's Homogeneous Linear Equation)

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = f(x) \qquad \ldots(1)$$

where $a_1, a_2, \ldots a_n$ are constants and $f(x)$ is a function of x is called a linear differential equation with variable coefficients.

Equation (1) can be reduced to a linear differential equation with constant co-efficients by putting the substitution

$$x = e^z \text{ (or) } z = \log x.$$

Here $z = \log x$

Now $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \cdot \frac{dy}{dz}$ $\Rightarrow \frac{dz}{dx} = \frac{d(\log x)}{dx} = \frac{1}{x}$]

i.e., $x \frac{dy}{dx} = \frac{dy}{dz}$ (or) $x \frac{dy}{dx} = D'y \qquad \ldots (2)$

i.e., $\boxed{D' = \frac{d}{dz}}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) \qquad \left[\because \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz}\right]$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) \qquad \text{[Applying product rule]}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} \qquad \text{[Multiply and divide by } dz\text{]}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2} \cdot \frac{1}{x} \qquad \left[\frac{dz}{dx} = \frac{1}{x}\right]$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

Notation:

$$D' = \frac{d}{dz}$$

$$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$= (D'^2 - D')\,y \text{ where } D' = \frac{d}{dz}$$

$$D'^2 = \frac{d^2}{dz^2}$$

$$\text{i.e.,} x^2 \frac{d^2y}{dx^2} = D'(D'-1)\,y \quad \ldots(3)$$

Similarly

$$x^3 \frac{d^3y}{dx^3} = D'(D'-1)(D'-2)\,y \quad \ldots(4)$$

$$x^4 \frac{d^4y}{dx^4} = D'(D'-1)(D'-2)(D'-3)\,y \quad \ldots(5)$$

and so on. Substituting (2), (3), (4), (5) and so on in (1) we get a differential equation with constant coefficients and can be solved by any one of the known methods.

■ EXAMPLE 1 ■

1. *Solve* $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \sin(\log x^2)$

Solution

The given differential equation can be written as

$$[x^2 D^2 + x D + 1]y = \sin(2\log x) \quad \dots (1)$$

$$\text{Put } x = e^z \quad \text{or} \quad z = \log x$$

$$\left.\begin{aligned} \therefore\ x D &\equiv D' \\ x^2 D^2 &\equiv D'(D'-1), \end{aligned}\right\} \quad \dots (2)$$

$$\text{where } D' \equiv \frac{d}{dz}$$

Substituting (2) in (1), we get

$$[D'(D'-1) + D' + 1]y = \sin(2z)$$

i.e.,

$$(D'^2 + 1)y = \sin(2z) \quad \dots (3)$$

Auxiliary equation of (3) is

$$m^2 + 1 = 0$$

i.e.,

$$m = \pm i$$

$$\therefore \text{Complementary function (C.F)} = c_1 \cos z + c_2 \sin z \quad \dots (4)$$

The particular integral $\text{P.I} = \frac{1}{D'^2 + 1} \sin 2z$

$$= \frac{1}{-4+1} \sin 2z$$

$$= \frac{-\sin 2z}{3} \qquad \ldots (5)$$

Therefore, complete solution is

$$y = \text{C.F} + \text{P.I}$$

$$= c_1 \cos z + c_2 \sin z - \frac{\sin 2z}{3}, \quad z = \log x$$

2. *Solve $(x^2 D^2 + x D + 4) y = \cos(\log x) + x \sin(\log x)$.*

Solution

The given differential equation is

$$[x^2 D^2 + x D + 4] y = \cos(\log x) + x \sin(\log x) \quad \dots (1)$$

$$\left.\begin{aligned} \text{Put } x &= e^z \text{ or } z = \log x \\ \therefore xD &\equiv D', \quad x^2 D^2 \equiv D'(D'-1) \end{aligned}\right\} \quad \dots (2)$$

Substituting (2) in (1), we get

$$[D'(D'-1) + D' + 4] y = \cos z + e^z \sin z$$

i.e.,
$$(D'^2 + 4) y = \cos z + e^z \sin z \quad \dots (3)$$

Auxiliary equation is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$\therefore$ Complementary function
$$\text{C.F} = (c_1 \cos 2z + c_2 \sin 2z) \quad \dots (4)$$

Now particular integral
$$\text{P.I} = \text{P.I}_1 + \text{P.I}_2 \quad \dots (5)$$

$$\text{P.I}_1 = \frac{1}{D'^2 + 4} \cos z$$

$$= \frac{1}{-1+4} \cos z = \frac{1}{3} \cos z \quad \dots (6)$$

$$\text{P.I}_2 = \frac{1}{D'^2 + 4} \cdot e^z \sin z$$

$$= e^z \frac{1}{(D'+1)^2 + 4} \sin z$$

$$= e^z \frac{1}{D'^2 + 2D' + 5} \sin z$$

$$= e^z \frac{1}{-1 + 2D' + 5} \sin z$$

$$= e^z \frac{1}{2D' + 4} \sin z$$

$$= \frac{e^z}{2} \cdot \frac{1}{D' + 2} \sin z$$

$$= \frac{e^z}{2} \cdot \frac{D' - 2}{D'^2 - 4} \sin z$$

$$= \frac{e^z}{2} \cdot \frac{\cos z - 2\sin z}{-5} \qquad \ldots (7)$$

Substituting (6) and (7) in (5), we get

$$\text{P.I} = \frac{1}{3}\cos z - \frac{e^z}{10}(\cos z - 2\sin z) \ldots (8)$$

$\therefore$ The complete solution is $\quad y = \text{C.F} + \text{P.I}$

i.e., $\quad y = (c_1 \cos 2z + c_2 \sin 2z) + \frac{1}{3}\cos z - \frac{e^z}{10}(\cos z - 2\sin z)$

where $\quad z = \log x$

1. *Solve :* $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = \cosh 2x$

● Solution

$$
\begin{aligned}
\text{A.E is} \quad m^2 + 2m + 1 &= 0 \\
(m+1)^2 &= 0 \\
m &= -1, -1 \\
\therefore \ \text{C.F} &= (c_1 x + c_2)\, e^{-x} \\
\text{P.I} &= \frac{1}{D^2 + 2D + 1} \cdot \cosh 2x \\
&= \frac{1}{D^2 + 2D + 1} \left(\frac{e^{2x} + e^{-2x}}{2} \right) \\
&= \frac{1}{2} \left[\frac{1}{D^2 + 2D + 1}\, e^{2x} + \frac{1}{D^2 + 2D + 1}\, e^{-2x} \right] \\
&= \frac{1}{2} \left[\frac{1}{4 + 4 + 1}\, e^{2x} + \frac{1}{4 - 4 + 1}\, e^{-2x} \right] \\
&= \frac{1}{2} \left[\frac{e^{2x}}{9} + e^{-2x} \right]
\end{aligned}
$$

$\therefore$ Complete solution is, $\quad y = \text{C.F} + \text{P.I}$

$$
= (c_1 x + c_2) + \frac{1}{2} \left(\frac{e^{2x}}{9} + e^{-2x} \right)
$$

2. *Solve :* $(D^2 + 5D + 6)\, y = e^{-7x} \cdot \sinh 3x$

● Solution

A.E is
$$m^2 + 5m + 6 = 0$$
$$(m+2)(m+3) = 0$$
$$m = -2, -3$$
$$\therefore \text{C.F} = c_1 e^{-2x} + c_2 e^{-3x}$$
$$\text{P.I} = \frac{1}{D^2 + 5D + 6} \cdot e^{-7x} \sinh 3x$$
$$= \frac{1}{D^2 + 5D + 6} \cdot e^{-7x} \left(\frac{e^{3x} - e^{-3x}}{2} \right)$$
$$= \frac{1}{2} \left[\frac{1}{D^2 + 5D + 6} e^{-4x} - \frac{1}{D^2 + 5D + 6} e^{-10x} \right]$$
$$= \frac{1}{2} \left[\frac{e^{-4x}}{16 - 20 + 6} - \frac{e^{-10x}}{10 - 50 + 6} \right]$$
$$= \frac{1}{2} \left[\frac{e^{-4x}}{2} + \frac{e^{-10x}}{34} \right]$$

$\therefore$ Complete solution is $\quad y = c_1 e^{-2x} + c_2 e^{-3x} + \dfrac{e^{-4x}}{4} + \dfrac{e^{-10x}}{68}$

4 Solve $\frac{d^2y}{dx^2} - 4y = x \sinh x$

● Solution

Auxiliary equation is $m^2 - 4 = 0$

$$m = \pm 2$$

$$\therefore \text{C.F} = c_1 e^{2x} + c_2 e^{-2x}$$

$$\text{P.I} = \frac{1}{D^2 - 4} x \sinh x$$

$$= \frac{1}{D^2 - 4} \cdot x \left(\frac{e^x - e^{-x}}{2} \right)$$

$$= \frac{1}{2} \left[\frac{1}{D^2 - 4} x e^x - \frac{1}{D^2 - 4} x e^{-x} \right]$$

$$= \frac{1}{2} \left[e^x \cdot \frac{1}{(D+1)^2 - 4} \cdot x - e^{-x} \cdot \frac{1}{(D-1)^2 - 4} \cdot x \right]$$

$$= \frac{1}{2} \left[e^x \cdot \frac{1}{D^2 + 2D - 3} - e^{-x} \cdot \frac{1}{D^2 - 2D - 3} \cdot x \right]$$

$$= \frac{1}{2} \left[\frac{e^x}{-3} \left\{ 1 - \left(\frac{2D}{3} + \frac{D^2}{3} \right) \right\}^{-1} \right] x$$

$$- \frac{1}{2} \left[\frac{e^{-x}}{-3} \left\{ 1 + \left(\frac{2D}{3} - \frac{D^2}{3} \right) \right\}^{-1} \right]$$

$$= -\frac{1}{6} \left[e^x \left(1 + \frac{2D}{3} + \dots \right) x - e^{-x} \left(1 - \frac{2D}{3} + \dots \right) x \right]$$

$$= -\frac{1}{6}\left[e^{x}\left(x+\frac{2}{3}\right)-e^{-x}\left(x-\frac{2}{3}\right)\right]$$

$$= -\frac{x}{3}\left(\frac{e^{x}-e^{-x}}{2}\right)-\frac{2}{9}\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$= \frac{-x}{3}\sinh x - \frac{2}{9}\cosh x$$

Hence the complete solution is,

$$y = \text{C.F} + \text{P.I}$$

$$= c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

5. ***Solve*** $(D^2 - 4D + 4)y = e^{2x}(x+1)^2$

● Solution

Auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$\text{C.F} = (c_1 x + c_2)e^{2x}$$

$$\text{P.I} = \frac{1}{D^2 - 4D + 4} e^{2x}(x+1)^2$$

$$= e^{2x} \cdot \frac{1}{(D+2)^2 - 4(D+2) + 4}(x+1)^2$$

$$= e^{2x} \cdot \frac{1}{D^2}(x+1)^2 = e^{2x} \cdot \frac{1}{D} \cdot \frac{(x+1)^3}{3} = e^{2x} \cdot \frac{(x+1)^4}{12}$$

$\therefore$ Complete solution is,

$$y = \text{C.F} + \text{P.I} = (c_1 x + c_2)e^{2x} + \frac{e^{2x}(x+1)^4}{12}$$

❑ EQUATIONS REDUCIBLE TO HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION (Legendre's Linear Equation)

The equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + K_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + K_n y = f(x) \quad \ldots (1)$$

can be reduced to homogeneous linear differential equation by putting the substitution

$$ax + b = e^z \quad \text{or} \quad z = \log(ax+b)$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{a}{ax+b} \cdot \frac{dy}{dz}$$

i.e.,

$$(ax+b)\frac{dy}{dx} = a D' y, \xrightarrow{} (2)$$

$$\Rightarrow (ax+b)Dy = aD'y.$$

$$\text{where } D' \equiv \frac{d}{dz}.$$

$$\boxed{\begin{aligned}(ax+b)D &= a \cdot D'.\\ (ax+b) \cdot D^2 &= a^2 \cdot D'(D'-1)\end{aligned}}$$

$$\text{Similarly } (ax+b)^2 \frac{d^2y}{dx^2} = a^2 D'(D'-1)y \quad \to (3)$$

$$(ax+b)^3 \frac{d^3y}{dx^3} = a^3 D'(D'-1)(D'-2) \quad \ldots (4)$$

and so on.

Substituting (2), (3) and (4) in (1), we get an ordinary linear differential equation with constant coefficients which can be solved easily.

1. *Solve :* $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4 \cos \log(1+x)$

● Solution

$$\text{Let } 1 + x = e^z \quad \text{(or)} \quad z = \log(1+x)$$

$$\therefore (1+x)\,D \equiv D'$$

$$(1+x)^2 D^2 \equiv D'(D'-1), \quad D' \equiv \frac{d}{dz}$$

$\therefore$ The given differential equation can be written as

$$[D'(D'-1) + D' + 1]\,y = 4 \cos z$$

$$(D'^2 + 1)\,y = 4 \cos z$$

A.E. is $\quad m^2 + 1 = 0$

$$m = \pm i$$

$$\therefore \text{C.F} = C_1 \cos z + C_2 \sin z$$

$$\text{P.I.} = \frac{1}{D'^2 + 1} 4 \cos z$$

Replace D'^2 by -1 $$= \frac{1}{-1+1} 4 \cos z$$

$$= 4 \frac{z}{2 D'} \cos z = 2z \cdot \sin z$$

$\therefore$ The solution is $$y = \text{C.F} + \text{P.I}$$
$$= C_1 \cos z + C_2 \sin z + 2z \sin z$$

where $z = \log(1+x)$

$4 \cdot \frac{1}{(D^2+1)} \cos z$ Put $D^2 = -1^2 = -1$

$4 \frac{1}{(-1+1)} \cos z = \frac{1}{0}$ form

$\therefore \frac{z}{D} \cos z$

$= 2z \cdot \frac{1}{D} \sin z$

$= 2z \cdot \sin z$

---

2. **Solve :** $(2x+3)^2 \dfrac{d^2y}{dx^2} - (2x+3)\dfrac{dy}{dx} - 12y = 6x$

● Solution

The given equation can be written as

$$\{ (2x+3)^2 D^2 - (2x+3) D - 12 \} y = 6x \qquad \ldots (1)$$

Put $$2x + 3 = e^z \text{ or } z = \log(2x+3)$$

$$(2x+3) D \equiv 2D' \qquad \ldots (2)$$

$$(2x+3)^2 D^2 \equiv 2^2 D'(D'-1) \qquad \ldots (3)$$

where $$D' \equiv \frac{d}{dz}$$

Substituting (2) and (3) in (1), we get

$$[4 D'(D'-1) - 2D' - 12] y = 3(e^z - 3)$$

$$(4D'^2 - 6D' - 12) y = 3e^z - 9$$

A.E. is $$4m^2 - 6m - 12 = 0$$

$$m = \frac{6 \pm \sqrt{36 + 192}}{8}$$

$$= \frac{6 \pm 2\sqrt{57}}{8} = \frac{3 \pm \sqrt{57}}{4}$$

$\therefore$ C.F $= C_1 e^{m_1 z} + C_2 e^{m_2 z}$, where

$$m_1 = \frac{3 + \sqrt{57}}{4}, \quad m_2 = \frac{-3 - \sqrt{57}}{2}$$

$$\text{P.I.} = \frac{1}{4D'^2 - 6D' - 12}(3\,e^z - 9\,e^{0z}) = \frac{1}{4-6-12}\,3\,e^z - \frac{9}{-12}$$

$$= \frac{-3}{14}\,e^z + \frac{3}{4}$$

$$\therefore\ y = \text{C.F} + \text{P.I}$$

$$= C_1\,e^{m_1 z} + C_2\,e^{m_2 z} - \frac{3}{14}\,e^z + \frac{3}{4}, \quad z = \log(1+x)$$

$$m_1 = \frac{3+\sqrt{57}}{4}, \quad m_2 = \frac{-3-\sqrt{57}}{4}$$

ORDINARY DIFFERENTIAL EQUATIONS

A differential equation of the type

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n y = X \quad - (1)$$

where $P_0, P_1, P_2, \dots P_n$ and X are function of x or constants is called a linear differential equation of nth order.

If the coefficients $P_0, P_1, P_2, \dots P_{n-1}, P_n$ of the derivatives are constants $a_0, a_1, a_2, \dots a_n$ then (1) reduces to

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = X \quad - (2)$$

this is a linear equations with constant coefficients.
The general solution of equation (2) is $y = C.F + P.I$.

Complementary function (C.F)

To find the C.F, we consider the equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \quad - (I)$$

If we use the differential operator symbols,

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}, \quad \dots, \quad D^n = \frac{d^n}{dx^n}.$$

Equation (2) becomes

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n) y = 0 \quad - (II)$$

First we're to find the auxillary equation (A.E) which is obtained by simply replacing D by m in the operator polynomial and then by equating it to zero.

ie $a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0$ — (III)

which is a polynomial equation in m of degree n.

By solving this we get n roots say $m_1, m_2, \dots, m_n$.

The solution of equation (I) depends on the nature of the roots.

The nature of roots of the A.E are as follows:

Case (i): when the roots $m_1, m_2, \dots m_n$ are all real and distinct then the solution ie C.F is given by $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$.

Case (ii): If any 2 roots are equal ie if $m_1 = m_2$ then the C.f is

$$y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}.$$

If any 3 roots are equal ie if $m_1 = m_2 = m_3$ then C.F is $y = (c_1 x^2 + c_2 x + c_3) e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$.

Case (iii): If any 2 roots are complex, let $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the C.f is

$$y = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right] + c_3 e^{m_3 x} + \dots$$

Case (iv): Two pairs of complex roots are equal, say, $m_1 = m_3 = \alpha + i\beta$ and $m_2 = m_4 = \alpha - i\beta$, then

$$y = e^{\alpha x}\left[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x\right] + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}.$$

Problems:

I.Q.B.

① Solve $(D^2-4D+3)y=0$.

Soln: A.E is $m^2-4m+3=0$

$(m-1)(m-3)=0 \Rightarrow m=1,3.$

$\therefore C.F = C_1e^{x}+C_2e^{3x}$

② Solve $2\frac{d^2y}{dx^2}+5\frac{dy}{dx}-12y=0$

Soln: The given Equation can be written in the form

$(2D^2+5D-12)y=0.$

A.E is $2m^2+5m-12=0.$

$(m+4)(m-3/2)=0 \Rightarrow m=3/2\,;-4$

$\therefore C.F = C_1e^{3/2\,x}+C_2e^{-4x}.$

3) Solve $\frac{d^2y}{dx^2}+3\frac{dy}{dx}-40y=0$

Soln $(D^2+3D-40)y=0$

A.E is $m^2+3m-40=0$

$\Rightarrow (m-5)(m+8)=0 \Rightarrow m=5,-8$

$\therefore C.f = C_1e^{5x}+C_2e^{-8x}$

Q.B ④

4) Solve $(D^2-6D+9)y=0$.

Soln: A.E is $m^2-6m+9=0$.

$(m-3)(m-3)=0$.

$\Rightarrow m=3,3$.

$\therefore C.F = (C_1x+C_2)e^{3x}$.

5.) Solve $(D^2+6D+9)y=0$.

Soln: A.E is $m^2+6m+9=0$.

$(m+3)(m+3)=0 \Rightarrow m=-3,-3$.

$\therefore C.F\ y=(C_1x+C_2)e^{-3x}$.

Q.B 6) $(D^2+8D+25)y=0$

$m^2+8m+25=0$

$m=\frac{-8\pm\sqrt{64-100}}{2}$

$=\frac{-8\pm 6i}{2}$

$=-4\pm 3i$

$\therefore y=e^{-4x}[A\cos 3x+B\sin 3x]$

Q.B ⑤ $(D^2+4D+4)y=0$

$m^2+4m+4=0$

$(m+2)^2=0$

$m=-2,-2$

$\therefore y=(A+Bx)e^{-2x}$

Q.B

1.) $(D^2-5D+6)y=0$

A.E is $m^2-5m+6=0$

$(m-3)(m-2)=0$

$m=2,3$

$\therefore y=Ae^{2x}+Be^{3x}$

3.) $(D^2-4)y=0$

$m^2-4=0$

$(m-2)(m+2)=0$

$m=2,-2$

$\therefore y=Ae^{2x}+Be^{-2x}$

6) Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 0$.

Soln: $(D^2 + 2D + 3)y = 0$.

A.E is $m^2 + 2m + 3 = 0$.

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4-12}}{2} = \frac{-2 \pm \sqrt{-8}}{2}.$$

$$= \frac{-2 \pm 2\sqrt{2}\,i}{2} = -1 \pm i\sqrt{2}.$$

$\therefore$ The roots are $-1+i\sqrt{2}$, $-1-i\sqrt{2}$.

$\therefore$ C.F is $e^{-x}\left[c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right]$.

7) Solve $(D^4 + 8D^2 + 16)y = 0$.

Soln: The eqn. is $(D^2+4)^2 = 0$.

The A.E is $(m^2+4)^2 = 0 \Rightarrow m^2 + 4 = 0$

$\Rightarrow m^2 = -4$

$\Rightarrow m = \pm 2i$ (Repeated twice)

$\therefore$ C.F $y = (c_1 x + c_2)\cos 2x + (c_3 x + c_4)\sin 2x$.

P.I Particular Integral:

When the RHS of the given equation is zero the general solution of the equation is only C.f.

When the RHS of the given differential equation is a function of x, the general solutions includes P.I also

There are different methods to find P.I.

Method of variation of parameters.

Consider the second order non-homogeneous linear Equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ — (1)

where P, q, r are functions of x or constants.

The homogeneous equation corresponding to Eqn. (1) is $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + qy = 0$ — (2)

ie $y'' + Py' + qy = 0$.

The general soln. of (2) is

$y = c_1 y_1(x) + c_2 y_2(x)$.

Replacing the constants c_1, c_2 by variable functions $u(x)$ and $v(x)$ we get,

$y = uy_1 + vy_2$.

where $u = -\int \frac{ry_2}{w}dx + K_1$

$v = \int \frac{ry_1}{w}dx + K_2$.

Here $w = y_1y_2' - y_2y_1'$ is called Wronskian of y_1 and y_2 $(w \neq 0)$.

1.) Solve $y''+y = \sec x$ by method of variation of parameters.

Soln: $y''+y = \sec x$ — (1)

The homogeneous equation corresponding to Eqn. (1) is

$y''+y = 0$.

A.E is $m^2+1=0 \Rightarrow m^2=-1 \Rightarrow m = \pm i$

$\therefore$ C.F is $y = c_1\cos x + c_2\sin x$

$\Rightarrow y = uy_1 + vy_2$

Hence $y_1 = \cos x$ $\quad y_2 = \sin x$

$y_1' = -\sin x$ $\quad y_2' = \cos x$.

$\therefore w = y_1y_2' - y_2y_1' = \cos^2 x + \sin^2 x = 1$

Now,

$$u = -\int \frac{r y_2}{w} dx + K_1 \qquad v = \int \frac{r y_1}{w} dx + K_2$$

$$= -\int \sec x \sin x \, dx + K_1 \qquad = \int \sec x \cos x \, dx + K_2$$

$$= -\int \tan x \, dx + K_1 \qquad = \int dx + K_2$$

$$= -\log \sec x + K_1 \qquad = x + K_2.$$

$$u = \log \cos x + K_1$$

$\therefore$ The general solution of Equation (1) is

$$y = (\log \cos x + K_1)\cos x + (x + K_2)\sin x.$$

2) $y'' + y = \tan x \quad -(1)$.

Soln: Homogeneous Eqn. of (1) is $y'' + y = 0$.

A.E is $m^2 + 1 = 0 \Rightarrow m = \pm i$

C.F is $y = C_1 \cos x + C_2 \sin x$.

$y_1 = \cos x \quad y_1' = -\sin x, \quad y_2 = \sin x, \quad y_2' = \cos x$

$\therefore w = y_1 y_2' - y_2 y_1' = \cos^2 x + \sin^2 x = 1$

$$u = -\int \tan x \sin x \, dx + K_1$$

$$= -\int \frac{\sin^2 x}{\cos x} dx + K_1$$

$$= -\int \frac{1 - \cos^2 x}{\cos x} dx + K_1$$

$$= -\left[\int \sec x \, dx - \int \cos x \, dx + K_1\right]$$

$$= -\left[\log(\sec x + \tan x) - \sin x\right] + K_1$$

$$u = \sin x - \log(\sec x + \tan x) + K_1$$

$$v = \int \tan x \cos x \, dx + K_2$$

$$= \int \sin x \, dx + K_2$$

$$v = -\cos x + K_2$$

$\therefore$ The general solution is

$y = \cos x\left[\sin x - \log(\sec x + \tan x) + K_1\right] + \sin x\,(-\cos x + K_2$

3) Solve $(D^2 + 2D + 5)y = e^{-x}\tan x$ by method of variation of parameters.

Soln: Homogeneous equation is $D^2 + 2D + 5 = 0$.

A.E is $m^2 + 2m + 5 = 0$.

$m = -1 \pm 2i$, $\alpha = -1$, $\beta = 2$.

C.F is $y = e^{-x}\left[C_1\cos 2x + C_2\sin 2x\right]$.

$y = C_1 e^{-x}\cos 2x + C_2 e^{-x}\sin 2x$.

P.I is $y = uy_1 + vy_2$

$y_1 = e^{-x}\cos 2x \qquad y_2 = e^{-x}\sin 2x$

$y_1' = -e^{-x}\cos 2x - 2e^{-x}\sin 2x \qquad y_2' = -e^{-x}\sin 2x + 2e^{-x}\cos 2x$

$W = y_1y_2' - y_2y_1'$

$= e^{-x}\cos 2x\left[2e^{-x}\cos 2x - e^{-x}\sin 2x\right]$

$\quad - e^{-x}\sin 2x\left[-2e^{-x}\sin 2x - e^{-x}\cos 2x\right]$

$= 2e^{-2x}\cos^2 2x - e^{-2x}\cos 2x\sin 2x + 2e^{-2x}\sin^2 2x + e^{-2x}\sin 2x\cos 2x$.

$= 2e^{-2x}(\sin^2 2x + \cos^2 2x)$

$\Rightarrow W = 2e^{-2x}$

$u = -\int \frac{ry_2}{W}\,dx + K_1$

$= -\int \frac{e^{-x}\tan x\, e^{-x}\sin 2x}{2e^{-2x}}\,dx + K_1$

$= -\frac{1}{2}\int \tan x\sin 2x\,dx + K_1$

$$= -\frac{1}{2}\int \frac{\sin x}{\cos x} \cdot 2\sin x \cos x \, dx + K_1 .$$

$$= -\int \frac{1-\cos 2x}{2} dx + K_1$$

$$= -\frac{1}{2}\left[\int dx - \int \cos 2x \, dx\right] + K_1$$

$$= -\frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + K_1$$

$$\Rightarrow u = \frac{\sin 2x}{4} - \frac{x}{2} + K_1 .$$

$$v = \int \frac{r y_1}{w} dx + K_2 .$$

$$= \int \frac{e^{-x}\tan x \, e^{-x}\cos 2x}{2e^{-2x}} dx + K_2 .$$

$$= \frac{1}{2}\int \tan x \left(2\cos^2 x - 1\right) dx + K_2$$

$$= \frac{1}{2}\int \left(2\sin x \cos x - \tan x\right) dx + K_2$$

$$= \frac{1}{2}\int \left(\sin 2x - \tan x\right) dx + K_2$$

$$= -\frac{1}{2}\frac{\cos 2x}{2} - \frac{1}{2}\log \sec x + K_2$$

$$= -\frac{\cos 2x}{4} - \frac{1}{2}\log \sec x + K_2$$

$$= \frac{1}{2}\log\cos x - \frac{\cos 2x}{4} + K_2 .$$

$\therefore$ The complete solution is

$$y = \left(\frac{\sin 2x}{4} - \frac{x}{2} + K_1\right)e^{-x}\cos 2x + \left(\frac{1}{2}\log\cos x - \frac{\cos 2x}{4} + K_2\right) e^{-x}\sin 2x$$

H/W.

4) $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

Hint: $m = \pm i$, $u = -\int \operatorname{cosec} x \sin x \, dx$ $\quad v = \int \operatorname{cosec} x \cos x \, dx$

$y_1 = \cos x, y_2 = \sin x$, $= -\int dx + k_1 = -x + k_1$ $\quad = \int \cot x \, dx$

$= \log \sin x + k_2$.

$W = 1$, C.F $= c_1 \cos x + c_2 \sin x + (-x + k_1)\cos x + (\log \sin x + k_2)\sin x$.

5) $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$.

Hint: $m = \pm 2i$, $u = \sin 2x - \log(\sec 2x + \tan 2x)$, $v = -\cos 2x$.

C.F $= c_1 \cos 2x + c_2 \sin 2x$ $\quad$ Soln: $(c_1 \cos 2x + c_2 \sin 2x) - \cos 2x - \log(\sec 2x + \tan 2x) + \sin 2x$.

$W = 2$

To find Particular Integral

when the RHS of the given differential equation is zero, the general soln of the eqn. is only complementary function.

when the RHS of the given differential equation is a function of x namely $e^{\alpha x}$, $\sin \alpha x$, $\cos \alpha x$, algebraic function, the general solution includes particular Integral also.

Type I :

1) Solve $(D^2 - 5D + 6)y = e^{-2x}$

Soln: A.E is $m^2 - 5m + 6 = 0$

$m = 2, 3$.

C.F is $c_1 e^{2x} + c_2 e^{3x}$

P.I $= \frac{1}{D^2 - 5D + 6} \cdot e^{-2x}$

Q.B

7) $(D^2 + 4D + 4)y = e^{-2x}$

P.I $= \frac{1}{D^2 + 4D + 4} \cdot e^{-2x}$

$= \frac{1}{(-2)^2 + 4(-2) + 4} \cdot e^{-2x}$

$= \frac{1}{0} e^{-2x} = \frac{x e^{-2x}}{2D + 4}$

$= \frac{x}{0} e^{-2x} = \frac{x^2 e^{-2x}}{2}$

$$= \frac{1}{(-2)^2-5(-2)+6} \cdot e^{-2x} \quad [\text{Replacing } D \text{ by } -2].$$

$$= \frac{1}{20} \cdot e^{-2x}.$$

$\therefore$ The complete solution is $y = C.F + P.I$

$$\therefore \; y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{20} e^{-2x}$$

2) Solve $(D^2 - 6D + 9)y = e^{3x}$.

<u>Soln</u>: A.E is $m^2 - 6m + 9 = 0 \Rightarrow m = 3, 3$.

C.F is $(C_1 x + C_2) e^{3x}$.

$$P.I = \frac{1}{D^2 - 6D + 9} \cdot e^{3x} = \frac{1}{9-18+9} \cdot e^{3x}$$

$$= \frac{1}{0} e^{3x} \quad [\text{This is not possible}]$$

$$= \frac{x}{2D-6} \cdot e^{3x} = \frac{x}{6-6} e^{3x} = \frac{x}{0} e^{3x} \quad [\;\;\text{"}\;\;]$$

$$= \frac{x^2}{2} \cdot e^{3x}$$

$\therefore$ The complete solution is $y = (C_1 x + C_2) e^{3x} + \frac{x^2}{2} e^{3x}$.

3) Solve $y'' + 3y' + 2y = 10e^{5x}$

<u>Soln</u>: A.E is $m^2 + 3m + 2 = 0$

$$(m+1)(m+2) = 0 \Rightarrow m = -1, -2.$$

$\therefore$ C.F is $C_1 e^{-x} + C_2 e^{-2x}$.

$$P.I = \frac{10}{D^2+3D+2} \cdot e^{5x} = \frac{10}{25+15+2} \cdot e^{5x} = \frac{10}{42} e^{5x}$$

$$= \frac{5}{21} e^{5x}.$$

$\therefore$ The complete solution is

$$y = C_1 e^{-x} + C_2 e^{-2x} + \frac{5}{21} e^{5x}.$$

4) Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 16$. Q.B. 5

Soln: A.E is $m^2 + 4m + 4 = 0 \Rightarrow m = -2, -2$. $(D^2+4D+4)y = 0$.

$\therefore$ C.F is $(c_1 x + c_2)e^{-2x}$.

$$P.I = \frac{1}{D^2+4D+4} . 16e^{0x} = \frac{16}{4}e^{0x} = 4.$$

$\therefore$ The complete solution is $y = (c_1x + c_2)e^{-2x} + 4$.

5) Solve $(D^3 - 3D^2 + 4D - 2)y = e^x$.

Soln. A.E is $m^3 - 3m^2 + 4m - 2 = 0$.

$(m-1)(m^2 - 2m + 2) = 0$.

$\Rightarrow m = 1$, (or) $m^2 - 2m + 2 = 0$.

$$m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

C.F is $c_1 e^x + e^x[c_2 \cos x + c_3 \sin x]$.

$$P.I = \frac{1}{D^3 - 3D^2 + 4D - 2} . e^x.$$

$$= \frac{1}{1-3+4-2} . e^x.$$

$$= \frac{1}{0} e^x = \frac{x}{3D^2 - 6D + 4} . e^x = \frac{x}{3-6+4} e^x = xe^x$$

$\therefore$ The complete solution is

$$y = c_1 e^x + e^x[c_2 \cos x + c_3 \sin x] + xe^x.$$

Q.B. 8) $(D^2+6D+5)y = e^{2x}$

6) $y'' + 3y' + 2y = e^{5x}$

Soln: $(c_1 e^{-2x} + c_2 e^{-x}) + \frac{1}{42} e^{5x}$

$$P.I = \frac{1}{D^2+6D+5} . e^{2x}.$$

$$= \frac{1}{4+12+5} . e^{2x}$$

$$= \frac{e^{2x}}{21}$$

7) $(D^2 + 2D + 1)y = e^{-x} + 3$.

TYPE - II

1) solve $(D^2-3D+2)y = 6e^{3x} + \sin 2x$.

Soln: A.E is $m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$.

C.F is $C_1 e^{x} + C_2 e^{2x}$.

$$P.I_1 = \frac{1}{D^2-3D+2} \cdot 6e^{3x} = \frac{6}{2} e^{3x} = 3e^{3x}.$$

$$P.I_2 = \frac{1}{D^2-3D+2} \cdot \sin 2x$$

$$= \frac{1}{-4-3D+2} \cdot \sin 2x \quad [\text{Replace } D^2 \text{ by } -2^2 = -4].$$

$$= \frac{1}{-2-3D} \cdot \sin 2x$$

$$= \frac{1}{-(2+3D)} \times \frac{(2-3D)}{(2-3D)} \cdot \sin 2x$$

$$= -\frac{[2-3D]}{4-9D^2} \cdot \sin 2x.$$

$$= -\frac{(2-3D)}{4+36} \cdot \sin 2x$$

$$= -\frac{1}{40}\left[2\sin 2x - 3D(\sin 2x)\right]$$

$$= -\frac{1}{40}\left[2\sin 2x - 6\cos 2x\right]$$

$$= -\frac{1}{20}\left[\sin 2x - 3\cos 2x\right].$$

$\therefore$ The complete solution is

$$y = C_1 e^{x} + C_2 e^{2x} + 3e^{3x} - \frac{1}{20}(\sin 2x - 3\cos 2x).$$

Q.B. 9) $(D^2+16)y = \cos 3x$

$$P.I = \frac{\cos 3x}{D^2+16} \quad D^2 = -a^2$$

$$= \frac{\cos 3x}{-9+16} = \frac{\cos 3x}{7}.$$

10) $(D^2+a^2)y = \sin ax$.

$$P.I = \frac{\sin ax}{D^2+a^2} = \frac{\sin ax}{-a^2+a^2}$$

$$= \frac{1}{0}\sin ax$$

$$= \frac{x \sin ax}{2D}$$

$$= \frac{x}{2} \cdot \int \sin ax \, dx$$

$$= \frac{x}{2} \cdot \frac{-\cos ax}{a} = \frac{-x\cos ax}{2a}.$$

2) Solve $(D^2-4D+3)y = \sin 3x \cos 2x$.

Soln: A.E is $m^2-4m+3=0 \Rightarrow m=1,3$.

$\therefore$ C.F is $c_1e^x + c_2e^{3x}$.

$\sin 3x \cos 2x = \frac{1}{2}[\sin 5x + \sin x] = \frac{1}{2}\sin 5x + \frac{1}{2}\sin x$.

$P.I_1 = \frac{1}{D^2-4D+3}\cdot(\frac{1}{2}\sin 5x)$

$= \frac{1}{2}\left[\frac{1}{-25-4D+3}\cdot\sin 5x\right]$

$= \frac{1}{2}\left[\frac{1}{-(22+4D)}\cdot\sin 5x\right]$

$= -\frac{1}{2}\left[\frac{(22-4D)}{484-16D^2}\cdot\sin 5x\right]$

$= -\frac{1}{2}\left[\frac{(22-4D)}{484-16(-25)}\cdot\sin 5x\right]$

$= -\frac{1}{2}\left[\frac{22\sin 5x - 4D(\sin 5x)}{484+400}\right]$

$= -\frac{1}{2}\left[\frac{22\sin 5x - 20\cos 5x}{884}\right]$

$= \frac{1}{884}[10\cos 5x - 11\sin 5x]$

$P.I_2 = \frac{1}{D^2-4D+3}(\frac{1}{2}\sin x)$

$= \frac{1}{2}\left[\frac{1}{-1-4D+3}\cdot\sin x\right]$

$= \frac{1}{2}\left[\frac{1}{2-4D}\cdot\sin x\right]$

$= \frac{1}{2}\left[\frac{(2+4D)}{4-16D^2}\cdot\sin x\right]$

$= \frac{1}{2}\left[\frac{(2+4D)}{4-16(-1)}\cdot\sin x\right]$

$= \frac{1}{2}\left[\frac{(2+4D)}{4+16}\cdot\sin x\right]$

$= \frac{1}{2}\left[\frac{2\sin x + 4D(\sin x)}{20}\right]$

$= \frac{1}{2}\left[\frac{2\sin x + 4\cos x}{20}\right]$

$= \frac{1}{20}[\sin x + 2\cos x]$

$\therefore$ The complete solution is

$$y = c_1e^x + c_2e^{3x} + \frac{1}{884}[10\cos 5x - 11\sin 5x] + \frac{1}{20}[2\cos x + \sin x]$$

H/W

3) Solve $(D^2-3D+2)y = \cos 3x \cos 2x$.

C.F is $c_1 e^{x} + c_2 e^{2x}$

$$\cos 3x \cos 2x = \frac{1}{2}\left[\cos 5x + \cos x\right] = \frac{1}{2}\cos 5x + \frac{1}{2}\cos x.$$

$$P.I_1 = \frac{1}{(D^2-3D+2)} \cdot \frac{1}{2}\cos 5x$$

$$= \frac{1}{2}\left[\frac{1}{-25-3D+2} \cdot \cos 5x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-(23+3D)} \cdot \cos 5x\right]$$

$$= -\frac{1}{2}\left[\frac{(23-3D)}{529-9D^2} \cdot \cos 5x\right]$$

$$= -\frac{1}{2}\left[\frac{(23-3D)}{529-9(-25)} \cdot \cos 5x\right]$$

$$= -\frac{1}{2}\left[\frac{23\cos 5x - 3D(\cos 5x)}{529+225}\right]$$

$$= -\frac{1}{2}\left[\frac{23\cos 5x + 15\sin 5x}{754}\right]$$

$$= -\left[\frac{23\cos 5x + 15\sin 5x}{1508}\right]$$

$$P.I_2 = \frac{1}{(D^2-3D+2)} \cdot \frac{1}{2}\cos x$$

$$= \frac{1}{2}\left[\frac{1}{-1-3D+2}\right] \cdot \cos x$$

$$= \frac{1}{2}\left[\frac{1}{1-3D}\right] \cdot \cos x$$

$$= \frac{1}{2}\left[\frac{1+3D}{1-9D^2}\right] \cdot \cos x$$

$$= \frac{1}{2}\left[\frac{(1+3D)}{1-9(-1)}\right] \cdot \cos x$$

$$= \frac{1}{2}\left[\frac{\cos x + 3D(\cos x)}{1+9}\right]$$

$$= \frac{1}{2}\left[\frac{\cos x - 3\sin x}{10}\right]$$

$$= \frac{\cos x - 3\sin x}{20}$$

$\therefore$ The complete solution is

$$y = c_1 e^{x} + c_2 e^{2x} - \frac{1}{1508}\left[23\cos 5x + 15\sin 5x\right] + \frac{1}{20}\left[\cos x - 3\sin x\right]$$

H/W.

4) Solve $(D^2+6D+8)y = e^{-2x} + \cos^2 x$.

Soln A.E is $m^2+6m+8=0 \Rightarrow (m+4)(m+2)=0 \Rightarrow m=-2,-4$

C.F is $C_1e^{-4x}+C_2e^{-2x}$.

$$\cos^2 x = \frac{1+\cos 2x}{2}.$$

$$P.I_1 = \frac{1}{D^2+6D+8}.e^{-2x}$$

$$= \frac{1}{4-12+8}.e^{-2x}$$

$$= \frac{1}{0}e^{-2x}$$

$$= \frac{x}{2D+6}.e^{-2x}$$

$$= \frac{x}{-4+6}.e^{-2x}$$

$$P.I_1 = \frac{x}{2}e^{-2x}$$

$$P.I_2 = \frac{1}{D^2+6D+8}.\frac{1}{2}e^{0x}$$

$$= \frac{1}{2}.\frac{1}{8}.e^{0x}$$

$$= \frac{1}{16}.$$

$$P.I_3 = \frac{1}{D^2+6D+8}.\frac{\cos 2x}{2}$$

$$= \frac{1}{2}\left[\frac{1}{D^2+6D+8}.\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-4+6D+8}.\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4+6D}.\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{(4-6D)}{16-36D^2}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{(4-6D)}{16-36(-4)}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{4\cos 2x - 6D\cos 2x}{16+144}\right]$$

$$= \frac{1}{2}\left[\frac{4\cos 2x + 12\sin 2x}{160}\right]$$

$$= \frac{1}{80}\left[\cos 2x + 3\sin 2x\right]$$

$\therefore$ The complete solution is

$$y = C_1e^{-4x} + C_2e^{-2x} + \frac{x}{2}e^{-2x} + \frac{1}{16} + \frac{1}{80}\left[\cos 2x + 3\sin 2x\right]$$

5) Solve $(D^2+4)y = \cos 3x$.

A.E is $m^2+4=0 \Rightarrow m = \pm 2i$.

C.f is $y = c_1 \cos 2x + c_2 \sin 2x$.

$$P.I = \frac{1}{D^2+4} \cdot \cos 3x = \frac{1}{-9+4} \cos 3x = \frac{1}{-5} \cos 3x$$

$\therefore$ The complete solution is $y = C.f + P.I$.

$$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{5} \cos 3x.$$

H/W.

6) Solve $(D^2+D+1)y = \sin 2x$.

Soln: $y = e^{-x/2}\left[c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right] - \frac{1}{13}\left[2\cos 2x + 3\sin 2x\right]$

TYPE III :

If $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ which is a polynomial in x of degrees n or algebraic function then

$$P.I = \frac{1}{\phi(D)} a_0x^n + a_1x^{n-1} + \dots + a_n$$

$$= [\phi(D)]^{-1}\left[a_0x^n + a_1x^{n-1} + \dots + a_n\right]$$

Expand $[\phi(D)]^{-1}$ by using binomial theorem in ascending power of D.

1.) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

2) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

3) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

4) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

Note : $\frac{1}{D} f(x) = \int f(x)$ wrt x.

1) Solve $(D^2 - 4D + 3)y = x^2$

Soln: A.E is $m^2 - 4m + 3 = 0 \Rightarrow m = 1, 3$.

$$C.f = C_1 e^{x} + C_2 e^{3x}.$$

$$P.I = \frac{1}{D^2 - 4D + 3} . x^2$$

$$= \frac{1}{3\left[1 + \left(\frac{D^2 - 4D}{3}\right)\right]} . x^2$$

$$= \frac{1}{3}\left[1 + \left(\frac{D^2 - 4D}{3}\right)\right]^{-1} . x^2.$$

$$= \frac{1}{3}\left[1 - \left(\frac{D^2 - 4D}{3}\right) + \left(\frac{D^2 - 4D}{3}\right)^2 - \ldots\right] . x^2$$

$$= \frac{1}{3}\left[1 - \frac{D^2}{3} + \frac{4D}{3} + \frac{D^4 - 8D^3 + 16D^2}{9} - \ldots\right] . x$$

$$= \frac{1}{3}\left[1 - \frac{D^2}{3} + \frac{4D}{3} + \frac{16D^2}{9}\right] . x^2$$

$$= \frac{1}{3}\left[x^2 - \frac{2}{3} + \frac{8x}{3} + \frac{32}{9}\right]$$

$$= \frac{1}{3}\left[x^2 + \frac{8x}{3} + \frac{26}{9}\right].$$

$\therefore$ The complete solution is $y = Cf + PI$

$$\Rightarrow y = C_1 e^{x} + C_2 e^{3x} + \frac{1}{3}\left[x^2 + \frac{8x}{3} + \frac{26}{9}\right]$$

2) Solve $(D^3 + 8)y = x^4 + 2x + 1$.

Soln: A.E is $m^3 + 8 = 0$.

$$(m+2)(m^2 - 2m + 4) = 0.$$

$m = -2, \; m = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3}.$

$C.F = e^{x}\left[C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right] + C_3 e^{-2x}.$

$P.I = \frac{1}{D^3+8} \cdot (x^4+2x+1)$

$= \frac{1}{8\left(1+\frac{D^3}{8}\right)} \cdot (x^4+2x+1)$

$= \frac{1}{8}\left[1+\frac{D^3}{8}\right]^{-1} (x^4+2x+1)$

$= \frac{1}{8}\left[1-\frac{D^3}{8}+\frac{D^6}{8} - \cdots\right] \cdot (x^4+2x+1)$

$= \frac{1}{8}\left[x^4+2x+1-\frac{(\overset{3}{\cancel{24}}x)}{\cancel{8}}\right]$

$= \frac{1}{8}\left[x^4-x+1\right].$

$\therefore$ The complete solution is $y = CF + PI$.

$\Rightarrow y = e^{x}\left[C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right] + \frac{1}{8}\left[x^4-x+1\right] + C_3 e^{-2x}$

3) solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + x^2 + x$

Soln: $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x.$

A.E is $m^3+2m^2+m = 0.$

$m(m^2+2m+1) = 0.$

$m(m+1)(m+1) = 0.$

$\Rightarrow m = 0, \; m = -1, \; m = -1.$

$C.F = C_1 e^{0x} + (C_2 x + C_3)e^{-x}.$

$$P.I_1 = \frac{1}{(D^3+2D^2+D)} \cdot e^{2x} = \frac{1}{8+8+2} \cdot e^{2x} = \frac{1}{18} e^{2x}.$$

$$P.I_2 = \frac{1}{(D^3+2D^2+D)} \cdot (x^2+x)$$

$$= \frac{1}{D\left[1+\left(\frac{D^3+2D^2}{D}\right)\right]} \cdot (x^2+x)$$

$$= \frac{1}{D}\left[1+(D^2+2D)\right]^{-1} \cdot (x^2+x)$$

$$= \frac{1}{D}\left[1-(D^2+2D)+(D^2+2D)^2-\cdots\right] \cdot (x^2+x)$$

$$= \frac{1}{D}\left[1-D^2-2D+4D^2+4D^3+D^4 \ldots\right] \cdot (x^2+x)$$

$$= \frac{1}{D}\left[x^2+x-2(2x+1)-2+4(2)\right]$$

$$= \frac{1}{D}\left[x^2+x-4x-2-2+8\right]$$

$$= \frac{1}{D}\left[x^2-3x+4\right] = \frac{x^3}{3} - \frac{3x^2}{2} + 4x.$$

The complete solution is $y = C.f + P.I.$

$$\therefore\ y = C_1 + (C_2x + C_3)e^{-x} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x.$$

H/W.

4) $(D^2-3D+2)y = 2x^2+1.$

Soln: A.E is $m^2-3m+2=0.$

$$\Rightarrow (m-2)(m-1)=0.$$

$$\Rightarrow m=1,2.$$

$\therefore$ $C.f = C_1e^x + C_2e^{2x}.$

$$P.I = \frac{1}{D^2-3D+2}\cdot(2x^2+1)$$

$$= \frac{1}{2\left[1+\left(\frac{D^2-3D}{2}\right)\right]^{-1}}\cdot(2x^2+1)$$

$$= \frac{1}{2}\left[1+\left(\frac{D^2-3D}{2}\right)\right]^{-1}\cdot(2x^2+1)$$

$$= \frac{1}{2}\left[1-\left(\frac{D^2-3D}{2}\right)+\left(\frac{D^2-3D}{2}\right)^2 - \cdots\right]\cdot(2x^2+1)$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}+\frac{3D}{2}+\frac{D^4-6D^3+9D^2}{4} - \cdots\right]\cdot(2x^2+1)$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}+\frac{3D}{2}+\frac{9D^2}{4} - \cdots\right]\cdot(2x^2+1)$$

$$= \frac{1}{2}\left[2x^2+1-\frac{4}{2}+\frac{3}{2}4x+\frac{9}{4}\cdot 4\right]$$

$$= \frac{1}{2}\left[2x^2+1-2+6x+9\right]$$

$$= \frac{1}{2}\left[2x^2+6x+8\right].$$

The complete solution is $y = C.f + P.I$.

$\Rightarrow \quad y = c_1e^{2x}+c_2e^{x}+x^2+3x+4.$

5) Solve $(D^2+5D+4)y = x^2+7x+9.$

TYPE - IV

If $f(x) = e^{\alpha x} \cdot X$ where X is any function of x.

$$P.I = \frac{1}{\phi(D)} . e^{\alpha x} . X.$$

$$= e^{\alpha x} . \frac{1}{\phi(D+\alpha)} . X \quad (\text{Replace } D \text{ by } D+\alpha).$$

1) Solve $(D^2 - 5D+6)y = e^x \cos 2x$.

Soln: A.E is $m^2 - 5m+6 = 0 \Rightarrow (m-2)(m-3)=0 \Rightarrow m=2,3$

$\therefore$ $C.F = C_1e^{2x} + C_2e^{3x}$.

$$P.I = \frac{1}{D^2-5D+6} . e^x \cos 2x.$$

$$= e^x . \frac{1}{(D+1)^2 - 5(D+1)+6} . \cos 2x$$

$$= e^x . \frac{1}{D^2+2D+1-5D-5+6} . \cos 2x$$

$$= e^x . \frac{1}{D^2-3D+2} . \cos 2x$$

$$= e^x . \frac{1}{-4-3D+2} . \cos 2x$$

$$= e^x . \frac{1}{-(2+3D)} \times \frac{(2-3D)}{(2-3D)} . \cos 2x$$

$$= -e^x . \frac{(2-3D)}{4-9D^2} . \cos 2x$$

$$= -e^{+x} . \frac{(2-3D)}{4-9(-4)} . \cos 2x$$

$$= -e^{+x}\ \frac{(2-3D)}{40}\ .\cos 2x.$$

$$= \frac{-e^{x}}{40}\left[2\cos 2x - 3D(\cos 2x)\right]$$

$$= -\frac{e^{x}}{40}\left[2\cos 2x + 6\sin 2x\right].$$

$\therefore$ The complete solution is $y = C.f + P.I.$

ie $y = c_1 e^{2x} + c_2 e^{3x} - \frac{e^{x}}{20}\left[\cos 2x + 3\sin 2x\right].$

2) Solve $(D^2 + 2D + 2)y = e^{-x}\sin x.$

<u>Soln</u>: A.E is $m^2 + 2m + 2 = 0.$

$$m = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$C.f = e^{-x}\left[c_1\cos x + c_2\sin x\right].$$

$$P.I = \frac{1}{(D^2+2D+2)}\ .e^{-x}\sin x.$$

$$= e^{-x}\ .\frac{1}{(D-1)^2 + 2(D-1) + 2}\ .\sin x$$

$$= e^{-x}\ .\frac{1}{D^2 - \cancel{2D} + 1 + \cancel{2D} - \cancel{2} + \cancel{2}}\ .\sin x.$$

$$= e^{-x}\ \frac{1}{D^2 + 1}\ .\sin x$$

$$= e^{-x}\ \frac{1}{-1+1}\ .\sin x = e^{-x}.\frac{1}{0}\sin x$$

(Not possible)

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$$= e^{-x} \cdot \frac{1}{2D} \cdot \sin x$$

$$= \frac{x}{2} e^{-x} (-\cos x) = -\frac{x e^{-x}}{2} \cos x$$

$\therefore$ The complete solution is $y = e^{-x}(c_1 \cos x + c_2 \sin x) - \frac{x e^{-x} \cos x}{2}$.

H/W 3.) $(D^4 - 1)y = \cos x \cosh x$.

Soln. A.E is $m^4 - 1 = 0$.

$$(m^2)^2 - 1 = 0.$$

$$(m^2 - 1)(m^2 + 1) = 0$$

$$(m-1)(m+1)(m^2+1) = 0 \Rightarrow m = 1, -1, \pm i$$

$\therefore$ C.F is $c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$.

$$\cos x \cosh x = \cos x \left(\frac{e^{x} + e^{-x}}{2}\right) = \frac{e^{x}}{2} \cos x + \frac{e^{-x}}{2} \cos x.$$

$$P.I_1 = \frac{1}{D^4 - 1} \cdot \frac{e^{x}}{2} \cos x.$$

$$= \frac{e^{x}}{2} \cdot \frac{1}{(D+1)^4 - 1} \cdot \cos x$$

$$= \frac{e^{x}}{2} \cdot \frac{1}{D^4 + 4D^3 + 6D^2 + 4D + 1 - 1} \cdot \cos x$$

$$= \frac{e^{x}}{2} \cdot \frac{1}{(D^2)^2 + 4D^2 \cdot D + 6D^2 + 4D} \cdot \cos x$$

$$= \frac{e^{x}}{2} \cdot \frac{1}{1 - \cancel{4D} - 6 + \cancel{4D}} \cdot \cos x$$

$$= \frac{e^{x}}{2} \cdot \frac{1}{-5} \cdot \cos x = -\frac{e^{x}}{10} \cos x$$

$$P.I_2 = \frac{1}{D^4-1} \cdot \frac{e^{-x}}{2} \cos x$$

$$= \frac{e^{-x}}{2} \cdot \frac{1}{(D-1)^4 - 1} \cdot \cos x$$

$$= \frac{e^{-x}}{2} \cdot \frac{1}{D^4 - 4D^3 + 6D^2 - 4D + 1 - 1} \cdot \cos x.$$

$$= \frac{e^{-x}}{2} \cdot \frac{1}{(D^2)^2 - 4D^2 \cdot D + 6D^2 - 4D} \cdot \cos x$$

$$= \frac{e^{-x}}{2} \cdot \frac{1}{1 + \cancel{4D} - 6 - \cancel{4D}} \cdot \cos x$$

$$= \frac{e^{-x}}{2} \cdot \frac{1}{-5} \cdot \cos x$$

$$= -\frac{1}{10} e^{-x} \cos x$$

$\therefore P.I = P.I_1 + P.I_2 = -\frac{1}{5}\left[\frac{e^{x} + e^{-x}}{2}\right] \cos x.$

$$= -\frac{1}{5} \cos x \cosh x.$$

$\therefore$ The complete solution is $y = C.F + P.I.$

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5} \cos x \cosh x.$$

H/W.

4.) Solve $\frac{d^4 y}{dx^4} - y = e^{x} \cos x.$

__Soln:__ $(D^4 - 1)y = e^{x} \cos x.$

A.E is $m^4 - 1 = 0 \Rightarrow (m^2)^2 - 1 = 0.$

$\Rightarrow (m^2 - 1)(m^2 + 1) = 0.$

$\Rightarrow (m-1)(m+1)(m^2+1) = 0$

$\Rightarrow m = 1, -1, \pm i$

C.F is $c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$.

$$P.I = \frac{1}{D^4 - 1} e^{x} \cos x.$$

$$= e^{x} \cdot \frac{1}{(D+1)^4 - 1} \cdot \cos x.$$

$$= e^{x} \cdot \frac{1}{(D+1)^2 (D+1)^2 - 1} \cdot \cos x$$

$$= e^{x} \cdot \frac{1}{(D^2 + 2D + 1)(D^2 + 2D + 1) - 1} \cdot \cos x.$$

$$= e^{x} \cdot \frac{1}{(\cancel{-1} + 2D + \cancel{1})(\cancel{-1} + 2D + \cancel{1}) - 1} \cdot \cos x$$

$$= e^{x} \cdot \frac{1}{4D^2 - 1} \cdot \cos x$$

$$= e^{x} \cdot \frac{1}{-4 - 1} \cdot \cos x$$

$$= e^{x} \cdot \frac{1}{-5} \cos x.$$

$\therefore$ The complete solution is

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + \frac{e^{x}}{-5} \cos x.$$

5) Solve $(D^3 - D)y = e^{x} \cdot x$

Soln. A.E is $m^3 - m = 0 \Rightarrow m(m^2 - 1) = 0 \Rightarrow m = 0, m = \pm 1$

C.F is $c_1 e^{0x} + c_2 e^{x} + c_3 e^{-x}$

$$P.I = \frac{1}{D^3 - D} \cdot e^{x} \cdot x$$

EULERS HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS.

The Equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X \quad -(1)$$

where $a_0, a_1, \ldots a_n$ are constants and X is a function of x is called Euler's homogeneous linear differential Equation. We can transform the Eqn (1) into an Equation with constant coefficients by using

$x = e^z$ ie $z = \log x$.

$$xD = x\frac{d}{dx} = D', \quad x^2 D^2 = x^2 \frac{d^2}{dx^2} = D'(D'-1).$$

$$x^3 D^3 = x^3 \frac{d^3}{dx^3} = D'(D'-1)(D'-2).$$

(1) Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x^2$

Soln: put $x = e^z \Rightarrow z = \log x$.

$$x\frac{d}{dx} = D', \quad x^2 \frac{d^2}{dx^2} = D'(D'-1)$$

Then the given equation becomes.

$$[D'(D'-1) - D' + 1]y = e^{2z}$$

$$[D'^2 - D' - D' + 1]y = e^{2z}$$

$$[D'^2 - 2D' + 1]y = e^{2z}$$

A.E is $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

$$C.F = (c_1 z + c_2)e^z.$$

$$P.I = \frac{1}{D'^2 - 2D' + 1} \cdot e^{2z}$$

$$= \frac{1}{4-4+1} \cdot e^{2z} = e^{2z}.$$

$\therefore$ The complete soln. is $y = (C_1 z + C_2)e^{z} + e^{2z}$.

ie $y = (C_1 \log x + C_2)x + x^2$.

2) Solve $y'' + \frac{1}{x} y' = \frac{12 \log x}{x^2}$

<u>Soln:</u> $x^2 y'' + x y' = 12 \log x$.

Put $x = e^z \Rightarrow z = \log x$

$$(x^2 D^2 + x D)y = 12 \log x.$$

$$[D'(D'-1) + D']y = \overset{12z}{\cancel{12\log}\ \cancel{12\log}}$$

$$[D'^2 - D' + D']y = 12z.$$

$$(D')^2 y = 12z.$$

A.E is $m^2 = 0 \Rightarrow m = 0, 0$.

$\therefore$ C.F $= (C_1 z + C_2)e^{0z}$

$\therefore$ C.F $= C_1 \log x + C_2$.

$$P.I = \frac{1}{D'^2} \cdot 12z = \frac{1}{D'} \cdot \int 12z\,dz = \frac{1}{D'} \cdot 12 \cdot \frac{z^2}{2}$$

$$= 6 \int z^2\,dz = 6 \cdot \frac{z^3}{3} = 2z^3.$$

$\therefore$ P.I $= 2(\log x)^3$.

$\therefore$ The complete solution is $y = C.F + P.I$.

$$y = C_1 \log x + C_2 + 2(\log x)^3.$$

H/W

3) Solve the equation $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = x^2 + \frac{1}{x^2}$

Soln: Put $x = e^z \Rightarrow z = \log x$.

$x\frac{d}{dx} = D'$, $x^2\frac{d^2}{dx^2} = D'(D'-1)$.

Then the given equation becomes.

$[D'(D'-1) + 4D' + 2]y = e^{2z} + e^{-2z}$.

$[D'^2 + 3D' + 2]y = e^{2z} + e^{-2z}$.

A.E is $m^2 + 3m + 2 = 0$

$\Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2$.

$\therefore$ C.F $= c_1 e^{-z} + c_2 e^{-2z}$.

$$P.I_1 = \frac{1}{D'^2 + 3D' + 2} \cdot e^{2z}$$

$$= \frac{1}{4+6+2} \cdot e^{2z}$$

$$= \frac{1}{12} e^{2z}$$

$$P.I_2 = \frac{1}{D'^2 + 3D' + 2} \cdot e^{-2z}$$

$$= \frac{1}{4-6+2} \cdot e^{-2z}$$

$$= \frac{1}{0} \cdot e^{-2z}$$

$$= \frac{z}{2D' + 3} \cdot e^{-2z}$$

$$= \frac{z}{-4+3} \cdot e^{-2z}$$

$$= -z e^{-2z}.$$

$\therefore P.I = P.I_1 + P.I_2 = \frac{1}{12} e^{2z} - \frac{z}{e^{2z}}$.

$\therefore$ The complete solution is $y = C.f + P.I$

$\Rightarrow y = c_1 e^{-z} + c_2 e^{-2z} + \frac{1}{12} e^{2z} - z e^{-2z}$.

$\Rightarrow y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{x^2}{12} - \frac{\log x}{x^2}$.

H/W 4) Solve $(x^2D^2 - xD + 1)y = \left(\frac{\log x}{x}\right)^2$

Soln. put $x = e^z \Rightarrow z = \log x$.

$xD = D'$, $x^2D^2 = D'(D'-1)$.

Then the given Eqn. becomes

$$[D'(D'-1) - D' + 1]y = \left(\frac{z}{e^z}\right)^2$$

$$[D'^2 - 2D' + 1]y = e^{-2z} \cdot z^2.$$

A.E is $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$

$\therefore$ C.F $= (c_1 z + c_2)e^z$.

$$P.I = \frac{1}{D^2 - 2D + 1} \cdot e^{-2z} \cdot z^2.$$

$$= \frac{1}{(D-1)^2} \cdot e^{-2z} \cdot z^2.$$

$$= e^{-2z} \cdot \frac{1}{(D-3)^2} \cdot z^2.$$

$$= e^{-2z} \cdot \frac{1}{(-3)^2\left[1 - \frac{D}{3}\right]^2} \cdot z^2.$$

$$= \frac{e^{-2z}}{9}\left[1 + \frac{2D}{3} + \frac{\cancel{3}D^2}{\cancel{9}3}\right]z^2$$

$$= \frac{e^{-2z}}{9}\left[z^2 + \frac{4z}{3} + \frac{2}{3}\right]$$

$$= \frac{e^{-2z}}{27}\left[3z^2 + 4z + 2\right].$$

The complete solution is

$$y = (c_1 z + c_2)e^z + \frac{e^{-2z}}{27}(3z^2+4z+2).$$

$$y = (c_1 \log x + c_2)x + \frac{1}{27x^2}\left[3(\log x)^2 + 4\log x + 2\right].$$

H/W

5) Solve $(x^2D^2 + xD + 1)y = \sin(2\log x)\sin(\log x)$

Soln: $c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{1}{16}\sin(3\log x) - \frac{1}{4}\log x \cos(\log x)$

6) solve $(x^2D^2 - 2xD - 4)y = 32(\log x)^2$

Soln: $c_1 x^4 + \frac{c_2}{x} - 8(\log x)^2 + 12\log x - 13.$

7) Solve $(x^2D^2 - xD + 4)y = x^2 \sin(\log x).$

Soln: $x\left[c_1\cos(\sqrt{3}\log x) + c_2\sin(\sqrt{3}\log x)\right] - \frac{1}{13}x^2\left[2\cos(\log x) - 3\sin(\log x)\right]$

LEGENDRE'S LINEAR DIFFERENTIAL EQUATIONS:

An Equation of the form

$$a_0(a+bx)^n\frac{d^ny}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = X.$$

where $a_0, \ldots, a_n$ are constant and X is a fn of x (1)

is called Legendre's Linear differential Equation

put $(a+bx) = e^z \Rightarrow x = \frac{e^z - a}{b}$

$$z = \log(a+bx)$$

$$(a+bx)\frac{dy}{dx} = b\,Dy$$

$$(a+bx)^2\frac{d^2y}{dx^2} = b^2 D(D-1)y.$$

1) Solve $(3x+2)^2 \frac{d^2y}{dx^2} + 5(3x+2)\frac{dy}{dx} - 3y = x^2+x+1$

Soln: put $3x+2 = e^z \Rightarrow z = \log(3x+2)$.

$\therefore x = \frac{e^z - 2}{3}$.

$(3x+2)\frac{d}{dx} = 3D$, $(3x+2)^2\frac{d^2}{dx^2} = 3^2 D(D-1) = 9D(D-1)$

$\therefore$ The given equation becomes

$$[9D(D-1) + 5\cdot 3D - 3]y = \left(\frac{e^z-2}{3}\right)^2 + \left(\frac{e^z-2}{3}\right) + 1$$

$$(9D^2 - 9D + 15D - 3)y = \frac{e^{2z} - 4e^z + 4}{9} + \frac{e^z}{3} - \frac{2}{3} + 1$$

$$(9D^2 + 6D - 3)y = \frac{e^{2z} - 4e^z + 4 + 3e^z - 6 + 9}{9}$$

$$(9D^2 + 6D - 3)y = \frac{e^{2z} - e^z + 7}{9}.$$

$$3(3D^2 + 2D - 1)y = \frac{e^{2z} - e^z + 7}{9}.$$

$$(3D^2 + 2D - 1)y = \frac{1}{27}\left[e^{2z} - e^z + 7\right].$$

A.E is $3m^2 + 2m - 1 = 0$.

$\Rightarrow m = -1, \frac{1}{3}$.

C.f is $c_1 e^{-z} + c_2 e^{z/3}$.

$P.I_1 = \frac{1}{3D^2+2D-1} \cdot \frac{e^{2z}}{27}$	$P.I_2 = \frac{1}{3D^2+2D-1} \cdot \frac{-e^{+z}}{27}$	$P.I_3 = \frac{1}{3D^2+2D-1}$
$= \frac{1}{27} \cdot \frac{1}{15} e^{2z}$	$= \frac{-1}{27} \cdot \frac{1}{4} e^z$	$= \frac{-7}{27} e^{0z}$
$= \frac{e^{2z}}{405}$	$= -\frac{1}{108} e^z$	$= \frac{-7}{27}$.

$\therefore$ P.I $= P.I_1 + P.I_2 + P.I_3$.

$$= \frac{e^{2z}}{405} - \frac{e^z}{108} - \frac{7}{27}.$$

$\therefore$ The complete solution is $y = C.F + P.I$.

$$y = c_1 e^{-z} + c_2 e^{z/3} + \frac{e^{2z}}{405} - \frac{e^z}{108} - \frac{7}{27}.$$

$$= c_1 (3x+2)^{-1} + c_2 (3x+2)^{1/3} + \frac{(3x+2)^2}{405} - \frac{(3x+2)}{108} - \frac{7}{27}$$

1W

2) Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$.

Soln: put $(1+x) = e^z \Rightarrow z = \log(1+x)$

$$\therefore x = e^z - 1$$

$$(1+x)\frac{d}{dx} = D, \quad (1+x)^2 \frac{d^2}{dx^2} = D(D-1).$$

Then the given Eqn. becomes.

$$[D(D-1) + D + 1]y = 2\sin z.$$

$$(D^2+1)y = 2\sin z.$$

A.E is $m^2+1 = 0 \Rightarrow m = \pm i$

C.F is $c_1 \cos z + c_2 \sin z$.

$$P.I = 2 \cdot \frac{1}{D^2+1}\sin z = 2 \cdot \frac{1}{-1+1}\sin z = 2 \cdot \frac{1}{0}\sin z.$$

$$= \frac{2z}{2D}\sin z = z(-\cos z).$$

The complete soln. is

$$y = c_1 \cos z + c_2 \sin z - z\cos z.$$

$$= c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] - \log(1+x).\cos[\log(1+x)]$$

H/W

3) Solve $(3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y=3x^2+4x+1$

Soln. put $(3x+2)\cancel{\frac{d}{dx}} = e^z \Rightarrow x=\frac{e^z-2}{3}$.

$\log(3x+2)=z$.

$(3x+2)\frac{d}{dx}=3D$, $(3x+2)^2\frac{d^2}{dx^2}=3^2D(D-1)=9D(D-1)$.

Then the given equation becomes,

$[9D(D-1)+3(3D)-36]y=3\left(\frac{e^z-2}{3}\right)^2+4\left(\frac{e^z-2}{3}\right)+1$.

$(9D^2-9D+9D-36)y=3\left[\frac{e^{2z}-4e^z+4}{9}\right]+\frac{4e^z}{3}-\frac{8}{3}+1$

$9(D^2-4)y=\frac{e^{2z}}{3}+\frac{4}{3}-\cancel{\frac{4e^z}{3}}+\cancel{\frac{4e^z}{3}}-\frac{8}{3}+1$.

$9(D^2-4)y=\frac{e^{2z}}{3}-\frac{1}{3}$.

$\Rightarrow (D^2-4)y=\frac{1}{27}[e^{2z}-1]$

A.E is $m^2-4=0 \Rightarrow m^2=4 \Rightarrow m=\pm 2$.

$C.F = c_1e^{2z}+c_2e^{-2z} = c_1(3x+2)^2+c_2(3x+2)^{-2}$.

$P.I = \frac{1}{27}\cdot\frac{1}{D^2-4}(e^{2z}-1)$.

$=\frac{1}{27}\left[\frac{1}{D^2-4}e^{2z}-\frac{1}{D^2-4}.e^{0z}\right]$.

$=\frac{1}{27}\left[\frac{1}{0}e^{2z}-\frac{1}{-4}.e^{0z}\right]=\frac{1}{27}\left[\frac{z}{2D}e^{2z}+\frac{1}{4}\right]$.

$=\frac{1}{27}\left[\frac{z}{4}e^{2z}+\frac{1}{4}\right]=\frac{1}{108}[ze^{2z}+1]$.

$=\frac{1}{108}[\log(3x+2)(3x+2)^2+1]$.

$y=c_1(3x+2)^2+c_2(3x+2)^{-2}+\frac{1}{108}[\log(3x+2)(3x+2)^2+1]$.

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H/W: 4) Solve $(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x$.

5) Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos[\log(1+x)]$.

SIMULTANEOUS DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS:

1) Solve $\frac{dx}{dt} + 2x - 3y = 5t$; $\frac{dy}{dt} - 3x + 2y = 2e^{2t}$.

Soln: The given Equations can be written as

$$Dx + 2x - 3y = 5t \Rightarrow (D+2)x - 3y = 5t \quad \text{— (1)}$$

$$Dy - 3x + 2y = 2e^{t} \Rightarrow -3x + (D+2)y = 2e^{2t} \quad \text{— (2)}$$

$$(1)\times 3 \Rightarrow 3(D+2)x - 9y = 15t.$$

$$(2)\times(D+2) = -3(D+2)x + (D+2)^2 y = 2(D+2)e^{2t}.$$

$$-9y + (D+2)^2 y = 2(D+2)e^{2t} + 15t.$$

$$\Rightarrow -9y + (D^2+4D+4)y = 2(2e^{2t} + 2e^{2t}) + 15t.$$

$$\Rightarrow (D^2+4D-5)y = 8e^{2t} + 15t \quad \text{— (I)}$$

A.E is $m^2 + 4m - 5 = 0 \Rightarrow m = 1, -5$.

C.F is $c_1 e^{t} + c_2 e^{-5t}$.

$$P.I_1 = \frac{1}{D^2+4D-5} \cdot 15t$$

$$= 15 \cdot \frac{1}{-5\left[1-\left(\frac{D^2+4D}{5}\right)\right]} \cdot t$$

$$= -\frac{15}{5}\left[1-\left(\frac{D^2+4D}{5}\right)\right]^{-1} \cdot t$$

$$= -3\left[1 + \frac{D^2+4D}{5}\right] \cdot t$$

$$= -3\left[t + \frac{4}{5}\right] = -3t - \frac{12}{5}$$

$$P.I_2 = \frac{1}{D^2+4D-5} \cdot 8e^{2t}$$

$$= 8 \cdot \frac{1}{4+8-5} e^{2t}$$

$$= \frac{8}{7} e^{2t}.$$

$$\therefore y = c_1 e^{t} + c_2 e^{-5t} - 3t - \frac{12}{5} + \frac{8}{7} e^{2t}.$$

Again,

$$\text{①} \times (D+2) \Rightarrow (D+2)^2 x - 3\cancel{(D+2)y} = 5(D+2)t.$$

$$\text{②} \times 3 \Rightarrow \quad -9x + \cancel{3(D+2)y} = 6e^{2t}.$$

$$(D+2)^2 x - 9x = 5(1+2t) + 6e^{2t}$$

$$\Rightarrow (D^2 + 4D + 4 - 9)x = 5 + 10t + 6e^{2t}.$$

$$\Rightarrow (D^2 + 4D - 5)x = 5(1+2t) + 6e^{2t}.$$

$$A.E = m^2 + 4m - 5 = 0 \Rightarrow m = 1, -5$$

$$C.F = c_1 e^{t} + c_2 e^{-5t}.$$

$$P.I_1 = \frac{1}{D^2 + 4D - 5} \cdot 5(1+2t)$$

$$= \frac{5}{-5}\left[1 - \left(\frac{D^2+4D}{5}\right)\right]^{-1} \cdot (1+2t)$$

$$= -1\left[1 + \left(\frac{D^2+4D}{5}\right)\right](1+2t)$$

$$= -\left[(1+2t) + \frac{8}{5}\right].$$

$$= -\left[2t + \frac{13}{5}\right]$$

$$P.I_2 = \frac{1}{D^2 + 4D - 5} \cdot 6e^{2t}$$

$$= 6 \cdot \frac{1}{4+8-5} e^{2t}$$

$$= \frac{6}{7} e^{2t}.$$

$$x = c_1 e^{t} + c_2 e^{-5t} - \left(\frac{13}{5} + 2t\right) + \frac{6}{7} e^{2t}.$$

2) Solve $\frac{d^2x}{dt^2} + y = \sin t$; $\frac{d^2y}{dt^2} + x = \cos t$.

Q.B. (14)

Soln. The given Equation can be written as.

$$D^2 x + y = \sin t \quad -①$$

$$D^2 y + x = \cos t \quad -②$$

$$① \times D^2 \Rightarrow D^4 x + \cancel{D^2 y} = D^2(\sin t)$$

$$② \Rightarrow \underset{(-)}{(-)x} \underset{(-)}{+} \cancel{D^2 y} = \underset{(\mp)}{\pm}\cos t$$

$D^2 \left[D(\sin t) \right]$

$D \left[\cos t \right]$

$$D^4 x - x = D^2(\sin t) - \cos t$$

$$(D^4 - 1)x = -\sin t - \cos t \quad — \quad (\mathrm{I})$$

$$A.E = m^4 - 1 = 0 \Rightarrow (m^2)^2 - 1 = 0$$

$$\Rightarrow (m^2-1)(m^2+1) = 0 \Rightarrow m^2 = 1,\ m^2 = -1 \Rightarrow m = \pm 1, \pm i$$

C.f is $c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$.

$$P.I_1 = \frac{1}{D^4 - 1}(-\sin t)$$

$$= \frac{-1}{+1-1}\sin t \cdot (\mathrm{NP})$$

$$= \frac{-t}{4D^3}\sin t$$

$$= \frac{-t}{-4D}\sin t$$

$$= \frac{t}{4}(-\cos t)$$

$$= -\frac{t\cos t}{4}$$

$$P.I_2 = \frac{1}{D^4 - 1}(-\cos t)$$

$$= \frac{-1}{1-1}\cos t$$

$$= \frac{-t}{4D^3}\cos t$$

$$= \frac{-t}{-4D}\cos t$$

$$= \frac{t}{4D}\cos t$$

$$= \frac{t}{4}\sin t$$

$$\therefore x = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t - \frac{t}{4}\cos t + \frac{t}{4}\sin t.$$

Again $y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t - \frac{t}{4}\cos t + \frac{t}{4}\sin t.$

$$① \Rightarrow \cancel{D^2 x} + y = \sin t$$

$$② \times D^2 \Rightarrow \underset{-}{\cancel{D^2}} x \underset{-}{+} D^4 y \underset{-}{=} D^2(\cos t)$$

$$(1 - D^4)y = \sin t - D^2(\cos t).$$

$$\Rightarrow -(D^4 - 1)y = \sin t + \cos t.$$

(SAME AS THE ABOVE).

$$\Rightarrow (D^4 - 1)y = -\sin t - \cos t \quad — \quad (\mathrm{II})$$

H/W.

3) Solve $\frac{d^2x}{dt^2} - 3x - 4y = 0$; $\frac{d^2y}{dt^2} + x + y = 0$.

$m = \pm 1, \pm 1$; $x = (c_1 t + c_2)e^{t} + (c_3 t + c_4)e^{-t}$. III^{ly} $y =$

4) Solve $\frac{dx}{dt} - \frac{dy}{dt} + 2y = \cos 2t$; $\frac{dx}{dt} + \frac{dy}{dt} - 2x = \sin 2t$

$m = 1 \pm i$; $x = e^{t}(c_1 \cos t + c_2 \sin t) - \frac{1}{2}\cos 2t$;

$y = e^{t}(c_1 \cos t + c_2 \sin t) - \frac{1}{2}\sin 2t$.

Formula :

1) $\int dx = x$.

2) $\int x^n dx = \frac{x^{n+1}}{n+1}$

3) $\int \sin mx \, dx = -\frac{\cos mx}{m}$

4) $\int \cos mx \, dx = \frac{\sin mx}{m}$

5) $\int \tan x \, dx = \log \sec x$.

$-\int \tan x \, dx = \log \cos x$.

6) $\int \cot x \, dx = \log \sin x$

7) $\int \sec x \, dx = \log(\sec x + \tan x)$

8) $\int \operatorname{cosec} x \, dx = -\log(\operatorname{cosec} x + \cot x)$.

UNIT - I . ODE.

Part - B. 16 marks Questions.

(11) Solve $(D^2-4D+4)y = e^{-4x} + 5\cos 3x$.

Soln. A.E is $m^2-4m+4=0$.

$(m-2)^2=0 \Rightarrow m=2,2$.

$\therefore$ C.f $= (C_1+C_2x)e^{2x}$.

$$P.I_1 = \frac{e^{-4x}}{D^2-4D+4} = \frac{e^{-4x}}{(-4)^2-4(4)+4)} = \frac{e^{-4x}}{36}.$$

$$P.I_2 = \frac{5\cos 3x}{D^2-4D+4} = \frac{5\cos 3x}{-9-4D+4} = \frac{5\cos 3x}{-5-4D}.$$

$$= \frac{-5\cos 3x(5-4D)}{(5+4D)(5-4D)} = \frac{-25\cos 3x+20D(\cos 3x)}{25-16D^2}.$$

$$= \frac{-25\cos 3x+20(-3\cdot\sin 3x)}{25-(16)(-9)}$$

$$= \frac{-25\cos 3x - 60\sin 3x}{25+144}$$

$$= \frac{-5}{169}\left[5\cos 3x + 12\sin 3x\right].$$

$\therefore y = C.f + P.I_1 + P.I_2$.

$$y = (C_1+C_2x)e^{2x} + \frac{e^{-4x}}{36} - \frac{5}{169}\left[5\cos 3x + 12\sin 3x\right].$$

(12) Solve by method of variation of parameters

$(D^2+4)y = \tan 2x$.

Soln: The A.E is $m^2+4=0$

$m^2=-4 \Rightarrow m = \pm 2i$.

$\therefore$ C.F $= C_1\cos 2x + C_2 \sin 2x$.

C.F $= C_1 y_1 + C_2 y_2$.

$\Rightarrow y_1 = \cos 2x$, $y_2 = \sin 2x$, here $r = \tan 2x$.

$y_1' = -2\sin 2x$, $y_2' = 2\cos 2x$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x.$$

$$= 2(\sin^2 2x + \cos^2 2x)$$

$$= 2(1) = 2.$$

$$u = -\int \frac{r y_2}{W}\, dx + k_1.$$

$$= -\int \frac{\sin 2x \cdot \tan 2x}{2}\, dx + k_1.$$

$$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\, dx + k_1 = \frac{-1}{2}\int\left(\frac{1-\cos^2 2x}{\cos 2x}\right)dx + k_1$$

$$= -\frac{1}{2}\left[\int \frac{1}{\cos 2x}\, dx - \int \cos 2x\, dx\right] + k_1$$

$$= -\frac{1}{2}\left[\int \sec 2x\, dx - \int \cos 2x\, dx\right] + k_1$$

$$= -\frac{1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right] + K_1$$

$$u = -\frac{1}{4}\left[\log(\sec 2x + \tan 2x) - \sin 2x\right] + K_1$$

$$v = \int \frac{r y_1}{W}\,dx + K_2$$

$$= \int \frac{\cos 2x \cdot \tan 2x}{2}\,dx + K_2 = \int \frac{\sin 2x}{2}\,dx + K_2$$

$$= \frac{1}{2}\left(\frac{-\cos 2x}{2}\right) + K_2 = -\frac{1}{4}\cos 2x + K_2$$

$$\therefore v = -\frac{1}{4}\cos 2x + K_2$$

$$\therefore P.I = u y_1 + v y_2$$

$$= -\frac{1}{4}\left[\log(\sec 2x + \tan 2x) - \sin 2x\right] \cdot \cos 2x$$

$$- \frac{1}{4}\cos 2x \sin 2x .$$

$$= -\frac{1}{4}\log(\sec 2x + \tan 2x)\cos 2x + \cancel{\frac{1}{4}\sin 2x \cos 2x}$$

$$- \cancel{\frac{1}{4}\sin 2x \cos 2x}$$

$$= -\frac{1}{4}\log(\sec 2x + \tan 2x)\cos 2x + K_1$$

(5) Solve the simultaneous Equations:

$$\frac{dx}{dt} + 2x - 3y = 5e^{-t}, \quad \frac{dy}{dt} - 3x + 2y = 0,$$

<u>Soln</u>. Equation can be written as

$$(D+2)x - 3y = 5e^{-t} \quad - (1)$$

$$-3x + (D+2)y = 0 \quad - (2)$$

$$(1) \times (D+2): \quad (D+2)^2 x - 3(D+2)y = 5(D+2)e^{-t}.$$

$$(2) \times 3: \quad -9x + 3(D+2)y = 0.$$

$$\left[(D+2)^2 - 9\right]x = 5(D+2)e^{-t}.$$

$$(D^2+4D-5)x = 5\left[D(e^{-t}) + 2e^{-t}\right]$$

$$= 5(-e^{-t} + 2e^{-t}) = 5e^{-t}.$$

$$\Rightarrow (D^2+4D-5)x = 5e^{-t}.$$

A.E is $m^2 + 4m - 5 = 0$.

$(m+5)(m-1) = 0$.

$\Rightarrow m = 1, -5$

C.F $x = c_1 e^{t} + c_2 e^{-5t}$.

$$P.I = \frac{5e^{-t}}{D^2+4D-5} = \frac{5e^{-t}}{1-4-5} = \frac{-5}{8}e^{-t}.$$

$$\therefore x = c_1 e^{t} + c_2 e^{-5t} - \frac{5}{8}e^{-t}.$$

$$(1) \times 3: \quad 3(D+2)x - 9y = 15e^{-t}.$$

$$(2) \times (D+2): \quad -3(D+2)x + (D+2)^2 y = 0.$$

$$\left[(D+2)^2 - 9\right]y = 15e^{-t}.$$

(13) Solve : $(x^2D^2 + xD + 1)y = 4\sin\log x$.

Soln. Put $x = e^z$, $\theta = \frac{d}{dz}$, $xD = \theta$

$\log x = z$. $\qquad x^2D^2 = \theta(\theta-1)$.

∴ Eqn. becomes

$[\theta(\theta-1) + \theta + 1]y = 4\sin z$.

$(\theta^2 - \theta + \theta + 1)y = 4\sin z$.

$(\theta^2 + 1)y = 4\sin z$.

A.E is $m^2+1=0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i$

$C.f = c_1\cos z + c_2\sin z$

$P.I = \frac{4\sin z}{\theta^2+1} = \frac{4\sin z}{-1+1} = \frac{4}{0}\sin z$

$\therefore P.I = \frac{\overset{2}{\cancel{4}}z\sin z}{\cancel{2}\theta} = 2z\cdot\frac{1}{\theta}(\sin z)$

$= 2z\int\sin z\,dz$

$= \cancel{2\int z\sin z\,dz}$ $\qquad = 2z(-\cos z) + C$.

$= \cancel{2[z(-\cos z) - (1)(-\sin z)]}$

$= \cancel{2[-z\cos z + \sin z]}$

$= \cancel{2[\sin z - z\cos z]}$.

$\therefore y = P.I + C.f = -2\log x\cos(\log x) + c_1\cos(\log x) + c_2\sin(\log x)$

$= \cancel{c_1\cos(\log x) + c_2\sin(\log x) + 2\sin(\log x) - 2\log x\cos(\log x)}$.

Unit - 3

Functions of Several Variables.

Partial Derivatives:

Let $z = f(x, y)$ be functions of two independent variables x & y. If we treat y as constant and find the derivative of $f(x, y)$ w.r.to x, then the derivative is called partial Derivative.

It is denoted by $\frac{\partial z}{\partial x}$, $\frac{\partial f}{\partial x}$, $f_x(x, y)$

$|||^{ly}$ the partial derivative of z w.r.t. 'y', keeping x as constant, then,

$\frac{\partial z}{\partial y}$, $\frac{\partial f}{\partial y}$, $f_y(x, y)$

If $z = f(x, y)$

$$\frac{\partial z}{\partial x} = f_x \quad , \quad \frac{\partial z}{\partial y} = f_y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx} \quad ; \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{yx} \quad ; \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{xy}$$

Problems:

1. If $u = \sin(y+ax)$ P.T $\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial y^2}$

Soln $u = \sin(y+ax)$

$$\frac{\partial u}{\partial x} = \cos(y+ax) \cdot a \; ; \quad \frac{\partial u}{\partial y} = \cos(y+ax).$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin(y+ax) \cdot a^2 \quad -(1) \; ; \quad \frac{\partial^2 u}{\partial y^2} = -\sin(y+ax) \quad -(2)$$

from (1) & (2) $\Rightarrow$ $\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial y^2}$

2) If $z = e^{ax+by} \sin(ax-by)$, S.T

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$$

Sln Gn., $z = e^{ax+by} \sin(ax-by)$

$$\frac{\partial z}{\partial x} = ae^{ax+by} \sin(ax-by) + ae^{ax+by} \cdot \cos(ax-by) \quad -(1)$$

$$\frac{\partial z}{\partial y} = be^{ax+by} \sin(ax-by) - be^{ax+by} \cos(ax-by) \quad -(2)$$

(1) $\times b$ + (2) $\times a$

$$\Rightarrow b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2ab\, e^{ax+by} \sin(ax-by)$$

$$= 2ab\, z \; //.$$

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3) If $u = \log(x^2+y^2+z^2)$ P.T $x\dfrac{\partial^2 u}{\partial y \partial z} = y\dfrac{\partial^2 u}{\partial z \partial x} = z\dfrac{\partial^2 u}{\partial x \partial y}$

Soln:, $u = \log(x^2+y^2+z^2)$

$$\frac{\partial u}{\partial y} = \frac{2y}{(x^2+y^2+z^2)} ; \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{2y}{(x^2+y^2+z^2)^2}(-2x)$$

$x^{-2} = (-2)x$

$$z\frac{\partial^2 u}{\partial y \partial z} = \frac{-4xyz}{(x^2+y^2+z^2)^2} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2+z^2} ; \quad \frac{\partial^2 u}{\partial z \partial x} = \frac{2x}{(x^2+y^2+z^2)^2}(-2z)$$

$$\therefore y\frac{\partial^2 u}{\partial z \partial x} = \frac{-4xyz}{(x^2+y^2+z^2)^2} \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial z} = \frac{2z}{(x^2+y^2+z^2)} = \frac{\partial^2 u}{\partial y \partial z} = \frac{2z}{(x^2+y^2+z^2)^2}(-2y)$$

$$x\frac{\partial^2 u}{\partial y \partial z} = \frac{-4xy}{(x^2+y^2+z^2)^2} \quad \text{--- (3)}$$

from (1), (2) & (3) we have $x\dfrac{\partial^2 u}{\partial y \partial z} = y\dfrac{\partial^2 u}{\partial z \partial x} = z\dfrac{\partial^2 u}{\partial x \partial y}$

4) If $u = \tan^{-1}(y/x)$ P.T $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Soln: Given, $u = \tan^{-1}(y/x)$

$$\frac{\partial u}{\partial x} = \frac{1}{1 + y^2/x^2}\left(-y/x^2\right)$$

$$= \frac{1}{\frac{x^2+y^2}{x^2}}\left(-y/x^2\right) = \frac{x^2}{x^2+y^2} \times \left(-y/x^2\right)$$

$$= -y/x^2+y^2$$

$$\frac{\partial^2 u}{\partial x^2} = -y\left(-(x^2+y^2)^{-2}\right) \cdot 2x$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+y^2/x^2} \cdot 1/x = \frac{x^2}{x^2+y^2} \times \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = x\left(-(x^2+y^2)^{-2}\right) \cdot 2y = \frac{-2xy}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

H.W.

1) If $u = y/z + z/x$, S.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

2) Verify that $f_{xy} = f_{yx}$ if $f(x,y) = \sin^{-1}(y/x)$.

Homogeneous Function :-

A function $f(x,y,z)$ is called a homogeneous function of degree n if $f(tx,ty,tz) = t^n f(x,y,z)$

ex: $f(x,y) = x^2 \sin(y/x)$ $\quad \frac{x^3+y^3}{x+y}$

$f(tx,ty) = (tx)^2 \sin\left(\frac{ty}{tx}\right) = t^2x^2 \sin(y/x)$

$= t^2 f(x,y)$

The given function is homogeneous fn., of degree 2.

Euler's Theorem:

If $f(x,y)$ is a homogeneous fn., of degree n in x & y then,

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.}$$

1) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

Soln.,

$$u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right) \Rightarrow \tan u = \frac{x^3+y^3}{x+y}.$$

i.e. Let $z = \tan u = f(x,y)$

Then z is a homogeneous fn., of x & y of degree 2. ∴ By Euler's Theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

$$x\frac{\partial}{\partial x}(\tan u) + y.\frac{\partial}{\partial y}(\tan u) = 2\tan u.$$

$$x\sec^2 u.\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u.$$

$$\sec^2 u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$= 2\left(\frac{\sin u}{\cos u}\right) \times \cos^2 u$$

$$= 2\sin u\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$$

2) If $u = \log \frac{x^2+y^2}{x+y}$, then P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$

Soln

$u = \log \frac{x^2+y^2}{x+y}$, $e^u = e^{\log \frac{x^2+y^2}{x+y}}$

$e^u = \frac{x^2+y^2}{x+y}$

Let $z = e^u = \frac{x^2+y^2}{x+y}$

Then z is a homogeneous fn., of x & y of degree 1.

$\therefore$ By Euler's Theorem,

$$x\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = 1\cdot z$$

$$x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = e^u$$

$$xe^u\cdot\frac{\partial u}{\partial x} + ye^u\cdot\frac{\partial u}{\partial y} = e^u$$

$$e^u\left[x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y}\right] = e^u$$

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = 1$$

3) If $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$ S.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Soln.,

$u = \sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} \Rightarrow \sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

$z = \sin u = f(x,y)$

Then z is a homogeneous fn., of x & y of degree 0.

$\therefore$ By Euler's Theorem,

$x\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = 0$

$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = 0$

$x\cos u\cdot\frac{\partial u}{\partial x} + y\cos u\cdot\frac{\partial u}{\partial y} = 0$

$\cos u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 0$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ //

4) If $u = e^{x^3+y^3}$, S.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u\log u$.

Soln $u = e^{x^3+y^3} \Rightarrow \log u = \log e^{x^3+y^3}$

$= x^3+y^3$

Let $z = \log u = f(x,y)$.

The z is a homogeneous fn., of degree 3 in x & y

$\therefore$ By Euler's Theorem

$$x \cdot \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 3z$$

$$x \frac{\partial}{\partial x}(\log u) + y \cdot \frac{\partial}{\partial y}(\log u) = 3\log u$$

$$x \cdot \frac{1}{u} \cdot \frac{\partial u}{\partial x} + y \cdot \frac{1}{u} \frac{\partial u}{\partial y} = 3\log u$$

$$\frac{1}{u}\left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right] = 3\log u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3u\log u.$$

H.W

1) If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ S.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$

2) If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ S.T

(Imp)

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{-\sin u \cos 2u}{4\cos^3 u}.$$

Total Derivative:

If $z = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$. Substituting the value of x & y in $f(x, y)$, then z can be expressed as a fn., of t. Then the derivative of z w.r to 't' is called the total derivative of z.

i.e., $dz = \partial f/\partial x \, dx + \partial f/\partial y \, dy$

Note: 1. Let $z = f(x, y)$; $x = \phi(t)$, $y = \psi(t)$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}$$

~~Note: 2~~ ~~If $z = f(x, y)$, $x = \phi(u, v)$, $y = \psi(u, v)$~~

~~then $\frac{\partial z}{\partial u}$~~

1) If $u = x^3 + y^3$, $x = a\cos t$, $y = b\sin t$.

Find $\frac{du}{dt}$.

Soln Gn., $u = x^3 + y^3$, $x = a\cos t$, $y = b\sin t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= 3x^2(-a\sin t) + 3y^2(b\cos t)$$

$$= -3(a^2\cos^2 t)\, a\sin t + 3\, b^2\sin^2 t\, b\cos t$$

$$\frac{du}{dt} = -3a^3\cos^2 t\sin t + 3b^3\sin^2 t\cos t$$ //

2) If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ S.T

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$$

Soln:

$u = f(r, s)$ $\quad r = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$

$s = \frac{z-x}{xz} = \frac{1}{x} - 1/z$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x}$$

$$= \frac{\partial u}{\partial r}\left(-1/x^2\right) + \frac{\partial u}{\partial s}\left(-1/x^2\right)$$

$$= -\frac{1}{x^2}\left[\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{1}{y^2} + \frac{\partial u}{\partial s}\cdot 0 = \frac{1}{y^2}\cdot\frac{\partial u}{\partial r}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}$$

$$= \frac{\partial u}{\partial r}\cdot 0 + \frac{\partial u}{\partial s}\cdot\frac{1}{z^2} = 1/z^2\,\frac{\partial u}{\partial s}$$

$$\therefore \; x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = x^2\left[-\frac{1}{x^2}\left(\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}\right)\right]$$

$$+ y^2 \cdot \frac{1}{y^2}\frac{\partial u}{\partial r} + z^2 \cdot \frac{1}{z^2}\frac{\partial u}{\partial s} \Bigg]$$

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} = 0 \text{ //.}$$

H.W

1) Find $\frac{du}{dt}$, when $u = x^2y$, $x = t^2$, $y = e^t$

2) Find $\frac{du}{dt}$, when $u = \sin x/y$, $x = e^t$, $y = t^2$

3) If $u = x^2+y^2$ & $x = a\cos t$, $y = b\sin t$
find $\frac{du}{dt}$.

Taylor's Series

If $f(x,y)$ possesses continuous partial derivatives of the n^{th} order in any neighbourhood of a point (x,y) & if (x_0+h, y_0+k) is any pt of this neighbourhood then,

$$f(x_0+h, y_0+k) = f(x_0,y_0) + \left[h f_x(x_0,y_0) + k f_y(x_0,y_0)\right] + \left[\frac{h^2}{2!} f_{xx}(x_0,y_0) + hk f_{xy}(x_0,y_0) + \frac{k^2}{2!} f_{yy}(x_0,y_0)\right] + \left[\frac{h^3}{3!} f_{xxx}(x_0,y_0) + \frac{3h^2k}{3!} f_{xxy}(x_0,y_0) + \frac{3hk^2}{3!} f_{xyy}(x_0,y_0)\right.$$

$+ \cdots$

Note: Putting $x_0 = 0$, $y_0 = 0$, $h = x$, $K = y$ in the above form, we've.

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \left[\frac{x^2}{2!} f_{xx}(0,0) + xy f_{xy}(0,0) + \frac{y^2}{2!} f_{yy}(0,0)\right] + \frac{x^3}{3!} f_{xxx}(0,0) + \frac{3x^2y}{3!} f_{xxy}(0,0) + \frac{3xy^2}{3!} f_{xyy}(0,0) + \frac{y^3}{3!} f_{yyy}(0,0)] + \cdots$$

which is known as Maclaurin's series for 2 variable.

1) Expand $e^x \cos y$ in power of x & y upto the 3rd degree at $(0,0)$

Soln Let $f(x,y) = e^x \cos y$, $f(0,0) = 1$

$f_x = e^x \cos y$ $\qquad f_x(0,0) = 1$

$f_y = -e^x \sin y$ $\qquad f_y(0,0) = 0$

$f_{xx} = e^x \cos y$ $\qquad f_{xx}(0,0) = 1$

$f_{xy} = -e^x \sin y$ $\qquad f_{xy}(0,0) = 0$

$f_{yy} = -e^x \cos y$ $\qquad f_{yy}(0,0) = -1$

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$f_{xxx} = e^x \cos y$ $\qquad f_{xxx}(0,0) = 1$

$f_{xxy} = -e^x \sin y$ $\qquad f_{xxy}(0,0) = 0$

$f_{xyy} = -e^x \cos y$ $\qquad f_{xyy}(0,0) = -1$

$f_{yyy} = e^x \sin y$ $\qquad f_{yyy}(0,0) = 0$

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{x^2}{2!} f_{xx}(0,0) + xy f_{xy}(0,0) + \frac{y^2}{2!} f_{yy}(0,0)] + \frac{x^3}{3!} f_{xxx}(0,0) + \frac{3x^2y}{3!} f_{xxy}(0,0) + \frac{3xy^2}{3!} f_{xyy} + \frac{y^3}{3!} f_{yyy}(0,0)] + \cdots$$

$$= 1 + x(1) + y(0) + \frac{x^2}{2!}(1) + xy(0) + \frac{y^2}{2!}(-1) + \frac{x^3}{3!}(1) + \frac{3x^2y}{3!}(0) + \frac{3xy^2}{3!}(-1) + \frac{y^3}{3!}(0) + \cdots$$

$$= 1 + x + \frac{x^2}{2!} - \frac{y^2}{2!} + \frac{x^3}{3!} - \frac{3xy^2}{3!} \cdots$$

$$+ 1 + x + \frac{1}{2!}(x^2 - y^2) + \frac{1}{3!}(x^3 - 3xy^2) + \cdots$$

2) Expand $e^x \log(1+y)$ in powers of x & y upto terms of 3rd degree, at $(0,0)$

Soln

Gn., $f(x,y) = e^x \log(1+y)$, $f(0,0) = \log 1 = 0$

$\therefore\ f_x = e^x \log(1+y)$ $\quad f_x(0,0) = 0$

$f_y = \dfrac{e^x \cdot 1}{1+y}$ $\quad f_y(0,0) = 1$

$f_{xx} = e^x \log(1+y)$ $\quad f_{xx} = 0$

$f_{xy} = \dfrac{e^x \cdot 1}{1+y}$ $\quad f_{xy}(0,0) = 1.$

$f_{yy} = \dfrac{-e^x}{(1+y)^2}$ $\quad f_{yy}(0,0) = -1$

$f_{xxx} = e^x \log(1+y)$ $\quad f_{xxx}(0,0) = 0$

$f_{xxy} = \dfrac{e^x \cdot 1}{1+y}$ $\quad f_{xxy}(0,0) = 1$

$f_{xyy} = \dfrac{-e^x}{(1+y)^2}$ $\quad f_{xyy}(0,0) = -1$

$f_{yyy} = \dfrac{2e^x}{(1+y)^3}$ $\quad f_{yyy}(0,0) = 2$

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$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] +$

$\frac{1}{2}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)]$

$+ \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \ldots$

$= 0 + x(0) + y(1) + \frac{1}{2}[x^2(0) + 2xy(1) + y^2(-1)]$

$+ \frac{1}{3!}[x^3(0) + 3x^2y(1) + 3xy^2(\cancel{0}^{-1}) + 2y^3]$

$= \cancel{0} + y + xy - \frac{y^2}{2} + \frac{1}{6}(3x^2y + 2y^3 - 3xy^2]$

$= y + xy - \frac{y^2}{2} + \frac{1}{2}x^2y + y^3/3 - \frac{xy^2}{2}$

3) Expand $x^3+y^3+xy^2$ in powers of $(x+1)$ & $(y-2)$ upto 2nd degree.

Soln Let $f(x,y) = x^3+y^3+xy^2$

Gn., $h = x+1$ $\quad$ $k = y-2$

$x = x_0 + h$ $\quad$ $y = y_0 + h$

$x = x_0 + x + 1$ $\quad$ $y = y_0 + y - 2$

$x_0 = -1$ $\quad$ $y_0 = 2$

$f(x,y) = x^3 + y^3 + xy^2$; $f(-1,2) = -1+8-4 = 3$

$f_x = 3x^2 + y^2$ $\quad$ $f_x(-1,2) = 3+4 = 7$

$f_y = 3y^2 + 2xy$ $\quad$ $f_y(-1,2) = 12 + 4 = 8$

$f_{xx} = 6x$ $\quad$ $f_{xx}(-1,2) = -6$

$f_{xy} = 2y$ $\quad$ $f_{xy}(-1,2) = 4$

$f_{yy} = 6y + 2x$ $\quad$ $f_{yy}(-1,2) = 10$

By Taylor's Expansion.

$$f(x,y) = 3 + [(x+1)7 + (y-2)\cdot 8] + \frac{(x+1)^2}{2!}(-6)$$

$$+ (x+1)(y-2)\cdot 4 + \frac{(y-2)^2}{2!}(10) + \cdots$$

$$x^3 + y^3 + xy^2 = 3 + (7x+1) + 8(y-2) - 3(x+1)^2 +$$
$$4(x+1)(y-2) + 5(y-2)^2$$

a.

H.W

4) Expand $e^x \sin y$ at $(0,0)$ upto the 2nd degree

5) Use Taylor's formula to expand the fns., $xy^2 + 2x - 3y$ in powers of $(x+2)$ & $(y-1)$

Jacobians:-

If u and v are fns. of the two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u, v with respect to x, y and is written as $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$

Thus $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

ii) $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

Properties:

1) If u and v are fns., of x & y, then

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

2) If u and v are fns., of p and q, where p and q are fns., of x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(p,q)} \cdot \frac{\partial(p,q)}{\partial(x,y)}$$

3) If functions u, v, w of three independent variables x, y, z are not independent variables, then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Problems:

1) If $u = x^2$, $v = y^2$ find $\frac{\partial(u,v)}{\partial(x,y)}$

Soln.:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix} = 4xy$$

2) If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$ & $\frac{\partial(r,\theta)}{\partial(x,y)}$ & also ~~find~~ verify that $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

Soln.,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

Gn., $x = r\cos\theta$ $y = r\sin\theta$; $x^2 = r^2\cos^2\theta$, $y^2 = r^2\sin^2\theta$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \qquad x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta)$$

$$\theta = \tan^{-1}(y/x) \qquad \therefore r^2 = x^2+y^2$$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+y^2/x^2}\left(-y/x^2\right) \qquad \frac{\partial r}{\partial x} = x/r$$

$$= -\frac{y}{x^2+y^2} = \frac{-y}{r^2}.$$

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$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$$

$$2r \cdot \frac{\partial r}{\partial y} = 2y$$

$$\frac{\partial r}{\partial y} = y/r$$

$$= \frac{x}{x^2+y^2} = \frac{x}{r^2}$$

$$\therefore \frac{\partial (r,\theta)}{\partial (x,y)} = \begin{vmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{vmatrix}$$

$$= \frac{x^2}{r^3} + \frac{y^2}{r^3} \Rightarrow \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = 1/r$$

$$\therefore \frac{\partial (x,y)}{\partial (r,\theta)} \cdot \frac{\partial (r,\theta)}{\partial (x,y)} = r \cdot 1/r = 1 \quad //.$$

Hence it is verified.

3) If $u = 2xy$ $V = x^2 - y^2$, $x = r\cos\theta$ $y = r\sin\theta$, find $\frac{\partial (u,v)}{\partial (r,\theta)}$

Soln:
$$\frac{\partial (u,v)}{\partial (r,\theta)} = \frac{\partial (u,v)}{\partial (x,y)} \cdot \frac{\partial (x,y)}{\partial (r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (-4y^2 - 4x^2)\,(r\cos^2\theta + r\sin^2\theta)$$

$$= -4(x^2+y^2) \cdot r$$

$$= -4r^3$$

$$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = -4r^3 \,//$$

4) If $p = 3x+2y-z$, $q = x-2y+z$

$r = x+2y-z$ P.T $\dfrac{\partial(p,q,r)}{\partial(x,y,z)} = 0$

Soln.,

$$\frac{\partial(p,q,r)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} & \frac{\partial q}{\partial z} \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 2 & -1 \\ 1 & -2 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= 3(2-2) - 2(-1-1) - 1(2+2)$$

$$= 0 + 4 - 4 = 0$$

H.W (NO).

5) If $x = uv$, $y = \frac{u}{v}$, P.T $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$

[Hint: $u^2 = xy$ $v^2 = x/y$]

5) If $x = uv$, $y = \frac{u+v}{u-v}$ find $\frac{\partial(u,v)}{\partial(x,y)}$.

6) If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$,

S.T $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$.

Maxima and Minima:

A fn., $f(x,y)$ is said to have a maximum value at $x=a$ & $y=b$ if $f(a,b) > f(a+h, b+k)$ $\forall$ small values of h & k.

A fn., $f(x,y)$ is said to have a minimum value at $x=a$ & $y=b$ if $f(a,b) < f(a+b, b+k)$ $\forall$ small values of h & k.

Necessary Condition for Maxima and Minima:

The necessary condition for existence of maxima and minima of $f(x,y)$ is $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$.

Find the values of $A = \frac{\partial^2 f}{\partial x^2}$, $B = \frac{\partial^2 f}{\partial x \partial y}$, $C = \frac{\partial^2 f}{\partial y^2}$ at $x=a$ & $y=b$

i) If $AC - B^2 > 0$ and $A > 0$, then $f(x,y)$ has a minimum value at (a,b)

(ii) If $AC - B^2 > 0$, & $A < 0$ then $f(x,y)$ has a maximum value at (a,b)

(iii) If $AC - B^2 < 0$ then $f(x,y)$ has neither a maximum nor a minimum at (a,b).

In this case the point (a,b) is called as Saddle point of $f(x,y)$

Extreme Value:-

$f(a,b)$ is said to be an extreme value of $f(x,y)$ if it is either a maximum or a minimum.

The maximum or minimum value of a function is called its extreme value

Stationary Point:

The point (a,b) at which $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ are called stationary points of the fns., $f(x,y)$. The value of $f(x,y)$ at the stationary points are called stationary values of $f(x,y)$.

1) Examine for maximum and minimum Values of the fn., $x^2+y^2+6x+12$

Soln., Let $f(x,y) = x^2+y^2+6x+12$

$\frac{\partial f}{\partial x} = 2x+6 \qquad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial^2 f}{\partial x \partial y} = 0 = B$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$\frac{\partial^2 f}{\partial x^2} = 2 = A, \quad \frac{\partial^2 f}{\partial y^2} = 2 = C$

for maxima & minima, $\frac{\partial f}{\partial x} = 0$, & $\frac{\partial f}{\partial y} = 0$

$2x+6=0 \qquad 2y=0$

$x = \frac{-6}{2} = -3 \qquad y = 0$

$\therefore$ The stationay points are $(-3,0)$, $A=2$, $B=0$, $C=2$

$AC-B^2 = 2(2)-0 = 4>0$ & $A>0$

$\therefore f(x,y)$ is Minimum at $(-3,0)$

$\therefore$ The minimum Value of $f(-3,0) =$

$9-18+12 = 3$

(2) In a plane triangle ABC find the maximum value of $\cos A \cos B \cos C$

Soln.,

In Δ^{le} ABC, $A+B+C=\pi$

let $f = \cos A \cos B \cos C$

$$= \cos A \cos B \cos [\pi - (A+B)]$$

$$= -\cos A \cos B \cos (A+B) \quad (\because \cos(180-\theta) = -\cos\theta)$$

$$\frac{\partial f}{\partial A} = -\cos B [\cos(A+B)(-\sin A) + \cos A [-\sin(A+B)]]$$

$$= \cos B [\cos(A+B)\sin A + \cos A \sin(A+B)]$$

$$= \cos B (\sin(A+A+B)) \quad [\because \sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\frac{\partial f}{\partial A} = \cos B \sin(2A+B)$$

$$\frac{\partial f}{\partial B} = -\cos A [\cos(A+B)(-\sin B) + \cos B(-\sin(A+B))]$$

$$= \cos A [\sin B \cos(A+B) + \cos B \sin(A+B)]$$

$$= \cos A [\sin(B+A+B)]$$

$$= \cos A \sin(A+2B)$$

For maxima (or) Minima

$$\frac{\partial f}{\partial A} = 0 \qquad \frac{\partial f}{\partial B} = 0$$

$\cos B \sin(2A+B) = 0$

$\therefore B = \pi/2$

(or) $2A+B = \pi$

$\cos A \sin(A+2B) = 0$

$A = \pi/2$

(or) $A+2B = \pi$

The cases $A = \pi/2$ & $B = \pi/2$ are impossible

$\therefore$ only possibility is $2A+B = \pi$ — (1)

$A+2B = \pi$ — (2)

Solving (1) & (2) $\Rightarrow$

$$\begin{aligned} 2A+B &= \pi \\ 2A+4B &= 2\pi \\ (-) \quad (-) & \quad (-) \\ \hline -3B &= -\pi \\ B &= \pi/3 \end{aligned}$$

Subt; $B = \pi/3$ in (1)

$$\begin{aligned} 2A+B &= \pi \\ 2A+\pi/3 &= \pi \\ 2A &= \pi - \pi/3 = 2\pi/3 \\ A &= \pi/3 \end{aligned}$$

$\therefore A = \pi/3, \; B = \pi/3$

$\therefore$ The stationary points are $(\pi/3, \pi/3)$

$$A = \frac{\partial^2 f}{\partial A^2} = \frac{\partial}{\partial A}\left(\frac{\partial f}{\partial A}\right)$$

$$= \frac{\partial}{\partial A}\left[\cos B \sin(2A+B)\right]$$

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$$= \cos B \cos(2A+B) \cdot 2$$

$$A = \frac{\partial^2 f}{\partial A^2} = 2\cos B \cos(2A+B)$$

$$B = \frac{\partial^2 f}{\partial A \partial B} = \frac{\partial}{\partial A}\left(\frac{\partial f}{\partial B}\right)$$

$$= \frac{\partial}{\partial A}\left[\cos A \sin(2B+A)\right]$$

$$= \cos A \cos(2B+A) + \sin(2B+A)(-\sin A)$$

$$= \cos A \cos(A+2B) - \sin A \sin(A+2B)$$

$$\left[\because \cos(A+B) = \cos A \cos B - \sin A \sin B\right]$$

$$B = \cos(A+A+2B) = \cos(2A+2B)$$

$$C = \frac{\partial^2 f}{\partial B^2} = \frac{\partial}{\partial B}\left(\frac{\partial f}{\partial B}\right) = \frac{\partial}{\partial B}\left[\cos A \sin(2B+A)\right]$$

$$= \cos A \cos(2B+A)(2) + \sin(2B+A)(0)$$

$$C = 2\cos A \cos(A+2B)$$

At $(\pi/3, \pi/3)$

$$\Rightarrow A = 2\cos(\pi/3)\cos(2(\pi/3)+\pi/3).$$

$$= 2\cos \pi/3 \cdot \cos\pi = 2(1/2)(-1) = -1$$

$$B = \cos\left(\frac{2\pi}{3}+\frac{2\pi}{3}\right) = \cos\left(\frac{4\pi}{3}\right) = -1/2$$

$$C = 2\cos(\pi/3)\cos(\pi/3 + 2\pi/3) = 2\cos\pi/3 \cos\pi$$

$$= 2\cos\pi/3 (-1)$$

$$= 2(1/2)(-1) = -1.$$

$AC - B^2 = (-1)(-1) - (-1/2)^2$

$= 1 - 1/4 = 3/4$

$A = -1$

$\therefore AC - B^2 > 0, \quad A < 0$

$\therefore f$ is maximum for $A = B = C = \pi/3$

$(\because A + B + C = \pi$

$\pi/3 + \pi/3 + C = \pi$

$C = \pi - 2\pi/3$

$= \pi/3)$

$\therefore$ Maximum value of

$\cos A \cos B \cos C = \cos \pi/3 \cos \pi/3 \cos \pi/3$

$= 1/2 \cdot 1/2 \cdot 1/2 = 1/8.$

3) Examine for maximum and minimum values for the fn., $x^3 + y^3 - 12x - 3y + 20$

Soln., $f(x, y) = x^3 + y^3 - 12x - 3y + 20$

$\frac{\partial f}{\partial x} = 3x^2 - 12, \quad \frac{\partial f}{\partial y} = 3y^2 - 3$

$A = \frac{\partial^2 f}{\partial x^2} = 6x, \quad B = \frac{\partial^2 f}{\partial x \partial y} = 0 = B$

$C = \frac{\partial^2 f}{\partial y^2} = 6y$

for Maxima & minima $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$

$3x^2 - 12 = 0$ $\qquad$ $3y^2 - 3 = 0$

$3x^2 = 12$ $\qquad$ $3y^2 = 3$

$x^2 = \frac{12}{3} \Rightarrow x^2 = 4$ $\qquad$ $y^2 = 3/3 = 1$

$x = \pm 2$

$y = \pm 1$

$\therefore$ The stationary points are $(2,1)$ $(-2,-1)$ $(2,-1)$ & $(-2,1)$

At $(2,1)$, $A = 6x = 6 \times 2 = 12$

$B = 0$

$C = 6y = 6 \times 1 = 6$

$\therefore AC - B^2 = 12 \times 6 - 0 = 72 > 0$

$A = 12 > 0$

$\therefore f(x,y)$ is minimum at $(2,1)$

$f(2,1) = (2)^3 + (1)^3 - 12(2) - 3(1) + 20 = 2$

At $(-2,-1)$, $A = 6x = 6(-2) = -12$

$B = 0$

$C = 6(-1) = -6$

$\therefore AC - B^2 = (-12)(-6) - 0 = 72 > 0$

$A = -12 < 0$

$\therefore f(x,y)$ is ~~minimum~~ max at $(-2,-1)$

$f(-2,-1) = (-2)^3 + (-1)^3 - 12(-2) - 3(-1) + 20 = 38$

At $(-2,+1)$, $A = 6(-2) = -12$, $B = 0$, $C = 6(1) = 6$

$AC - B^2 = \cancel{36(-2)} -72 < 0$

$A < 0$

$\therefore (-2,1)$ is a saddle pt.

At $(+2,-1)$, $AC - B^2 = -72 < 0$ $\qquad$ $A = 6(2) = 12 > 0$

$\Rightarrow (2,-1)$ is a saddle pt.

4) Examine for maximum and minimum value of $x^3y^2(12-3x-4y)$

(Ans:- pts : $(2,1)$ - max.)

5) Find the maximum & minimum value for x^2+y^2+4x+8. (Ans:- $(-3,0)$, - Mini)

UNIT-I

MATRICES

CAYLEY HAMILTON THEORM

Every square matrices satisfies its own charecteristic equation

$$|A-\lambda I| = 0$$

Results:-

consider 3x3 matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$$

the charecteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = sum of leading diagonal elements = $a_{11}+a_{22}+a_{33}$ (trace)

S_2 = sum of minors of its leading diagonals $= |\ \ | + |\ \ | + |\ \ |$

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$S_3 = |A|$

= determinat of A

① Verify cayley Hamilton theory for $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ also Find A^{-1}

let the charecteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

S_1 = trace of A = $1+2+1 = 4$

$$S_2 = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = -20.$$

$$S_3 = |A| = \begin{vmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix} = 1(2-6) - 3(4-3) + 7(8-2)$$

$$= +35$$

then the charecteristic equation is.

$$\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$$

By cayley hamilton theory

$$A^3 - 4A^2 - 20A - 35I = 0.$$

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20+92+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28 & 70+27+14 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$$A^3 - 4A^2 - 20A - 35I = \begin{vmatrix} 1 & 3 & 7 \\ 4 & \cancel{2} & 3 \\ 1 & 2 & 1 \end{vmatrix} \cancel{4}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4 \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20 \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\cancel{= 135\,(18093 - 15808)} \qquad = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (0)$$

Hence verified

(2) Verify cayley Hamilton theorem for $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}_{2\times 2}$ $\quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

let the charecteristic equatoon is

$$|A - \lambda I| = 0 \qquad \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2-0 \\ 3-0 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) - 6 = 0$$

$$\lambda^2 - 6\lambda - 1 = 0$$

the charecteristic equation is $A^2 - 6A - I = 0$.

$$A^2 - 6A - I = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} - 6 \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix} - 6 \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = (0)$$

③ Verify Hamilton theorem for $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

let the charecteristic equation

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

S_1 = trace of A = $2+2+2 = 6$

$$S_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4-1) + (4-2) + (4-1) = 3+2+3 = 8$$

$$S_3 = \begin{vmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 2(4-1) + 1(-2+1) + 2(1-2)$$

$$= 6-1-2 = 3$$

$$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

requried charecteristic equation $A^3 - 6A^2 + 8A - 3I = 0$

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 4+1+2 & -2-2-2 & 4+1+4 \\ -2-2-1 & 1+4+1 & -2-2-2 \\ 2+1+2 & -1-2-2 & 2+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -2 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 14+6+9 & -7-12-9 & 18-9+18 \\ -10-6-6 & 5+12+6 & -10-6-12 \\ 10+2+7 & -5-4-7 & 10+2+14 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 27 \\ -22 & 23 & -28 \\ 19 & -16 & 26 \end{bmatrix}$$

$$A^3 - 6A^2 + 8A - 3I = \begin{bmatrix} 29 & -28 & 27 \\ -22 & 23 & -28 \\ 19 & -16 & 26 \end{bmatrix} - 6\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -2 & 7 \end{bmatrix} + 8\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (0)$$

③ Verify cayley hamilton theory for $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ and also find A^4 and A^{-1}

Sol

let the characteristic equation

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

$$S_1 = 1+1+1 = 3$$

$$S_2 = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 0-2+1 = -1$$

$$S_3 = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 1(0) - 0 + 3(-2-1) = -9.$$

required characteristic equation.

$$\lambda^3 - 3\lambda^2 - 1\lambda + 9 = 0 \qquad A^3 - 3A^2 - 1A + 9I = 0$$

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4-6+12 & 0-3-6 & 12+3+6 \\ 3+4+8 & 0+2-4 & 9-2+4 \\ 0-4+10 & 0-2-5 & 0+2+5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix}$$

$$A^3 - 3A^2 - A + 9I = \begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix} - 3\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (0)$$

verified.

$$A^4 = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 16-9+0 & -12-6-12 & 24-12+30 \\ 12+6+0 & -9+4-8 & 18+8+20 \\ 0-6+0 & 0-4-10 & 0-8+25 \end{bmatrix} = \begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix}$$

$$A^3 A^{-1} - 3A^2 \cdot A^{-1} - IA \cdot A^{-1} + 9A^{-1} = 0$$

$$A^2 - 3A - I + 9A^{-1} = 0$$

$$9A^{-1} = I + 3A - A^2$$

$$A^{-1} = \frac{1}{9}\left[I + 3A - A^2\right]$$

$$= \frac{1}{9}\left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}\right]$$

$$A^{-1} = 1/9 \begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -2 \end{bmatrix}$$

To find A^4

$$A^3 - 3A^2 - A + 9I = A^4 - 3A^3 - A^2 + 9A = 0$$

$$A^4 = 3A^3 + A^2 - 9A$$

$$= 3\begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix} + \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -15 & -14 & 17 \end{bmatrix}$$

Result:-

use of cayley Hamilton theory

i) To find inverse of 'A'

ii) Any positive integral power of 'A' can be expressed.

② Verify cayley hamilton theory for $A = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}$ find A^{-1}

let the characteristic equation is

$$\lambda^2 - S_1\lambda + S_2 = 0.$$

$S_1 = \text{trace} = -1+4 = 3$ $\quad$ $S_2 = -4-6 = -10$

required characteristic equation

$$\lambda^2 - 3\lambda - 10 = 0 \qquad A^2 - 3A - 10I = 0.$$

$A^2 - 3A - 10I = 0 \Rightarrow A - 3I - 10A^{-1} = 0$

$10A^{-1} = A - 3I$

$$= \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{10}\begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1+6 & -3+12 \\ -2+8 & 6+16 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 6 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 9 \\ 6 & 22 \end{bmatrix} - 3\begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = (0)$$

Verified

① Verify cayley Hamilton theory for $A = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$ and A^{-1}

Sol

let the characteristic equation be

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 2+1-3 = 0$

$$S_2 = \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} 2 & -7 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= (-3-2) + (-6) + (2-4) = -5-6-2 = -13$$

$$S_3 = \begin{vmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{vmatrix} = 2(-3-2) - 2(-6-0) - 7(2)$$

$$= -10 + 12 - 14 = -12$$

the required characteristic equation is

$$\lambda^3 - 13\lambda + 12 = 0$$

$$A^3 - 13A + 12I = 0.$$

$$A^2 = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 4+4+0 & 4+2-7 & -14+4+21 \\ 4+2+0 & 4+1+2 & -14+2-6 \\ 0+2+0 & 0+1-3 & 0+2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -1 & 11 \\ 6 & 7 & -18 \\ 2 & -2 & 11 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 8 & -1 & 11 \\ 6 & 7 & -18 \\ 2 & -2 & 11 \end{bmatrix} \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 16-2+0 & 16-1+11 & -56-2-33 \\ 12+14+0 & 12+7-18 & -42+14+54 \\ 4-4+0 & 4-2+11 & -14-4-33 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 26 & -91 \\ 26 & 1 & 26 \\ 0 & 13 & -51 \end{bmatrix}$$

$$A^3 - 13A + 12I = \begin{bmatrix} 14 & 26 & -91 \\ 26 & 1 & 26 \\ 0 & 13 & -51 \end{bmatrix} - 13 \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix} + 12 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (0)$$

Verified.

$$A^3 \cdot A^{-1} - 13A \cdot A^{-1} + 12A^{-1} = 0 \Rightarrow A^2 - 13I + 12A^{-1} = 0$$

$$12A^{-1} = 13I - A^2 = 13 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 8 & -1 & 11 \\ 6 & 7 & -18 \\ 2 & -2 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 & -11 \\ -6 & -6 & 18 \\ -2 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{12} \begin{bmatrix} 5 & 1 & -11 \\ -6 & -6 & 18 \\ -2 & 2 & 2 \end{bmatrix}$$

(*) use cayley hamilton theory $A = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}$ and find A^3

Sol let the charecteristic equation $\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 = -1+4 = 3$ $\quad S_2 = \begin{vmatrix} -1 & 3 \\ 2 & 4 \end{vmatrix} = -4-6 = -10$

required charecteristic equation $\lambda^2 - 3\lambda - 10 = 0$

$A^2 - 3A - 10I = 0$ $\quad A^2 \cdot A - 3 \cdot A^2 - 10 \cdot A = 0$

$A^3 - 3A^2 - 10A = 0$

$A^3 = 3A^2 + 10A.$

$$A^2 = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1+6 & -3+12 \\ -2+8 & 6+16 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 6 & 22 \end{bmatrix}$$

$$A^3 = 3\begin{bmatrix} 7 & 9 \\ 6 & 22 \end{bmatrix} + 10\begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 11 & 57 \\ 38 & 106 \end{bmatrix}$$

$$A^2 - 3A - 10I = \begin{bmatrix} 7 & 9 \\ 6 & 22 \end{bmatrix} - 3\begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = (0)$$

Hence verified.

(1) use cayley hamilton theorem to find the value of the matrix given by

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

If the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

charecteristic equation

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 2+1+2 = 5$ $\quad S_2 = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$

$= 2-0+4-1+2-0 = 7$

$$S_3 = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 2(2-0) - 1(0-0) + 1(0-1)$$

$= 4-1 = 3$

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

$\boxed{A^3 - 5A^2 + 7A - 3I = 0}$ — (1)

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^5(A^3 - 5A^2 + 7A - 3I) + A^4 - 5A^3 + 8A^2 - 2A + I$

$= 0 + A^4 - 5A^3 + 7A^2 + A^2 - 3A + A + I$

$= A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$

$= A^2 + A + I$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

(2) Use cayley hamilton theorm. $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ to express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ an linear form.

$\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 = 1 + 3 = 4$ $\quad S_2 = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 3 + 2 = 5$

$\lambda^2 - 4\lambda + 5 = 0$

$A^2 - 4A + 5I = 0$

$A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$

$= A^4(A^2 - 4A + 5I) + 3A^4 - 12A^3 + 14A^2$

$= 3A^4 - 12A^3 + 14A^2$

$= 3A^4 - 12A^3 + 15A^2 - A^2 = 3A^2(A^2 - 4A + 5I) - A^2$

$= 0 - A^2$

$= 5I - 4A$

$$5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ 4 & -7 \end{bmatrix}$$

③ Given $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix}$ find Adj(A) using cayley hamilton

$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$

Charecteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 1-3-2 = -4$ $\quad S_2 = \begin{vmatrix} -3 & 1 \\ 1 & -2 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix}$

$$= (6-1) + (-2+2) + (-3-6) = -4$$

$$S_3 = \begin{vmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{vmatrix} = 1(6-1) - 2(-6-2) - 1(3+6)$$

$$= 5+16-9 = 12$$

$$\lambda^3 + 4\lambda^2 - 4\lambda - 12 = 0$$

$$A^3 + 4A^2 - 4A - 12I = 0$$

$$A^2 + 4A - 4I - 12A^{-1} = 0$$

$$12A^{-1} = A^2 + 4A - 4I$$

$$A^2 = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1+6-2 & 2-6-1 & -1+2+2 \\ 3-9+2 & 6+9+1 & -3-3-2 \\ 2+3-4 & 4-3-2 & -2+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ +1 & -1 & 3 \end{bmatrix}$$

$$12A^{-1} = \begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} + 4\begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 & -1 \\ 8 & 0 & -4 \\ 9 & 3 & -9 \end{bmatrix}$$

$$A^{-1} = \frac{1}{12}\begin{bmatrix} 5 & 3 & -1 \\ 8 & 0 & -4 \\ 9 & 3 & -9 \end{bmatrix}$$

$$\operatorname{Adj}(A) = \begin{bmatrix} 5 & 3 & -1 \\ 8 & 0 & -4 \\ 9 & 3 & -9 \end{bmatrix}$$

~~UNIT-1~~ CHAPTER-2

EIGUN VALUES AND EIGEN VECTOR

definition: let $A=[a_{ij}]$ be a square matrix of order n. If there exist a non-zero column vector x and scalar λ such that $Ax=\lambda x$, then λ is called an eigen value of the matrix A and x is called an eigen value vector corresponding to eigen value λ

eigen vector equation

$$[A-\lambda I]x=0$$

Result

the roots of the charecteristic equation is called eigen values.

① Find eigen value and eigen vector for $A=\begin{bmatrix}1&1&3\\1&5&1\\3&1&1\end{bmatrix}$

Let the characteristic equation is

$$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$$

$S_1 = 1+5+1 = 7$ $\qquad S_2=\begin{vmatrix}5&1\\1&1\end{vmatrix}+\begin{vmatrix}1&3\\3&1\end{vmatrix}+\begin{vmatrix}1&1\\1&5\end{vmatrix}$

$= 4+-8+4=0$

$S_3 = 1(5-1)-1(1-3)+3(1-15)$

$= \cancel{4}+2-42 = -36$

$$\boxed{\lambda^3-7\lambda^2+36=0}$$

To find eigen values.

-2	1	-7	0	36
	0	-2	+18	-36
	+1	-9	18	0

$\lambda=-2$ is a root

$\lambda^2-9\lambda+18=0$

$(\lambda-6)(\lambda-3)=0$

$\lambda = 3,6$

$\therefore$ the eigen values are $-2, 3, 6$.

To eigen vector $\qquad [A-\lambda I]x=0$

$$\begin{bmatrix}1-\lambda&1&3\\1&5-\lambda&1\\3&1&1-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

② ~~Fixed~~ $(1-\lambda)x_1+x_2+3x_3=0$ — ①

$x_1+(5-\lambda)x_2+1x_3=0$ — ②

$3x_1+x_2+(1-\lambda)x_3=0$ — ③

case-I $\lambda=-2$

$3x_1+x_2+3x_3=0$

$x_1+7x_2+x_3=0$

$3x_1+x_2+3x_3=0$.

m	l	f	m
1	3	3	1
7	1	1	7

$\frac{x_1}{1-21}=\frac{x_2}{3-3}=\frac{x_3}{21-1} \Rightarrow \frac{x_1}{-20}=\frac{x_2}{0}=\frac{x_3}{20}$

$\Rightarrow \frac{x_1}{+1}=\frac{x_2}{0}=\frac{x_3}{-1}$ $\quad x_1=1 \quad x_2=0 \quad x_3=-1$

the eigen vector is $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ the corresponding value of $\lambda=-2$.

case :2 $\lambda=3$

$-2x_1+x_2+3x_3=0$

$x_1+2x_2+x_3=0$

$3x_1+x_2-2x_3=0$

m	l	f	m
1	-2	-2	3
2	1	1	2

$\frac{x_1}{1+4}=\frac{x_2}{-2+2}=\frac{x_3}{-4-3} \Rightarrow \frac{x_1}{5}=\frac{x_2}{0}=\frac{x_3}{-12}$

the eigen vector is $\begin{bmatrix}5\\0\\-12\end{bmatrix}$ for corresponding $\lambda=3$

case-3: $\lambda=6$

$-5x_1+x_2+3x_3=0$

$x_1+-1x_2+x_3=0$

$3x_1+x_2-5x_3=0$

m	l	f	m
1	3	-5	1
-1	1	1	-1

$\frac{x_1}{1-3}=\frac{x_2}{3+5}=\frac{x_3}{5-1} \Rightarrow \frac{x_1}{-2}=\frac{x_2}{8}=\frac{x_3}{4}$

$\Rightarrow \frac{x_1}{1}=\frac{x_2}{-4}=\frac{x_3}{-2}$

let the vector is $x=\begin{bmatrix}1\\-4\\-2\end{bmatrix}$ at $\lambda=6$.

② find eigen values and eigen vectors of $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

Sol let characteristic equation be

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 6 \qquad S_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$

$= 4 + 4 - 1 + 4 = 11$

$S_3 = 2(4-0) - 1(+2) = 8 - 2 = 6$

$$\boxed{\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0}$$

1	1	-6	11	-6
	0	1	-5	6
	1	-5	6	0

$\lambda^2 - 5\lambda + 6 = 0$

$\lambda = 2, 3$

the eigen values are $1, 2, 3$

~~case~~ To find eigen vector.

$$\begin{bmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(2-\lambda)x_1 + 0x_2 - x_3 = 0 \qquad (2-\lambda)x_1 - x_3 = 0$ — ①

$0x_1 + (2-\lambda)x_2 + 0x_3 = 0 \qquad (2-\lambda)x_2 = 0$ — ②

$-x_1 + 0x_2 + (2-\lambda)x_3 = 0 \qquad -x_1 + (2-\lambda)x_3 = 0$ — ③

Case-1 let $\lambda = 1$

$x_1 - x_3 = 0$

$x_2 = 0$

$-x_1 + x_3 = 0$

m	l	f	m
0	-1	1	0
0	0	-1	0

m	l	f	m
0	-1	1	0
1	0	0	1

$\dfrac{x_1}{1} = \dfrac{x_2}{0} = \dfrac{x_3}{1}$

$x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Case-2 let $\lambda = 2$

$-x_3 = 0$

$0 = 0$

$-x_1 + 0 = 0$

m	l	f	m
0	-1	0	0
0	0	-1	0

$\dfrac{x_1}{0} = \dfrac{x_2}{1} = \dfrac{x_3}{0} \qquad x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Case-3: $\lambda = 3$

$-x_1 - x_3 = 0$

$-x_2 = 0$

$-x_1 - x_3 = 0$

$$\begin{array}{cccc} m & l & f & m \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{array}$$

$\frac{x_1}{0-1} = \frac{x_2}{0} = \frac{x_3}{1}$

$x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

eigen values 1, 2, 3

eigen vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

③ find the eigen values and eigen vectors for $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Sol let the charecteristic equation

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = \text{trace} = 0+0+0 = 0$

$S_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = (0-1) + 0-1 + 0-1 = -3$

$S_3 = -1\,(0-1) + 1\,(1-0) = 2$

$\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0 \qquad \lambda^3 - 3\lambda - 2 = 0$

$$\begin{array}{c|cccc} -1 & 1 & 0 & -3 & -2 \\ & 0 & -1 & +1 & +2 \\ \hline & +1 & -1 & -2 & 0 \end{array}$$

$+\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda^2 - \lambda - 2 = 0$

$\lambda^2 - 2\lambda + \lambda - 2 = 0 \qquad (\lambda - 2) + (\lambda + 1) = 0$

$\lambda = 2, -1$

eigen values $-1, -1, 2$

To eigen vector

$$\begin{bmatrix} (0-\lambda) & 1 & 1 \\ 1 & (0-\lambda) & 1 \\ 1 & 1 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\cancel{-2x_1 + x_2 + x_3} = 0$

$\cancel{x_1 - 2x_2 + x_3} = 0$

$-\lambda x_1 + x_2 + x_3 = 0$ — ①

$x_1 - x_2\lambda + x_3 = 0$ — ②

$x_1 + x_2 - \lambda x_3 = 0$ — ③

case-I : $\lambda = -1$

$$x_1 + x_2 + x_3 = 0$$
$$x_1 + x_2 + x_3 = 0$$
$$x_1 + x_2 + x_3 = 0 .$$

the system of equation reduced single equation $x_1+x_2+x_3=0$

if $x_2 = 0$ $\quad x_1 + x_3 = 0 \rightarrow x_1 = -x_3$

$$\frac{x_1}{1} = \frac{x_3}{-1}$$

the eigen vector $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

take $x_3 = 0$ $\quad x_1 + x_2 = 0$

$x_1 = -x_2 \quad X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

case-II

$$-2x_1 + x_2 + x_3 = 0$$
$$x_1 - 2x_2 + x_3 = 0$$
$$x_1 + x_2 - 2x_3 = 0 .$$

m l f m

1 $\cancel{1}$ -2 1

-2 1 1 -2

$$\frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1} \Rightarrow \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence the eigen values are $-1, -1, 2$

and eigen vectors are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(4) $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

sol let the charecteristic equation

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 2+1-1 = 2 \qquad S_2 = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}$

$= -1-3 + \cancel{-2}-\cancel{2} + \cancel{2}+\cancel{2} = -4$

$S_3 = 2\,(-1-3) + 2\,(-1-1) + 2\,(3-1)$

$= -8-4+4 = -8$

$$\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$$

$$\begin{array}{r|rrrr} 2 & 1 & -2 & -4 & 8 \\ & 0 & 2 & 0 & -8 \\ \hline & 1 & 0 & -4 & 0 \end{array}$$

$$\lambda^2 - 4 = 0$$
$$\lambda = \pm 2$$

the eigen values are $-2, 2, 2$

To find eigen vectors

$$\begin{bmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(2-\lambda)x_1 - 2x_2 + 2x_3 = 0 \quad -① $

$x_1 + (1-\lambda)x_2 + x_3 = 0 \quad -②$

$x_1 + 3x_2 + -(1+\lambda)x_3 = 0 \quad -③$

Case-1 $\lambda = -2$

$4x_1 - 2x_2 + 2x_3 = 0$

$x_1 + 3x_2 + x_3 = 0$

$x_1 + 3x_2 + x_3 = 0$

m	l	f	m
-2	2	4	-2
3	1	1	3

$\frac{x_1}{-2-6} = \frac{x_2}{2-4} = \frac{x_3}{12+2}$

$\frac{x_1}{-8} = \frac{x_2}{-2} = \frac{x_3}{14}$

$\Rightarrow \frac{x_1}{4} = \frac{x_2}{1} = \frac{x_3}{-7}$ $\quad X_1 = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$

Case-2 $\lambda = 2$.

$0x_1 - 2x_2 + 2x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$x_1 + 3x_2 - 8x_3 = 0$

m	l	f	m
-2	2	0	-2
-1	1	1	-1

$\frac{x_1}{-2+2} = \frac{x_2}{2-0} = \frac{x_3}{0+2} \Rightarrow \frac{x_1}{0} = \frac{x_2}{2} = \frac{x_3}{2}$

$x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

m	l	f	m
-1	1	1	-1
3	-3	1	3

$\frac{x_1}{3-3} = \frac{x_2}{1+3} = \frac{x_3}{3+1}$

$x_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

the eigen values are $-2, 2, 2$

eigen vectors are

$\begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

5/7

(5) find the eigen values and eigen vectors for $A = \begin{bmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{bmatrix}$

sol let the characteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 6 - 13 + 4 = -3$

$$S_2 = \begin{vmatrix} -13 & 10 \\ -6 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 5 \\ 7 & 4 \end{vmatrix} + \begin{vmatrix} 6 & -6 \\ 14 & -13 \end{vmatrix}$$

$= -52 + 60 + 24 - 35 - 78 + 84 = 144 + 24 - 87 - 78$

$= 168 - 165 = 3$

$S_3 = 6(-52+60) + 6(56-70) + 5(-84+91)$

$= 6(8) + 6(-14) + 5(7)$

$= 48 - 84 + 35 = 83 - 84 = -1$

$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$.

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 3 & 1 \\ & 0 & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

$\lambda^2 + 2\lambda + 1 = 0$

$\lambda^2 + \lambda + \lambda + 1 = 0$

$\lambda(\lambda+1) + 1(\lambda+1) = 0 \Rightarrow \lambda = -1, -1$

$\therefore$ the eigen values are $-1, -1, -1$

To find eigen vectors.

$$\begin{bmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(6-\lambda)x_1 - 6x_2 + 5x_3 = 0$ — (1)

$14x_1 - (13+\lambda)x_2 + 10x_3 = 0$ — (2)

$7x_1 - 6x_2 + (4-\lambda)x_3 = 0$ — (3)

case-1: $\lambda = -1$

$7x_1 - 6x_2 + 5x_3 = 0$

$14x_1 - 12x_2 + 10x_3 = 0$

$7x_1 - 6x_2 + 5x_3 = 0$.

the system of the equation $7x_1 - 6x_2 + 5x_3 = 0$.

Let $x_2=0$ $\quad 7x_1-6x_2+5x_3=0$

$7x_1=-5x_3$

$\frac{x_1}{5}=\frac{x_3}{-7}$ $\quad X_1=\begin{bmatrix}5\\0\\-7\end{bmatrix}$

$x_3=0$ $\quad 7x_1-6x_2=0$

$7x_1=6x_2$

$\frac{x_1}{6}=\frac{x_2}{7}$ $\quad x_2=\begin{bmatrix}6\\7\\0\end{bmatrix}$

$x_1=0$ $\quad -6x_2+5x_3=0$

$6x_2=5x_3$ $\quad X_3=\begin{bmatrix}0\\5\\6\end{bmatrix}$

$\frac{x_2}{5}=\frac{x_3}{6}$

$\therefore$ the eigen values are $-1,-1,-1$

and eigen vectors $\begin{bmatrix}5\\0\\-7\end{bmatrix},\begin{bmatrix}6\\7\\0\end{bmatrix},\begin{bmatrix}0\\5\\6\end{bmatrix}$

ii) $A=\begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$

sol let the characteristic equation is

$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$

$S_1=6$ $\quad S_2=\begin{vmatrix}2&1\\0&2\end{vmatrix}+\begin{vmatrix}2&0\\0&2\end{vmatrix}+\begin{vmatrix}2&1\\0&2\end{vmatrix}$

$=(4-0)+(4-0)+(4-0)=12$

$S_3=2(4-0)-1(0-0)-0=8$

$\boxed{\lambda^3-6\lambda+12\lambda-8=0}$

$$\begin{array}{r|rrrr} 1 & 1 & -4 & 12 & -8 \\ & 0 & 1 & -3 & 9 \\ \hline & 1 & -3 & 9 & 1 \end{array}$$

$$\begin{array}{r|rrrr} +2 & 1 & -4 & 12 & -8 \\ & 0 & -2 & 12 & \\ \hline & 1 & -6 & & \end{array}$$

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 12 & -8 \\ & 0 & -1 & 5 & \\ \hline & 1 & -5 & 17 & \end{array}$$

The eigen values 2, 2, 2

To eigen vectors.

$$\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(2-\lambda)x_1 + x_2 + 0 = 0$ — (1)

$(2-\lambda)x_2 + x_3 = 0$ — (2)

$(2-\lambda)x_3 = 0$ — (3)

Case I : $\lambda = 2$

$1x_2 = 0 \quad 1x_3 = 0$

$0 = 0$

m	l	f	m
1	0	0	1
0	1	0	1

$$\frac{x_1}{1-0} = \frac{x_2}{0} = \frac{x_3}{0} \qquad x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

the eigen vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ the eigen values are $\lambda = 2, 2, 2$

properties

* Find the eigen values of $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

Let the charm - eq.

$$\lambda^2 - S_1\lambda + S_2 = 0$$

$S_1 = 0 \qquad S_2 = -1-3 = -4$

$$\lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$$

To find eigen vectors

$$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(1-\lambda)x_1 + x_2 = 0$ — (1) $\qquad 3x_1 - (1+\lambda)x_2 = 0$

Case I $\lambda = 2$

$-x_1 + x_2 = 0 \qquad x_1 - x_2 = 0$

$3x_1 - 3x_2 = 0 \qquad x_1 - x_2 = 0 \qquad x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

case $\lambda = -2$

$3x_1 + x_2 = 0 \qquad \frac{x_1}{1} = \frac{x_2}{-3} \qquad x = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$3x_1 + x_2 = 0$

ii) $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Sol let the charecteristic equation

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

S_1 = trace of A = $3+5+3 = 11$

$$S_2 = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= 15-1+9-1+15-1 = 39-3 = 36.$$

$$S_3 = |A| = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 3(15-1)+1(-3+1)+1(1-5)$$

$$= 3\times14 - 2 - 4$$

$$= 42-6 = 36.$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

2	1	-11	36	-36
	0	2	-18	36
	1	-9	18	0

$$\lambda^2 - 9\lambda + 18 = 0$$

$$\lambda^2 - 3\lambda - 6\lambda + 18 = 0$$

$$(\lambda-3)(\lambda-6) = 0$$

$$\lambda = 3, 6$$

$\therefore$ the eigen values are $2, 3, 6$

Case To find eigen vectors

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(3-\lambda)x_1 - x_2 + x_3 = 0$ — (1)

$-1x_1 + (5-\lambda)x_2 + -x_3 = 0$ — (2)

$x_1 - x_2 + (3-\lambda)x_3$ — (3)

case-I $\lambda = 2$

$x_1 - x_2 + x_3 = 0$

$-x_1 + 3x_2 - x_3 = 0$

$x_1 - x_2 + x_3 = 0$.

m	l	f	m
-1	1	1	-1
3	-1	-1	3

$$\frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2} \Rightarrow X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

case-2 $\lambda = 3$

$0x_1 - x_2 + x_3 = 0$

$-x_1 + 2x_2 - x_3 = 0$

$x_1 - x_2 + 0x_3 = 0$

m	l	f	m
-1	1	0	-1
2	-1	-1	2

$$\frac{x_1}{1-2} = \frac{x_2}{-1+0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1} \qquad X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

case-3 $\lambda = 6$

$-3x_1 - x_2 + x_3 = 0$

$-x_1 - x_2 - x_3 = 0$

$x_1 - x_2 - 3x_3 = 0$

m	l	f	m
-1	1	-3	-1
-1	-1	-1	-1

$$\frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1}$$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2} \qquad X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

3) $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ +2 & -1 & 3 \end{bmatrix}$

Let the charecteristic equation

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 .$$

$S_1 = 6 + 3 + 3 = 12$

$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$

$= 9 - 1 + 18 - 4 + 18 - 4 = 45 - 9 = 36$

$S_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 6(9-1) + 2(-6+2) + 2(2-6)$

$= 48 - 8 - 8 = 32$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$\lambda^3 - 12\lambda^2 + 40\lambda - 32 = 0$

$$\begin{array}{c|cccc} 1 & 1 & -12 & 40 & -32 \\ & 0 & 1 & -11 & \\ \hline & 1 & -11 & & \end{array}$$

$$\begin{array}{c|cccc} +2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & 0 \end{array}$$

$\lambda^2 - 10\lambda + 16 = 0$

$\lambda^2 - 10\lambda + 16 = 0$

$\lambda^2 - 8\lambda - 2\lambda + 16 = 0$

$\lambda(\lambda - 8) - 2(\lambda - 8) = 0$

$(\lambda - 2)(\lambda - 8) = 0$

$\lambda = 2, 2, 8$

To find eigen vectors

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(6-\lambda)x_1 - 2x_2 + 2x_3 = 0$ — (1)

$-2x_1 + (3-\lambda)x_2 - x_3 = 0$ — (2)

$2x_1 - x_2 + (3-\lambda)x_3 = 0$ — (3)

Case-1 $\lambda = 2$

$4x_1 - 2x_2 + 2x_2 = 0$

$-2x_1 + x_2 - x_3 = 0$

$2x_1 - x_2 + x_3 = 0$.

case-i $x_1 = 0$ $\quad x_2 = x_3$ $\quad x_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$x_2 = 0$ $\quad -2x_1 = x_3$ $\quad x_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

$\frac{x_1}{1} = \frac{x_3}{-2}$

Case-2 $\lambda = 8$

$-2x_1 - 2x_2 + 2x_3 = 0$.

$-2x_1 - 5x_2 - x_3 = 0$

$2x_1 - x_2 - 5x_3 = 0$.

m	l	f	m
-2	2	-2	2
-5	-1	-2	-5

$\frac{x_1}{2+10} = \frac{x_2}{-4+2} = \frac{x_3}{10+4}$

$\frac{x_1}{12} = \frac{x_2}{-2} = \frac{x_3}{14}$

$x_3 = \begin{pmatrix} 6 \\ -1 \\ 7 \end{pmatrix}$

DIAGONALISATION BY ORTHOGONAL TRANSFORMATION

diagonalisation of Matrix $\boxed{D = N^T A N}$

where N is normalised modal matrix

Modal Matrix

let A be a square matrix (symmetrix matrix)

the Model Matrix M is a matrix whose column are Eigen values of Matrix A

① diagonalise the matrix $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$ by using orthogonal transformation

Sol let the charecteristic equation

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 2+6+2 = 10$

$S_2 = \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 6 \end{vmatrix}$

$= 12-0 + 4-16+12-0 = 28-16 = 12$

$S_3 = \begin{vmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{vmatrix} = 2(12-0)+0+4(0-24)$

$= 24-96 = -72$

$$\lambda^3 - 10\lambda^2 + 12\lambda + 72 = 0$$

-2	1	-10	12	72
	0	-2	24	-72
	1	-12	36	0

$$\lambda^2 - 12\lambda + 36 = 0$$

$$\lambda^2 - 6\lambda - 6\lambda + 36 = 0$$

$$\lambda(\lambda-6) - 6(\lambda-6) = 0$$

$$\lambda = 6, 6$$

eigen values $-2, 6, 6$.

To find eigen vectors

$$\begin{bmatrix} 2-\lambda & 0 & 4 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(2-\lambda) x_1 + 0x_2 + 4x_3 = 0$ — ①

$0x_1 + (6-\lambda) x_2 + 0x_3 = 0$ — ②

$4x_1 + 0x_2 + (2-\lambda) x_3 = 0$ — ③

Case-1 $\lambda = -2$

$4x_1 + 0x_2 + 4x_3 = 0$

$0x_1 + 8x_2 + 0x_3 = 0$

$4x_1 + 0x_2 + 4x_3 = 0$

$$\begin{array}{cccc} m & l & f & m \\ 0 & 4 & 4 & 0 \\ 8 & 0 & 0 & 8 \end{array}$$

$$\frac{x_1}{0-32} = \frac{x_2}{0-0} = \frac{x_3}{32-0}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$|x_1| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$

case-2 $\lambda = 6$

$-4x_1 + 0x_1 + 4x_3 = 0$

$0x_1 + 0x_2 + 0x_3 = 0$

$4x_1 + 0x_2 - 4x_3 = 0$

$4x_1 + 0x_2 - 4x_3 = 0$

$4x_1 = 4x_3 \Rightarrow \frac{x_1}{1} = \frac{x_3}{1}$

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \cancel{x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$$

$|x_2| = \sqrt{1+0+1} = \sqrt{2}$

let $x_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

If $\bar{a}$ & $\bar{b}$ are orthogonal then $\bar{a} \cdot \bar{b} = 0$.

Here since x_3 is $\perp$ to both x_1 and x_2

$a + 0b - c = 0$

$a + 0b + c = 0$

$$\begin{array}{cccc} m & l & f & m \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}$$

$$\frac{a}{0} = \frac{b}{-2} = \frac{c}{0} \qquad \frac{a}{0} = \frac{b}{1} = \frac{c}{0}$$

$$X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad |x| = \sqrt{1} = 1$$

Hence modal matrix

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

To Find Normalised matrix

$$N = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \qquad N^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$N^TAN = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}+0-2\sqrt{2} & \sqrt{2}+0+2\sqrt{2} & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+6+0 \\ 2\sqrt{2}+0-\sqrt{2} & 2\sqrt{2}+0+\sqrt{2} & 0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\sqrt{2} & 3\sqrt{2} & 0 \\ 0 & 0 & 6 \\ \sqrt{2} & 3\sqrt{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1+0-1 & 3+0-3 & 0+0+0 \\ -1+0+1 & 3+0+3 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+6+0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$N^TAN = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Quadratic form:- $Q = X^T A X$

- A homogenious polynomial of the second degree in number of variables is called quadratic form.
- linear Transformation of a Quadratic form $Q = X^T A X$ is $X = NY$
- Reduce quadratic form $Q = X^T A X$ into a canonical form $= Y^T D Y$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

Note

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

① Reduce the quadratic form $2x_1^2 + 6x_2^2 + 2x_3^2 + 8x_1x_3$ into canonical form by orthogonal reduction

Sol let $Q: 2x_1^2 + 6x_2^2 + 2x_3^2 + 0x_1x_2 + 0x_2x_3 + 8x_1x_3$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$a_{11} = \text{co-eff } x_1^2 = 2$	$a_{12} = a_{21} = \frac{1}{2}[\text{coeff } x_1x_2] = \frac{1}{2} \times 0 = 0$
$a_{22} = \text{coeff } x_2^2 = 6$	$a_{13} = a_{31} = \frac{1}{2}[\text{coeff } x_{13}] = \frac{1}{2} \times 8 = 4$
$a_{33} = \text{coeff } x_3^2 = 2$	$a_{23} = a_{23} = \frac{1}{2}[\text{coeff } x_{23}] = \frac{1}{2} \times 0 = 0$

$$\therefore A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

the eigen values $-2, 6, 6$

the eigen vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$D = N^T A N = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

let the linear transformat $X = NY$ Transform the quadratic form Q into canonical form $= Y^T D Y$

$$= [y_1 \; y_2 \; y_3] \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= -2y_1^2 + 6y_2^2 + 6y_3^2$$

② Reduce Quadratic form Q :- $2x_1^2+x_2^2+x_3^2+2x_1x_2-2x_1x_3-4x_2x_3$ into canonical form by using orthogonal form

Sol

$Q: 2x_1^2+x_2^2+x_3^2+2x_1x_2-2x_1x_3-4x_2x_3$

$a_{11}=2 \qquad a_{12}=a_{21}=\frac{1}{2}(2)=1$

$a_{22}=1 \qquad a_{23}=a_{32}=\frac{1}{2}(-4)=-2$

$a_{33}=1 \qquad a_{13}=a_{31}=\frac{1}{2}(-2)=-1$

$$A=\begin{bmatrix}2 & 1 & -1\\ 1 & 1 & -2\\ -1 & -2 & 1\end{bmatrix}$$

let the charecteristic equation

$$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$$

$S_1=2+1+1=4$

$$S_2=\begin{vmatrix}1 & -2\\ -2 & 1\end{vmatrix}+\begin{vmatrix}2 & -1\\ -1 & 1\end{vmatrix}+\begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix}$$

$$=\cancel{1}-\cancel{4}+\cancel{2}-\cancel{1}+\cancel{2}-1=-1$$

$$S_3=\begin{vmatrix}2 & 1 & -1\\ 1 & 1 & -2\\ -1 & -2 & 1\end{vmatrix}=2(1-4)-1(1-2)-1(-2+1)$$

$$=-6+1+1=-4$$

$$\lambda^3-4\lambda^2-\lambda+4=0$$

+1	1	-4	-1	4
	0	+1	-3	-4
	1	-3	-4	0

$$\lambda^2-3\lambda-4=0$$

$$\lambda^2-4\lambda+\lambda-4=0$$

$$\lambda(\lambda-4)+1(\lambda-4)=0$$

$$(\lambda-4)(\lambda+1)=0$$

$$\lambda=-1,4$$

the eigen values are $-1,1,4$

To find eigen vectors.

$$\begin{bmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(2-\lambda)x_1 + x_2 - x_3 = 0$ —— ①

$x_1 + (1-\lambda)x_2 - 2x_3 = 0$ —— ②

$-x_1 - 2x_2 + (1-\lambda)x_3 = 0$ —— ③

Case-1 $\lambda = -1$

$3x_1 + x_2 - x_3 = 0$

$x_1 + 2x_2 - 2x_3 = 0$

$-x_1 - 2x_2 + 2x_3 = 0$

m	l	f	m
1	-1	3	1
2	-2	1	2

$$\frac{x_1}{-2+2} = \frac{x_2}{-1+6} = \frac{x_3}{6-1}$$

$$\frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\|x_1\| = \sqrt{2}$

Case-2 $\lambda = 1$

$x_1 + x_2 - x_3 = 0$

$x_1 + 0x_2 - 2x_3 = 0$

$-x_1 - 2x_2 + 0x_3 = 0$

m	l	f	m
1	-1	1	1
0	-2	1	0

$$\frac{x_1}{-2+0} = \frac{x_2}{-1+2} = \frac{x_3}{0-1}$$

$$\frac{x_1}{+2} = \frac{x_2}{-1} = \frac{x_3}{+1}$$

$X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$\|x_2\| = \sqrt{6}$

Case-3 $\lambda = 4$

$-2x_1 + x_2 - x_3 = 0$

$x_1 - 3x_2 - 2x_3 = 0$

$-x_1 - 2x_2 - 3x_3 = 0$

m	l	f	m
1	-1	-2	1
-3	-2	1	-3

$$\frac{x_1}{-2-3} = \frac{x_2}{-1-4} = \frac{x_3}{6-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{5}$$

$X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$x_3 = \sqrt{3}$

$$M = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix}$$

$$N^T = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$$

$$N^T A N = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 0+1/\sqrt{2}-1/\sqrt{2} & 4/\sqrt{6}-1/\sqrt{6}-1/\sqrt{6} & 2/\sqrt{3}+1/\sqrt{3}+1/\sqrt{3} \\ 0+1/\sqrt{2}-2/\sqrt{2} & 2/\sqrt{6}-1/\sqrt{6}-2/\sqrt{6} & +1/\sqrt{3}+1/\sqrt{3}-2/\sqrt{3} \\ 0-2/\sqrt{2}+1/\sqrt{2} & -2/\sqrt{6}+2/\sqrt{6}+1/\sqrt{6} & -1/\sqrt{3}-2/\sqrt{3}-1/\sqrt{3} \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

canonical form $= -y_1^2 + y_2^2 + 4y_3^2$

① Reduce Quadratic form $Q: 6x^2+3y^2+3z^2-4xy-2yz+4zx$ into canonical form by using orthogonal transformation also find rank, index, signature and nature of canonical matrix.

Sol Given $Q: 6x^2+3y^2+3z^2-4xy-2yz+4zx$.

$a_{11} = 6$ $\quad a_{12} = a_{21} = \frac{1}{2}(\text{coefficient of } xy) = -2$

$a_{22} = 3$ $\quad a_{23} = a_{32} = \frac{1}{2}(-2) = -1$

$a_{33} = 3$ $\quad a_{13} = a_{31} = \frac{1}{2}(4) = 2$

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

let the charecteristic equation

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

S_1 = trace of $A = 6+3+3 = 12$

$$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= 9-1+18-4+18-4 = 45-9 = 36$$

$$S_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 6(9-1)+2(-6+2)+2(2-6)$$

$$= 48-8-8 = 32$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\begin{array}{c|cccc} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & 0 \end{array}$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$\lambda^2 - 8\lambda - 2\lambda + 16 = 0$$

$$\lambda(\lambda-8) - 2(\lambda-8) = 0$$

$$\lambda = 2, 8$$

the eigen values are $2, 2, 8$

To find eigen vectors

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(6-\lambda)x_1 - 2x_2 + 2x_3 = 0$ — (1)

$-2x_1 + (3-\lambda)x_2 - x_3 = 0$ — (2)

$2x_1 - x_2 + (3-\lambda)x_3 = 0$ — (3)

case-I $\lambda = 8$

$-2x_1 - 2x_2 + 2x_3 = 0$

$-2x_1 - 5x_2 - x_3 = 0$

$2x_1 - x_2 - 5x_3 = 0$

m	l	f	m
-2	2	-2	-2
-5	-1	-2	-5

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad \|X_1\| = \sqrt{6}$$

18
18
9

45

Case-II $\lambda = 2$

$4x_1 - 2x_2 + 2x_3 = 0$

$-2x_1 + x_2 - x_3 = 0$

$2x_1 - x_2 + x_3 = 0$

the system of equations are same so we consider.

$2x_1 - x_2 + x_3 = 0$

let $x_2 = 0$ $\quad 2x_1 = -x_3$ $\quad X_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

$\frac{x_1}{1} = \frac{x_3}{-2}$ $\quad |x_2| = \sqrt{5}$

let the matrix $x_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$2a - b + c = 0$

$a + 0b - 2c = 0$

$\begin{matrix} m & l & f & m \\ -1 & 1 & 2 & -1 \\ 0 & -2 & 1 & 0 \end{matrix}$

$\frac{x_1}{2-0} = \frac{x_2}{1+4} = \frac{x_3}{0+1}$ $\quad x_3 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$

$\frac{x_1}{2} = \frac{x_2}{5} = \frac{x_3}{1}$ $\quad |x_3| = \sqrt{30}$

~~the~~ $M = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 0 & 5 \\ 1 & -2 & 1 \end{bmatrix}$

$\cancel{N} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 2 & 5 & 1 \end{bmatrix}$

$D = N^TAN = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 0 & 5 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 2 & 5 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 & 2 \\ -1 & 0 & 5 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 12-2+4 & -6+0+10 & 6+4+2 \\ -4+3-2 & 2+0-5 & -2-6-1 \\ 4-1+6 & -2+0+15 & 2+2+3 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 & 2 \\ -1 & 0 & 5 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 14 & 4 & 12 \\ -3 & -3 & -9 \\ 9 & 13 & 7 \end{bmatrix}$ $\cancel{*}$ $\quad \cancel{28-3+18}$

$$D = N^T A N = \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{30} & 5/\sqrt{30} & 1/\sqrt{30} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{5} & 2/\sqrt{30} \\ -1/\sqrt{6} & 0/\sqrt{5} & 5/\sqrt{30} \\ 1/\sqrt{6} & -2/\sqrt{5} & 1/\sqrt{30} \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{30} & 5/\sqrt{30} & 1/\sqrt{30} \end{bmatrix} \begin{bmatrix} 12/\sqrt{6} + 2/\sqrt{6} + 2/\sqrt{6} & 6/\sqrt{5} + 0 - 4/\sqrt{5} & 12/\sqrt{30} - 10/\sqrt{30} + 2/\sqrt{30} \\ -4/\sqrt{6} - 3/\sqrt{6} - 1/\sqrt{6} & -2/\sqrt{5} + 0 + 2/\sqrt{5} & -4/\sqrt{30} + 15/\sqrt{30} - 1/\sqrt{30} \\ 4/\sqrt{6} + 1/\sqrt{6} + 3/\sqrt{6} & 2/\sqrt{5} + 0 - 6/\sqrt{5} & 4/\sqrt{30} - 5/\sqrt{30} + 3/\sqrt{30} \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{30} & 5/\sqrt{30} & 1/\sqrt{30} \end{bmatrix} \begin{bmatrix} 15/\sqrt{6} & 2/\sqrt{5} & 4/\sqrt{30} \\ -8/\sqrt{6} & 0 & 10/\sqrt{30} \\ 8/\sqrt{6} & -4/\sqrt{5} & 2/\sqrt{30} \end{bmatrix}$$

$$= \begin{bmatrix} 30/\cancel{6} + 8/\cancel{6} + 8/\cancel{6} & 0 & 0 \\ 0 & 2/\cancel{5} + 8/\cancel{5} & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

equation $= Y^T D Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 8y_1^2 + 2y_2^2 + 2y_3^2$

Rank = No. of non-zero eigen value

$r = 3$

Index = No. of the square terms in a canonical form

$s = 3$

Signature $= 2s - r$

$= 3$

nature : positive definite.

4) Reduce $Q: 2x_1x_2 + 2x_2x_3 + 2x_3x_1$ into canonical form by using orthogonal transformation

(4) Reduce Quadratic equation $2x_1x_2 + 2x_2x_3 + 2x_3x_1$ into canonical form by using orthogonal transformation and also find out rank, index, Signature and nature.

Sol let the quadratic equation be

$$0x_1^2 + 0x_2^2 + 0x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_3x_1$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

a_{11} = coefficient of $x_1^2 = 0$ $a_{12} = a_{21} = \frac{1}{2}$(coefficient of x_1x_2) $= \frac{1}{2} \times 2 = 1$

a_{22} = coefficient of $x_2^2 = 0$ $a_{13} = a_{31} = \frac{1}{2}$(coefficient of x_1x_3) $= \frac{1}{2} \times 2 = 1$

a_{33} = coefficient of $x_3^2 = 0$. $a_{23} = a_{32} = \frac{1}{2}$(coefficient of x_2x_3) $= \frac{1}{2} \times 2 = 1$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

let the charecteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

S_1 = trace of $A = 0+0+0 = 0$

S_2 = Sum of the minors of leading diagonals

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 0-1 + 0-1 + 0-1 = -3$$

$$S_3 = |A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0(0-1) - 1(0-1) + 1(0-1)$$

$$= 0+1+1 = 2$$

$$\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$$

$$\begin{array}{r|rrrr} -1 & 1 & +0 & -3 & -2 \\ & 0 & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

$$\lambda^2 - \lambda - 2 = 0$$

$$\lambda^2-\lambda-2=0$$

$$\lambda^2-2\lambda+\lambda-2=0$$

$$\lambda(\lambda-2)+1(\lambda-2)=0$$

$$(\lambda-2)(\lambda+1)=0$$

$$\lambda=2 \text{ or } -1$$

the eigen values are $-1,-1,2$

To find vectors eigen

$$\begin{bmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(0-\lambda)x_1+x_2+x_3=0 \quad —① $$

$$x_1+(0-\lambda)x_2+x_3=0 \quad —②$$

$$x_1+x_2+(0-\lambda)x_3=0 \quad —③$$

case-I $\lambda=2$

$$-2x_1+x_2+x_3=0$$

$$x_1-2x_2+x_3=0$$

$$x_1+x_2-2x_3=0$$

m	l	f	m
1	1	-2	1
-2	1	1	-2

$$\frac{x_1}{1+2}=\frac{x_2}{1+2}=\frac{x_3}{4-1} \Rightarrow \frac{x_1}{3}=\frac{x_2}{3}=\frac{x_3}{3}$$

$$X_1=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad |x_1|=\sqrt{3}$$

case-II $\lambda=-1$

$$x_1+x_2+x_3=0$$

$$x_1+x_2+x_3=0$$

$$x_1+x_2+x_3=0.$$

$$x_1=0 \qquad x_2=-x_3 \Rightarrow \frac{x_2}{1}=\frac{x_3}{-1} \qquad X_2=\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad |x_2|=\sqrt{2}$$

$$X_3=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

X_3 is perpendicular to x_1 and x_2

$$a+b+c=0$$

$$0a+b-c=0$$

m	l	f	m
1	1	1	1
1	-1	0	1

$$\frac{x_1}{-1-1}=\frac{x_2}{0+1}=\frac{x_3}{1-0} \Rightarrow \frac{x_1}{-2}=\frac{x_2}{1}=\frac{x_3}{1}$$

$$X_3=\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \qquad |x_3|=\sqrt{6}$$

$$M=\begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$N=\begin{bmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \qquad N^T=\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$$D=N^TAN=\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}=\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

linear equation $x = NY$

equation $= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 2y_1^2 - y_2^2 - y_3^2$

Rank = No. of non-zero eigen values

$r = 1$

Index = No. of the square terms in a canonical form

$s = 0$

Signature $= 2s - r = -1$

---

① diagonalise $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ By using orthogonal transformation.

Sol Let the characteristic equation

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 8 + 7 + 3 = 18$

$S_2 = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$

$= 21 - 16 + 24 - 4 + 56 - 36$

$= 101 - 56 = 45$

$S_3 = 8(21-16) + 6(-18+8) + 2(24-14)$

$= 8(5) + 6(-10) + 2(10) = 40 - 60 + 20 = 0$

$$\lambda^3 - 18\lambda^2 + 45\lambda + 0 = 0$$

3	1	-18	45	0
	0	3	45	0
	1	-15	0	0

$\lambda^2 - 15 = 0$

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\lambda = 0$ or $\lambda^2 - 18\lambda + 45 = 0$

$(\lambda - 3)(\lambda - 15) = 0$

$\lambda = 3, 15$

the eigen values are $0, 3, 15$

To find eigen values..

$[A-\lambda I]x = 0$

$(8-\lambda)x_1 - 6x_2 + 2x_3 = 0$ — (1)

$-6x_1 + (7-\lambda)x_2 - 4x_3 = 0$ — (2)

$\cancel{2x_1 + (4-\lambda)x_2 + 3x_3 = 0}$ — (3)

$2x_1 - 4x_2 + (3-\lambda)x_3 = 0$

case (i) when $\lambda = 0$

$8x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 7x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 3x_3 = 0$

m	l	f	m
-6	2	8	-6
7	-4	-6	7

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20} \qquad X_1 = \begin{bmatrix}1\\2\\2\end{bmatrix}$$

case (ii) $\lambda = 3$

$|X_1| = 3$

$5x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 4x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 0x_3$

m	l	f	m
4	-4	-6	4
-4	0	2	-4

$$\frac{x_1}{0-16} = \frac{x_2}{-8+0} = \frac{x_3}{24-8}$$

$$\frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{16} \qquad X_2 = \begin{bmatrix}2\\1\\-2\end{bmatrix}$$

$|X_2| = \sqrt{9} = 3$

case (iii) $\lambda = 15$

$-8x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 - 8x_2 - 4x_3 = 0$

$2x_1 - 4x_2 - 12x_3 = 0$

$3x_1 + 4x_2 + 2x_3 = 0$

m	l	f	m
4	2	3	4
-2	-6	1	-2

$$\frac{x_1}{-24+4} = \frac{x_2}{2+18} = \frac{x_3}{-6-4}$$

$$\frac{x_1}{-20} = \frac{x_2}{20} = \frac{x_3}{-10} \qquad X_3 = \begin{bmatrix}2\\-2\\1\end{bmatrix}$$

$$M = \begin{bmatrix}1 & 2 & 2\\2 & 1 & -2\\2 & -2 & 1\end{bmatrix}$$

$$N = \begin{bmatrix}1/3 & 2/\sqrt{6} & 2/3\\2/3 & 1/\sqrt{6} & -2/3\\2/3 & -2/\sqrt{6} & 1/3\end{bmatrix}$$

$|X_3| = 3$

$$D = N^T A N = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/\sqrt{6} & 1/\sqrt{6} & -4/\sqrt{6} \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/\sqrt{6} & 2/3 \\ 2/3 & 1/\sqrt{6} & -2/3 \\ 2/3 & -2/\sqrt{6} & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/\sqrt{6} & 2/\sqrt{6} & -4/\sqrt{6} \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 8/3 - 12/3 + 4/3 & 16/\sqrt{6} - 12/\sqrt{6} + 4/\sqrt{6} & 16/3 + 12/3 + 2/3 \\ -6/3 + 14/3 - 8/3 & -12/\sqrt{6} + 14/\sqrt{6} - 8/\sqrt{6} & -12/3 - 14/3 - 4/3 \\ 2/3 - 8/3 + 6/3 & 4/\sqrt{6} - 8/\sqrt{6} + 8/\sqrt{6} & 4/3 + 8/3 + 3/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/\sqrt{6} & 2/\sqrt{6} & -4/\sqrt{6} \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 8/\sqrt{6} & 10 \\ 0 & -6/\sqrt{6} & -10 \\ 0 & 2/\sqrt{6} & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$$0y_1^2 + 3y_2^2 + 15y_3^2$$

---x---

ii, Diagonalised $A = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$ by using orthogonal transfo.—

Sol

$$\lambda^2 - S_1\lambda + S_2 = 0$$

$S_1 = 6+3 = 9$ $\quad S_2 = 18-4$

$= 14$

$$\lambda^2 - 9\lambda + 14 = 0$$

$$\lambda^2 - 7\lambda - 2\lambda + 14 = 0$$

$$(\lambda - 7)(\lambda - 2) = 0$$

$$\lambda = 2, 7$$

To find vector

$$\begin{bmatrix} 6-\lambda & -2 \\ -2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(6-\lambda)x_1 - 2x_2 = 0 \quad (1)$$

$$-2x_1 + (3-\lambda)x_2 = 0 \quad (2)$$

$\lambda = 2$

$4x_1 - 2x_2 = 0$

$-2x_1 + x_2 = 0$

$2x_1 = x_2$

$X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad |x_1| = \sqrt{5}$

$\lambda = 7$

$-x_1 - 2x_2 = 0$

$-2x_1 - 4x_2 = 0$

$-x_1 = 2x_2$

$\frac{x_1}{2} = \frac{x_2}{-1}$ $\quad X_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$|x_2| = \sqrt{5}$

$$M = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$N = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \qquad N^T = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

$$D = N^T A N = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 6/\sqrt{5} - 4/\sqrt{5} & 12/\sqrt{5} + 2/\sqrt{5} \\ -2/\sqrt{5} + 6/\sqrt{5} & -4/\sqrt{5} - 3/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 14/\sqrt{5} \\ 4/\sqrt{5} & -7/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 2/5 + 8/5 & 14/5 - 14/5 \\ 4/5 - 4/5 & 28/5 + 7/5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$$

HW.

Reduce Q.F into canonical form $2x_1x_2 + 2x_2x_3 - 2x_1x_3$.

Sol let the quadratic equation be

$$0x_1^2 + 0x_2^2 + 0x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_1x_3$$

then the square matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$S_1 = 0+0+0 = 0$ $\qquad S_3 = 0 - 1(0+1) - 1(0+1)$

$= -1-1 = -2$

$S_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0-1+0-1+0-1$

$= -3.$

$\lambda^3 + 0\lambda^2 - 3\lambda + 2 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & 2 \\ & 0 & -1 & 1 & \\ \hline & 1 & -1 & & \end{array}$$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$S_1 = 0+0+0 = 0$ $\qquad$ $S_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -3.$

$$S_3 = \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} = 0(0-1) - 1(0+1) - 1(1-0)$$
$$= -1-1 = -2$$

obtained charecteristic equation

$$\lambda^3 - 0\lambda - 3\lambda + 2 = 0$$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

$\lambda^2 + \lambda - 2 = 0$

$\lambda^2 + 2\lambda - \lambda - 2 = 0$

$\lambda(\lambda+2) - (\lambda+2) = 0$

$(\lambda+2)(\lambda-1) = 0$

$\lambda = 1, 1, -2$

To find eigen vectors

$$\begin{bmatrix} 0-\lambda & 1 & -1 \\ 1 & 0-\lambda & 1 \\ -1 & 1 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(0-\lambda)x_1 + x_2 - x_3 = 0$ — (1)

$x_1 + (0-\lambda)x_2 + x_3 = 0$ — (2)

$-x_1 + x_2 + (0-\lambda)x_3 = 0$ — (3)

let $\lambda = -2$

$2x_1 + x_2 - x_3 = 0$

$x_1 + 2x_2 + x_3 = 0$

$-x_1 + x_2 + 2x_3 = 0$

m l f m

1 -1 2 1

2 1 1 2

$$\frac{x_1}{1+2} = \frac{x_2}{-1-2} = \frac{x_3}{4-1}$$

$X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$|x_1| = \sqrt{3}.$

let $\lambda = 1$ $\quad$ $-1x_1 + x_2 - x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$-x_1 + x_2 - x_3 = 0$

m l f m

1 -1 -1 1

-1 1 1 -1

$$\frac{x_1}{1-1} = \frac{x_2}{-1+1} \neq \frac{x_3}{1-1}$$

$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$|x_2| = \sqrt{2}$

$x_1 = 0$

$$X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$a - b + c = 0$

$0a + b + c = 0$

m l f m

-1 1 1 -1

1 1 0 1

$$\frac{x_1}{-1-1} = \frac{x_2}{0-1} = \frac{x_3}{1+0}$$

$X_3 = \begin{bmatrix} +2 \\ +1 \\ -1 \end{bmatrix}$

$|x_3| = \sqrt{6}$

$$M = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad N = \begin{bmatrix} 1/\sqrt{3} & 0/\sqrt{2} & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

$$D = N^{\dagger}AN = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 0-1/\sqrt{3}-1/\sqrt{3} & 0+1/\sqrt{2}-1/\sqrt{2} & 0+1/\sqrt{6}+1/\sqrt{6} \\ 1/\sqrt{3}+0+1/\sqrt{3} & 0+0+1/\sqrt{2} & 2/\sqrt{6}+0-1/\sqrt{6} \\ -1/\sqrt{3}-1/\sqrt{3}+0 & 0+1/\sqrt{2}+0 & -2/\sqrt{6}+1/\sqrt{6}+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} -2/\sqrt{3} & 0 & 2/\sqrt{6} \\ 2/\sqrt{3} & 1/\sqrt{2} & 3/\sqrt{6} \\ -2/\sqrt{3} & 1/\sqrt{2} & +1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} -2/3-2/3-2/3 & 0-1/\sqrt{6}+1/\sqrt{6} & 2/\sqrt{18}-\frac{3}{\sqrt{18}}+\frac{1}{\sqrt{18}} \\ 0 & +1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

canonical form $= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$= -2y_1^2 + y_2^2 + y_3^2$.

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Circle of curvature

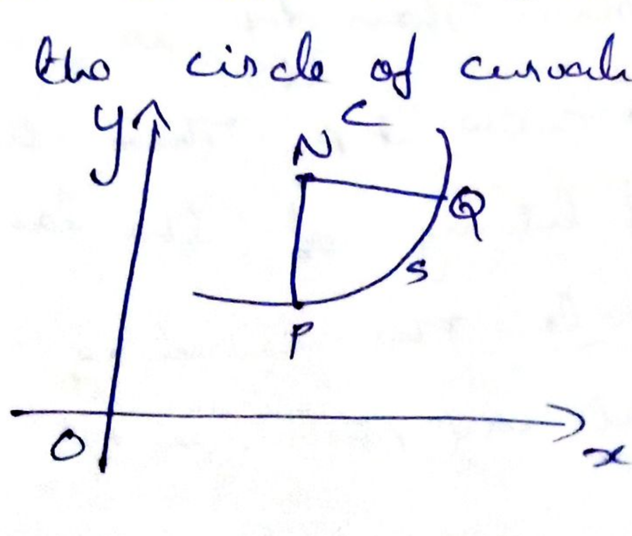
The curvature at any pt of a curve is equal to the curvature of the circle which passes thro' P and two close pts on the curve on either side of P. Such a circle exists for each pt. of the curve. It is called the circle of curvature of the curve at the pt. The radius of this circle is called the radius of curvature of the curve at that pt.

Centre of curvature

Let P be a gn. pt on a gn. curve, and Q be any other pt. on. it. Let the normals at P and Q intersect at N. If N tends to a definite position C as Q tends to P, then C is called the centre of curvature of the curve at P.

The reciprocal of the distance CP is called the curvature of the curve at P.

The circle with its centre at C and radius CP is called the circle of curvature of the curve at P.



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Radius of curvature

1. Cartesian form

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

2. Parametric form

$$\rho = \frac{\left(f'^2+g'^2\right)^{3/2}}{f'g''-f''g'}$$

where $f' = \frac{df}{dt}$ $\quad g' = \frac{dg}{dt}$

3. Polar form

$$\rho = \frac{\left(r^2+r_1^2\right)^{3/2}}{r^2+2r_1^2-rr_2}$$

$r_1 = \frac{dr}{d\theta}$, $r_2 = \frac{d^2r}{d\theta^2}$

4. Centre of curvature $(\bar{x}, \bar{y})$

$$\bar{x} = x - \frac{\frac{dy}{dx}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}}$$

$$\bar{y} = y + \left\{\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right\}$$

5. Chord of curvature thro' the origin

$2\rho\sin\phi$

6. Equation of the circle of curvature

$$(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$$

Cartesian form

1. Find the radius of curvature of the curve $y=e^x$ at $(0,1)$

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$y=e^x$ at $(0,1)$

$\frac{dy}{dx}=e^x$ $\quad e^0=1$

$\frac{d^2y}{dx^2}=e^x$ $\quad e^0=1$

$\therefore \rho=2\sqrt{2}$

2. Find the radius of curvature at the pt $(\frac{1}{4},\frac{1}{4})$ on the curve $\sqrt{x}+\sqrt{y}=1$

An. $\sqrt{x}+\sqrt{y}=1$

$\sqrt{y}=1-\sqrt{x}$

$y=(1-\sqrt{x})^2=1+x-2\sqrt{x}$

$\frac{dy}{dx}=1-2\cdot\frac{1}{2\sqrt{x}}$ $\quad \left.\frac{dy}{dx}\right)_{\frac{1}{4},\frac{1}{4}}=-1$

$\frac{d^2y}{dx^2}=\frac{1}{2}x^{-3/2}$ $\quad \left.\frac{d^2y}{dx^2}\right)_{\frac{1}{4},\frac{1}{4}}=\cancel{8}\,4$

3. S.T. the radius of curvature at any pt of the catenary $y = c\cosh(x/c)$ is y^2/c. (5)

Also find ρ at $(0,c)$

Gn. $y = c\cosh(x/c)$

$$\frac{dy}{dx} = c\left[\sinh\frac{x}{c}\right]\cdot\frac{1}{c}$$

$$\frac{d^2y}{dx^2} = \left(\cosh\frac{x}{c}\right)\frac{1}{c}$$

$$\therefore \rho = \frac{\left[1+\sinh^2(x/c)\right]^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= \frac{c\left(\cosh^2\frac{x}{c}\right)^{3/2}}{\cosh\frac{x}{c}}$$

$$= c\cosh^2\frac{x}{c} = c\frac{y^2}{c^2} = \frac{y^2}{c}$$

$$\text{At } (0,c) : \rho = \frac{c^2}{c} = c$$

4. Find the radius of curvature at the pt. (c,c) on the curve $xy = c^2$

$$\frac{dy}{dx} = \frac{-c^2}{x^2}, \quad \left.\frac{dy}{dx}\right)_{(c,c)} = -1$$

$$\frac{d^2y}{dx^2} = \frac{2c^2}{x^3} \quad \left.\frac{d^2y}{dx^2}\right)_{(c,c)} = 2/c$$

$$\rho = c\sqrt{2}$$

What is the radius of curvature at $(3,4)$ on the curve $x^2+y^2=25$

Gn. $x^2+y^2=25$

$\Rightarrow (x-0)^2+(y-0)^2=5^2$

which is a circle whose centre $(0,0)$ & radius is 5

$\therefore$ Radius of curvature $\rho=5$ for any pt

$\Rightarrow$ Radius of " $\rho=5$ for $(3,4)$

6. If ρ is the radius of curvature of any pt (x,y) on the curve $y=\frac{ax}{a+x}$ show that $\left(\frac{2\rho}{a}\right)^{2/3}=\left(\frac{x}{y}\right)^2+\left(\frac{y}{x}\right)^2$

Gn. $y=\frac{ax}{a+x}$

$$\frac{dy}{dx}=ax\left(\frac{-1}{(a+x)^2}\right)+\left(\frac{1}{a+x}\right)a$$

$$=\frac{-ax+a(a+x)}{(a+x)^2}$$

$$=\frac{a^2}{(a+x)^2}$$

$$\frac{d^2y}{dx^2}=a^2\left[\frac{-2}{(a+x)^3}\right]$$

Gn. $y=\frac{ax}{a+x}\Rightarrow y(a+x)=ax$

$a+x=\frac{ax}{y}$

$$\frac{dy}{dx} = \frac{a^2}{\left(\frac{ax}{y}\right)^2} = \frac{y^2}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{-2a^2}{\left(\frac{ax}{y}\right)^3} = \frac{-2y^3}{ax^3}$$

$$\therefore \rho = \frac{\left[1+\left(\frac{y}{x}\right)^4\right]^{3/2}}{\left(\frac{-2y^3}{ax^3}\right)}$$

$$\rho^2 = \frac{\left[1+\left(\frac{y}{x}\right)^4\right]^3}{\frac{2^2}{a^2}\left[\left(\frac{y}{x}\right)^3\right]^2}$$

$$\frac{(2\rho)^2}{a^2} = \frac{\left[1+\left(\frac{y}{x}\right)^4\right]^3}{\left[\left(\frac{y}{x}\right)^2\right]^3}$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \frac{1+\left(\frac{y}{x}\right)^4}{\left(\frac{y}{x}\right)^2}$$

$$= \cdot$$

Parametric form

7. Find the radius of curvature at any pt $P(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$x = a\cos\theta \qquad y = b\sin\theta$$

$$x' = \frac{dx}{d\theta} = -a\sin\theta \qquad y' = \frac{dy}{d\theta} = b\cos\theta$$

$$x'' = \frac{d^2x}{d\theta^2} = -a\cos\theta \qquad y'' = \frac{d^2y}{d\theta^2} = -b\sin\theta$$

$$\rho = \frac{\left[(x')^2+(y')^2\right]^{3/2}}{x'y''-y'x''}$$

$$= \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$$

(*) 8. Find the radius of curvature at any pt $x = a\cos^3\theta$, $y = a\sin^3\theta$ of the curve $x^{2/3}+y^{2/3}=a^{2/3}$. Also S.T. $\rho^3 = 27axy$

$$x = a\cos^3\theta \qquad y = a\sin^3\theta$$

$$\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta) \qquad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}(-\tan\theta) \cdot \frac{d\theta}{dx}$$

$$= \frac{-\sec^2\theta \cdot}{-3a\cos^2\theta\sin\theta} = \frac{1}{3a\cos^4\theta\sin\theta}$$

$$\therefore \rho = \frac{(1+\tan^2\theta)^{3/2}}{\left[\frac{1}{3a\cos^4\theta\sin\theta}\right]}$$

$$= (\sec^2\theta)^{3/2} \cdot 3a\cos^4\theta\sin\theta$$

$$= \sec^3\theta \cdot 3a\cos^4\theta\sin\theta$$

$$= 3a\cos\theta\sin\theta$$

$$= \frac{3}{2}a\sin 2\theta$$

$$\rho^3 = (3a\sin\theta\cos\theta)^3$$

$$= 27a^3\sin^3\theta\cos^3\theta$$

$$= 27a^3\left(\frac{y}{a}\right)\left(\frac{x}{a}\right)$$

$$= 27axy$$

9. Prove that the radius of curvature at any pt. of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ is $4a\cos\theta/2$

Gn. $x = a(\theta + \sin\theta)$

$x' = a(1+\cos\theta)$

$y = a(1-\cos\theta)$

$y' = a(\sin\theta)$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$= \frac{\sin \theta/2}{\cos \theta/2} = \tan \theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \frac{\frac{1}{2}\sec^2\theta/2}{a(1+\cos\theta)} = \frac{1/2 \sec^2\theta/2}{a2\cos^2\theta/2}$$

$$= \frac{1}{4a\cos^4\theta/2}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\tan^2\theta/2)^{3/2}}{\left[\frac{1}{4a\cos^4\theta/2}\right]}$$

$$= (\sec^2\theta/2)^{3/2} \cdot 4a\cos^4\theta/2$$

$$= 4a\cos\theta/2$$

Implicit form

10. Find the curvature and radius of curvature of $2x^2+2y^2+2x-5y+1=0$ at any pt.

$$2x^2+2y^2+2x-5y+1=0$$

$$f(x,y) = 2x^2+2y^2+2x-5y+1$$

$f_x = 4x+2$ $\quad f_y = 4y-5$ $\quad f_{xy}=0$

$f_{xx} = 4$ $\quad f_{yy} = 4$

Radius of curvature $\rho = \dfrac{(f_x^2+f_y^2)^{3/2}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}$

$$= \frac{[(4x+2)^2+(4y-5)^2]^{3/2}}{4[(4y-5)^2+(4x+2)^2]}$$

$$= \frac{1}{4}\left[(4x+2)^2+(4y-5)^2\right]^{3/2-1}$$

$$= \frac{1}{4}\sqrt{16x^2+4+16x+16y^2+25-40y}$$

$$= \frac{1}{4}\sqrt{8(2x^2+2y^2+2x-5y)+29}$$

$$= \frac{1}{4}\sqrt{8(-1)+29} \quad [\text{by gn. fn}]$$

$$\rho = \frac{\sqrt{21}}{4}$$

$$\therefore \text{curvature} = 1/\rho = \frac{4}{\sqrt{21}}$$

11. $2x^2+2y^2+5x-2y+1=0$

$$x^2+y^2+\frac{5x}{2}-\frac{\cancel{2}}{\cancel{2}}y+\frac{1}{2}=0$$

This eqn. is of the form

$$x^2+y^2+2gx+2fy+c=0$$

$$r = \sqrt{g^2+f^2-c}$$

Here $g = \frac{5}{4}$, $f = -\frac{1}{2}$, $c = \frac{1}{2}$

$$\therefore r = \sqrt{\frac{21}{4}}$$

Radius of curvature $\rho = \frac{\sqrt{21}}{4}$

$$\therefore \text{Curvature} = 1/\rho = \frac{4}{\sqrt{21}}$$

12. $x^2+y^2-4x-6y+10=0$

$$\rho = \sqrt{3}$$

$$1/\rho = 1/\sqrt{3}$$

13. Find the radius of curvature at $(a,0)$ of the curve $xy^2 = a^3 - x^3$

Let $f(x,y) = xy^2 - a^3 + x^3$

At $(a,0)$

$f_x = y^2 + 3x^2$ $f_x = 3a^2$

$f_{xx} = 6x$ $f_{xx} = 6a$

$f_y = 2xy$ $f_y = 0$

$f_{yy} = 2x$ $f_{yy} = 2a$

$f_{xy} = 2y$ $f_{xy} = 0$

$$\rho = \frac{[f_x^2 + f_y^2]^{3/2}}{f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2}$$

$$= \frac{3}{2} a$$

14. Find the radius of curvature at the pt $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the curve $x^3 + y^3 = 3axy$

$f(x,y) = x^3 + y^3 - 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$f_x = 3x^2 - 3ay$ $f_x = \frac{9}{4}a^2$

$f_{xx} = 6x$ $f_{xx} = 9a$

$f_y = 3y^2 - 3ax$ $f_y = \frac{9}{4}a^2$

$f_{yy} = 6y$ $f_{yy} = 9a$

$f_{xy} = -3a$ $f_{xy} = -3a$

$$\rho = \frac{3a}{8\sqrt{2}}$$

15. S.T. the numerical value of the radius of curvature of the curve $x^2y = a(x^2+y^2)$ at $(-2a, 2a)$ is $2a$.

16. Find ρ at $(1,1)$ on $x^3+y^3=2$

Polar form

7. For the cardioid $r = a(1+\cos\theta)$ P.T. $\frac{\rho^2}{r}$ is constant

Gn. $r = a(1+\cos\theta)$

$$r_1 = \frac{dr}{d\theta} = -a\sin\theta$$

$$r_2 = \frac{d^2r}{d\theta^2} = -a\cos\theta$$

$$r^2 + r_1^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2[1+\cos^2\theta + 2\cos\theta] + a^2\sin^2\theta$$

$$= a^2 + a^2 + 2a^2\cos\theta$$

$$= 2a^2[1+\cos\theta]$$

$$= 4a^2\cos^2\theta/2$$

$$r^2 + 2r_1^2 - rr_2 = a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta + a^2\cos^2\theta\,(1+\cos\theta)$$

$$= 6a^2\cos^2\theta/2$$

$$\therefore \rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} = \frac{[4a^2\cos^2\theta/2]^{3/2}}{6a^2\cos^2\theta/2}$$

$$= \frac{4a}{3}\cos\theta/2$$

$\therefore p^2 = \frac{16a^2}{9}\cos^2\theta/2$

$\frac{p^2}{r} = \frac{16a}{9}\frac{\cos^2\theta/2}{2\cos^2\theta/2} = \frac{8a}{9}$

1, Sushma

2, Harsha vardhan

3, Aswin.

Centre of curvature & circle of curvature

18. Find the centre of curvature of $y = x^2$ at the origin

$y = x^2$ at $(0,0)$

$\frac{dy}{dx} = 2x$ 0

$\frac{d^2y}{dx^2} = 2$ 2

$\therefore \bar{x} = x - \frac{\frac{dy}{dx}\left(1+\left(\frac{dy}{dx}\right)^2\right)}{\frac{d^2y}{dx^2}} = 0 - \frac{0(1+0)^2}{2}$

$= 0$

$\bar{y} = y + \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = \frac{1}{2}$

$\therefore$ Centre of curvature is $(0, \frac{1}{2})$

19, Find the circle of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the pt $(\frac{a}{4}, \frac{a}{4})$

$\sqrt{x} + \sqrt{y} = \sqrt{a}$

$\sqrt{y} = \sqrt{a} - \sqrt{x}$

$y = (\sqrt{a} - \sqrt{x})^2 = a - 2\sqrt{a}\sqrt{x} + x$

at $(\frac{a}{4}, \frac{a}{4})$

$\frac{dy}{dx} = -\frac{2\sqrt{a}}{2\sqrt{x}} + 1$ $1 - \frac{\sqrt{a}}{\sqrt{a/4}} = -1$

$\frac{d^2y}{dx^2} = +\frac{\sqrt{a}}{2(x)^{+3/2}}$ $\frac{\sqrt{a}}{2(a/4)^{3/2}} = 4/a$

$$\therefore \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{a}{\sqrt{2}}$$

$\bar{x} = \frac{3a}{4}$ $\qquad \bar{y} = \frac{3a}{4}$

Centre of curvature is $\left(\frac{3a}{4}, \frac{3a}{4}\right)$

Hence the eqn. of the circle of curvature is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2}$$

20. Find the eqn. of the circle of curvature at (c, c) on $xy = c^2$

21. Find the eqn. of the circle of curvature of $\sqrt{x} + \sqrt{y} = 1$ at $(1/4, 1/4)$

$$\rho = \frac{1}{\sqrt{2}}$$

$$\left(x - 3/4\right)^2 + \left(y - 3/4\right)^2 = 1/2$$

(*) Find the eqn. of circle of curvature of the rectangular hyperbola $xy = 12$ at the pt $(3, 4)$.

23. Find the eqn. of the circle of curvature of the parabola $y^2 = 12x$ at the pt $(3, 6)$

Ans: $(x-15)^2 + (y+6)^2 = 288$

Evolutes

Involutes and Evolutes

The locus of the centre of curvature of the gn. curve is called the evolute of the curve. The gn. curve is called the involute of its evolute.

Evolute of a curve is the envelope of its normals.

24. Find the eqn. of the evolute of the parabola $y^2 = 4ax$

The parametric eqns. of the parabola is

$x = at^2$, $y = 2at$

$\frac{dx}{dt} = 2at$ $\qquad$ $\frac{dy}{dt} = 2a$

$\frac{d^2x}{dt^2} = 2a$ $\qquad$ $\frac{d^2y}{dt^2} = 0$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{1}{t}\right)\frac{1}{2at} = \frac{-1}{t^2}\cdot\frac{1}{2at} = \frac{-1}{2at^3}$$

If (X, Y) is the centre of curvature at t

then $X = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{d^2y/dx^2} = 3at^2 + 2a$

$$X = at^2 - \frac{\left(\frac{1}{t}\right)\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2at^3}}$$

$$= at^2 + 2at^2\left(1+\frac{1}{t^2}\right)$$

$$X = 3at^2 + 2a$$

$$Y = y + \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]$$

$$= 2at - 2at^3\left[1+\frac{1}{t^2}\right]$$

$$Y = -2at^3$$

Now we have to eliminate t between X & Y

$$X = 3at^2 + 2a$$

$$t^2 = \frac{X-2a}{3a}$$

$$(t^2)^3 = \left(\frac{X-2a}{3a}\right)^3 \quad - \text{①}$$

$$Y = -2at^3$$

$$t^3 = \frac{-Y}{2a}$$

$$(t^3)^2 = \frac{Y^2}{4a^2} \quad - \text{②}$$

① & ②

$$\left(\frac{X-2a}{3a}\right)^3 = \left(\frac{Y}{2a}\right)^2$$

$$4(X-2a)^3 = 27aY^2$$

Changing X & Y to x & y the locus of (X, Y) becomes $4(x-2a)^3 = 27ay^2$

Find the eqn. of the evolute of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The parametric eqns of the ellipse are

$$x = a\cos\theta \qquad y = b\sin\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$$

$$y_1 = \frac{dy}{dx} = \frac{-b\cot\theta}{a}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}\left(-\tfrac{b}{a}\cot\theta\right)\left(\frac{-1}{a\sin\theta}\right)$$

$$= +\tfrac{b}{a}\operatorname{cosec}^2\theta\left(\frac{-1}{a\sin\theta}\right)$$

$$= \frac{-b}{a^2}\operatorname{cosec}^3\theta$$

21/9

Harsh

Let (X, Y) be the coordinates of the centre of curvature.

$$X = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

$$= a\cos\theta + \frac{\frac{b}{a}\cot\theta\left[1 + \frac{b^2}{a^2}\cot^2\theta\right]}{-\frac{b}{a^2}\operatorname{cosec}^3\theta}$$

$$= a\cos\theta - a\cot\theta\,\sin^3\theta\left[a^2 + b^2\cot^2\theta\right]$$

$$= a\cos\theta - a\overset{\cos\theta}{n}\sin^2\theta\left[a^2 + b^2\cot^2\theta\right]$$

$$= a\cos\theta - a^3\sin^2\theta - ab^2\cos^2\theta]$$

$$= a\cos\theta - a^3(1-\cos^2\theta) - ab^2\cos^2\theta$$

$$= a\cos^3\theta - \tfrac{b^2}{a}\cos^3\theta$$

$$X = \left(\frac{a^2-b^2}{a}\right)\cos^3\theta$$

$$Y = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

$$= \left(\frac{b^2-a^2}{b}\right)\sin^3\theta$$

To find the eqn. of the evolute we have to eliminate θ between X & Y

$$aX = (a^2-b^2)\cos^3\theta$$

$$(aX)^{2/3} = (a^2-b^2)^{2/3}\cos^2\theta$$

$$bY = (b^2-a^2)\sin^3\theta$$

$$(bY)^{2/3} = (b^2-a^2)^{2/3}\sin^2\theta$$

$$(aX)^{2/3}+(bY)^{2/3} = (a^2-b^2)^{2/3}(\sin^2\theta+\cos^2\theta)$$

$$= (a^2-b^2)^{2/3}$$

Locus of (X, Y) is $(ax)^{2/3}+(by)^{2/3} = (a^2-b^2)^{2/3}$

6. Find the evolute of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$x = a\sec\theta \qquad y = b\tan\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta \qquad \frac{dy}{d\theta} = b\sec^2\theta$

$$\frac{dy}{dx} = \frac{b}{a\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}\left(\frac{b}{a}\operatorname{cosec}\theta\right)\frac{1}{a\sec\theta\tan\theta}$$

$$= -\frac{b}{a}\frac{\operatorname{cosec}\theta\cot\theta}{a\sec\theta\tan\theta} = -\frac{b}{a^2}\cot^3\theta$$

$$\bar{X} = x - \frac{\frac{dy}{dx}\left(1+\left(\frac{dy}{dx}\right)^2\right)}{\frac{d^2y}{dx^2}}$$

$$= a^2 \sec\theta + a\sec^3\theta - a\sec\theta + \frac{b^2}{a}\sec^3\theta$$

$$a\bar{X} = a^2\sec^3\theta + b^2\sec^3\theta$$

$$= (a^2+b^2)\sec^3\theta$$

$$\bar{Y} = y + \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

$$= b\tan\theta - \frac{a^2}{b}\tan^3\theta - b\tan\theta\sec^2\theta$$

$$b\bar{Y} = -(a^2+b^2)\tan^3\theta$$

$$(a\bar{x})^{2/3} - (b\bar{y})^{2/3} = (a^2+b^2)^{2/3}\left[\sec^2\theta - \tan^2\theta\right]$$

$$= (a^2+b^2)^{2/3}$$

$\therefore$ Locus of $(\bar{x}, \bar{y})$ is $(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$

27. Find the evolute of the rectangular hyperbola $xy = c^2$

$$x = ct \qquad y = c/t$$

$$\frac{dx}{dt} = c \qquad \frac{dy}{dt} = -c/t^2$$

$$\frac{dy}{dx} = -1/t^2$$

$$\frac{d^2y}{dx^2} = 2/ct^3$$

Let (X, Y) be the coordinates of centre of curvature, then

$$X = ct + \frac{ct}{2}\left(1+\frac{1}{t^4}\right)$$

$$= \frac{3ct}{2} + \frac{c}{2t^3}$$

$$Y = \frac{c}{t} + \frac{1+\frac{1}{t^4}}{\left(\frac{2}{ct^3}\right)} = \frac{3}{2}\frac{c}{t} + \frac{ct^3}{2}$$

$$X+Y = \frac{c}{2}\left[t+\frac{1}{t}\right]^3$$

$$(X+Y)^{2/3} = \left(\frac{c}{2}\right)^{2/3}\left(1+\frac{1}{t}\right)^2$$

$$X-Y = -\frac{c}{2}\left[t-\frac{1}{t}\right]^3$$

$$(X-Y)^{2/3} = \left(-\frac{c}{2}\right)^{2/3}\left(t-\frac{1}{t}\right)^2 = \left(\frac{c}{2}\right)^{2/3}\left(t-\frac{1}{t}\right)^2$$

$$(X+Y)^{2/3} - (X-Y)^{2/3} = \left(\frac{c}{2}\right)^{2/3}\left[\left(t+\frac{1}{t}\right)^2 - \left(t-\frac{1}{t}\right)^2\right]$$

$$= 4\left(\frac{c}{2}\right)^{2/3}$$

The locus is $(x+y)^{2/3} - (x-y)^{2/3} = (4c)^{2/3}$

28. Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$ $y = a(1-\cos\theta)$ is another equal cycloid.

29. Show that the evolute of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ is a circle.

Envelopes

A curve which touches each member of a family of curve is called the envelope of that family curves.

Envelope of a family of curves
The locus of the ultimate

Type I (parameter) (Q.F)

30. Find the envelope of the curve $y = mx + \frac{1}{m}$ where m is the parameter

$$y = mx + \frac{1}{m}$$

$$my = m^2x + 1$$

$m^2x - my + 1 = 0$ which is a Q.E in the parameter m.

So the envelope is $B^2 - 4AC = 0$

i.e. $y^2 = 4x$

31. Find the envelope of the family of lines $y = mx + \frac{a}{m}$ where a is a constant

$$y = mx + \frac{a}{m}$$

$$m^2x - ym + a = 0$$

So the envelope is $B^2 - 4AC = 0$

$$y^2 = 4ax$$

32 Find the envelope of $y = mx + \sqrt{a^2m^2+b^2}$ where m is a parameter

$$y - mx = \pm\sqrt{a^2m^2+b^2}$$

$$m^2(x^2-a^2) - 2mxy + y^2 - b^2 = 0$$

which is a Q.E in m

Hence the envelope is $B^2 - 4AC = 0$

$$x^2y^2 - (x^2-a^2)(y^2-b^2) = 0$$

$$\Rightarrow x^2y^2 - [x^2y^2 - b^2x^2 - a^2y^2 + a^2b^2] = 0$$

$$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

33. Find the envelope of $y = mx + \sqrt{1+m^2}$ where m is a parameter (Ans: $x^2+y^2=1$)

Show that the envelope of the family of circles described on the double ordinates of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as diameters, is the ellipse

$$\frac{x^2}{a^2+b^2} + \frac{y^2}{b^2} = 1$$

The double ordinates on the envelope $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $(a\cos t, b\sin t)$ & $(a\cos t, -b\sin t)$ with t as the parameter.

we know that the diameter form of the eqn. of a circle is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

$$\therefore (x - a\cos t)^2 + y^2 - b^2\sin^2 t = 0$$

$$x^2+a^2\cos^2t-2ax\cos t+y^2-b^2\sin^2t=0$$

$$(a^2+b^2)\cos^2t-2ax\cos t+(x^2+y^2-b^2)=0$$

which is a Quadratic in $\cos t$

$$\therefore B^2-4AC=0 \Rightarrow a^2x^2=(a^2+b^2)x^2+(a^2+b^2)y^2-(a^2+b^2)b^2$$

$$\div (a^2+b^2)b^2 \Rightarrow \frac{x^2}{(a^2+b^2)}+\frac{y^2}{b^2}=1$$

$x^2+y^2-2yb\sin t+b^2\sin^2t-a^2\cos^2t=0$

$(2yb)^2-4(b^2+a^2)(x^2+y^2-a^2)=0$

Type II (eliminating the parameter)

34. Find the envelope of $x\cos\theta+y\sin\theta=a$, θ is a Para.

$$x\cos\theta+y\sin\theta=a \quad -(1)$$

Diff. w.r.t θ

$$-x\sin\theta+y\cos\theta=0 \quad -(2)$$

eliminate θ from (1) & (2)

$$(x\cos\theta+y\sin\theta)^2+(-x\sin\theta+y\cos\theta)^2=a^2$$

$$x^2+y^2=a^2$$

35. Find the envelope of the family of st. lines

$$y\cos\alpha-x\overset{\sin\alpha}{\cos\alpha}=a\cos2\alpha$$

$$y\cos\alpha-x\cos\alpha=a\cos2\alpha \quad -(1)$$

Diff. w.r.t α

$$-y\sin\alpha-x\cos\alpha=-2a\sin2\alpha \quad -(2)$$

Eliminate between (1) & (2)

(1) $\times\sin\alpha$
(2) $\times\cos\alpha$ & add

$$x=a[\sin3\alpha+3\sin\alpha\cos^2\alpha]$$

(1) $\times\cos\alpha$
(2) $\times\sin\alpha$ & add

$$y=a[\cos^3\alpha+3\sin^2\alpha\cos\alpha]$$

$x+y = a[\sin\alpha+\cos\alpha]^3$

$x-y = a[\sin\alpha-\cos\alpha]^3$

$(x+y)^{2/3}+(x-y)^{2/3} = 2a^{2/3}$

36. Find the envelope of the family of st. lines

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2-b^2$$

Type III (Envelope of two parameter family of curves)

37. Find the envelope of $\frac{x}{a}+\frac{y}{b}=1$ sub. to $a+b=c$, c is a constant.

Gn. $\frac{x}{a}+\frac{y}{b}=1$

& $a+b=c$

$b = c-a$

$\therefore \frac{x}{a}+\frac{y}{c-a}=1$ Here a is the parameter

$a^2+a(y-x-c)+cx=0$

The envelope is $B^2-4AC=0$

$(y-x-c)^2-4cx=0$

$\sqrt{x}+\sqrt{y}=\sqrt{c}$

$x = a\sin\alpha(2\cos^2\alpha+1)$

$y = 2\cos\alpha(2\sin^2\alpha+1)$

Find the envelope of $\frac{x}{a}+\frac{y}{b}=1$ where a & b are Connected by $a^2+b^2=c^2$

$$\frac{x}{a}+\frac{y}{b}=1$$

Taking differentials $-\frac{x}{a^2}da-\frac{y}{b^2}db=0$

$$\frac{da}{db}=\frac{-a^2y}{b^2x}$$

Giv. $a^2+b^2=c^2$

$$2a\,da+2b\,db=0$$

$$\frac{da}{db}=-b/a$$

$$\therefore \frac{-a^2y}{b^2x}=-b/a$$

$$\frac{x}{a^3}=\frac{y}{b^3}$$

$$\frac{\frac{x}{a}}{a^2}=\frac{\frac{y}{b}}{b^2}$$

$$\Rightarrow \frac{\frac{x}{a}}{a^2}=\frac{\frac{y}{b}}{b^2}=\frac{\frac{x}{a}+\frac{y}{b}}{a^2+b^2}=\frac{1}{c^2}$$

$$\therefore \frac{x}{a^3}=\frac{1}{c^2} \quad \& \quad \frac{y}{b^3}=\frac{1}{c^2}$$

$$a=(xc^2)^{1/3}, \quad b=(yc^2)^{1/3}$$

Sub. in $a^2+b^2=c^2$

$$(xc^2)^{2/3}+(yc^2)^{2/3}=c^2$$

$$c^{4/3}[x^{2/3}+y^{2/3}]=c^2$$

$$x^{2/3}+y^{2/3}=c^{2/3}$$

39 Find the envelope of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where the two parameters are connected by the relation $a+b=c$ where c is a constant.

40 Find the envelope of the st. line $\frac{x}{a}+\frac{y}{b}=1$, where a and b are connected by the relation $ab=c^2$.

Properties of Envelopes and Evolutes

1 The normal at any pt. of a curve is a tangent to its evolute touching at the corresponding centre of curvature.

2 The difference between the radii of curvatures at two points of a curve is equal to the length of the arc of the evolute between the two corresponding points.

3 There is one evolute, but an infinite no. of involutes.

Evolute as the envelope of normals

The evolute of a curve can also be considered as the envelope of the normals to the curve.

Find the evolute of $y^2 = 4ax$ considering it as the envelope of the normals.

The parametric coordinates $x = at^2$ & $y = 2at$

$$\frac{dy}{dx} = \frac{1}{t} = m$$

The normal eqn. is $y - y_1 = -\frac{1}{m}(x - x_1)$

$$\Rightarrow y + xt = at^3 + 2at \quad (1)$$

Diff. (1) partially w.r.t. t,

$$x = 3at^2 + 2a$$

$$t^2 = \frac{x - 2a}{3a}$$

$$(1) \Rightarrow y = at^3 + 2at - xt$$

$$= t(at^2 + 2a - x]$$

$$= -t[-at^2 + (x - 2a)]$$

$$= -t[(x - 2a) - a\left(\frac{x - 2a}{3a}\right)$$

$$= -t\left[\frac{2}{3}(x - 2a)\right]$$

$$y^2 = \frac{4}{9}(x - 2a)^2(t^2)$$

$$= \frac{4}{9}(x - 2a)^2\left(\frac{x - 2a}{3a}\right)$$

$$= \frac{4}{9}\frac{(x - 2a)^3}{3a}$$

$$27y^2 = 4(x - 2a)^3$$

42. Considering the evolute as the envelope of normals find the evolute of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The parametric eqn. are $x=a\cos\theta$, $y=b\sin\theta$

$$\frac{dy}{dx}=\frac{-b}{a}\cot\theta$$

eqn. of the normal is $y-y_1=\frac{-1}{m}(x-x_1)$

$$y-b\sin\theta=\frac{a}{b}\frac{\sin\theta}{\cos\theta}(x-a\cos\theta)$$

$$by\cos\theta-b^2\sin\theta\cos\theta=ax\sin\theta-a^2\sin\theta\cos\theta$$

$\div by\sin\theta\cos\theta$

$$\Rightarrow \frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \qquad -(1)$$

Diff. partially w.r.t. θ

$$\frac{ax\sin\theta}{\cos^2\theta}+\frac{by\cos\theta}{\sin^2\theta}=0$$

$$\frac{ax\sin\theta}{\cos^2\theta}=-\frac{by\cos\theta}{\sin^2\theta} \Rightarrow \frac{ax}{\cos^3\theta}=\frac{-by}{\sin^3\theta}$$

$$\frac{\frac{ax}{\cos\theta}}{\cos^2\theta}=\frac{\frac{-by}{\sin\theta}}{\sin^2\theta}=\frac{\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}}{\cos^2\theta+\sin^2\theta}=\frac{a^2-b^2}{1}$$

$$\frac{ax}{\cos^3\theta}=a^2-b^2 \quad \& \quad \frac{-by}{\sin^3\theta}=a^2-b^2$$

$$\cos^3\theta=\frac{ax}{a^2-b^2} \qquad \sin^3\theta=\frac{-by}{a^2-b^2}$$

$$\cos\theta=\left(\frac{ax}{a^2-b^2}\right)^{1/3} \qquad \sin\theta=\left(\frac{-by}{a^2-b^2}\right)^{1/3}$$

Sub in (1)

$$\Rightarrow \frac{ax}{\left(\frac{ax}{a^2-b^2}\right)^{1/3}}-\frac{by}{\left(\frac{-by}{a^2-b^2}\right)^{1/3}}=a^2-b^2$$

$$\Rightarrow (ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$$

Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ is another cycloid

Any point on the cycloid is given by

$x = a(\theta - \sin\theta)$ and $y = a(1-\cos\theta)$

$\frac{dx}{d\theta} = a(1-\cos\theta)$ $\qquad \frac{dy}{d\theta} = a\sin\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} / \frac{d\theta}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\theta/2 \cos\theta/2}{2\sin^2\theta/2} = \cot\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}(\cot\theta/2)\frac{1}{a(1-\cos\theta)}$$

$$= -\operatorname{cosec}^2\theta/2 \cdot \frac{1}{2}\frac{1}{2a\sin^2\theta/2} = \frac{-1}{4a\sin^4\theta/2}$$

$$\bar{x} = x - \frac{\frac{dy}{dx}\left(1+\left(\frac{dy}{dx}\right)^2\right)}{\frac{d^2y}{dx^2}} = a(\theta - \sin\theta) - \frac{\cot\theta/2\left(1+\cot^2\theta/2\right)}{\frac{-1}{4a\sin^4\theta/2}}$$

$$= a(\theta - \sin\theta) + \cot\theta/2 \, 4a\sin^4\frac{\theta}{2}\operatorname{cosec}^2\theta/2$$

$$= a(\theta - \sin\theta) + \frac{\cos\theta/2}{\cancel{\sin\theta/2}}\sin^{\cancel{4}}\theta/2 \cdot \frac{1}{\cancel{\sin^2\theta/2}} \cdot 4a$$

$$= a(\theta - \sin\theta) + 4a\sin\theta/2\cos\theta/2$$

$$= a(\theta - \sin\theta) + 2a\sin\theta$$

$$= a(\theta + \sin\theta)$$

$$\bar{y} = y + \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)}{\frac{d^2y}{dx^2}} = a(1-\cos\theta) + \frac{\left(1+\cot^2\theta/2\right)}{\frac{-1}{4a}\sin^4\theta/2}$$

$$= a(1-\cos\theta) - 4a\sin^4\theta/2\operatorname{cosec}^2\theta/2$$

$$= a(1-\cos\theta) - 4a\sin^2\theta/2$$

$$= a(1-\cos\theta) - 2a(1-\cos\theta)$$

$$= -a(1-\cos\theta)$$

UNIT - II

Sequences and Series

Sequences: Definition and examples - Series: Types and Convergence - Series of positive terms - Tests of convergence: Comparison test, Integral test and D' Alembert's ratio test - Alternating series - Leibnitz's test - Series of positive and negative terms - Absolute and conditional convergence

Definitions

Sequence

Let $f: N \to R$ be a fn. and let $f(n) = a_n$. Then $a_1, a_2 \ldots a_n \ldots$ is called the sequence in R determined by the fn. f and is denoted by $\{a_n\}$ or (a_n). a_n is called the n^{th} term of the sequence.

An ordered set of real nos. $a_1, a_2, a_3 \ldots a_n$ is called a sequence and is denoted by $\{a_n\}$

Finite sequence

A finite sequence has a finite no. of terms

Infinite sequence

A sequence, which is not finite, is an infinite sequence.

Real sequence

A sequence whose range is a subset of R is called a real sequence.

Limit of a sequence

A sequence is tend to a limit l, if for every $\epsilon > 0$, a value N of n can be found $\ni |a_n - l| < \epsilon$ for $n \geq N$

ie $\lim \{a_n\} = l$

Convergence and divergence

If a sequence $\{a_n\}$ has a finite limit, it is called a convergent sequence. If $\{a_n\}$ is not convergent, it is said to be divergent

Bounded sequence

A sequence (a_n) is said to be bounded, if there exists a no. K such that $a_n < K$ for every n.

Monotonic Sequence

The sequence (a_n) is said to increase steadily or to decrease steadily according as $a_{n+1} \geq a_n$ or $a_{n+1} \leq a_n$, for all values of n.

Both increasing and decreasing sequence are called monotonic sequences.

Note:

A sequence which is monotonic and bounded is convergent.

Eg:

i) $1, 2, 2, 3, 3, 3, 4, 4, 4, 4 \ldots$ is a monotonic increasing sequence.

ii) $1, 2, 3, 4 \ldots n$ is a strictly monotonic increasing sequence.

iii) $1, \frac{1}{2}, \frac{1}{3} \ldots \frac{1}{n}$ is a strictly monotonic decreasing sequence.

iv) $1, -1, 1, -1, 1$ is neither monotonic increasing nor decreasing. Hence $\{a_n\}$ is not a monotonic sequence.

1. Prove that the sequence $\left\{\frac{n}{n^2+1}\right\}$ is convergent.

Here $a_n = \frac{n}{n^2+1}$, $a_{n+1} = \frac{n+1}{(n+1)^2+1}$

$a_{n+1} - a_n = \frac{-n^2-n+1}{((n+1)^2+1)(n^2+1)} < 0$ for all n

$\Rightarrow a_{n+1} - a_n < 0$

$\Rightarrow a_{n+1} < a_n \quad \forall n$

& $\{a_n\}$ is bounded below by 0

Hence the gn. seq. is convergent.

Also $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{n^2+1}$

$= \lim\limits_{n\to\infty} \frac{1/n}{1+1/n^2} = 0$

2, Prove that $\lim\limits_{n\to\infty} \frac{n+1}{n} = 1$

$\lim\limits_{n\to\infty} \frac{n+1}{n} = \lim\limits_{n\to\infty} \frac{\cancel{n}(1+1/n)}{\cancel{n}} = \lim\limits_{n\to\infty} (1+1/n)$

$= 1$

3, Show that the sequence $\{(1+1/n)^n\}$ is monotonic increasing.

Here $a_n = (1+1/n)^n$

Assume that $a_{n-1} < a_n$

$\Rightarrow \left(1+\frac{1}{n-1}\right)^{n-1} < \left(1+\frac{1}{n}\right)^n$

$\Rightarrow \left(1+\frac{1}{n-1}\right)^n \cdot \left(1+\frac{1}{n-1}\right)^{-1} < \left(1+\frac{1}{n}\right)^n$

$\Rightarrow \frac{n-1}{n} < \left(\frac{1+1/n}{1+1/(n-1)}\right)^n$

$\Rightarrow \frac{n-1}{n} < \left(\frac{\frac{n+1}{n}}{\frac{n}{n-1}}\right)^n$

$$\Rightarrow 1-\frac{1}{n} < \left(\frac{n^2-1}{n^2}\right)^n$$

$$\Rightarrow 1-\frac{1}{n} < \left(1-\frac{1}{n^2}\right)^n$$

which is true by using Bernouli's inequality.

Series: Types and Convergence

Def:

If $u_1, u_2, u_3 \ldots u_n \ldots$ be an infinite sequence of real nos. then $u_1+u_2+\ldots+u_n+\ldots\infty$ is called an infinite series.

Convergent, Divergent and Oscillatory series

Consider the infinite series

$$\Sigma u_n = u_1+u_2+\ldots\infty$$

$$S_n = u_1+\ldots+u_n$$

i) If S_n tends to a finite no. as $n\to\infty$ the series Σu_n is said to be convergent. ($\lim_{n\to\infty} u_n = N$)

ii) If S_n tends to infinity as $n\to\infty$, the series Σu_n is said to be divergent. (is $\lim_{n\to\infty} u_n=\infty$)

iii) If S_n does not tend to a unique limit, finite or infinite, the series Σu_n is called oscillatory or non-convergent.

Note:

i) $S_n = 1+2+3\ldots+n = \frac{n(n+1)}{2}$ [AP]

ii) $S_n = 1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\ldots\left(\frac{1}{2}\right)^{n-1} = a\left(\frac{1-r^n}{1-r}\right)$ [GP]

iii) $S_n = 1+3+\ldots(2n-1) = \frac{n}{2}[2a+(n-1)d]$ [AP]

iv) $S_n = 1+3^2+5^2+\ldots(2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

v) $S_n = 1^2+2^2+\ldots n^2 = \frac{n(n+1)(2n+1)}{6}$

4. Test the convergence of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots\infty$

Let $S_n = 1+\frac{1}{2}+\frac{1}{4}+\dots$

$= 1+\frac{1}{2}+(\frac{1}{2})^2+\dots$ [series is in G.P]

$= 1/(1-\frac{1}{2})$ $\qquad S_n = \frac{a}{1-r}$

$\underset{n\to\infty}{lt}\ S_n = \frac{1}{1/2} = 2$

$\therefore$ The series is convergent.

5. Discuss the nature of the series $6-5-1+6-5-1+\dots\infty$

Let $S_n = 6-5-1+6-5-1+\dots$

$= 1, 6, \dots$ if n is even

$= 6, 0, \dots$ if n is odd

$\therefore$ S_n does not tend to a unique lt.

$\therefore$ The series is oscillatory

6. Examine the convergence of the series

i) $1+3+5+7+\dots\infty$ & ii) $1+3^2+5^2+\dots(2n-1)^2+\dots\infty$

iii) $1-\frac{1}{3}+\frac{1}{3^2}-\frac{1}{3^3}+\dots\infty$ & iv) $\sum\frac{1}{n(n+2)}$

i) $1+3+5+7+\dots\infty$

The series is in A.P. $\qquad \therefore t_n = a+(n-1)d$

$a=1,\ d=2$

$\therefore t_n = 1+(n-1)2$

$= 2n-1$

$\therefore S_n = 1+3+\dots+(2n-1)$

$= \frac{n}{2}[2a+(n-1)d]$

$= \frac{n}{2}[2+2n-2] = n^2$

$\underset{n\to\infty}{lt}\ S_n = \underset{n\to\infty}{lt}\ n^2 = \infty$

$\therefore$ The series is divergent.

ii) $1+3^2+5^2+\cdots(2n-1)^2+\cdots\infty$

$$S_n = \frac{1}{3}n(2n-1)(2n+1)$$

$$= \frac{n(4n^2-1)}{3} = \frac{4n^3-n}{3}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{4n^3-n}{3} = \infty$$

$\therefore$ The series is divergent.

iii) $1-\frac{1}{3}+\frac{1}{3^2}-\frac{1}{3^3}+\cdots\infty$

$$S_n = \frac{1-r^n}{1-r}, \; r=\frac{1}{3}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{1-r}-\frac{r^n}{1-r}\right]$$

$\therefore \lim_{n\to\infty} r^n = 0$ where

$$= \left[\frac{1}{1-r}-0\right] = \frac{1}{1-r}$$ The series is convergent

iv) $\sum \frac{1}{n(n+2)}$

$$S_n = \sum\frac{1}{n(n+2)}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\sum\frac{1}{n(n+2)}$$

$$= \sum\lim_{n\to\infty}\frac{1}{n^2(1+2/n)} = \frac{1}{\infty} = 0$$

$\therefore$ The series is convergent.

v) S.T. $\lim_{n\to\infty}\frac{3n^2+2n+5}{6n^2+4n+7} = \frac{1}{2}$

vi) S.T $\lim_{n\to\infty}\left(\frac{1^2+2^2+\cdots+n^2}{n^3}\right) = \frac{1}{3}$

$$1^2+2^2+\cdots n^2 = \frac{n(n+1)(2n+1)}{6}$$

7. Examine the convergence of the series

$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}\cdots\infty$

$$S_n=\sum\frac{1}{n(n+1)}=\sum\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

Here $U_n=\frac{1}{n}-\frac{1}{n+1}$

$U_1=1-\frac{1}{2}$

$U_2=\frac{1}{2}-\frac{1}{3}$

$U_3=\frac{1}{3}-\frac{1}{4}$

$\vdots$

$U_n=\frac{1}{n}-\frac{1}{n+1}$

$S_1=1-\frac{1}{2}$

$S_2=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}=1-\frac{1}{3}$

$S_3=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}=1-\frac{1}{4}$

$\vdots$

$S_n=U_1+U_2+\cdots U_n=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}\cdots\frac{1}{n}-\frac{1}{n+1}$

$=1-\frac{1}{n+1}$ [sequence convergence $\therefore$ the series convergence]

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1-0=1$$

$\therefore$ The series is convergent.

8. Test the convergence or divergence of infinite series. $\sum[(n+1)^{1/3}-n^{1/3}]$

$S_n=\sum[(n+1)^{1/3}-n^{1/3}]$

$u_n=(n+1)^{1/3}-n^{1/3}$

$u_1=2^{1/3}-1$

$u_2=3^{1/3}-2^{1/3}$

$\vdots$

$u_{n-1}=n^{1/3}-(n-1)^{1/3}$

$u_n=(n+1)^{1/3}-n^{1/3}$

$S_n=u_1+u_2+\cdots u_n=(n+1)^{1/3}-1$

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}[(n+1)^{1/3}-1]=\infty$$

$\therefore$ The series is divergent.

Series of positive terms

Properties

1. Convergence of a series remains unchanged by the replacement, inclusion or omission of a finite no. of terms.
2. A series remains convergent, divergent or oscillatory when each term of it is multiplied by a fixed no. other than zero.
3. A series of positive terms either converges or diverges to $+\infty$ if omitting the negative terms, the sum of first n terms tends to either a finite limit or $+\infty$.
4. Every finite series is a convergent

Note:

i) If all terms after few negative terms in an infinite series are positive, such a series is a positive term series.

eg: $-10-6-1+5+12+20+\ldots$

ii) Necessary condition for convergence.

If a positive term series $\sum u_n$ is convergent then $\lim\limits_{n\to\infty} u_n = 0$

iii) Test for divergence

If $\lim\limits_{n\to\infty} u_n \neq 0$, the series $\sum u_n$ must be divergent.

Test for convergence

1. Comparison test for convergence

If two positive term series Σu_n and Σv_n be such that

i) Σv_n converges

ii) $u_n \leq v_n$ for all values of n, then Σu_n is also converges.

2. Comparison test for divergence.

If two positive term series Σu_n and Σv_n be such that

i) Σv_n diverges

ii) $u_n \geq v_n$ for all values of n, then Σu_n also diverges.

3. Limit Comparison test

If two positive term series Σu_n and Σv_n be such that $\lim_{n \to \infty} \frac{u_n}{v_n}$ = finite quantity ($\neq 0$), then Σu_n and Σv_n converge or diverge together.

Behaviours of Auxillary series

i) The P- series

The harmonic series $\frac{1}{1^p} + \frac{1}{2^p} + \dots + \frac{1}{n^p} + \dots = \Sigma \frac{1}{n^p}$

i) Converges if $p > 1$, ii) diverges if $p \leq 1$

ii) The geometric series

The geometric series $1 + x + x^2 + \dots x^n + \dots = \Sigma x^n$

i) converges if $x < 1$, ii) diverges if $x \geq 1$

iii) order test

$\frac{1}{n^{p-q}}$ is the order of u_n

If $p-q=k$ then, u_n is of the order $\frac{1}{n^k}$ then

i) Σu_n is convergent if $k>1$, ii) Σu_n divergent if $k \le 1$

Problems based on Comparison test for convergence

9. Examine the convergence of the series

$$1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots$$

Let $\Sigma u_n = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots$

Consider $\Sigma v_n = 1+1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 2}+\frac{1}{1\cdot 2\cdot 2\cdot 2}+\cdots$

$= 1+1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots$

$= 1+[1+r+r^2+\cdots]$ where $r=\frac{1}{2}$

$= 1+$ geometric series for which $|r|<1$

$\Rightarrow \Sigma v_n$ is convergent

$\Sigma u_n = 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\cdots$

$\Sigma v_n = 1+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$

From Σu_n & Σv_n

$u_1 = 1 = v_1 = 1$

$u_2 = 1 = v_2 = 1$

$u_3 = \frac{1}{2} = v_3 = \frac{1}{2}$

$u_4 = \frac{1}{6} < v_4 = \frac{1}{4}$

$\vdots$

is $u_n \le v_n \ \forall n$.

$\therefore$ by the comparison test Σa_n is convergent

is Σv_n convergent $\Rightarrow \Sigma u_n$ is convergent

10. Test for convergence the series $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$

$$u_n = \frac{1}{3^n+1}$$

Let $v_n = \frac{1}{3^n}$

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty} \frac{1}{3^{n+1}} \times 3^n$$

$$= \lim_{n\to\infty} \frac{1}{3^{-n}+1} = 1 \neq 0$$

$\sum \frac{1}{3^n}$ is convergent $\Rightarrow \sum u_n$ is also convergent.

11. Test the convergence of the series $\frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \dots$

Let $\sum u_n = \frac{1}{1^1} + \frac{1}{2^2} + \dots$

Consider $\sum v_n = 1 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \dots = [1 + r + r^2 + \dots]$ where $r = \frac{1}{2}$

$\sum v_n$ is convergent

$$u_1 = 1 = v_1 = 1$$
$$u_2 = \frac{1}{4} < v_2 = \frac{1}{2}$$
$$u_3 = \frac{1}{27} < v_3 = \frac{1}{4}$$

$\therefore u_n \leq v_n \ \forall n$

by comparison test $\sum u_n$ is convergent

Note:

i) $S = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots$ is a geometric series [convergent]

ii) $S = 1 + \frac{1}{2} + \frac{1}{2} + \dots$ is a geometric series [divergent]

iii) $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ is a p series [divergent]

$= \frac{1}{1^p} + \frac{1}{2^p} + \dots$

iv) $S = 1 + \frac{1}{2^2} + \frac{1}{3^2}$ is [convergent]

(X) Problems based on Comparison test for divergence

P.T. the harmonic series is divergent.

13. Test the convergence of the series $1+\frac{1}{2}+\frac{1}{3}+\cdots$

$$\sum u_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots$$

$$= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots$$

Consider $\sum v_n = 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots$

$= 1+\frac{1}{2}+\frac{1}{2}+\cdots$

$= 1+a+a+\cdots$ where $r=1$

$= 1+$ a geometric series for $r=1$

$=$ divergent.

$u_1 = 1 = v_1 = 1$

$u_2 = \frac{1}{2} = v_2 = \frac{1}{2}$

$u_3 = \frac{1}{3} > v_3 = \frac{1}{4}$

$u_4 = \frac{1}{4} = v_4 = \frac{1}{4}$

$u_n \geq v_n \ \forall n$

by Comparison test $\sum v_n$ divergent $\Rightarrow \sum u_n$ divergent.

14.

15 Problems based on comparison test for convergence or divergence together - Limit form

15. Test the convergence of the series

$$\frac{1}{1\cdot2\cdot3}+\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}+\cdots\infty$$

$$\text{Gn. } \Sigma u_n = \frac{1}{1\cdot2\cdot3}+\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}+\cdots\infty$$

n^{th} term of u_n is $\dfrac{2n-1}{n(n+1)(n+2)}$

To find v_n

$v_n = \dfrac{1}{n^{p-q}}$ where p is the highest power of n is in the denominator $=3$

q is the highest power of n in the numerator $=1$

$$\therefore v_n = \frac{1}{n^{3-1}} = \frac{1}{n^2}$$

To find $\underset{n\to\infty}{lt} \dfrac{u_n}{v_n}$

$$\frac{u_n}{v_n} = \frac{\dfrac{2n-1}{n(n+1)(n+2)}}{1/n^2} = \frac{n^2(2n-1)}{n(n+1)(n+2)}$$

$$= \frac{(2-1/n)}{(1+1/n)(1+\frac{2}{n})}$$

$$\underset{n\to\infty}{lt}\frac{u_n}{v_n} = \underset{n\to\infty}{lt}\frac{2-1/n}{(1+1/n)(1+2/n)}$$

$$= 2$$

$\therefore$ by comparison test Σu_n & Σv_n are convergent or divergent together.

$\Sigma v_n = \Sigma \dfrac{1}{n^2}$ is of the form $\Sigma \dfrac{1}{n^p}$ where $p=2$ is $p>1$

$\therefore \Sigma v_n$ convergent

Hence Σu_n is also convergent.

16. Test for convergence of the series

(x) $\frac{1}{4\cdot7\cdot10}+\frac{4}{7\cdot10\cdot13}+\frac{9}{10\cdot13\cdot16}+\cdots$

Let $\Sigma u_n = \frac{1}{4\cdot7\cdot10}+\frac{4}{7\cdot10\cdot13}+\cdots$

The nth term of Σu_n is $\frac{n^2}{(3n+1)(3n+4)(3n+7)}$

To find v_n

$v_n = \frac{1}{n^{p-q}}$, where p power of n in the denominator - 3
q power of n in numerator - 2

$= \frac{1}{n^{3-2}} = \frac{1}{n}$

To find $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{\frac{n^2}{(3n+1)(3n+4)(3n+7)}}{\frac{1}{n}}$

$= \frac{n^3}{(3n+1)(3n+4)(3n+7)} = \frac{1}{(3+1/n)(3+\frac{4}{n})(3+7/n)}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(3+1/n)(3+\frac{4}{n})(3+7/n)} = \frac{1}{27} \neq 0$

$\therefore$ by comparison test Σu_n & Σv_n are convergent or divergent together.

but $\Sigma v_n = \Sigma \frac{1}{n}$, $\Sigma\frac{1}{n^p}$ where $p=1$ is $p \le 1$

$\Rightarrow \Sigma v_n$ is divergent

$\Rightarrow \Sigma u_n$ is also divergent

17. Find the nature of the series $\frac{1}{\sqrt{10}}+\frac{1}{\sqrt{20}}+\frac{1}{\sqrt{30}}+\cdots$

Let $\Sigma u_n = \frac{1}{\sqrt{10}}+\frac{1}{\sqrt{20}}+\cdots$

nth term $= \frac{1}{\sqrt{10n}}$

$v_n = \frac{1}{n^{p-q}}$, $p = 1/2$, $q = 0$

$\therefore v_n = \frac{1}{n^{1/2}} = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{\frac{1}{\sqrt{10n}}}{\frac{1}{\sqrt{n}}} = \frac{\sqrt{n}}{\sqrt{10n}} = \frac{n^{1/2}}{\sqrt{10}\cdot n^{1/2}} = \frac{1}{\sqrt{10}} \neq 0$$

$\therefore$ by comparison test $\sum u_n$ & $\sum v_n$ are convergent or divergent together.

But $\sum v_n = \sum \frac{1}{n^{1/2}}$ is of the form $\sum \frac{1}{n^p}$ where $p = \frac{1}{2} < 1$

$\Rightarrow \sum v_n$ is divergent

Hence $\sum u_n$ is also divergent.

8. Test for convergence of the series

$$\frac{1+a}{2+b} + \frac{1+2a}{2+8b} + \frac{1+3a}{2+27b} + \dots \quad a, b \neq 0$$

Let $\sum u_n = \frac{1+a}{2+b} + \frac{1+2a}{2+8b} + \dots$

$$n\text{th term} = \frac{1+na}{2+n^3b}$$

To find v_n

$$\sum v_n = \frac{1}{n^{p-q}}, \quad p = 3, \ q = 1$$

$$= \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1+na}{2+n^3b}}{1/n^2} = \lim_{n\to\infty} n^2\left(\frac{1+na}{2+n^3b}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\frac{1}{n}+a}{\frac{2}{n^3}+b}\right) = a/b$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = a/b \neq 0$$

$\therefore$ By comparison test $\sum u_n$ & $\sum v_n$ are convergent or divergent together

$$\sum v_n = \frac{1}{n^2}, \quad p = 2 > 1$$

$\therefore \sum v_n$ is convergent

Hence $\sum u_n$ is convergent.

19. Discuss the nature of the series

i) $\sum_{n=0}^{\infty} \frac{2n^3+5}{4n^5+1}$ ii) $\sum_{n=1}^{\infty}\left[\sqrt{n^4+1} - \sqrt{n^4-1}\right]$

i) $u_n = \frac{2n^3+5}{4n^5+1}$

$v_n = \frac{1}{n^{p-q}}$, $p=5$, $q=3$

$v_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{2n^3+5}{4n^5+1} \times \frac{n^2}{1}\right)$

$= \lim_{n\to\infty} \frac{2+\frac{5}{n^3}}{4+\frac{1}{n^5}} = \frac{1}{2} \neq 0$

$\Rightarrow \sum u_n$ & $\sum v_n$ are convergent or divergent together

$\sum v_n = \frac{1}{n^2}$ $p=2>1$

$\Rightarrow \sum v_n$ is convergent

Hence $\sum u_n$ is also convergent.

ii) $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

$= \sqrt{n^4+1} - \sqrt{n^4-1} \times \left[\frac{\sqrt{n^4+1}+\sqrt{n^4-1}}{\sqrt{n^4+1}+\sqrt{n^4-1}}\right]$

$= \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$

$v_n = \frac{1}{n^{p-q}} = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}} = 1$

$\Rightarrow \sum u_n$ & $\sum v_n$ are convergent or divergent together

but $\sum v_n = \frac{1}{n^2}$ convergent

$\Rightarrow \sum u_n$ is convergent.

Integral test

Cauchy's integral test

If $\sum u_n$ is a series of positive terms and if $u_x = f(x)$ be such that i) $f(x)$ is continuous in $1 < x < \infty$
ii) $f(x)$ decreases as x increases then, the series $\sum u_n$ is convergent or divergent according as the integral $\int_1^{\infty} f(x)\,dx$ is finite or infinite.

20. Apply Cauchy's integral test to discuss the nature of the harmonic series (P series) $\sum_{n=1}^{\infty} \frac{1}{n^P}$

Let $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n^P}$

To find $f(x) = \frac{1}{x^P}$ which is decreases as x increases

$\therefore$ By the integral test the gn. series is Convergent or divergent according as the integral

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^P}\,dx \quad -①$$

Case i) If $P=1$ then ① $\Rightarrow \int_1^{\infty} \frac{1}{x}\,dx = (\log x)_1^{\infty} = \infty - 0 = \infty$

$\sum u_n$ is divergent if $P=1$

case ii) If $P>1$ then ① $\Rightarrow \int_1^{\infty} \frac{1}{x^P}\,dx = \left(\frac{x^{-P+1}}{-P+1}\right)_1^{\infty}$

$$= \frac{1}{1-P}\left(\frac{1}{x^{-(1-P)}}\right)_1^{\infty}$$

$$= 0 - \frac{1}{1-P} = \frac{1}{P-1} \neq 0$$

Thus $\sum u_n$ is convergent if $P>1$

case iii) If $P<1$ then ① $\Rightarrow \int_1^{\infty} \frac{1}{x^P}\,dx = \left[\frac{x^{1-P}}{1-P}\right]_1^{\infty} = \infty - \frac{1}{1-P} = \infty$

Thus $\sum u_n$ is divergent if $P<1$

21, Examine the convergence of $\sum_{n=2}^{\infty} \frac{1}{n\log n}$

9 $f(x) = \frac{1}{x\log x}$ which decreases as x increases

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x\log x}\,dx = \int_{\log 2}^{\infty} \frac{1}{t}\,dt$$

Put $t = \log x$

$dt = \frac{1}{x}\,dx$

$x \to 2 \Rightarrow t \to \log 2$

$x \to \infty \Rightarrow t \to \infty$

$$= \left[\log t\right]_{\log 2}^{\infty}$$

$$= \infty - \log(\log 2)$$

$$= \infty$$

$\Rightarrow \sum u_n$ is a divergent series.

22, i) $\sum_{n=1}^{\infty} \sqrt{\frac{3^n-1}{2^n+1}}$

Let $u_n = \sqrt{\frac{3^n-1}{2^n+1}}$ $\quad v_n = \left(\sqrt{\frac{3}{2}}\right)^n$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \sqrt{\frac{3^n-1}{2^n+1}} \times \frac{\sqrt{2}^n}{\sqrt{3}^n} \Rightarrow \lim_{n\to\infty} \frac{\sqrt{3^n}\sqrt{\left(1-\frac{1}{3^n}\right)2^n}}{\sqrt{2^n\left(1+\frac{1}{2^n}\right)\cdot 3^n}}$$

$\Rightarrow 1 \Rightarrow \sum u_n$ & $\sum v_n$ are convergent or divergent together

but $\sum v_n = \left(\sqrt{\frac{3}{2}}\right)^n$ is $r = \sqrt{\frac{3}{2}} > 1$ $\therefore$ divergent

$\Rightarrow \sum u_n$ is divergent.

ii). $\sum_{n=1}^{\infty} \frac{n+1}{n^p}$, $p > 0$

23, Examine the convergence of $\sum_{n=1}^{\infty} n e^{(-n^2)}$

Here $f(x) = x e^{-x^2}$ which decreases as x increases

Now $\int_1^{\infty} x e^{-x^2}\,dx = \int_1^{\infty} x e^{-t} \frac{dt}{2x}$

Put $x^2 = t$

$2x\,dx = dt$

$dx = \frac{dt}{2x}$

when $x = 1 \Rightarrow t = 1$

$x = \infty \Rightarrow t = \infty$

$= \frac{1}{2e}$ which is finite

Hence the gn. series is convergence

D'Alembert's ratio test

The series Σu_n of positive terms is convergent if $\underset{n\to\infty}{lt} \frac{u_{n+1}}{u_n} < 1$ is divergent if $\underset{n\to\infty}{lt} \frac{u_{n+1}}{u_n} > 1$

If $\underset{n\to\infty}{lt} \frac{u_{n+1}}{u_n} = 1$, the test fails.

Alternative form of Ratio test

If Σu_n is a positive term series such that

$$\underset{n\to\infty}{lt} \left(\frac{u_n}{u_{n+1}}\right) = l$$

then the series Σu_n i) converges if $l > 1$ ii) diverges if $l < 1$ iii) $l = 1$ the series may converge or diverge.

Cauchy's integral test

23) ii) Find the nature of the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ by cauchy's integral test.

24, Discuss the convergence of the series

(x)

$$1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\cdots$$

Here $u_n = \frac{n^2}{n!}$

$$u_{n+1} = \frac{(n+1)^2}{(n+1)!}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{\frac{(n+1)^2}{(n+1)!}}{n^2/n!} = \frac{(n+1)^2}{(n+1)!}\cdot\frac{n!}{n^2}$$

$$= \frac{n+1}{n^2}$$

$$\lim_{n\to\infty}\frac{n+1}{n^2} = 0 < 1$$

Hence by ratio test Σu_n is convergent.

25, Find the nature of the series $\frac{2}{1}+\frac{2\cdot5\cdot8}{1\cdot5\cdot9}+\frac{2\cdot5\cdot8\cdot11}{1\cdot5\cdot9\cdot13}+\cdots$

$$u_n = \frac{2\cdot5\cdot8\cdots(3n+5)}{1\cdot5\cdot9\cdots(4n+5)}$$

$$u_{n+1} = \frac{2\cdot5\cdot8\cdots(3(n+1)+5)}{1\cdot5\cdot9\cdots(4(n+1)+5)} = \frac{2\cdot5\cdot8\cdots(3n+5)(3n+8)}{1\cdot5\cdot9\cdots(4n+5)(4n+9)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{2\cdot5\cdots(3n+5)(3n+8)}{1\cdot5\cdot9\cdots(4n+5)(4n+9)}\times\frac{1\cdot5\cdot9\cdots(4n+5)}{2\cdot5\cdot8\cdots(3n+5)}$$

$$= \lim_{n\to\infty}\frac{3n+8}{4n+9} = 3/4 < 1$$

$\therefore \Sigma u_n$ is convergent.

26. $$\left(\frac{1}{3}\right)^2+\left(\frac{1\cdot2}{3\cdot5}\right)^2+\cdots$$

27. $$\frac{1^2}{4^2}+\frac{1^2\cdot5^2}{4^2\cdot8^2}+\cdots$$

28. Discuss the convergence of the series

i) $\sum \frac{n!\,2^n}{n^n}$ ii) $\left(\frac{n^2}{2^n}+\frac{1}{n^2}\right)$ iii) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

i) $\sum \frac{n!\,2^n}{n^n}$

ii) $\left(\frac{n^2}{2^n}+\frac{1}{n^2}\right)$

Let $u_n = \left(\frac{n^2}{2^n}+\frac{1}{n^2}\right)$

$u_{n+1} =$

x. iv) Test the convergence of the series $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\cdots$ by D'Alembert's ratio test.

Here $u_n = \frac{n^p}{n!}$, $u_{n+1} = \frac{(n+1)^p}{(n+1)!}$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^p}{(n+1)!} \times \frac{n!}{n^p} = \frac{\cancel{n^p}(1+\frac{1}{n})^p}{\cancel{n!}\,(n+1)} \cdot \frac{\cancel{n!}}{\cancel{n^p}} = \frac{(1+\frac{1}{n})^p}{n+1}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^p}{n+1} = \frac{1}{\infty} = 0 < 1$$

Hence by ratio test $\sum u_n$ is convergent.

29. Test for the convergence of the series.

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \cdots (x>0) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$u_n = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$u_{n+1} = \frac{x^{2(n+1)}}{(n+3)\sqrt{n+2}}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{\cancel{x^{2n}}\cdot x^2}{(n+3)\sqrt{n+2}}}{\frac{\cancel{x^{2n}}}{(n+2)\sqrt{n+1}}} = \frac{(n+2)\sqrt{n+1}\,x^2}{(n+3)\sqrt{n+2}}$$

$$= \frac{\sqrt{n+2}\sqrt{n+2}\sqrt{n+1}}{(n+3)\sqrt{n+2}}x^2$$

$$= \frac{\sqrt{(n+1)(n+2)}\,x^2}{n+3}$$

$$= \frac{n\sqrt{\left(1+\frac{2}{n}\right)\left(1+\frac{1}{n}\right)}\,x^2}{n\left(1+\frac{3}{n}\right)}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x^2$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x^2$$

If $x^2<1$ then $\sum u_n$ is convergent

If $x^2>1$ then $\sum u_n$ is divergent.

If $x^2=1$ then the test fails.

If $x^2=1 \Rightarrow u_n = \frac{1}{(n+2)(\sqrt{n+1})}$

u_n is of order $\frac{1}{n^{3/2}}$ is of the form $\frac{1}{n^p}$, $p=\frac{3}{2}>$

Hence $\sum u_n$ is convergent.

$\therefore \sum u_n$ is convergent if $x^2 \le 1$ & divergent if $x^2>1$

30. Test for the convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{1+n^2x^{2n}}$$

$$u_n = \frac{x^n}{1+n^2x^{2n}}$$

$$u_{n+1} = \frac{x^{n+1}}{1+(n+1)^2x^{2n+2}}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x(1+n^2x^{2n})}{1+(n+1)^2x^{2n+2}}$$

Case i) If $x<1$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{x} \qquad \because \lim_{n\to\infty} n^2x^{2n} = 0 = \lim_{n\to\infty}(n+1)^2x^{2n+2}$$

$x<1$

$\therefore \sum u_n$ is convergent

ii) If $x>1 \Rightarrow \frac{1}{x}<1$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x\left(\frac{1}{n^2x^{2n}}+1\right)}{\frac{1}{n^2x^{2n}}+\left(1+\frac{1}{n}\right)^2x^2} = \lim_{n\to\infty} \frac{x}{x^2} = \frac{1}{x}<1$$

$\therefore \sum u_n$ is convergent

iii) If $x=1$ then $u_n = \frac{1}{1+n^2}$

This is of the form $\frac{1}{n^p}$ where $p=2>1$

$\therefore \sum u_n$ is convergent

$\therefore \sum u_n$ is convergent for all values of x.

31. Discuss the convergence of the series

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \dots \infty$$

$$u_n = \frac{n^nx^n}{n!}$$

$$u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n x^n}$$

$$= \frac{(n+1)^{n+1} x^{\cancel{n}+1}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{n^n x^n}$$

$$= \frac{(n+1)^n \cdot x}{n^n} = \frac{\cancel{n^n}\left(1+\frac{1}{n}\right)^n \cdot x}{\cancel{n^n}}$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n x = ex \qquad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Case i) If $ex < 1$ then Σu_n is convergent

ii) If $ex > 1$ then Σu_n is divergent

iii) If $ex = 1$ then the ratio test fails.

Apply Log test (Here $ex = 1$) $\therefore e = 1/x$

Now $\frac{u_n}{u_{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n \cdot x} = \frac{e}{\left(1+\frac{1}{n}\right)^n}$

$$\log \frac{u_n}{u_{n+1}} = \log\left(\frac{e}{(1+1/n)^n \cancel{x}}\right)$$

$$= \log e - \log\left(\left(1+1/n\right)^n\right)\cancel{x}$$

$$= \log e - \log\left(1+1/n\right)^n \cancel{\log x}$$

$$= 1 - \left(\log\left(1+1/n\right)^n + \cancel{\log x}\right)$$

$$= 1 - \left[n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right)\right]$$

$$= \frac{1}{2n} - \frac{1}{3n^2} + \cdots$$

$$n \log \frac{u_n}{u_{n+1}} = n\left[\frac{1}{2n} - \frac{1}{3n^2} + \cdots\right]$$

$$= \left[\frac{1}{2} - \frac{1}{3n} + \cdots\right]$$

lt log $\frac{u_n}{\ldots}$ ... the series diverges

Alternating Series - Leibnitz's test

A series in which the terms are alternately positive or negative is called an alternating series.

Leibnitz's rule

An alternating series $u_1 - u_2 + \dots$ Converges if

i) $u_n - u_{n-1} < 0$ (Consider numerical value of u_n & u_{n-1})

ii) $\lim_{n \to \infty} u_n = 0$ (" " " of u_n)

Note :

i) The alternating series will not be convergent if any one of the Leibnitz's condition is not satisfied.

ii) If $\lim_{n \to \infty} u_n \neq 0$ then $\lim_{n \to \infty} S_{2n} \neq \lim_{n \to \infty} S_{2n+1}$, the gn. series is oscillatory.

32. Discuss the convergence of the series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$$

The terms of the gn. series are alternately positive and negative.

$$u_n = \frac{1}{\sqrt{n}}$$

$$u_{n-1} = \frac{1}{\sqrt{n-1}}$$

$$u_n - u_{n-1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n-1}} = \frac{\sqrt{n-1} - \sqrt{n}}{\sqrt{n}\sqrt{n-1}} < 0$$

$$\Rightarrow u_n - u_{n-1} < 0 \quad \text{--- (1)}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \quad -(2)$$

From (1) & (2) Leibnitz's rule satisfied.

$\therefore$ The gn. series is convergent.

33. Discuss the convergence of the series

$$\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \cdots \quad 0<x<1$$

The terms of the gn. series are alternately positive and negative.

$$u_n = \frac{x^n}{x^n+1}, \quad u_{n-1} = \frac{x^{n-1}}{1+x^{n-1}}$$

$$u_n - u_{n-1} = x^n\left[\frac{1}{1+x^n} - \frac{1}{x+x^n}\right]$$

$$= x^n\left[\frac{x-1}{(1+x^n)(x+x^n)}\right] < 0 \quad -(1) \quad [\because 0<x<1]$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{x^n}{1+x^n} = 0 \quad -(2)$$

$\because x<1,\ x^n \to 0$ as $n\to\infty$

From (1) & (2) Leibnitz's rule satisfied

$\therefore$ the gn. series is convergent.

34. Discuss the convergence of the series

$$\left(\frac{1}{2} - \frac{1}{\log 2}\right) - \left(\frac{1}{2} - \frac{1}{\log 3}\right) + \left(\frac{1}{2} - \frac{1}{\log 4}\right) - \cdots$$

$$u_n = \frac{1}{2} - \frac{1}{\log(n+1)}$$

$$u_{n-1} = \frac{1}{2} - \frac{1}{\log(n-1+1)} = \frac{1}{2} - \frac{1}{\log n}$$

$$u_n - u_{n-1} = \left[\frac{1}{2} - \frac{1}{\log(n+1)}\right] - \left[\frac{1}{2} - \frac{1}{\log n}\right]$$

$$= \frac{1}{\log n} - \frac{1}{\log(n+1)} = \frac{\log(n+1) - \log(n)}{(\log n)[\log(n+1)]} \nless 0$$

$\Rightarrow u_n - u_{n-1} \nless 0$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left[\frac{1}{2} - \frac{1}{\log(n+1)}\right] = \frac{1}{2} \neq 0 \quad - (2)$$

From (1) & (2) both the conditions of Leibnitz's rule not satisfied.

Hence the series is oscillatory and it oscillates between $-\infty$ & ∞

35. Examine the character of the series.

i) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2^{n-1}}$ ii) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

i) The terms of the gn. series are alternatively positive and negative.

$$u_n = \frac{n}{2n-1}$$

$$u_{n-1} = \frac{n-1}{2(n-1)-1} = \frac{n-1}{2n-3}$$

$$u_n - u_{n-1} = \frac{-1}{(2n-1)(2n-3)} < 0 \quad - (1)$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{n}{n(2-1/n)} = \frac{1}{2-\frac{1}{n}} = \frac{1}{2} \neq 0 \quad (2)$$

From (1) + (2) Leibnitz's rule not satisfied.

$\therefore$ The gn. series is not convergent

$\Rightarrow$ The gn. series is oscillatory.

ii) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

Series of positive and negative terms - Absolute and Conditional Convergence

Absolutely convergent

If the series of arbitrary terms $u_1 + u_2 + \cdots u_n +$ be such that the series $|u_1| + |u_2| + \cdots + |u_n| +$ is convergent then the series $\sum u_n$ is said to be absolutely convergent.

Conditionally convergent

If $\sum |u_n|$ is divergent but $\sum u_n$ is convergent, then $\sum u_n$ is said to be conditionally convergent.

Note:

i) An absolutely convergent series is necessarily convergent but not conversely.

ii) The series $\sum u_n$ is absolutely convergent if $\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} < 1$ and is divergent if $\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} > 1$

The test fails when the lt. is 1

iii) If $\sum |u_n|$ is convergent then $\sum u_n$ is absolutely conv

iv) If $\sum u_n$ is convergent and $\sum |u_n|$ diverges, then $\sum u_n$ is said to be conditionally convergent.

v) The p series is $\sum \frac{1}{n^p}$, converges if $p > 1$ & diverges if $p \leq 1$

vi) The G.P is $\sum r^n$, converges if $r < 1$, diverges if $r >$

vii) order test

$u_n = \frac{1}{n^{p-q}} = 1/n^k$ where $k = p - q$

$\sum u_n$ is convergent if $k > 1$ & divergent if $k \leq 1$

36. Show that the series $1+\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}-\dots$ is absolutely convergent

$\Sigma u_n = 1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}\dots$

$\Rightarrow \Sigma|u_n| = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots$

T.P. Σu_n is absolutely convergent

i.e. T.P $\Sigma|u_n|$ is convergent

$\Sigma|u_n|$ is of the form $\frac{1}{n^p}$ where $p=2>1$

$\Rightarrow \Sigma|u_n|$ is convergent

$\Rightarrow \Sigma u_n$ is absolutely convergent.

37. Prove that the alternating series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$ is conditionally convergent.

$\Sigma u_n = 1-\frac{1}{2}+\frac{1}{3}+\dots$

$\Sigma|u_n| = 1+\frac{1}{2}+\frac{1}{3}+\dots$

T.P. Σu_n is conditionally convergent

i.e. T.P i) Σu_n is convergent

ii) $\Sigma|u_n|$ is divergent

$\Sigma u_n = 1-\frac{1}{2}+\frac{1}{3}-\dots$ is an alternating series

Apply Leibnitz's rule,

$u_{n-1} = \frac{1}{n-1}$ $\quad u_n = \frac{1}{n}$

$u_n - u_{n-1} = \frac{-1}{n(n-1)} < 0$ — (1)

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$ — (2)

∴ from (1) & (2) the gn. series is convergent

$\sum |u_n| = 1 + \frac{1}{2} + \ldots = \sum \frac{1}{n}$ is of the form $\sum \frac{1}{n^p}$ where p

$\Rightarrow \sum |u_n|$ diverges

∴ $\sum u_n$ is conditionaly convergent.

38. Test for conditional Convergence of the following series. $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \ldots \infty$

$$\sum u_n = \sum \frac{1}{(n+1)^3}(1+2+\ldots n)(-1)^{n-1}$$

$$= \sum \frac{1}{(n+1)^3} \frac{n(n+1)}{2}(-1)^{n-1} = \sum \frac{1}{2} \frac{n(-1)^{n-1}}{(n+1)^2}$$

$$\sum |u_n| = \sum \frac{1}{2} \frac{n}{(n+1)^2}$$

i) To find $\sum |u_n|$ is convergent or divergent

Apply order test

$u_n = \frac{1}{2} \frac{n}{(n+1)^2} = \frac{1}{n^k}$ where $k = p - q = 2 - 1 = 1$

∴ $\sum |u_n|$ is divergent

ii) To find $\sum u_n$ is convergent or divergent

Apply Liebnitz's test

$u_n = \frac{1}{2} \frac{n}{(n+1)^2}$, $u_{n-1} = \frac{1}{2} \frac{n-1}{n^2}$

$$u_n - u_{n-1} = \frac{1}{2}\left[\frac{-n^2+n+1}{n^2(n^2+2n+1)}\right] < 0, \ (n > 1) \ \text{---} (1)$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2} \frac{n}{(n+1)^2} = 0$$

∴ The gn. series is convergent

$\Rightarrow \sum u_n$ is conditionally convergent

Test the convergence and absolute convergence of the series $\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} - \cdots$

$$\Sigma u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n+1}+1}$$

$$\Sigma |u_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+1}$$

To find $\Sigma |u_n|$ is convergent or not

Apply order test

$$u_n = \frac{1}{n^{p-q}} = \frac{1}{n^K} = \frac{1}{n^{1/2}} \; ; \; K < 1$$

$\therefore \Sigma |u_n|$ is divergent

To find Σu_n is convergent or not

To find u_n & u_{n-1} (numerically)

$$u_{n-1} = \frac{1}{\sqrt{n}+1}$$

$$u_n - u_{n-1} = \frac{1}{\sqrt{n+1}+1} - \frac{1}{\sqrt{n}+1} = \frac{\sqrt{n}-\sqrt{n+1}}{(\sqrt{n+1}+1)(\sqrt{n}+1)} < 0$$

$$u_n - u_{n-1} < 0 \quad -(1)$$

$$\Rightarrow \lim_{n\to\infty} u_n = \frac{1}{\sqrt{n+1}+1} = 0 \quad -(2)$$

from (1) & (2) The gn. series is convergent

$\Rightarrow \Sigma u_n$ is conditionally convergent.

40. Test $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2}$ for convergence and absolute convergence

$$\Sigma u_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2} \qquad \Sigma |u_n| = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$$

To find $\Sigma |u_n|$ is convergent or not

$$u_n = \frac{1}{n^{p-q}} = \frac{1}{n^3} \; , \; K > 1$$

$\therefore \Sigma |u_n|$ is convergent $\Rightarrow \Sigma u_n$ is absolutely convergent.

41. S.T. the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{\sqrt{2n+1}}$ is absolutely conv

for $|x|<1$, conditionally convergent for $x=1$ & divergent for $x=-1$

$$\Sigma u_n = \frac{(-1)^{n-1}x^n}{\sqrt{2n+1}}$$

$$\Sigma |u_n| = \frac{x^n}{\sqrt{2n+1}}$$

To find $\Sigma|u_n|$ convergent or not

Apply Ratio test

To find u_n & u_{n+1}

$$u_{n+1} = \frac{x^{n+1}}{\sqrt{2n+3}}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{\sqrt{2n+3}} \times \frac{\sqrt{2n+1}}{x^n} = \frac{x\sqrt{2+1/n}}{\sqrt{2+3/n}}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = x$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = |x|$$

If $|x|<1$ the $\Sigma|u_n|$ is convergent

$\Rightarrow \Sigma u_n$ is absolutely convergent.

If $x=1 \Rightarrow \Sigma u_n = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \dots$ which is conditionally convergent

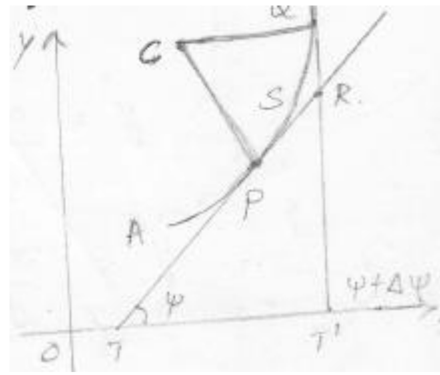
If $x=-1 \Rightarrow \Sigma u_n = -\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{7}} + \dots$

$\Rightarrow \Sigma u_n$ is divergent.

If $x=1$, $\Sigma|u_n|$ is divergent & Σu_n is convergent

$\Rightarrow \Sigma u_n$ is conditionally convergent for $x=1$

UNIT IV



Curvature:

The rate of bending of a curve in any interval is called the curvature of the curve in that interval.

$$\therefore \ \frac{1}{\rho} = \frac{d\psi}{ds}$$

Note:

1. The curvature of a circle at any point on it is the sence and is equal to the reciprocal of its radius.
2. The curvature of a straight line is zero.
3. The radius of curvature ρ is positive.

Radius of Curvature:

The reciprocal of the curvature of a curve at any point is called the radius of curvature at the point and it is denoted by ρ .

$$\text{Curavature} = \frac{1}{\text{radius of curvature}}$$

$$\text{Radius of Curuvature} = \frac{1}{\text{curvature}}$$

$$\therefore \ \rho = \frac{ds}{d\psi}.$$

Cartesian form:

Curve	Radius of Curuvature
(i) $Y = f(x)$ at (x,y)	$\rho = \dfrac{\left(1+\left((y')^2\right)^{3/2}\right)}{y''}$
(ii) $X = f(y)$ at (x,y)	$\rho = \dfrac{\left(1+\left((x')^2\right)^{3/2}\right)}{x''}$

1. Find the radius of curvature at $x=\frac{\pi}{2}$ **on the curve** $y=4\sin x$**.**

Sol:

Given: Y= 4 sin x at $x=\frac{\pi}{2}$.

$$y^{'}=4\cos x=4\cos\frac{\pi}{2}=0$$

$$y^{''}=-4\sin x=-4\sin\frac{\pi}{2}=-4\times 1=-4\ .$$

$$\therefore \rho=\frac{\left(1+(y^{'})^{2}\right)^{3/2}}{y^{''}}=\frac{(1+0)^{3/2}}{-4}=\frac{1^{3/2}}{-4}=-\frac{1}{4}.\ =1/4$$

2. Find the radius of curvature at any point (x, y) on $y=c\log\sec\frac{x}{c}$.

Sol:

Given $y=c\log\sec\frac{x}{c}$

$$y^{'}=\cancel{c}\,\frac{1}{\sec\frac{\cancel{x}}{c}}\tan\frac{x}{c}\sec\frac{\cancel{x}}{c}\,\frac{1}{\cancel{c}}=\tan\frac{x}{c}$$

$$y^{''}=\frac{1}{c}\sec^{2}\frac{x}{c}$$

$$\therefore \rho=\frac{\left(1+\left(y^{'}\right)^{2}\right)^{3/2}}{y^{''}}=\frac{\left(1+\left(\tan^{2}\frac{x}{c}\right)\right)^{3/2}}{\frac{1}{c}\sec^{2}x/c}$$

$$=\frac{c\left(\sec^{2}x/c\right)^{3/2}}{\sec^{2}x/c}=\frac{c\,\cancel{\sec^{3}}\,x/c}{\cancel{\sec^{2}}\,x/c}$$

$$\rho=c\sec\frac{x}{c}$$

3. Find the radius of curvature at the point $\left(\frac{1}{4},\frac{1}{4}\right)$ **on the curve** $\sqrt{x}+\sqrt{y}$ **= 1.**

Sol:

Given $\sqrt{x}+\sqrt{y}=1$ at $\left(\frac{1}{4},\frac{1}{4}\right)$.

Here, x = $\frac{1}{4}$, y = $\frac{1}{4}$.

$$\sqrt{x}+\sqrt{y}=1$$

i.e., $\sqrt{y}=1-\sqrt{x}\Rightarrow y=\left(1-\sqrt{x}\right)^2\left[\because x^n=n\,x^{n-1}\right]$

$$\Rightarrow y=1+x-2\sqrt{x}.$$

$$y^{'}=1-\cancel{2}\,x^{1/2-1}\,\frac{1}{\cancel{2}}=1-x^{-1/2}=1-\frac{1}{\sqrt{x}}=1-\frac{1}{\sqrt{1/4}}$$

$$=1-\frac{1}{1/2}=1-2=-1$$

$$y^{'}=-1$$

$$y^{''}=-x^{-1/2-1}\left(-\frac{1}{2}\right)=\frac{1}{2}x^{-3/2}=\frac{1}{2}\left(\frac{1}{4}\right)^{-3/2}=\frac{1}{2}\left(\frac{1}{4}\right)^{-1}\left(\frac{1}{4}\right)^{-1/2}$$

$$=\frac{1}{2}4\sqrt{4}=\frac{1}{\cancel{2}}4.\cancel{2}=4$$

$$y^{''}=4$$

$$\therefore\rho=\frac{\left(1+(y^{'})^2\right)^{3/2}}{y^{''}}=\frac{\left(1+(-1)^2\right)^{3/2}}{4}=\frac{(1+1)^{3/2}}{4}=\frac{2^{3/2}}{4}$$

$$=\frac{\cancel{2}\sqrt{2}}{\cancel{4}\,2}=\frac{\cancel{\sqrt{2}}}{\sqrt{2}\,\cancel{\sqrt{2}}}=\frac{1}{\sqrt{2}}$$

$$\rho=\frac{1}{\sqrt{2}}.$$

4. **Show that the radius of curvature at any point of the catenary y = c** $\cosh\left(\frac{x}{c}\right)$ **is** $\frac{y^2}{c}$ **also find P at (o, c**).

Sol:

Given y = c cosh $\left(\frac{x}{c}\right)$ $\rightarrow(1)$

$$y^{'}=\cancel{c}\,\sinh\left(\frac{x}{c}\right)\left(\frac{1}{\cancel{c}}\right)=\sinh\left(\frac{x}{c}\right).$$

$$y^{''}=\cosh\left(\frac{x}{c}\right)\left(\frac{1}{c}\right)=\left(\frac{1}{c}\right)\cosh\left(\frac{x}{c}\right).$$

$$\therefore \rho = \frac{\left(1+(y')^2\right)^{3/2}}{y''} = \frac{\left(1+\left(\sinh\left(\frac{x}{c}\right)^2\right)\right)^{3/2}}{\left(\frac{1}{c}\cosh\left(\frac{x}{c}\right)\right)}$$

$$= \frac{\left(1+\sinh^2\left(\frac{x}{c}\right)\right)^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= \frac{c\left(\cosh^2\left(\frac{x}{c}\right)\right)^{3/2}}{\cosh\frac{x}{c}} = c\frac{\cosh^{\cancel{3}}\left(\frac{x}{c}\right)}{\cancel{\cosh\left(\frac{x}{c}\right)}}$$

$$= c\cosh^2\left(\frac{x}{c}\right) \qquad \left[\frac{y}{c} = \cosh\left(x/c\right)\right]$$

$$c\left(\frac{y^2}{c^2}\right) = \frac{y^2}{c}$$

$$\rho = \frac{y^2}{c}$$

At (o, c) $\Rightarrow \rho = \frac{c^{\cancel{2}}}{\cancel{c}} = c$

$\rho = c.$

5. Show that the measure of curvatures of the curve $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ at any point (x, y) on it is

$$\frac{ab}{2\left(ax+by\right)^{3/2}}.$$

<u>Sol.</u>

Given $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1 \qquad \rightarrow (1)$

$$\Rightarrow \frac{\sqrt{x}}{\sqrt{a}} + \frac{\sqrt{y}}{\sqrt{b}} = 1$$

$$\Rightarrow \frac{\sqrt{y}}{\sqrt{b}} = 1 - \frac{\sqrt{x}}{\sqrt{a}} \qquad \rightarrow (2)$$

$$\Rightarrow \sqrt{y} = \sqrt{b}\left(1 - \frac{\sqrt{x}}{\sqrt{a}}\right)$$

$$\Rightarrow y = \sqrt{b}^{2}\left(1-\frac{\sqrt{x}}{\sqrt{a}}\right)^{2}$$

$$y^{'} = 2\left(\sqrt{b}\right)^{2}\left(1-\frac{\sqrt{x}}{\sqrt{a}}\right)^{1}\left(\frac{-1}{2\sqrt{x}\sqrt{a}}\right)$$

$$= \cancel{2}\left(\sqrt{b}\right)^{\cancel{2}}\left(\frac{\sqrt{y}}{\cancel{\sqrt{b}}}\right)\left(\frac{-1}{\cancel{2}\sqrt{x}\sqrt{a}}\right)\left(by\,(2)\right)$$

$$y^{'} = -\frac{\sqrt{b}\sqrt{y}}{\sqrt{a}\sqrt{x}}$$

$$y^{''} = -\frac{\sqrt{b}}{\sqrt{a}}\left(-\frac{1}{2}\right)x^{3/2}\sqrt{b}\left(1-\frac{\sqrt{x}}{\sqrt{a}}\right)^{1-1}$$

$$= \frac{\sqrt{b}\sqrt{b}}{2\sqrt{a}\,x^{3/2}}\left(1-\frac{\sqrt{x}}{\sqrt{a}}\right)^{0}$$

$$= \frac{b}{2\sqrt{a}\,x^{3/2}}$$

$$\rho = \frac{\left(1+\left(y^{'}\right)^{2}\right)^{3/2}}{y^{''}} = \frac{\left(1+\left(-\frac{\sqrt{b}\sqrt{y}}{\sqrt{a}\sqrt{x}}\right)^{2}\right)^{3/2}}{\left(\frac{b}{2\sqrt{a}\,x^{3/2}}\right)}$$

$$= \frac{\left(1+\frac{by}{ax}\right)^{3/2}}{\left(\frac{b}{2\sqrt{a}\,x^{3/2}}\right)} = \frac{\left(ax+by\right)^{3/2}}{\left(\frac{b}{2\sqrt{a}\,x^{3/2}}\right)a^{3/2}\,x^{3/2}}$$

$$\sqrt{a} = a^{1/2}$$

$$= a^{1/2}a^{1}\,a^{-1}$$

$$= a^{3/2}a^{-1}$$

$$= \frac{\left(ax+by\right)^{3/2}}{\left(ax\right)^{3/2}}\left(\frac{2\sqrt{a}\,x^{3/2}}{b}\right)$$

$$= \frac{(ax+by)^{3/2}}{\cancel{a^{3/2}}\ \cancel{x^{3/2}}} \frac{\left(2\,\cancel{a^{3/2}}\ \cancel{x^{3/2}}\right)}{ab}$$

$$\rho = \frac{2}{ab}(ax+by)^{3/2}$$

$$\therefore \text{ Curvature } = \frac{1}{\rho} = \frac{ab}{2(ax+by)^{3/2}}.$$

6.Find the radius of curvature at the point (c, c) on the curve $xy = c^2$.

Sol.

Given: $xy = c^2$

$$y = \frac{c^2}{x} = c^2\, x^{-1}$$

$$y' = \frac{dy}{dx} = -1c^2\, x^{-2} = -\frac{c^2}{x^2} = -c^2\, x^{-2}$$

$$y'' = \frac{d^2y}{dx^2} = +2c^2\, x^{-3} = \frac{2c^2}{x^3}$$

$$at\,(c,c)\,\frac{dy}{dx} = -\frac{c^2}{c^2} = -1$$

$$at\,(c,c)\,\frac{d^2y}{dx^2} = \frac{2c^2}{c^3} = \frac{2}{c}$$

$$\therefore \rho = \frac{\left(1+y'^2\right)}{y''} = \frac{\left(1+(-1)^2\right)^{3/2}}{2/c}$$

$$= \frac{c(2)^{3/2}}{2} = c\,\frac{\cancel{2}\sqrt{2}}{\cancel{2}}$$

$$\rho = c\sqrt{2}$$

Note: If $\frac{dy}{dx} = \infty$ Then the radius of curvature can be taken as

$$\rho = \frac{\left(1+\left(\frac{dx}{dy}\right)^2\right)^{3/2}}{\frac{d^2x}{dy^2}}$$

7.Find the radius of curvature of the curve $xy^2 = a^3 - x^3$ $at\,(a,0)$.

Sol:

Given: $xy^2 = a^3 - x^3 \qquad \rightarrow (1)$

Differenting (1) w.r.to x we get,

$$x.2y\frac{dy}{dx} + y^2 = -3x^2$$

$$\Rightarrow 2xy\frac{dy}{dx} + y^2 = -3x^2$$

$$\Rightarrow 2xy\frac{dy}{dx} = -3x^2 - y^2$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\left(3x^2 + y^2\right)}{2xy} \qquad \rightarrow (2)$$

$$\therefore \left(\frac{dy}{dx}\right)_{(a,0)} = -\frac{\left(3a^2 + 0\right)}{0} = \infty$$

Here, $\frac{dy}{dx}$ $at\,(a,0)\,is\,\infty.$

$$\therefore \rho = \frac{\left(1+\left(\frac{dx}{dy}\right)^2\right)^{3/2}}{\frac{d^2x}{dy^2}}$$

From (2) we get, $\frac{dx}{dy} = \frac{-2xy}{3x^2 + y^2} = 0.$

$$\frac{dx}{dy} = 0$$

$$\frac{d^2x}{dy^2} = \frac{\left(3x^2 + y^2\right)(-2)\left(x + y\frac{dx}{dy}\right) + 2xy\left(6x\frac{dx}{dy} + 2y\right)}{\left(3x^2 + y^2\right)^2}$$

$$\left(\frac{d^2x}{dy^2}\right)_{(a,0)} = \frac{-2\left(3a^2 + 0\right)(a + 0) + 2(a)(0)\left(6a(0) + 2(0)\right)}{\left(3a^2 + 0\right)^2}$$

$$= \frac{(-2)\left(3a^2\right)(a)}{\left(3a^2\right)^2} = \frac{-\cancel{6}^{2}\,\cancel{a^3}}{\cancel{9}^{3}\,\cancel{a^4}} = \frac{-2}{3a}$$

$$\rho = \frac{\left(1+0^2\right)^{3/2}}{-2/3\,a} = -\frac{3a}{2}\times 1^{3/2} = -\frac{3a}{2}$$

$$\therefore \rho = \frac{3a}{2}$$

8.Find ρ for the curve $x^3 + y^3 = 3axy$ $at\left(\frac{3a}{2},\frac{3a}{2}\right)$

Sol:

Given: $x^3 + y^3 = 3axy \qquad \rightarrow (1)$

Differentiating w.r.to x we get,

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx} + y\right)$$

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 3ax\frac{dy}{dx} + 3ay$$

$$\Rightarrow 3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay - 3yx^2$$

$$\Rightarrow \frac{dy}{dx}\left(3y^2 - 3ax\right) = 3ay - 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3ay - 3x^2}{3y^2 - 3ax} = \frac{ay - x^2}{y^2 - ax}$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{a\frac{3a}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - a\frac{3a}{2}} = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}}$$

$$= -\frac{\cancel{3a^2}}{\cancel{4}}\times\frac{\cancel{4}}{\cancel{3a^2}}$$

$$= -1$$

Differentiating (1) w.r.to x we get,

$$\frac{d^2y}{dx^2} = \frac{\left(y^2 - ax\right)\left(a\frac{dy}{dx} - 2x\right) - \left(ay - x^2\right)\left(2y\frac{dy}{dx} - a\right)}{\left(y^2 - ax\right)^2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)\left[a(-1) - 2\frac{3a}{2}\right] - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)\left(\frac{6a}{2}(-1) - a\right)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{\left(\frac{3a^2}{4}\right)(-a-3a)-\left(\frac{-3a^2}{4}\right)(-3a-a)}{\left(\frac{3a^2}{4}\right)^2}$$

$$= \frac{\left(\frac{3a^2}{4}\right)(-4a)+\left(\frac{3a^2}{4}\right)(-4a)}{\left(\frac{9a^4}{16}\right)}$$

$$= \frac{2\left(\frac{3a^2}{\cancel{4}}\right)(-\cancel{4}a)}{\frac{9a^4}{16}} = -\cancel{6}^2\,\cancel{a^3} \times \frac{16}{\cancel{9}_3\,\cancel{a^4}}$$

$$= \frac{-32}{3a}$$

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1+(-1)^2\right)^{3/2}}{\frac{-32}{3a}}$$

$$= \frac{-2^{3/2}\times 3a}{32} = -\frac{\cancel{2}\sqrt{2}\,3a}{\cancel{32}_{16}} = -\frac{3a\sqrt{2}}{16}$$

$$\rho = \frac{3a\sqrt{2}}{16}$$

Parametric form Curve Radius of curvature

$x = f(t)\, and\ y = g(t)$

$$\rho = \frac{\left[f'^2+g'^2\right]^{3/2}}{f'g''-f''g'}\ (or)$$

$$\rho = \frac{\left[x'^2+y'^2\right]^{3/2}}{x'y''-y'x''}$$

$(derivative\ w.r.to\ 't')$

10.Find ρ at any point $p(at^2, 2at)$ on the parabola $y^2 = 4ax.$

Sol:

Given $y^2 = 4ax$ at $p\left(at^2, 2at\right)$

Here, $x = at^2$ $y = 2at$

$$x' = \frac{dx}{dt} = 2at, \; y' = \frac{dy}{dt} = 2a$$

$$x'' = \frac{d^2x}{dt^2} = 2a, \; y'' = \frac{d^2y}{dt^2} = 0$$

$$\rho = \frac{\left[\left(x'\right)^2 + \left(y'\right)^2\right]^{3/2}}{x'y'' - y'x''} = \frac{\left[\left(2at\right)^2 + \left(2a\right)^2\right]^{3/2}}{\left(2at\right)\left(0\right) - \left(2a\right)\left(2a\right)}$$

$$= \frac{\left[4a^2t^2 + 4a^2\right]^{3/2}}{0 - 4a^2} = \frac{\left(4a^2\right)^{3/2}\left(t^2+1\right)^{3/2}}{-4a^2}$$

$$= \frac{\cancel{4}\sqrt{4}\,a^{\cancel{3}}\left(t^2+1\right)^{3/2}}{-\cancel{4}\,\cancel{a^2}} = -\sqrt{4a}\left(t^2+1\right)^{3/2}$$

$$= -2a\left(t^2+1\right)^{3/2}$$

$$\left|\rho\right| = 2a\left(1+t^2\right)^{3/2}.$$

12.Find the radius of curvature at any point $p\left(a\cos\theta, b\sin\theta\right)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

<u>Sol:</u>

Given that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ at $p\left(a\cos\theta, b\sin\theta\right)$

Here,

$x = a\cos\theta$ $\qquad$ $y = b\sin\theta$

, $x' = \frac{dx}{d\theta} = -a\sin\theta$ $\qquad$ $y' = \frac{dy}{dx} = b\cos\theta$

$x'' = \frac{d^2x}{d\theta^2} = -a\cos\theta$ $\qquad$ $y'' = \frac{d^2x}{d\theta^2} = b\sin\theta$

$$\therefore \rho = \frac{\left[\left(x'\right)^2 + \left(y'\right)^2\right]^{3/2}}{x'y'' - y'x''}$$

$$= \frac{\left[\left(-a\sin\theta\right)^2 + \left(b\cos\theta\right)^2\right]^{3/2}}{\left(-a\sin\theta\right)\left(-b\sin\theta\right) - \left(b\cos\theta\right)\left(-a\cos\theta\right)}$$

$$\frac{\left(a^2\sin^2\theta\right)+\left(b^2\cos^2\theta\right)^{3/2}}{ab\sin^2\theta+ab\cos^2\theta}$$

$$=\frac{\left(a^2\sin^2\theta+b^2\cos^2\theta\right)^{3/2}}{ab}$$

$$=\frac{\left(a^3\sin^3\theta+b^3\cos^3\theta\right)}{ab}$$

$$=\frac{\left(a^{\cancel{3}}\sin^3\theta\right)}{\cancel{a}\,b}+\frac{b^{\cancel{3}}\cos^3\theta}{a\,\cancel{b}}=\frac{a^2\sin^3\theta}{b}+\frac{b^2\cos^3\theta}{a}$$

Parametric to Cartesian form:

1**. Find the radius of curvature at any point** $x=a\cos^3\theta, y=a\sin^3\theta$ **of the curve** $x^{2/3}+y^{2/3}=a^{2/3}$, **also show that** $\rho^3=27axy.$

Sol:

Given: $x=a\cos^3\theta \quad \rightarrow(1) \qquad y=a\sin^3\theta \quad \rightarrow(2)$

$$y'=\frac{dy}{dx}=\frac{dy}{d\theta}\frac{d\theta}{dx}=\frac{\left(\frac{dy}{d\theta}\right)=\frac{d}{d\theta}\left(a\sin^3\theta\right)}{\left(\frac{dx}{d\theta}\right)\frac{d}{d\theta}\left(a\cos^3\theta\right)}$$

$$\frac{dy}{dx}=\frac{\cancel{3}\,\cancel{a}\sin^{\cancel{2}}\theta\,\cancel{\cos\theta}}{\cancel{-3a}\cos^{\cancel{2}}\theta\,\cancel{\sin\theta}}=-\frac{\sin\theta}{\cos\theta}=-\tan\theta$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\tan\theta\right)$$

$$=\frac{d}{d\theta}\left(-\tan\theta\right)\frac{d\theta}{dx}=\frac{\frac{d}{d\theta}\left(-\tan\theta\right)}{\frac{dx}{d\theta}}=\frac{\frac{d}{d\theta}\left(-\tan\theta\right)}{\frac{d}{d\theta}\left(a\cos^3\theta\right)}$$

$$=\frac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta}$$

$$=\frac{\sec^2\theta}{3a\cos^2\theta\sin\theta} \qquad \left[\because \sec\theta=\frac{1}{\cos\theta}\right]$$

$$y'' = \frac{1}{3a\cos^4\theta\sin\theta} \qquad 1+\tan^2\theta = \sec^2\theta$$

$$1+\cot^2\theta = \operatorname{cosec}^2\theta$$

$$\rho = \frac{\left(1+y'^2\right)^{3/2}}{y''} = \frac{\left(1+(-\tan\theta)^2\right)^{3/2}}{\left(\frac{1}{3a\cos^4\theta\sin\theta}\right)}$$

$$= \left(1+\tan^2\theta\right)^{3/2}\left(3a\cos^4\theta\sin\theta\right)$$

$$= \left(\sec^2\theta\right)^{3/2}\left(3a\cos^4\theta\sin\theta\right)$$

$$\sec^3\theta\, 3a\cos^4\theta\sin\theta$$

$$= 3a\sin\theta\cos^{\cancel{4}}\theta\frac{1}{\cos^{\cancel{3}}\theta} \qquad \left[\because \sin 2\theta = 2\sin\theta\cos\theta, \frac{\sin 2\theta}{2} = \sin\theta\cos\theta\right]$$

$$= 3a\sin\theta\cos\theta$$

$$= 3a\frac{\sin 2\theta}{2}$$

$$\rho = \frac{3}{2}a\sin 2\theta$$

To find ρ^3;

$$\rho^3 = (3a\sin\theta\cos\theta)^3 \qquad (1) \Rightarrow x = a\cos^3\theta$$

$$= 27a^3\sin^3\theta\cos^3\theta \qquad \frac{x}{a} = \cos^3\theta$$

$$= 27a^{\cancel{3}}\left(\frac{y}{\cancel{a}}\right)\left(\frac{x}{\cancel{a}}\right) \qquad (2) \Rightarrow y = a\sin^3\theta$$

$$\rho^3 = 27axy. \qquad \frac{y}{a} = \sin^3\theta$$

2.Prove that the radius of curvature at any point of the cycloid

$x = a(\theta+\sin\theta), y = a(1-\cos\theta)$ *is* $4a\cos\frac{\theta}{2}$.

Sol:

Given,

$$x = a(\theta + \sin\theta), \qquad y = a(1 - \cos\theta)$$

$$\Rightarrow x' = \frac{dx}{d\theta} = a(1 + \cos\theta) \qquad y' = \frac{dy}{d\theta} = (a(0 + \sin\theta))$$

$$= a(1 + \cos\theta) \qquad = a\sin\theta$$

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cancel{a}\sin\theta}{\cancel{a}(1 + \cos\theta)} = \frac{\sin\theta}{1 + \cos\theta}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cancel{1} + 2\cos^2\frac{\theta}{2} - \cancel{1}} = \frac{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}} \qquad \left[\begin{array}{l} \because \sin^2\theta + \cos^2\theta = 1 \\ \Rightarrow \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1 \end{array}\right]$$

$$= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$y' = \tan\frac{\theta}{2}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}\left(\tan\frac{\theta}{2}\right)}{\frac{dx}{d\theta}}$$

$$=\frac{\sec^2\frac{\theta}{2}\left(\frac{1}{2}\right)}{a(1+\cos\theta)}=\frac{\frac{1}{2}\sec^2\frac{\theta}{2}}{a\,2\cos^2\left(\frac{\theta}{2}\right)}$$

$$y''=\frac{1}{4a\cos^4\frac{\theta}{2}}$$

$$\rho=\frac{\left(1+y'^2\right)^{3/2}}{y''}=\frac{\left(1+\tan^2\frac{\theta}{2}\right)^{3/2}}{\left(\frac{1}{4a\cos^4\frac{\theta}{2}}\right)}$$

$$=\left(1+\tan^2\frac{\theta}{2}\right)^{3/2}\times 4a\cos^4\frac{\theta}{2}$$

$$=\left(\sec^2\frac{\theta}{2}\right)^{3/2}4a\cos^4\frac{\theta}{2}$$

$$=\sec^3\frac{\theta}{2}(4a)\cos^4\frac{\theta}{2}$$

$$=4a\sec^3\frac{\theta}{2}\cos^4\frac{\theta}{2}$$

$$=4a\cos^{\cancel{4}}\frac{\theta}{2}\times\frac{1}{\cos^{\cancel{3}}\frac{\theta}{2}}$$

$$=4a\cos\frac{\theta}{2}$$

$$\rho=4a\cos\frac{\theta}{2}$$

Polar form:

Curve **Radius of curvature**

$$r = f(\theta) \qquad \rho = \frac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1^2 - r_1 r_2}$$

$(Derivative\ w.r.to\ \theta)$

1.For the cardioid $r = a(1+\cos\theta)$

Sol:

Given,

$$r = a(1+\cos\theta) = a + a\cos\theta$$

$$r_1 = \frac{dr}{d\theta} = -a\sin\theta \Rightarrow r_1^2 = a^2\sin^2\theta$$

$$r_2 = \frac{d^2r}{d\theta^2} = -a\cos\theta \Rightarrow r_2^2 = a^2\cos^2\theta.$$

$$r^2 = a^2 + a^2\cos^2\theta = a^2(1+\cos\theta)^2$$

$$w.k.t \quad \rho = \frac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1^2 - r\,r_2}$$

$$Consider, r^2 + r_1^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2\left(1+\cos^2\theta + 2\cos\theta\right) + a^2\sin^2\theta$$

$$= a^2 + a^2\cos^2\theta + 2a^2\cos\theta + a^2\sin^2\theta$$

$$= a^2 + a^2\left(\cos^2\theta + \sin^2\theta\right) + 2a^2\cos\theta$$

$$= a^2 + a^2 + 2a^2\cos\theta$$

$$= 2a^2 + 2a^2\cos\theta$$

$$= 2a^2(1+\cos\theta)$$

$$= 2a^2\left(2\cos^2\frac{\theta}{2}\right)$$

$$= 4a^2\cos^2\frac{\theta}{2}$$

$$Consider, r^2 + 2r_1^2 - r\,r_2 = a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta + (-a\cos\theta)a(1+\cos\theta)$$

$$= a^2 + a^2\cos^2\theta + 2a^2\cos\theta + 2a^2\sin^2\theta + a^2\cos\theta + a^2\cos^2\theta$$

$$= a^2 + 2a^2\cos^2\theta + 2a^2\cos\theta + 2a^2\sin^2\theta + a^2\cos\theta + a^2\cos^2\theta$$

$$= a^2\left[1+\cos^2\theta + 2\cos\theta + 2\sin^2\theta + \cos\theta + \cos^2\theta\right]$$

$$=a^2\left[1+1+1+3\cos\theta\right]$$
$$=a^2\left[\ 3+3\cos\theta\right]$$
$$=3a^2\left(1+\cos\theta\right)$$
$$=6a^2\cos^2\left(\frac{\theta}{2}\right)$$

$$\rho=\frac{\left(r^2+r_1^2\right)^{3/2}}{r^2+2r_1^2-r\,r_2}$$

$$=\frac{\left(4a^2\cos^2\frac{\theta}{2}\right)^{3/2}}{6a^2\cos^2\frac{\theta}{2}}=\frac{\left(2a\cos\frac{\theta}{2}\right)^3}{6a^2\cos^2\frac{\theta}{2}}$$

$$=\frac{\cancel{8}^{4}\,a^{\cancel{3}}\cos^{\cancel{3}}\frac{\theta}{2}}{\cancel{6}_{3}\,a^{\cancel{2}}\cos^{\cancel{2}}\frac{\theta}{2}}=\frac{4a}{3}\cos\frac{\theta}{2}$$

$$\rho=\frac{4a}{3}\ \cos\frac{\theta}{2}\Rightarrow\rho^2=\frac{16a^2}{9}\cos^2\frac{\theta}{2}$$

$$\frac{\rho^2}{r}=\frac{\left(\frac{16a^2}{9}\right)\cos^2\frac{\theta}{2}}{a\left(1+\cos\theta\right)}=\frac{\cancel{16}^{8}\,a}{9}\,\frac{\cancel{\cos^2\frac{\theta}{2}}}{\cancel{2}\,\cancel{\cos^2\frac{\theta}{2}}}=\frac{8a}{9}$$

$$\frac{\rho^2}{r}=\frac{8a}{9}$$

2.Find the radius of curvature at any point θ on the curve $r^2=a^2\cos 2\theta$.

Sol:

Given: $r^2=a^2\cos 2\theta$

$$2r\frac{dr}{d\theta}=-2a^2\sin 2\theta$$

$$\therefore \frac{dr}{d\theta}=r_1=\frac{-\cancel{2}a^2\sin 2\theta}{\cancel{2}r}=\frac{-a^2\sin 2\theta}{r}$$

$$r\frac{dr}{d\theta}=-a^2\sin 2\theta \qquad \frac{d^2r}{d\theta^2}=\frac{-a^2 2}{r}\cos\theta$$

$$r^2=a^2\cos 2\theta\, r_1^2 \quad \frac{a^4\sin^2 2\theta}{r^2}$$

$$r\frac{d^2r}{d\theta^2}=\frac{-\cancel{r}\,a^2\cos 2\theta}{\cancel{r}}=-2a^2\cos 2\theta$$

$$r\frac{d^2r}{d\theta^2}+\left(\frac{dr}{d\theta}\right)^2=-2a^2\cos 2\theta=-2r^2$$

$$r\,r_2=r\frac{d^2r}{d\theta^2}=-2r^2-\left(\frac{dr}{d\theta}\right)^2=-2r^2-\frac{a^4\sin^2 2\theta}{r^2}$$

$$\rho=\frac{\left(r^2+r_1^2\right)^{3/2}}{r^2+2r_1^2-r\,r_2}=\frac{\left\{r^2+\frac{a^4\sin^2 2\theta}{r^2}\right\}^{3/2}}{r^2+\frac{2a^4\sin^2 2\theta}{r^2}+2r^2+\frac{a^4\sin^2 2\theta}{r^2}}$$

$$=\frac{\frac{\left[r^4+a^4\sin^2 2\theta\right]^{3/2}}{r^2}}{\frac{3r^2+a^4\sin^2 2\theta}{r^2}+\frac{2a^4\sin^2 2\theta}{r^2}}=\frac{\left(r^4+a^4\sin^2 2\theta\right)^{3/2}\times\cancel{r^2}}{3r\left(r^4+a^4\sin^2 2\theta\right)\times\cancel{r^3}}$$

$$=\frac{\left(r^4+a^4\sin^2 2\theta\right)^{3/2}}{3r}\left(r^4+a^4\sin^2 2\theta\right)^{-1}$$

$$=\frac{1}{3r}\left(r^4+a^4\sin^2 2\theta\right)^{1/2}=\frac{1}{3r}\left(a^4\cos^2 2\theta+a^4\sin^2 2\theta\right)^{1/2}$$

$$=\frac{a^2}{3r}\left(\cos^2 2\theta+\sin^2 2\theta\right)^{1/2}=\frac{a^2}{3r}$$

Centre of curvature:

The co-ordinates of the centre of curvature in the Cartesian form is,

$$\bar{x}=x-\frac{y'}{y''}\left(1+y'^2\right) \qquad \bar{y}=y+\frac{1+y'^2}{y''}$$

1.Find the centre of curvature of $y=x^2$ at the origin

Sol:

Given: $y=x^2$

$$y' = 2x,\ at\,(0,0) \Rightarrow y' = 0$$
$$y'' = 2,\ at\,(0,0) \Rightarrow y'' = 2$$

w.k.t the centre of curvature is,

$$\therefore \bar{x} = x - \frac{y'}{y''}\left(1 + y'^2\right)$$
$$= 0 - \frac{0}{2}\left(1 + 0^2\right) = 0$$

$$\bar{y} = y + \frac{1 + y'^2}{y''} = 0 + \frac{1+0}{2} = \frac{1}{2}$$
$$\bar{y} = \frac{1}{2}$$

$\therefore$ Centre of curvature is $\left(0, \frac{1}{2}\right)$.

2.Fid the coordinate of the centre of curvature on the parabola $y^2 = 4ax$ at any point (x, y).

<u>Sol:</u>

Given: $y^2 = 4ax$

$$2y\frac{dy}{dx} = 4a \qquad\qquad y^2 = 4ax$$
$$\Rightarrow \frac{dy}{dx} = \frac{\cancel{4}^{2}a}{\cancel{2}y} = \frac{2a}{y} \qquad\qquad y = 2a^{1/2}x^{1/2}$$
$$x^{-1/2} = \frac{2a^{1/2}a^{-1/2}a^{1/2}}{y} \qquad = \frac{2a^{1/2+1/2}a^{-1/2}}{y}$$
$$x^{-1/2} = \frac{2a}{y\,a^{1/2}} \Rightarrow a^{1/2}x^{-1/2} = \frac{2a}{y}$$

$$y' = \frac{dy}{dx} = a^{1/2}\, x^{-1/2}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{-1}{2} a^{1/2}\, x^{-3/2}$$

The centre of curvature $\bar{x}$ and $\bar{y}$ is given by,

$$\bar{x} = x - \frac{y'}{y''}\left(1 + y'^2\right)$$

$$= x - \frac{a^{1/2}\, x^{-1/2}\left(1 + \left(a^{1/2}\, x^{-1/2}\right)^2\right)}{\frac{-1}{2} a^{1/2}\, x^{-3/2}}$$

$$= x + \frac{2\, \cancel{a^{1/2}}\, \cancel{x^{-1/2}}\left(1 + a\, x^{-1}\right)}{\cancel{a^{1/2}}\, \cancel{x^{-1/2}}\, x^{-1}}$$

$$= x + 2\left(1 + \frac{a}{x}\right)x = x + \frac{(2x + 2a)\cancel{x}}{\cancel{x}}$$

$$= 3x + 2a$$

$$\bar{y} = y + \frac{1 + y'^2}{y''} \qquad \left(\begin{matrix} y^2 = 4ax \\ y = 2\, a^{1/2}\, x^{1/2} \end{matrix}\right)$$

$$= y + \frac{1 + \left(a^{1/2}\, x^{-1/2}\right)^2}{-\frac{1}{2} a^{1/2}\, x^{-3/2}} = 2a^{1/2}\, x^{1/2} - 2\frac{\left(1 + ax^{-1}\right)}{a^{1/2}\, x^{-3/2}}$$

$$= \frac{\left[2a^{1/2}\, x^{1/2} - 2\left(1 + \frac{a}{x}\right)\right]}{a^{1/2}\, x^{-3/2}} = \frac{2a^{1/2}\, x^{1/2} - \left(\frac{2x + 2a}{x}\right)}{a^{1/2}\, x^{-3/2}}$$

$$= 2a^{1/2}\, x^{1/2} - (2x + 2a)\, a^{-1/2}\, x^{3/2}$$

$$=2a^{1/2}x^{3/2}-2x\,a^{-1/2}x^{3/2}-2a^{1}a^{1/2}x^{3/2}$$

$$=2a^{1/2}\cancel{x^{3/2}}-2x\,a^{-1/2}x^{3/2}-2a^{1/2}\cancel{x^{3/2}}$$

$$=2a^{1/2}x^{1/2}-\left(2+2ax^{-1}\right)a^{-1/2}x^{3/2}=2a^{1/2}x^{1/2}-2a^{-1/2}x^{3/2}-2a^{-1/2}a\,x^{-1}x^{3/2}$$

$$=2a^{1/2}\cancel{x^{1/2}}-2a^{-1/2}x^{3/2}-2a^{1/2}\cancel{x^{1/2}}=2a^{-1/2}x^{3/2}.$$

<u>Circle of curvature:</u>

The equation of the circle of curvature is $\left(x-\bar{x}\right)^2+\left(y-\bar{y}\right)^2=\rho^2.$

1.Find the centre and circle of curvature of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}\ at\left(\frac{a}{4},\frac{a}{4}\right).$

Sol:

Given: $\sqrt{x}+\sqrt{y}=\sqrt{a}$ Differentiating w.r.to ‘x’ we get,

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

$$\frac{d^2y}{dx^2}=-\frac{\left[\sqrt{x}\frac{1}{2\sqrt{y}}\frac{dy}{dx}-\sqrt{y}.\frac{1}{2\sqrt{x}}\right]}{x}$$

$$y'=\left(\frac{dy}{dx}\right)_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{-\sqrt{\frac{a}{4}}}{\sqrt{\frac{a}{4}}}=-1$$

$$y''=\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{\left[\cancel{\sqrt{\frac{a}{4}}}\frac{1}{2\cancel{\sqrt{\frac{a}{4}}}}\left(\cancel{-}1\right)-\cancel{\sqrt{\frac{a}{4}}}\frac{1}{2\cancel{\sqrt{\frac{a}{4}}}}\right]}{\frac{a}{4}}$$

$$= -\frac{\left[-\frac{1}{2}-\frac{1}{2}\right]}{\frac{a}{4}} = \frac{4}{a}$$

$$y'' = \frac{4}{a}$$

$$\bar{x} = x - \frac{y'\left(1+y'^2\right)}{y''}$$

$$= \frac{a}{4} - \frac{(-1)\left(1+(-1)^2\right)}{\frac{4}{a}} = \frac{a}{4} + \frac{2a}{4} = \frac{3a}{4}$$

$$\bar{x} = \frac{3a}{4}$$

$$\bar{y} = \frac{y+\left(1+y^2\right)}{y''}$$

$$= \frac{a}{4} + \frac{(1+1)}{\frac{4}{a}} = \frac{a}{4} + \frac{2a}{4} = \frac{3a}{4}$$

$$\bar{y} = \frac{3a}{4}$$

$$\rho = \frac{\left(1+(-1)^2\right)^{3/2}}{\frac{4}{a}} = \frac{a}{\sqrt{2}} \Rightarrow \rho^2 = \frac{a^2}{2}$$

$\therefore$ The centre of curvature is $\left(\frac{3a}{4}, \frac{3a}{4}\right)$

The equation of the circle of curvature is,

$$\left(x-\bar{x}\right)^2 + \left(y-\bar{y}\right)^2 = \rho^2$$

$$\left(x-\frac{3a}{4}\right)^2 + \left(y-\frac{3a}{4}\right)^2 = \rho^2$$

$$\left(x-\frac{3a}{4}\right)^2 + \left(y-\frac{3a}{4}\right)^2 = \frac{1}{2}a^2$$

2.Find the equation of the circle of curvature of $\sqrt{x}+\sqrt{y}=1 at \left(\frac{1}{4},\frac{1}{4}\right)$

Sol:

$$y^{'}=-1, y^{''}=4$$

$$\rho=\frac{\left(1+(-1)^2\right)^{3/2}}{4}=\frac{(2)^{3/2}}{4}=\frac{\cancel{2}\sqrt{2}}{\cancel{4}2}=\frac{1}{\sqrt{2}}.$$

$$\bar{x}=x-\frac{y^{'}}{y^{''}}\left(1+y^{'2}\right)=\frac{1}{4}+\frac{1(1+1)}{4}=\frac{1}{4}+\frac{2}{4}=\frac{3}{4}$$

$$\bar{y}=y+\frac{1+y^{'2}}{y^{''}}=\frac{1}{4}+\frac{1+(-1)^2}{4}=\frac{1}{4}+\frac{2}{4}=\frac{3}{4}$$

$\therefore$ The centre of curvature is $\left(\frac{3}{4},\frac{3}{4}\right)$ the equation of the circle of curvature is

$$\left(x-\bar{x}\right)^2+\left(y-\bar{y}\right)^2=\rho^2$$

$$\left(x-\frac{3}{4}\right)^2+\left(y-\frac{3}{4}\right)^2=\frac{1}{2}.$$

Evolutes:

The locus of the centre of curvature of the given curve is called the evolute of the curve. The given curve is called the involute of its centre.

Evolute of a curve in two ways:

1.Evolute of a curve is the locus of the centre of curvature at any general point on the curve

2.Evolute of a curve is the envelope of its normals.

Curve	Cartesian equation	Parametric equation
Parabola	$y^2=4ax$ x^2 =4ay	$x=at^2, y=2at$ $x=2at, y=at^2$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$x=a\cos\theta, y=b\sin\theta$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$x=a\sec\theta, y=b\tan\theta$
Rectangular hyperbola	$xy=c^2$	$x=ct, y=\frac{c}{t}$

Asteroid	$x^{2/3}+y^{2/3}=a^{2/3}$	$x=a\cos^3\theta$ $y=a\sin^3\theta$

1.Find the equation of the evolute of the parabola $y^2=4ax$

Sol:

The parametric equations of the parabola $y^2=4ax$ are $x=at^2$, $y=2at$.

$x=at^2$ $\qquad\qquad$ $y=2at$

$\frac{dx}{dt}=2at$ $\qquad\qquad$ $\frac{dy}{dt}=2a$

$$y'=\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\cancel{2a}}{\cancel{2a}t}=\frac{1}{t}$$

$$y'=\frac{1}{t}$$

$$y''=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$=\frac{d}{dt}\left(\frac{1}{t}\right)\frac{dt}{dx}=\frac{\frac{d}{dt}\left(\frac{1}{t}\right)}{\frac{dx}{dt}}$$

$$=\frac{\left(\frac{-1}{t^2}\right)}{2at}=\frac{-1}{2at\,t^2}$$

$$y''=-\frac{1}{2at\,t^3}$$

Let $\left(\bar{x},\bar{y}\right)$ be the centre of curvature at t,

$$\overline{X}=x-\frac{y^{'}\left(1+y^{'2}\right)}{y^{''}}$$

$$=at^2-\frac{\left(\frac{1}{t}\right)\left(1+\frac{1}{t^2}\right)}{\frac{-1}{2at^3}}=at^2+2a\,\cancel{t^3}\,\frac{1}{\cancel{t}}\left(\frac{\left(t^2+1\right)}{\cancel{t^2}}\right)$$

$$=at^2+2at^2+2a=3at^2+2a$$

$$\overline{X}=3at^2+2a \qquad \rightarrow(1)$$

$$\overline{Y}=y+\frac{1+y^{'2}}{y^{''}}$$

$$=2at+\frac{\left(1+\frac{1}{t^2}\right)}{\left(\frac{-1}{2at^3}\right)}=2at-2at^{\cancel{3}}\,\frac{\left(t^2+1\right)}{\cancel{t^2}}$$

$$=2at-2at\left(t^2+1\right)=\cancel{2at}-2at^3-\cancel{2at}$$

$$\overline{Y}=-2at^3 \qquad \rightarrow(2)$$

Now eliminate‘t’ we get,

From (1)

$$\Rightarrow\overline{x}=3at^2+2a$$

$$\Rightarrow 3t^2a=\overline{x}-2a$$

$$\Rightarrow t^2=\frac{\overline{x}-2a}{3a}$$

$$\left(t^2\right)^3=\left(\frac{\overline{x}-2a}{3a}\right)^3$$

$$t^6=\frac{\left(\overline{x}-2a\right)^3}{27a^3} \qquad \rightarrow(3)$$

From (2) $\Rightarrow\overline{y}=-2at^3$

$$t^3 = \frac{\bar{y}}{-2a}$$

$$\left(t^3\right)^2 = \left(\frac{\bar{y}}{-2a}\right)^2 = \frac{\bar{y}^2}{4a^2}$$

$$t^6 = \frac{\bar{y}^2}{4a^2} \qquad \rightarrow (4)$$

From (3) & 4

$$\frac{\bar{Y}^2}{4\cancel{a^2}} = \frac{\left(\bar{x}-2a\right)^3}{27a^{\cancel{3}}}$$

$$\frac{\bar{Y}^2}{4} = \frac{\left(\bar{X}-2a\right)^3}{27a}$$

$$27a\bar{Y}^2 = 4\left(\bar{X}-2a\right)^3$$

$\therefore$ The locus of $\left(\bar{X}, \bar{Y}\right)$ is,

$$27a\, y^2 = 4\left(x-2a\right)^3 \quad \text{(Or)} \quad 4\left(x-2a\right)^3 = 27ay^2$$

2. **Find the equation of the evolute of the ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$**.**

<u>Sol:</u> The parametric equation of the ellipse are

$$x = a\cos\theta \qquad y = b\sin\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$$

$$y' = \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b\cos\theta}{-a\sin\theta} = \frac{-b}{a}\cot\theta$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{d\theta}\left(\frac{-b}{a}\cot\theta\right)}{\frac{dx}{d\theta}} = \frac{\frac{b}{a}\operatorname{cosec}^2\theta}{-a\sin\theta}$$

$$y'' = \frac{-b}{a^2}\operatorname{cosec}^3\theta \qquad \left[\because \operatorname{cosec}\theta = \frac{1}{\sin\theta}\right]$$

Let $(\bar{x}, \bar{y})$ *be the coordinates of the centre of curvature,*

$$X = x - \frac{y'\left(1+y'^2\right)}{y''}$$

$$= a\cos\theta - \frac{\left(\frac{-b}{a}\cot\theta\right)\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{\frac{-b}{a^2}\,cocec^3\theta}$$

$$= a\cos\theta + \frac{\cancel{b}}{\cancel{a}}\cot\theta \times \frac{-\cancel{a^2}}{\cancel{b}}\sin^3\theta\left(1+\frac{b^2}{a^2}\cot^2\theta\right)$$

$$= a\cos\theta - a\sin^{\cancel{3}}\theta\,\frac{\cos\theta}{\cancel{\sin\theta}}\left(1+\frac{b^2}{a^2}\,\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= a\cos\theta - a\sin^2\theta\cos\theta\left(1+\frac{b^2}{a^2}\,\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= a\cos\theta - a\sin^2\theta\cos\theta - \cancel{a}\,\cancel{\sin^2\theta}\cos\theta\,\frac{b}{\cancel{a^2}}\,\frac{\cos\theta}{\cancel{\sin^2\theta}}$$

$$= a\cos\theta - a\sin^2\theta\cos\theta - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta\left(1-\sin^2\theta\right) - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta\cos^2\theta - \frac{b^2}{a}\cos^3\theta$$

$$X = \left(\frac{a^2-b^2}{a}\right)\cos^3\theta \qquad \rightarrow (1)$$

$$Y=y+\frac{1+y'^2}{y''}$$

$$=b\sin\theta+\frac{1+\frac{b^2\cot^2\theta}{a^2}}{-\frac{b}{a^2}\operatorname{co}ces^3\theta}$$

$$=b\sin\theta-\frac{a^2}{b}\sin^3\theta\left(\frac{a^2+b^2\frac{\cos^2\theta}{\sin^2\theta}}{a^2}\right)$$

$$=b\sin\theta-\frac{\cancel{a2}}{b}\sin^{\cancel{3}}\left(\frac{a^2\sin^2\theta+b^2\cos^2\theta}{\cancel{a^2}\ \cancel{\sin^2\theta}}\right)$$

$$=b\sin\theta-\frac{\sin\theta}{b}\left(a^2\sin^2\theta+b^2\cos^2\theta\right)$$

$$=b\sin\theta-\frac{a^2}{b}\sin^3\theta-b\cos^2\theta\sin\theta$$

$$=b\sin\theta\left(1-\cos^2\theta\right)-\frac{a^2}{b}\sin^3\theta$$

$$=b\sin^3\theta-\frac{a^2}{b}\sin^3\theta$$

$$=\left(b-\frac{a^2}{b}\right)\sin^3\theta=\left(\frac{b^2-a^2}{b}\right)\sin^3\theta$$

$$Y=\left(\frac{b^2-a^2}{b}\right)\sin^3\theta \qquad \rightarrow(2)$$

$$(1)\Rightarrow \quad X=\left(\frac{a^2-b^2}{a}\right)\cos^3\theta$$

$$aX=\left(a^2-b^2\right)\cos^3\theta$$

$$\left(aX\right)^{2/3}=\left(a^2-b^2\right)^{2/3}\cos^3\theta$$

$$(2) \Rightarrow \quad Y = \left(\frac{b^2 - a^2}{b}\right)\sin^3\theta$$

$$bY = \left(b^2 - a^2\right)\sin^3\theta$$

$$\left(bY\right)^{2/3} = \left(b^2 - a^2\right)^{2/3}\sin^2\theta$$

$$\left(bY\right)^{2/3} = \left(a^2 - b^2\right)^{2/3}\sin^2\theta$$

$$\left(aX\right)^{2/3} + \left(bY\right)^{2/3} = \left(a^2 - b^2\right)^{2/3}\left(\sin^2\theta + \cos^2\theta\right)$$

$$\left(aX\right)^{2/3} + \left(bY\right)^{2/3} = \left(a^2 - b^2\right)^{2/3}$$

$$Locus\,of\ (X, Y)\,is\left(ax\right)^{2/3} + \left(by\right)^{2/3} = \left(a^2 - b^2\right)^{2/3}.$$

3. **Find the evolute of the hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Sol:

The parametric equations of the hyperbola are,

$$x = a\sec\theta \qquad\qquad y = b\tan\theta$$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta \qquad\qquad \frac{dy}{d\theta} = b\sec^2\theta$$

$$y' = \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b\sec^{2}\theta}{a\,\cancel{\sec\theta}\,\tan\theta} = \frac{b\sec\theta}{\frac{a\sin\theta}{\cos\theta}}$$

$$= \frac{b\,\cancel{\cos\theta}}{a\sin\theta\,\cancel{\cos\theta}} = \frac{b}{a\sin\theta}$$

$$y' = \frac{b}{a}\cos ec\,\theta$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{d\theta}\left(\frac{b}{a}\cos ec\,\theta\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{\frac{-b}{a}\cos ec\,\theta\cot\theta}{a\sec\theta\tan\theta}$$

$$= \frac{-b}{a^2}\cos ec\,\theta.\cot\theta \times \frac{1}{\sec\theta\tan\theta}$$

$$=\frac{-b}{a^2}\times\frac{\cos^3\theta}{\sin^3\theta}=\frac{-b}{a^2}\cot^3\theta$$

$$y''=\frac{-b}{a^2}\cot^3\theta$$

Let (X,Y) be the coordinates of the centre of curvature,

$$X=x-\frac{y'(1+y'^2)}{y''}$$

$$=a\sec\theta-\frac{\left(\frac{b}{a}\operatorname{cosec}\theta\right)\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2\theta\right)}{-\frac{b}{a^2}\cot^3\theta}$$

$$=a\sec\theta+\frac{\frac{\cancel{b}}{a}\operatorname{cosec}\theta\times\frac{\cancel{a^2}}{\cancel{b}}\left(\frac{a^2+b^2\operatorname{cosec}^2\theta}{\cancel{a^2}}\right)}{\cot^3\theta}$$

$$=a\sec\theta+\frac{1}{a}.\frac{\operatorname{cosec}\theta\left(a^2+b^2\operatorname{cosec}^2\theta\right)}{\cot^3\theta}$$

$$=a\sec\theta+\frac{1}{a}.\frac{1}{\cancel{\sin\theta}}\ \frac{\sin^{\cancel{3}}\theta}{\cos^3\theta}\left(a^2+b^2\frac{1}{\sin^2\theta}\right)$$

$$=a\sec\theta+\frac{1}{a}.\frac{\sin^2\theta}{\cos^3\theta}\left(a^2+\frac{b^2}{\sin^2\theta}\right)$$

$$=a\sec\theta+\frac{\cancel{a^2}\sin^2\theta}{\cancel{a}\cos^3\theta}+\frac{b^2\cancel{\sin^2\theta}}{a\cancel{\sin^2\theta}\cos^3\theta}$$

$$=a\sec\theta+\frac{a\left(1-\cos^2\theta\right)}{\cos^3\theta}+\frac{b^2}{a\cos^3\theta}$$

$$=a\sec\theta+a\sec^3\theta-\sec\theta\, a+\frac{b^2}{a}-\sec^3\theta$$

$$aX=\cancel{a^2\sec\theta}+a^2\sec^3\theta-\cancel{a^2\sec\theta}+b^2\sec^3\theta$$

$$aX=\left(a^2+b^2\right)\sec^3\theta$$

$$\left(aX\right)^{2/3}=(a^2+b^2)^{2/3}\sec^2\theta$$

$$Y = y + \frac{1+y'^2}{y''} = b\tan\theta + \frac{1+\frac{b^2}{a^2}\cos ec^2\,\theta}{-\frac{b}{a^2}\cot^3\theta}$$

$$= b\tan\theta - \frac{a^2 + b^2\,\frac{1}{\sin^2\theta}}{b\,\frac{\cos^3\theta}{\sin^3\theta}}$$

$$= b\tan\theta - \frac{\left(a^2 + b^2\,\frac{1}{\sin^2\theta}\right)\sin^3\theta}{b\cos^3\theta}$$

$$= b\tan\theta - \frac{a^2}{b}\,\frac{\sin^3\theta}{\cos^3\theta} - \frac{b^{\cancel{2}}\,\sin^{\cancel{3}}\theta}{\cancel{b}\,\cos^3\theta}\,\frac{1}{\sin^{\cancel{2}}\theta} \qquad \left[\therefore \frac{1}{\cos^2\theta} = \left(\sec^2\theta\right)\right]$$

$$= b\tan\theta - \frac{a^2}{b}\tan^3\theta - b^2\tan\theta\left(1+\tan^2\theta\right)$$

$$bY = \cancel{b^2\tan\theta} - a^2\tan^3\theta - \cancel{b^2\tan\theta} - b^2\tan^3\theta$$

$$bY = -\left(a^2+b^2\right)\tan^3\theta$$

$$\left(bY\right)^{2/3} = \left(a^2+b^2\right)^{2/3}\tan^2\theta$$

$$\left(aX\right)^{2/3} - \left(bY\right)^{2/3} = \left(a^2+b^2\right)^{2/3}\left(\sec^2\theta - \tan^2\theta\right)$$

$$= \left(a^2+b^2\right)^{2/3}$$

Locus of $\left(X,Y\right)$ is

$$\left(ax\right)^{2/3} - \left(by\right)^{2/3} = \left(a^2+b^2\right)^{2/3}$$

4. Find the evolute of the rectangular hyperbola $xy = c^2$

<u>Sol:</u>

The parametric equation for $xy = c^2$ is,

$$x = ct, \qquad y = \frac{c}{t}$$

$$\frac{dx}{dt} = c \quad , \qquad \frac{dy}{dt} = c\left(\frac{-1}{t^2}\right) = \frac{-c}{t^2}$$

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \left(\frac{-c/t^2}{c}\right) = \frac{-\cancel{c}}{\cancel{c}t^2} = \frac{-1}{t^2}$$

$$y' = \frac{-1}{t^2}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{\frac{d}{dt}\left(\frac{-1}{t^2}\right)}{\frac{dx}{dt}} = \frac{\left(\frac{2}{t^3}\right)}{c} = \frac{2}{t^3}.\frac{1}{c} = \frac{2}{ct^3}$$

$$y'' = \frac{2}{ct^3}$$

Let $\left(\overline{X}, \overline{Y}\right)$ be the co-ordinates of the centre of curvature, then

$$\overline{X} = x - \frac{y'\left(1 + y'^2\right)}{y''}$$

$$= ct - \frac{-\left(\frac{-1}{t^2}\right)\left(1 + \frac{1}{t^4}\right)}{\left(\frac{2}{ct^3}\right)}$$

$$= ct + \frac{1}{\cancel{t^2}}\frac{ct^{\cancel{3}}}{2}\left(1 + \frac{1}{t^4}\right)$$

$$= ct + \frac{ct}{2}\left(1 + \frac{1}{t^4}\right) = ct + \frac{ct}{2}\left(\frac{t^4 + 1}{t^4}\right)$$

$$= ct + \frac{ct^{\cancel{5}}}{2t^{\cancel{4}}} + \frac{c\cancel{t}}{2t^{\cancel{4}}} = ct + \frac{ct}{2} + \frac{c}{2t^3}$$

$$= \frac{2ct + ct}{2} + \frac{c}{2t^3} = \frac{3ct}{2} + \frac{c}{2t^3}$$

$$\overline{X} = \frac{3ct}{2} + \frac{c}{2t^3} \qquad \rightarrow \quad (1)$$

$$\overline{Y} = y + \frac{1+y'^2}{y''} = \frac{c}{t} + \frac{1+\frac{1}{t^4}}{\frac{2}{ct^3}} = \frac{c}{t} + \frac{ct^3}{2}\left(1+\frac{1}{t^4}\right)$$

$$= \frac{c}{t} + \frac{ct^3}{2}\left(\frac{t^4+1}{t^4}\right) = \frac{c}{t} + \frac{ct^3\cancel{t^4}}{2\cancel{t^4}} + \frac{c\cancel{t^3}}{2t^{\cancel{4}}}$$

$$= \frac{c}{t} + \frac{ct^3}{2} + \frac{c}{2t} = \frac{2c+c}{2t} + \frac{ct^3}{2}$$

$$\overline{Y} = \frac{3c}{2t} + \frac{ct^3}{2} \qquad \rightarrow \qquad (2)$$

Adding (1) & (2) we get,

$$\overline{X} + \overline{Y} = \frac{3ct}{2} + \frac{c}{2t^3} + \frac{3c}{2t} + \frac{ct^3}{2}$$

$$= \frac{c}{2}\left[3t + \frac{1}{t^3} + \frac{3}{t} + t^3\right]$$

$$= \frac{c}{2}\left[\frac{3t^4+1+3t^2+t^6}{t^3}\right]$$

$$= \frac{c}{2}\frac{\left[\left(t^2+1\right)^3\right]}{t^3} \qquad = \frac{c}{2}\left[\frac{\left(t^2+1\right)}{t}\right]^3$$

$$= \frac{c}{2}\left(\frac{t^{\cancel{2}}}{\cancel{t}} + \frac{1}{t}\right)^3 \quad = \frac{c}{2}\left(t+\frac{1}{t}\right)^3$$

$$\overline{X} + \overline{Y} = \frac{c}{2}\left(t+\frac{1}{t}\right)^3$$

$$\left(\overline{X}+\overline{Y}\right)^{2/3} = \left(\frac{c}{2}\right)^{2/3}\left(t+\frac{1}{t}\right)^2 \qquad \rightarrow \qquad (3)$$

Subtracting (1) & (2) we get,

$$\overline{X} - \overline{Y} = \frac{3ct}{2} + \frac{c}{2t^3} - \frac{3c}{2t} - \frac{ct^3}{2}$$

$$= \frac{c}{2}\left[3t + \frac{1}{t^3} - \frac{3}{t} - t^3\right]$$

$$=\frac{-c}{2}\left[t^3-3t+\frac{3}{t}-\frac{1}{t^3}\right]=\frac{-c}{2}\left[\frac{\left(t^2-1\right)^3}{t^3}\right]=\frac{-c}{2}\left(\frac{t^2-1}{t}\right)^3$$

$$\overline{X}-\overline{Y}=\frac{-c}{2}\left(t-\frac{1}{t}\right)^3$$

$$\left(\overline{X}-\overline{Y}\right)^{2/3}=\left(\frac{-c}{2}\right)^{2/3}\left(t-\frac{1}{t}\right)^2$$

$$\left(\overline{X}-\overline{Y}\right)^{2/3}=\left(\frac{c}{2}\right)^{2/3}\left(t-\frac{1}{t}\right)^2 \qquad \rightarrow \qquad (4)$$

Subtracting (3) & (4) we get,

$$\left(\overline{X}+\overline{Y}\right)^{2/3}-\left(\overline{X}-\overline{Y}\right)^{2/3}=\left(\frac{c}{2}\right)^{2/3}\left[\left(t+\frac{1}{t}\right)^2-\left(t-\frac{1}{t}\right)^2\right]$$

$$=\left(\frac{c}{2}\right)^{2/3}\left[\left(t^2+\frac{1}{t^2}+2\right)-\left(t^2+\frac{1}{t^2}-2\right)\right]$$

$$=\left(\frac{c}{2}\right)^{2/3}\left[\cancel{t^2}+\cancel{\frac{1}{t^2}}+2-\cancel{t^2}-\cancel{\frac{1}{t^2}}+2\right]$$

$$=\left(\frac{c}{2}\right)^{2/3}[4]=2^2\left(\frac{c}{2}\right)^{2/3}$$

$$=2^2\frac{c^{2/3}}{2^{2/3}}=c^{2/3}\,2^{2-2/3}$$

$$=c^{2/3}\,2^{4/3}$$

$$=c^{2/3}\,(4)^{2/3}$$

$$\therefore\left(\overline{X}+\overline{Y}\right)^{2/3}-\left(\overline{X}-\overline{Y}\right)^{2/3}=(4c)^{2/3}$$

$\therefore$ The Locus of $\left(\overline{X},\overline{Y}\right)$ is,

$$(x+y)^{2/3}-(x-y)^{2/3}=(4c)^{2/3}.$$

5. **Find the equation of the evolute of the curve** $x^{2/3}+y^{2/3}=a^{2/3}$ **.**

<u>Sol:</u>

Given: $x^{2/3}+y^{2/3}=a^{2/3}$

The parametric equations of the given curve are,

$$x = a\cos^3\theta \qquad y = a\sin^3\theta$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \qquad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$y' = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{3a}\sin^{\cancel{2}}\theta\,\cancel{\cos\theta}}{\cancel{-3a}\cos^{\cancel{2}}\theta\,\cancel{\sin\theta}} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

$$y' = -\tan\theta$$

$$y'' = \frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(-\tan\theta)}{-3a\cos^2\theta\sin\theta} = \frac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta}$$

$$y'' = \frac{1}{3a\sin\theta\cos^4\theta}$$

Let $(\overline{X}, \overline{Y})$ be the coordinates of the centre of curvature.

$$\overline{X} = x - \frac{y'\left(1+y'^2\right)}{y''}$$

$$= a\cos^3\theta - \frac{(-\tan\theta)\left(1+\tan^2\theta\right)}{\left[\frac{1}{3a\sin\theta\cos^4\theta}\right]}$$

$$= a\cos^3\theta + 3a\sin\theta\cos^{\cancel{4}}\theta\,\frac{\sin\theta}{\cancel{\cos\theta}}\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right)$$

$$= a\cos^3\theta + 3a\sin^2\theta\cos^3\theta\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right)$$

$$= a\cos^3\theta + 3a\sin^2\theta\cos^3\theta + 3a\sin^2\theta\cos^{\cancel{3}}\theta\,\frac{\sin^2\theta}{\cancel{\cos^2\theta}}$$

$$= a\cos^3\theta + 3a\sin^2\theta\cos^3\theta + 3a\sin^4\theta\cos\theta$$

$$= a\cos^3\theta + 3a\sin^2\theta\cos\theta\left(\cos^2\theta+\sin^2\theta\right)$$

$$\overline{X} = a\cos^3\theta + 3a\sin^2\theta\cos\theta \qquad \rightarrow \qquad (1)$$

$$\overline{Y} = y + \frac{1+y'^2}{y''}$$

$$= a\sin^3\theta + \frac{1+\tan^2\theta}{\left[\frac{1}{3a\sin\theta\cos^4\theta}\right]}$$

$$= a\sin^3\theta + 3a\sin\theta\cos^4\theta\left(1+\tan^2\theta\right)$$

$$= a\sin^3\theta + 3a\sin\theta\cos^4\theta\left(\sec^2\theta\right)$$

$$= a\sin^3\theta + 3a\sin\theta\cos^{\cancel{4}}\theta . \frac{1}{\cancel{\cos^2\theta}}.$$

$$= a\sin^3\theta + 3a\sin\theta\cos^2\theta$$

$$\overline{Y} \ = a\sin\theta(\sin^2\theta + 3\cos^2\theta) \qquad \rightarrow \qquad (2)$$

To find the equation of the evolute we have to eliminate θ from (1) and (2).

Adding (1) & (2) we get,

$$\overline{X}+\overline{Y} = a\cos\theta\left(\cos^2\theta + 3\sin^2\theta\right) + a\sin\theta\left(\sin^2\theta + 3\cos^2\theta\right)$$

$$= a\cos^3\theta + 3a\cos\theta\sin^2\theta + a\sin^3\theta + 3a\sin\theta\cos^2\theta$$

$$= a\left[\cos^3\theta + \sin^3\theta + 3\cos\theta\sin^2\theta + 3\sin\theta\cos^2\theta\right]$$

$$\overline{X}+\overline{Y} = a\left(\cos\theta + \sin\theta\right)^3$$

$$\left(\overline{X}+\overline{Y}\right)^{2/3} = a^{2/3}\left(\cos\theta + \sin\theta\right)^2$$

$$= a^{2/3}\left(\cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta\right)$$

$$\left(\overline{X}+\overline{Y}\right)^{2/3} = a^{2/3}\left(1 + 2\cos\theta\sin\theta\right) \qquad \rightarrow \qquad (3)$$

Subtracting (1) & (2) we get,

$$\overline{X}-\overline{Y} \ = a\cos\theta\left(\cos^2\theta + 3\sin^2\theta\right) - a\sin\theta\left(\sin^2\theta + 3\cos^2\theta\right)$$

$$= a\cos^3\theta + 3a\cos\theta\sin^2\theta - a\sin^3\theta - 3a\sin\theta\cos^2\theta$$

$$= a\left[\cos^3\theta - 3\cos^2\theta\sin\theta + 3\sin^2\theta\cos\theta - \sin^3\theta\right]$$

$$\overline{X}-\overline{Y} = a\left[\cos\theta - \sin\theta\right]^3$$

$$\left(\overline{X}-\overline{Y}\right)^{2/3} = a^{2/3}\left(\cos\theta - \sin\theta\right)^2$$

$$= a^{2/3}\left(\cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta\right)$$

$$\left(\overline{X}-\overline{Y}\right)^{2/3} = a^{2/3}\left(1 - 2\cos\theta\sin\theta\right) \qquad \rightarrow \qquad (4)$$

$$(3)+(4) \quad \Rightarrow \left(\overline{X}+\overline{Y}\right)^{2/3} + \left(\overline{X}-\overline{Y}\right)^{2/3}$$

$$= a^{2/3}\left[1 + \cancel{2\cos\theta\sin\theta} + 1 - \cancel{2\cos\theta\sin\theta}\right]$$

$$= a^{2/3}(1+1) = 2a^{2/3}.$$

The locus of $\left(\overline{X},\overline{Y}\right)$ is, $(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$.

6. **Show that the evolute of the cycloid** $x=a(\theta-\sin\theta)$ $y=a(1-\cos\theta)$ **is another cycloid**.

Sol:

Given the parametric form of the given cycloid is,

$$x=a(\theta-\sin\theta) \qquad y=a(1-\cos\theta)$$

$$\frac{dx}{d\theta}=a(1-\cos\theta) \qquad \frac{dy}{d\theta}=a\sin\theta$$

$$y'=\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{\cancel{a}\sin\theta}{\cancel{a}(1-\cos\theta)}=\frac{\cancel{2\sin\frac{\theta}{2}}.\cos\frac{\theta}{2}}{\cancel{2}\sin^{\cancel{2}}\frac{\theta}{2}}$$

$$=\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=\cot\frac{\theta}{2}$$

$$y'=\cot\frac{\theta}{2}$$

$$y''=\frac{d^2y}{dx^2}=\frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}=\frac{\frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right)}{a(1-\cos\theta)}$$

$$=\frac{-\cos ec^2\frac{\theta}{2}\left(\frac{1}{2}\right)}{a(1-\cos\theta)}=\frac{-\cos ec^2\frac{\theta}{2}}{2a\left(2\sin^2\frac{\theta}{2}\right)}$$

$$y''=\frac{-1}{4a\sin^4\frac{\theta}{2}}$$

Let $(\overline{X},\overline{Y})$ be the co-ordinates of the centre of curvature

$$\overline{X}=\frac{x-y'(1+y'^2)}{y''}$$

$$=a(\theta-\sin\theta)-\frac{\left(\frac{\cos\theta/2}{\sin\theta/2}\right)\left(1+\frac{\cos^2\theta/2}{\sin^2\theta/2}\right)}{-\frac{1}{4a\sin^4\theta/2}}$$

$$=a(\theta-\sin\theta)+4a\sin^4\theta/2\,\frac{\cos\theta/2}{\sin\theta/2}\left[\frac{\sin^2\theta/2+\cos^2\theta/2}{\sin^2\theta/2}\right]$$

$$=a(\theta-\sin\theta)+4a\sin^{\cancel{4}}\theta/2\,\frac{\cos\theta/2}{\cancel{\sin\theta/2}}\left[\frac{1}{\cancel{\sin^2\theta/2}}\right]$$

$$=a(\theta-\sin\theta)+4a\sin\theta/2\cos\theta/2$$

$$=a(\theta-\sin\theta)+2a\left[2\sin\theta/2\,\cos\theta/2\right]$$

$$=a(\theta-\sin\theta)+2a\sin\theta=a\theta-a\sin\theta+2a\sin\theta$$

$$=a[\theta-\sin\theta+2\sin\theta]$$

$$=a[\theta+\sin\theta]\qquad\rightarrow\qquad(1)$$

$$\overline{Y}=y+\frac{1+y'^2}{y''}$$

$$=a(1-\cos\theta)+\frac{1+\cot^2\theta/2}{\left[\frac{-1}{4a\sin^4\theta/2}\right]}$$

$$=a(1-\cos\theta)-4a\sin^4\theta/2\left[1+\frac{\cos^2\theta/2}{\sin^2\theta/2}\right]$$

$$=a(1-\cos\theta)-4a\sin^{\cancel{4}}\theta/2\left[\frac{\sin^2\theta/2+\cos^2\theta/2}{\cancel{\sin^2\theta/2}}\right]$$

$$=a(1-\cos\theta)-4a\sin^2\theta/2$$

$$=a(1-\cos\theta)-4a\left[\frac{1-\cos\theta}{2}\right]$$

$$=a(1-\cos\theta)-\frac{\cancel{4}\,a}{\cancel{2}}+\frac{\overset{2}{\cancel{4}}\,a\cos\theta}{\cancel{2}}$$

$$=a(1-\cos\theta)-2a+2a\cos\theta$$

$$=a[1-\cos\theta-2+2\cos\theta]=a[-1+\cos\theta]$$

$$Y=a[-1+\cos\theta] \qquad \rightarrow \qquad (2)$$

$$Y=a(\cos\theta-1)$$

<u>Envelope:-</u>

A curve which touches all the curves of a family is called the envelope of the family

The family of curves is represented by $f(x,y,\alpha)=0$ where α is a parameter**.**

<u>Case (i)</u>

If the equation of the given curve is of the form $A\alpha^2+B\alpha+C=0$ α is the parameters then the equation of the envelope is given by $B^2-4AC=0$

1. **Find the envelope of the family of straight lines $y=mx+am^2$, m being the parameters**.

<u>SOL:</u>

Given $y=mx+am^2$

$$am^2+mx-y=0 \qquad (1) \qquad [ax^2+bx+c=0]$$

Here A = a, B = x, C = -Y

$\therefore The\,envelope\,is, B^2-4AC=0$

$$x^2-4a(-y)=0$$

$$x^2+4ay=0$$

2. **Find the envelope of the family of lines y = mx $+\frac{a}{m}$ where a is a constant**.

<u>Sol:</u>

Given y = mx$+\dfrac{a}{m}$ (1)

$(1)\times m\Rightarrow y=m^2x+a$

$=m^2x-ym+a=0.$

This is a quadratic in m. A = x

` B = -y

$\therefore$ The Envelope is $B^2-4AC=0.$ C = a

$$y^2 - 4xa = 0$$
$$ie., \qquad y^2 = 4ax.$$

3. **Find the envelope of the family of straight lines** $x\cos\alpha + y\sin\alpha = a\sec\alpha, \alpha$ **being the parameter.**

Sol:

Given:

$$x\cos\alpha + y\sin\alpha = a\sec\alpha$$

$$\div\cos\alpha \qquad \Rightarrow x + \tan\alpha = a\sec^2\alpha$$

$$x + y\tan\alpha = a\left(1 + \tan^2\alpha\right)$$

$$x + y\tan\alpha = a + a\tan^2\alpha$$

$$a\tan^2\alpha - y\tan\alpha + a - x = 0$$

Which is a quadratic in $\tan\alpha$

Here, $A = a, B = -y, c = a - x$

$\therefore$ The envelope is $B^2 - 4AC = 0$

$$(-y)^2 - 4a(a - x) = 0$$

$$\Rightarrow \quad y^2 - 4a\,(a - x) = 0$$

$$ie, \quad y^2 = 4a\,(a - x).$$

4. Find the envelope of $y = mx + \sqrt{a^2m^2 + b^2}$ **where** m **is a parameter.**

Sol:

Given:

$$y = mx + \sqrt{a^2m^2 + b^2}$$

$$(y - mx) = \sqrt{a^2m^2 + b^2}$$

$$(y - mx)^2 = a^2m^2 + b^2$$

$$y^2 + m^2x^2 - 2mxy = a^2m^2 + b^2$$

$$y^2 + m^2x^2 - 2mxy - a^2m^2 - b^2 = 0$$

$$m^2\left(x^2 - a^2\right) - 2mxy + y^2 - b^2 = 0$$

Which is a quadratic in m

Here, $A = \left(x^2 - a^2\right), B = (-2xy), c = y^2 - b^2$

$\therefore$ The envelope is, $B^2 - 4ac = 0$

$$(-2xy)^2-4(x^2-a^2)(y^2-b^2)=0$$
$$4x^2y^2-4(x^2-a^2)(y^2-b^2)=0$$
$$x^2y^2-(x^2-a^2)(y^2-b^2)=0$$
$$x^2y^2-\left[x^2y^2-b^2x^2-a^2y^2+a^2b^2\right]=0$$
$$ie,\ \cancel{x^2y^2}-\cancel{x^2y^2}+b^2x^2+a^2y^2-a^2b^2=0$$
$$b^2x^2+a^2y^2=a^2b^2$$
$$ie,\frac{x^2}{a^2}+\frac{y^2}{a^2}=1.$$

5. **Find the envelope of** $\frac{x}{a}+\frac{y}{b}=1$ **where the parameter** a,b **are related by** $ab=c^2$ **, where** c **is known. (Or) Find the envelope of straight lines which move such that the product of the intercepts on the two axis is constant.**

<u>Sol:</u>

Given: $b=\frac{c^2}{a}$

The straight line is, $\frac{x}{a}+\frac{y}{b}=1$

$$\Rightarrow\frac{x}{a}+\frac{ay}{c^2}=1$$
$$\Rightarrow c^2x+a^2y=ac^2$$
$$\Rightarrow a^2y-ac^2+c^2x=0$$ is a quadratic in a .

Here, $A=y, B=-c^2, C=c^2x$

$\therefore$ The envelope is, $B^2-4AC=0$

6. **Find the envelope of** $y=mx+\frac{1}{m}$ **where** m **is the parameter.**

<u>Sol:</u> Given: $y=mx+\frac{1}{m}$

$$my=m^2x+1$$
$$m^2x-my+1=0$$

Here, $A=x, B=-y, C=1$

$\therefore$ The envelope is $B^2-4AC=0$

$$(-y)^2-4x=0$$
$$ie, y^2=4x$$

7. Find the envelope of the curve $y=mx+\frac{3}{2m}$

Sol: Given $y=mx+\frac{3}{2m}$

$2my=2m^2x+3$

$2m^2x-2ym+3=0$

Here, $A=2x, B=-2y, C=3.$

$B^2-4AC=0$

$4y^2=8(x)(3)=0$

$4y^2-24x=0$

$y^2-6x=0$

$y^2=6x$

Case (ii)

The envelope is got by eliminating c between the equations $f(x,y,c)=0$ and $\frac{\partial f}{\partial c}(x,y,c)=0$

1. Find the envelope of $x\cos\theta+y\sin\theta=a,$ **where** θ **is a parameter.**

Sol:

Given $x\cos\theta+y\sin\theta=a \qquad \rightarrow \qquad (1)$

Differentiating w.r to θ

$-x\sin\theta+y\cos\theta=0 \qquad \rightarrow \qquad (2)$

Eliminate θ **between (1) and (2)**

$(x\cos\theta+y\sin\theta)^2+(-x\sin\theta+y\cos\theta)^2=a^2+0^2$

$x^2\cos^2\theta+y^2\sin^2\theta+\cancel{2xy\sin\theta\cos\theta}+x^2\sin^2\theta+y^2\cos^2\theta-\cancel{2xy\sin\theta\cos\theta}=a^2$

$x^2(\cos^2\theta+\sin^2\theta)+y^2(\sin^2\theta+\cos^2\theta)=a^2$

$x^2+y^2=a^2$

2. Find the envelope of the family of curves $\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1$ **where** α **being the parameter and** a,b **constants.**

Sol:

Given: $\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1 \qquad \rightarrow \qquad (1)$

Differentiating. w.t to α .

$\frac{-x\sin\alpha}{a}+\frac{y\cos\alpha}{b}=0 \qquad \rightarrow \qquad (2)$

$(1)^2+(2)^2 \Rightarrow$

$$\left[\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}\right]^2+\left[\frac{-x\sin\alpha}{a}+\frac{y\cos\alpha}{b}\right]^2=1^2+0^2$$

$$\frac{x^2\cos^2\alpha}{a^2}+\frac{y^2\sin^2\alpha}{b^2}+\cancel{\frac{2xy\sin\alpha\cos\alpha}{ab}}+\frac{x^2\sin^2\alpha}{a^2}+\frac{y^2\cos^2\alpha}{b^2}-\cancel{\frac{2xy\sin\alpha\cos\alpha}{ab}}=1$$

$$\frac{x^2}{a^2}\left[\cos^2\alpha+\sin^2\alpha\right]+\frac{y^2}{b^2}\left[\cos^2\alpha+\sin^2\alpha\right]=1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

3. Find the envelope of the family of straight lines $y\cos\alpha-x\sin\alpha=a\cos 2\alpha$ where α is a parameter.

<u>Sol:</u>

Given :

$y\cos\alpha-x\sin\alpha=a\cos 2\alpha \qquad \rightarrow \qquad (1)$

Differentiating.w.r to α

$-y\sin\alpha-x\cos\alpha=-2a\sin 2\alpha \qquad \rightarrow \qquad (2)$

<u>Eliminate y between (1) and (2),</u>

$(1)\times\sin\alpha \Rightarrow \cancel{y\sin\alpha\cos\alpha}-x\sin^2\alpha=a\sin\alpha\cos 2\alpha$

$(2)\times\cos\alpha \Rightarrow \cancel{-y\sin\alpha\cos\alpha}-x\cos^2\alpha=-2a\sin 2\alpha\cos\alpha$

$-x\left[\sin^2\alpha+\cos^2\alpha\right]=a\left[\cos 2\alpha.\sin\alpha-2\sin 2\alpha\cos\alpha\right]$

$-x=a\left[\left(\cos^2\alpha-\sin^2\alpha\right)\sin\alpha-2.2\sin\alpha\cos\alpha\cos\alpha\right]$

$-x=a\left[\cos^2\alpha\sin\alpha-\sin^3\alpha-4\sin\alpha\cos^2\alpha\right]$

$-x=a\left[-3\sin\alpha\cos^2\alpha-\sin^3\alpha\right] \qquad \left[\because \sin 2A=2\sin A\cos A\right]$

$\left[\cos 2A=\cos^2 A-\sin^2 A\right]$

$x=a\left[\sin^3\alpha+3\sin\alpha\cos^2\alpha\right] \qquad \rightarrow \qquad (3)$

<u>Eliminate x between (1) and (2),</u>

$$(1)\times\cos\alpha \Rightarrow y\cos^2\alpha - \cancel{x\sin\alpha\cos\alpha} = a\cos 2\alpha\cos\alpha$$

$$(2)\times\sin\alpha \Rightarrow \underset{(+)}{-} y\sin^2\alpha \underset{(+)}{-} \cancel{x\sin\alpha\cos\alpha} = \underset{(+)}{-} 2a\sin 2\alpha\sin\alpha$$

$$y\left(\cos^2\alpha+\sin^2\alpha\right) = a\left[\cos 2\alpha\cos\alpha + 2\ \sin 2\alpha\sin\alpha\right]$$

$$y = a\left[\left(\cos^2\alpha-\sin^2\alpha\right)\cos\alpha + 2\times 2\sin\alpha\cos\alpha\sin\alpha\right]$$

$$y = a\left[\cos^3\alpha - \sin^2\alpha\cos\alpha + 4\sin^2\alpha\cos\alpha\right]$$

$$y = a\left[\cos^3\alpha + 3\sin^2\alpha\cos\alpha\right]$$

$$y = a\left[\cos^3\alpha + 3\sin^2\alpha\cos\alpha\right] \qquad \rightarrow \qquad (4)$$

$$x+y = a\left[\sin^3\alpha + 3\sin\alpha\cos^2\alpha\right] + a\left[\cos^3\alpha + 3\sin^2\alpha\cos\alpha\right]$$

$$x+y = a\left[\sin^3\alpha + 3\sin\alpha\cos^2\alpha + \cos^3\alpha + 3\sin^2\alpha\cos\alpha\right]$$

$$x+y = a\left[\sin\alpha+\cos\alpha\right]^3$$

$$\therefore (x+y)^{2/3} = a^{2/3}\left(\sin\alpha+\cos\alpha\right)^2$$

$$= a^{2/3}\left(\sin^2\alpha+\cos^2\alpha+2\sin\alpha\cos\alpha\right)$$

$$(x+y)^{2/3} = a^{2/3}\left(1+2\sin\alpha\cos\alpha\right) \qquad \rightarrow \qquad (5)$$

$$x-y = a\left[\sin^3\alpha + 3\sin\alpha\cos^2\alpha - \cos^3\alpha - 3\sin^2\alpha\cos\alpha\right]$$

$$= a\left(\sin\alpha-\cos\alpha\right)^3$$

$$(x-y)^{2/3} = a^{2/3}\left(\sin\alpha-\cos\alpha\right)^2$$

$$= a^{2/3}\left(\sin^2\alpha+\cos^2\alpha-2\sin\alpha\cos\alpha\right)$$

$$(x-y)^{2/3} = a^{2/3}\left(1-2\sin\alpha\cos\alpha\right) \qquad \rightarrow \qquad (6)$$

$$(5)+(6) \Rightarrow (x+y)^{2/3} + (x-y)^{2/3} = a^{2/3}\left(1+\cancel{2\sin\alpha\cos\alpha}+1-\cancel{2\sin\alpha\cos\alpha}\right)$$

$$= a^{2/3}\,2$$

$$(x+y)^{2/3} + (x-y)^{2/3} \quad = 2a^{2/3}$$

4. Find the envelope of the family of straight lines $\dfrac{ax}{\cos\,\theta} - \dfrac{by}{\sin\theta} = a^2-b^2$

Sol:

Given $\dfrac{ax}{\cos\alpha} - \dfrac{by}{\sin\theta} = a^2-b^2$

$$a\,x\sec\theta - b\,y\,\mathrm{cosec}\,\theta = a^2-b^2$$

Differentiating w.r to θ

$$ax[\sec\theta\tan\theta]-by[-\cos ec\theta\cot\theta]=0$$

$$ax\sec\theta\frac{\sin\theta}{\cos\theta}+by\cos ec\theta\frac{\cos\theta}{\sin\theta}=0$$

$$\frac{ax\sin\theta}{\cos^2\theta}+\frac{by\cos\theta}{\sin^2\theta}=0$$

$$\frac{ax\sin\theta}{\cos^2\theta}=\frac{-by\cos\theta}{\sin^2\theta}\Rightarrow\frac{ax}{\cos^3\theta}=\frac{-by}{\sin^3\theta}$$

$$\frac{\left(\frac{ax}{\cos\theta}\right)}{\cos^2\theta}=\frac{-\left(\frac{by}{\sin\theta}\right)}{\sin^2\theta}=\frac{\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}}{\cos^2\theta+\sin^2\theta}=\frac{a^2-b^2}{1}$$

$$\therefore\frac{ax}{\cos^3\theta}=\frac{-by}{\sin^3\theta}=a^2-b^2$$

$$ax=\left(a^2-b^2\right)\cos^3,\ by=-\left(a^2-b^2\right)\sin^3\theta$$

$$(ax)^{2/3}=\left(a^2-b^2\right)^{2/3}\cos^2\theta,\ (by)^{2/3}=\left(a^2-b^2\right)^{2/3}\sin^2\theta$$

$$(ax)^{2/3}+(by)^{2/3}=\left(a^2-b^2\right)^{2/3}\left(\cos^2\theta+\sin^2\theta\right)$$

$$=\left(a^2-b^2\right)^{2/3}(1)$$

$$(ax)^{2/3}+(by)^{2/3}=\left(a^2-b^2\right)^{2/3}$$

5. Find the envelope of the family of straight lines $y=mx-2am-am^3$, where m is the parameter.

<u>Sol:</u>

Given: $y=mx-2am-am^3 \quad\rightarrow\quad (1)$

Differentiating w.r to m

$$0=x-2a-3am^2 \quad\rightarrow\quad (2)$$

$$x-2a=3am^2$$

<u>Eliminate $'m'$ from (1) and (2),</u>

$$y = m\left[x - 2a - am^2\right]$$

$$= m\left[(x-2a) - \frac{(x-2a)}{3}\right] \qquad \left[by(2)\right]$$

$$= m\left[\frac{2}{3}(x-2a)\right]$$

$$y = \tfrac{2}{3}\, m(x-2a)$$

$$m = \frac{3y}{2(x-2a)}$$

$$(2) \Rightarrow \quad 0 = (x-2a) - 3a\left[\frac{3y}{2(x-2a)}\right]^2$$

$$0 = (x-2a) - 3a\frac{9y^2}{4(x-2a)^2}$$

$$(x-2a) = 3a\frac{9y^2}{4(x-2a)^2}$$

$$(x-2a)^3 = \frac{27ay^2}{4}$$

$$ie, \qquad 27ay^2 = 4(x-2a)^3$$

<u>Case (iii)</u>

Envelope of two parameter family of curves (Eliminate m from $f(x,y,m)=0$ and $\partial/\partial m\, f(x,y,m)=0$ which gives the equation of envelope)

1. **Find the envelope of $\frac{x}{a}+\frac{y}{b}=1$ subject to $a+b=c$ where c is a known constant.**

<u>Sol:</u>

Given:

$$\frac{x}{a}+\frac{y}{b}=1 \qquad \rightarrow \qquad (1)$$

$$a+b=c \qquad \rightarrow \qquad (2)$$

Differentiating (1) w.r to b , we get,

$$x\left(\frac{-1}{a^2}\right)\frac{da}{db}+y\left(\frac{-1}{b^2}\right)\left(\frac{db}{db}\right)=0$$

$$\frac{-x}{a^2}\frac{da}{db}-\frac{y}{b^2}=0$$

$$\frac{da}{db}=\frac{-y}{x}\cdot\frac{a^2}{b^2} \qquad \rightarrow \qquad (3)$$

Differentiating (2) w.r.to b weget,

$$\frac{da}{db}+1=0$$

$$\frac{da}{db}=-1 \qquad \rightarrow \qquad (4)$$

Now,

(3) = (4)

$$\frac{-y}{x}\frac{a^2}{b^2}=-1 \qquad -y\,a^2=-x\,b^2$$

$$\frac{x}{a^2}=\frac{y}{b^2}$$

$$\frac{\frac{x}{a}}{a}=\frac{\frac{y}{b}}{b}=\frac{\frac{x}{a}+\frac{y}{b}}{a+b}=\frac{1}{c}$$

$$\therefore \frac{x}{a^2}=\frac{1}{c}\ \frac{y}{b^2}=\frac{1}{c}$$

$$\frac{x}{a^2}=\frac{1}{c} \qquad \frac{y}{b^2}=\frac{1}{c}$$

$$a^2=cx \qquad b^2=yc$$

$$a=\sqrt{cx} \qquad b=\sqrt{cy}$$

$sub's\,a\,b\,in(2),$

$$a+b=c$$

$$\sqrt{cx}+\sqrt{cy}=c$$

$$c^{1/2}x^{1/2}+c^{1/2}y^{1/2}=c$$

$$\sqrt{x}+\sqrt{y}=\frac{c}{\sqrt{c}}$$

$$\sqrt{x}+\sqrt{y}=\sqrt{c}.$$

2. Find the envelope of the lines $\frac{x}{a}+\frac{y}{b}=1$ where a and b connected by the relation

$a^n+b^n=c^{n.}$

Sol:

Given: $\frac{x}{a}+\frac{y}{b}=1$ (1)

$a^n + b^n = c^{n.}$ (2)

Differentiating (1) &(2) w.r.to b, we get,

$$\frac{-x}{a^2}\frac{da}{db}-\frac{y}{b^2}=0$$

$$\frac{-x}{a^2}\frac{da}{db}=\frac{y}{b^2}$$

$$\frac{da}{db}=-\frac{y}{b^2}\frac{a^2}{x} \qquad \Rightarrow (3)$$

$$n\,a^{n-1}\frac{da}{db}+n\,b^{n-1}=0$$

$$n\,a^{n-1}\frac{da}{db}=-n\,b^{n-1}$$

$$\frac{da}{db}=\frac{-nb^{n-1}}{-na^{n-1}}=\frac{-b^{n-1}}{-a^{n-1}} \qquad \Rightarrow (4).$$

<u>Now (3) = (4)</u>

$$\frac{-y}{b^2}\;\frac{a^2}{x}=\frac{-b^{n-1}}{a^{n-1}}$$

$$\frac{y}{x}\frac{a^2}{b^2}=\frac{b^{n-1}}{a^{n-1}}$$

$$\frac{y}{x} = \frac{b^{n-1}b^2}{a^{n-1}a^2} = \frac{b^{n-1+2}}{a^{n-1+2}} = \frac{b^{n+1}}{a^{n+1}}$$

$$\frac{y}{x} = \frac{b^{n+1}}{a^{n+1}}$$

ie, $$\frac{x}{a^{n+1}} = \frac{y}{b^{n+1}}$$

$$\frac{x}{a^n . a} = \frac{y}{b^n . b}$$

$$\frac{x/a}{a^n} = \frac{y/b}{b^n} = \frac{\frac{x}{a}+\frac{y}{b}}{a^n+b^n} = \frac{1}{c^n}$$

ie, $$\frac{x}{a^{n+1}} = \frac{y}{b^{n+1}} = \frac{1}{c^n}$$

$$\Rightarrow \frac{x}{a^{n+1}} = \frac{1}{c^n} \quad , \quad \frac{y}{b^{n+1}} = \frac{1}{c^n}$$

$$x c^n = a^{n+1} \quad , y c^n = b^{n+1}$$

$$a = \left(x c^n\right)^{1/n+1}, b = \left(y c^n\right)^{1/n+1}$$

Sub's $a \& b$ in (2) we get,

$$x^n + b^n = c^n$$

$$\left(x c^n\right)^{n/n+1} + \left(y c^n\right)^{n/n+1} = c^n$$

$$x^{n/n+1} c^{n^2/n+1} + y^{n/n+1} c^{n^2/n+1} = c^n$$

$$c^{n^2/n+1}\left(x^{n/n+1} + y^{n/n+1}\right) = c^n$$

$$x^{n/n+1} + y^{n/n+1} = c^n . c^{-n^2/n+1} = c^{\frac{\cancel{n^2}+n-\cancel{n^2}}{n+1}} = c^{n/n+1}$$

3. Find the envelope of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **such that** $a+b=c$

<u>Sol:</u>

Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \rightarrow \quad (1) \qquad a+b=c \quad \rightarrow \quad (2)$

$$(1) \Rightarrow x^2\left(\frac{-2}{a^3}\right)\frac{da}{db} + y^2\left(\frac{-2}{b^3}\right) = 0$$

$$\frac{-2x^2}{a^3}\frac{da}{db} + \frac{-2y^2}{b^3} = 0$$

$$\frac{-2x^2}{a^3}\frac{da}{db} = \frac{2y^2}{b^3}$$

$$\frac{da}{db} = \frac{-\cancel{2}y^2}{b^3} \times \frac{a^3}{\cancel{2}x^2}$$

$$\therefore \frac{da}{db} = \frac{-a^3}{b^3}\frac{y^2}{x^2} \qquad \rightarrow \qquad (3)$$

$$(2) \Rightarrow \frac{da}{db} + 1 = 0$$

$$\frac{da}{db} = -1 \qquad \rightarrow \qquad (4)$$

Now (3) = (4)

$$\frac{-a^3}{b^3}\frac{y^2}{x^2} = -1 \quad \Rightarrow \quad \frac{y^2}{x^2} = \frac{b^3}{a^3} \Rightarrow \frac{y^2}{b^3} = \frac{x^2}{a^3}$$

$$\frac{x^2}{a^3} = \frac{y^2}{b^3}$$

$$\frac{\frac{x^2}{a^2}}{a} = \frac{\frac{y^2}{b^2}}{b} = \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2}}{a+b} = \frac{1}{c}$$

$$\therefore \frac{x^2}{a^3} = \frac{y^2}{b^3} = \frac{1}{c}$$

$$\frac{x^2}{a^3} = \frac{1}{c} \qquad \frac{y^2}{b^3} = \frac{1}{c}$$

$$x^2 c = a^3 \qquad y^2 c = b^3$$

$$a = \left(x^2 c\right)^{1/3} \qquad b = \left(y^2 c\right)^{1/3}$$

Sub's $a \& b$ in (2) we get,.

$$\left(x^2 c\right)^{1/3} + \left(y^2 c\right)^{1/3} = c$$

$$x^{2/3} c^{1/3} + y^{2/3} c^{1/3} = c$$

$$c^{1/3}\left(x^{2/3} + y^{2/3}\right) = c$$

$$x^{2/3} + y^{2/3} = \frac{c}{c^{1/3}} = c^1 c^{-1/3} = c^{1-1/3} = c^{2/3}$$

$$x^{2/3} + y^{2/3} = c^{2/3}$$

4. Find the envelope of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **where** $a^n + b^n = c^n$

Sol:

Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \rightarrow \quad (1)$

$a^n + b^n = c^n \quad \rightarrow \quad (2)$

Differentiating (1) and (2) w r to b we get

$$(1) \Rightarrow \frac{-2x^2}{a^3}\frac{da}{db}\frac{-2y^2}{b^3} = 0$$

$$\frac{da}{db} = \frac{-2y^2}{b^3} \times \frac{a^3}{2x^2}$$

$$\frac{da}{db} = \frac{-y^2}{x^2}\frac{a^3}{b^3} \quad \rightarrow \quad (3)$$

$$(2) \Rightarrow n a^{n-1}\frac{da}{db} + n b^{n-1} = 0$$

$$n a^{n-1}\frac{da}{db} = -n b^{n-1}$$

$$\frac{da}{db} = -\frac{\cancel{n} b^{n-1}}{\cancel{n} a^{n-1}} = -\frac{b^{n-1}}{a^{n-1}} \quad \rightarrow \quad (4)$$

Now (3) = (4)

$$-\frac{y^2}{x^2}\frac{a^3}{b^3}=-\frac{b^{n-1}}{a^{n-1}}$$

$$\frac{y^2}{x^2}=\frac{b^{n-1}b^3}{a^{n-1}a^3}=\frac{b^{n+2}}{a^{n+2}}$$

$$\Rightarrow \qquad \frac{x^2}{a^{n+2}}=\frac{y^2}{b^{n+2}}$$

$$\frac{\frac{x^2}{a^2}}{a^n}=\frac{y^2/b^2}{b^n}=\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}}{a^n+b^n}=\frac{1}{c^n}$$

$$\therefore \frac{x^2}{a^{n+2}}=\frac{1}{c^n} \qquad \frac{y^2}{b^{n+2}}=\frac{1}{c^n}$$

$$a^{n+2}=c^n x^2 \qquad b^{n+2}=c^n y^2$$

$$a=\left(c^n x^2\right)^{1/n+2} \qquad b=\left(c^n y^2\right)^{1/n+2}$$

Sub's $a \& b$ in (2) $a^n+b^n=c^n$

$$\left(c^n x^2\right)^{n/n+2}+\left(c^n y^2\right)^{n/n+2}=c^n$$

$$c^{n^2/n+2}x^{2n/n+2}+c^{n^2/n+2}y^{2n/n+2}=c^n$$

$$x^{2n/n+2}+y^{2n/n+2}=c^n c^{-n^2/n+2}=c^{\frac{\cancel{n^2}+2n-\cancel{n^2}}{n+2}}=c^{2n/n+2}$$

$$x^{2n/n+2}+y^{2n/n+2}=c^{2n/n+2}$$

5.**Find the envelope of** $\frac{x}{a}+\frac{y}{b}=1$ **where** $a \& b$ **are connected by** $a^2+b^2=c^2, c$ **is being a constant .**

Ans:

$$x^{2/3}+y^{2/3}=c^{2/3}$$