Divergence of a vector:-

(1) The Divergence of a vector point function $\bar{F}$ is denoted by "div $\bar{F}$" (or) $\nabla \cdot \bar{F}$ and is defined as

$$\text{div}\,\bar{F} = \nabla \cdot \bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \bar{F}$$

$$\text{div}\,\bar{F} = \bar{i} \cdot \frac{\partial \bar{F}}{\partial x} + \bar{j} \cdot \frac{\partial \bar{F}}{\partial y} + \bar{k} \cdot \frac{\partial \bar{F}}{\partial z} = \sum_{\substack{i,j,k \\ x,y,z}} \bar{i} \cdot \frac{\partial \bar{F}}{\partial x}$$

(2) if $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$\nabla \cdot \bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \left(F_1\bar{i} + F_2\bar{j} + F_3\bar{k}\right)$$

$$\boxed{\nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}} \; *$$

(3) solenoidal vector function:-

A vector point function $\bar{F}$ is said to be solenoidal if $\boxed{\text{div}\,\bar{F} = 0}$ *: for such vector, there is no loss (or) gain of fluid.

(4) $\nabla \cdot \bar{F} \neq \bar{F} \cdot \nabla$

(5)

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Problems:-

1) if $\bar{A} = x^2 z\bar{i} - 2y^3 z^2 \bar{j} + xy^2 z\bar{k}$, find $\nabla \cdot \bar{A}$ at the point $(1,-1,1)$.

Sol^n:- if $\bar{A} = A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$ then $\nabla \cdot \bar{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$.

$$\nabla \cdot \bar{A} = \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-2y^3 z^2) + \frac{\partial}{\partial z}(xy^2 z)$$

$$\nabla \cdot \bar{A} = 2xz - 6y^2 z^2 + xy^2$$

$$(\nabla \cdot \bar{A})_{at\,(1,-1,1)} = 2(1)(1) - 6(-1)^2(1)^2 + 1(-1)^2 = 2-6+1 = -3.$$

2) show that $\bar{A} = 3y^4 z^2 \bar{i} + 4x^3 z^2 \bar{j} - 3x^2 y^2 \bar{k}$ is solenoidal.

Sol^n:- given $\bar{A} = 3y^4 z^2 \bar{i} + 4x^3 z^2 \bar{j} - 3x^2 y^2 \bar{k}$

$$\nabla \cdot \bar{A} = \frac{\partial}{\partial x}(3y^4 z^2) + \frac{\partial}{\partial y}(4x^3 z^2) + \frac{\partial}{\partial z}(-3x^2 y^2)$$

$$= 0 + 0 + 0$$

$$= 0.$$

$\therefore$ $\bar{A}$ is solenoidal.

3) Find 'a' such that $(x+3y)\bar{i} + (y-2z)\bar{j} + (x+az)\bar{k}$ is solenoidal.

Sol^n:- let $\bar{F} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+az)\bar{k}$,

since $\bar{F}$ is solenoidal,

$$\nabla \cdot \bar{F} = 0.$$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0.$$

$$1 + 1 + a = 0.$$

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$\boxed{a = -2}$

(4) Show that $r^n \bar{r}$ is solenoidal vector if $n=-3$

(or)

prove that $\operatorname{div}(r^n \bar{r}) = 0$ if $n=-3$.

Proof:- let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

i.e- $\boxed{r^2 = x^2+y^2+z^2}$

p.d. it w.r.to x, y, z

$2r \cdot \frac{\partial r}{\partial x} = 2x$ | $2r \cdot \frac{\partial r}{\partial y} = 2y$ | $2r \cdot \frac{\partial r}{\partial z} = 2z$

$\frac{\partial r}{\partial x} = \frac{x}{r}$ | $\frac{\partial r}{\partial y} = \frac{y}{r}$ | $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$r^n \bar{r} = r^n(x\bar{i} + y\bar{j} + z\bar{k}) = r^n x\bar{i} + r^n y\bar{j} + r^n z\bar{k}$

$(F_1 i + F_2 j + F_3 k)$

we know that $\boxed{\nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$

$\operatorname{div}(r^n \bar{r}) = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$

$= \sum \frac{\partial}{\partial x}(r^n x)$

$\sum_{x,y,z}$

$= \sum\left[n \cdot r^{n-1} \frac{\partial r}{\partial x} \cdot x + r^n \cdot 1\right]$.

$\frac{\partial}{\partial x}(uv) = u'v + uv'$

$= \sum n \cdot r^{n-1} \cdot \frac{x}{r} \cdot x + \sum r^n$.

$= n \cdot r^{n-2} \sum x^2 + (r^n + r^n + r^n)$.

$$= n \cdot r^{n-2}(x^2+y^2+z^2) + 3r^n$$

$$= n \cdot r^{n-2} r^2 + 3r^n$$

$$= n \cdot r^n + 3r^n$$

$$\operatorname{div}(r^n \bar{r}) = (n+3)r^n \quad \text{---} \quad (*)$$

if $n=-3$, then $\operatorname{div}(r^n \bar{r}) = 0$.

$\therefore r^n \bar{r}$ is solenoidal vector.

(or)

i.e. if $n=-3$ in $(*)$

$$\operatorname{div}(r^{-3}\bar{r}) = (-3+3)r^n = 0.$$

$$\operatorname{div}\left(\frac{\bar{r}}{r^3}\right) = 0.$$

$\therefore \frac{\bar{r}}{r^3}$ is solenoidal always.

H.W.

(5) if $\bar{A} = (ax^2y+yz)\bar{i} + (xy^2-xz^2)\bar{j} + (2xyz-2x^2y^2)\bar{k}$ is solenoidal, find constant 'a'

(6) prove that $\operatorname{div} \bar{r} = 3$.

(7) Find 'b' such that $\bar{A} = (bx+4y^2z)\bar{i} + (x^3\sin z-3y)\bar{j} - (e^x+4\cos x^2y)\bar{k}$ is solenoidal.

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"curl" of vector point function.

1. The curl of a vector point function $\bar{F}$ is ~~defined~~ denoted by curl $\bar{F}$ (or) $\nabla \times \bar{F}$ and is defined as

$$\text{curl}\,\bar{F} = \nabla \times \bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times (F_1\bar{i} + F_2\bar{j} + F_3\bar{k}).$$

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

(or)

$$\text{curl}\,\bar{F} = \nabla \times \bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times \bar{F}$$

$$= \bar{i} \times \frac{\partial \bar{F}}{\partial x} + \bar{j} \times \frac{\partial \bar{F}}{\partial y} + \bar{k} \times \frac{\partial \bar{F}}{\partial z}$$

$$\text{curl}\,\bar{F} = \sum \bar{i} \times \frac{\partial \bar{F}}{\partial x}.$$

2. Irrotational vector function (conservative feild).

A vector point function $\bar{F}$ is said to be irrotational if curl $\bar{F} = 0$.

3. if $\bar{F}$ is irrotational vector $\Leftrightarrow$ then there exist a scalar function ϕ such that $\boxed{\bar{F} = \nabla \phi}$.

4. curl$(\nabla\phi) = 0$. i.e. curl (grad ϕ) $= 0$. $\forall\ \phi$ - scalar fn.

MRIT

Problems:-

① Find curl of the vector $\bar{F} = xyz\bar{i} + 3x^2y\bar{j} + (xz^2 - y^2z)\bar{k}$ at the point $(1,-1,1)$.

Soln:- given, $\bar{F} = xyz\bar{i} + 3x^2y\bar{j} + (xz^2 - y^2z)\bar{k}$

$$\text{curl}\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$\text{curl}\bar{F} = \bar{i}\left[\frac{\partial}{\partial y}(xz^2 - y^2z) - \frac{\partial}{\partial z}(3x^2y)\right] - \bar{j}\left[\frac{\partial}{\partial x}(xz^2 - y^2z) - \frac{\partial}{\partial z}(xyz)\right] + \bar{k}\left[\frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial y}(xyz)\right]$$

$$\text{curl}\bar{F} = \bar{i}(-2yz - 0) - \bar{j}\left[z^2 - xy\right] + \bar{k}\left[6xy - xz\right]$$

$$(\text{curl}\bar{F})_{at\,(1,-1,1)} = \bar{i}\left[-2(-1)(1)\right] - \bar{j}\left[1^2 - (1)(-1)\right] + \bar{k}\left[6(1)(-1) - (1)(1)\right]$$

$$\text{curl}\,\bar{F} = 2\bar{i} - 2\bar{j} - 7\bar{k}$$

② Find the values of a, b, c so that $\bar{F} = (x + y + az)\bar{i} + (bx + 2y - z)\bar{j} + (-x + cy + 2z)\bar{k}$ is irrotational.

Soln:- given $\bar{F} = (x+y+az)\bar{i} + (bx + 2y - z)\bar{j} + (-x + cy + 2z)\bar{k}$ is irrotational.

$$\text{curl}\bar{F} = \bar{0}$$

$$\text{curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+az) & (bx+2y-z) & (-x+cy+2z) \end{vmatrix} = \bar{0} = 0.\bar{i} + 0\bar{j} + 0.\bar{k}$$

3) Prove that $\bar{F} = (6xy + z^3)\bar{i} + (3x^2 - z)\bar{j} + (3xz^2 - y)\bar{k}$ is irrotational vector and find the scalar potential such that $\bar{F} = \nabla\phi$.

Sol^n:- given $\bar{F} = (6xy + z^3)\bar{i} + (3x^2 - z)\bar{j} + (3xz^2 - y)\bar{k}$.

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$\nabla \times \bar{F} = \bar{i}\left[\frac{\partial}{\partial y}(3xz^2-y) - \frac{\partial}{\partial z}(3x^2-z)\right] - \bar{j}\left[\frac{\partial}{\partial x}(3xz^2-y) - \frac{\partial}{\partial z}(6xy+z^3)\right]$$
$$+ \bar{k}\left[-\frac{\partial}{\partial y}(6xy+z^3) + \frac{\partial}{\partial x}(3x^2-z)\right]$$

$$\nabla \times \bar{F} = \bar{i}\left[(-1)-(-1)\right] - \bar{j}\left[3z^2 - 3z^2\right] + \bar{k}\left[-6x + 6x\right].$$

$\nabla \times \bar{F} = 0$. $\therefore$ $\bar{F}$ is irrotational vector.

given $\bar{F} = \nabla\phi$

$$(6xy+z^3)\bar{i} + (3x^2-z)\bar{j} + (3xz^2-y)\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}.$$

Equating co-efficients of $\bar{i}, \bar{j}, \bar{k}$

$$\frac{\partial\phi}{\partial x} = 6xy + z^3 \text{ ——— } (1)$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z \text{ ——— } (2).$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y \text{ ——— } (3)$$

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integrate ① w.r.to 'x'

$$\phi = 6y \cdot \frac{x^2}{2} + z^3 x + c_1$$

$$\phi = 3yx^2 + xz^3 + c_1 \quad \text{---} \quad (4)$$

integrate ② w.r.to 'y'

$$\phi = 3x^2 y - zy + c_2 \quad \text{---} \quad (5)$$

Integrate ③ w.r.to 'z'

$$\phi = 3x\frac{z^3}{3} - yz + c_3$$

$$\phi = xz^3 - yz + c_3 \quad \text{---} \quad (6)$$

∴ from ④, ⑤, ⑥, required scalar potential function.

$$\boxed{\phi = 3x^2 y + xz^3 - yz + c}$$

④ prove that $r^n \bar{r}$ is an irrotational vector.

(or)

prove that $\text{curl}(r^n \bar{r}) = 0$.

Soln:- $\bar{F} = r^n \bar{r} = r^n(x\bar{i} + y\bar{j} + z\bar{k})$

$$\bar{F} = r^n x\,\bar{i} + r^n y\,\bar{j} + r^n z\,\bar{k}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$\nabla \times \bar{F} = \bar{i}\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right] - \bar{j}\left[\frac{\partial}{\partial x}(r^n z) - \frac{\partial}{\partial z}(r^n x)\right] + \bar{k}\left[\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right].$$

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$$\nabla \times \bar{F} = \sum \bar{i}\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right]$$

$x \to y \to z$

$$= \sum \bar{i}\left[z n . r^{n-1} \cdot \frac{\partial r}{\partial y} - y n . r^{n-1} \cdot \frac{\partial r}{\partial z}\right].$$

$\frac{\partial r}{\partial y} = \frac{y}{r}$

$\frac{\partial r}{\partial z} = \frac{z}{r}$

$$= \sum \bar{i}\left[z n . r^{n-1} \cdot \frac{y}{r} - y n . r^{n-1} \cdot \frac{z}{r}\right].$$

$$= \sum \bar{i}\left[y z n . r^{n-2} - y z n . r^{n-2}\right]$$

$$\nabla \times \bar{F} = 0 .$$

$\therefore$ $\bar{F} = r^n \bar{r}$ is irrotational vector//.

H.W's

(5) if $\bar{F} = xz^3\bar{i} - 2xy\bar{j} + xz\bar{k}$, find div $\bar{F}$ & curl $\bar{F}$ at $(1,2,0)$.

(6) find div $\bar{F}$ & curl $\bar{F}$ where $\bar{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

(7) show that $\bar{F} = (x^2-y^2-x)\bar{i} - (2xy+y)\bar{j}$ is irrotational & find it's scalar potential.

(8) Find a, b such that $(2xy+3yz)\bar{i} + (x^2+axz-4z^2)\bar{j} + (3xy+2byz)\bar{k}$ is irrotational vector (or) conservative field.

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Laplacian operator: $(\nabla^2) = \nabla \cdot \nabla$.

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called "Laplacian operator"

$$\boxed{\nabla^2 \phi = \nabla \cdot \nabla \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}}$$

$$\boxed{\nabla^2 \phi = \nabla \cdot \nabla \phi = div(\nabla \phi) = div(grad\, \phi)}$$

* if $\boxed{\nabla^2 \phi = 0}$ then ϕ is called "harmonic function"

it is called "Laplacian Equation".

Theorem: if $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, prove that $\boxed{div(grad\, r^n) = n(n+1) r^{n-2}}$

$$\boxed{div(grad\, r^n) = n(n+1) \cdot r^{n-2}}$$

proof: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$$\boxed{r^2 = x^2 + y^2 + z^2}$$

p.d.w.r to x, y, z

$2r \cdot \frac{\partial r}{\partial x} = 2x$	$2r \cdot \frac{\partial r}{\partial y} = 2y$	$2r \cdot \frac{\partial r}{\partial z} = 2z$
$\frac{\partial r}{\partial x} = \frac{x}{r}$	$\frac{\partial r}{\partial y} = \frac{y}{r}$	$\frac{\partial r}{\partial z} = \frac{z}{r}$

w.k.that $\boxed{\nabla \phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z} = grad\, \phi = \sum \bar{i}\frac{\partial \phi}{\partial x}}$

$$grad(r^n) = \sum \bar{i}\frac{\partial}{\partial x}(r^n)$$

$$= \sum \bar{i}\, n r^{n-1} \frac{\partial r}{\partial x}$$

* $\boxed{\text{if } F = f_1\bar{i} + f_2\bar{j} + f_3\bar{k} \quad div F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}}$

$$= \sum \bar{i}\, n \cdot r^{n-1} \cdot \frac{x}{r}$$

$$= n \cdot r^{n-2} \sum x\bar{i}$$

$$= n \cdot r^{n-2} \cdot (x\bar{i} + y\bar{j} + z\bar{k})$$

$$\text{grad } r^n = n \cdot \left[r^{n-2}x\,\bar{i} + r^{n-2}y\,\bar{j} + r^{n-2}z\,\bar{k} \right]$$

$\xrightarrow{\text{compare}}$ $f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\text{div}(\text{grad } r^n) = n \cdot \left[\frac{\partial}{\partial x}(r^{n-2}x) + \frac{\partial}{\partial y}(r^{n-2}y) + \frac{\partial}{\partial z}(r^{n-2}z) \right]$$

$$= n \cdot \sum \frac{\partial}{\partial x}(\underset{u}{r^{n-2}}\underset{v}{x})$$

$\sum_{x,y,z}$

$$= n \sum \left[1 \cdot r^{n-2} + x \cdot (n-2) \cdot r^{n-3} \cdot \frac{\partial r}{\partial x} \right]$$

$$= n \cdot \left[\sum r^{n-2} + \sum x(n-2) \cdot r^{n-3} \cdot \frac{x}{r} \right]$$

$$= n \cdot \left[(r^{n-2} + r^{n-2} + r^{n-2}) + r^{n-4}(n-2) \sum x^2 \right]$$

$$= n \cdot \left[3 \cdot r^{n-2} + (n-2) r^{n-4} \cdot r^2 \right]$$

$\left[\because \sum x^2 = x^2 + y^2 + z^2 = r^2 \right]$

$$= n \cdot \left[3 \cdot r^{n-2} + (n-2) \cdot r^{n-2} \right]$$

$$= n \cdot r^{n-2} \left[3 + n - 2 \right]$$

$$\text{div}(\text{grad } r^n) = n(n+1) r^{n-2}$$

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*Prove that $\nabla^2 f(r) = \mathrm{div}[\mathrm{grad}\, f(r)] = f''(r) + \frac{2}{r} f'(r)$.

(or)

$$\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\cdot\frac{df}{dr} \quad \text{where } \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

Proof: w.k.t $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$.

$\boxed{r^2 = x^2 + y^2 + z^2}$, $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$$\nabla^2 f(r) = \nabla\cdot\nabla[f(r)] = \mathrm{div}[\mathrm{grad}\, f(r)].$$

$$\mathrm{grad}[f(r)] = \sum \bar{i}\frac{\partial}{\partial x}[f(r)]$$

$$= \sum \bar{i}\, f'(r)\cdot\frac{\partial r}{\partial x}$$

$$= f'(r)\sum \bar{i}\,\frac{x}{r}$$

$$= \frac{f'(r)}{r}\cdot(x\bar{i} + y\bar{j} + z\bar{k})$$

$$\mathrm{grad}[f(r)] = \frac{x f'(r)}{r}\bar{i} + \frac{y f'(r)}{r}\bar{j} + \frac{z f'(r)}{r}\bar{k}.$$

$$\mathrm{div}[\mathrm{grad}\, f(r)] = \frac{\partial}{\partial x}\left[\frac{x\cdot f'(r)}{r}\right] + \frac{\partial}{\partial y}\left[\frac{y\cdot f'(r)}{r}\right] + \frac{\partial}{\partial z}\left[\frac{z\cdot f'(r)}{r}\right]$$

$$= \sum_{x,y,z}\frac{\partial}{\partial x}\left[\frac{x\cdot f'(r)}{r}\right]$$

$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}}$

$$= \sum\frac{r\cdot\frac{\partial}{\partial x}[x\cdot f'(r)] - x\cdot f'(r)\cdot\frac{\partial r}{\partial x}}{r^2}$$

$$= \sum\frac{1}{r^2}\left[r\left(1\cdot f'(r) + x\cdot f''(r)\cdot\frac{\partial r}{\partial x}\right) - x\cdot f'(r)\cdot\frac{x}{r}\right]$$

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$$= \sum \frac{1}{r}\left[f'(r) + x \cdot f''(r) \cdot \frac{x}{r}\right] - \sum \frac{x^2}{r^3} f'(r)$$

$$= \sum \frac{1}{r} f'(r) + \sum \frac{x^2}{r^2} f''(r) - \sum \frac{x^2}{r^3} f'(r)$$

$$= \frac{3}{r} \cdot f'(r) + \frac{f''(r)}{r^2} \sum x^2 - \frac{f'(r)}{r^3} \sum x^2.$$

$$= \frac{3}{r} \cdot f'(r) + \frac{f''(r)}{r^2} \cdot r^2 - \frac{f'(r)}{r^3} (r^2)$$

$$= f''(r) + \frac{3}{r} f'(r) - \frac{1}{r} \cdot f'(r)$$

$$\boxed{\operatorname{div}[\operatorname{grad} f(r)] = f''(r) + \frac{2}{r} \cdot f'(r)} \; **$$

* Prove that $\nabla\left[\nabla \cdot \frac{\bar{r}}{r}\right] = -\frac{2}{r^3} \bar{r}$

$\nabla \cdot \bar{F} = \operatorname{div} \bar{F}$
$\nabla \phi = \operatorname{grad} \phi$

proof:- $\frac{\bar{r}}{r} = \frac{x\bar{i} + y\bar{j} + z\bar{k}}{r} = \frac{x}{r}\bar{i} + \frac{y}{r}\bar{j} + \frac{z}{r}\bar{k}$

$$\nabla \cdot \frac{\bar{r}}{r} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\underset{u}{x} \cdot \underset{v}{\frac{1}{r}}\right)$$

$$= \sum \left[1 \cdot \frac{1}{r} + x \cdot \frac{-1}{r^2} \cdot \frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{1}{r} - \sum\left(\frac{x}{r^2} \cdot \frac{x}{r}\right)$$

$$= \left(\frac{1}{r} + \frac{1}{r} + \frac{1}{r}\right) - \frac{1}{r^3} \cdot \sum x^2$$

$\left[\because \sum x^2 = r^2\right]$

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$$\nabla \cdot \frac{\bar{r}}{r} = \frac{3}{r} - \frac{r^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}.$$

$$\nabla \cdot \frac{\bar{r}}{r} = \frac{2}{r}.$$

$$\nabla \phi = \sum \bar{i} \frac{\partial \phi}{\partial x}$$

$$\left(\nabla \cdot \frac{\bar{r}}{r}\right) = \nabla\left(\frac{2}{r}\right) = \sum \bar{i} \frac{\partial}{\partial x}\left(\frac{2}{r}\right).$$

$$= \sum \bar{i} \, -\frac{2}{r^2} \cdot \frac{\partial r}{\partial x}$$

$$= -\frac{2}{r^2} \cdot \sum \bar{i} \cdot \frac{x}{r}$$

$$= -\frac{2}{r^3} \sum x\bar{i}$$

$$= -\frac{2}{r^3}(x\bar{i} + y\bar{j} + z\bar{k})$$

$$\nabla\left[\nabla \cdot \frac{\bar{r}}{r}\right] = -\frac{2}{r^3}\bar{r}$$

Important formulae

① $\nabla = \sum \bar{i} \frac{\partial}{\partial x}$

② $\nabla \phi = \sum \bar{i} \frac{\partial \phi}{\partial x} = \text{grad}\, \phi$

③ $\nabla \cdot \bar{f} = \sum \bar{i} \cdot \frac{\partial \bar{f}}{\partial x} = \text{div}\, \bar{f}$

④ $\nabla \times \bar{f} = \sum \bar{i} \times \frac{\partial \bar{f}}{\partial x} = \text{curl}\, \bar{f}$

⑤ $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$

⑥ $\bar{a} \times \bar{b} = -(\bar{b} \times \bar{a})$

⑦ $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$

⑧ $\bar{a} \cdot (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \cdot \bar{c}$

⑨ $\frac{\partial}{\partial x}(\bar{a} \cdot \bar{b}) = \frac{\partial \bar{a}}{\partial x} \cdot \bar{b} + \bar{a} \cdot \frac{\partial \bar{b}}{\partial x}$

⑩ $\frac{\partial}{\partial x}(\bar{a} \times \bar{b}) = \frac{\partial \bar{a}}{\partial x} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial x}$

Theorems:-

① Prove that $\text{div}(\phi \bar{a}) = (\text{grad}\, \phi) \cdot \bar{a} + \phi(\nabla \cdot \bar{a})$ (or)

$$\nabla \cdot (\phi \bar{a}) = (\nabla \phi) \cdot \bar{a} + \phi(\nabla \cdot \bar{a})$$

Proof:-

$$\text{div}(\phi \bar{a}) = \nabla \cdot (\phi \bar{a}) = \sum \bar{i} \frac{\partial}{\partial x} \cdot (\phi \bar{a}) = \sum \bar{i} \cdot \frac{\partial}{\partial x}(\phi \bar{a})$$

$$\text{div}(\phi \bar{a}) = \sum \bar{i} \cdot \left(\frac{\partial \phi}{\partial x} \bar{a} + \phi \frac{\partial \bar{a}}{\partial x}\right)$$

$$= \sum \bar{i} \cdot \left(\frac{\partial \phi}{\partial x} \bar{a}\right) + \sum \bar{i} \cdot \left(\phi \frac{\partial \bar{a}}{\partial x}\right)$$

$$= \sum \bar{i} \frac{\partial \phi}{\partial x} \cdot \bar{a} + \phi \sum \bar{i} \cdot \frac{\partial \bar{a}}{\partial x}$$

$$= \nabla \phi \cdot \bar{a} + \phi(\nabla \cdot \bar{a})$$

$$\text{div}(\phi \bar{a}) = \text{grad}\, \phi \cdot \bar{a} + \phi\, \text{div}\, \bar{a}$$

② prove that $curl(\phi\bar{a}) = (grad\,\phi)\times\bar{a} + \phi\, curl\,\bar{a}$

(or)

$$\nabla\times(\phi\bar{a}) = (\nabla\phi)\times\bar{a} + \phi(\nabla\times\bar{a}).$$

proof: $curl(\phi\bar{a}) = \nabla\times(\phi\bar{a}) = \sum \bar{i}\frac{\partial}{\partial x}\times(\phi\bar{a}) = \sum \bar{i}\times\frac{\partial}{\partial x}(\phi\bar{a}).$

$$curl(\phi\bar{a}) = \sum \bar{i}\times\left(\frac{\partial\phi}{\partial x}\bar{a} + \phi\frac{\partial\bar{a}}{\partial x}\right)$$

$$= \sum \bar{i}\times\left(\frac{\partial\phi}{\partial x}\bar{a}\right) + \sum \bar{i}\times\left(\phi\frac{\partial\bar{a}}{\partial x}\right)$$

$$= \sum \bar{i}\frac{\partial\phi}{\partial x}\times\bar{a} + \phi\sum \bar{i}\times\frac{\partial\bar{a}}{\partial x}$$

$$= \nabla\phi\times\bar{a} + \phi(\nabla\times\bar{a})$$

$$curl(\phi\bar{a}) = (grad\,\phi)\times\bar{a} + \phi\, curl\,\bar{a}$$

③ prove that $grad(\bar{a}\cdot\bar{b}) = (\bar{b}\cdot\nabla)\bar{a} + (\bar{a}\cdot\nabla)\bar{b} + \bar{b}\times curl\,\bar{a} + \bar{a}\times curl\,\bar{b}$

proof: consider, $\bar{a}\times curl\,\bar{b} = \bar{a}\times(\nabla\times\bar{b})$

$$= \bar{a}\times\left(\sum \bar{i}\frac{\partial}{\partial x}\times\bar{b}\right)$$

$$= \bar{a}\times\sum \bar{i}\times\frac{\partial\bar{b}}{\partial x}$$

$$= \sum \bar{a}\times\left(\bar{i}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$= \sum\left\{\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right)\bar{i} - (\bar{a}\cdot\bar{i})\frac{\partial\bar{b}}{\partial x}\right\} \quad [\because ⑦]$$

$$= \sum \bar{i}\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right) - \sum\left(\bar{a}\cdot\bar{i}\frac{\partial}{\partial x}\right)\bar{b}$$

$$= \sum \bar{i}\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right) - \left(\bar{a}\cdot\sum\bar{i}\frac{\partial}{\partial x}\right)\bar{b}$$

$$\bar{a}\times curl\,\bar{b} = \sum \bar{i}\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right) - (\bar{a}\cdot\nabla)\bar{b} \quad \text{--- ①}$$

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$\bar{b} \times \text{curl}\,\bar{a} = \sum \bar{i}\left(\bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right) - (\bar{b}\cdot\nabla)\bar{a}$ — (2)

(1) + (2)

$\bar{a}\times\text{curl}\,\bar{b} + \bar{b}\times\text{curl}\,\bar{a} = \sum \bar{i}\left(\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}\right) + \sum \bar{i}\left(\bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right) - (\bar{a}\cdot\nabla)\bar{b} - (\bar{b}\cdot\nabla)\bar{a}$

$\bar{a}\times\text{curl}\,\bar{b} + \bar{b}\times\text{curl}\,\bar{a} + (\bar{a}\cdot\nabla)\bar{b} + (\bar{b}\cdot\nabla)\bar{a} = \sum \bar{i}\left(\bar{a}\cdot\frac{\partial \bar{b}}{\partial x} + \bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right)$

$= \sum \bar{i}\,\frac{\partial}{\partial x}(\bar{a}\cdot\bar{b})$

$= \text{grad}(\bar{a}\cdot\bar{b})$

(4) Prove that $\text{div}(\bar{a}\times\bar{b}) = \bar{b}\cdot\text{curl}\,\bar{a} - \bar{a}\cdot\text{curl}\,\bar{b}$

proof:

$\text{div}(\bar{a}\times\bar{b}) = \nabla\cdot(\bar{a}\times\bar{b}) = \sum \bar{i}\frac{\partial}{\partial x}\cdot(\bar{a}\times\bar{b}) = \sum \bar{i}\cdot\frac{\partial}{\partial x}(\bar{a}\times\bar{b})$

$\text{div}(\bar{a}\times\bar{b}) = \sum \bar{i}\cdot\left(\frac{\partial \bar{a}}{\partial x}\times\bar{b} + \bar{a}\times\frac{\partial \bar{b}}{\partial x}\right)$

$= \sum \bar{i}\cdot\left(\frac{\partial \bar{a}}{\partial x}\times\bar{b}\right) + \sum \bar{i}\cdot\left(\bar{a}\times\frac{\partial \bar{b}}{\partial x}\right)$

$= \sum\left(\bar{i}\times\frac{\partial \bar{a}}{\partial x}\right)\cdot\bar{b} - \sum \bar{i}\cdot\left(\frac{\partial \bar{b}}{\partial x}\times\bar{a}\right)$ [∵ (8)] [∵ (6)]

$= \left(\sum \bar{i}\times\frac{\partial \bar{a}}{\partial x}\right)\cdot\bar{b} - \left(\sum \bar{i}\times\frac{\partial \bar{b}}{\partial x}\right)\cdot\bar{a}$ [∵ (8)]

$\text{div}(\bar{a}\times\bar{b}) = \bar{b}\cdot\text{curl}\,\bar{a} - \bar{a}\cdot\text{curl}\,\bar{b}$

(8) prove that $\nabla\times(\nabla\times\bar{a}) = \nabla(\nabla\cdot\bar{a}) - \nabla^2\bar{a}$

proof:- $\nabla\times(\nabla\times\bar{a}) = \sum i\frac{\partial}{\partial x}\times(\nabla\times\bar{a}) = \sum \bar{i}\times\frac{\partial}{\partial x}(\nabla\times\bar{a})$

now, $i\times\frac{\partial}{\partial x}(\nabla\times\bar{a}) = i\times\frac{\partial}{\partial x}\left(i\times\frac{\partial\bar{a}}{\partial x} + j\times\frac{\partial\bar{a}}{\partial y} + \bar{k}\times\frac{\partial\bar{a}}{\partial z}\right)$

$= \bar{i}\times\left(i\times\frac{\partial^2\bar{a}}{\partial x^2} + j\times\frac{\partial^2\bar{a}}{\partial x\partial y} + \bar{k}\times\frac{\partial^2\bar{a}}{\partial x\partial z}\right)$

$= \bar{i}\times\left(i\times\frac{\partial^2\bar{a}}{\partial x^2}\right) + \bar{i}\times\left(j\times\frac{\partial^2\bar{a}}{\partial x\partial y}\right) + \bar{i}\times\left(\bar{k}\times\frac{\partial^2\bar{a}}{\partial x\partial z}\right)$

$= \left(i\cdot\frac{\partial^2\bar{a}}{\partial x^2}\right)\bar{i} - \frac{\partial^2\bar{a}}{\partial x^2} + \left(i\cdot\frac{\partial^2\bar{a}}{\partial x\partial y}\right)j - 0 + \left(i\cdot\frac{\partial^2\bar{a}}{\partial x\partial z}\right)\bar{k} - 0$

$[\because \bar{i}\cdot i = 1,\ \bar{i}\cdot\bar{j} = \bar{i}\cdot\bar{k} = 0]$

$= i\frac{\partial}{\partial x}\left(i\cdot\frac{\partial\bar{a}}{\partial x}\right) + j\frac{\partial}{\partial y}\left(i\cdot\frac{\partial\bar{a}}{\partial x}\right) + \bar{k}\frac{\partial}{\partial z}\left(i\cdot\frac{\partial\bar{a}}{\partial x}\right) - \frac{\partial^2\bar{a}}{\partial x^2}$

$= \nabla\left(i\cdot\frac{\partial\bar{a}}{\partial x}\right) - \frac{\partial^2\bar{a}}{\partial x^2}$ $\quad [\because (2)] = \text{grad}\,\phi$

$\sum i\times\frac{\partial}{\partial x}[\nabla\times\bar{a}] = \nabla\left[\sum\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right] - \sum\frac{\partial^2\bar{a}}{\partial x^2} = \nabla(\nabla\cdot\bar{a}) - \left(\frac{\partial^2\bar{a}}{\partial x^2} + \frac{\partial^2\bar{a}}{\partial y^2} + \frac{\partial^2\bar{a}}{\partial z^2}\right)$

$= \nabla(\nabla\cdot\bar{a}) - \nabla^2\bar{a}$

$\therefore \boxed{\nabla\times(\nabla\times\bar{a}) = \nabla(\nabla\cdot\bar{a}) - \nabla^2\bar{a}}$

Laplacian operator:- (∇^2)

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

ϕ, $\quad \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

$\nabla^2\phi = \nabla\cdot\nabla\phi$

$\nabla^2\phi = \nabla\cdot(\text{grad}\,\phi)$

$\nabla^2\phi = \text{div}(\text{grad}\,\phi)$

⑥ prove that $\boxed{\text{div curl } \bar{f} = 0}$.

Proof:- Let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\text{curl } \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{curl } \bar{f} = \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \bar{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$\text{div}(\text{curl } \bar{f}) = \nabla \cdot (\text{curl } \bar{f}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (\text{curl } \bar{f})$$

$$\text{div}(\text{curl } \bar{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 f_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 f_2}{\partial x \partial z}} - \cancel{\frac{\partial^2 f_3}{\partial x \partial y}} + \cancel{\frac{\partial^2 f_1}{\partial y \partial z}} + \cancel{\frac{\partial^2 f_2}{\partial x \partial z}} - \cancel{\frac{\partial^2 f_1}{\partial y \partial z}}$$

$\text{div}(\text{curl } \bar{f}) = 0$ //

⑦ prove that $\boxed{\text{curl}(\text{grad } \phi) = 0}$

Proof:- we know that $\nabla\phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$

$$\text{curl}(\nabla\phi) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z}\right] - \bar{j}\left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z}\right] + \bar{k}\left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right] = 0$$

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Theorem

(8) Prove that

$$\text{curl}(\bar{a}\times\bar{b}) = \bar{a}\,\text{div}\,\bar{b} - \bar{b}\,\text{div}\,\bar{a} + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$$

(or)

$$\nabla\times(\bar{a}\times\bar{b}) = \bar{a}(\nabla\cdot\bar{b}) - \bar{b}(\nabla\cdot\bar{a}) + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$$

proof:-

$$\text{curl}(\bar{a}\times\bar{b}) = \sum i\times\frac{\partial}{\partial x}(\bar{a}\times\bar{b}) = \sum i\times\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$= \sum \bar{i}\times\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right) + \sum \bar{i}\times\left(\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$\text{curl}(\bar{a}\times\bar{b}) = \sum\left\{(i\cdot\bar{b})\frac{\partial\bar{a}}{\partial x} - \left(i\cdot\frac{\partial\bar{a}}{\partial x}\right)\bar{b}\right\} + \sum\left\{\left(i\cdot\frac{\partial\bar{b}}{\partial x}\right)\bar{a} - (i\cdot\bar{a})\frac{\partial\bar{b}}{\partial x}\right\}$$

$$= \sum(\bar{b}\cdot i)\frac{\partial\bar{a}}{\partial x} - \sum\left(i\cdot\frac{\partial\bar{a}}{\partial x}\right)\bar{b} + \sum\left(i\cdot\frac{\partial\bar{b}}{\partial x}\right)\bar{a} - \sum(\bar{a}\cdot i)\frac{\partial\bar{b}}{\partial x}$$

$[\because \bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}]$

$$= \left(\bar{b}\cdot\sum i\frac{\partial}{\partial x}\right)\bar{a} - (\nabla\cdot\bar{a})\bar{b} + (\nabla\cdot\bar{b})\bar{a} - \left(\bar{a}\cdot\sum i\frac{\partial}{\partial x}\right)\bar{b}$$

$$= (\bar{b}\cdot\nabla)\bar{a} - \bar{b}\,\text{div}\,\bar{a} + \bar{a}\,\text{div}\,\bar{b} - (\bar{a}\cdot\nabla)\bar{b}$$

(1) prove that $\nabla[f(r)] = \frac{f'(r)}{r}\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

proof:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

p.d. it w.r.to x, y, z

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla[f(r)] = \sum i \frac{\partial}{\partial x}[f(r)] = \sum i f'(r) \cdot \frac{\partial r}{\partial x}$$

$$= \sum \bar{i} f'(r) \cdot \frac{x}{r}$$

$$= \frac{f'(r)}{r} \cdot \sum x\bar{i}$$

$$= \frac{f'(r)}{r} (xi + yj + zk)$$

$$\therefore \boxed{\nabla[f(r)] = \frac{f'(r)}{r} \bar{r}}$$

2) prove that $\boxed{\nabla(r^n) = n r^{n-2} \bar{r}}$

$$\nabla(r^n) = \sum i \frac{\partial}{\partial x}(r^n) = \sum i \, n \cdot r^{n-1} \frac{\partial r}{\partial x}$$

$$= n \cdot r^{n-1} \sum i \frac{x}{r}$$

$$= n \cdot r^{n-2} \sum x\bar{i}$$

$$= n r^{n-2} (xi + yj + zk)$$

$$= n \cdot r^{n-2} \bar{r}$$

Q. (3) $\boxed{\text{Prove that } r^n \bar{r} \text{ is solenoidal if } n = -3.}$

Proof:- $r^n \bar{r} = r^n (xi + yj + z\bar{k}) = r^n x i + r^n y \bar{j} + r^n z \bar{k}$

$$\operatorname{div}(r^n \bar{r}) = \nabla \cdot (r^n \bar{r}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left(r^n x i + r^n y j + r^n z \bar{k}\right)$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z).$$

$$= \sum_{x,y,z} \frac{\partial}{\partial x}(r^n x)$$

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$$div(r^n \bar{r}) = \sum \left(1 \cdot r^n + x \cdot n\, r^{n-1} \frac{\partial r}{\partial x}\right)$$

$$= \sum r^n + \sum x \cdot n r^{n-1} \frac{x}{r}$$

$$= r^n \sum 1 + n r^{n-2} \sum x^2$$

$$= r^n (1+1+1) + n \cdot r^{n-2} (x^2+y^2+z^2)$$

$$= 3r^n + n \cdot r^{n-2} r^2$$

$$= 3r^n + n \cdot r^n$$

$$div(r^n \bar{r}) = (3+n) r^n$$

if $n = -3$, $div(r^n \bar{r}) = 0$

i.e. $div(r^{-3} \bar{r}) = 0$.

$\therefore \frac{\bar{r}}{r^3}$ is solenoidal.

Prove that $\nabla^2 (r^m) = m \cdot (m+2) \cdot r^{m-}$

(or)

$div(grad\, r^m) = m(m+2) r^{m-}$

v.v. Imp :- 3, 4, 5, 8