# GRAPH THEORY

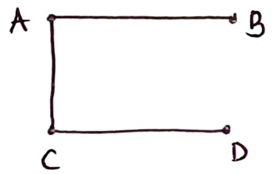
**Def:** A graph is a pair $(V, E)$, where $V$ is a non-empty set and $E$ is a set of unordered pairs of elements taken from the set $V$.

The elements of $V$ are called vertices (or) nodes

" " " $E$ " " undirected edges (or) edges

The graph $(V, E)$ is also denoted by $G = (V, E)$

**Ex:**



In above graph the vertex set $V = \{A, B, C, D\}$

edge set is $E = \{AB, AC, CD\} = \{\{A,B\}, \{A,C\}, \{C,D\}\}$

A graph/digraph containing no edges is called a null graph.

A null graph with only one vertex is called a trivial graph.

## Finite graph/digraph:

A graph with only a finite number of vertices as well as a finite number of edges is called a finite graph/digraph.

otherwise it is called an infinite graph/digraph.

## Order and Size

The number of vertices in a graph is called the order of the graph, it is denoted by $|V|$.

The number of edges in it is called the size, it is denoted by $|E|$

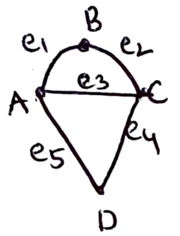
A graph of order $n$ and size $m$ is called a $(n, m)$ graph.

## End vertices:

If $v_i$ and $v_j$ denote two vertices of a graph and if $e_k$ denotes the edge joining $v_i$ and $v_j$ then $v_i$ and $v_j$ are called the end vertices. of $e_k$.

This symbolically written as $e_k = \{v_i, v_j\}$

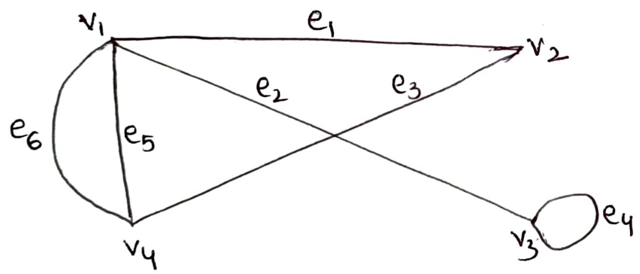
Ex:




$e_1 = \{A, B\} = AB.$

**Loop :-** An edge $e_i$ has the same vertex $v_i$ as both of its end vertices then it is called a loop.

## parallel edges and Multiple edges




Two edges $e_5$ and $e_6$ have the same end vertices $v_1, v_4$. Edges such as these are called parallel edges.

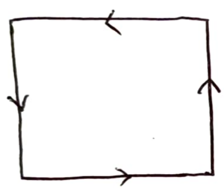
If in a graph there are two or more edges with the same end vertices, the edges are called multiple edges.

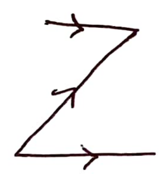
## Directed Graph (or Digraph)

Let $G = (V, E)$ be a graph. If the elements of E are ordered pairs of vertices, then the graph G is called a directed graph.

If e is an edge of a directed graph G, denoted by $e = (u, v)$ then e is a directed edge in G. The vertex u is called the origin or initial point and v is called the destination point or terminal point of e.

Ex:

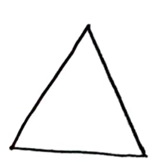


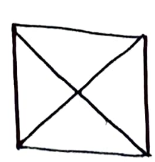


## Non-Directed Graph

Let $G = (V, E)$ be a graph. If the elements of E are unordered pairs of vertices of G then G is called a non-directed graph.

If e is an edge of non-directed graph G, connecting the vertices u and v of G, then it is denoted by $e = \{u, v\}$. The points u and v are called the end points of the edge e.





# Adjacency Matrix of a Digraph

Let G be a digraph with $n$ vertices, containing non-parallel edges. The adjacency matrix $X(G)$ of the digraph G is $n \times n$ matrix defined by

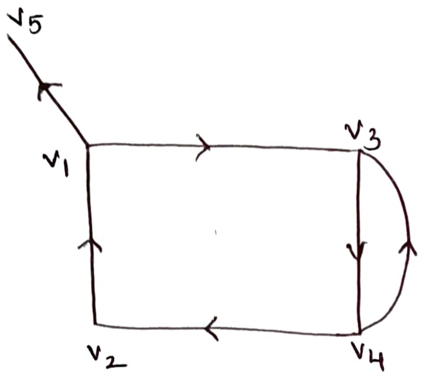
$$X(G) = [m_{ij}]_{n \times n}$$

where $x_{ij} = 1$ if there is an edge directed from $v_i$ to $v_j$

$= 0$ otherwise

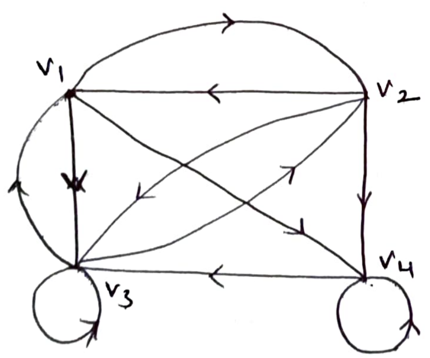
Ex:

1)



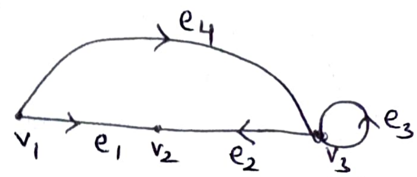

$$X(G) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2)




$$X(G) = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

3) $X(G) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$



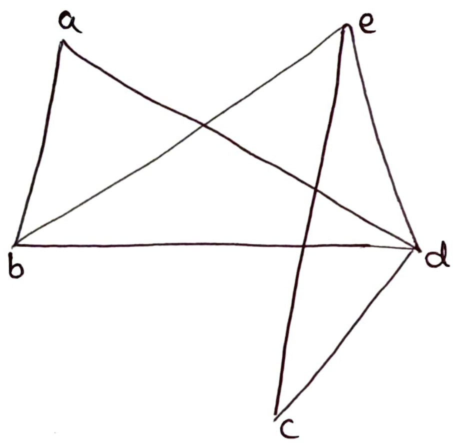
# Adjacency Matrix of a Non-Directed Graph

$$X(G) = [m_{ij}]_{n \times n}$$

$m_{ij} = 1$ if $v_i$ and $v_j$ in $G$ are adjacent

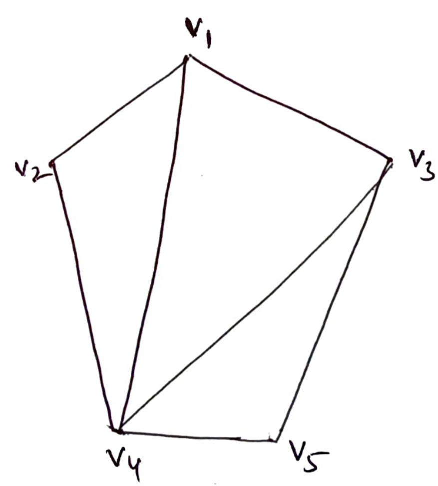
$= 0$ if $v_i$ and $v_j$ in $G$ are not adjacent.

**Ex 1:**




$$X(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$



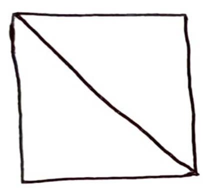
## Simple Graph, Multigraph, General graph

A graph which does not contain loops and multiple edges is called a simple graph.

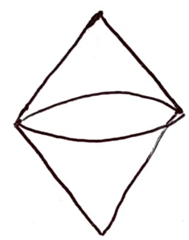
A graph which contains multiple edges but no loops is called a multigraph.

A graph which contains multiple edges or loops is called a general graph.

**Ex:**



Simple Graph



Multigraph

## Incidence :-

When a vertex v of a graph G is an end vertex of an edge e of the graph G, we say that the edge e is incident to the vertex v.

**NOTE:** 1) edge is incident only on two vertices but a vertex may be incident with any number of edges.

2) If two edges have a common vertex then they are called adjacent edges.

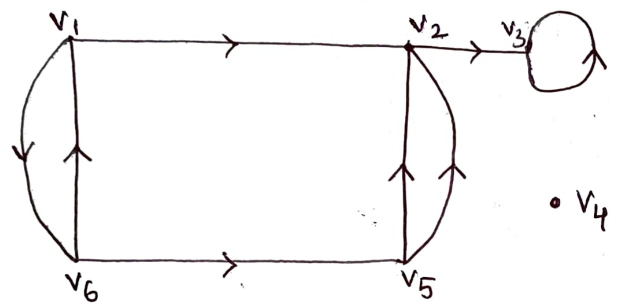
3) Two vertices are said to be adjacent vertices if there is an edge joining them.

**Ex:**

## In-degree and out-degree

If 'v' is a vertex of a digraph D, the number of edges for which v is the initial vertex is called the out-degree of v and the number of edges for which v is the terminal vertex is called the in-degree of v.

The out-degree of v is denoted by $d^+(v)$ and the in-degree of v is denoted by $d^-(v)$.




$d^+(v_1) = 2$ ; $d^-(v_1) = 1$

$d^+(v_2) = 1$ ; $d^-(v_2) = 3$

$d^+(v_3) = 1$ ; $d^-(v_3) = 2$

$d^+(v_4) = 0$ ; $d^-(v_4) = 0$

$d^+(v_5) = 2$ ; $d^-(v_5) = 1$

$d^+(v_6) = 2$ ; $d^-(v_6) = 1$

$\sum_{i=1}^{6} d^+(v_i) = 8$ ; $\sum_{i=1}^{6} d^-(v_i) = 8$

1) In every digraph D, the sum of the out-degrees of all vertices is equal to sum of the in-degrees of all vertices, each sum being equal to the number of edges in D. (First theorem of the Digraph theory)

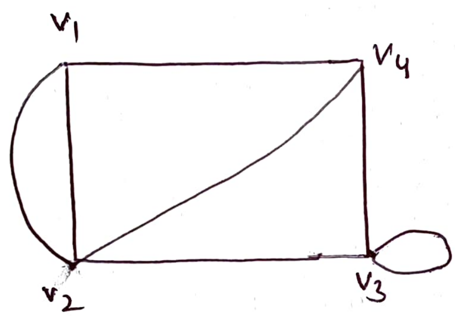
2) If $d^+(v) = d^-(v) = 0$ if v is an isolated vertex.

## vertex degree

Let $G = (V, E)$ be a graph and $v$ be a vertex of $G$. Then, the number of edges of $G$ that are incident on $v$ with the loops counted twice is called the degree of the vertex $v$ and is denoted by $\deg(v)$ or $d(v)$.

→ The degrees of the vertices of a graph arranged in non-decreasing order is called the degree sequence of the graph.

→ the minimum of the degrees of vertices of a graph is called the degree of the graph.




$d(v_1) = 3$ ; $d(v_2) = 4$ ; $d(v_3) = 4$, $d(v_4) = 3$

degree sequence → 3, 3, 4, 4.

degree of the above graph is $\underline{3}$

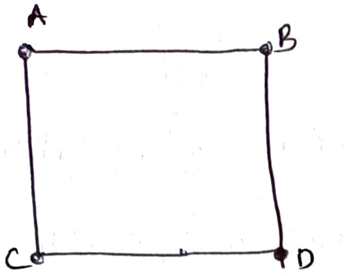
## Pendant vertex

A vertex of degree 1 is called a pendant vertex. An edge incident on a pendant vertex is called a pendant edge.

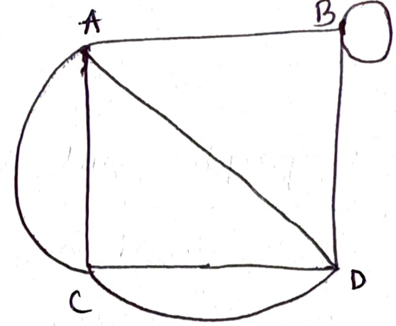
## Regular graph

A graph in which all the vertices are of the same ~~order~~ degree K is called a regular graph of degree K or a K-regular graph.

Ex:




2-regular graph



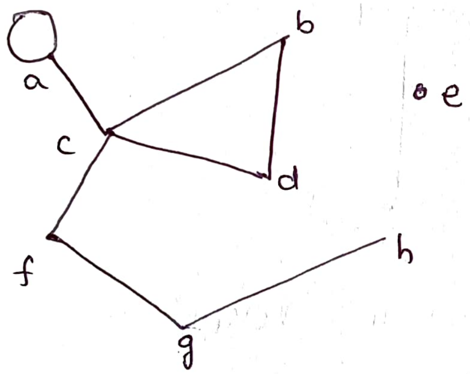

4-regular graph.

## Hand Shaking property

The sum of the degrees of all the vertices in a graph is an even number, and this number is equal to twice the number of edges in the graph.

i.e., $$\sum_{v \in V} \deg(v) = 2|E|$$

Ex:




$d(a) = 3$, $d(b) = 2$, $d(c) = 4$, $d(d) = 2$, $d(e) = 0$, $d(f) = 2$, $d(g) = 2$

$d(h) = 1$

$$\sum \deg(v) = 3+2+4+2+0+2+2+1 = 16 \text{ even}$$

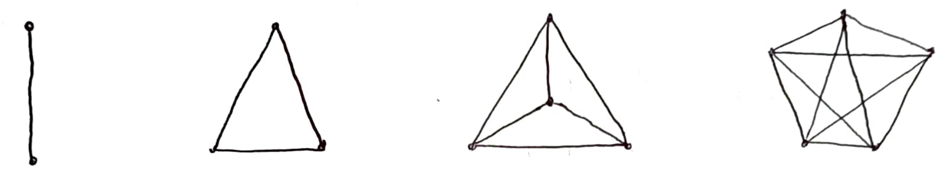
$$= 2 \times 8 \quad (\text{no. of edges } 8)$$

This verifies the handshaking property for the given graph

## Complete graph

A simple graph of order $\geq 2$ in which there is an edge between every pair of vertices is called a complete graph or full graph. In other words, a complete graph is a simple graph in which every pair of distinct vertices are adjacent.

A complete graph with $n \geq 2$ vertices is denoted by $K_n$



a) $K_2$ b) $K_3$ c) $K_4$ d) $K_5$

Complete graph $K_5$ is also called Kuratowski's first graph.

## Bipartite graph

A graph $G = (V, E)$ is a ~~Bi~~ bipartite graph, if the vertex set $V$ can be partitioned into subsets $V_1$ and $V_2$ such that every edge in $E$ connects to a vertex in $V_1$ and a vertex in $V_2$ (no edge in $G$ connects either two vertices in $V_1$ or two vertices in $V_2$) is called a Bipartite graph.

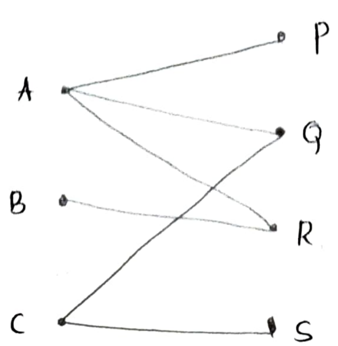
NOTE: 1) A bipartite graph does not contain any loops

2) A bipartite graph can be denoted by $K_{m,n}$

$m \rightarrow$ contains the vertices in subset $V_1$

$n \rightarrow$ contains the vertices in subset $V_2$

Ex:




$V_1 = \{A, B, C\}$

$V_2 = \{P, Q, R, S\}$

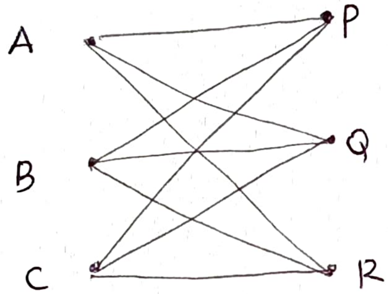
It is denoted by $K_{3,4}$

# Complete Bipartite graph

A bipartite graph $G = (V_1, V_2 ; E)$ is called a complete bipartite graph if there is an edge between every vertex in $V_1$ and every vertex in $V_2$.

Ex: 1) $K_{2,3}$ (above example) is a bipartite graph.

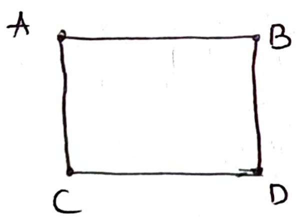
2)




$K_{3,3}$ is a complete bipartite graph (Kuratowski's second graph).

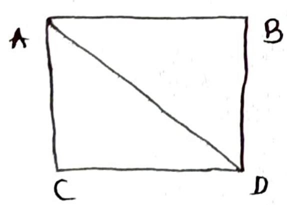
3) A complete bipartite graph is denoted by $K_{m,n}$ then $K_{m,n}$ has $m+n$ vertices and $mn$ edges.

4)




It is a ~~pa~~ bipartite graph with $V_1 = \{A, D\}$ and $V_2 = \{B, C\}$

5)

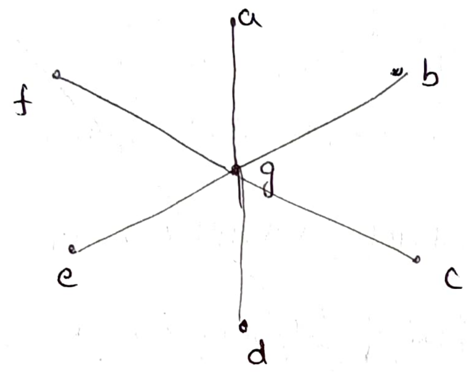



Not bipartite

6) The graph $K_{4,7}$ has $4+7=11$ vertices and $4 \times 7 = 28$ edges.

## Star graph

A complete bipartite graph $K_{1,n}$ is called a star graph.



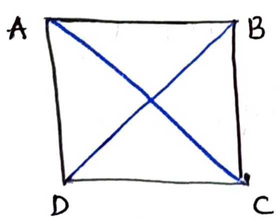

Star graph $K_{1,6}$.

## Isomorphism

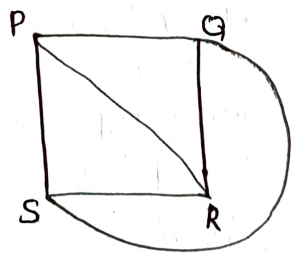
Consider two graphs $G = (V, E)$ and $G' = (V', E')$. Suppose there exists a function $f : V \longrightarrow V'$ such that (i) $f$ is one-to-one and onto, and (ii) for all vertices $A, B$ of $G$, the edge $\{A, B\} \in E$ if and only if the edge $\{f(A), f(B)\} \in E'$. Then $f$ is called an isomorphism between $G$ and $G'$, and we say that $G$ and $G'$ are isomorphic graphs.

In other words, two graphs $G$ and $G'$ are said to be isomorphic if there is a one-to-one correspondence between their vertices and between their edges such that the adjacency of vertices is preserved.

Ex:




$\underline{\underline{G}}$




$\underline{\underline{G'}}$

one-to-one correspondence between the vertices of these two graphs.

$A \leftrightarrow P$, $B \leftrightarrow Q$, $C \leftrightarrow R$, $D \leftrightarrow S$

under this correspondence, the edges in the two graphs correspond with each other; as indicated below.

$\{A,B\} \leftrightarrow \{P,Q\}$, $\{A,C\} \leftrightarrow \{P,R\}$, $\{A,D\} \leftrightarrow \{P,S\}$

$\{B,C\} \leftrightarrow \{Q,R\}$, $\{B,D\} \leftrightarrow \{Q,S\}$, $\{C,D\} \leftrightarrow \{R,S\}$

above indicated one-to-one correspondence between the vertices/edges of the two graphs preserve the adjacency of the vertices.

2)

These two graphs have the same number of vertices but different number of edges, there cannot be a one-to-one correspondence between the edges.

NOTE: i) If two graphs are isomorphic then they must have

a) Both graphs have the same number of vertices

b) Both graphs have the same number of edges.

c) An equal number of vertices with a given degree

These conditions are necessary but not sufficient.

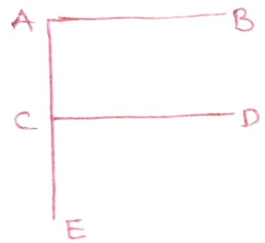
2) Let $G=(V,E)$ and $G'=(V',E')$ be any two graphs and $f: G \to G'$ is an isomorphism. If $v \in V$ then $\deg(v) = \deg(f(v))$

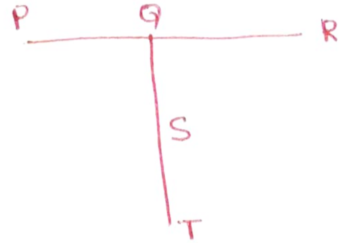
3) Two complete graphs with n vertices ($K_n$) are isomorphic

4) Two complete bipartite graphs ($K_{r,s}$) are isomorphic

5) If $\chi(G) \neq \chi(G')$, then $f: G \to G'$ is not an isomorphism.

Check whether the following graphs are isomorphic or not




No. of vertices : 5

No. of edges : 4

Degree sequence

(1, 1, 1, 2, 3)

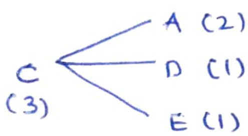
No. of vertices = 5

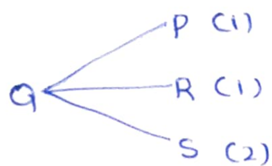
No. of edges = 4

Degree sequence

(1, 1, 1, 2, 3)

<u>Vertex preserving</u>






A ⟷ S

B ⟷ T

C ⟷ Q

D ⟷ P

E ⟷ R

<u>Edges preserving</u>

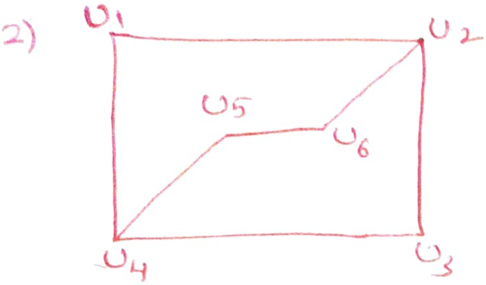
Adjacency Matrix for given graphs

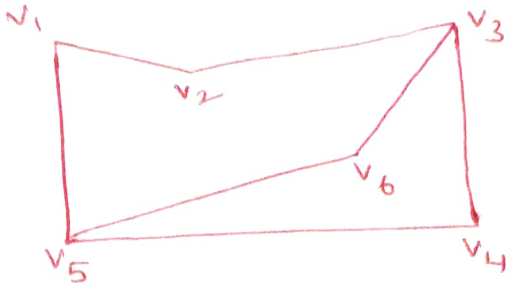
| | A | B | C | D | E |
|---|---|---|---|---|---|
| A | 0 | 1 | 1 | 0 | 0 |
| B | 1 | 0 | 0 | 0 | 0 |
| C | 1 | 0 | 0 | 1 | 1 |
| D | 0 | 0 | 1 | 0 | 0 |
| E | 0 | 0 | 1 | 0 | 0 |

| | S | T | Q | P | R |
|---|---|---|---|---|---|
| S | 0 | 1 | 1 | 0 | 0 |
| T | 1 | 0 | 0 | 0 | 0 |
| Q | 1 | 0 | 0 | 1 | 1 |
| P | 0 | 0 | 1 | 0 | 0 |
| R | 0 | 0 | 1 | 0 | 0 |

∴ Given graphs are isomorphic

2)




No. of vertices : 6

No. of edges : 7

Degree sequence

2, 3, 2, 3, 2, 2,

No. of vertices : 6

No. of edges : 7

Degree sequence

2, 2, 3, 2, 3, 2

Vertex preserving

$u_1$ (2) < $u_2$ (3), $u_4$ (3)

$u_2$ (2) < $u_1$ (2), $u_3$ (2), $u_6$ (2)

$u_3$ (2) < $u_2$ (3), $u_4$ (3)

$u_4$ (3) < $u_1$ (2), $u_3$ (2), $u_5$ (2)

$u_5$ (2) < $u_4$ (3), $u_6$ (2)

$u_6$ (2) < $u_2$ (3), $u_5$ (2)

$v_1$ (2) < $v_2$ (2), $v_5$ (3)

$v_2$ (2) < $v_3$ (3), $v_1$ (2)

$v_3$ (3) < $v_1$ (2), $v_4$ (2), $v_6$ (2)

$v_4$ (2) < $v_3$ (3), $v_2$ (2)

$v_5$ (3) < $v_1$ (2), $v_4$ (2), $v_6$ (2)

$v_6$ (2) < $v_3$ (3), $v_5$ (3)

$u_1 \leftrightarrow v_6$

$u_2 \leftrightarrow v_3$

$u_3 \leftrightarrow v_4$

$u_4 \leftrightarrow v_5$

$u_5 \leftrightarrow v_1$

$u_6 \leftrightarrow v_2$

Edges preserving

adjacency Matrix

| | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ |
|---|---|---|---|---|---|---|
| $u_1$ | 0 | 1 | 0 | 1 | 0 | 0 |
| $u_2$ | 1 | 0 | 1 | 0 | 0 | 1 |
| $u_3$ | 0 | 1 | 0 | 1 | 0 | 0 |
| $u_4$ | 1 | 0 | 1 | 0 | 1 | 0 |
| $u_5$ | 0 | 0 | 0 | 1 | 0 | 1 |
| $u_6$ | 0 | 1 | 0 | 0 | 1 | 0 |

| | $v_6$ | $v_3$ | $v_4$ | $v_5$ | $v_1$ | $v_2$ |
|---|---|---|---|---|---|---|
| $v_6$ | 0 | 1 | 0 | 1 | 0 | 0 |
| $v_3$ | 1 | 0 | 1 | 0 | 0 | 1 |
| $v_4$ | 0 | 1 | 0 | 1 | 0 | 0 |
| $v_5$ | 1 | 0 | 1 | 0 | 1 | 0 |
| $v_1$ | 0 | 0 | 0 | 1 | 0 | 1 |
| $v_2$ | 0 | 1 | 0 | 0 | 1 | 0 |

$\therefore$ given graphs are isomorphic.

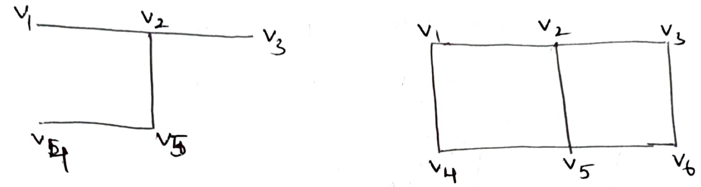
## Subgraph

If H is a subgraph of G

a) All the vertices and all the edges of H are in G

b) each edge of H has the same end vertices in G as in H

Ex:



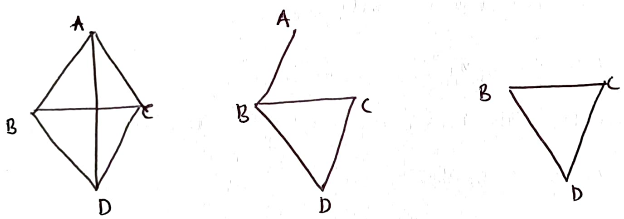

H G

H is a subgraph of G.

## Spanning subgraph

A subgraph H of a graph G is called a spanning subgraph of G if $V(H) = V$ i.e., H contains all the vertices of G.

Ex:

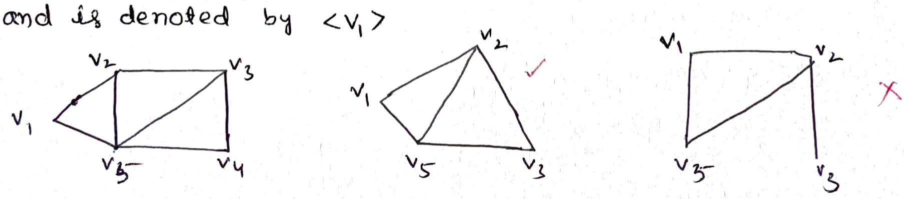



G $H_1$ $H_2$

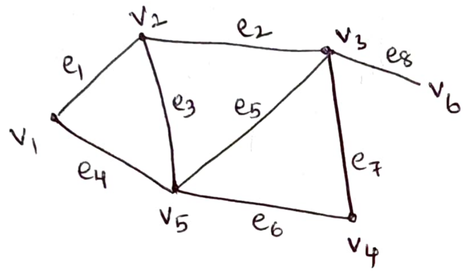
$H_1$ is a spanning subgraph of G and $H_2$ is a subgraph but not a spanning subgraph.

## Induced subgraph

Given a graph $G = (V, E)$, suppose there is a subgraph $G_1 = (V_1, E_1)$ of G such that every edge $\{A, B\}$ of G, where $A, B \in V_1$ is an edge of $G_1$ also. Then $G_1$ is called a subgraph of G induced by $V_1$ and is denoted by $\langle V_1 \rangle$



Walk: Consider a finite, alternating sequence of vertices and edges of the form $v_i\, e_j\, v_{i+1}\, e_{j+1}\, v_{i+2} \cdots e_k\, v_m$ which begins and ends with vertices and which is such that each edge in the sequence is incident on the vertices preceding and following it in the sequence. Such a sequence is called a walk in G.




- The sequence $v_1 e_1 v_2 e_2 v_3 e_8 v_6$ is a walk of length 3

- Closed walk: A walk that begins and ends at the same vertex is called a closed walk.

  Ex: $v_1 e_1 v_2 e_3 v_5 e_4 v_1$

Open walk: A walk that is not closed is called an open walk

Ex: $v_1 e_1 v_2 e_2 v_3 e_5 v_5$

Trail: If in an open walk no edge appears more than once, then the walk is called a trail.

Ex: $v_1 e_4 v_5 e_3 v_2 e_2 v_3 e_5 v_5 e_6 v_4$

Circuit: A closed walk in which no edge appears more than once is called a circuit.

Ex: $v_1 e_1 v_2 e_3 v_5 e_5 v_3 e_7 v_4 e_6 v_5 e_4 v_1$

Path: A trail in which no vertex appears more than once is called a path.

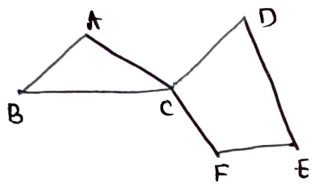
Ex: $v_1 e_1 v_2 e_3 v_5 e_5 v_3 e_7 v_4$

**Cycle:** A circuit with no other repeated vertices except its end points is called a cycle.

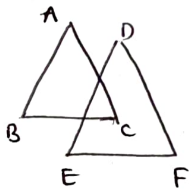
Ex: $v_2 e_2 v_3 e_5 v_5 e_3 v_2$

**Connected Graph:** A graph G is said to be connected if there is at least one path between every two distinct vertices in G.

Ex:




connected




Disconnected (no path from A to D $v_1$ to $v_4$)

## Euler Circuits and Euler Trails

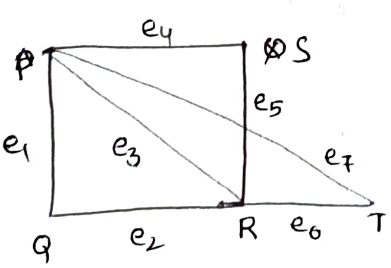
Consider a connected graph G. If there is a circuit in G that contains all the edges of G, then that circuit is called an Euler circuit.

If there is a trail in G that contains all the edges of G then that trail is called Euler trail.

A connected graph that contains an Euler circuit is called an Euler graph or Eulerian graph.

A connected graph that contains an Euler trail is called a semi-Euler graph.

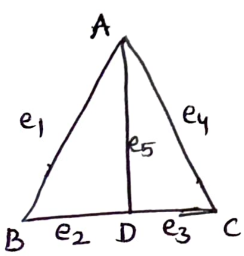
Ex:




The closed walk

$P e_1 Q e_2 R e_3 P e_4 S e_5 R e_6 T e_7 P$

is an Euler circuit. Therefore, this graph is Euler graph.




This graph has no Euler circuits. But this is a semi-Euler graph because Euler-trail exist $A e_1 B e_2 D e_3 C e_4 A e_5 D$

NOTE: 1) A connected graph G has an Euler circuit if and only if all vertices of G are of even degree.

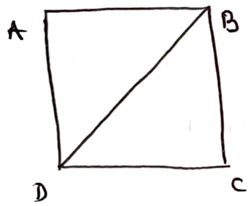
2) A connected graph contains an Euler trail iff it has exactly zero or two vertices of odd degree.

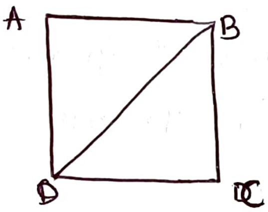
## Hamilton cycle

Let G be a connected graph. If there is a cycle in G that contains all the vertices of G then that cycle is called a Hamilton cycle in G.

A graph that contains a Hamilton cycle is called a Hamilton graph.

Ex: 1)



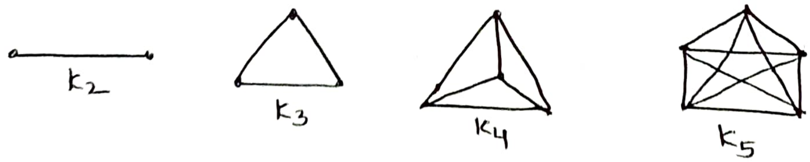

ABCDA is a Hamilton cycle

ABADC → Hamilton path.

NOTE: The complete graph $K_n$, $n \geq 3$ is a Hamilton graph.

## Planar graph

A graph which can be represented by at least one plane drawing in which the edges meet only at the vertices is called a planar graph.



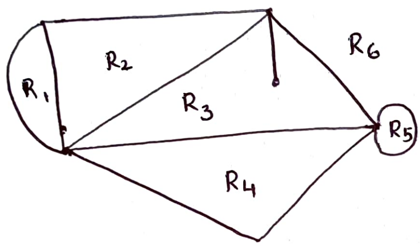

$K_2, K_3, K_4$ are planar but $K_5$ is ~~not~~ non-planar graph.

# Euler's Formula

A connected planar graph G with $|V| = n$ vertices and $|E| = m$ edges has exactly $m-n+2$ regions in all of its diagrams

$$r = m - n + 2 \quad \text{or} \quad n - m + r = 2$$

1) verify Euler's formula for the planar graph ~~shown~~ given below




The given graph has $n=6$ vertices, $m=10$ edges & $r=6$ regions

Thus, $n - m + r = 6 - 10 + 6 = 2$

the Euler's formula is verified.