The Blasius Function: Computations Before Computers, the Value of Tricks, Undergraduate Projects, and Open Research Problems[*]

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Abstract. The Blasius flow is the idealized flow of a viscous fluid past an infinitesimally thick, semi-infinite flat plate. The Blasius function is the solution to $2f_{xxx} + ff_{xx} = 0$ on $x \in [0, \infty]$ subject to $f(0) = f_x(0) = 0, f_x(\infty) = 1$. We use this famous problem to illustrate several themes. First, although the flow solves a nonlinear partial differential equation (PDE), Toepfer successfully computed highly accurate numerical solutions in 1912. His secret was to combine a Runge–Kutta method for integrating an ordinary differential equation (ODE) initial value problem with some symmetry principles and similarity reductions, which collapse the PDE system to the ODE shown above. This shows that PDE numerical studies were possible even in the precomputer age. The truth, both a hundred years ago and now, is that mathematical theorems and insights are an arithmurgist's best friend, and they can vastly reduce the computational burden. Second, we show that special tricks, applicable only to a given problem, can be as useful as the broad, general methods that are the fabric of most applied mathematics courses: the importance of "particularity." In spite of these triumphs, many properties of the Blasius function $f(x)$ are unknown. We give a list of interesting projects for undergraduates and another list of challenging issues for the research mathematician.

Key words. boundary layer, fluid mechanics, Blasius

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> Thinking is the cheapest and one of the most effective long-range weapons.
> –Field Marshal Sir William Slim, 1st Viscount Slim (1891–1970)

1. Introduction. Slim's maxim applies as well to science as to war. The Blasius problem of hydrodynamics is a good illustration of how cunning can triumph over brute force.

The Blasius flow of a steady fluid current past a thin plate is the solution of the partial differential equation (PDE)

$$(1.1) \qquad \psi_{YYY} + \psi_X \, \psi_{YY} - \psi_Y \psi_{XY} = 0, \qquad X \in [-\infty, \infty], \ Y \in [0, \infty],$$

where subscripts with respect to a coordinate denote differentiation with respect to the

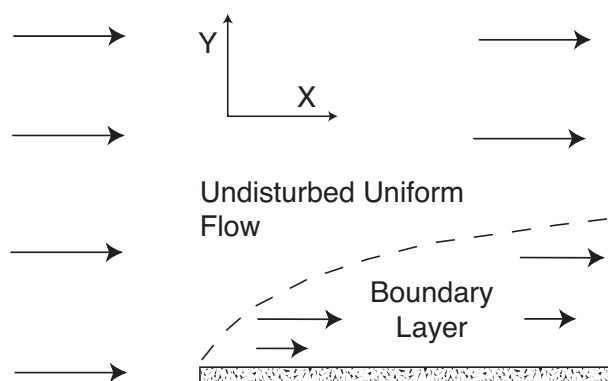[†]Department of Atmospheric, Oceanic and Space Science, University of Michigan, 2455 Hayward Avenue, Ann Arbor, MI 48109 (jpboyd@engin.umich.edu, http://www.engin.umich.edu/~jpboyd/).

Fig. 1.1 *The Blasius flow is the result of the interaction of a current that is spatially uniform for large negative X (left part of diagram) with a solid plate (thin shaded rectangle), which is idealized as being infinitely thin and extending infinitely far to the right as $X \to \infty$. Because all fluid flows must be zero at a solid boundary, the velocity must slow rapidly to zero in a "boundary layer," which thickens as $X \to \infty$. The region of velocity change ("shear") is called a "boundary layer" because in fluids of low viscosity, such as air and water, the shear layer is very thin.*

coordinate (three subscripts for a third derivative, and so on), subject to the boundary conditions

$$(1.2) \quad \psi(X,0) = 0, \ \ X \in [-\infty, 0], \qquad \psi_Y(X,0) = 0, \ \ X \in [0,\infty], \qquad \psi_Y(X,\infty) = 1,$$

where $\psi(X,Y)$ is the so-called streamfunction; the fluid velocities are just the spatial derivatives of ψ. Figure 1.1 illustrates the flow schematically. The fluid mechanics background is given in all graduate and most undergraduate texts in hydrodynamics.

Although the geometry is idealized, all flows past a solid body have thin "boundary layers" similar to the Blasius flow. Air rushing past a bird or an airplane, ocean currents streaming past an undersea mountain, a brook babbling through rapids made by a boulder and fallen trees, even the blood and breath flowing through our own bodies—all have boundary layers. The Blasius problem is as fundamental to fluid mechanics as the tangent function to trigonometry.

The Blasius problem has developed a vast bibliography [10]. Even though the problem is almost a century old, recent papers that employ the Blasius problem as an example include [2, 1, 5, 6, 11, 15, 16, 21, 18, 17, 23, 25, 26, 27, 28, 29, 30, 32, 33, 34, 36].

In this age of fast workstations, it is an almost irresistible temptation to blindly apply a two-dimensional finite difference or finite element method and then, a few billion floating point operations later, make a gaudy full-color contour plot of the answer. *Voilá*!

Unfortunately, numbers and pictures are meaningless by themselves. A supercomputer is like a ten-foot slide rule: it adds a little more accuracy to the results of less ambitious hardware, but it is still not a substitute for thinking. For all computational problems, intricate contour lines and isosurface plots advance science no more than a toddler's fingerpainting unless guided by a physicist's intuition and ability to plan, supervise, and analyze. For the Blasius problem, the additional payoffs of thinking (or rethinking) are not merely an enormous reduction in the computational burden, but also a drastic simplification of the conceptual burden: it is easier to understand a function of one variable than a function of two.

2. Symmetry Reductions.

2.1. PDE to ODE. Blasius himself noted in his 1908 paper that if $\psi(X, Y)$ is a solution, then so is

$$(2.1) \qquad \Psi(X, Y) \equiv c\,\psi(c^2 X, cY),$$

where c is an arbitrary constant. In other words, this implies that the problem has a continuous group invariance. The streamfunction is not a function of X and Y separately, but rather must be a univariate function of the "similarity" variable

$$(2.2) \qquad x \equiv \frac{Y}{\sqrt{X}}.$$

Any other dependence on the coordinates would disrupt the group invariance. There is some flexibility in the sense that one could replace x by x^2 or any other smooth function of x as the similarity variable, but the most convenient choice, and the historical choice, is

$$(2.3) \qquad \psi(X, Y) = \sqrt{X}\, f(Y/\sqrt{X})$$

for some univariate function f.

Thus, Blasius showed that the PDE (1.1) could be converted to the ODE

$$(2.4) \qquad 2f_{xxx} + f f_{xx} = 0$$

subject to the boundary conditions

$$(2.5) \qquad f(0) = f_x(0) = 0, \qquad f_x(\infty) = 1.$$

Figure 2.1 illustrates the Blasius function and its derivatives.

Blasius himself [7] derived both power series and asymptotic expansions and patched them together at finite x to obtain an approximation which agrees quite satisfactorily with later treatments. In particular, he computed the value of the second derivative at the origin to about one part in a thousand.

Note that, in some problems (but not the Blasius problem), "symmetry-breaking" solutions are possible. Also, some boundary layers and other steady solutions (but not the Blasius flow) are unstable. These possible complications, which are common throughout physics, reiterate how important it is for physical thinking to guide mathematical problem-posing.
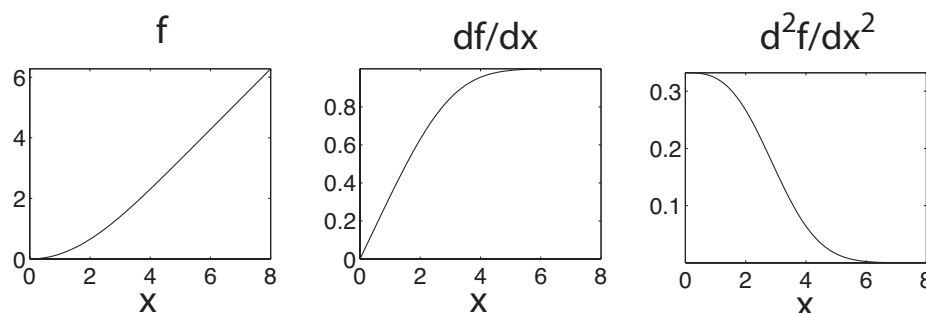

Fig. 2.1 *A plot of the Blasius function (left) and its first two derivatives (middle and right).*

2.2. Conversion of the Boundary Value Problem to an Initial Value Problem. Boundary value problems must be solved at all points simultaneously (a "jury" problem), whereas an initial value problem can be solved by a stepwise procedure (a "marching" problem). In this sense, initial value problems are easier. The Blasius ODE could be converted into an initial value problem and integrated by marching from $x = 0$ to large x if only

$$(2.6) \qquad \qquad \kappa = f_{xx}(0)$$

were known. Karl Toepfer [35] realized that knowledge of κ is in fact *unnecessary*. The reason is that there is a second group invariance such that if $g(x)$ denotes the solution to the Blasius equation with $g(0) = 0$, $g_x(0) = 0$, and its second derivative is arbitrarily set equal to one, then the solution with $f_{xx}(0) = K$ is

$$(2.7) \qquad \qquad f(x; K) = K^{1/3} g(K^{1/3} x).$$

It therefore suffices to compute $g(x)$ and then *rescale* both the horizontal and vertical axes of the graph of $g(x)$ so that the rescaled function has the desired asymptotic behavior at large x, namely, $f_x(\infty) = 1$. The true value of the second derivative at the origin is then

$$(2.8) \qquad \qquad \kappa = \lim_{x \to \infty} g_x(x)^{-3/2}.$$

With Toepfer's trick, it is only necessary to solve the differential equation as an initial value problem *once*. At two large but finite x_j, ordered so that $x_j > x_{j-1}$, compute $\kappa_j \equiv (1/g_x(x_j))^{3/2}$. If the κ_j closely agree, κ is approximately equal to the common value of the κ_j; if they are far apart, march to still larger x and try again. Using a Runge–Kutta method with a grid step of $1/10$ and the endpoints of $x_1 = 4$ and $x_2 = 6$, Toepfer was thus able to determine κ correctly to about one part in 110,000!

Freshman physics books are populated with analytical solutions, but none has ever been found for the Blasius equation. The first general-purpose electronic computer, ENIAC, was more than a third of a century in the future when Toepfer did his work. Nevertheless, by exploiting group invariances, he reduced the Blasius problem to perhaps a couple of hundred multiplications and additions. Though he does not record his paper-and-pencil computing time, it was likely only an afternoon.

3. Lessons from Symmetry Reductions and the Numerical Solution.

3.1. When Computers Were Human.

> In conducting extensive arithmetical operations, it would be natural to avail oneself of the skill of professional [human] computers. But unfortunately the trained computer, who is also a mathematician, is rare. I have thus found myself compelled to forego the advantage of the rapidity and accuracy of the computer, for the higher qualities of mathematical knowledge and judgment.
>
> –Sir George C. Darwin (1845–1912) [14, p. 101]

The Blasius–Toepfer numerical work was only a tiny part of the vast computations performed before the advent of electronic computers. Much of the number-crunching was done by full-time human calculators known as "computers." Grier [20], Croarken [12], and others [3, 13] have written very readable accounts of the heroic age of number-crunching. A substantial portion of the "computers" were female—none of this "girls can't do math" nonsense in the eighteenth or nineteenth century.

Weyl [39, p. 385] wrote that the Blasius problem "was the first boundary-layer problem to be numerically integrated . . . [in] 1907." However, the history of numerical solutions to differential equations is much older and richer. More than two decades before Blasius, John Couch Adams devised what are now called the Adams–Bashforth and Adams–Moulton methods for numerically integrating an initial value problem [4]. Toepfer numerically integrated the Blasius problem using a Runge–Kutta method published by Runge in 1895 and greatly refined by Kutta in 1901.

Aspray notes in his preface to *Computing Before Computers* [3], "Wherever we turn we hear about the 'Computer Revolution' and our 'Information Age'. . . . With all of this attention to the computer we tend to forget that computing has a rich history that extends back beyond 1945." Similarly, Gear and Skeel [19] note that the post–World War II development of electronic computers "has, of course, affected the algorithms used, but this has resulted in surprisingly few innovations in numerical techniques." What they mean is that although numerical algorithms have been greatly improved and advanced since the first electronic computers were built in the late 1940s, the building blocks—Runge–Kutta, Lagrangian interpolation, finite differences, etc.— were all created in the *precomputer* age. For example, Lanczos published his great paper, which is the origin of both Chebyshev pseudospectral methods and the tau method, in 1938 [24]. Rosenhead's point vortex paper was published even earlier [31]. George Boole's book [8] on finite differences first appeared in 1860!

Howarth's 1938 article, which contains an extensive table of the Blasius $f(x)$ and its first two derivatives, contains the acknowledgment, "I wish to express my gratitude to the Air Ministry for providing me with a [human] computer to perform much of the mechanical labour" [22]. But his calculations by government-funded human computer had long antecedents. Sir George Darwin, legendary for his prodigious numerical calculations in celestial mechanics and the equilibria of self-gravitating stars and planets, independently reinvented some of Adams' methods and used them to compute periodic orbits in 1897 [14]. He noted sadly that his previous human computer had died, but he found two new helpers at Cambridge and did much of the calculating himself. Foreshadowing Howarth, he thanked a grant from the Royal Society for funding two-thirds of the cost of this monumental number-crunching.

This brief history, though omitting many other examples, shows that computing has never been primarily about electronics, but always primarily about mathematics, algorithms, and organization, plus money for the he/she/it which is the "computer."

3.2. Precomputing.

Computing is a temptation that should be resisted as long as possible.
–John P. Boyd in the first edition of his book
Chebyshev and Fourier Spectral Methods
(Springer-Verlag, 1989)

Presoaking is a sound strategy for washing pots and pans with burnt or dried residues. In a similar way, successful computing is dependent on precomputing: identifying symmetries and other mathematical principles that can greatly reduce the scope of the problem. Engineers have relied for generations on dimensional analysis and the Buckingham pi theorem, which assert that physics must be independent of the system of units. Thus, a problem with three dimensional parameters can be reduced to the computation of a function that depends on only one parameter or perhaps no parameters at all.

Good algorithms are vital, but intelligent formulation of the numerical problem is equally important. Unfortunately, the vast expansion of numerical algorithms, soft-

ware management, and parallel computing has rather crowded out of the curriculum the tools of twentieth century applied mathematics: singular perturbation theory, group theory, special functions, and transformations. The Blasius problem is but one of many examples where these fading tools reduced the computational burden by an order of magnitude and greatly simplified the description of results. A century of progress in floating point hardware has not reduced the need for problem-specific thinking, but rather merely increased our number-crunching ambitions.

4. Known Properties.

> It is important to make friends with the function.
> –Tai Tsun Wu, Gordon McKay Professor of Applied Physics, Harvard

Before we can dive into the virtues of "particularity," we need to "make friends" with the Blasius function. As archaeologists reconstruct a lost civilization one pottery shard at a time, so applied mathematics accumulates understanding through a slow accretion of individual properties.

4.1. Power Series. The power series begins

$$(4.1) \qquad f(x) \approx \frac{\kappa}{2}x^2 - \frac{\kappa^2}{240}x^5 + \frac{11}{161280}\kappa^3 x^8 - \frac{5}{4257792}\kappa^4 x^{11} + \cdots,$$

where κ is the second derivative of the function at the origin, given to very high precision in Table 4.1. Only every *third* coefficient is different from zero. As proved in introductory college calculus, a power series converges only within the largest disk, centered on $x = 0$, which is free of singularities of $f(x)$. Although it would take us too far afield to recapitulate the proof here, the Blasius function has a singularity on the negative real axis [10] at $x = -S$, where no closed form for S is known. The series converges for $|x| < S$, where $S \approx 5.69$, and is given to high precision in the table.

A closed form for the coefficients is not known. However, denoting the general series by $f = \sum_{j=0}^{\infty} a_j x^{3j+2}$, the coefficients can be computed from $a_1 = \kappa/2$ plus the recurrence

$$(4.2) \qquad a_m = -\frac{1}{(3m-1)(3m-2)(3m-3)} \sum_{j=1}^{m-1}(3j-1)(3j-2)a_j a_{m-j}.$$

The limitation of a finite radius of convergence can be overcome by constructing, from the power series, either Padé approximants or an Euler-accelerated series, which both apparently converge for all positive real x [10, 9].

4.2. Asymptotic Approximations. For large positive x,

$$(4.3) \qquad f(x) \sim B + x + \text{exponentially decaying terms}, \qquad x \gg 1,$$

and

$$(4.4) \qquad f_{xx} = Q \exp\left\{-(1/4)x(x+2B)\right\},$$

Table 4.1 *Blasius constants to high precision (for benchmarking).*

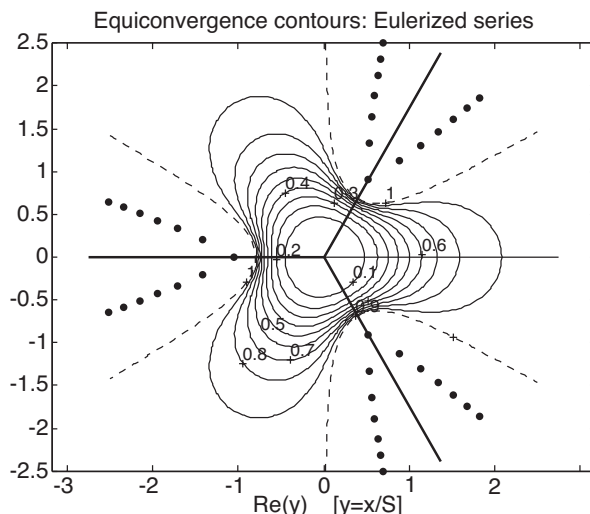| Symbol	Definition	Numerical value
κ	$f_{xx}(0)$	0.33205733621519630
S	power series radius of convergence	5.6900380545
B	$\lim_{x\to\infty}(f(x)-x)$	−1.720787657520503
Q	$\lim_{x\to\infty}\exp\left\{[1/4]x(x+2B)\right\}f(x)$	0.233727621285063

Fig. 4.1 *The Blasius function in the complex plane. The coordinate is scaled by S, the distance from the origin to the nearest singularity. The three thick rays are the symmetry axes in the complex plane. The black dots denote known singularities of the Blasius function. The contours are the "equiconvergence" contours of the Euler-accelerated power series; everywhere along the contour labeled "0.8," the $n + 1$th term is smaller than the nth term by a factor of 0.8. The Eulerized series appears to converge everywhere within the region bounded by the three dashed parabolas (including the entire positive real axis, which is the physical domain for the flow), but a rigorous proof is lacking.*

where $Q \approx 0.234$ and $B \approx -1.72$. (These constants are given to high precision in Table 4.1.)

The linear polynomial $B + x$, which is the leading order asymptotic approximation for $f(x)$, is an exact solution to the differential equation for all x, but fails to satisfy the boundary conditions. Physically, the Blasius velocity f_x is constant at large distances from the plate, and this implies that $f(x)$ should asymptote to a linear function of x far from the plate. Near the surface of the plate at $x = 0$, the streamfunction f curves away from the straight line to satisfy the boundary conditions at $x = 0$, creating a region of rapid variation called a "boundary layer."

4.3. Symmetry. The function f_{xx} has a power series which contains every third power of x. This implies that this function has a C_3 symmetry in the complex plane. That is, f_{xx} is the same on the rays $\arg = \pm(2/3)\pi$ as on the positive real axis. Similarly, the singularity on the negative real x-axis which limits the convergence of the power series of f is replicated on the rays $\arg(x) = \pm\pi/3$. A contour plot of the function is unchanged by rotating it about the origin through any multiple of 120 degrees, as in Figure 4.1. However, unlike the group invariances described earlier, the C_3 complex plane symmetry does not simplify the problem.

However, because of this symmetry, the singularities of the Blasius function, described in the next subsection, must also always occur in *threes*: If there is a singularity at $x = \rho \exp(i\theta)$, then there must also be identical singularities at $x = \rho \exp(i[\theta \pm (2/3)\pi])$.

4.4. Singularities. If we look for a singularity of the form

(4.5) $$f \sim r(x + S)^{\nu} + \text{higher order terms},$$

and substitute this into (2.4), we find by matching exponents and leading terms that the dominant singularity must be a simple pole with a residue of precisely 6. (This analysis is rigorous, but cannot exclude the possibility of singularities of more exotic form, though none has been identified for the Blasius function.) The location S of the singularity is much harder and can be determined only by numerical integration [10]. Through a mixture of analysis and numerical calculations too complex to be repeated here, near the singularity,

$$(4.6) \qquad f \approx \frac{6}{x - S} - 1.5036(1 + x/S) \cos(\sqrt{2} \log(1 + x/S) + 0.54454)$$
$$+ (1 + x/S)^3 \left(-0.1152 \cos \left(\sqrt{8} \log \left(1 + \frac{x}{S} \right) + 1.0891 \right) \right.$$
$$\left. + 0.021323 \sin \left(\sqrt{8} \log \left(1 + \frac{x}{S} \right) + 1.0891 \right) + 0.2144 \right) + \cdots.$$

The complicated cosine-of-logarithm corrections to the pole show that the singularities of analytic functions can be more far more complex than the simple poles-logarithms-fractional powers of most textbooks.

Because of the C_3 symmetry noted above, the singularities closest to the origin lie at the three points $x = -S, -S \exp(i(2/3)\pi), -S \exp(-i(2/3)\pi)$. There seems to be a countable infinity of singularities more remote than these three, as illustrated in [10] and Figure 4.1.

4.5. Euler-Accelerated Power Series. Boyd [10] shows that the power series can be accelerated by Euler's summation. The Eulerized approximation is

$$(4.7) \qquad f(y) = (x/S)^2 \sum_{j=1}^{\infty} b_j \, \zeta^{j-1},$$

where, respecting the C_3 symmetry in the complex plane,

$$(4.8) \qquad \zeta \equiv \frac{2x^3}{S^3 + x^3},$$

where the Euler coefficients b_j are calculated from those of the power series a_j by a recurrence given in [10]. Figure 4.1 shows the equiconvergence contours for the Eulerized expansion.

Herman Weyl, who was a very gifted mathematician, criticized Blasius's work because he matched a power series with a finite radius of convergence to an asymptotic approximation, which technically has no radius of convergence at all. It is ironic that the defects of Blasius's series could have been fixed by using a device created by Euler long before either Blasius or Weyl was born!

5. The Importance of Particularity.

> The more specialized a trick is, the more useful it is.
> –Tai T. Wu, Harvard University lecture, 1973

The Blasius problem is a good illustration of Wu's proverb. Applied mathematics is often taught as a collection of broad techniques; a good method is one that applies to a lot of problems. However, often the best method is a device that applies narrowly to only a single problem.

One such trick is the Eulerized series of the previous section: Euler's acceleration is a general artifice, but choosing the variable so as to respect the C_3 symmetry in the complex plane, which implies that only every third power series coefficient is nonzero, is a trick peculiar to the Blasius problem.

5.1. Blasius's Integrodifferential Equation. Another narrow trick of great power for the Blasius problem begins with a broad method of great generality. A *linear*, *first order* ODE has the explicit solution

$$(5.1) \qquad v_x + p(x)v = 0 \qquad \longrightarrow \qquad v(x) = v(0) \exp\left(-\int_0^x p(y)dy\right).$$

This can be applied to the Blasius problem by identifying $v = f_{xx}$ and $p(x) = f$:

$$(5.2) \qquad f_{xx}(x) = \kappa \, \exp\left(-F(x)\right),$$

where

$$(5.3) \quad F(x) \equiv \frac{1}{2}\int_0^x f(y)dy \approx \frac{\kappa}{12}x^3 - \frac{\kappa^2}{2880}x^6 + \frac{11}{2903040}\kappa^3 x^9 - \frac{5}{102187008}\kappa^4 x^{12} + \cdots.$$

This converts the differential equation of third order into an integrodifferential equation of second order. By substituting local approximations for $F(x)$, we can extract insights and approximations.

For example, as noted earlier, $f \sim B + x$ for some constant B as $x \to \infty$. This implies that $F(x)$ is asymptotically a quadratic polynomial and therefore $f_{xx} \sim Q \exp\{-(1/4)x(x + 2B)\}$ as $x \to \infty$, as quoted without derivation in (4.4).

This bit of insight was known to Blasius himself, who systematically derived higher order corrections. Later workers found, however, that one can gain as much or more by exploiting the power series approximations of $f(x)$.

In 1925, Bairstow observed that substituting just the lowest term in the power series yields

$$(5.4) \qquad f_{xx} \approx \kappa \, \exp\left(-\frac{\kappa}{12}\,x^3\right).$$

Although this is qualitatively wrong as $x \to \infty$ in the sense that f_{xx} should decay as an exponential of a *quadratic* function of x instead of a cubic, this error only happens when the second derivative of $f(x)$ is very tiny anyway. The maximum pointwise error in Bairstow's approximation on $x \in [0, \infty]$ is only 0.0077, which is smaller than the maximum of f_{xx} (which is $\kappa = f_{xx}(0) \approx 0.33$) by a factor of forty-three. For most engineering purposes, this is a very good approximation.

5.2. Analytical Approximation to the Second Derivative. Unfortunately, it is impossible to straightforwardly generalize Bairstow's approximation to higher order because the power series for $F(x)$ has only a finite radius of convergence, and we need to approximate f_{xx} for all positive x. However, Parlange, Braddock, and Sander [29] found a way around this more than half a century later—another specialized strategy that works only for the Blasius problem.

Their key observation is that, as shown by Darboux in the late nineteenth century, the power series coefficients a_n of a function $f(x)$ *asymptote* as $n \to \infty$ to those of the simplest function—or any function—that contains the same convergence-limiting singularity. For the Blasius function, the asymptotic power coefficients will be those generated by a trio of first order poles with a residue of 6 at $x = -S, -S\exp(i(2/3)\pi)$, $-S\exp(-i(2/3)\pi)$ (plus smaller corrections (4.6) which Parlange, Braddock, and Sander ignore). Because of these cosine-of-logarithm corrections, and also the infinite number of singularities at larger x, it is not possible to completely remove all singularities by subtracting the poles from f_{xx}. However, Parlange, Braddock, and

Sander realized that full singularity subtraction was unnecessary since Bairstow after all obtained a decent approximation without even knowing the type or location of the singularities!

The coefficients of $f(x)$ are asymptotically those of

$$(5.5) \qquad \sigma(x) = \frac{18x^2}{184.2237031 + x^3},$$

from whence it follows by integration that the coefficients of $F(x) = (1/2)\int_0^x f(y)dy$ will asymptote to those of

$$(5.6) \qquad \Xi \equiv 3 \log\left(1 + \frac{x^3}{184.2237031}\right).$$

It is helpful to define

$$(5.7) \qquad y \equiv \frac{\kappa}{12}x^3 \qquad \longleftrightarrow \qquad x = \left(\frac{12y}{\kappa}\right)^{1/3},$$

$$(5.8) \qquad F(x[y]) = \sum_{n=0}^{\infty}(-1)^n B_n\, y^{n+1},$$

where

$$(5.9) \qquad B_0 = 1, \qquad B_1 = (1/20), \qquad B_2 = (11/1680), \qquad B_3 = (5/4928),$$

and

$$(5.10) \qquad \Xi = 3\log(1 + \beta y) = \sum_{n=0}^{\infty}(-1)^{n+1}3\beta^{n+1}/(n+1)\,y^{n+1}, \qquad \beta = 0.196165513.$$

Unlike $F(y)$, the coefficients of $\Xi(y)$ are known analytically as given in (5.10). Asymptotically, $B_n \sim 3\beta^{n+1}/(n+1)$, which are the power series coefficients of $\Xi(y)$ displayed in (5.10).

Parlange, Braddock, and Sander's idea is to rewrite $F(x)$ by adding and subtracting Ξ—adding Ξ in its analytic, logarithmic form and subtracting Ξ in the form of its power series truncated to the same number of terms as the series for F. Taking just the first term gives

$$(5.11) \qquad F(x[y]) \approx \tilde{F}(y) \equiv (1 - .588496)y + 3\log(1 + \beta y),$$

$$(5.12) \qquad f_{xx}(x[y]) = \frac{1}{(1 + \beta y)^3}\exp(-0.4115y).$$

The maximum absolute error on all $x \in [0, \infty]$ is 0.0013, which is an error relative to κ of less than 1 part in 250.

However, the power series expansion for $\tilde{F}(y)$, which is

$$(5.13) \qquad \tilde{F}(y) \approx y + 0.05772y^2 + 0.007549y^3 - 0.001111y^4,$$

does not match the second term of the exact power series of $F(y)$, which is

$$(5.14) \qquad F(y) = 1 - 0.0500y^2 + 0.006548y^3.$$

To produce a better match, Parlange, Braddock, and Sander empirically modified the leading approximation to

$$(5.15) \qquad F(x[y]) \approx \hat{F}(y) \equiv (1 - 3\beta_1)y + 3\log(1 + \beta_1 y),$$

where β_1 is chosen to match the second coefficient of the power series of $\hat{F}(x[y])$ to that of $F(x[y])$, i.e.,

$$(5.16) \qquad \hat{F}(y) \approx 1 - 0.0500y^2 + 0.006086y^3.$$

Their improved approximation is

$$(5.17) \qquad f_{xx}(x[y]) = \frac{1}{(1 + 0.182574y)^3} \, \exp(-0.452277y),$$

where $\beta_1 = \sqrt{1/30} \approx 0.182574$. Although this is most definitely a heuristic fix—indeed, the whole idea of adding a function in analytical form and subtracting it in the form of its power series coefficients, and then applying the result of the series over all x, would give many a mathematics professor apoplexy—the result, which in terms of the original coordinate is

$$(5.18) \qquad f_{xx} \approx \kappa \, \frac{1}{(1 + 0.00542818x^3)^3} \, \exp(-0.0125152x^3),$$

has a maximum pointwise error for f_{xx} of only 0.0001, which is only 1 part in 3200 relative to κ!

This is really an extraordinarily accurate approximation. Unfortunately, the approximation, simple as it is, cannot be integrated in closed form. Thus, the problem of finding a simple, uniform, explicit analytical approximation for f and f_x remains elusive, nearly a century after Blasius.

6. Undergraduate Projects. Because of its simplicity, many aspects of the Blasius function can be explored by undergraduates. At the same time, because the problem is nonlinear and cannot be solved in terms of standard special functions, it is not trivial.

To compute errors in various approximations, one can use the short MATLAB program of [10], which evaluates $f(x)$ and its first three derivatives to about ten decimal places of accuracy.

None of the questions posed below in the undergraduate projects has a published answer.

Project One. Repeat Toepfer's 1912 study by setting $f_{xx}(0) = 1$, integrating the Blasius equation as an initial value problem to large x, and stopping when f_x has asymptoted to a constant. How large a grid spacing is necessary to obtain a given accuracy with a particular method such as the fourth order Runge–Kutta scheme? How far must one go in x to obtain an accurate approximation to f_x and, from this, using Toepfer's group invariance, to the true value of the second derivative at the origin, κ?

Project Two. Hermann Weyl [37, 38, 39] strongly criticized Blasius for matching the power series, which has only a finite radius of convergence, to the asymptotic expansion, which likely has no radius of convergence at all (though the first term or two is a good approximation for large x). Boyd [10] showed, however, that the Euler

transformation of the power series appears to converge for all positive real x. (This is equivalent to making a conformal map of the complex x-plane and then computing the power series of the transformed function.) The Euler transformation was known in 1908 and thus could have been applied by Blasius himself. "Rehabilitate" the power series by examining the convergence of the Euler-accelerated expansion. Can one obtain accurate values for κ from the Eulerized power series alone? (Boyd [10] does a similar analysis for another method of series extension known in 1908, Padé approximation. However, Euler's method is much better suited to hand computation, and would have been Blasius' preferred choice.)

Project Three. The power series coefficients of a function with a simple pole are

$$(6.1) \qquad \frac{r}{1 + x/S} = r \sum_{n=0}^{\infty} (-1)^N \frac{1}{S^n} x^n,$$

where S is a constant which is also the radius of convergence of the series. What are the power series coefficients for a singularity of the form of the second worst singularity that appears in the Blasius function,

$$(6.2) \qquad h(x) \equiv (1 + x) \, \cos(\log(1 + x)) = \sum_{n=0}^{\infty} h_n x^n ?$$

Why is there no loss of generality in placing the singularity at $x = -1$?

It is easy to observe, by using the command for computing a power series in Maple or *Mathematica* or a similar symbolic manipulation system, that h_n decreases roughly as $1/n^2$. Can one be more precise?

7. Open Research Problems. Although the Blasius problem is simple enough for fooling around with by undergraduates, it still poses unresolved challenges for the research mathematician. A few of these include the following, some from [10].

1. A proof that the Eulerized power series converges for all positive real x (strongly supported by numerical evidence).
2. A proof that diagonal Padé approximations converge for all positive real x (strongly supported by numerical evidence).
3. A proof that $f(x)$ is free of singularities everywhere in the sector $\arg(x) \leq \pi/4$ (supported by the asymptotic approximation, which is accurate and singularity-free in this sector).
4. A rigorous and complete analysis of the essential singularities that are superimposed upon the simple pole at $x = -5.69$.
5. A simple, uniformly accurate analytical approximation to $f(x)$, similar to that already known for f_{xx}.
6. A theory for the asymptotic approximation of the power series coefficients for functions with singularities of the form $(1 + x) \cos(\log(1 + x))$, as occurs in the Blasius function.

8. Summary. One maxim for good number-crunching is: Never solve a PDE when an ODE will do, and never solve a boundary value problem when an initial value problem will do. Blasius and Toepfer successfully applied this maxim using mathematical tools which students are no longer taught very often. The film director Sir Alfred Hitchcock said his most important and enjoyable work was all done before filming even began: the meticulous planning of each shot in his mind. Similarly, the most important part of computing is what happens before the code is written.

Numerical analysis classes do a good job of explaining the importance of choosing an efficient numerical scheme. They do not usually do a good job of explaining the nonnumerical Blasius-and-Toepfer-like "precomputing."

The second theme is that many problems can be attacked with special tricks. This runs counter to the ever-generalize, even-more-abstract prevalent trend of pure mathematics. In contrast, applied mathematicians are plumbers, always adapting general strategies, like copper pipes and traps and plumber's putty, to the idiosyncrasies of the problem at hand, never too proud to hammer a pipe into alignment, never too proud to use a trick that only works for a narrow, specific problem.

The third theme is that even though the Blasius function has been intensively studied for a century, there are still many challenging open problems at all levels: some that are educational fun for an undergraduate, others that test the skills of a postdoctoral mathematician. The high school treatment of the "scientific method" as a sequence of ever-more-complex problems, vanquished and left behind, is unrealistic. The history of the Blasius problem is more typical: a major problem is never completely solved. Instead, science is like lunar exploration: we return again and again with different satellites and landers, slowly making the map of knowledge denser.

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