Saturday 19-11-2022

• **Rank of a matrix:** Rank of a matrix is the number of non-zero rows in the matrix.

- Let A is an $m\times n$ matrix. If A is null matrix we define its rank as '0' (zero).
- If A is a non-zero matrix, we say that 'r' is the rank of 'A', if:
  - (a) every $(r+1)$th order, minor is zero
  - (b) there exist atleast one (r)th order, minor of A which is not zero.
- Rank of 'A' is denoted by ~~$\rho(A)$~~ '$\rho(A)$'.

note:
- every matrix will have a rank
- rank of a matrix is always unique.
- Rank of $A > r=1$, when A is non-zero matrix.
- If 'A' is a matrix of order $m\times n$, then $\rho(A) \leq \min(m,n)$.
- Rank of the identity matrix I_n is n.
- If 'A' is a matrix of order $m\times n$ and 'A' is non-singular matrix, then rank of A is equal to n. $\rho(A)=n$
- $\rho(A) = \rho(A)^T \rightarrow$ A transpose.
- If 'A' and 'B' are equivalent matrices, then rank of A is equal to B $\rho(A)=\rho(B)$.

▸ **Zero and non-zero row:** If all the elements in a row of a matrix are zeros then it is called as 'zero row'.

- If there is atleast 1 non-zero element in a row, then it is called 'non-zero row'.

eg. $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 - \text{zero row} \end{matrix}$ } non-zero rows $\boxed{\rho(A)=2}$

▸ **methods to find rank of a matrix:**

1. Echelon form (or) Triangular form
2. Normal form
3. Normal form of the type PAQ.

• **To find the rank by using Echelon form:-**

step 1: In the given matrix, first element is always 1 (a_{11}).

step 2: By using only row operations, given matrix has to be changed into upper triangular matrix.

step 3: After completing the row operations, we calculate the rank.

- **note:** In the matrix, when $(a_{11}) \neq 1$, by interchanging the rows we can replace the (a_{11}) as 1.

$\rightarrow (a_{11})$ pivotal element.

Q1: reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}_{4\times4}$
$R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$, $R_4 \rightarrow R_4 - 6R_1$;

$\therefore A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$ $R_2 \leftrightarrow R_3$

$\therefore A \approx \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$ $R_4 \rightarrow R_4 - R_2$

$\therefore A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$ $R_4 \rightarrow R_4 - R_3$

$\therefore A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank of A = no. of non-zero rows

$\boxed{\rho(A)=3}$

Q2. Find the rank of matrix by using Echlon form. $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

• **Solution:**

$\therefore R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 4R_1$, $R_4 \rightarrow R_4 - 4R_1$; $A \approx \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$

$\therefore R_4 \rightarrow R_4 - 3R_3$;

$A \approx \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank of A = no. of non-zero rows

$\boxed{\rho(A)=3}$

Q3. Find rank of $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ by Echolon form.

• **Solution:**

$R_1 \leftrightarrow R_2$

$A \approx \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ $R_3 \rightarrow R_3 - 3R_1$, $R_4 \rightarrow R_4 - R_1$

$\therefore A \approx \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$ $R_3 \rightarrow R_3 - R_2$, $R_4 \rightarrow R_4 - R_2$

$\therefore A \approx \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank of A = no. of non-zero rows.

$\boxed{\therefore \rho(A)=2}$

Q4. Find the rank of $A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$ By Echolon form?

Q4. Soln:

$R_2 \rightarrow R_2 + 2R_1$, $R_3 \rightarrow R_3 + R_1$, $R_4 \rightarrow R_4 + 3R_1$ $\therefore A \approx \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 10 & 10 & 0 & 10 \\ 0 & 15 & 15 & 0 & 15 \end{bmatrix}$

$R_3 \rightarrow R_3 - 2R_2$, $R_4 \rightarrow R_4 - 3R_2$ $\therefore A \approx \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Rank of A = no. of non-zero rows.

$\boxed{\therefore \rho(A)=2}$

Q5. Find the value of K, if the rank of "A" is 2 where $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & K & 0 \end{bmatrix}$

• **Solution:** $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & K & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$\therefore A \approx \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & K & 0 \end{bmatrix}$

$R_3 \rightarrow R_3 - 3R_1$, $R_4 \rightarrow R_4 - R_1$ $\therefore A \approx \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & (k-1) & -1 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_2$, $R_4 \rightarrow R_4 - R_2$ $\therefore A \approx \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (K+2) & 0 \end{bmatrix}$

$\boxed{\rho(A)=2}$ ....(given)

$\therefore (a_{43}) = 0$ ....($\because R_3$ and R_4 are zero rows)

$\therefore (K+2) = 0$

$\boxed{\therefore K=-2}$

$\therefore$ The value of K in the given matrix is -2.

... To be continued

Q6 $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & K & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ Find the value of K if $\rho(A) = 2$.

• **Soln:** $R_2 \leftrightarrow R_3$

$\therefore A \approx \begin{bmatrix} 1 & 1 & -1 & 1 \\ 3 & 1 & 0 & 1 \\ 1 & -1 & K & -1 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$, $\therefore A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & -2 & (K+1) & -2 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\therefore A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & (K-2) & 0 \end{bmatrix}$

$\because \rho(A) = 2$ ....(given)

$\therefore$ R_3 is a zero row.

$\therefore (a_{33}) = 0$

$\therefore K - 2 = 0$ $\boxed{\therefore K = 2}$

$\therefore$ The value of K in the given matrix is 2.

• **Normal form:-**

Step 1: Given matrix can be checked if its in correct order or not.

Step 2: If its not in correct order, by interchanging rows and columns, we can correct it.

Step 3: By applying row and column operations, we can reduce the given matrix into identity matrix.

Step 4: Normal form of the matrix is $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where 'r' is the rank of the matrix.

Q1. Find the rank of matrix A, by reducing it into its Normal form where A.

$A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$

• **Soln:** $C_1 \leftrightarrow C_2$

$\therefore A \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 0 & 4 & 1 \\ 3 & 2 & 7 & 5 \\ 5 & 2 & 11 & 6 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 5R_1$ $A \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -5 & -11 \\ 0 & -4 & -2 & -7 \\ 0 & -8 & -4 & -14 \end{bmatrix}$

$C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$, $C_4 \to C_4 - 4C_1$ $A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & -11 \\ 0 & -4 & -2 & -7 \\ 0 & -8 & -4 & -14 \end{bmatrix}$

$C_2 \to \frac{C_2}{-2}$, $C_3 = \frac{C_3}{-1}$, $C_4 = \frac{C_4}{-1}$

$A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 5 & 11 \\ 0 & 2 & 2 & 7 \\ 0 & 4 & 4 & 14 \end{bmatrix}$

$R_3 \to 3R_3 - 2R_2$, $R_4 \to 3R_4 - 4R_2$ $A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 5 & 11 \\ 0 & 0 & -4 & -1 \\ 0 & 0 & -8 & -2 \end{bmatrix}$

$C_3 \to 3C_3 - 5C_2$, $C_4 \to 3C_4 - 11C_2$ $A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -12 & -3 \\ 0 & 0 & -24 & -6 \end{bmatrix}$

$C_3 \Rightarrow \frac{C_3}{-12}$ $C_4 \to \frac{C_4}{-3}$ $C_2 \to \frac{C_2}{3}$

$A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

$R_4 \to R_4 - 2R_2$ $A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_4 \to C_4 - C_3$

$\therefore A \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is the normal form of A i.e.

$A \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ $\because r = 3$

$\boxed{\therefore \rho(A) = 3}$

The rank of the given matrix A is 3.

Q2. By reducing the matrix $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & 7 \end{bmatrix}$ into normal form and find its rank.

• **Soln:** $R_2 \leftrightarrow R_1$

$A \approx \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & 7 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 6R_1$ $A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 31 \end{bmatrix}$

$C_2 \to C_2 + C_1$, $C_3 \to C_3 + 2C_1$, $C_4 \to C_4 + 4C_1$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 31 \end{bmatrix}$

$R_3 \to 5R_3 - 4R_2$, $R_4 \to 5R_4 - 9R_2$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 92 \end{bmatrix}$

$C_3 \to \frac{C_3}{3}$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 1 & 7 \\ 0 & 0 & 11 & 22 \\ 0 & 0 & 11 & 92 \end{bmatrix}$

$R_3 \to \frac{R_3}{11}$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 1 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 11 & 92 \end{bmatrix}$

$C_3 \to 5C_3 - C_2$, $C_4 \to 5C_4 - 7C_2$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 55 & 460 \end{bmatrix}$

$C_2 \to \frac{C_2}{5}$ $C_3 \to \frac{C_3}{5}$, $C_4 \to \frac{C_4}{10}$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 11 & 46 \end{bmatrix}$

$R_4 \to R_4 - 11R_3$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 35 \end{bmatrix}$

$C_4 \to C_4 - C_3$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 35 \end{bmatrix}$

$R_4 \to \frac{R_4}{35}$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore$ Normal form of $A = [I_4]$

$\boxed{\therefore \rho(A) = 4}$

Q3. (Q.8 H.W): Use normal form and find rank.

• **Soln:** $A = \begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

$R_1 \leftrightarrow R_2$ $A \sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 2 & -4 & 3 & -1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_4 \to R_4 - 4R_1$ $A \sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 8 & 12 & -3 \end{bmatrix}$

$C_2 \to C_2 + 2C_1$, $C_3 \to C_3 + C_1$, $C_4 \to C_4 + 4C_1$, $C_5 \to C_5 - 2C_1$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 8 & 12 & -3 \end{bmatrix}$

$R_2 \leftrightarrow R_3$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 1 & 8 & 12 & -3 \end{bmatrix}$

$R_4 \to R_4 - R_2$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 0 & 9 & 9 & -4 \end{bmatrix}$

$C_3 \to C_3 + C_2$, $C_4 \to C_4 - 3C_2$, $C_5 \to C_5 - C_2$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 0 & 9 & 9 & -4 \end{bmatrix}$

$C_5 \to \frac{C_5}{-4}$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 7 & 1 \\ 0 & 0 & 9 & 9 & 1 \end{bmatrix}$

$R_4 \to 5R_4 - 9R_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 7 & 1 \\ 0 & 0 & 0 & -18 & -4 \end{bmatrix}$$

$C_4 \to 5C_4 - 7C_3$
$C_5 \to 5C_5 - C_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 90 & -20 \end{bmatrix}$$

$C_3 \to \frac{C_3}{5}$ $\quad R_4 \to \frac{R_4}{10}$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 9 & 2 \end{bmatrix}$$

$C_4 \to \frac{C_4}{9}$ $\quad C_5 \to \frac{C_5}{2}$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$C_5 \to C_5 - C_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Normal form $= \begin{bmatrix} I_4 & 0 \\ 0 & 0 \end{bmatrix}$

$\boxed{\therefore\ \rho(A) = 4}$

- **Normal form of the type PAQ:-**

If A is an $m \times n$ matrix of rank (r) then there exists 2 non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ then its called PAQ form of matrix A.

Step 1 - We write $A_{m\times n} = I_m A I_n$

Step 2 - reduce the matrix A on LHS to normal form by applying elementary operations.

Step 3 - Every elementary row operation on A must be accompanied by same operation on the prefactor of RHS.

Step 4 - Every elementary column operation on A must be accompanied by same operation on the post factor of RHS.

Q1 Find non-singular matrices P and Q so that PAQ is of normal form where

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}_{3\times 4}$$

- soln: We write $A_{3\times4} = I_3 A I_4$

$$\begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 10 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 + 2C_1$, $C_3 \to C_3 - 3C_1$, $C_4 \to C_4 - 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 10 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \leftrightarrow C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 10 & 10 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -4 & -3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -4 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$R_2 \to \frac{R_2}{5}$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{1}{5} & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -4 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$ $\quad \boxed{\rho(A) = 2}$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{1}{5} & 0 \\ -3 & -2 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & -4 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

Q2. If $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ find non-singular matrix with PAQ in normal form.

- Soln: $A_{3\times4} = I_3 A I_4$

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -5 & -1 & -2 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_1$, $C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & -1 & -2 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 5 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to \frac{C_2}{-1}$ $\quad C_3 \to \frac{C_3}{-1}$ $\quad C_4 \to \frac{C_4}{-1}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 1 & 2 \\ 0 & 1 & 5 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 5 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 5 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 5 \\ -2 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q2. Soln: $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}_{3\times4}$

$A_{3\times4} = I_3 A I_4$

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -5 & -1 & -2 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, $C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & -1 & -2 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & -10 \\ 0 & -5 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to \frac{R_2}{-1}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & -5 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & 24 & 48 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -5 & 1 & 8 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - 5C_2$ $\quad C_4 \to C_4 - 10C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 24 & 48 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -5 & 1 & 8 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 8 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \to C_4 - 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 24 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -5 & 1 & 8 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to \frac{C_3}{24}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -5 & 1 & 8 \end{bmatrix} A \begin{bmatrix} 1 & -1 & \frac{1}{6} & 0 \\ 0 & 1 & -\frac{5}{24} & 0 \\ 0 & 0 & \frac{1}{24} & -2 \\ 0 & 0 & \frac{1}{4} & 1 \end{bmatrix}$$

The Normal form of $A = [I_3]$

$\boxed{\therefore\ \rho(A) = 3}$

Also,

$$P = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -5 & 1 & 8 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & -1 & 1/6 & 0 \\ 0 & 1 & -5/24 & 0 \\ 0 & 0 & 1/24 & -2 \\ 0 & 0 & 1/4 & 1 \end{bmatrix}$$

• Inverse of a matrix by elementary operations [Gauss-Jordan method]:

- Suppose A is a non-singular square matrix of order 'n'. we write $A_{n\times n} = I_n A$.
- Now we apply elementary row operations only to the matrix A and the prefactor 'I_n' of the RHS.
- We will do this process till we get an equation of the form $I_n = BA$.

$B = \frac{I_n}{A}$ $\boxed{\therefore B = A^{-1}}$

Q1. Find the inverse of A, my Gauss Jordan method:

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}_{3\times 3}$$

$$A_{3\times3} = I_3 A$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 1$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 3R_2$, $\cancel{R}$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$$\therefore \frac{I_3}{A} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Q2. Find inverse of matrix A by Gauss-Jordan's method.

$$A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}_{4\times 4}$$

$$A_{4\times4} = I_4 A$$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & -3 & 3 & -1 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 + R_1$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & -7 & 4 & -3 \\ 0 & 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} A$$

$R_3 \to 2R_3 - 7R_2$, $R_4 \to R_4 + R_2$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -7 & -11 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ -7 & -11 & 2 & 0 \end{bmatrix} A$$

$R_4 \to R_4 + 6R_3$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$R_2 \to R_2 + R_4$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 6 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_3$, $R_1 \to R_1 + R_3$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 1 \\ -2 & -2 & 2 & 4 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$R_2 \to \frac{R_2}{-2}$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$R_1 \to R_1 - R_2$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$$\frac{I_4}{A} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}.$$

~~Q3. Find inverse of matrix by Gauss Jordan.~~

A = ~~1 -2~~

02-12-2022

• System of linear equations:

1. Homogenous linear equations.
2. Non-homogenous linear equations.

1. Homogenous linear equations:

Consider the system of eqns.

$a_1x + b_1y + c_1z = 0$

$a_2x + b_2y + c_2z = 0$

$a_3x + b_3y + c_3z = 0$

The above system of eqns can be written in matrix form i.e. $\boxed{ax = 0}$ where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ - co-efficient matrix.}$$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ - variable matrix}$$

$$O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ null matrix}$$

- The system $Ax = 0$ is always consistent (solution exists) this solution is called Trivial solution.

2. Non-homogenous linear equations:

- Consider the system of equations.

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

The above system of equations in matrix form i.e. $\boxed{Ax = B}$ where

A = co-efficient matrix

x = variable matrix

$$B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ - constant matrix.}$$

• **Agumented matrix**: combination of co-efficient matrix and constant matrix is called agumented matrix.

$$[A|B] \text{ or } [A:B] = \begin{bmatrix} a_1 & b_1 & c_1 : d_1 \\ a_2 & b_2 & c_2 : d_2 \\ a_3 & b_3 & c_3 : d_3 \end{bmatrix}$$

• Types of solutions for non-homogenous linear equations:

1) If $\boxed{\rho(A) = \rho(A|B) = n}$ then the system is consistent and has unique solution. [n = no. of variables]
2) If $\boxed{\rho(A) = \rho(A|B) < n}$ then the system is consistant and has infinite solutions.
3) If $\boxed{\rho(A) \neq \rho(A|B)}$ then the system is inconsitent and has no solution.

• Solve the system of linear equations:

1] $x+2y+2z=2,$
$3x-2y-z=5,$
$2x-5y+3z=-4,$
$x+4y+6z=0.$

• Soln: The above system of equation we write in matrix form $AX=B$

$$\begin{bmatrix}1&2&2\\3&-2&-1\\2&-5&3\\1&4&6\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\5\\-4\\0\end{bmatrix}$$

The agumented matrix $[A|B]$ is

$$[A|B]=\left[\begin{array}{ccc:c}1&2&2&2\\3&-2&-1&5\\2&-5&3&-4\\1&4&6&0\end{array}\right]$$

Apply row equations to get upper triangular matrix.

$R_2\to R_2-3R_1,$
$R_3\to R_3-2R_1,$
$R_4\to R_4-R_1.$
$$[A|B]=\left[\begin{array}{ccc:c}1&2&2&2\\0&-8&-7&-1\\0&-9&-1&-8\\0&2&4&-2\end{array}\right]$$

$R_3\to 8R_3-9R_2$
$R_4\to 4R_4+R_2,$
$$[A|B]=\left[\begin{array}{ccc:c}1&2&2&2\\0&-8&-7&-1\\0&0&55&-55\\0&0&9&-9\end{array}\right]$$

$R_3\to \frac{R_3}{55}$, $R_4\to\frac{R_4}{9}$

$$[A|B]=\left[\begin{array}{ccc:c}1&2&2&2\\0&-8&-7&-1\\0&0&1&-1\\0&0&1&-1\end{array}\right]$$

$R_4\to R_4-R_3$

$$[A|B]=\left[\begin{array}{ccc:c}1&2&2&2\\0&-8&-7&-1\\0&0&1&-1\\0&0&0&0\end{array}\right]$$ This is in Echolon Form.

$\boxed{\rho(A|B)=3}$ ....(no. of non-zero rows).

Now, we write $[A|B]$ in $AX=B$ form.

$\boxed{\rho(A)=3}$ $\boxed{\text{no. of variables}=3}$

$\boxed{\therefore\ \rho(A|B)=\rho(A)=n=3}$

It has unique solution.
↳ The system of equation.

Now we write $(A|B)$ in $AX=B$ form

$$\begin{bmatrix}1&2&2\\0&-8&-7\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\-1\\-1\\0\end{bmatrix}$$

using matrix multiplication, we get.

$x+2y+2z=2$ ....(1)
$-8y-7z=-1$ ....(2)
$\boxed{z=-1}$ ....(3)

put $z=-1$ in eqn (2), we get
$-8y-7(-1)=-1$
$-8y+7=-1$
$-8y=-8$
$\boxed{\therefore y=1}$

put $z=-1$, $y=1$ in eqn (1)
$x+2(1)+2(-1)=2$
$x+2-2=2$
$\boxed{x=2}$

Hence, the solution for the given system of equations is $x=2,\ y=1,\ z=-1$.

2] $x+y+z=6$
$2x+3y-2z=2$
$5x+y+2z=13.$

•Soln: The above system of equation can be written in matrix form as $AX=B$

$$\begin{bmatrix}1&1&1\\2&3&-2\\5&1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\2\\13\end{bmatrix}$$

The agumented matrix $[A|B]$ is

$$\left[\begin{array}{ccc:c}1&1&1&6\\2&3&-2&2\\5&1&2&13\end{array}\right]=[A|B].$$

Applying R.T. to get echolon form,

$R_2\to R_2-2R_1$
$R_3\to R_3-5R_1$
$$[A|B]\approx\left[\begin{array}{ccc:c}1&1&1&6\\0&1&-4&-10\\0&-4&-3&-17\end{array}\right]$$

$R_3\to R_3+4R_2$
$$[A|B]\approx\left[\begin{array}{ccc:c}1&1&1&6\\0&1&-4&-10\\0&0&-19&-57\end{array}\right]$$

$R_3\to\frac{R_3}{-19}$
$$[A|B]\approx\left[\begin{array}{ccc:c}1&1&1&6\\0&1&-4&-10\\0&0&1&3\end{array}\right]$$

~~$R_4\to 4R_4+R$~~ This is in Echolon for

$\therefore\ \rho(A|B)=3$
$\rho(A)=3$
no. of variables $=3$
$\boxed{\therefore\ \rho(A|B)=\rho(A)=n=3}$

∴ The system of equations has a unique soln.

Writing $(A|B)$ in $AX=B$ form,

$$\begin{bmatrix}1&1&1\\0&1&-4\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\-10\\3\end{bmatrix}$$

using matrix multiplication, we get.

$x+y+z=6$ ... (1)
$y-4z=-10$ ... (2)
$\boxed{z=3}$ ... (3)

put $z=3$ in eqn (2).
$y-4(3)=-10$
$\therefore y-12=-10$
$\boxed{\therefore y=2}$

put $y=2$ $z=3$ in eq (1).
$x+2+3=6$
$x+5=6$
$\boxed{x=1}$

∴ The solution for given system of equation is $(x,y,z)=(1,2,3)$.

3] $x+2y+z=2$
$3x+y-2z=1$
$4x-3y-z=3$
$2x+4y+2z=4$

Ans. $x=1$
$y=0$
$z=1$

•Soln: The above system in matrix form is written as $AX=B$

$$\begin{bmatrix}1&2&1\\3&1&-2\\4&-3&-1\\2&4&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\1\\3\\4\end{bmatrix}$$

This in agumented form is,

$$[A|B]=\left[\begin{array}{ccc:c}1&2&1&2\\3&1&-2&1\\4&-3&-1&3\\2&4&2&4\end{array}\right]$$

$R_4\to\frac{R_4}{2}$
$$[A|B]=\left[\begin{array}{ccc:c}1&2&1&2\\3&1&-2&1\\4&-3&-1&3\\1&2&1&2\end{array}\right]$$

$R_2\to R_2-3R_1$
$R_3\to R_3-4R_1$
$R_4\to R_4-R_1$
$$[A|B]=\left[\begin{array}{ccc:c}1&2&1&2\\0&-5&-5&-5\\0&-11&-5&-5\\0&0&0&0\end{array}\right]$$

$R_2\to\frac{R_2}{-5}$
$$[A|B]=\left[\begin{array}{ccc:c}1&2&1&2\\0&1&1&1\\0&-11&-5&-5\\0&0&0&0\end{array}\right]$$

$R_3\to R_3+11R_2$
$$[A|B]=\left[\begin{array}{ccc:c}1&2&1&2\\0&1&1&1\\0&0&6&6\\0&0&0&0\end{array}\right]$$

$R_3\to\frac{R_3}{6}$
$$[A|B]=\left[\begin{array}{ccc:c}1&2&1&2\\0&1&1&1\\0&0&1&1\\0&0&0&0\end{array}\right]$$

$\rho(A)=3$ $\rho(A|B)=3$ $n=3$

$\boxed{\therefore\ \rho(A)=\rho(A|B)=n}$

The system is consistent and has unique solution.

This can be written in matrix form as,

$$\begin{bmatrix}1&2&1\\0&1&1\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\1\\1\end{bmatrix}$$

By matrix multiplication,

$x+2y+z=2,$ ... (1)
$y+z=1$ ... (2)
$\boxed{z=1}$

put $z=1$ in eqn (2)
$y+1=1$
$\boxed{y=0}$

put $y=0$ $z=1$ in eq (1)
$x+2(0)+1=2$
$x+0=2-1$
$\boxed{x=1}$

∴ The solution of the given system is $(x,y,z)=(1,0,1)$.

• Find whether the following system of equations are consistent, if so, solve 'em:

1] $5x+3y+7z=4;$
$3x+26y+2z=9;$
$7x+2y+10z=5;$

• Soln: This system of eqns can be written in matrix form as $AX=B$

$$\begin{bmatrix}5&3&7\\3&26&2\\7&2&10\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\9\\5\end{bmatrix}$$

The agumented matrix is:

$$[A|B]=\left[\begin{array}{ccc:c}5&3&7&4\\3&26&2&9\\7&2&10&5\end{array}\right]$$

$R_2\to 5R_2-3R_1$, $R_3\to 5R_3-7R_1$
$$[A|B]\approx\left[\begin{array}{ccc:c}5&3&7&4\\0&121&-11&33\\0&-11&1&-3\end{array}\right]$$

~~$R_3\to 11R_3+$~~ $R_2\to \frac{R_2}{11}$
$$[A|B]\approx\left[\begin{array}{ccc:c}5&3&7&4\\0&11&-1&3\\0&-11&1&-3\end{array}\right]$$

$R_3\to R_3+R_2$
$$[A|B]\approx\left[\begin{array}{ccc:c}5&3&7&4\\0&11&-1&3\\0&0&0&0\end{array}\right]$$

This is in Echolon form

$\rho(A)=\cancel{3}2 \quad \rho(A|B)=2 \quad n=3$

$\boxed{\therefore \rho(A)=\rho(A|B)<n}$

∴ The system is consistent and has infinite solutions.

Write $[A|B]$ in $AX=B$ form.

$$\therefore \begin{bmatrix}5&3&7\\0&11&-1\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\3\\0\end{bmatrix}$$

By using matrix multiplication

$5x+3y+7z=4$ .... ①

$11y-z=3$ ..... ②

~~$z=11y-3$~~

Put $z=K$ in eqn ③

eq ② $\Rightarrow 11y-K=3$

$$\boxed{\therefore y=\frac{3+K}{11}}$$

put $z=k$ $y=\frac{3+k}{11}$ in eq ①

$5x+3\left[\frac{3+k}{11}\right]+7k=4$

$\therefore 5x+\frac{9+3k}{11}+7k=4$

$\therefore 5x+\frac{80k+9}{11}=4$

$\therefore 5x=4-\left(\frac{80k+9}{11}\right)$

$\therefore 5x=\frac{44-80k-9}{11}$

$\therefore 5x=\frac{-80k-35}{11}$

$$\boxed{\therefore x=\frac{-16k-7}{11}}$$

∴ The ~~solution is~~ infinite solutions of the given system are

$x=\frac{-16k-7}{11}$, $y=\frac{3+k}{11}$, $z=k$.

2] $3x-y+4z=3;$
$x+2y-3z=-2;$
$6x+5y-5z=-3;$

This system has infinite no. of solutions.

• Soln: The system of eqns can be written in matrix form as $AX=B$

$$\begin{bmatrix}3&-1&4\\1&2&-3\\6&5&-5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\-2\\-3\end{bmatrix}$$

The agumented matrix is:

$$[A|B]=\left[\begin{array}{ccc:c}3&-1&4&3\\1&2&-3&-2\\6&5&-5&-3\end{array}\right]$$

$R_1\leftrightarrow R_2$
$$[A|B]=\left[\begin{array}{ccc:c}1&2&-3&-2\\3&-1&4&3\\6&5&-5&-3\end{array}\right]$$

$R_2\to R_2-3R_1$, $R_3\to R_3-6R_1$
$$[A|B]=\left[\begin{array}{ccc:c}1&2&-3&-2\\0&-7&13&9\\0&-7&13&9\end{array}\right]$$

$R_3\to R_3-R_2$
$$[A|B]=\left[\begin{array}{ccc:c}1&2&-3&-2\\0&-7&13&9\\0&0&0&0\end{array}\right]$$

This is in Echolon form

$\therefore \rho(A)=2 \quad \rho[A|B]=2 \quad n=3$

$\boxed{\therefore \rho(A)=\rho(A|B)<n}$

∴ The system has infinite solutions

The matrix $[A|B]$ is written in $AX=B$ again.

$$\begin{bmatrix}1&2&-3\\0&-7&13\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2\\9\\0\end{bmatrix}$$

By matrix multiplication, we get

$x+2y-3z=-2$ .... ①

$-7y+13z=9$ ...... ②

put $z=k$ in eq ②

$-7y+13K=9$

$-7y=9-13K$

$$\boxed{y=\frac{-9+13k}{7}}$$

put $y=\frac{-9+13k}{7}$ $z=k$ in eq ①

$x+2\left(\frac{-9+13k}{7}\right)-3k=-2$

$x-\frac{18+26k}{7}-3k=-2$

$x-\frac{18+26k-21k}{7}=-2$

$x-\frac{18+5k}{7}=-2$

$x=-2+\frac{18+5k}{7}$

$x=\frac{-14+18+5k}{7}$

$$\boxed{\therefore x=\frac{4+5k}{7}}$$

∴ The solution of the given system of eqns. is $x=\frac{4+5k}{7}$,

$y=\frac{-9+13k}{7}$, $z=k$.

3] Test for the consistency of the system,

$x+y+z=1,$
$x-y+2z=1,$
$x-y+2z=5,$
$2x-2y+3z=1.$

~~given system of eqns are~~

• Soln: The above system of eqns can be written into matrix form

$\boxed{AX=B}$

$$\begin{bmatrix}1&1&1\\1&-1&2\\1&-1&2\\2&-2&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\1\\5\\1\end{bmatrix}$$

The agumented form is,

$$[A|B]=\left[\begin{array}{ccc:c}1&1&1&1\\1&-1&2&1\\1&-1&2&5\\2&-2&3&1\end{array}\right]$$

$R_2\to R_2-R_1$, $R_3\to R_3-R_1$, $R_4\to R_4-2R_1$
$$[A|B]=\left[\begin{array}{ccc:c}1&1&1&1\\0&-2&1&0\\0&-2&1&4\\0&-4&1&-1\end{array}\right]$$

$R_3\to R_3-R_2$, $R_4\to R_4-2R_2$
$$[A|B]=\left[\begin{array}{ccc:c}1&1&1&1\\0&-2&1&0\\0&0&0&4\\0&0&-1&-1\end{array}\right]$$

$R_3\leftrightarrow R_4$
$$[A|B]=\left[\begin{array}{ccc:c}1&1&1&1\\0&-2&1&0\\0&0&-1&-1\\0&0&0&4\end{array}\right]$$

$\boxed{\rho(A)=3 \quad \rho(A|B)=4}$

$\boxed{\therefore \rho(A)\neq\rho(A|B)}$

The system is inconsistent and has no solution.

4] Find whether the following system of equations are consistent or not.

$x-4y+7z=14$
$3x+8y-2z=13$
$7x-8y+26z=5$

• Soln: The given system in matrix form $AX=B$ can be written as

$$\begin{bmatrix}1&-4&7\\3&8&-2\\7&-8&26\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}14\\13\\5\end{bmatrix}$$

This in agumented form is

$$[A|B]=\left[\begin{array}{ccc:c}1&-4&7&14\\3&8&-2&13\\7&-8&26&5\end{array}\right]$$

$R_2\to R_2-3R_1$, $R_3\to R_3-7R_1$
$$[A|B]=\left[\begin{array}{ccc:c}1&-4&7&14\\0&20&-23&-29\\0&20&-23&-93\end{array}\right]$$

$R_3 \to R_3 - R_2$ $[A|B] = \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 0 & 20 & -23 & : & 29 \\ 0 & 0 & 0 & : & -64 \end{bmatrix}$

~~$R_3 \to \frac{R_3}{-14}$ $[A|B] = \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 0 & 20 & -23 & : & 29 \\ 0 & 0 & 0 & : & 64 \end{bmatrix}$~~

~~This again in matrix form is~~

$\rho(A) = 3 \quad \rho(A|B) = 4$

$\boxed{\therefore \rho(A) \neq \rho(A|B)}$

∴ The system is Inconsistent and has no solutions.

Q. Discuss for what value λ, μ the simultaneous equations have no solutions, Infinite no. of solutions and unique solution.

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$.

•Soln: The above system of equations In matrix form $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

The agumented matrix is,

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ $[A|B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & (\lambda-1) & : & (\mu-6) \end{bmatrix}$

$R_3 \to R_3 - R_2$ $[A|B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & (\lambda-3) & : & (\mu-10) \end{bmatrix}$

This is in Echolon form.

① Case I. $\lambda = 3$, $\mu = 10$, assumption.

$$[A|B] \approx \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\rho(A) = 2$, $\rho(A|B) = 2$ $n = 3$

$\boxed{\rho(A) = \rho(A|B) < n}$

∴ The system is consistent and has infinite no. of solutions when $\lambda = 3$, $\mu \neq 10$.

② Case II : let $\lambda = 3$, $\mu \neq 10$

$$[A|B] \approx \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & 0 & : & (\mu-10) \end{bmatrix}$$

$\rho(A) = 2 \quad \rho(A|B) = 3$

$\boxed{\rho(A) \neq \rho(A|B)}$

∴ The system is inconsistent and has no solutions when $\lambda = 3$, $\mu \neq 10$.

③ Case III let $\lambda \neq 3$, $\mu \neq 10$.

$$[A|B] \approx \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & (\lambda-3) & : & (\mu-10) \end{bmatrix}$$

$\rho(A) = 3$, $\rho(A|B) = 3$ $n = 3$

$\boxed{\therefore \rho(A) = \rho(A|B) = n}$

∴ The system is consistent and has a unique solution.

Q. Find the values of p and q such that the eqns ~~are~~ have no solution, infinite solutions or unique solution.

$2x + 3y + 5z = 9$

$7x + 3y + 2z = 8$

$2x + 3y + pz = q$;

•Soln: The above system of eqn in matrix form $AX = B$,

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & 2 \\ 2 & 3 & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ q \end{bmatrix}$$

Agumented form is,

$$[A|B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & 2 & : & 8 \\ 2 & 3 & p & : & q \end{bmatrix}$$

$R_2 \to 2R_2 - 7R_1$, $R_3 \to R_3 - R_2$ $[A|B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -15 & -31 & : & -47 \\ 0 & 0 & (p-5) & : & (q-9) \end{bmatrix}$

This is in Echolon form.

• Case - I: Let $p = 5$, $q = 9$.

$$[A|B] \approx \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -15 & -31 & : & -47 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\rho(A) = 2 \quad \rho(A|B) = 2 \quad n = 3$

$\boxed{\rho(A) = \rho(A|B) < n}$

∴ The system is consistent and has infinite solutions when $p = 5, q = 9$.

• Case - II: Let $p = 5$ $q \neq 9$.

$$[A|B] \approx \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -15 & -31 & : & -47 \\ 0 & 0 & 0 & : & (q-9) \end{bmatrix}$$

$\rho(A) = 2 \quad \rho(A|B) = 3 \quad n = 3$

$\boxed{\rho(A) \neq \rho(A|B)}$

∴ The solution is inconsitent and has no solution when $p = 5$, $q \neq 9$.

• Case - III : Let $p \neq 5$, $q \neq 9$.

$$[A|B] \approx \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -15 & -31 & : & -47 \\ 0 & 0 & (p-5) & : & (q-9) \end{bmatrix}$$

$\rho(A) = 3$, $\rho(A|B) = 3$ $n = 3$

$\boxed{\therefore \rho(A) = \rho(A|B) = n = 3}$

∴ The system is consistent and has a unique soln; when $p \neq 5, q \neq 9$.

★ Homogeneous - Linear equations :-

Q. Solve the system of equations

$2x - y + 3z = 0$

$3x + 2y + z = 0$

$x - 4y + 5z = 0$

•Soln: The above system in matrix form is $\boxed{AX = 0}$

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing matrix A into Echolon form.

$R_3 \leftrightarrow R_1$

$$A \approx \begin{bmatrix} 1 & -4 & 5 \\ 3 & 2 & 1 \\ 2 & -1 & 3 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$ $A \approx \begin{bmatrix} 1 & -4 & 5 \\ 0 & 14 & -14 \\ 0 & 7 & -7 \end{bmatrix}$

$R_2 \to \frac{R_2}{14}$, $R_3 \to \frac{R_3}{7}$ $A \approx \begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $A \approx \begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

~~Matrix A is in Echolon form.~~

~~writing again in matrix form~~

Matrix A is in Echolon form

$\boxed{\rho(A) = 2}$ variables $= n = 3$

No. of non-zero solutions $= (n - r) = (3 - 2) = \underline{1}$

∴ The system has unique non-zero solution.

writing in matrix form $AX = 0$

$$\begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

using matrix multiplication, we get equations,

$x - 4y + 5z = 0$ ... eqn①

$y - z = 0$ .... eqn②

$\boxed{\therefore y = z}$ ... eqn③

substituting $y = z$ in eqn①

$x - 4y + 5y = 0$ ...

$x + y = 0$ .... eq④.

substituting $y = z = k$ in eq④.

$x + k = 0$

$\boxed{x = -k}$

The non-zero soln is $x = -k$, $y = z = k$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

2] $x + y - 2z + 3w = 0$

$x - 2y + z - w = 0$

$4x + y - 5z + 8w = 0$

$5x - 7y + 2z - w = 0$

• Soln: The above system of eqns in matrix form is $\boxed{AX=0}$

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

* Reducing matrix A to Echolon form.

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 5R_1$

$$A \approx \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & 4 \\ 0 & -12 & 12 & -16 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, $R_4 \to R_4 - 4R_2$

$$A \approx \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in echolon form

$\boxed{\rho(A) = 2}$ $n = 4$

$(n - r) = (4 - 2) = 2$

The system has 2 non-zero soln.

we rewrite in matrix form $Ax = 0$

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By matrix multiplication, we get

$x + y - 2z + 3w = 0$ ... eqn ①

$-3y + 3z - 4w = 0$ ... eqn ②.

put $z = k_1$, $w = k_2$ in eq ②

$-3y + 3k_1 - 4k_2 = 0$

$-3y = -3k_1 + 4k_2$

$$\boxed{y = \frac{3k_1 - 4k_2}{3}}$$

put $y = \frac{3k_1 - 4k_2}{3}$ $z = k_1$, $w = k_2$ in eqn ①

$$x + \left(\frac{3k_1 - 4k_2}{3}\right) - 2k_1 + 3k_2 = 0$$

$$\frac{3x + 3k_1 - 4k_2 - 6k_1 + 9k_2 = 0}{3}$$

$3x - 3k_1 + 5k_2 = 0$

$3x = 3k_1 - 5k_2$

$$\boxed{x = \frac{3k_1 - 5k_2}{3}}$$

The two non-zero solutions for the given homogenous system is

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} \frac{3k_1 - 5k_2}{3} \\ \frac{3k_1 - 4k_2}{3} \\ k_1 \\ k_2 \end{bmatrix}.$$

• **Gauss Elimination method:**

1] Solve the system of eqn by Gauss-Elimination method.

$2x_1 + x_2 + 2x_3 + x_4 = 6$

$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$

$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$

$2x_1 + 2x_2 - x_3 + x_4 = 10$

• Soln: The agumented matrix of given system

$$[A|B] = \left[\begin{array}{cccc:c} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right]$$

$R_2 \to \frac{R_2}{6}$

$$[A|B] \sim \left[\begin{array}{cccc:c} 2 & 1 & 2 & 1 & 6 \\ 1 & -1 & 1 & 2 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$[A|B] \sim \left[\begin{array}{cccc:c} 1 & -1 & 1 & 2 & 6 \\ 2 & 1 & 2 & 1 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 2R_1$

$$[A|B] \sim \left[\begin{array}{cccc:c} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 7 & -1 & -11 & -25 \\ 0 & 4 & -3 & -3 & -2 \end{array}\right]$$

$R_2 \to \frac{R_2}{3}$

$$[A|B] = \left[\begin{array}{cccc:c} 1 & -1 & 1 & 2 & 6 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 7 & -1 & -11 & -25 \\ 0 & 4 & -3 & -3 & -2 \end{array}\right]$$

$R_3 \to R_3 - 7R_2$, $R_4 \to R_4 - 4R_2$

$$[A|B] = \left[\begin{array}{cccc:c} 1 & -1 & 1 & 2 & 6 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -4 & -11 \\ 0 & 0 & -3 & 1 & 6 \end{array}\right]$$

$R_4 \to R_4 - 3R_3$

$$[A|B] = \left[\begin{array}{cccc:c} 1 & -1 & 1 & 2 & 6 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -4 & -11 \\ 0 & 0 & 0 & 13 & 39 \end{array}\right]$$

$R_4 \to \frac{R_4}{13}$

$$[A|B] = \left[\begin{array}{cccc:c} 1 & -1 & 1 & 2 & 6 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -4 & -11 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

By using backward substitution method,

$\boxed{x_4 = 3}$ ... ①

$-x_3 - 4x_4 = -11$ ... ②

$x_2 - x_4 = -2$ ... ③

$x_1 - x_2 + x_3 + 2x_4 = 6$ ... ④

Substituting $x_4 = 3$ in eq ③ and ②

eq ② $\Rightarrow -x_3 - 4(3) = -11$

$-x_3 - 12 = -11$

$-x_3 = -11 + 12$

$-x_3 = 1$

$\boxed{x_3 = -1}$

eq ③ $\Rightarrow x_2 - 3 = -2$

$x_2 = -2 + 3$

$\boxed{x_2 = 1}$

put $x_2 = 1$, $x_3 = -1$, $x_4 = 3$ in eq ④

eq ④ $\Rightarrow x_1 - 1 - 1 + 2(3) = 6$

$x_1 - 2 + 6 = 6$

$x_1 = 6 - 6 + 2$

$\boxed{x_1 = 2}$

∴ The soln of system of eqn

$x_1 = 2$, $x_2 = 1$, $x_3 = -1$, $x_4 = 3$.

• **Gauss-Seidal Iteration Method:**

► working rule:

- consider the system of eqns

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$;

The above equation we can write as

$$x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right] \; \ldots ①$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right] \; \ldots ②$$

$$x_3 = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right] \; \ldots ③$$

Iterations = No. of approximation.

Take $x_2 = 0$, $x_3 = 0$ as initial value

to be continued

first approximation: $\left.\begin{matrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{matrix}\right\}$ put $x_2 = 0$, $x_3 = 0$ respectively.

second approx: $\left.\begin{matrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{matrix}\right\}$ put $x_1 = x_1^{(1)}$, $x_2 = 0$

Third iteration value: $\left.\begin{matrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{matrix}\right\}$ $x_1 = x_1^{(2)}$, $x_2 = x_2^{(1)}$

Then compare ② and ③ approx values

...continuation

Take $x_2^{(0)} = 0$ $x_3^{(0)}$ as initial values to find $x_1^{(1)}$. For I Iteration

put $x_2 = 0$ and $x_3 = 0$ in eq ①

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right]$$

substituting $x_1 = x_1^{(1)}$, $x_3 = 0$ in eqn ②

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}\right]$$

put $x_1 = x_1^{(1)}$ $x_2 = x_2^{(1)}$ in eq ③

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right]$$

2nd iteration values put

$= x_2^{(1)}$ $x_3 = x_3^{(1)}$ in eq ①

$$x_1^{(2)} = \frac{1}{a_{11}}\left[b - a_{12}x_2^{(1)} - a_{13}x_3^{(1)}\right]$$

...ing putting $x_1 = x_1^{(2)}$ $x_3 = x_3^{(1)}$

$$x_2^{(2)} = \frac{1}{a_{22}}\left[b - a_{21}x_1^{(2)} - a_{23}x_3^{(1)}\right]$$

...t $x_1' = x_1^{(2)}$, $x_2 = x_2^{(2)}$

$$x_3^{(2)} = \frac{1}{a_{33}}\left[b - a_{31}x_1^{(2)} - a_{32}x_2^{(2)}\right]$$

For III iteration

put $x_2 = x_2^{(2)}$ $x_3 = x_3^{(2)}$ in eq ①

$$x_1^{(3)} = \frac{1}{a_{11}}\left[b - a_{12}x_2^{(2)} - a_{13}x_3^{(2)}\right]$$

put $x_1 = x_1^{(3)}$ $x_3 = x_3^{(2)}$ in eq ②

$$x_2^{(3)} = \frac{1}{a_{22}}\left[b - a_{21}x_1^{(3)} - a_{23}x_3^{(2)}\right]$$

put $x_1 = x_1^{(3)}$ $x_2 = x_2^{(3)}$ in eq ③

$$x_3^{(3)} = \frac{1}{a_{33}}\left[b - a_{31}x_1^{(3)} - a_{32}x_2^{(3)}\right]$$

• Upto III iteration is a must.

Hence we get

I iteration	II iteration		III iteration
$x_1^{(1)}$	$x_1^{(2)}$	=	$x_1^{(3)}$
$x_2^{(1)}$	$x_2^{(2)}$	=	$x_2^{(3)}$
$x_3^{(1)}$	$x_3^{(2)}$	=	$x_3^{(3)}$

- If II iteration = III iteration, stop the method and you get your solution
- If any one of the variables x_1, x_2 and x_3 of both iterations are unequal go for further iterations unless you get equal values for successive iteration

Q. Solve the system of eqns by Gauss-Seidel iteration method.

1] $27x + 6y - z = 85$

$6x + 15y + 2z = 72$

$x + y + 54z = 110$;

• Soln: The above system of equations can be written as,

$$x = \frac{1}{27}\left[85 - 6y + z\right] \quad \dots ①$$

$$y = \frac{1}{15}\left[72 - 6x - 2z\right] \quad \dots ②$$

$$z = \frac{1}{54}\left[110 - x - y\right] \quad \dots ③$$

I iteration:

put $y = 0$, $z = 0$ as initial values

put $y = 0$, $z = 0$ in eqn ①, we get

$$x^{(1)} = \frac{1}{27}\left[85 - 6(0) + 0\right]$$

$$x^{(1)} = \frac{1}{27} \times 85 = \frac{85}{27}$$

$\boxed{x^{(1)} = 3.148}$

now put $x = x^{(1)} = 3.148$ $z = 0$ in ②

$$y^{(1)} = \frac{1}{15}\left[72 - 6(3.148) - 2(0)\right]$$

$$y^{(1)} = \frac{1}{15}\left[72 - 18.888 - 0\right]$$

$$y^{(1)} = \frac{1}{15}\left[53.112\right]$$

$\boxed{y^{(1)} = 3.540}$

now put $x^{(1)} = 3.148$, $y^{(1)} = 3.540$ in ③

$$z^{(1)} = \frac{1}{54}\left[110 - 3.148 - 3.540\right]$$

$$z^{(1)} = \frac{1}{54}(103.312)$$

$\boxed{z^{(1)} = 1.913}$

II iteration :-

put $y^{(1)} = 3.540$ $z^{(1)} = 1.913$ in eq ①

$$x^{(2)} = \left[\frac{1}{27}\left[85 - 6(3.540) + 1.913\right]\right]$$

$\boxed{x^{(2)} = 2.432}$

put $x^{(2)} = 2.432$ $z^{(1)} = 1.913$ in eq ②

$$y^{(2)} = \frac{1}{15}\left[72 - 6(2.432) - 2(1.913)\right]$$

$\boxed{y^{(2)} = 3.572}$

put $x^{(2)}$ and $y^{(2)}$ in eq ③

$$z^{(2)} = \frac{1}{54}\left[110 - 2.432 - 3.572\right]$$

$\boxed{z^{(2)} = 1.925}$

III iteration :-

put $y^{(2)}$ and $z^{(2)}$ in eqn ①

$$x^{(3)} = \frac{1}{27}\left[85 - 6(3.572) + 1.925\right]$$

$\boxed{x^{(3)} = 2.425}$

put $x^{(3)}$ and $z^{(2)}$ in eqn ②

$$y^{(3)} = \frac{1}{15}\left[72 - 6(2.425) - 2(1.925)\right]$$

$\boxed{y^{(3)} = 3.573}$

put $x^{(3)}$ and $y^{(3)}$ in eqn ③

$$z^{(3)} = \frac{1}{54}\left[110 - 2.425 - 3.573\right]$$

$\boxed{z^{(3)} = 1.925}$

~~III iterat~~

IV iteration:

put $y^{(3)}$ and $z^{(3)}$ in eqn ①

$$x^{(4)} = \frac{1}{27}\left[85 - 6(3.573) + 1.925\right]$$

$\boxed{x^{(4)} = 2.425}$

put $x^{(4)}$ and $z^{(3)}$ in eq ②

$$y^{(4)} = \frac{1}{15}\left[72 - 6(2.425) - 2(1.925)\right]$$

$\boxed{y^{(4)} = 3.573}$

put $x^{(4)}$ and $y^{(4)}$ in eqn ③

$$z^{(4)} = \frac{1}{54}\left[110 - 2.425 - 3.573\right]$$

$\boxed{z^{(4)} = 1.925}$

III iteration = IV iteration

$x^{(3)} = x^{(4)}$

$y^{(3)} = y^{(4)}$

$z^{(3)} = z^{(4)}$

∴ The solution to the given system is

$$\boxed{\begin{aligned} x &= 2.425 \\ y &= 3.573 \\ z &= 1.925 \end{aligned}}$$

Solve the following system of equations by Gaus-Seidal iteration method.

$$8x_1 - 3x_2 + 2x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$
$$6x_1 + 3x_2 + 12x_3 = 36;$$

Soln:

The above system can also be written as,

$$x_1 = \frac{1}{8}\left[20 + 3x_2 - 2x_3\right] \quad \ldots (1)$$
$$x_2 = \frac{1}{11}\left[33 - 4x_1 + x_3\right] \quad \ldots (2)$$
$$x_3 = \frac{1}{12}\left[36 - 6x_1 - 3x_2\right] \quad \ldots (3)$$

I iteration:

put $x_2 = 0$ $x_3 = 0$ in eqn (1) as initial values

$$x_1^{(1)} = \frac{1}{8}\left[20 + 3(0) - 2(0)\right]$$
$$x_1^{(1)} = \frac{1}{8}[20]$$
$$\boxed{x_1^{(1)} = 2.5}$$

put $x_1^{(1)}$ $x_3 = 0$ in eqn (2),

$$x_2^{(1)} = \frac{1}{11}\left[33 - 4(2.5) + (0)\right]$$
$$= \frac{1}{11}\left[33 - 10\right]$$
$$= \frac{1}{11}(23)$$
$$\boxed{x_2^{(1)} = 2.090}$$

put $x_1^{(1)}$ and $x_2^{(1)}$ in eqn (3),

$$x_3^{(1)} = \frac{1}{12}\left[36 - 6(2.5) - 3(2.090)\right]$$
$$= \frac{1}{12}\left[14.73\right]$$
$$\boxed{x_3^{(1)} = 1.227}$$

II iteration

put $x_2^{(1)}$ and $x_3^{(1)}$ in eq (1)

$$x_1^{(2)} = \frac{1}{8}\left[20 + 3(2.090) - 2(1.227)\right]$$
$$x_1^{(2)} = \frac{1}{8}(23.816)$$
$$\boxed{x_1^{(2)} = 2.977}$$

put $x_1^{(2)}$ and $x_3^{(1)}$ in eq (2)

$$x_2^{(2)} = \frac{1}{11}\left[33 - 4(2.977) + 1.227\right]$$
$$x_2^{(2)} = \frac{1}{11}(22.319)$$
$$\boxed{x_2^{(2)} = 2.029}$$

$x_1^{(2)}$ and $x_2^{(2)}$ we put in eq (3)

$$x_3^{(2)} = \frac{1}{12}\left[36 - 6(2.977) - 3(2.029)\right]$$
$$= \frac{1}{12}(12.051)$$
$$\boxed{x_3^{(2)} = 1.004}$$

III iteration

put $x_2^{(2)}$ and $x_3^{(2)}$ in eq (1).

$$x_1^{(3)} = \frac{1}{8}\left[20 + 3(2.029) - 2(1.004)\right]$$
$$= \frac{1}{8}(24.079)$$
$$\boxed{x_1^{(3)} = 3.009}$$

put $x_1^{(3)}$ and $x_3^{(2)}$ in eq (2)

$$x_2^{(3)} = \frac{1}{11}\left(33 - 4(3.009) + 1.004\right)$$
$$x_2^{(3)} = \frac{1}{11}(21.968)$$
$$\boxed{x_2^{(3)} = 1.997}$$

put $x_1^{(3)}$ and $x_2^{(3)}$ in eq (3)

$$x_3^{(3)} = \frac{1}{12}\left[36 - 6(3.009) - 3(1.997)\right]$$
$$= \frac{1}{12}(11.955)$$
$$\boxed{x_3^{(3)} = 0.996}$$

IV iteration:

put $x_2^{(3)}$ and $x_3^{(3)}$ in eqn (1)

$$x_1^{(4)} = \frac{1}{8}\left(20 + 3(1.997) - 2(0.996)\right)$$
$$= \frac{1}{8}(23.999)$$
$$\boxed{x_1^{(4)} = 2.999} = 3$$

put $x_1^{(4)}$ and $x_3^{(3)}$ in eq (2)

$$x_2^{(4)} = \frac{1}{11}\left[33 - 4(2.999) + 0.996\right]$$
$$= \frac{1}{11}(22)$$
$$\boxed{x_2^{(4)} = 2.000}$$

put $x_1^{(4)}$ and $x_2^{(4)}$ in eq (3).

$$x_3^{(4)} = \frac{1}{12}\left[36 - 6(2.999) - 3(2)\right]$$
$$= \frac{1}{12}(12.006)$$
$$\boxed{x_3^{(4)} = 1.000}$$

V iteration:

put $x_2^{(4)}$ and $x_3^{(4)}$ in eq (1).

$$x_1^{(5)} = \frac{1}{8}\left[20 + 3(2) - 2(1)\right]$$
$$= \frac{1}{8}[24]$$
$$\boxed{x_1^{(5)} = 3}$$

put $x_1^{(5)}$ and $x_3^{(4)}$ in eq (2)

$$x_2^{(5)} = \frac{1}{11}\left[33 - 4(3) + 1\right]$$
$$= \frac{1}{11}(22)$$
$$\boxed{x_2^{(5)} = 2}$$

put $x_1^{(5)}$ and $x_2^{(5)}$ in eq (3)

$$x_3^{(5)} = \frac{1}{12}\left[36 - 6(3) - 3(2)\right]$$
$$= \frac{1}{12}(12)$$
$$\boxed{x_3^{(5)} = 1}$$

~~VI iteration~~

IV iteration = V iteration

$$x_1^{(4)} = x_1^{(5)}$$
$$x_2^{(4)} = x_2^{(5)}$$
$$x_3^{(4)} = x_3^{(5)}$$

$\therefore$ The solution for the given system is $\boxed{x_1 = 3,\ x_2 = 2,\ x_3 = 1}$.

	I	II	III	IV	V
x_1	2.5	2.977	3.009	3	3
x_2	2.090	2.029	1.997	2	2
x_3	1.227	1.004	0.996	1	1

IV = V

$$\boxed{\therefore x_1 = 3 \quad x_2 = 2 \quad x_3 = 1}$$