Flat output:

$$y = (x \; y \; z \; \phi \; \theta \; \psi \; p \; q \; r)^T$$

$$x = \varphi(y, \dot{y}) = (x \; y \; z \; \phi \; \theta \; \psi \; \dot{x} \; \dot{y} \; \dot{z} \; p \; q \; r)^T$$

$$= (y_1 \; y_2 \; y_3 \; y_4 \; y_5 \; y_6 \; \dot{y}_1 \; \dot{y}_2 \; \dot{y}_3 \; y_7 \; y_8 \; y_9)^T$$

$$\ddot{x} = g_{71} u_1 \quad \ddot{y} = g_{81} u_1 \quad \ddot{z} = g + g_{91} u_1$$

$$g_{71}^2 + g_{81}^2 = \frac{1}{m^2}\left(-\sin^2\phi \sin^2\theta + \sin^2\phi + \sin^2\theta\right)$$

$$g_{91}^2 = \frac{1}{m^2}\cos^2\phi \cos^2\theta = \frac{1}{m^2}\left[(1-\sin^2\phi)(1-\sin^2\theta)\right]$$

$$= \frac{1}{m^2}(1 - \sin^2\phi - \sin^2\theta + \sin^2\phi \sin^2\theta)$$

$$\Rightarrow \quad m^2(g_{71}^2 + g_{81}^2 + g_{91}^2) = 1$$

$$u = \alpha(y, \dot{y}, \ddot{y}) = \left(\sqrt{\ddot{x}^2 + \ddot{y}^2 + (\ddot{z} - g)^2}, \; \frac{\dot{p} I_x}{(I_y - I_z) q r}, \; \frac{\dot{q} I_y}{(I_z - I_x) p r}, \; \frac{\dot{r} I_z}{(I_x - I_y) p q}\right)$$

$$= \left(\sqrt{\ddot{y}_1^2 + \ddot{y}_2^2 + (\ddot{y}_3 - g)^2}, \; \frac{\dot{y}_7 I_x}{(I_y - I_z) y_8 y_9}, \; \frac{\dot{y}_8 I_y}{(I_z - I_x) y_7 y_9}, \; \frac{\dot{y}_9 I_z}{(I_x - I_y) y_7 y_8}\right)$$

$$g_{71} = -\frac{1}{m}(\sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi)\sin(\theta))$$

$$g_{81} = -\frac{1}{m}(\cos(\psi)\sin(\phi) - \cos(\phi)\sin(\psi)\sin(\theta))$$

$$g_{91} = -\frac{1}{m}(\cos(\phi)\cos(\theta))$$

$$\dot{s} = f(x) + g(x)u$$

$$f(x) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ p + q\sin(\phi)\tan(\theta) + r\cos(\phi)\tan(\theta) \\ q\cos(\phi) - r\sin(\phi) \\ q\frac{\sin(\phi)}{\cos(\theta)} + r\frac{\cos(\phi)}{\cos(\theta)} \\ 0 \\ 0 \\ g \\ \frac{(I_y - I_z)}{I_x} qr \\ \frac{(I_z - I_x)}{I_y} pr \\ \frac{(I_x - I_y)}{I_z} pq \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_{71} & 0 & 0 & 0 \\ g_{81} & 0 & 0 & 0 \\ g_{91} & 0 & 0 & 0 \\ 0 & \frac{1}{I_x} & 0 & 0 \\ 0 & 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & 0 & \frac{1}{I_z} \end{bmatrix}$$

Block Backstepping

$$\frac{d}{dt}\underbrace{\begin{pmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \end{pmatrix}}_{\dot\eta} = \underbrace{0}_{f(\eta)} + \underbrace{\begin{pmatrix} I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & \begin{matrix} 1 & s\phi t\theta & c\phi t\theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{matrix} \end{pmatrix}}_{G(\eta)} \underbrace{\begin{pmatrix} \dot x \\ \dot y \\ \dot z \\ p \\ q \\ r \end{pmatrix}}_{\xi}$$

$$\frac{d}{dt}\underbrace{\begin{pmatrix} \dot x \\ \dot y \\ \dot z \\ p \\ q \\ r \end{pmatrix}}_{\dot\xi} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ g \\ (I_y - I_z)qr/I_x \\ (I_z - I_x)pr/I_y \\ (I_x - I_y)pq/I_z \end{pmatrix}}_{f_a(\eta,\xi)} + \underbrace{\begin{pmatrix} g_{71} & & 0_{3\times3} & \\ g_{81} & & & \\ g_{91} & & & \\ 0 & 1/I_x & 0 & 0 \\ 0 & 0 & 1/I_y & 0 \\ 0 & 0 & 0 & 1/I_z \end{pmatrix}}_{G_a(\eta,\xi)} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}}_{u}$$

Assume $\xi = \Phi(\eta)$ s.t. $\dot\eta = G\xi$ is AS

$$V_0 = \frac{1}{2}\eta^T\eta$$

$$\dot V_0 = \eta^T\dot\eta = \Phi_1 x + \Phi_2 y + \Phi_3 z + \phi(\Phi_4 + \Phi_5 \sin\phi\tan\theta + \Phi_6\cos\phi\tan\theta)$$
$$+\theta(\Phi_5\cos\phi - \Phi_6\sin\phi) + \psi\left(\Phi_5\frac{\sin\phi}{\cos\theta} + \Phi_6\frac{\cos\phi}{\cos\theta}\right)$$

$$\Phi_1 = -k_1 x \quad \Phi_2 = -k_1 y \quad \Phi_3 = -k_1 z$$

$$\left\{\begin{aligned} &\Phi_4 + \Phi_5\sin\phi\tan\theta + \Phi_6\cos\phi\tan\theta = -k_1\phi \\ &\Phi_5\cos\phi - \Phi_6\sin\phi = -k_1\theta \\ &\Phi_5\frac{\sin\phi}{\cos\theta} + \Phi_6\frac{\cos\phi}{\cos\theta} = -k_1\psi \end{aligned}\right. \Rightarrow \left\{\begin{aligned} &\Phi_4 = -k_1(\phi - \psi\sin\theta) \\ &\Phi_5 = -k_1(\psi\sin\phi\cos\theta + \theta\cos\phi) \\ &\Phi_6 = -k_1(\psi\cos\phi\cos\theta - \theta\sin\phi) \end{aligned}\right.$$

$$z = \xi - \Phi$$

$$V = V_0 + \frac{1}{2}z^Tz$$

$$u = G_a^{-1}\left[\nabla\Phi^T(f + G(\Phi + z)) - f_a - G^T\nabla V_0 - k_2 z\right]$$