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UNIT-2

Basics of Counting

(1) Sum rule.

(2) Product rule.

1. Sum rule:- Principle of disjunctive counting.

→ If 'x' is a set then $|X|$ denotes the no. of elements in X. If the X is the union of disjoint non empty subsets $X_1, X_2, X_3, \ldots X_n$ then $|X|$ is

$$|X| = |X_1| + |X_2| + |X_3| + \cdots + |X_n|$$

→ the subsets $X_1, X_2, X_3, \ldots X_n$ must have no elements in common.

→ Since, $X = X_1 \cup X_2 \cup \ldots \cup X_n$ each element of X is in exactly one of the subsets 'X_i'.

Eg: $X = \{1, 2, 3, \ldots 10\}$

$|X| = 10$

$$\left.\begin{aligned} X_1 &= \{3,4,5\} \\ X_2 &= \{1,6,7\} \\ X_3 &= \{2,8,9,10\} \end{aligned}\right\} \text{ } X_1, X_2, X_3 \text{ are different sets}$$

$|X| = |X_1| + |X_2| + |X_3|$

$10 = 3 + 3 + 4$

$10 = 10$.

example:

1. In how many ways we can draw (a) a ~~pa~~ heart or spade from a deck of ~~plain~~ playing cards.
(b) A heart or Ace
(c) Ace or king

Sol:
(a) $13 + 13 = 26$ ways
(b) $13 + 3 = 16$ ways
(c) $4 + 4 = 8$ ways

2\. a. How many ways we can get a sum of 4 or 8 when two distinguisable dice are rolled.

b. How many ways we can get an even sum.

Sol: (a)

	Dice 1	Dice 2		
4	1	3	(1, 3)	3 ways
	2	2	(2, 2)	
	3	1	(3, 1)	
8	3	5	(3, 5)	5 ways
	2	6	(2, 6)	
	4	4	(4, 4)	
	5	3	(5, 3)	
	6	2	(6, 2)	

⇒ 5+3
= 8 ways

(b)

2, 4, 6, 8, 10, 12

= 1+3+5+5+3+1

= 18 ways.

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3\. a) How many ways can we get a sum of 8 when two indistinguisable dice are rolled?

b) An even sum

Sol: (a)

8: (2, 6), (3, 5), (4, 4) } 3 ways

(b) 2, 4, 6, 8, 10, 12

= (1,1), (1,3), (2,2), (1,5), (2,4), (3,3), (2,6), (3,5), (4,4), (4,6), (5,5), (6,6)

= 12 ways.

2. Product rule :- The Principle of sequential counting.

→ If $S_1, S_2, \ldots\ldots, S_n$ are non empty sets then the no. of elements in the Cartesian product

$S_1 \times S_2 \times \ldots\ldots \times S_n$ is the product.

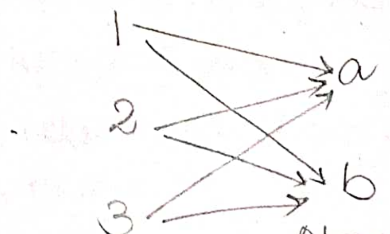
i.e, $\prod_{i=1}^{n} |S_i|$

$$|S_1 \times S_2 \times \ldots\ldots \times S_n| = \prod_{i=1}^{n} |S_i|$$

Eg:

$A = \{1, 2, 3\} \quad B = \{a, b\}$

$\underset{\text{Cartesian product.}}{\underline{A \times B}} = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$



Tree diagram

Eg:

1. if Two distinguisable dice are rolled in how many ways can it fall.

b) if 5 distinguisable dice are rolled, how many possible outcomes are there.

c) How many, if 100 distinguisable dice are rolled?

Sol:

a. 1 dice can be rolled in 6^1 ways.

2 dice can be rolled in 6^2 ways.

b. 5 dice can be rolled in 6^5 ways

c. 100 dice can be rolled in 6^{100} ways.

2. How many 3 digits number can be formed using the digits 1, 3, 4, 5, 6, 8, 9.

b) How many can be formed, if no digit can be repeated.

Sol: a)

$\underline{7}\ \underline{7}\ \underline{7} = 7^3$ ways

b)

$\underline{7}\ \underline{6}\ \underline{5} = 7\times6\times5 = 210$ ways.

3) How many numbers can be formed using the digits 1, 3, 4, 5, 6, 8, 9. if no repitition is allowed.

Sol: $7 + 7\times6 + 7\times6\times5 + \cdots\cdots$ ways.

4) How many different lisence plates are there that involve 1, 2, or 3 letters followed by four digits.

Sol:

$26^1 \times 10^4$

$26^2 \times 10^4$

$26^3 \times 10^4$

$\Rightarrow 26\times10^4 + 26^2\times10^4 + 26^3\times10^4$

$[26 + 26^2 + 26^3]\times10^4$ ways.

5) How many 3 digit no's are there which are even and have no repeated digits (0 to 9)

Sol: $\underline{\ \ }\ \underline{\ \ }\ \underline{\ \ } \to 0, 2, 4, 6, 8$

i) 3 digit no. end with 0

first place fill in 9 ways

second place fill in 8 ways

$\Rightarrow 9\times8$ ways.

ii) 3 digit no. end with 2, 4, 6, 8.

$\underline{8}\times\underline{8}\times\underline{4}$

$\Rightarrow 8\times8\times4$ ways

$(9\times8 + 8\times8\times4)$ ways.

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* Combinations and Permutations :-

A combination of 'n' objects taken 'r' at a time is an unordered selection of 'r' of the objects.
A combination is selection way.

A Permutation of 'n' objects taken 'r' at a time is an ordered selection of 'r' of the objects.
Permutation is arranging.

Eg:

1) suppose that 5 objects from which selections are to be made. a, a, a, b, c.
find the 3 combinations and 3 Permutations

Sol: 3- combinations:
aaa, aab, aac, abc

3 - permutations :-
aaa, aba, aab, baa, aac, aca, caa, abc, acb, cab, cba, bac, bca.

2) find the 3 combinations of {3a, 2b, 5c}

Sol: aaa, aab, aac, bbc, abb, acc, ccc, abc, bcc

3) find the 3 combinations of {3a, 2b, 1d, 2c}

Sol: aaa, aab, aac, aad, bbc, bbd, abb, abc, abd, bcd, cca, ccb, ccd, acd

4) find the 2 combinations and 2 Permutations of
{α.a, α.b, α.c, α.d}

Sol: 2- Combinations:
aa, ab, bc, cd, bb, cc, dd, da, ac, bd = 10.

2 - Permutations:
aa, ab, ba, ac, ca, ad, da, bb, bc, cd, bd, db, cc, cd, dc, dd. = 16

2 Combination without repetition	2 Permutation without repetition
ab	ab, ba
ac	ac, ca
ad	ad, da
bc	bc, cb
bd	bd, db
cd	cd, dc.
6	12

* Enumeration of Combinations and Permutations :-

Theorem 1: Enumerating 'r' (at a time)* without repetitions.
*Permutations

$$P(n,r) = \frac{n!}{(n-r)!} \quad ; n \neq r$$

$$P(n,r) = n! \quad ; n = r.$$

Eg 1. The no. of 2 permutations of $\{a,b,c,d,e\}$ is

Sol. $r = 2 \;;\; n = 5$

$$P(5,2) = \frac{5!}{(5-2)!} = \frac{5 \times 4 \times 3!}{3!} = 20.$$

2. The no. of 5 letter words using the letter $\{a,b,c,d,e\}$ is

Sol. $r = 5 \;;\; n = 5$

$$P(5,5) = 5! = 120$$

3. In how many ways can 7 women & 3 men be arranged in a row if the 3 men must always stand next to each other.

Sol. 3 men can arranged in 3! ways. $\text{---} \rightarrow x$

If i consider 3 men as one entity

$\times \ \omega_1 \ \omega_2 \ \omega_3 \ \omega_4 \ \omega_5 \ \omega_6 \ \omega_7$

$\Rightarrow 8!$ ways.

$\therefore 3!\,8!$ ways.

<u>Theorem 2</u>: Enumerating all 'r' combinations without repetitions

$$C(n,r) = \frac{n!}{(n-r)!\,r!} \quad ; n \neq r$$

Eg:

1. In how many ways can ~~the~~ a hand of 5 cards be selected from a deck of 52 cards.

<u>Sol</u>:

$r=5, n=52$

$$C(52,5) = \frac{52!}{(52-5)!\,5!} \text{ ways.}$$

$$= \frac{52\times51\times50\times49\times48\times\cancel{47!}}{\cancel{47!}\,5!} = \frac{\cancel{52}\times\cancel{51}\times\cancel{50}\times49}{\cancel{120}}$$

$= 13\times17\times49$

$= 2598960$

2. How many 5 card hands consists of only hearts.

<u>Sol</u>:

$r=5, n=13$

$$C(13,5) = \frac{13!}{(13-5)!\,5!} = \frac{\cancel{8!}\times12\times13\times11\times10\times9}{\cancel{8!}\,5!}$$

$= 1287$ ways.

3. How many 5 cards hands have 2 clubs & 3 hearts.

<u>Sol</u>:

clubs: $n=13$, $r=2$

hearts: $n=13$, $r=3$

$C(13,2)\,C(13,3)$

4) How many 3 card hands contains 2 Aces 3 kings

Sol:

Aces	kings
$n=4$	$n=4$
$r=2$	$r=3$
$C(4,2)$	$C(4,3)$

5) In how many ways can a commitee of 5 teachers & 4 students can be selected from 9 teachers & 15 students.

Sol:

5T	4S
9T	15S
$C(9,5)$	$C(15,4)$
=	=

6) There are ~~26~~ 21 consonants and 5 vowels in english Alphabet. Consider only 8 letters word with 3 different vowels & 5 different consonants. How many such words can be formed.

Sol: 21C 5V $\Rightarrow C(5,3)\,C(21,5)\,8!$.

_ _ _ _ _ _ _ _

$C(5,3)$ $C(21,5)$. $8!$.

(a) How many such words containing the letter 'a'.

a _ _ _ _ _ _ _ _

C(4,2) C(21,5) 8!

(b) How many contains 'a' & 'b'.

a b _ _ _ _ _ _ _

C(4,2) C(20,4) 8!.

(c) How many contains the letters 'b' & 'c'.

C(5,3) C(19,3) 8!

b c _ _ _ _ _ _ _

(d) How many begin with 'a' and end with 'b'.

a _ _ _ _ _ _ _ b

C(4,2) b(20,4) 6!.

(e) How many begin with 'b' and end with 'c'.

b _ _ _ _ _ _ _ c

C(5,3) C(19,3) 6!.

1) There are 30 females and 35 males in the junior class. while there are 25 females and 20 males in the senior class. In how many ways can a committee of 10 be choosen such that there are exactly 5 females and 3 junior in the committee.

Sol:

Junior Female	Junior Male	Senior Female	Senior Male
0	3	5	2
1	2	4	3
2	1	3	4
3	0	2	5

NO. of ways of selecting

$C(30,0)$ $C(35,3)$ $C(25,5)$ $C(20,2)$

$C(30,1)$ $C(35,2)$ $C(25,4)$ $C(20,3)$

$C(30,2)$ $C(35,1)$ $C(25,3)$ $C(20,4)$

$C(30,3)$ $C(35,0)$ $C(25,2)$ $C(20,5)$

Total no. of ways $= {}^{30}C_0 \; {}^{35}C_3 \; {}^{25}C_5 \; {}^{20}C_2 +$
${}^{30}C_1 \; {}^{35}C_2 \; {}^{25}C_4 \; {}^{20}C_3 + {}^{30}C_2 \; {}^{35}C_1 \; {}^{25}C_3 \; {}^{20}C_4 + {}^{30}C_3 \; {}^{35}C_0 \; {}^{25}C_2 \; {}^{20}C_5$

2) In how many ways can 30 distinguishable books be distributed among 3 people A, B, C so that

a) A, B together exactly twice as many books as C.

b) C receives atleast two books, B receives atleast twice as many books as C and A receives atleast 3 times as many books as B.

Sol: (a) $C(30,10)\,2^{20}$

(b)

NO. of books C has	NO. of books B has	NO. of books A has
2	4	24

2	5	23
2	6	22
2	7	21
2	6	21

$C(30,2)\,C(28,4) + C(30,2)\,C(28,5) + C(30,2)\,C(28,6) +$
$C(30,2)\,C(28,7) + C(30,3)\,C(27,6)$

3) A collection of 100 light bulbs contains 8 defective ${}^{92}C_6\ {}^{8}C_4$

a) In how many ways can a sample of 10 bulbs can be selected? (Ans: $C(100,10)$)

b) In how many ways can a sample of 10 bulbs be selected which 6 good one and 4 defective
(Ans: $C(92,6)\,C(8,4)$)

c) In how many ways a sample of 10 bulbs can be selected so that either the sample contain 6 good bulbs and 4 defective one or 5 good bulbs and 5 defective ones?
(Ans: $C(92,6)\,C(8,4) + C(92,5)\,C(8,5)$)

* **Enumerating Combinations and Permutations with Repetations:**

Theorem 3:- Enumerating 'r' permutations with unlimited repetation.

$$U(n,r) = n^r$$

Ex:1. There are 25 True or false questions in an examinations. How many diff ways can student give the examination if he/she can also choose to leave the answer blank.

Sol: 3^{25}

2. The results of 50 football games (win, lose, tie) are to be predicted. How many diff forecasts can

contain exactly 28 correct results.

Sol: $C(50,28)\,2^{22}$

* Enumerating 'r' combinations with unlimited repetitions.

$V(n,r) =$

1) The no. of 'r' combinations of 'n' distinct objects with unlimited repetation.
2) The no. of non-negative integral solutions to $x_1+x_2+\dots+x_n=r$.
3) The no. of ways of distributing 'r' similar balls into 'n' numbered boxes.
4) The no. of binary numbers with $(n-1)$'s 1's and 'r' 0's.

$\Rightarrow C(n-1+r, r)$

Ex: 1) The no. of 4 combinations of $\{\alpha.a_1, \alpha.a_2, \alpha.a_3, \alpha.a_4, \alpha.a_5\}$ is

Sol: $n=5$, $r=4$ $\Rightarrow {}^{n+r-1}C_r$

$\Rightarrow {}^{5+4-1}C_4 = {}^8C_4$.

2) The no. of non-negative integral solutions to $x_1+x_2+x_3\dots x_{50}=50$.

Sol: $r=50$, $n=5$

$C(n+r-1, r) = C(5+50-1, 50)$
$= C(54, 50)$
$= {}^{54}C_{50}$
$= {}^{54}C_{44}$.

3) The no. of ways of placing 10 similar balls in 6 numbered boxes.

Sol: $r=10$, $n=6$

$C(n+r-1, r) = C(6+10-1, 10)$
$= C(15, 10) = {}^{15}C_{10} = {}^{15}C_5$

4) The no. of binary numbers with 10 1's, and 5 zeros.

Sol: $n=11$ $\quad C(11+5-1,5)$

$r=5$ $\quad C(15,5) = {}^{15}C_5$.

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5) How many integral solutions are there to $x_1+x_2+x_3+x_4+x_5=20$ where each $x_i \geq 2$.

Sol: $x_1+x_2+x_3+x_4+x_5=20$

$2+2+2+2+2=10$

$y_1+y_2+y_3+y_4+y_5=10$ $\quad (\because 20-10=10)$

$n=5 \quad r=10$

$C(n+r-1,r) = C(14,10) = {}^{14}C_{10}$

6) How many integral solutions are there to $x_1+x_2+x_3+x_4+x_5=20$ where $x_1 \geq 3$, $x_2 \geq 2$, $x_3 \geq 4$ $\quad x_4 \geq 6$ $\quad x_5 \geq 0$

Sol: $x_1+3+x_2+2+x_3+4+x_4+6+x_5+0=20$

$x_1+x_2+x_3+x_4+x_5=5$

$n=5 \quad r=5$

$C(5+5-1,5) = C(9,5) = {}^9C_5$

7) How many integral solutions are there to $x_1+x_2+x_3+x_4+x_5=20$ where $x_1 \geq -3$, $x_2 \geq 0$, $x_3 \geq 4$, $x_4 \geq 2$, $x_5 \geq 2$

Sol: $x_1+x_2+x_3+x_4+x_5=20$

$-3+0+4+2+2=5$

$y_1+y_2+y_3+y_4+y_5=15$ $\quad (\because 20-5=15)$

$C(15+5-1,15) = C(19,15) = {}^{19}C_5 = {}^{19}C_4$

* Enumerating Permutations with constrained repetitions

Theorem 5: Enumerating 'n' Permutations with constrained repetitions.

$$P(n;q_1,q_2,\ldots\ldots q_t) = \frac{n!}{q_1!\,q_2!\cdots q_t!}$$

1) The no. of arrangement of letters in the word TALLAHASSEE

Sol: $n = 11$, $T = 1$, $A = 3$, $L = 2$, $H = 1$, $S = 2$, $E = 2$

$$P(11; 1,3,2,1,2,2) = \frac{11!}{1!\,3!\,2!\,1!\,2!\,2!} = \frac{11!}{(1!)^2\, 3!\,(2!)^3}$$

2) The no. of arrangements of these letters that begin with T and end with E.

Sol:

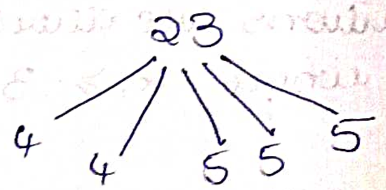
T _ _ _ _ _ _ _ _ _ E

$A = 3$, $L = 2$, $H = 1$, $S = 2$, $E = 1$

$$P(9; 3,2,1,2,1) = \frac{9!}{3!\,2!\,1!\,2!\,1!} = \frac{9!}{3!\,(1!)^2(2!)^2}$$

3) In how many ways can 23 diff. books be given to 5 students so that 2 of the students will have 4 books each other 3 will have 5 books each.

Sol:



$$C(5,2)\,.\,P(23; 4,4,5,5,5) = 5C_2 \cdot \frac{23!}{4!\,4!\,5!\,5!\,5!}$$

4) Find the no. of 5 combinations and the no. of 5 permutations of $\{5a, 3b, 2c, 3d, 2e, 1f, 4g\}$

Sol: The diff ways of selecting 5 letters may be classified in the following table:

Types of selection	no. of 5 combi	No. of arrangements from each selection	No. of 5 permutations
1) All 5 like	1	$5!/5! = 1$	1
2) 4 like 1 diff	12	$\frac{5!}{4!\,1!} = 5$	60

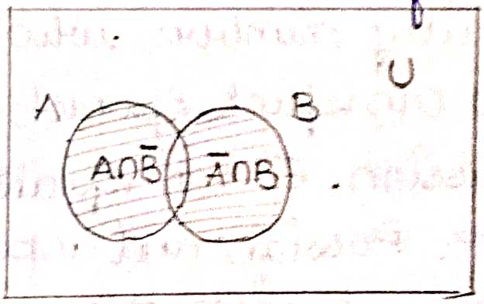
3) 3 like 2 others like	20	$\frac{5!}{3!2!} = 10$	200
4) 3 like 2 others diff	10	$\frac{5!}{3!1!1!} = 20$	1200
5) 2 like, 2 others like 1 diff	75	$\frac{5!}{3!2!} =$	2250
6) 2 like 3 diff	120	$\frac{5!}{2!1!1!1!} = 60$	7200
7) All 5 diff	$C(7,5) = 21$	$5!$	2520.

* **Principle of Inclusion and Exclusion :-**

i) If A and B are subsets of universal set U then

$|A \cup B| = |A| + |B| - |A \cap B|$

(ii) In general $A \cup B$ is union of 3 disjoint sets



$$|A\cup B| = |A\cap\bar{B}| + |A\cap B| + |\bar{A}\cap B|$$

$$|A| = |A\cap\bar{B}| + |A\cap B| \longrightarrow ①$$

$$|B| = |\bar{A}\cap B| + |A\cap B| \longrightarrow ②$$

$$① + ② = |A| + |B| = |A\cap\bar{B}| + |\bar{A}\cap B| + |A\cap B| + |A\cap B|$$

$$|A| + |B| = |A\cup B| + |A\cap B|$$

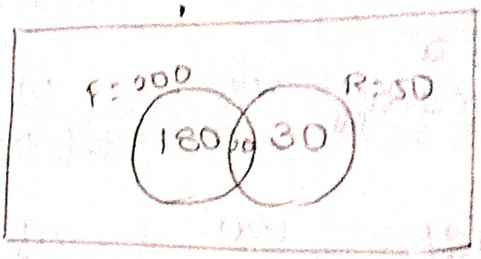
$$\Rightarrow |A\cup B| = |A| + |B| - |A\cap B|$$

1) Suppose there are 200 faculty members who speak french and 50 speak Russian while 20 can speak both French and Russian. How many faculty members can speak french or Russian?

Sol: $|F| = 200$

$|R| = 50$

$|F \cap R| = 20$



$|F \cup R| = |F| + |R| - |F \cap R|$

$= 200 + 50 - 20$

$= 230$

b) There are 1000 faculty all together how many can speak neither french nor Russian.

Sol: $1000 - 230 = 770$

$U - |F \cup R|$

2) If there are 200 faculty members who speak French, 50 speak Russian, 100 can speak Spanish and 20 speak french and Russian, 60 can speak French & spanish. 35 can speak Russian and spanish, 10 speak French, Russian, spanish. How many speak either French, Russian, spanish.

Sol: $|F \cup R \cup S| = |F| + |R| + |S| - |F \cap R| - |F \cap S| - |R \cap S| + |F \cap R \cap S|$

$|F \cup R \cup S| = 200 + 50 + 100 - 20 - 60 - 35 + 10$

$= 245$ //