★ UNIT-2:

• **Eigen values and eigen vectors** :-

- Eigen values and eigen vectors.
- Cayley - Hamilton theorem.
- Diagonalization
- Orthogonalization
- Quadratic form (or) canonial form

Let $A = (a_{ij})$ be an $n \times n$ square matrix. A non-zero vector 'x' is said to be a characteristic vector of 'A' if there exists a scalar 'λ' such that $Ax = \lambda x$ $(x \neq 0)$, We say that 'X' is Eigen vector or characteristic vector of 'A' corresponding to the Eigen values (or) characteristic values of 'λA'.

• To find Eigen values and vectors of a matrix:

- Consider the matrix of order 3×3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Let us take lambda (λ) as a scalar which is subtracted from the diagonal elements, we get characteristic matrix.

i.e. $|A - \lambda I| = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$

$\boxed{\lambda I = \lambda}$

- The characteristic polynomial of 'A' is a determinant of A minus lambda I i.e $|A - \lambda I|$ and is equated to 0. we get characteristic equation. i.e $|A - \lambda I| = 0$.

- Solving the characteristic eqn we get the λ values, these values are called Eigen values or latent values or proper values.

note:

1. By using characteristic equation $\boxed{|A - \lambda I| = 0}$ we have to calculate Eigen values
2. By using matrix equation $\boxed{(A - \lambda I) X = 0}$ we have to calculate Eigen vectors.

• **Find the characteristic roots of the matrix:**

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

The characteristic eqn of A is $|A - \lambda I| = 0$.

$$|A - \lambda I| = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

$\therefore (2-\lambda)\left[(3-\lambda)(2-\lambda) - 2\right] - 2(2-\lambda-1) + \left[2-(3-\lambda)\right] = 0$

$\therefore (2-\lambda)\left[6 - 3\lambda - 2\lambda + \lambda^2 - 2\right] - 2(1-\lambda) + (-1+\lambda) = 0$

$\therefore (2-\lambda)(\lambda^2 - 5\lambda + 4) - 2 + 2\lambda - 1 + \lambda = 0$

$\therefore 2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda - 3 + 3\lambda = 0$

$\therefore -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$

$\therefore \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

using synthetic division; let $\lambda = 1$

1	1	-7	11	-5
	0	1	-6	5
1	1	-6	5	$\boxed{0}$
	0	1	-5	
5	1	-5	$\boxed{0}$	
	0	5		
	1	$\boxed{0}$		

$\therefore (\lambda-1)(\lambda-1)(\lambda-5) = 0$

$\lambda - 1 = 0 \Rightarrow \lambda = 1$

$\lambda - 1 = 0 \Rightarrow \lambda = 1$

$\lambda - 5 = 0 \Rightarrow \lambda = 5$

$\therefore$ The characteristic roots of A are $(1, 1, 5)$.

Q. Find the Eigen values corresponding to the Eigen vectors of matrix:

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

• soln: The characteristic eqn of A is ~~$|A-\lambda I| = 0$~~ $\boxed{A - \lambda I = 0}$

$$\therefore |A - \lambda| = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$$

$(8-\lambda)\left[(7-\lambda)(3-\lambda) - 16\right] + 6\left[-6(3-\lambda) + 8\right] + 2\left[24 - 2(7-\lambda)\right] = 0$

$\therefore (8-\lambda)\left[21 - 7\lambda - 3\lambda + \lambda^2 - 16\right] + 6(-18 + 6\lambda + 8) + 2(24 - 14 + 2\lambda) = 0$

$\Rightarrow (8-\lambda)(\lambda^2 - 10\lambda + 5) + 6(6\lambda - 10) + 2(10 + 2\lambda) = 0$

$\Rightarrow 8\lambda^2 - 80\lambda + 40 - \lambda^3 + 10\lambda^2 - 5\lambda + 36\lambda - 60 + 20 + 4\lambda = 0$

$\Rightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$-\lambda\left[\lambda^2 - 18\lambda + 45\right] = 0$

Here $-\lambda = 0$ and $\lambda^2 - 18\lambda + 45 = 0$

$\boxed{\lambda = 0}$

$\lambda^2 - 15\lambda - 3\lambda + 45 = 0$

$\lambda(\lambda - 15) - 3(\lambda - 15) = 0$

$(\lambda - 15)(\lambda - 3) = 0$

$\lambda - 15 = 0$ or $\lambda - 3 = 0$

$\boxed{\lambda = 15}$ or $\boxed{\lambda = 3}$

The Eigen values of A are 0, 3, 15.

To find Eigen vectors:

Eigen vector corresponding to $\boxed{\lambda = 0}$

The matrix eqn is

$(A - \lambda I)x = 0$

$(A - 0I)x = 0$

put $\lambda = 0$ in matrix $|A - \lambda I|$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$8x - 6y + 2z = 0$ ..... ①

$-6x + 7y - 4z = 0$ ..... ②

$2x - 4y + 3z = 0$ ..... ③

Solving eq ① and ② by cross-multiplication method.

y	z	x	y
-6	2	8	-6
7	-4	-6	7

$$\frac{y}{24-14} = \frac{z}{-12+32} = \frac{x}{56-36}$$

$$\frac{y}{10} = \frac{z}{20} = \frac{x}{20}$$

$x = 2 \quad y = 1 \quad z = 2$

The eigen vector corresponding to $\lambda = 0$ is $X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Eigen vectors corresponding to $\lambda = 3$

The matrix eqn is $|A - \lambda I| x = 0$

$|A - 3I| x = 0$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x - 6y + 2z = 0$ ..... ①

$-6x + 4y - 4z = 0$ ..... ②

$2x - 4y = 0$ ..... ③

Solving eq ① and ② by cross-multiplication method.

~~y z x y~~
~~6 -6 2 5~~

y	z	x	y
-6	2	5	-6
4	-4	-6	4

$$\frac{y}{24-18} = \frac{z}{-12+20} = \frac{x}{20-36}$$

$$\frac{y}{\cancel{16}_2} = \frac{z}{\cancel{-8}_1} = \frac{x}{\cancel{-16}_2}$$

$x=-2,\ y=-1,\ z=2$

The eigen vectors corresponding to $\lambda = 3$ is

$$X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}.$$

Eigen vectors corresponding to $\lambda = 15$.

The matrix eqn is $|A - \lambda I|X = 0$

$|A - 15I|x = 0$

$$\therefore \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$-7x - 6y + 2z = 0 \ \ldots\ldots ①$

$-6x - 8y - 4z = 0$

$2x - 4y - 12z = 0$

$3x + 4y + 2z = 0 \ \ldots\ldots ②$

$x - 2y - 6z = 0 \ \ldots\ldots ③$

Using cross-multiplication method on eq ② & ③

y	z	x	y
4	2	3	4
-2	-6	1	-2

$$\frac{y}{-24+4} = \frac{z}{2+18} = \frac{x}{-6-4}$$

$$\frac{y}{-20} = \frac{z}{20} = \frac{x}{-10}$$

$x = -10,\ y = -2,\ z = 2$

The Eigen vectors corresponding to $\lambda = 15$. is $X_3 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$

$\therefore$ The Eigen vectors corresponding to λ for given ~~equation~~ matrix are

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \quad X_3 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

• Find Eigen values and eigen vectors of the given ~~vector~~ matrix.

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Soln

The characteristic eqn of A is

$|A - \lambda I| = 0$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

$\therefore (6-\lambda)[(3-\lambda)(3-\lambda)-1] + 2[-2(3-\lambda)+2] - 2[-2-2(3-\lambda)] = 0$

$\therefore (6-\lambda)[\lambda^2 - 6\lambda + 9 - 1] + 2[6 - 2\lambda + 2] + 2[-2 - 6 + 2\lambda] = 0$

$\therefore (6-\lambda)[\lambda^2 - 6\lambda + 8] + 2[8 - 2\lambda] + 2[-8 + 2\lambda] = 0$

$\therefore 6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - \cancel{8\lambda} + 16 + \cancel{4\lambda} + 16 + \cancel{4\lambda} = 0$

$\therefore -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$

$\therefore \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

• Synthetic division:

2	1	-12	36	-32
	0	2	-20	32
2	1	-10	16	0
	0	2	-16	
8	1	-8	0	
	0	8		
	1	0		

$(\lambda-2)(\lambda-2)(\lambda-8) = 0$

$\lambda - 2 = 0 \Rightarrow \lambda = 2$

$\lambda - 2 = 0 \Rightarrow \lambda = 2$

$\lambda - 8 = 0 \Rightarrow \lambda = 8$

$\therefore$ The eigen values are 2, 2, 8.

• To find Eigen vectors:

The Eigen vector corresponding to $\boxed{\lambda = 2}$ the matrix eqn is

$(A - \lambda I)X = 0$

$(A - 2I)X = 0$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$4x - 2y + 2z = 0 \ \ldots\ldots ①$

$-2x + y - z = 0 \ \ldots\ldots ②$

$2x - y + z = 0 \ \ldots\ldots ③$

Here eqn ① ② and ③ are same equations, we can apply substitution method

put $y = k_1,\ z = k_2$ in eq ③

$2x - k_1 + k_2 = 0$

$\Rightarrow 2x = k_1 - k_2$

$$\boxed{x = \frac{k_1 - k_2}{2}}$$

The eigen vectors corresponding to $\lambda = 2$ is

$$X = \begin{bmatrix} \frac{k_1 - k_2}{2} \\ k_1 \\ k_2 \end{bmatrix} = k_1\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

The Eigen value corresponding to $\boxed{\lambda = 8}$

The matrix eqn is $(A - \lambda I)x = 0$

$|A - 8I|x = 0$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x - 2y + 2z = 0 \ \ldots\ldots ④$

$-2x - 5y - z = 0 \ \ldots\ldots ⑤$

$2x - y - 5z = 0 \ \ldots\ldots ⑥$

Solving eq ⑤ and ⑥ by cross multiplication method.

y	z	x	y
-5	-1	-2	-5
-1	-5	2	-1

$$\frac{y}{25-1} = \frac{z}{-2-10} = \frac{x}{2+10}$$

$$\frac{y}{\cancel{24}_2} = \frac{z}{\cancel{-12}_1} = \frac{x}{\cancel{12}_1}$$

$x = 1,\ y = 2,\ z = -1$

The eigen vectors corresponding to $\lambda = 8$ is

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

The Eigen values are 2, 2, 8 and the Eigen vectors are

$$X = \begin{bmatrix} \frac{k_1 - k_2}{2} \\ k_1 \\ k_2 \end{bmatrix} = k_1\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

• Practice:

Q. Find Eigen values and corresponding Eigen vectors of the matrices.

(1): $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

• DIAGONALIZATION : of a matrix :-

A matrix 'A' is diagonalizable if there exists an invertible matrix 'P'. such that $\boxed{P^{-1}AP = D}$ then D is called the 'diagonal matrix'.

• Working rule:

Step 1: Find Eigen values

Step 2: Find Eigen vectors

Step 3: Write modal matrix 'P'. i.e $P = [X_1\ X_2\ X_3]$

Step 4: Find the inverse of P.

[If given matrix is symmetric, we can write $\boxed{P^{-1} = P^T}$]

-If given matrix is not symmetric, we have to find P^{-1} value i.e $P^{-1} = \frac{1}{|P|}\cdot Adj(P)$

Step 5: Check diagonalization. i.e $\boxed{P^{-1}AP = D}$

Q1. Determine the modal matrix 'P' for $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and hence diagonalize 'A'.

• Soln: The characteristic eqn for A is $|A - \lambda I| = 0$

$\therefore |A - \lambda I| = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$

$\therefore (1-\lambda)[(5-\lambda)(1-\lambda) - 1] - 1[(1-\lambda)3] + 3[1 - 3(5-\lambda)] = 0$

$\therefore (1-\lambda)(5 - 5\lambda - \lambda + \lambda^2 - 1) - (3 - 3\lambda) + 3(1 - 15 + 3\lambda) = 0$

$\therefore (1-\lambda)(\lambda^2 - 6\lambda - 4) - 3 + 3\lambda + 3(-14+3\lambda)$

$\therefore \lambda^2 - 6\lambda - 4 - \lambda^3 + 6\lambda^2 + 4\lambda - 3 + 3\lambda - 42 - 9\lambda$

$\therefore -\lambda^3 + 7\lambda^2 - 36 = 0$

$\therefore \lambda^3 - 7\lambda^2 + 36 = 0$ $(x+2)(x-3)(x-6) = 0$

$\therefore x_1 = -2$ $x_2 = 6$ $x_3 = 3$

$\therefore$ The Eigen values for the matrix A is are (-2, 6 and 3).

The Eigen vectors:

Corresponding to $\lambda = -2$

The matrix eqn is $|A - \lambda I|x = 0$

$\therefore |A + 2I|x = 0$

$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3x + y + 3z = 0$ .... (1)

$x + 7y + z = 0$ ...... (2)

$3x + y + 3z = 0$ ...... (3)

solving eq (1) and (2) by cross multiplication method,

$\begin{matrix} y & z & x & y \\ 1 & 3 & 3 & 1 \\ 7 & 1 & 1 & 7 \end{matrix}$

$\frac{y}{1-21} = \frac{z}{3-3} = \frac{x}{21-1}$

$\frac{y}{-20} = \frac{z}{0} = \frac{x}{20}$

$\therefore x = 1$ $y = -1$ $z = 0$

$\therefore X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

• corresponding to $\lambda = 6$

The matrix eqn is $|A - 6I|x = 0$

$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-5x + y + 3z = 0$ ..... (4)

$x - y - z = 0$ ..... (5)

$3x + y - 5z = 0$ .... (6)

Solving eq (4) and (5) by cross multiplication method.

$\begin{matrix} y & z & x & y \\ 1 & 3 & -5 & 1 \\ -1 & -1 & 1 & -1 \end{matrix}$

$\frac{y}{-1+3} = \frac{z}{3-5} = \frac{x}{5-1}$

$\frac{y}{2} = \frac{z}{-2} = \frac{x}{4}$

$x = 1$ $y = 0$ $z = -2$

$\therefore X_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.

• Corresponding to $\lambda = 3$

The matrix eqn is $|A - 3I|x = 0$

$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2x + y + 3z = 0$ .... (7)

$x + 2y + z = 0$ ..... (8)

$3x + y - 2z = 0$ ..... (9)

solving eq (7) and (8) by cross multiplication method.

$\begin{matrix} y & z & x & y \\ 1 & 3 & -2 & 1 \\ 2 & 1 & 1 & 2 \end{matrix}$

$\frac{y}{1-6} = \frac{z}{3+2} = \frac{x}{-4-1}$

$\frac{y}{\cancel{-5}_1} = \frac{z}{\cancel{5}_1} = \frac{x}{\cancel{-5}_1}$

$X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

The modal matrix $P = [X_1, X_2, X_3]$

$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$

• Find P^{-1}:

Given matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$\boxed{\because A = A^T}$ The given matrix is a symmetric matrix.

Now $\boxed{P^{-1} = P^T}$ ... ($\because$ A is symmetric matric)

$P^{-1} = P^T$

Now $P^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

• check diagonalization:

Now $P^{-1}AP = P^TAP$

$= \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = D$

$\therefore$ The diagonal matrix of A is

$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

- orthogonal reduction of real symmetric matrix.

Step 1: Eigen values

Step 2: Eigen vectors

Step 3: Normalized eigen vectors

i.e. $\frac{X_1}{||X_1||}, \frac{X_2}{||X_2||}, \frac{X_3}{||X_3||}$

where $||X|| = \sqrt{x^2+y^2+z^2}$

Step 4: Write modal matrix

$$P = \left[\frac{X_1}{||X_1||} \quad \frac{X_2}{||X_2||} \quad \frac{X_3}{||X_3||}\right]$$

Step 5: Find P^{-1}

Here given matrix is real symmetric matrix. So we can write

$P^{-1} = P^T$

Step 6: Check diagonalization

$\boxed{D = P^{-1}AP}$

Q. Diagonalize the matrix where A

$$A = \cancel{\begin{bmatrix} 7 & 4 & 4 \\ 4 & -8 & 1 \\ -4 & 1 & 8 \end{bmatrix}} \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

- Soln: The characteristic eqn is

$|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 2-\lambda & 0 & 4 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{vmatrix} = |A-\lambda I| = 0$$

$\therefore (2-\lambda)[(6-\lambda)(2-\lambda)] + 4[-4(6-\lambda)] = 0$

$\therefore (2-\lambda)(12-6\lambda-2\lambda+\lambda^2) + 4(-24+4\lambda) = 0$

$(2-\lambda)(\lambda^2-8\lambda+12) - 96 + 16\lambda = 0$

$\therefore 2\lambda^2 - \cancel{16\lambda} + 24 - \lambda^3 + 8\lambda^2 - 12\lambda - 96 + \cancel{16\lambda} = 0$

$\therefore -\lambda^3 + 10\lambda^2 - 12\lambda - 72 = 0$

$\therefore \lambda^3 - 10\lambda^2 + 12\lambda + 72 = 0$

$\lambda = -2, \ \lambda = 6, \ \lambda = 6$

The eigen vectors for $\lambda = -2$ is:

The characteristic eqn is

$|A + 2I|X = 0$

$$\begin{bmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4x + 4z = 0$

$x + z = 0$ — ②

$x = -z$ — ①

$8y = 0$

$y = 0$ — ②

$4x + 4z = 0$

$x + z = 0$ — ③

put $z = k$ in eq ①

$x = -k$

$y = 0$

$z = k$

$$X_1 = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The eigen vectors for $\lambda = 6$ is:

The characteristic eqn $|A - 6I|X = 0$

$$\begin{bmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4x + 4z = 0$

$x - z = 0$

$x = z$ — ①

$4x - 4z = 0$

$x - z = 0$ — ②

$x = z$ — ②

put $x = z = k$

$\therefore x = k \quad y = 0$

$z = k$

$$\therefore X_2 = \begin{bmatrix} k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$X_3 = k_2\begin{bmatrix} 0 \\ 1 \\ \end{bmatrix}$ put $y = k_2$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$||X_1|| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

$||X_2|| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$

$||X_3|| = \sqrt{0^2 + (1)^2 + 0^2} = \sqrt{1} = 1$

$$\frac{X_1}{||X_1||} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\frac{X_2}{||X_2||} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\frac{X_3}{||X_3||} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- We write the modal matrix P.

$$P = \left[\frac{X_1}{||X_1||} \quad \frac{X_2}{||X_2||} \quad \frac{X_3}{||X_3||}\right]$$

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

Matrix A is symmetric

$\therefore A^{-1} = A^T$

similarly $\boxed{P^{-1} = P^T}$

$$P^{-1} = P^T = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

checking diagonalization:

$D = P^{-1}AP$

$$= \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

$$\times \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

★ Caley Hamilton theorem:

Every square matrix satisfies its own characteristic equation.

Q. Verify Caley Hamilton theorem for the matrix $A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

- Soln:
- Given: matrix

$$A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

The characteristic eqn of matrix A

$|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 8-\lambda & -8 & 2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (8-\lambda)[(-3-\lambda)(1-\lambda) - 8] + 8[4(1-\lambda)+6]$

$+ 2[-16 - 3(-3-\lambda)] = 0$

$\Rightarrow (8-\lambda)(-3+3\lambda-\lambda+\lambda^2-8) + 8(4-4\lambda+6)$

$+ 2(-16+9+3\lambda) = 0$

$\Rightarrow (8-\lambda)(\lambda^2+2\lambda-11) + 8(-4\lambda+10)$

$+ 2(3\lambda-7) = 0$

$\Rightarrow 8\lambda^2 + 16\lambda - 88 - \lambda^3 - 2\lambda + 11\lambda - 24\lambda$

$+ 80 + 6\lambda - 14 = 0$

$\Rightarrow -\lambda^3 + 6\lambda^2 + \lambda - 22 = 0$

$\lambda^3 - 6\lambda^2 - \lambda + 22 = 0$

By Caley Hamilton theorem,

put $\boxed{\lambda = A}$

$\therefore A^3 - 6A^2 - A + 22I = 0$

is the characteristic equation.

$$A^2 = \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix}$$

$$6A^2 = \begin{bmatrix}228 & -288 & 204\\ 84 & -90 & 72\\ 66 & -96 & 90\end{bmatrix}$$

Substituting them in characteristic eqn

$$A^3 - 6A^2 - A + 22I = 0$$

$$\begin{bmatrix}214 & -296 & 206\\ 88 & -115 & 70\\ 69 & -100 & 69\end{bmatrix} - \begin{bmatrix}228 & -288 & 204\\ 84 & -90 & 72\\ 66 & -96 & 90\end{bmatrix}$$

$$- \begin{bmatrix}8 & -8 & 2\\ 4 & -3 & -2\\ 3 & -4 & 1\end{bmatrix} + \begin{bmatrix}22 & 0 & 0\\ 0 & 22 & 0\\ 0 & 0 & 22\end{bmatrix} = 0$$

$$\therefore \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = 0$$

∴ The matrix A satisfies it characteristic equation.
Hence, the Caley Hamilton theorem is proved.

• Find A^{-1}:-

By using eqn ①

$$A^3 - 6A^2 - A + 22I = 0$$

Multiplying the eqn ① by A^{-1}.

$$A^{-1}[A^3 - 6A^2 - A + 22I] = 0A^{-1} = 0$$

$$\therefore A^2 - 6A - I + 22A^{-1} = 0$$

$$22A^{-1} = I - A^2 + 6A$$

$$A^{-1} = \frac{1}{22}[I - A^2 + 6A]$$

$$A^{-1} = \frac{1}{22}\left\{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}38 & -48 & 34\\ 14 & -15 & 12\\ 11 & -16 & 15\end{bmatrix} + \begin{bmatrix}48 & -48 & 12\\ 24 & -18 & -12\\ 18 & -24 & 6\end{bmatrix}\right\}$$

$$A^{-1} = \frac{1}{22}\begin{bmatrix}1-38+48 & 0+48-48 & 0-34+12\\ 0+14+24 & 1+15-18 & 0-12-12\\ 0-11+18 & 0+16-24 & 1-15-6\end{bmatrix}$$

$$A^{-1} = \frac{1}{22}\begin{bmatrix}11 & 0 & 46\\ 10 & -2 & -24\\ 7 & -8 & -8\end{bmatrix}$$

Q Verify Caley-Hamilton theorem of $A = \begin{bmatrix}1 & 2 & -1\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$ and find A^4.

• Soln: The characteristic eqn is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix}1-\lambda & 2 & -1\\ 2 & 1-\lambda & -2\\ 2 & -2 & 1-\lambda\end{vmatrix} = 0$$

$$\therefore (1-\lambda)[(1-\lambda)(1-\lambda)-4] - 2[2(1-\lambda)+4] - 1[-4-2(1-\lambda)] = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda-\lambda+\lambda^2-4) - 2(2-2\lambda+4) - (-4-2+2\lambda) = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2-2\lambda-3) - 2(-2\lambda+6) + 6 - 2\lambda = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 - \lambda^3 + 2\lambda^2 + 3\lambda + 4\lambda - 12 + 6 - 2\lambda = 0$$

$$\Rightarrow -\lambda^3 + 3\lambda^2 + 3\lambda - 9 = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - 3\lambda + 9 = 0$$

By Caley Hamilton theorem, put $\lambda = A$.

$$A^3 - 3A^2 - 3A + 9I = 0 \quad \text{— ①}$$

$$A^3 = \begin{bmatrix}3 & 24 & -21\\ 6 & 21 & -24\\ 6 & -6 & 3\end{bmatrix}$$

$$A^2 = \begin{bmatrix}3 & 6 & -6\\ 0 & 9 & -6\\ 0 & 0 & 3\end{bmatrix}$$

$$3A^2 = \begin{bmatrix}9 & 18 & -18\\ 0 & 27 & -18\\ 0 & 0 & 9\end{bmatrix}$$

$$3A = \begin{bmatrix}3 & 6 & -3\\ 6 & 3 & -6\\ 6 & -6 & 3\end{bmatrix}$$

substitute these values in eqn ①

$$\begin{bmatrix}3 & 24 & -21\\ 6 & 21 & -24\\ 6 & -6 & 3\end{bmatrix} - \begin{bmatrix}9 & 18 & -18\\ 0 & 27 & -18\\ 0 & 0 & 9\end{bmatrix} - \begin{bmatrix}3 & 6 & -3\\ 6 & 3 & -6\\ 6 & -6 & 3\end{bmatrix} + \begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix} = 0$$

$$\begin{bmatrix}3-9-3+9 & 24-18-6+0 & -21+18+3+0\\ 6-0-6+0 & 21-27-3+9 & -24+18+6+0\\ 6-0-6+0 & -6+6+0-0 & 3-9-3+9\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = 0$$

∴ Caley Hamiton theorem is verified.

Now $A^4 = \cancel{\begin{bmatrix}9 & 72 & -72\\ 0 & 81 & -72\\ 0 & 0 & 9\end{bmatrix}}$

using eqn ①

$$A^3 - 3A^2 - 3A + 9I = 0$$

Multiplying both sides with A

$$A^4 - 3A^3 - 3A^2 + 9A = 0$$

$$A^4 = 3A^3 + 3A^2 - 9A$$

$$A^4 = 3\begin{bmatrix}3 & 24 & -21\\ 6 & 21 & -24\\ 6 & -6 & 3\end{bmatrix} + 3\begin{bmatrix}3 & 6 & -6\\ 0 & 9 & -6\\ 0 & 0 & 3\end{bmatrix}$$

$$\cancel{\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}} - 9\begin{bmatrix}1 & 2 & -1\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$$

$$A^4 = \begin{bmatrix}9+9-9 & 72+18-18 & -63-18+9\\ 18+0-18 & 63+27-9 & -72-18+18\\ 18+0-18 & -18+0+18 & 9+9-9\end{bmatrix}$$

$$A^4 = \begin{bmatrix}9 & 72 & -72\\ 0 & 81 & -72\\ 0 & 0 & 9\end{bmatrix}$$

Q. If $A=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}$ find the value of matrix

$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I=0$.

• Soln :

The characteristic eqn of matrix A is $|A-\lambda I|=0$

$\therefore \begin{vmatrix}2-\lambda&1&1\\0&1-\lambda&0\\1&1&2-\lambda\end{vmatrix}=0$

$\Rightarrow (2-\lambda)\left[(1-\lambda)(2-\lambda)\right]+1(-1+\lambda)=0$

$\Rightarrow (2-\lambda)(2-\lambda-2\lambda+\lambda^2)-1+\lambda=0$

$\cancel{\Rightarrow 4-2\lambda-4\lambda+2\lambda^2-2\lambda+\lambda^2+4\lambda-\lambda^3}$

$\cancel{-1+\lambda=0}$

$\cancel{\Rightarrow -\lambda^3+3\lambda^2-3\lambda-1=0}$

$\Rightarrow (2-\lambda)(\lambda^2-3\lambda+2)-1+\lambda=0$

$2\lambda^2-6\lambda+4-\lambda^3+3\lambda^2-2\lambda-1+\lambda=0$

$\Rightarrow -\lambda^3+5\lambda^2-7\lambda+3=0$

$\Rightarrow \lambda^3-5\lambda^2+7\lambda-3=0$

By Caley Hamilton theorem:

put $\boxed{\lambda=A}$

$A^3-5A^2+7A-3=0 \;\ldots\ldots (1)$

so,

$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$

$=A^5\left[A^3-5A^2+7A-3I\right]+A^4-5A^3+8A^2-2A+I$

$=A^5(0)+A^4-5A^3+7A^2+A^2-3A+A+I$

$=A(A^4-5A^3+7A^2-3A)+A^2+A+I$

$=A(A^3-5A^2+7A-3I)+A^2+A+I$

$=A(0)+A^2+A+I$

$=A^2+A+I$

$A^2=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}$

$A^2+A+I=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}+\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$=\begin{bmatrix}8&5&5\\0&3&0\\5&5&8\end{bmatrix}$

• Quadratic form:

$\Rightarrow$ Quadratic form notation.

$Q=a_{11}x_1^2+a_{12}x_1x_2+a_{13}x_1x_3+\ldots\ldots+a_{1n}x_1x_n$

$+a_{21}x_2x_1+a_{22}x_2x_2+a_{23}x_2x_3+\ldots\ldots a_{2n}x_2x_n$

$+\ldots\ldots$

$+\ldots\ldots$

$+a_{n1}x_nx_1+a_{n2}x_nx_2+a_{n3}x_nx_3+\ldots\ldots+a_{nn}x_n^2$

$$\boxed{Q.f=\sum_{i=1}^{n}\cdot\sum_{j=1}^{n} a_{ij}\,x_i\,x_j}$$

- where a_{ij} are the co-efficients (constants).

→ Every quadratic form in n variable can be expressed in the form X^TAX.

where, A is symmetric matrix

x is variable matrix.

The quadratic form are 5 types:

1. Positive definite: All the eigen values are positive.
eg: 1, 2, 3.

2. Positive semi definite: At least one eigen value is zero and remaining all are positive. eg: 0, 1, 2

3. Negative definite: All the eigen values are negative.
eg: −1, −2, −3

4. Negative semi definite: At least one eigen values is zero remaining all are negative.
eg: 0, −1, −2.

5. Indefinite: If the eigen values are positive and negative.
eg: −1, 2, −3.

• Write the quadratic form into matrix form:

x	x	y
x	x^2	xy
y	yx	y^2

x	x	y	z
x	x^2	xy	xz
y	yx	y^2	yz
z	zx	zy	z^2

1. Write the matrix of the given quadratic form is

$x^2+y^2+2xy+4yz+2zx$

Sol: Given, $Q.f=x^2+y^2+2xy+4yz+2zx$

$=x^2+y^2+xy+xy+2yz+2yz+zx+zx$

x	x	y	z
x	1	1	1
y	1	1	2
z	1	2	0

$A=\begin{bmatrix}1&1&1\\1&1&2\\1&2&0\end{bmatrix}$

• Write the matrix of a given quadratic form

$ax^2+by^2+cz^2+2hxy+2gyz+2fzx$

• soln: Given $Q.f=$

$ax^2+by^2+cz^2+2hxy+2gyz+2fzx$

$=ax^2+by^2+cz^2+hxy+hxy+gyz+gyz+fzx+fzx$

x	x	y	z
x	a	h	f
y	h	b	g
z	f	g	c

$A=\begin{bmatrix}a&h&f\\h&b&g\\f&g&c\end{bmatrix}$

Write the matrix of the given quadratic form

$x_1^2 + x_2^2 - x_3^2 - 2x_1x_2 - 3x_2x_3$

Soln: Given:

$Q.F = x_1^2 + x_2^2 - x_3^2 - 2x_1x_2 - 3x_2x_3$

$= x_1^2 + x_2^2 - x_3^2 - x_1x_2 - x_1x_2 - \frac{3}{2}x_2x_3 - \frac{3}{2}x_2x_3$

x	x_1	x_2	x_3
x_1	1	-1	0
x_2	-1	1	-3/2
x_3	0	-3/2	-1

$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -3/2 \\ 0 & -3/2 & -1 \end{bmatrix}$

• Change the given matrix into quadratic form. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & 2 \end{bmatrix}$

• Given: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & 2 \end{bmatrix}_{3\times3}$

Let us consider,

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow X^T = [x_1 \; x_2 \; x_3]$

Now, $X^TAX = [x_1 \; x_2 \; x_3] \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= [x_1+2x_2+3x_3 \quad 2x_1-x_2 \quad 3x_1+2x_3]_{1\times3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= x_1(x_1+2x_2+3x_3) + x_2(2x_1-x_2) + x_3(3x_1+2x_3)$

$= x_1^2 + 2x_1x_2 + 3x_1x_2 + 2x_2x_1 - x_3^2 + 3x_1x_3 + 2x_3^2$

QF: $x_1^2 - x_2^2 + 2x_3^2 + 4x_1x_2 + 6x_1x_3$

• Change the given matrix into quadratic form: $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

• Soln: Given, $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}_{2\times2}$

Let us consider,

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x^{-1}[x_1 \; x_2]$

Now,

$X^TAX = [x_1 \; x_2] \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= [x_1+2x_2 \quad 2x_1+5x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= x_1(x_1+2x_2) + x_2(2x_1+5x_2)$

$= x_1^2 + 2x_1x_2 + 2x_1x_2 + 5x_2^2$

$QF = x_1^2 + 5x_2^2 + 4x_1x_2$

• **Reduction of Q.F. into canonical form:**

Quadratic form: combination of product terms and power terms.

canonical form: only power terms

→ We can change quadratic form into canonical form by using 3 methods:

1. Orthogonalization
2. Diagonalization
3. Lagrange's reduction.

• **Reduction of Q.F into C.F by orthogonalization:**

▸ working rule:

Step 1: Write symmetric matrix

Step 2: Find eigen values

Step 3: Find eigen vectors

Step 4: Write the modal matrix

$P = \left[\frac{X_1}{\|X_1\|} \quad \frac{X_2}{\|X_2\|} \quad \frac{X_3}{\|X_3\|}\right]$

Which is formed by normalized eigen vectors and D = diagonal matrix which is formed by eigen values.

Step 5: Find P^{-1}.

'P' is orthogonal matrix $[PP^T = I]$

$\Rightarrow P^{-1} = P^T$

Step 6: $D = P^{-1}AP$.

Step 7: Canonical form $\cancel{X^TDX}$ Y^TDY

i.e $\cancel{X^T[P^{-1}AP]X}$

$Y^T[P^{-1}AP]Y$ $\boxed{\text{METHOD I}}$

Q. Reduce the given quadratic form into canonical form by orthogonalization.

$x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$

• Soln: Given $QF = x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$

• Step I write symmetric matrix

x	x_1	x_2	x_3
x_1	1	0	0
x_2	0	3	-1
x_3	0	-1	3

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

• Step 2. Find eigen values

The characteristic equation is

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$

$\cancel{(1-\lambda)(3-\lambda) = 0}$

$(1-\lambda)[(3-\lambda)^2 - 1] = 0$

$(1-\lambda)(9 - 6\lambda + \lambda^2 - 1) = 0$

$(1-\lambda)(\lambda^2 - 6\lambda + 8) = 0$

$\lambda^2 - 6\lambda + 8 - \lambda^3 + 6\lambda^2 - 8\lambda = 0$

$-\lambda^3 + 7\lambda^2 - 14\lambda + 8 = 0$

$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$

By solving the equation we get,

$\lambda = 1, 2, 4$

∴ The Eigen values are 1, 2, 4

• Step 3. To find Eigen vectors

The Eigen vectors corresponding to $\boxed{\lambda = 1}$

The matrix eqn is given by

$(A - I)X = 0$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

corresponding equations are

$2x_2 - x_3 = 0$ .... (1)

$-x_2 + 2x_3 = 0$ .... (2)

solving eq (1) and (2) by crossmultipl

x_2	x_3	x_1	x_2
2	-1	0	2
-1	2	0	-1

$\frac{x_1}{4-1} = \frac{x_2}{0} = \frac{x_3}{0}$

$\frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{0}$ $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

The eigen vectors corresponding to

$\boxed{\lambda = 2}$ $\qquad |A-\lambda I|X = 0$

The matrix eqn is $|A-2I|X=0$

$$\therefore \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding eqns are

$-2x_1 = 0$

$x_1 = 0 \; \ldots\ldots$ (4)

$x_2 - x_3 = 0$

$x_2 = x_3 \; \ldots\ldots$ (5)

$-x_2 + x_3 = 0$

$x_3 = x_2 \; \ldots\ldots$ (3)

put $x_3 = k$ in eq (5)

$x_2 = x_3 = k$

$$X_2 = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix} = k\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The eigen vectors corresponding to $\boxed{\lambda = 4}$

The matrix eqn is $|A-\lambda I|X=0$

$|A-4I|X=0$

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding eqns are

$-3x_1 = 0$

$x_1 = 0 \; \ldots\ldots$ (7)

$-x_2 - x_3 = 0$

$x_2 = -x_3 \; \ldots\ldots$ (8)

$-x_2 - x_3 = 0 \; \ldots$ (9)

put $x_3 = k$ in eq (8)

$x_2 = -x_3 = -k$

$$X_3 = \begin{bmatrix} 0 \\ k \\ -k \end{bmatrix} = k\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Step 3:

The normalized vectors are

$\cancel{\|X_1\| = \sqrt{0^2+1^2+1^2} = \sqrt{2}}$

$\|X_1\| = \sqrt{1^2+0^2+0^2} = \sqrt{1} = 1$

$$\boxed{\frac{X_1}{\|X_1\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$\|X_2\| = \sqrt{0^2+1^2+1^2} = \sqrt{2}$

$$\frac{X_2}{\|X_2\|} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$\|X_3\| = \sqrt{0^2+1^2+(-1)^2} = \sqrt{2}$

$$\frac{X_3}{\|X_3\|} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

- Write modal matrix (step 4)

$$P = \begin{bmatrix} \frac{X_1}{\|X_1\|} & \frac{X_2}{\|X_2\|} & \frac{X_3}{\|X_3\|} \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

(Step 5)

$\because P$ is orthogonal matrix $[PP^T = I]$

$\boxed{\therefore P^{-1} = P^T}$

$$P^{-1} = P^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

- Step 6: $D = P^{-1}AP$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Hence $D = P^{-1}AP$

- Step 7: Canonical form $= Y^TDY$

Let $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow Y^T = [y_1 \; y_2 \; y_3]$

Now, $Y^TDY =$

$$[y_1 \; y_2 \; y_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$[y_1 \; 2y_2 \; 4y_3]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$C.F = y_1^2 + 2y_2^2 + 4y_3^2$ //

METHOD II

Q. Reduce the Q.F into C.F. by diagonalization.

- Working rule:
- Step 1: Write the symmetric matrix
- Step 2: Write $A = I_m A I_n$ form.

Apply row operation in prefactor matrix and column operation in post-factor matrix.

The given matrix we can change into diagonal matrix.

$D = PAQ$

$D = PAP^T$

- Step 3: Canonical Form $= Y^TDY$.

Q. Reduce the quadratic equation into Canonical form by diagonalization

$6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$

- Soln: The given quadratic eqn is

$6x_1^2 + 3x_2^2 + 3x_3^3 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$

- Step 1: Write symmetric matrix

x	x_1	x_2	x_3
x_1	6	-2	2
x_2	-2	3	-1
x_3	2	-1	3

$$\therefore A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

- Step 2: Write $A = I_M A I_N$

$A = I_3 A I_3$

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to 3R_2 + R_1, \; R_3 \to 3R_3 - R_1$

$$\begin{bmatrix} 6 & -2 & 2 \\ 0 & 7 & -1 \\ 0 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 0 & 3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to \cancel{2}7R_3 + R_2$

$$\begin{bmatrix} 6 & -2 & 2 \\ 0 & 7 & -1 \\ 0 & 0 & 48 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -6 & 3 & 21 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to 3C_2 + C_1, \; C_3 \to 3C_3 - C_1$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & -3 \\ 0 & 0 & 144 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -6 & 3 & 21 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$C_3 \longrightarrow 7C_3 + C_2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 1008 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -6 & 3 & 21 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & 21 \end{bmatrix}$$

$D = PAQ$

$\boxed{D = PAP^T.}$

- Step 3: Canonical form $= Y^TDY$

Let $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad Y^T = [y_1 \; y_2 \; y_3]$

Now $Y^TDY = [y_1 \; y_2 \; y_3]\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 1008 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$= \begin{bmatrix} 6y_1 & 21y_2 & 1008y_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$C.F = 6y_1^2 + 21y_2^2 + 1008y_3^2$$

Q. Reduce the QF into C.F. By

• orthogonalization.

1. $5x^2+26y^2+10z^2+4yz+14zx+6xy$
2. $3x^2+2y^2+3z^2-2xy-2yz$
3. $2x_1^2+2x_2^2+2x_3^2+2x_2x_3$

• Diagonalization:

1. $x_1^2+2x_2^2+2x_3^2-2x_1x_2+2x_1x_3-2x_2x_3$
2. $8x^2-7y^2+3z^2+12xy+4xz-8yz$

• Reduce QF into C.F by Lagrange's reduction:

• Working rule:

- Step 1: Take the common terms from the product form of QF
- Step 2: Make perfect square by re-grouping the terms.
- Step 3: The resulting relation gives Quadratic form.

Q1. Reduce the QF into C.F. by Lagrange's reduction.

$x_1^2+2x_2^2-7x_3^2-4x_1x_2+8x_1x_3$

• Soln: Given quadratic form is

$QF = x_1^2+2x_2^2-7x_3^2-4x_1x_2+8x_1x_3$

$\Rightarrow x_1^2+2x_2^2-7x_3^2-4x_1[x_2-2x_3]$

$\Rightarrow x_1^2-4x_1[x_2-2x_3]+2x_2^2-7x_3^2$

$\Rightarrow (x_1)^2-(2x_1)[2(x_2-2x_3)]+2x_2^2-7x_3^2$

$\Rightarrow [x_1^2-2x_1[2(x_2-2x_3)]+4(x_2-2x_3)^2]$ $\cancel{+2x_2^2-7x_3^2}$

$\cancel{\Rightarrow [x_1-2(x_2-2x_3)]^2+2x_2^2-7x_3^2}$

$-4(x_2-2x_3)^2+2x_2^2-7x_3^2$

$\Rightarrow [x_1-2(x_2-2x_3)^2]-4[x_2^2-4x_2x_3+4x_3^2]+2x_2^2-7x_3^2$

$\Rightarrow [x_1-2(x_2-2x_3)^2]-4x_2^2+16x_2x_3-16x_3^2+2x_2^2-7x_3^2$

$\Rightarrow [x_1-2(x_2-2x_3)^2]-2x_2^2+16x_2x_3-23x_3^2$

$\Rightarrow [x_1-2(x_2-2x_3)^2]-2[x_2^2-8x_2x_3]-23x_3^2$

$\Rightarrow [x_1-2(x_2-2x_3)^2]-2[x_2^2-(2x_2)(4x_3)+16x_3^2]+32x_3^2-23x_3^2$

$\Rightarrow [x_1-2(x_2-2x_3)^2]-2[x_2-4x_3]^2+9x_3^2$

$C.F = [x_1-2(x_2-2x_3)^2]-2[x_2-4x_3]^2+9x_3^2$

Q2. By Lagrange's reduction transform the QF X^TAX to sum of squares for $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix}$

• Soln: Given matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix}$$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $X^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$

Now, $X^TAX =$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1+2x_2+4x_3 & 2x_1+6x_2-2x_3 & 4x_1-2x_2+18x_3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$= x_1(x_1+2x_2+4x_3)+x_2(2x_1+6x_2-2x_3)+x_3(4x_1-2x_2+18x_3)$

$= x_1^2+2x_1x_2+4x_1x_3+2x_1x_2+6x_2^2-2x_2x_3+4x_1x_3-2x_2x_3+18x_3^2$

$= x_1^2+6x_2^2+18x_3^2+4x_1x_2+8x_1x_3-4x_2x_3$

$QF = x_1^2+6x_2^2+18x_3^2+4x_1x_2+8x_1x_3-4x_2x_3$

$\Rightarrow x_1^2+6x_2^2+18x_3^2+4x_1(x_2+2x_3)-4x_2x_3$

$\Rightarrow x_1^2+6x_2^2+18x_3^2+4x_1(x_2+2x_3)-4x_2x_3$

$\Rightarrow [x_1^2+2(x_1).2(x_2+2x_3)+4(x_2+2x_3)^2]$
$-4(x_2+2x_3)^2+6x_2^2+18x_3^2-4x_2x_3$

$\Rightarrow [x_1+2(x_2+2x_3)]^2-4[x_2^2+4x_2x_3+4x_3^2]+6x_2^2+18x_3^2-4x_2x_3$

$\Rightarrow [x_1+2(x_2+2x_3)]^2-4x_2^2-16x_2x_3-16x_3^2+6x_2^2+18x_3^2-4x_2x_3$

$\Rightarrow [x_1+2(x_2+2x_3)]^2+2x_2^2-20x_2x_3+2x_3^2$

$\Rightarrow [x_1+2(x_2+2x_3)]^2+2[x_2^2-10x_2x_3]+2x_3^2$

$\Rightarrow [x_1+2(x_2+2x_3)]^2+2[x_2^2-(2x_2)(5x_3)+25x_3^2]-50x_3^2+2x_3^2$

$\Rightarrow [x_1+2(x_2+2x_3)]^2+2[x_2-5x_3]^2-48x_3^2 = C.F.$