★ UNIT IV:

[Multivariable Calculas]

• Partial differentiation:

ordinary derivates - $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dt}$

dy — dependant on x

dx - independant variable.

Ordinary partial differentiation

↳ one independent variable

• Partial differential equation:

An equation containing one or more partial derivatives of an unknown function of two or more independent variables is called a partial differential equation.

eg: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

→ $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial t}$ .... are first order derivatives (partial)

$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2}$ ... second order derivatives (partial)

$\frac{\partial^3 u}{\partial x^3}, \frac{\partial^3 u}{\partial y^3}, \frac{\partial^3 u}{\partial z^3}$ ... third order derivatives (partial)

$\frac{\partial^n u}{\partial x^n}, \frac{\partial^n u}{\partial y^n}, \frac{\partial^n u}{\partial z^n}$ ... nth order derivative (partial)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right]$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right]$$

Q.1. Find first and second order partial derivatives of $ax^2+2hxy+by^2$ and also verify $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

• Soln:

Given function: $f = ax^2+2hxy+by^2$ (1)

• First order partial derivatives

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ values

diff. eqn (1) partially w.r.t x

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[ax^2+2hxy+by^2]$$

$$\frac{\partial f}{\partial x} = 2ax + 2hy(1) + 0$$

$$\frac{\partial f}{\partial x} = 2ax+2hy \quad \text{..... (2)}$$

diff eqn (1) partially wrt y.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[ax^2+2hxy+by^2]$$

$$= 0 + 2hx(1) + 2by$$

$$\frac{\partial f}{\partial y} = 2hx + 2by \quad \text{..... (3)}$$

• Second order partial derivatives

$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}$ values

diff eqn (2) partially wrt x

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}[2ax+2hy]$$

$$\frac{\partial^2 f}{\partial x^2} = 2a(1) + 0$$

$$\frac{\partial^2 f}{\partial x^2} = 2a \quad \text{..... (4)}$$

diff eqn (3) partially wrt y

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}[2hx + 2by]$$

$$\frac{\partial^2 f}{\partial y^2} = 0 + 2b(1) = 2b \quad \text{.... (5)}$$

To verify $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

⇒ diff eqn (2) wrt y

$$\frac{\partial f}{\partial x} = 2ax + 2hy$$

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}[2ax+2hy]$$

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = 0 + 2h(1) = 2h \quad \text{.... (6)}$$

diff eqn (3) wrt x

$$\frac{\partial f}{\partial y} = 2by + 2hx$$

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\{2by + 2hx]$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0 + 2h(1) = 2h \quad \text{.... (7)}$$

from eqn (6) and (7)

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ is verified.

Q2. If $\omega = (y-z)(z-x)(x-y)$ then find value of $\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} + \frac{\partial \omega}{\partial z}$

• Soln: given function is

$$\omega = (y-z)(z-x)(x-y) \quad \text{..... (1)}$$

diff eqn (1) partially wrt x

$$\frac{\partial \omega}{\partial x} = \frac{\partial}{\partial x}[(y-z)(z-x)(x-y)]$$

$$= (y-z)\frac{\partial}{\partial x}[(z-x)(x-y)]$$

$$= (y-z)[(z-x)(1) + (x-y)(-1)]$$

$$= (y-z)[(z-x) - x + y]$$

$$= (y-z)[(z+y) - 2x]$$

$$\frac{\partial \omega}{\partial x} = y^2 - z^2 - 2x(y-z) \quad \text{..... (1)}$$

Similarly,

$$\frac{\partial \omega}{\partial y} = z^2 - x^2 - 2y(z-x) \quad \text{.... (2)}$$

$$\frac{\partial \omega}{\partial z} = x^2 - y^2 - 2z(x-y) \quad \text{.... (3)}$$

Now eqn (1) + eqn (2) + eqn (3)

$$\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} + \frac{\partial \omega}{\partial z}$$

$$= \cancel{y^2} - \cancel{z^2} - 2x(y-z) + \cancel{z^2} - \cancel{x^2} - 2y(z-x) + \cancel{x^2} - \cancel{y^2} - 2z(x-y)$$

$$= -\cancel{2xy} + \cancel{2xz} - \cancel{2yz} + \cancel{2xy} - \cancel{2xz} + \cancel{2zy} = 0$$

$$\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} + \frac{\partial \omega}{\partial z} = 0$$

Q3. If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$, $x^2+y^2+z^2 \neq 0$ then prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

• ~~Given~~ Soln: The given function

$$u = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-1/2}$$

$$u = (x^2+y^2+z^2)^{-1/2} \quad \text{— (1)}$$

diff eqn (1) partially wrt x

$$\frac{\partial u}{\partial x} = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2} \cdot \cancel{2}x$$

$$\frac{\partial u}{\partial x} = -x(x^2+y^2+z^2)^{-3/2} \quad \text{— (2)}$$

similarly by partially diff. eq (1) wrt to y and z respectively we get

$$\frac{\partial u}{\partial y} = -y(x^2+y^2+z^2)^{-3/2} \quad \text{— (3)}$$

$$\frac{\partial u}{\partial z} = -z(x^2+y^2+z^2)^{-3/2} \quad \text{— (4)}$$

diff. eqn (2) partially w.r.t x

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}\left[-x.(x^2+y^2+z^2)^{-3/2}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = -\left[x.-\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-5/2}.\cancel{2}x\right] + (x^2+y^2+z^2)^{-3/2}(1)\Big]$$

$$= -\left[3x^2.(x^2+y^2+z^2)^{-5/2} + (x^2+y^2+z^2)^{-3/2}\right]$$

$$= -(x^2+y^2+z^2)^{-3/2}\left[\frac{-3x^2}{(x^2+y^2+z^2)} + 1\right]$$

$$= -(x^2+y^2+z^2)^{-3/2}\left[\frac{-3x^2+(x^2+y^2+z^2)}{x^2+y^2+z^2}\right]$$

$$= -(x^2+y^2+z^2)^{-3/2}.(x^2+y^2+z^2)^{-1}.(-2x^2+y^2+z^2)$$

$$\frac{\partial^2 u}{\partial x^2} = -(x^2+y^2+z^2)^{-5/2}.(-2x^2+y^2+z^2) \quad \text{—(3)}$$

similarly,

$$\frac{\partial^2 u}{\partial y^2} = -(x^2+y^2+z^2)^{-5/2}.(-2y^2+x^2+z^2) \quad \text{—(4)}$$

$$\frac{\partial^2 u}{\partial z^2} = -(x^2+y^2+z^2)^{-5/2}.(-2z^2+x^2+y^2) \quad \text{—(5)}$$

Now adding eqn (3) (4) (5) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= -(x^2+y^2+z^2)^{-5/2}\left[-\cancel{2y^2}+\cancel{x^2}+\cancel{z^2} -\cancel{2x^2}+\cancel{y^2}+\cancel{z^2} -\cancel{2z^2}+\cancel{x^2}+\cancel{y^2}\right]$$

$$= -(x^2+y^2+z^2)^{-5/2}(0)$$

$$= 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Q4. If $u = \log(x^3+y^3+z^3-3xyz)$ then P.T. $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

• Soln. The given function is

$$u = \log(x^3+y^3+z^3-3xyz) \quad \text{—(1)}$$

diff eqn (1) w.r.t x

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\log(x^3+y^3+z^3-3xyz)\right]$$

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz} \quad \text{—(2)}$$

similarly.

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz} \quad \text{—(3)}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz} \quad \text{—(4)}$$

Now eqn (3)+(4)+(5)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-yz-xy-xz)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\cancel{(x^2+y^2+z^2-yz-xy-xz)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-xz-yz)}}$$

$$= \frac{3}{x+y+z}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

Now $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$

$$= \left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right]$$

$$= \left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{3}{x+y+z}\right] + \frac{\partial}{\partial y}\left[\frac{3}{x+y+z}\right] + \frac{\partial}{\partial z}\left[\frac{3}{x+y+z}\right]$$

$$= \frac{(-3)}{(x+y+z)^2}.1 + \frac{(-3)}{(x+y+z)^2}.1 + \frac{(-3)(1)}{(x+y+z)^2}$$

$$= \frac{-3-3-3}{(x+y+z)^2} = \frac{-9}{(x+y+z)^2}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Hence, proved ....

Q7. If $x^x y^y z^z = e$, show that at $x=y=z$, $\frac{\partial^2 z}{\partial x \partial y} = -[x\log ex]^{-1}$.

Soln: Given function is

$x^x y^y z^z = e$

taking log on both sides

$\log[x^x y^y z^z] = \log e$

$\log x^x + \log y^y + \log z^z = 1$

$x\log x + y\log y + z\log z = 1$

$z\log z = 1 - x\log x - y\log y$ — ①

diff eqn ① partially wrt x and y

$\frac{\partial}{\partial x}[z\log z] = \frac{\partial}{\partial x}[1 - x\log x - y\log y]$

$\left[\cancel{z}\cdot\frac{1}{\cancel{z}} + \log z(1)\right]\frac{\partial z}{\partial x} = -\left[\cancel{x}\cdot\frac{1}{\cancel{x}} + \log x\right]$

$(1+\log z)\frac{\partial z}{\partial x} = -(1+\log x)$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-(1+\log x)}{(1+\log z)}}$$

$\frac{\partial}{\partial y}[z\log z] = \frac{\partial}{\partial y}[1 - x\log x - y\log y]$

$\cancel{z}\cdot\frac{1}{\cancel{z}} + \log z(1)\frac{dz}{dy} = \left[0 - \left(\cancel{y}\cdot\frac{1}{\cancel{y}} + \log y \cdot 1\right)\right]$

$(1+\log z)\frac{dz}{dy} = -(1+\log y)$

$\frac{dz}{dy} = \frac{-(1+\log y)}{(1+\log z)}$ — ②

Given condition $x=y=z$

at $x=y=z$, we have

$\frac{\partial z}{\partial x} = \frac{-(1+\cancel{\log x})}{1+\cancel{\log x}} = -1$ $\boxed{\frac{\partial z}{\partial x} = -1}$

and $\frac{\partial z}{\partial y} = \frac{-(1+\log y)}{1+\log x} = \frac{-(1+\cancel{\log y})}{1+\cancel{\log y}} = -1$

$$\boxed{\therefore \frac{\partial z}{\partial y} = -1}$$

To find the value of $\frac{\partial^2 z}{\partial x \partial y}$

Now, diff eqn ② partially wrt x

$\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{-(1+\log y)}{1+\log z}\right]$

$= (1+\log y)\frac{\partial}{\partial x}\left[\frac{1}{(1+\log z)}\right]$

$= -\frac{(1+\log y)(-1)}{(1+\log z)^2}\left[0+\frac{1}{z}\right]\frac{\partial z}{\partial x}$

$= \frac{1+\log y}{(1+\log z)^2}\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x}$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{-(1+\log y)}{z(1+\log z)^2}$.

at $x=y=z$,

$\frac{\partial^2 z}{\partial x \partial y} = \frac{-\cancel{(1+\log x)}}{x(1+\log x)^{\cancel{2}}} = -\frac{1}{x(1+\log x)}$

$= x^{-1}(1+\log x)^{-1}$

$= -[x(1+\log x)]^{-1}$

$= [-x[\log e + \log x]]^{-1}$

$= -[x(\log ex)]^{-1}$

$$\boxed{\therefore \frac{\partial^2 z}{\partial x \partial y} = -[x\log ex]^{-1}}$$

Q8. verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ when

$u = \log[y\sin x + x\sin y]$

Soln: The given function is

$u = \log[y\sin x + x\sin y]$ — ①

diff both sides w.r.t x

$\frac{\partial u}{\partial x} = \frac{1}{y\sin x + x\sin y}(y\cos x + \sin y(1))$

$\frac{\partial u}{\partial x} = \frac{y\cos x + \sin y}{y\sin x + x\sin y}$ — ②

diff eqn ② partially wrt. y

$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial y}\left[\frac{y\cos x + \sin y}{y\sin x + x\sin y}\right]$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{(y\sin x + x\sin y)(\cos x \cdot (1) + \cos y) - [y\cos x + \sin y][\sin x \cdot (1) + x(\cos y)]}{(y\sin x + x\sin y)^2}$$

$$= \frac{y\sin x\cancel{\cos x} + y\sin x\cos y + x\sin y\cos x + x\cancel{\sin y}\cos y - y\cancel{\cos x}\sin x - xy\cos x\cos y - \sin x\sin y - x\sin y\cancel{\cos y}}{(y\sin x + x\sin y)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{y\sin x\cos y + x\cos x\sin y - xy\cos x\cos y - \sin x\sin y}{(y\sin x + x\sin y)^2}$$

diff. eqn ① wrt 'y'.

$$\frac{\partial u}{\partial y} = \frac{\sin x + x\cos y}{y\sin x + x\sin y} \quad \text{—③}$$

diff eqn ③ wrt. x

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{\sin x + x\cos y}{y\sin x + x\sin y}\right]$$

$$= \frac{(y\sin x + x\sin y)(\cos x + \cos y) - (\sin x + x\cos y)(y\cos x + \sin y)}{(y\sin x + x\sin y)^2}$$

$$= y\sin x\cancel{\cos x} + y\sin x\cos y + x\sin y\cos x + x\sin y\cancel{\cos y} - y\sin x\cancel{\cos x} - \sin x\cos y - xy\cos y\cos x - x\cos y\cancel{\sin y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{y\sin x\cos y + x\sin y\cos x - \sin x\cos y - xy\cos y\cos x}{(y\sin x + x\sin y)^2}$$

$$\boxed{\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}}$$

Hence, proved.

• Jacobian:

- Let $u = u(x,y)$, $v = v(x,y)$ then these two simultaneous relations constitutes a transformation from ~~(u,v) to~~ (x,y) to (u,v).
- The determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \text{ (or) } \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

is called the 'Jacobian' of (u,v) wrt (x,y).

- The determinant value is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ (or) $J\left[\frac{u,v}{x,y}\right]$ and is defined as

$$\frac{\partial(u,v)}{\partial(x,y)} = J\left[\frac{u,v}{x,y}\right] = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

• note 1: If $x = x(u,v)$ $y = y(u,v)$ then $J\left[\frac{x,y}{u,v}\right] = \frac{\partial(x,y)}{\partial(u,v)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

• note 2: If $u = u(x,y,z)$ $v = v(x,y,z)$ ~~then~~ $\omega = \omega(x,y,z)$

$$J\left[\frac{(u,v,\omega)}{(x,y,z)}\right] = \frac{\partial(u,v,\omega)}{\partial(x,y,z)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial y} & \frac{\partial \omega}{\partial z} \end{vmatrix}$$

• note 3: $x = x(u,v,\omega)$ $y = y(u,v,\omega)$ $z = z(u,v,\omega)$ then

$$J\left[\frac{(x,y,z)}{(u,v,\omega)}\right] = \frac{\partial(x,y,z)}{\partial(u,v,\omega)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial \omega} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial \omega} \end{vmatrix}$$

★ Properties of Jacobians:

1] If $J = \frac{\partial(x,y)}{\partial(u,v)}$ and $J' = \frac{\partial(u,v)}{\partial(x,y)}$

Then $J \cdot J' = 1$ or

$$\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1$$

(2) Jacobian Chain rule:

If (u,v) are functions of (r,s) and (r,s) are functions of (x,y)

$$\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$$

Q1. If $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $\omega = \frac{xy}{z}$ then show that $\frac{\partial(u,v,\omega)}{\partial(x,y,z)} = 4$

• Soln: Given functions are

$u = \frac{yz}{x}$ $v = \frac{xz}{y}$ $\omega = \frac{xy}{z}$

partially diff. each function wrt x,y,z

$\frac{\partial u}{\partial x} = \frac{-yz}{x^2}$ $\frac{\partial v}{\partial x} = \frac{z}{y}$

$\frac{\partial u}{\partial y} = \frac{z}{x}$ $\frac{\partial v}{\partial y} = \frac{-xz}{y^2}$

$\frac{\partial u}{\partial z} = \frac{y}{x}$ $\frac{\partial v}{\partial z} = \frac{x}{y}$

$\frac{\partial \omega}{\partial x} = \frac{y}{z}$ $\frac{\partial \omega}{\partial y} = \frac{x}{z}$ $\frac{\partial \omega}{\partial z} = \frac{-xy}{z^2}$

Now,

$$J\left[\frac{(u,v,\omega)}{(x,y,z)}\right] = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= -\frac{yz}{x^2}\left[\frac{-xz}{y^2}\cdot\frac{-xy}{z^2} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[\frac{-xyz}{yz^2} - \frac{xy}{yz}\right] + \frac{y}{z}\left[\frac{xz}{yz} + \frac{xyz}{y^2z}\right]$$

$$= \frac{-yz}{x^2}\left[\frac{x^2}{yz} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[\frac{-x}{z} - \frac{x}{z}\right] + \frac{y}{z}\left[\frac{x}{y} + \frac{x}{y}\right]$$

$$= 0 + \frac{xz}{xz} + \frac{xz}{xz} + \frac{xy}{zy} + \frac{xy}{zy}$$

$$= 0 + 1 + 1 + 1 + 1 = 4$$

$$\boxed{\therefore J\left[\frac{(u,v,\omega)}{(x,y,z)}\right] = \frac{\partial(u,v,\omega)}{\partial(x,y,z)} = 4}$$

Q2. If $u = x^2 - 2y$ $v = x + y + z$ $\omega = x - 2y + 3z$. Find $\frac{\partial(u,v,\omega)}{\partial(x,y,z)}$.

• Soln: The given functions are

$u = x^2 - 2y$

$v = x + y + z$

$\omega = x - 2y + 3z$

diff. all the functions partially wrt x, y and z respectively

$\frac{\partial u}{\partial x} = 2x$ $\frac{\partial u}{\partial y} = -2$ $\frac{\partial u}{\partial z} = 0$

$\frac{\partial v}{\partial x} = 1$ $\frac{\partial v}{\partial y} = 1$ $\frac{\partial v}{\partial z} = 1$

$\frac{\partial \omega}{\partial x} = 1$ $\frac{\partial \omega}{\partial y} = -2$ $\frac{\partial \omega}{\partial z} = 3$

Now, $\frac{\partial(u,v,\omega)}{\partial(x,y,z)}$

$$= \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= 2x(3+2) + 2(3-1) + 0$$

$$= 2x(5) + 2(2) + 0$$

$$= 10x + 4$$

$$\boxed{\frac{\partial(u,v,\omega)}{\partial(x,y,z)} = 10x + 4}$$

Q3. If $x = u(1+v)$ $y = v(1+u)$ prove that $\frac{\partial(x,y)}{\partial(u,v)} = 1 + u + v$.

• Soln: Given functions

$x = u(1+v)$, $y = v(1+u)$

$x = u + uv$ $y = v + uv$

diff. the above functions wrt u and v

$\frac{\partial x}{\partial u} = 1 + v$ $\frac{\partial x}{\partial v} = 0 + u = u$

$\frac{\partial y}{\partial u} = v$ $\frac{\partial y}{\partial v} = 1 + u$

now,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv = 1 + u + v + uv - uv$$
$$= 1 + u + v \text{ proved.}$$

Q4. If $x = r\cos\theta$ $y = r\sin\theta$ $z = z$

Find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$

• Soln: The given functions are

$x = r\cos\theta$ $y = r\sin\theta$ $z = z$

diff the above functions partially wrt r, θ, z.

$\frac{\partial x}{\partial r} = \cos\theta$ $\frac{\partial x}{\partial \theta} = -r\sin\theta$ $\frac{\partial x}{\partial z} = 0$

$\frac{\partial y}{\partial r} = \sin\theta$ $\frac{\partial y}{\partial \theta} = r\cos\theta$ $\frac{\partial y}{\partial z} = 0$

$\frac{\partial z}{\partial r} = 0$ $\frac{\partial z}{\partial \theta} = 0$ $\frac{\partial z}{\partial z} = 1$.

now, $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} =$

$$\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= 1(r\cos^2\theta + r\sin^2\theta)$

$= r(\cos^2\theta + \sin^2\theta)$

$= r(1)$

$= r$

$$\boxed{\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r}$$

$\frac{\partial(r,\theta,z)}{\partial(x,y,z)} = \frac{1}{r}$ ... By 1st property of Jacobian

.... $[J \times J' = 1]$

Q5 If $x = uv$ $y = \frac{u}{v}$ then find $\frac{\partial(x,y)}{\partial(u,v)}$ and also verify $\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1$

• Soln: The given functions are.

$x = uv$ $y = \frac{u}{v}$

diff. the above functions partially by

$\frac{\partial x}{\partial u} = v$ $\frac{\partial x}{\partial v} = u$

$\frac{\partial y}{\partial u} = \frac{1}{v}$ $\frac{\partial y}{\partial v} = \frac{u}{v^2}$

$$\begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = \frac{-uv}{v^2} - \frac{u}{v} =$$

$= \frac{uv^2 - u}{v}$

$= \frac{u(v^2-1)}{v}$

$\frac{\partial x}{\partial u} = v$ $\frac{\partial x}{\partial v} = u$

$\frac{\partial y}{\partial u} = \frac{1}{v}$ $\frac{\partial y}{\partial v} = -\frac{u}{v^2}$

$$\begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{uv}{v^2} - \frac{u}{v}$$

$= \frac{-u-u}{v} = \frac{-2u}{v}$

Q6. If $x+y+z = u$, $y+z = uv$, $z = uvw$, then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

• Soln: $z = uvw$ —— ①

$y+z = uv$

$\Rightarrow y + uvw = uv$

$\Rightarrow y = uv - uvw$ —— ②

$x+y+z = u$

$x + uv = u$

$x = u - uv$ —— ③

diff. eqn ③ partially wrt to u, v, w.

$\frac{\partial x}{\partial u} = 1-v$ $\frac{\partial x}{\partial v} = -u$ $\frac{\partial x}{\partial w} = 0$

diff eqn ② partially wrt u, v, w

$\frac{\partial y}{\partial u} = v - vw$ $\frac{\partial y}{\partial v} = u - uw$ $\frac{\partial y}{\partial w} = -uv$

diff eqn ① partially wrt u, v, w

$\frac{\partial z}{\partial u} = vw$ $\frac{\partial z}{\partial v} = uw$ $\frac{\partial z}{\partial w} = uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} (1-v) & -u & 0 \\ (v-vw) & (u-uw) & -uv \\ vw & uw & uv \end{vmatrix}$$

$\cancel{= (1-v)[(u-uw)uv + uv(uw)]}$

$\cancel{+ u[uv(v-vw) - vw(u-uv)]}$

$\cancel{= (1-v)(u^2v - u^2vw + u^2vw) +}$

$\cancel{u[uvw - uvw^2 - uvw + uv^2w}$

$(1-v)[(uv)(u-uw) + u^2vw]$

$+ u[uv(v-vw) + uv^2w \cancel{(u-uv)}]$

$= (1-v)(u^2v - \cancel{u^2vw} + \cancel{u^2vw})$

$+ u(uv^2 - \cancel{uv^2w} \cancel{- uvw} + \cancel{uv^2w})$

$= (1-v)(u^2v) + u(uv^2)$

$= u^2v - \cancel{u^2v^2} + \cancel{u^2v^2}$

$= u^2v$

$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$

Q7. If $x = r\cos\theta$ $y = r\sin\theta$ find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$ also show that $\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

• Soln: $x = r\cos\theta$ ① $y = r\sin\theta$ ② given

diff the above functions wrt r, θ.

$\frac{\partial x}{\partial r} = \cos\theta$ $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$ $\frac{\partial y}{\partial \theta} = r\cos\theta$ —— ④

Now, $①^2 + ②^2 \Rightarrow$

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

$= r^2(1)$

$r^2 = x^2 + y^2$ ..... ③

eqn ②/① $\Rightarrow$ $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$

$\frac{y}{x} = \tan\theta$

$\theta = \tan^{-1}(y/x)$ ..... ④

diff eqn ③ partially wrt x, y.

$$r^2 = x^2 + y^2$$

$$2r\frac{\partial r}{\partial x} = 2x \qquad \cancel{\frac{\partial r}{\partial y}} = 2r\frac{\partial r}{\partial y} = 2y$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r}$$

diff eqn ④ partially wrt x, y,

$$\cancel{\frac{y}{x}} = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{-y}{x^2}$$

$$= \frac{1}{\frac{x^2+y^2}{x^2}}\cdot\frac{-y}{x^2}$$

$$= \frac{\cancel{x^2}}{x^2+y^2}\cdot\frac{-y}{\cancel{x^2}}$$

$$\frac{\partial \theta}{\partial x} = -\frac{y^2}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= \frac{1}{\frac{x^2+y^2}{x^2}}\cdot\frac{1}{x}$$

$$= \frac{x^2}{x^2+y^2}\cdot\frac{1}{\cancel{x}}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$\boxed{\frac{\partial(x,y)}{\partial(r,\theta)} = r} \quad \ldots\ldots ⑤$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{r(x^2+y^2)} + \frac{y^2}{r(x^2+y^2)}$$

$$= \frac{\cancel{(x^2+y^2)}}{r\cancel{(x^2+y^2)}},$$

$$\boxed{\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}} \quad — ⑥$$

Multiplying eqn ⑤ and ⑥

$$\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)}$$

$$= r \times \frac{1}{r} = 1$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

Hence, proved.

Q. If $u = \frac{x+y}{1-xy}$, $V = \tan^{-1}x + \tan^{-1}y$

find $\frac{\partial(u, v)}{\partial(x,y)}$

• Soln: given functions are

$$u = \frac{x+y}{1-xy} \qquad V = \tan^{-1}x + \tan^{-1}y$$

differentiating the above functions partially wrt x and y.

$$\frac{\partial u}{\partial x} = \frac{(1-xy)(1) - (x+y)(-y)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial x} = \frac{1+y^2}{(1-xy)^2}$$

Similarly

$$\frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial V}{\partial x} = \frac{1}{1+x^2} + 0$$

$$\frac{\partial V}{\partial x} = \frac{1}{1+x^2}$$

similarly

$$\frac{\partial V}{\partial y} = \frac{1}{1+y^2}$$

Now, $\frac{\partial(u,v)}{\partial(x,y)}$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{\cancel{1+y^2}}{(1-xy)^2} \times \frac{1}{\cancel{1+y^2}} - \frac{\cancel{1+x^2}}{(1-xy)^2} \cdot \frac{1}{\cancel{1+x^2}}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

$$\boxed{\therefore \frac{\partial(u,v)}{\partial(x,y)} = 0}$$

• Polar co-ordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

★ Q. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ } spherical co-ordinates. S.T.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$ and also

find $\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$

• Soln: The given functions are

$$x = r\sin\theta\cos\phi$$

diff. it partially wrt r, θ, ϕ

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

$$y = r\sin\theta\sin\phi$$

diff it partially wrt r, θ, ϕ:

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$$

$$\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$

$$z = r\cos\theta$$

diff it partially wrt r, θ, ϕ.

$$\frac{\partial z}{\partial r} = \cos\theta$$

$$\frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial z}{\partial \phi} = 0$$

Now, $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

Now, $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= \cos\theta\left[r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\right] + r\sin\theta\left[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right]$

$= \cos\theta\left[r^2\sin\theta\cos\theta(1)\right] + r\sin\theta\left[r\sin^2\theta(1)\right]$

$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta$

$= r^2\sin\theta(\cos^2\theta + \sin^2\theta)$

$= r^2\sin\theta(1)$

$= r^2\sin\theta$

$$\boxed{\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta}$$

By using property of Jacobian

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} \times \frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = 1$$

$$r^2\sin\theta \times \frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = 1$$

$$\boxed{\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \frac{1}{r^2\sin\theta}}$$

Q If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$ S.T. $\dfrac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$

• Soln The given functions are $u = x^2 - y^2$, $v = 2xy$

diff. above functions partially w.r.t x, y

$\dfrac{\partial u}{\partial x} = 2x$ $\quad \dfrac{\partial u}{\partial y} = -2y$

$\dfrac{\partial v}{\partial x} = 2y$ $\quad \dfrac{\partial v}{\partial y} = 2x$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$= 4x^2 + 4y^2$

$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = 4(x^2+y^2)}$$

The given functions are $x = r\cos\theta$, $y = r\sin\theta$

diff partially w.r.t r, θ

$\dfrac{\partial x}{\partial r} = \cos\theta$, $\quad \dfrac{\partial x}{\partial \theta} = -r\sin\theta$

$\dfrac{\partial y}{\partial r} = \sin\theta$ $\quad \dfrac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta$

$= r(\sin^2\theta + \cos^2\theta)$

$= r$

$$\boxed{\frac{\partial(x,y)}{\partial(r,\theta)} = r}$$

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(r,\theta)}$$

$4(x^2+y^2) \times r = 4(r^2)r$

$$\boxed{\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3}$$

$(\because r = \sqrt{x^2+y^2})$.

Q. If $x = e^r\sec\theta$, $y = e^r\tan\theta$

P.T. $\dfrac{\partial(x,y)}{\partial(r,\theta)} \times \dfrac{\partial(r,\theta)}{\partial(x,y)} = 1$

• Soln: Given functions

$x = e^r\sec\theta$ — ①

$y = e^r\tan\theta$ — ②

eqn ①^2 − ②^2 ⇒ $x^2 - y^2 = e^{2r}\sec^2\theta - e^{2r}\tan^2\theta$

$= e^{2r}(\sec^2\theta - \tan^2\theta)$

$x^2 - y^2 = e^{2r}$

$e^{2r} = x^2 - y^2$ — ③

eqn ② ÷ ① ⇒ $\dfrac{y}{x} = \dfrac{e^r\tan\theta}{e^r\sec\theta}$

$\dfrac{y}{x} = \sin\theta$

$\theta = \sin^{-1}(y/x)$ — ④

diff eqn ① partially w.r.t r and θ

$\dfrac{\partial x}{\partial r} = e^r\sec\theta$ $\quad \dfrac{\partial x}{\partial \theta} = e^r\sec\theta\tan\theta$

diff eqn ② partially w.r.t r and θ

$\dfrac{\partial y}{\partial r} = e^r\tan\theta$ $\quad \dfrac{\partial y}{\partial \theta} = e^r\sec^2\theta$

diff eqn ③ and ④ partially w.r.t x and y

$e^{2r} = x^2 - y^2$ — ⑤

$e^{2r}\cdot(2)\cdot\dfrac{\partial r}{\partial x} = 2x$

$\dfrac{\partial r}{\partial x} = \dfrac{x}{e^{2r}}$ $\quad \dfrac{\partial r}{\partial y} = \dfrac{-y}{e^{2r}}$

$$\frac{\partial \theta}{\partial x} = \frac{-\frac{y}{x^2}}{\sqrt{1-\frac{y^2}{x^2}}} = \frac{-\frac{y}{x^2}}{\frac{\sqrt{x^2-y^2}}{x}}$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x\sqrt{x^2-y^2}}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{\frac{\sqrt{x^2-y^2}}{x}} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= \frac{\frac{1}{x}}{\frac{\sqrt{x^2-y^2}}{x}} = \frac{1}{\sqrt{x^2-y^2}}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{\sqrt{x^2-y^2}}$$

Now, $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} e^r\sec\theta & e^r\sec\theta\tan\theta \\ e^r\tan\theta & e^r\sec^2\theta \end{vmatrix}$$

$= e^{2r}\sec^3\theta - e^{2r}\sec\theta\tan^2\theta$

$= e^{2r}\sec\theta(\sec^2\theta - \tan^2\theta)$

$\dfrac{\partial(x,y)}{\partial(r,\theta)} = e^{2r}\sec\theta$ — ⑥

and $\dfrac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{vmatrix}$

$$\Rightarrow \begin{vmatrix} \dfrac{x}{x^2-y^2} & \dfrac{-y}{x^2-y^2} \\ \dfrac{-y}{x\sqrt{x^2-y^2}} & \dfrac{1}{\sqrt{x^2-y^2}} \end{vmatrix}$$

$$\Rightarrow \frac{x}{x^2-y^2\sqrt{x^2-y^2}} - \frac{y^2}{x(x^2-y^2)\sqrt{x^2-y^2}}$$

$$\Rightarrow \frac{1}{x^2-y^2\sqrt{x^2-y^2}} \cdot \left[x - \frac{y^2}{x}\right]$$

$$\Rightarrow \frac{1}{\cancel{x^2-y^2}\sqrt{x^2-y^2}} \cdot \frac{\cancel{x^2-y^2}}{x}$$

$$= \frac{1}{x\sqrt{x^2-y^2}}$$

$$= \frac{1}{(e^{2r})8^{1/2}\cdot e^r\sec\theta}$$

$$= \frac{1}{e^{2r}\sec\theta}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{e^{2r}\sec\theta}$$

Now, $\dfrac{\partial(x,y)}{\partial(r,\theta)} \times \dfrac{\partial(r,\theta)}{\partial(x,y)}$

$$= \cancel{e^{2r}\sec\theta} \times \frac{1}{\cancel{e^{2r}\sec\theta}} = 1$$

Hence, proved.

★ **Functional dependance:**

If the functions u and v of the independant variables x and y are functionally dependent then the jacobian vanishes

i.e. ~~$\partial(x,y)$~~ $\dfrac{\partial(u,v)}{\partial(x,y)} = 0$

note:

• If the Jacobian $J\left[\dfrac{u,v}{x,y}\right] \neq 0$ then u and v are said to functionally independent.

Q1. Show that the functions

$u = xy + yz + zx$

$v = x^2 + y^2 + z^2$

$\omega = x + y + z$ are functionally dependent. Find the relation between them.

• Sol^n: The given functions are

$u = xy + yz + zx$ —— ①

$v = x^2 + y^2 + z^2$ —— ②

$\omega = x + y + z$ —— ③

diff eqn ① ② ③ partially wrt x, y and z.

$u = xy + yz + zx$

$\dfrac{\partial u}{\partial x} = y + z \quad \dfrac{\partial u}{\partial y} = x + z \quad \dfrac{\partial u}{\partial z} = z + y$

$\dfrac{\partial v}{\partial x} = 2x \quad \dfrac{\partial v}{\partial y} = 2y \quad \dfrac{\partial v}{\partial z} = 2z$

$\dfrac{\partial \omega}{\partial x} = 1 \quad \dfrac{\partial \omega}{\partial y} = 1 \quad \dfrac{\partial \omega}{\partial z} = 1$

Now, $\dfrac{\partial(u,v,\omega)}{\partial(x,y,z)}$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial \omega}{\partial x} & \dfrac{\partial \omega}{\partial y} & \dfrac{\partial \omega}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} y+z & x+z & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$= (y-z)(2y-2z) - (x+z)(2x-2z) + (x+y)(2x-2y)$

$= \cancel{2y^2} - \cancel{2xy} - \cancel{2zy} + \cancel{2z^2} - \cancel{2x^2} + \cancel{2zx} - \cancel{2zx} + \cancel{2z^2} + \cancel{2x^2} - \cancel{2xy} + \cancel{2xy} - \cancel{2y^2}$

$= 0$ //

$\therefore \dfrac{\partial(u,v,\omega)}{\partial(x,y,z)} = 0$ //

$\therefore$ u, v and ω are functionally dependent.

• **Relation**: $u = xy + yz + xz$

$v = x^2 + y^2 + z^2$

$\omega = x + y + z$

$\omega = x + y + z$

$\omega^2 = (x+y+z)^2$

$\omega^2 = x^2 + y^2 + z^2 + 2(xy + yz + xz)$

$$\boxed{\omega^2 = v + 2u}$$

Q2. Prove that $u = \dfrac{x^2-y^2}{x^2+y^2}$, $v = \dfrac{2xy}{x^2+y^2}$ and check their dependancy and relation.

Diff. 'u' partially wrt x and y.

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(2x) - (x^2-y^2)(2x)}{(x^2+y^2)^2}$$

$$= \frac{2x(\cancel{x^2} + y^2 - \cancel{x^2} + y^2)}{(x^2+y^2)^2}$$

$$= \frac{2x(2y^2)}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{4xy^2}{(x^2+y^2)^2}}$$

~~Diff 'y' partially wrt x and~~

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(-2y) - (x^2-y^2)(+2y)}{(x^2+y^2)^2}$$

$$= \frac{2y\left[-(x^2+y^2) - (x^2-y^2)\right]}{(x^2+y^2)^2}$$

$$= \frac{2y(-\cancel{x^2} - \cancel{y^2} - x^2 + \cancel{y^2})}{(x^2+y^2)^2}$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{-4x^2y}{(x^2+y^2)^2}}$$

Diff 'v' partially wrt x and y.

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)(2y) - 2xy(2x)}{(x^2+y^2)^2}$$

$$= \frac{2y\left[x^2+y^2-2x^2\right]}{(x^2+y^2)^2}$$

$$\boxed{\frac{\partial v}{\partial x} = \frac{-2y(x^2-y^2)}{(x^2+y^2)^2}}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(2x) - 2xy(2y)}{(x^2+y^2)^2}$$

$$= 2x(x^2+y^2-2y^2)/(x^2+y^2)$$

$$\boxed{\frac{\partial v}{\partial y} = \frac{2x(x^2-y^2)}{(x^2+y^2)^2}}$$

Now, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} \frac{4xy^2}{(x^2+y^2)^2} & \frac{-4x^2y}{(x^2+y^2)^2} \\ \frac{-2y(x^2+y^2)}{(x^2+y^2)^2} & \frac{2x(x^2-y^2)}{(x^2+y^2)^2} \end{vmatrix}$$

$$= \frac{8x^2y^2(x^2-y^2)}{(x^2+y^2)^4} + \frac{8x^2y^2(-x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{8x^2y^2}{(x^2+y^2)^4}\left[\cancel{x^2}-\cancel{y^2}-\cancel{x^2}+\cancel{y^2}\right]$$

$= 0$

$$\boxed{\therefore \frac{\partial(u,v)}{\partial(x,y)} = 0}$$

$\therefore$ u and v are functionally dependent

• Relation:

$$u^2+v^2 = \left(\frac{x^2-y^2}{x^2+y^2}\right)^2 + \left(\frac{2xy}{x^2+y^2}\right)^2$$

$$= \frac{(x^2-y^2)^2+(2xy)^2}{(x^2+y^2)^2}$$

$$= \frac{x^4+y^4-2x^2y^2+4x^2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^4+y^4+2x^2y^2}{(x^2+y^2)^2}$$

$$u^2+v^2 = \frac{(x^2+y^2)^2}{(x^2+y^2)^2} = 1 \qquad \boxed{u^2+v^2=1}$$

P.T. given functions are functionally dependent then find the relation between them.

1] $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$

2] $u = e^x \sin y$, $v = e^x \cos y$ F.I. $\neq 0$

3] $u = \frac{x}{y}$, $v = \frac{x+y}{x-y}$

4] $u = x+y+z$, $v = x^2+y^2+z^2-2xy-2yz-2xz$.

$w = x^3+y^3+z^3-3xyz$ F.I $\neq 0$

5] $u = 2x-y+3z$, $v = 2x-y-z$,

$w = 2x-y+z$

6] $u = \frac{x+y}{x-y}$, $v = \frac{xy}{(x-y)^2}$

7] $x = u\sqrt{1-v^2} + v\sqrt{1-u^2}$

$y = \sin^{-1}u + \sin^{-1}v$

8] $u = \frac{x-y}{x+y}$ $\quad v = \frac{xy}{(x-y)^2}$

1] Soln: Given functions are

$u = \frac{x+y}{1-xy}$ $\quad v = \tan^{-1}x + \tan^{-1}y$

diff the above ~~equat~~ functions partially w.r.t x and y.

$$\frac{\partial u}{\partial x} = \frac{(1-xy)(1)-(x+y)(-y)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{1+y^2}{(1-xy)^2}}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)(1)-(x+y)(-x)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+x^2+\cancel{xy}}{(1-xy)^2}$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}}$$

$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}$ $\quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1+y^2}{(1-xy)} & \frac{1+x^2}{(1-xy)} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{\cancel{1+y^2}}{(1-xy)}\cdot\frac{1}{\cancel{1+y^2}} - \frac{\cancel{1+x^2}}{(1-xy)}\cdot\frac{1}{\cancel{1+x^2}}$$

$$= \frac{1}{1-xy} - \frac{1}{1-xy} = 0$$

$$\boxed{\therefore \frac{\partial(u,v)}{\partial(x,y)} = 0}$$

$\therefore$ The function u and v are functionally dependent.

• Relation:

$v = \tan^{-1}x + \tan^{-1}y$

$v = \tan^{-1}\left[\frac{x+y}{1-xy}\right]$

now $\because u = \frac{x+y}{1-xy}$

$$\boxed{\therefore v = \tan^{-1}(u)}$$

2] Soln: $u = e^x \sin y$ $\quad v = e^x \cos y$

diff. the above functions partially w.r.t. x and y.

$\frac{\partial u}{\partial x} = e^x \sin y$ $\quad \frac{\partial u}{\partial y} = e^x \cos y$

$\frac{\partial v}{\partial x} = e^x \cos y$ $\quad \frac{\partial v}{\partial y} = -e^x \sin y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{vmatrix}$$

$$= -e^{2x}\sin^2 y - e^{2x}\cos^2 y$$

$$= -e^{2x}(\sin^2 y + \cos^2 y)$$

$$= -e^{2x}(1)$$

$$= -e^{2x}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = -e^{2x} \neq 0$$

$\therefore$ The functions u and v are independent.

$\therefore$ No relation can be derived between them.

3] Soln: $u = \frac{x}{y}$, $v = \frac{x+y}{x-y}$

diff. the above functions partially w.r.t. x and y.

$\frac{\partial u}{\partial x} = \frac{1}{y}$ $\quad \frac{\partial u}{\partial y} = -\frac{x}{y^2}$

$$\frac{\partial v}{\partial x} = \frac{(x-y)(1)-(x+y)(1)}{(x-y)^2} = \cancel{x}-y-\cancel{x}-y$$

$$\boxed{\frac{\partial v}{\partial x} = \frac{-2y}{(x-y)^2}}$$

$$\frac{\partial v}{\partial y} = \frac{(x-y)(1)-(x+y)(-1)}{(x-y)^2}$$

$$= \frac{x-\cancel{y}+x+\cancel{y}}{(x-y)^2} = \frac{2x}{(x-y)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \end{vmatrix}$$

$$= \frac{2x}{y(x-y)^2} - \frac{2y^2 x}{y^2(x-y)^2}$$

$$= \frac{2x-2x}{y(x-y)^2} = 0$$

$$\boxed{\therefore \frac{\partial(u,v)}{\partial(x,y)} = 0}$$

$\therefore$ u and v are functionally dependent.

$u = \frac{x}{y} \qquad v = \frac{x+y}{x-y}$

$$v = \frac{y\left[\frac{x}{y}+1\right]}{y\left[\frac{x}{y}-1\right]}$$

$$\boxed{v = \frac{u+1}{u-1}}$$

...[with the help of theorem of K. Joshua (B6).] ★

4] soln: $u = x+y+z$,

$v = x^2+y^2+z^2 - 2xy - 2yz - 2xz$

$w = x^3+y^3+z^3 - 3xyz$

diff the above functions, partially w.r.t x and y.

$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = 1 \qquad \frac{\partial u}{\partial z} = 1$

$\frac{\partial v}{\partial x} = 2x-2y-2z \qquad \frac{\partial v}{\partial y} = 2y-2x-2z$

$\frac{\partial v}{\partial z} = 2z-2y-2x$

$\frac{\partial w}{\partial x} = 3x^2 - 3yz$

$\frac{\partial w}{\partial y} = 3y^2 - 3xz$

$\frac{\partial w}{\partial z} = 3z^2 - 3xy$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x-2y-2z & 2y-2x-2z & 2z-2y-2x \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$\cancel{= 2(y-x-z)(3z^2-3xy) - 2(x-y-z)(3z^2-3xy)}$

$\cancel{[(2y-2x-2z)(3z^2-3xy) - (3y^2-3xz)(2z-2y-2x)]}$ −

$=$

$3x^2-3yz\,[2z-2y-\cancel{2x} - 2y + \cancel{2x} + 2z]$

$-3y^2-3xz\,[2z-\cancel{2y}-2x-2x+\cancel{2y}+2z]$

$3z^2-3xy\,[2y-2x-\cancel{2z}-2x+2y+\cancel{2z}]$

$= (3x^2-3yz)[4z-4y] - (3y^2-3xz)(4z-4x] + (3z^2-3xy)(4y-4x)$

$= \cancel{12x^2z} - \cancel{12x^2y} - \cancel{12xyz^2} + \cancel{12y^2z}$
$-\cancel{12y^2z} + \cancel{12xy^2} + \cancel{12xz^2} - \cancel{12x^2z}$
$+ \cancel{12z^2y} - \cancel{12xz^2} - \cancel{12xy^2} + \cancel{12x^2y}$

$= 0$

$$\boxed{\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0}$$

$u^3 = (x+y+z)^3$

$= (x^3+y^3+z^3) + 3\,[(x+y+z)(xy+yz+xz) - xyz]$

$= (x^3+y^3+z^3) + 3(x+y+z)(xy+yz+xz) - 3xyz$

$u^3 = x^3+y^3+z^3 - 3xyz +$

$u^3 = w +$

$u \times v$

$(x+y+z)(x^2+y^2+z^2-2xy-2yz-2xz)$

$x^3+y^3+z^3$ ~~$+xy^2+xz^2+2y^2+yz^2$~~
~~$x^2z+y^2z-2x^2y-2xyz$~~
~~$-2x^2z-2xy^2-2y^2z-2xyz$~~
~~$-2xyz-2yz^2-2xz^2$~~

$x^3+y^3+z^3 - 6xyz - xy^2 - xz^2$
$- x^2y - yz^2 - x^2z - y^2z$

$x^3+y^3+z^3 - 6xyz - x(y^2+z^2)$
$- y(x^2+z^2) - z(x^2+y^2)$