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UNIT-V

Algebraic Structures

Algebraic structure – closure

Semigroup – closure + associative

monoid – closure + associative + identity

group – closure + associative + identity + inverse.

Abilian of group – closure + Associative + identity + inverse + Commutative.

→ **Algebraic structures :-**

A non empty set 'S' is called a Algebraic structure with respect to binary operation * if $(a*b) \in S, \forall (a,b) \in S$.

Eg :- $(N,+)$ is closed under + operation

$5+6=11$

$(N,-)$ is not closed under – operation

$3-5=-2$ ×

$(N,\times)$ is closed under × operation

$(N,\div)$ is not closed under ÷ operation.

$(Z,+)$ is closed under + operation.

$(Z,-)$ is closed under – operation.

$(R,\div)$ is closed under ÷ operation.

$(R,-), (R,\times), (R,+)$ is closure.

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→ **Semi group :-** A Algebraic structure $(S,*)$ is called Semi group if it follows Associative property.

$$(a*b)*c = a*(b*c)$$

Eg: $(N,+)$ 2, 6, 7

$(2+6)+7 = 2+(6+7)$

$8+7 = 2+13$

$15 = 15$ ✓

$(N,*), (N,+)$ is a semigroup.

$(Z,-)$ 5, 2, 4

$(5-2)-4 = 5-(2-4)$

$3-4 = 5-(-2)$

$-1 = 7$ ×

$(Z,*), (Z,+)$ is a semigroup but $(Z,-)$ is not a semigroup.

1) $2^n / n$ is a integer is a semi group w.r.t '$\times$'

$2^3 = 8$

$(2^2 \times 2^3) \times 2^4 = 2^2 \times (2^3 \times 2^4)$

$(2^5) \times 2^4 = 2^2 \times (2^7)$

$2^9 = 2^9$

* **Monoid :-**

A semigroup $(S, *)$ is a monoid if there exist an element $e \in S$ such that $a * e = e * a = a$, $\forall a \in S$.

then element e is called identity element of S w.r.t $*$.

Eg: (i) $(Z, +)$

$a + e = e + a = a \quad \Rightarrow a = 5$

$5 + \underline{0} = 5$

0 is called Additive identity element.

(ii) $(N, \times)$

$a * e = e \times a = a$

$a = 5$

$5 \times \underline{1} = 5$

1 is called multiplicative identity element.

2) $(R, \div)$ is a monoid or not.

$R = Q + IR$

$Q = 1, 0$

$(Q, \div)$ is failed bcoz $1/0$ is not possible

So, $(R, \div)$ is not a monoid.

3) $2^n / n$ is a integer is a monoid or not.

$2^3 \times \underline{2^0} = 2^3$

$2^3 \times 1 = 2^3$ $\therefore$ It is a monoid.

* **Group:-** A monoid $(S,*)$ with identity element 'e' is called Group, if each element $(a \in S)$ there exist an element $(b \in S)$ such that $a * b = b * a = e$. then b is called the inverse of a. denoted as a^{-1}.

Eg: $(\mathbb{Z},+)$ $(\mathbb{Z},\times)$

$5+(-5)=0$ $5 \times \frac{1}{5} = 1$ 1/5 is a multiplicative inverse (identity) of 5.

It is a group. It is a group

$(N,\times)$ is not a group.

$3 \times (1/3) = 1$

↳ not a natural number

* **Abelian group:-** A group $(G,*)$ said to be Abelian, if $a * b = b * a \quad \forall (a,b) \in G$. (commutative property).

Eg: $(\mathbb{Z},+)$

$5, -3$

$\Rightarrow 5-3 = -3+5$

$\Rightarrow 2 = 2$.

It is a Abelian group.

$(R^+,\times)$ is a Abelian group.

$(N,\times)$ is not a Abelian group.

1) the set $(N,*)$ where N is a set of Natural Numbers $a * b = a^b$ is a semigroup or not.

Sol:

closure :- $\Rightarrow 2 \times 3 = 2^3$

$= 8 \in N$

Associative :- $\Rightarrow (a \times b) \times c = a \times (b \times c)$

$a^b \times c = a \times b^c$ 2, 3, 4

$a^{bc} \neq a^{b^c}$

$2^{3(4)} \neq 2^{3^4}$

$2^{12} \neq 2^{81}$

It is not a semigroup bcoz, it is not satisfying Associative property.

2) The set $(Z, *)$ is a set of integers where $a*b = \max(a,b)$ is a semigroup or not.

Sol: Closure :- $2*3 = \max(2,3)$

$= 3$ ✓

associative :- $(a*b)*c = a*(b*c)$

$\max(a,b)*c = a*\max(b,c)$

$\max(2,3)*5 = 2*\max(3,5)$

$3*5 = 2*5$

$\max(3,5) = \max(2,5)$

$5 = 5$ ✓

It is a semigroup as it satisfies all properties.

3) If Q^+ is set of all rational numbers and $a*b = \frac{ab}{3}$ then $(Q^+, *)$ is an Abelian group. which of the following are not true.

(a) $e = 3$ (b) $a^{-1} = 9/a$ (c) $(2/3)^{-1} = 6$ (d) $3^{-1} = 3$

Sol:

(a) $a*e = e*a = a$

$e*e = a$ — (1)

$a*e = \frac{ae}{3}$ — (2)

$a = \frac{ae}{3}$

$\boxed{e = 3}$

(b) $a*a^{-1} = e$

$a*a^{-1} = \frac{aa^{-1}}{3}$

$3 = \frac{aa^{-1}}{3}$

$9 = aa^{-1}$

$\boxed{a^{-1} = 9/a}$

(c) $3^{-1} = 9/3 \Rightarrow$ from (b)

$\boxed{3^{-1} = 3}$

(d) $(2/3)^{-1} = 9 \times \frac{3}{2}$

$\frac{3}{2} = 9 \times \frac{3}{2}$

$\boxed{1 \neq 9}$

It is not true.

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* **Finite Group:-** A group with finite no. of elements is called finite group.

Eg: $S = \{0, 1\}$, $+$

+	0	1
0	0	1
1	1	(2)

$0, 1 \in S$
$0+1 \in S$
$1+0 \in S$
$0+0 \in S$
$1+1 \notin S$

closure property is failed.
∴ It is not a group.

$S = \{0, 1\}$, $\times$

×	0	1
0	0	0
1	0	1

Closure is satisfied
Associative is also satisfied
Inverse is not satisfied.
Hence, It is not a group.

$S = (-1, 1)$, $\times$

×	-1	1
-1	1	-1
1	-1	1

all are satisfied
∴ it is a group & finite group

* Find the cube root of unity.

$\{1, \omega, \omega^2\}$, $\times$

×	1 (inverse)	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	$\omega^3 = 1$
ω^2	ω^2	1	ω

Closure is satisfied.
Associative, Inverse, Identity are also satisfied.
∴ It is a finite group

* 4th root of unity

$\{1, -1, i, -i\}$, $\times$

×	1 (inverse of 1)	-1	i	-i (inverse of i)
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

All properties are satisfied.
∴ It is a finite group.

* Addition Modulo :-

m → modulo.

$$a +_m b = \begin{cases} a+b, & \text{if } (a+b<m) \\ (a+b)-m, & \text{if } (a+b \geq m) \end{cases}$$

m=4

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$(a+b)+c = a+(b+c)$

$(1+2)+3 = 1+(2+3)$

$6 = 6$

∴ It is a finite group.

a +

A + e = a

e = a - a

e = 0

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

* Multiplication Modulo :-

$$a \times_m b = \begin{cases} a \times b & \text{if } (a \times b) < m \\ (a \times b) \%_m & \text{if } (a \times b \geq m) \end{cases}$$

(a×b) %. m

m=5

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

∴ It is a finite group.

5) 8 (1
5
3

5) 6 (1
5
1

m=5

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

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* Subgroup:-

If 'H' is a non empty subset of a group 'G' then 'H' is a subgroup of G. If H is a group under the same operation as G.

→ H is a subset of G.
→ H is a group.
→ H and G has to use same binary operation.

Eg: 1) $G = \langle Z, + \rangle$

$H = \langle 5Z, + \rangle$

(i) $H \subset G$

H is subset of G but $H \neq G$

(ii) H is a group

(iii) H and G use the same binary operation '+'

∴ H is a subgroup of G.

2) $G = \langle R, + \rangle$

$H = \langle R^*, \times \rangle$

R^+ except 0 All no's has to be included.

(i) $H \subset G$

(ii) H is a group.

(iii) H and G are not using same binary operation.

∴ H is not subgroup of G.

3) $G = \langle Z, +_4 \rangle$

$H = \langle \{0, 2\}, +_4 \rangle$

G

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

H

+	0	2
0	0	2
2	2	0

(i) $H \subset G$

(ii) H is a group

(iii) H, G has same binary operations.

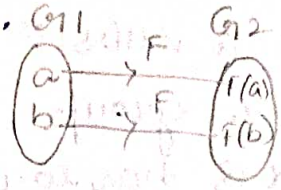
∴ H is a subgroup of G.

* Homomorphism :-

Let $<G_1, o>$ & $<G_2, *>$ be two groups then F is a function $f : G_1 \to G_2$ defined by

$F(a \circ b) = F(a) * F(b)$ $\forall a, b \in G_1$ then

F is said to be Homomorphism.



→ $F: G_1 \to G_2$

1) One to One

2) onto

3) homomorphism, then we say isomorphism if it satisfies all the 3 rules.

Eg: 1) $G_1 = (R, +)$ $\quad F(x) = e^x$.

$G_2 = (R, \times)$

Sol:

$x, y \in R$ i.e. G_1

$F(x+y) = e^{x+y} = e^x \cdot e^y$

$= e^x \cdot e^y$.

$F(x+y) = f(x) \times f(y)$

$G_1 \qquad G_2$

$\therefore$ F is a homomorphism.

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2) $G = \{-1, 1\}$ be a multiplicative group and $(Z, +), (G, \cdot)$

$$F(x) = \begin{cases} 1 & \text{if } x \text{ is even} \\ -1 & \text{if } x \text{ is odd.} \end{cases}$$

Sol:

Case 1: $x, y \in Z$

when x and y are even

$F(x+y) = 1$

$= 1 \cdot 1$

$= F(x) \cdot F(y)$

Case 2: when x and y are odd

$F(x+y) = 1$

$= -1 \cdot -1$

$= F(x) \cdot F(y)$

Case 3: when x is even & y is odd

$F(x+y) = -1$

$= 1 \cdot -1$

$= F(x) \cdot F(y)$

name --

Case 4: when x is odd and y is even

$F(x+y) = -1$

$= -1 \cdot 1$

$= F(x) \cdot F(y)$.

$F: G_1 \to G_2$ is +homomorphism.

* Isomorphism:-

Let (G_1, o) and (G_1', o') are two groups then a function $f: (G_1, o) \to (G_1', o')$ is called an isomorphism if

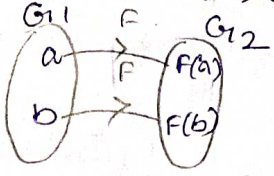
1) f is homomorphism

2) f is one to one and onto

then G_1' is called isomorphic image of G_1.

Sol: $a, b \in G_{11}$

$F(a o b) = F(a) o' (F(b))$



all rules are satisfied. So, it is a isomorphism.

Eg: 1) Let 'R' be the additive group of Real numbers and R^+ be the multiplicative group of +ve Real numbers and f is from $F: R \to R^+$ defined by $F(x) = e^x \ \forall x \in R$ show that $R \cong R^+$

Sol: $G_{11} = (R, +)$

$G_{12} = (R^+, \times)$

$F: R \to R^+$

$F(x) = e^x$

① $x, y \in R$

$f(x+y) = F(x) \cdot F(y)$

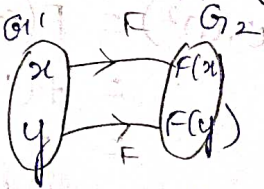
LHS $F(x+y) = e^{x+y}$

$= e^x \cdot e^y$

$= F(x) \cdot F(x)$

$=$ RHS

② One to one mapping:-



$F(x) = F(y)$

$e^x = e^y$ (apply logs)

$x = y$

Continue.....

1) Let z belongs to R^+ be an arbitrary element then there exits atleast one element, $\log z \in R$, such that

$F(\log z) = e^{\log z}$

$\therefore$ F is onto mapping.

$\therefore$ F is isomorphic.

Hence, $R \cong R^+$

2) Let $G_1 = \{1, \omega, \omega^2\}$, $\times$ show that $G_1 \cong G_1'$

$G_1' = \{0, 1, 2, 3\}$, $+_3$

Sol:

G_1

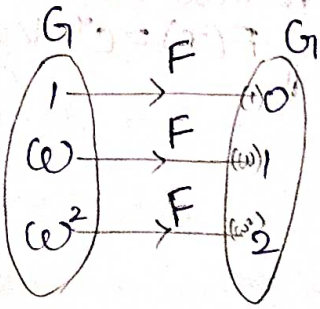
$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

G_1'

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$1 \to 0$ $\quad \omega \to 1$ $\quad \omega^2 \to 2$

$F(1) = 0$ $\quad F(\omega) = 1$ $\quad F(\omega^2) = 2$



It is one to one mapping.

$\omega, \omega^2 \in G_1$

$F(\omega, \omega^2) = F(\omega^3)$

$= f(1)$

$= 0$

$= 1 + 2$

$= F(\omega) + F(\omega^2)$

It is homomorphism.

$\therefore$ Hence $G_1 \cong G_1'$ and F is isomorphic

Rings and types of Rings :-

An Algebraic system $(R, +, \cdot)$ is called Ring if it satisfies (R is set; + and · are binary operations)

i) $(R, +)$ is an abelian group with identity 0.

ii) $(R, \cdot)$ is an semi group.

iii) Multiplication is distributive over addition.

$a \cdot (b+c) = a \cdot b + a \cdot c$ → left distributive

$(b+c) \cdot a = b \cdot a + c \cdot a$ → right distributive

→ Properties for a non empty set are R is a ring with respect to binary operation $(\cdot, +)$

$(R, +)$ is an abelian group.

i) closure :- $\forall a, b \in R$

$a+b \in R$

ii) Associative :- $\forall a, b, c \in R$

$(a+b)+c = a+(b+c)$

iii) identity : $a \in R$

$a+e = e+a = a$

iv) inverse : $a + a^{-1} = a^{-1} + a = e$

v) Commutative : $\forall a, b \in R$

$a+b = b+a$

$(R, \cdot)$ is a semi group.

i) closure :- $\forall a, b \in R$

$a \cdot b \in R$

ii) Associative : $\forall a, b, c \in R$

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

→ Types of Rings :-

1) Commutative Ring
2) Ring with unity element
3) Ring with '0' divisor
4) Ring without zero divisor

1) Commutative Ring :- If multiplication composition in R is commutative $ab = ba \quad \forall a, b \in R$.

2) Ring with unity element :- for any $a \in R$ is called unity element if there exist $b \in R$ such that $ab = ba = 1$ $\forall a, b \in R$

3) Ring with zero divisor :- A ring element a is not equal to zero $(a \neq 0)$ is called a zero divisor if there exist an element $b, b \neq 0$ in the Ring such that either $ab = 0$, $ba = 0$

4) Ring without zero divisor :- if the product of no. of two non zero elements is 0 i.e, if $ab = 0$ either $a = 0$ / $b = 0$ / both $a = b = 0$

Eg:- In the ring $(\mathbb{Z}, +, \cdot)$ and (1 and -1) are the unity elements in which cases it is possible.

Sol:

$\mathbb{Z} = \{1, -1\}$

$ab = 1$

$1 \cdot 1 = 1$ ✓

$-1 \cdot -1 = 1$ ✓

$\cdot$	1	-1
1	1 ✓	-1
-1	-1	1 ✓

Eg: In the ring $(\{0,1,2,3,4\}, +_5, \times_5)$ Show the ring with unity

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

6½.5 6½.5 10½.5

$1 \times_5 1 = 1$

$2 \times_5 3 = 1$

$3 \times_5 2 = 1$

$4 \times_5 4 = 1$