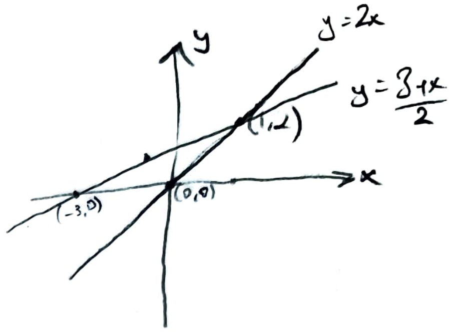
Linear Algebra {18-06}

Lec-1:

$2x - y = 0$
$-x + 2y = 3$ $\rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$

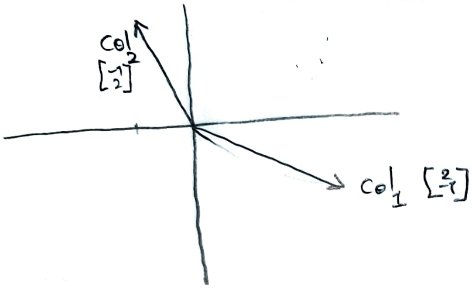
$A \quad x \quad = \quad b$

Raw Picture

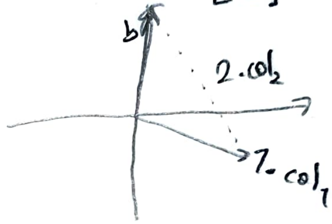


$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

linear combination of columns.



$1 \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$



Let's say

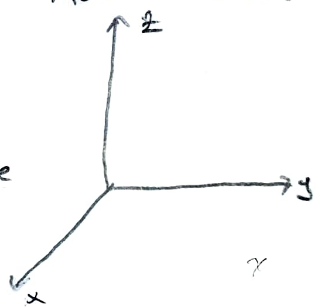
$2x - y = 0$

$-x + 2y - z = -1$

$-3y + 4z = 4$

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$

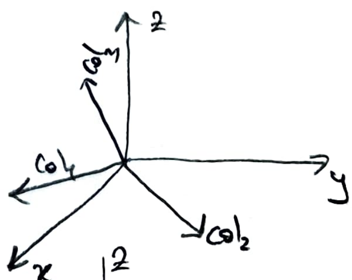
Row Picture



Solution of this eq. will be planes and those planes will have a point that satisfies both 3 eq. We can't do it by row picture, cause its hard to solve. So we will use matrix.

$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} \Rightarrow col_3 = b$

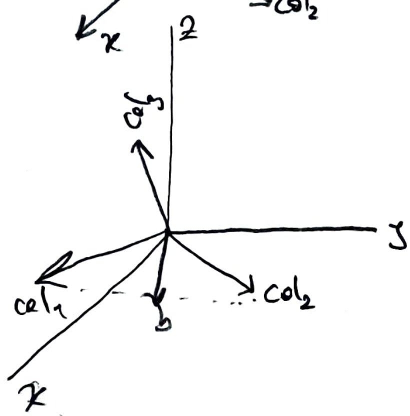
$x=0, \quad y=0, \quad z=1$



Let us take different right hand side

$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

$x=1 \quad y=1 \quad z=0$



We need to answer a question.

Can I solve $Ax = b$ for every b?

Do the linear comb. of the columns fill 3d-space?

For this A, the answer is Yes.

If the three vectors both lie on the same plane - say xy, zy or xz - their linear combinations also will lie on the same plane.

$Ax = b$, Ax is a linear comb. of columns of A.

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

Lec-2 Elimination with Matrixes:

$$\begin{aligned} x + 2y + z &= 2 \\ 3x + 8y + z &= 12 \\ 4y + z &= 2 \end{aligned} \rightarrow A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

1st pivot, 2nd pivot

$$\underbrace{\begin{matrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{matrix}}_{A}\ \underbrace{\begin{matrix} 2 \\ 12 \\ 2 \end{matrix}}_{b} \rightarrow \begin{matrix} 1 & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{matrix}\ \begin{matrix} 2 \\ 6 \\ 2 \end{matrix} \rightarrow \underbrace{\begin{matrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & \boxed{5} \end{matrix}}_{U}\ \underbrace{\begin{matrix} 2 \\ 6 \\ -10 \end{matrix}}_{c}$$

3th pivot

Now back substitution

$$\begin{aligned} x + 2y + z &= 2 \\ 2y - 2z &= 6 \\ 5z &= -10 \end{aligned} \rightarrow \begin{aligned} z &= -2 \\ y &= 1 \\ x &= 2 \end{aligned}$$

Now I will show by elimination matrixes.
But first need to show row multiplication.

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = 3 \times col_1 + 4 \times col_2 + 5 \times col_3$$

matrix x column = column

$$\underbrace{\begin{bmatrix} 1 & 2 & 7 \end{bmatrix}}_{1 \times 3}\underbrace{\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}}_{3 \times 3} = 1 \times row_1 + 2 \times row_2 + 7 \times row_3$$

Matrices: Substract $3 \times row_1$ from row_2 / Step 2: Subtract $2 \times row_2$ from row_3.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{2\text{-}1}}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \qquad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{E_{3\text{-}2}}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$E_{2\text{-}1}$ = Elemination "elementary" matrices.

So now I want to express that we done today:

$$E_{3-2}\,(E_{2-1}\cdot A) = U$$

$$(E_{3-2}\,E_{2-1})\cdot A = U \quad \{\text{assoiative law allows.}\}$$

Permutation:

Exchange rows 1 and 2?

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Also suppose that we want to change columns 1 and 2;

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

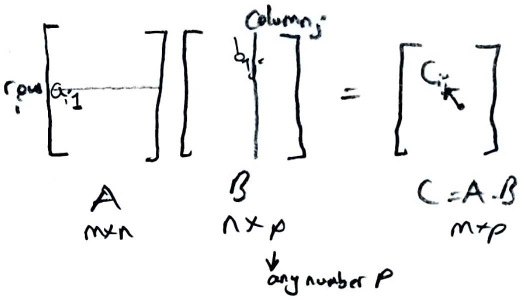
{commutative law can't be used here.}

If I want to get U from A just with one single matrix operation, we can do it multiplying E_{3-2} and E_{2-1} but I also can do from trying to get A from U.

Inverses:

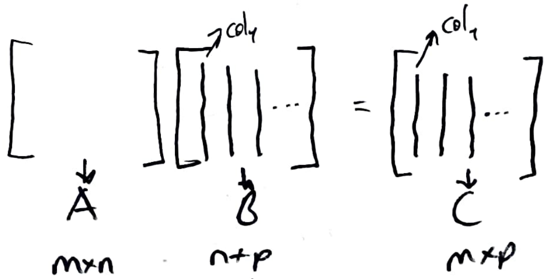
$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E^{-1}}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} \rightarrow \text{clear to see.}$$

Lec 3, Inverses and Multiplication on Matrices

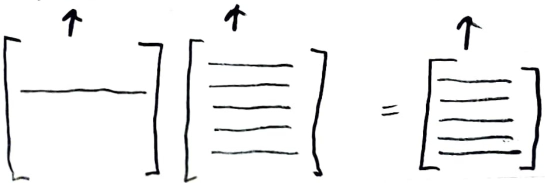


$C_{ij} = (\text{row } i \text{ of } A)\cdot(\text{column } j \text{ of } B)$ {1st Way}

$$= a_{i1}\cdot b_{1j} + a_{i2}\cdot b_{2j} + \dots = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$



Columns of C are combinations of columns of A.
{2nd Way}



Rows of C are combinations of rows of B. {3th Way}

2

Columns of A × rows of B

$m \times 1$ $\quad$ $1 \times p$ $\Rightarrow \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$

AB = sum of (colmns of A) × (rows of B) {4th way}

$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$ If we draw a picture of this rows, they both lie on the same line.

Block Multiplication:

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] = \left[\begin{array}{c|c} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ \hline A_3B_1 + A_3B_3 & A_3B_2 + A_3B_4 \end{array}\right]$$

Inverses:

First we do with square matrices.

Let have a matrix A which is invertible, say A^{-1}. If A^{-1} exists;

$A^{-1} \cdot A = I$. equal to identity matrix. A is invertible, non-singular.

$A \cdot A^{-1} = I$

Singular case {no inverse}

$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ It has no inverse because you can find a vector x with $Ax = 0$

$$\underset{A}{Ax} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \underset{x}{\begin{bmatrix} -3 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If we multiply Ax with A^{-1} we will get $Ix = 0$ since I is not zero x has to be zero but we both know that x is not zero which makes that A has no inverse.

Now let's solve a invertible matrix.

$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \Rightarrow A \times$ column j of A^{-1} = column j of I.

Gauss-Jordan (Solve 2 eq. at once)

$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$\downarrow I \quad \downarrow A^{-1}$

1 3 1 0
2 7 0 1

$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Lec. 4 Gaussian Elimination:

Let A and B inversible. What is the inverse of AB

$$(AB)(B^{-1}A^{-1}) = I \quad , \quad B^{-1}A^{-1}AB = I.$$

$$AA^{-1} = I \quad , \quad (A^{-1})^T A^T = I$$

$$\downarrow$$

$$(A^T)^{-1}$$

$$\begin{array}{ccc} E_{2-1} & A & U \\ \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = & \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \end{array} \quad , \quad \begin{array}{ccc} A & = L & U \\ \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = & \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \end{array} \Rightarrow LU = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

$E_{3-2}E_{3-1}E_{2-1}A = U$ (no row exchanges)

$A = E_{2-1}^{-1}E_{3-1}^{-1}E_{3-2}^{-1}U = LU$ (inverses reverse order)

$$\underset{E_{3-2}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}}\underset{E_{2-1}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix} = E \Rightarrow EA = U$$

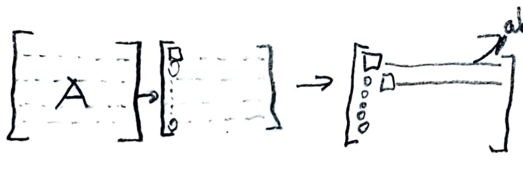
$A = LU$
If no row exchanges, then multipliers directly go into L.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = L \Rightarrow A = LU$$

So now, how many operations do take on $n \times n$ matrix A?

Say $n = 100$

We will have n^2 operation for every row and column (operation = multiply + subtract)



$$1^2 + 2^2 + \cdots + (n-1)^2 + n^2 \approx \frac{n^3}{3}$$

$$\sum_{k=1}^{n} k^2 \xrightarrow{\lim} \int_1^n x^2 dx = \frac{x^3}{3}$$

But also there are extra columns, actually just one column, b.

Cost of $b = n^2$

For the permutations of I matrix 3×3; $P^{-1} = P^T$

Lec5. Transposes, Permutations, Spaces R^n:

Permutations: P execute row exchanges

$A = LU = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ 0 & & \end{bmatrix}$ becomes $PA = LU$ // any invertible A.

Permutations

$P =$ identity matrix with reordered rows.

$n!$ number of counts of reordering, it counts all $n \times n$ permutations and both are invertible because they both can be brough back to P.

The matrix that does inverse is the same as transpose matrix $\Rightarrow P^{-1} = P^T$, $P^T P = I$.

$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 4 \end{bmatrix}$, Now transpose $(A^T)_{ij} = A_{ji}$ { symmetric matrices }

$A^T = A \Rightarrow$ example: $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$

$R^T R$ is always symmetric

$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 17 \end{bmatrix}$ $\Rightarrow$ Why? Take transpose again.

$(R^T R)^T = R^T R$.

Vector Spaces:

Subspaces of R^n

$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$ Columns in R^3 all their combinations from a subspace. Called column space $C(A)$

If $n = 2$,

1-) any line through origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2-) zero vector

3-) space itself

If $n > 2$

then

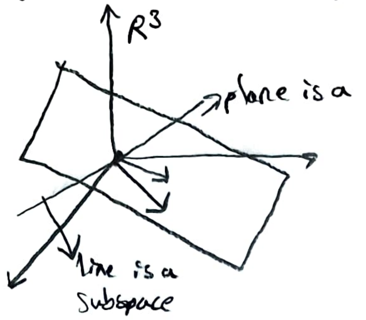
1-) any plane that covers origin

2-) zero vector

3-) plane itself

Lec 6 - Vector Spaces and Null Space:

$v+w$ and cv are in the space, all combn. of $cv+dw$ also are in the space.



2 subspaces: P and L

$P \cup L$ is not a subspace

$P \cap L$ is a subspace because it only contains zero vector.

Subspaces S and T,

also their intersect is a subspace $\Rightarrow$ $S \cap T$ is a subspace

Column Space of A is a subspace of R^4

$$A = \begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix} = \text{all linear combinations of columns}$$

we need to make a connection with $Ax=b$.

Does $Ax=b$ have a solution for every b?

There are 4 eq and 3 unknowns. You can't solve 4 eq. with 3 unknowns, but sometimes you do.

$$Ax = \begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$$

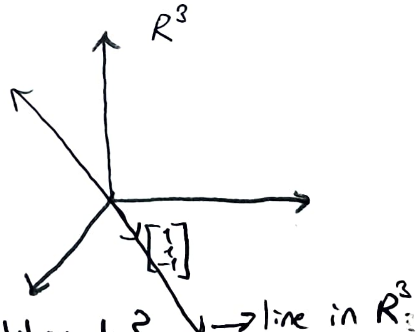
I want the same solutions that will solve the matrix, we can't find all of them because the solution space won't span the whole R^4 space. So which b's allow this system to be solved?

Column 3 of A dependent with 1, 2nd columns each other, means each will fill up the same space, will not add anything new to our subspace, so we can describe the column space of this matrix as a 2 dimensional subspace of R^4

Null Space:

Solutions to $Ax=0$ creates the null space of A.

$$Ax = \begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \Rightarrow N(A) \text{ contains } \begin{bmatrix}c\\c\\-c\end{bmatrix}$$



Check that solutions to $Ax=0$ always give a subspace

If $Av=0$, $Aw=0$, then $A(v+w)=0=Av+Aw$ {distributive law}

Lec. 7:

$A=\begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix} \to \begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&2&4\end{bmatrix} \to \begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix} = U \Rightarrow$ It has 2 pivots means the rank of A.

echelon form

pivot columns

free columns

free means that I can give any number to x_2 and x_4, then we can solve x_1 and x_3 for the x's we've chosen.

Our eq.s are:

$x_1+2x_2+2x_3+2x_4=0$
$2x_3+4x_4=0$ $\Rightarrow x=\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} \to x=c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}$ a infinitely long line in R^4

$x=d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$ → special solutions.

The null space $\{Ux=0\}$ contains all the comb. of the special solutions.

There is a special solution for every free variables.

In a matrix $m\times n$ with rank are, we have $n-r$ free variables. In our case $4-2=2$ free var.

R = reduced row echelon form.

$\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}$ → zeros above and below pivots.

↓

$\begin{bmatrix}1&2&0&-2\\0&0&2&4\\0&0&0&0\end{bmatrix} \to \begin{bmatrix}1&2&0&-2\\0&0&1&2\\0&0&0&0\end{bmatrix} = R = \text{rref}(A)$

$\begin{array}{cc} I & F \\ \begin{bmatrix}1&0\\0&1\end{bmatrix} & \begin{bmatrix}2&-2\\0&2\end{bmatrix} \\ \text{pivot cols} & \text{free cols}\end{array}$

rref form

$R=\begin{bmatrix}I&F\\0&0\end{bmatrix} \Rightarrow Rx=0 \Rightarrow$ I will create a nullspace matrix and its columns will be the special sol.

$RN=0$

$N=\begin{bmatrix}-F\\I\end{bmatrix}$

$Rx=0$

$\begin{bmatrix}I&F\end{bmatrix}\begin{bmatrix}x_{pivot}\\x_{free}\end{bmatrix}=0 \to x_{pivot}=-Fx_{free}$.

$A=\begin{bmatrix}1&2&3\\2&4&6\\2&6&8\\2&8&10\end{bmatrix} \to \begin{bmatrix}1&2&3\\0&2&2\\0&0&0\\0&0&0\end{bmatrix}$ $\quad x_1+2x_2+3x_3=0$, $2x_2+2x_3=0$ $\to \begin{bmatrix}1&0&1\\0&1&1\\0&0&0\\0&0&0\end{bmatrix}=R$ $\quad x=c\begin{bmatrix}-F\\I\end{bmatrix}$ → N

r=2 → U

$x=c\begin{bmatrix}-1\\-1\\1\end{bmatrix}$

Lecture 8. Complete Solution of $Ax=b$.

$$\left[\begin{array}{cccc|c} 1&2&2&2&b_1\\ 2&4&6&8&b_2\\ 3&6&8&10&b_3\end{array}\right] \rightarrow \left[\begin{array}{cccc|c} \boxed{1}&2&2&2&b_1\\ 0&0&\boxed{2}&4&b_2-2b_1\\ 0&0&2&4&b_3-3b_1\end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1&2&2&2&b_1\\ 0&0&2&4&b_2-2b_1\\ 0&0&0&0&b_3-b_2-b_1\end{array}\right] \quad 0=b_3-b_2-b_1 \Rightarrow b_3=b_2+b_1$$

Augmented matrix $= [A\ b]$

Solvability Condition on b.

$Ax=b$ solvable when b is in $C(A)$. If a combination of rows of A gives zero row, then the same combination of entries of b must give $\vec{0}$.

To find complete sol'n to $Ax=b$

1-) $x_{particular}$: Set all free variables to zero. Then solve $Ax=b$ for the pivot variables.
$\searrow x_2, x_4$ $\quad\quad$ $\searrow x_1, x_3$

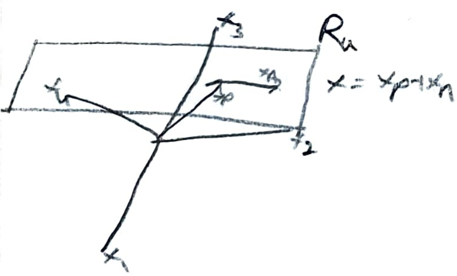
$x_1+2x_3=1 \Rightarrow x_3=\frac{3}{2}$, $x_1=-2$.
$2x_3=3$

$$x_p=\begin{bmatrix}-2\\0\\3/2\\0\end{bmatrix} \qquad Ax_p=b$$

2-) $x_{nullspace}$ $\qquad Ax_n=0$

$x=x_p+x_n$. $\qquad$ one of the special sol'n.

$$A(x_p+x_n)=b\ ,\quad x_{complete}=\begin{bmatrix}-2\\0\\3/2\\0\end{bmatrix}+c_1\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$



$x=x_p+x_n$

m by n matrix A of rank r ($r\le m$, $r\le n$). Full column rank means $r=n$: no free variables

$N(A)=\{\text{zero vector}\}$, solution to $Ax=b$: $x=x_p$ unique sol'n if it exists (0 or 1 solution) $A=\begin{bmatrix}1&3\\2&1\\6&1\\5&1\end{bmatrix}$, $R=\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}$

Full row rank means $r=m$.

Can solve $Ax=b$ for every b ⇒ Exists
Left with $n-r$ free variables $\quad A=\begin{bmatrix}1&2&6&5\\3&1&1&1\end{bmatrix}$, $R=\begin{bmatrix}1&0&F\\0&1&\end{bmatrix}$ $\qquad r=m=n$

$A=\begin{bmatrix}1&2\\3&1\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}$, $R=[I]$
$N(A)=\{\text{zero vector}\}$

If $r=m=n$, $R=I$ (1 solution to $Ax=b$)
If $r=n<m$ $R=\begin{bmatrix}I\\0\end{bmatrix}$ (0 or 1 sol'n)
If $r=m<n$ $R=[I\ F]$ (1 or ∞ sol'n)
If $r<m,n$. $R=\begin{bmatrix}I&F\\0&0\end{bmatrix}$ (0 or ∞ sol'n)

Lecture 9 - Independence, Span:

Suppose A is m by n with $m < n$. Then there are nonzero solutions to $Ax = 0$.

Vectors $x_1, x_2, \dots, x_n$ are independent if no combination gives zero vector (except the zero comb. all $c_i = 0$)

$$c_1 x_1 + c_2 x_2 + \dots + c_n x_n \neq 0$$

If a matrix A on R^n has m vectors where $m > n$, means that there will be some comb. of that will give dependent with $m-n$ free variables.

$$A = \begin{bmatrix} 1 & 2 & 2.5 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

They are independent if nullspace of A is $\{$zero vector$\}$ rank $= n \to$ no free variables $N(A) = \{0\}$

They are dependent if $Ac = 0$ for some $c \neq 0$. rank $< n \to$ has free variables

Vectors $v_1, \dots, v_\ell$ span a space means the space consists of all comb. of those vectors.

Basis for a vector space is a sequence of vectors $v_1, v_2, \dots, v_d$ with two properties.

1. They are independent
2. They span the space.

Example:

Space is R^3 $\quad R^n$, n vectors give basis if the $n \times n$ matrix with those columns is invertiable.

One basis is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Another basis

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix},$$

The two columns $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$ creates a basis for R^2,

If we add a vector that comb. of those two vectors the plane will span the space but won't be basis since it is not independent.

> Note: In a given space R^n, every basis for the space has the same number of vectors. This number is dimension of space.

The rank of A = # pivot columns = dimension of A.

$$\dim C(A) = r$$

what about the null space?

$$\dim N(A) = \text{\# free variables} = n - r$$

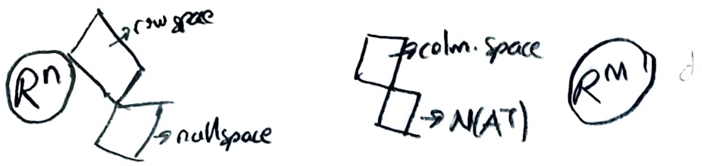
Lecture-10 The Four Fundamental ~~Basis:~~ Subspaces:

column space $C(A)$ in R^m — A is $m \times n$

null space $N(A)$ in R^n

row space = all comb. of rows = all comb. of the columns of $A^T = C(A^T)$ in R^n

null space of $A^T = N(A^T)$ = the left nullspace of A (also called) in R^m



basis of $C(A)$ = pivot colms., basis $C(A^T) = r$, basis $N(A)$ = special sol. $= n-r$

dim $C(A)$ = rank r, dim $C(A^T) = r$, dim $N(A) = n-r$

dim $N(A^T) = m-r$

What about 4th guy the $N(A^T)$?

$$A^T y = 0$$

$$\begin{bmatrix} A^T \end{bmatrix}\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \rightarrow y^T A = 0$$

$$\begin{bmatrix} y^T \end{bmatrix}\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$ → We are multiplying A from left because of that called left nullspace.

Say $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$

$$\underset{E}{\text{rref}}\begin{bmatrix} A_{m\times n} & I_{m\times m} \end{bmatrix} \rightarrow \begin{bmatrix} R_{m\times n} & E_{m\times m} \end{bmatrix}$$

$EA = R$, at previous lectures, R was I for some square matrix. Then E was A^{-1}

$$\underset{E}{\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}\underset{A}{\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Lecture 11 - Matrix Spaces:

Basis of ~~new~~ new vector spaces $\Rightarrow M =$ all 3by3 matrices , $\dim(M) = 9$

$\rightarrow$ Symmetric $3\times3 = S$, $\dim S = 6$ $\Rightarrow$ subspace for M

upper triangular $3\times3 = U$, $\dim U = 6$ $\Rightarrow$ subspace for M

$S \cap U$ = symmetric and upper triangular = diagonal 3×3 , $\dim(S \cap U) = 3$ $\Rightarrow$ subspace for M

$S \cup U$ = symmetric or upper triangular or both. $\Rightarrow$ not subspace

$S + U$ = any element of S + any element of U = all 3×3 matrices. , $\dim(S+U) = 9$

$$\dim S + \dim U = \dim(S \cap U) + \dim(S+U)$$

Suppose I have a diff. eq. like

$$\frac{d^2y}{dx^2} + y = 0 \Rightarrow \text{complete sol'ns: } y = c_1\cos x + c_2\sin x$$

$y = \cos x, \sin x$

Rank 1 Matrices:

Let A be 2×3 matrix

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} \quad \dim C(A) = \text{rank } r = \dim C(A^T)$$

$$= \underset{2\times1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \underset{1\times3}{\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}} \Rightarrow \text{Rank 1 matrix,} \quad A = u \cdot v^T$$

Rank 1 matrices are like build blocks for any $m\times n$ matrix, and rank r we say that r rank 1 matrix will need to build $m\times n$ matrix.

Let M be a 5×17 matrices, subset of rank 4 matrices, is that a subspace?

Well could be but the rank can't be bigger then 5. We can't know exactly.

In R^4 $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$, S = all v in R^4 with $v_1 + v_2 + v_3 + v_4 = 0$ $\quad Av = 0$

= null space of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ $\quad$ rank $= 1 = r$

$\dim N(A) = n - r = 4 - 1 = 3$

Basis for S

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\dim C(A^T) = 1$

$\dim C(A) = 1$

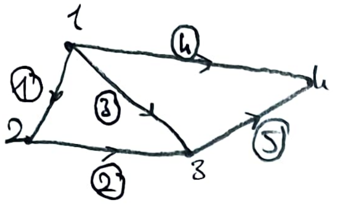
$\dim N(A^T) = \{0\} = 0$

$\dim C(A^T) + \dim N(A) = 4 = n$

$\dim C(A) + \dim N(A^T) = 1 = m$

Lecture. 12 : Applications of Linear Algebra {21th Centuries most important Math Workspace}

Graphs in linear algebra produced by nodes and edges.



$n = 4$, nodes

$m = 5$, edges

$$A = \begin{array}{c} \text{node } 1\ 2\ 3\ 4 \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \text{edge } 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

edge 1, 2, 3 } loop

$A \downarrow$ Incidence Matrix

$$Ax = 0 = A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \dim N(A) = 1$$

$x = x_1, x_2, x_3, x_4, x_5$, potentials at the nodes. r rank $= 3$

$\downarrow A$, $\boxed{e = Ax}$

$x_2 - x_1$ etc, their potential differences $\xrightarrow{\text{ohm's law}}$ $\boxed{y = Ce}$ currents on edges y_1, y_2, y_3, y_4, y_5

Kirchoff Current Law

$A^T \uparrow$ $\boxed{A^T y = 0}$

$A^T y = f$

If we have zero potential difference, then all potentials at edges must be equal.

$A^T y = 0$, $\dim N(A^T) = m - r = 5 - 3 = 2$

$\downarrow$

$$\underset{4\times 5}{n \times m} \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -y_1 - y_3 - y_4 = 0 \\ y_1 - y_2 = 0 \\ y_2 + y_3 - y_4 = 0 \\ y_4 + y_5 = 0 \end{array}$$

Basis for $N(A^T)$

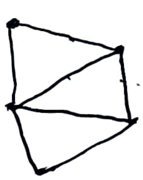
$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ +1 \end{bmatrix}$$

Tree: Graphs with no loops.

$\dim C(A^T) = 3$

$\dim N(A^T) = m - r$

$\#\text{loops} = \#\text{edges} - (\#\text{nodes} - 1) \Rightarrow \underset{\text{2-dim}}{\#\text{loops}} + \underset{\text{0-dim}}{\#\text{nodes}} - \underset{\text{1-dim}}{\#\text{edges}} = 1 \Rightarrow$ Euler's Formula {Again that guy}



$\Rightarrow 5 - 7 + 3 = 1$

In conclusion

$A^T \cdot C A x = f$

Linear Algebra Review for 12 lectures:

1-) Suppose u, v and w in $\mathbb{R}^7$, they span a vector space. What are the possible dim of subspace?

Answer: 0, 1, 2, 3.

2-) 5×3 matrix U, in echelon form, $r=3$.
what's the null space?

Ans: $N(U) = \{0\}$. Columns are independent.

3-) $B = \begin{bmatrix} R \\ 2R \end{bmatrix}$, what's the echelon form of it?

$C = \begin{bmatrix} R & R \\ R & 0 \end{bmatrix}$ " " " " " " ?

Ans:

$B = \begin{bmatrix} R \\ 2R \end{bmatrix} \Rightarrow \begin{bmatrix} R \\ 0 \end{bmatrix} \Rightarrow r=3$

$\underset{10\times 6}{C} = \begin{bmatrix} R & R \\ R & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} R & R \\ 0 & -R \end{bmatrix} \Rightarrow \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \Rightarrow r=6$, $\dim N(C^T) = 10-6 = 4$

4-) $Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ $\qquad x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

3×3

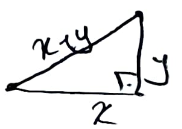
Ans: $\dim N(A) = 2$ $\qquad$ $Ax=b$ can be solved if b has the form $b = c\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Note: Don't forget $r=m$ or $r=n$

$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

Lec-14 Orthogonal Subspaces and Vectors

Orthogonal Vectors



$x^T y = 0$ ✓

Pythagoras : $\|x\|^2 + \|y\|^2 = \|x+y\|^2$

$$x^T x + y^T y = (x+y)^T (x+y) \Rightarrow \text{true when right angle}$$

$$\cancel{x^T x} + \cancel{y^T y} = \cancel{x^T x} + \cancel{y^T y} + x^T y + y^T x$$

$$0 = x^T y + y^T x \Rightarrow x^T y = y^T x = 0$$

Subspace S is orthogonal to subspace T, means that every vector in S is orthogonal to every vector in T.

Row space is orthogonal to the null space because;

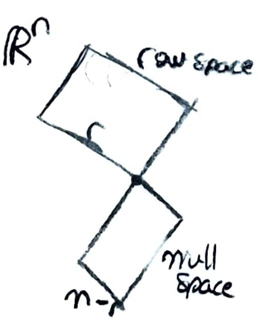
$Ax = 0$ $\quad \begin{bmatrix} \text{row } 1 \\ \text{row } 2 \\ \vdots \\ \text{row } m \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\quad$ (row)$\cdot$(x) $= 0$ is a dot product, dot product equals zero if and only if $x = 0$ or row and x is perpendicular to each other.

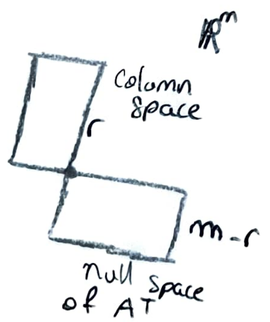
$c_1 (\text{row}_1)^T x = 0$

$c_2 (\text{row}_2)^T x = 0$

$\vdots$

$(c_1 \text{row}_1 + c_2 \text{row}_2 + \dots)^T x = 0$





$A = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

$n = 3$

$r = 1 \qquad \dim N(A) = 2$

So $C(A^T) \perp N(A)$ $\qquad$ Null space and row space are orthogonal complements in R^n.

$C(A) \perp N(A^T)$ $\qquad$ Null space contains all vectors $\perp$ row space.

Now we have another problem to discuss and solve

Solve $Ax=b$ when there is no solutions.
$m > n$

Some measurement have noise, some error rate on them and sometimes you make a thousand measurement for just a few unknowns.

$\underset{n\times m}{A^T}\underset{m\times n}{A}$ → $n\times n$ symmetric $(A^TA)^T = A^TA$

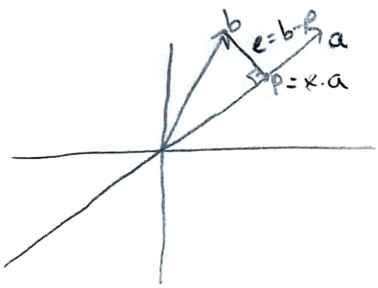
Let us give the good eq. for $Ax=b$ is;

$$A^TA\hat{x} = A^Tb$$

I also interested in that A^TA is invertible or not.

$N(A^TA) = N(A)$
rank of A^TA = rank of A. Conclusion: A^TA is invertiable exactly if A has independent columns.

Lec. 15 Projections onto Subspaces



$a^T(b-xa)=0$

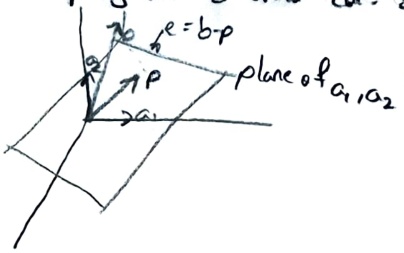
$xa^Ta = a^Tb$

$x = \frac{a^Tb}{a^Ta}$, $p = ax = a\frac{a^Tb}{a^Ta}$, proj $p = P.b$

$P = \frac{aa^T}{a^Ta}$, $C(P)$ = line through a, rank$(P)=1$

* $P^T=P$, If a project twice I still get the same point so $P^2=P$

First of all why do I want to project?
Because $Ax=b$ may have no solution
Solve $A\hat{x}=p$ instead.
↓ proj. of b onto col. space



= col-space of $A = [a_1, a_2]$

e is perpendicular to the plane.
$p = x_1a_1 + x_2a_2 = A\hat{x}$

$p = A\hat{x}$, find $\hat{x}$

Key: $b-A\hat{x}$ is $\perp$ to the plane.

$a_1^T(b-A\hat{x})=0$, $a_2^T(b-A\hat{x})=0$

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}\begin{bmatrix} b-A\hat{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(A^T)\underbrace{(b-A\hat{x})}_{e} = 0$$

e in $N(A^T)$, $e \perp C(A)$

$(A^T)(b-A\hat{x})=0 \Rightarrow A^TA\hat{x}=A^Tb \Rightarrow (A^TA)^{-1}A^Tb$

Lecture 16 - Projections Matrices and Least Squares

$p=A\hat{x}=A(A^TA)^{-1}A^Tb$

projection matrix $P=A(A^TA)^{-1}A^T$

I also want to write P as;

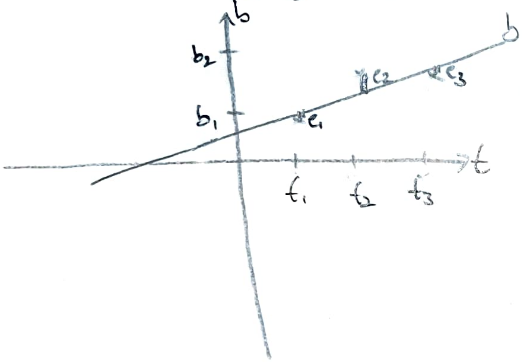
$$A(A^TA)^{-1}A^T = AA^{-1}A^{T^{-1}}A^T = I$$, can't be true because A is not a square matrix.

If A is a nice, square invertible matrix than, yes that's okay.

$P^T=P$, true for both $n\times n$, $n\times m$.

$P^2=P \quad A(A^TA)^{-1}\cancel{A^TA(A^TA)^{-1}}A^T$

Least squares fitting by a line



$b=C+Dt$

Find the best fitting line for those points.

$C+t_1D=b_1$
$C+t_2D=b_2$
$C+t_3D=b_2$

$$\underbrace{\begin{bmatrix}1 & t_1\\1 & t_2\\1 & t_3\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}C\\D\end{bmatrix}}_{x}=\underbrace{\begin{bmatrix}b_1\\b_2\\b_2\end{bmatrix}}_{b}$$

Minimize $\|Ax-b\|^2=\|e\|^2=e_1^2+e_2^2+e_3^2$

$C+t_1D-b_1=e_1$
$C+t_2D-b_2=e_2$
$C+t_3D-b_2=e_3$

Find $\hat{x}=\begin{bmatrix}\hat{C}\\\hat{D}\end{bmatrix}$, P

$$A^TA\hat{x}=A^Tb \quad \underbrace{\begin{bmatrix}1 & 1 & 1\\t_1 & t_2 & t_3\end{bmatrix}}_{A^T}\underbrace{\begin{bmatrix}1 & t_1\\1 & t_2\\1 & t_3\end{bmatrix}}_{A}=\begin{bmatrix}3 & t_1+t_2+t_3\\t_1+t_2+t_3 & t_1^2+t_2^2+t_3^2\end{bmatrix}$$

$$A^Tb=\begin{bmatrix}1 & 1 & 1\\t_1 & t_2 & t_3\end{bmatrix}\begin{bmatrix}1 & t_1 & b_1\\1 & t_2 & b_2\\1 & t_3 & b_2\end{bmatrix}=\begin{bmatrix}3 & t_1+t_2+t_3 & b_1+2b_2\\t_1+t_2+t_3 & t_1^2+t_2^2+t_3^2 & t_1b_1+t_2b_2+t_3b_2\end{bmatrix}$$

$3C+(t_1+t_2+t_3)D=b_1+2b_2$

$(t_1+t_2+t_3)C+(t_1^2+t_2^2+t_3^2)D=t_1b_1+t_2b_2+t_3b_2$

Say $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$A^T A \hat{x} = A^T b$

$$\begin{array}{cc} A^T & A \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \end{array} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$\begin{array}{cc} A^T & b \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix} \end{array} = \begin{bmatrix} 3 & 6 & 5 \\ 6 & 14 & 11 \end{bmatrix} \rightarrow \begin{matrix} 3C + 6D = 5 \\ 6C + 14D = 11 \end{matrix} \Rightarrow 2D = 1 \Rightarrow D = \frac{1}{2}$$

$C = \frac{2}{3}$

$Ax = (C + D - 1)^2 + (C + 2D - 2)^2 + (C + 3D - 2)^2$

We can also get this eq. from Calculus by partial derivatives. Which I does not know yet.

Best line is $\frac{2}{3} + \frac{t}{2} = y$, $b = pe$

$e_1 = 1/6$
$e_2 = -1/6$
$e_3 = 1/6$

If A has independent columns then $A^T A$ is invertible

Proof: Suppose $A^T A x = 0$, x must be zero.

$x^T A^T A x = 0 = (Ax)^T (Ax) = \|Ax\|^2 \Rightarrow Ax = 0 \overset{A \text{ has ind. columns.}}{\Rightarrow} x = 0$

Lec 17. Orthogonality:

Orthonormal vectors

$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \rightarrow q = \hat{u}$ $\quad Q = \begin{bmatrix} q_1 \cdots q_n \end{bmatrix}$ $\quad Q^T Q = \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} \begin{bmatrix} q_1 \cdots q_n \end{bmatrix} = I$

If Q is square then $Q^TQ = I$ tells us that

$Q^T = Q^{-1}$

Exp:

perm $Q = \begin{bmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{bmatrix} \begin{bmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{bmatrix} = I$, $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $Q = \frac{1}{2} \begin{bmatrix} 1&1&1&1 \\ 1&-1&1&-1 \\ 1&1&-1&-1 \\ 1&-1&-1&1 \end{bmatrix}$, $Q = \frac{1}{3} \begin{bmatrix} 1&-2&2 \\ 2&-1&-2 \\ 2&2&1 \end{bmatrix}$

what changes if a have an orthogonal matrix?

Say Q has orthonormal columns, project onto its column space

$P = Q(Q^TQ)^{-1}Q^T \rightarrow$ If that has orthonormal basis, then $= QQ^T$

If Q is square then $P = I$

$Q^TQ\hat{x} = Q^Tb = \hat{x} \Rightarrow Q^TQ = I$

$\hat{x}_i = q_i^Tb$

Grahama-Schmidt

vectors a,b,c are independent. Goal is get orthogonal vectors A,B,C from a,b,c Then get q's from them.

$q_1 = \frac{A}{\|A\|} \quad q_2 = \frac{B}{\|B\|} \quad q_3 = \frac{C}{\|C\|}$

$a = A$ $\qquad A \perp B$

$B = b - \frac{A^Tb}{A^TA}A$, $A^TB = A^T\left(b - \frac{A^Tb}{A^TA}A\right) = 0$

$C = c - \frac{A^Tc}{A^TA}A - \frac{B^Tc}{B^TB}B$ $\qquad C \perp A$
$\qquad C \perp B$

10

Lecture 18. Properties of Determinants

First we need to know the first three properties of determinants so we can figure out what is it.

1-) $\det I = 1 = |I|$

2-) If we make row exchange in a matrix, then its determinant will change sign.

So we can say;

$\det P = 1 \Rightarrow$ if # exchanged rows = even
$\quad\quad\;\; -1 \Rightarrow$ if # exchanged rows = odd

3-)
a-) If we multiply a row of matrix with a constant t, then its determinant will be;

$t \cdot \det A$

b-) If we add some constants on a row, then the final det. will be the sum of the det's.

Say $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

It sustains for linearity in "EACH" row seperately.

4-) 2 equal rows $\Rightarrow \det = 0$
Exchange those rows $\Rightarrow$ same matrix, but prop. 2 says there must be a sign change so $\det = 0$

5-) Subtract $\ell \times \text{row}_i$ from row_k. Det does not change.

$$\begin{vmatrix} a & b \\ c-\ell a & d-\ell b \end{vmatrix} \overset{(3b)}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \ell \overbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}^{(4)\ \to 0} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

6-) Row of zeros $\Rightarrow \det A = 0$

$t \cdot \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} t.0 & t.0 \\ c & d \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix}$, if t multipl of something equals itself then it must be only zero.

7-) $\mathcal{U} = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \dots & d_n \end{bmatrix}$, $\det \mathcal{U} = d_1 \times d_2 \times \dots \times d_n$

product of pivots.

$d_n \times d_{n-1} \times \dots \times d_1 \begin{vmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \dots \end{vmatrix}$ $\Rightarrow$ from Rule 5, 3a and 1.

8-) $\det A = 0$ when A is singular. So we can say;

$\det A \neq 0$ when A is invertible $\Rightarrow U \Rightarrow 0 \Rightarrow d_1 \cdot d_2 \dots d_n$

9-) $\det AB = (\det A)(\det B)$, $\det A^2 = (\det A)^2$

$\det 2A = 2^n (\det A)$

$\det A^{-1} = 1 / \det A$

$A^{-1}A = I \Rightarrow (\det A^{-1})(\det A) = 1$

10-) $\det A^T = \det A$

If we conclude the 10 properties we saw nothing special about row 1 or two because we can exchange them, also with the 10th prop, we see that thing that was special for rows are also valid for columns since the det of transpose equals to the original matrix.

Proof:

\# 10 $|A^T| = |A|$

$\downarrow \qquad \downarrow$

$|U^T L^T| = |LU| \rightarrow |U^T||L^T| = |L||U|$

L and L^T both triangular so det of them is 1

Lecture 19. Formulation of Determinants and Cofactors

We're gonna try to find a formula for computing det. from first three properties.

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \cancel{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}_{0} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} , \cancel{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}_{0} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$$

$$= ad \overbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}^{1} + bc \overbrace{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}^{-1} = ad - bc$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (a_{11}a_{22}a_{33}) + (-a_{11}a_{23}a_{32}) + (-a_{21}a_{12}a_{33}) + (a_{21}a_{13}a_{32}) + (a_{31}a_{12}a_{23}) + (-a_{31}a_{13}a_{22})$$

So we need to generalize it to $n \times n$

$$\det A = \sum_{n!\ \text{terms}} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega}$$

$(\alpha, \beta, \gamma, \dots \omega) = P$ of $(1, 2, \dots, n)$

Example:

$$\begin{vmatrix} \underline{1} & \underline{-1} & \underline{1} & \underline{-1} \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 0 \text{, singular}$$

Cofactors 3x3

$$\det = a_{11}(a_{22}a_{33} - a_{23}a_{32})$$
$$- a_{12}(a_{21}a_{33} - a_{23}a_{31})$$
$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31}$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Cofactors of $a_{ij} = C_{ij}$

$\pm \det$ ($n-1$ matrix with row i column j erased)

$+$: $i+j$ even, $-$: $i+j$ odd

$$\begin{vmatrix} + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \end{vmatrix}$$

$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$ (cofactor along row_1)

Exp.

$$A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ & & 1 & 1 \end{vmatrix}$$

$|A_1| = 1$, $|A_2| = 0$, $|A_3| = -1$

$|A_4| = 1 \cdot |A_3| - 1|A_2|$

$|A_n| = |A_{n-1}| - |A_{n-2}| \rightarrow$ true for all n.

Lecture 20. Cramer's Rule, Inverse Matrix and Volume

$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, $A^{-1} = \frac{1}{\det A} C^T$ → product of $n-1$ entries

$\det A$ → product of n entries

Check $AC^T = (\det A) I$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & & a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \vdots \\ C_{n1} & & C_{nn} \end{bmatrix} = \begin{bmatrix} \det A & & \\ & \det A & \\ & & \ddots & \\ & & & \det A \end{bmatrix} \checkmark$$

$Ax = b$

$x = A^{-1}b = \frac{1}{\det A} C^T b$

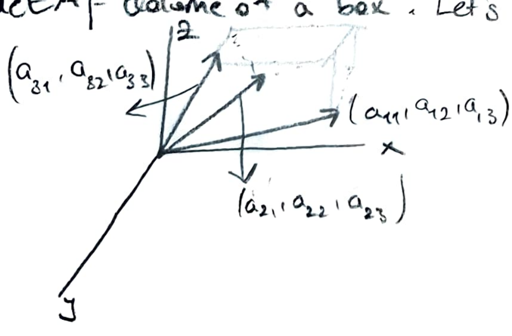
Cramer's Rule

$x_1 = \frac{\det B_1}{\det A}$ $\quad B_1 = \begin{bmatrix} b & \begin{matrix} n-1 \\ \text{columns} \\ \text{of } A \end{matrix} \end{bmatrix}$ = A with column 1 replaced by b.

$\vdots$

$x_j = \frac{\det B_j}{\det A}$ $\quad B_j$ = A with column j replaced by b.

$|\det A|$ = volume of a box. Let's talk on 3-d so we can visulize

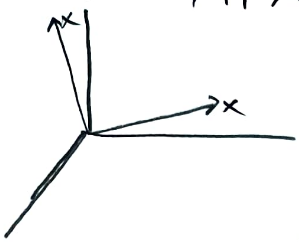


Note: If you want to see why true watch lec 20, 36th minute.

Lecture. 21 Eigenvectors - Eigenvalues:

Ax parallel to x. Eigenvalue

$Ax = \lambda x$, If A is singular , $\lambda = 0$ is eigenvalue



what are x's and λ's for projection matrix?

Any x in plane: $Px = x$, $\lambda = 1$
Any $x \perp$ plane: $Px = 0 \cdot x, \lambda = 0$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda = 1$

$x = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$ $\quad Ax = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \lambda = -1$

Fact: Sum of λ's

$= a_{11} + a_{22} + \cdots + a_{nn}$

How to solve $Ax = \lambda x$

Rewrite: $(A - \lambda I) \cdot x = 0$, $\det(A - \lambda I) = 0$

$\downarrow$
Singular

Find λ first.

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = 0 = (\lambda - 4)(\lambda - 2)$

$\lambda_1 = 4 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad$ If $Ax = \lambda x$, $(A + 3I)x = \lambda x + 3x = (\lambda + 3)x$

$\lambda_2 = 2 \Rightarrow x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Not so great for $A+B$, AB

If $Ax = \lambda x$, B has eigenvalue $\alpha_1, \dots, \alpha_n$
$Bx = \alpha x$

then

$(A+B)x = (\alpha + \lambda)x \Rightarrow$ not true since, eigenvector of B does not have to equal to the ones in A.

$Ax = \lambda x$
$By = \alpha y$

Example:

90° rotation. $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ trace $= 0+0=0 = \lambda_1+\lambda_2$

det $Q = 1 = \lambda_1 \lambda_2$

$$\det(Q-\lambda I) = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2+1=0$$

$$\Rightarrow \lambda_1 = i$$
$$\lambda_2 = -i$$

Example:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}, \det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2$$

$$\Rightarrow \lambda_1 = 3, \lambda_2 = 3$$

$$(A-\lambda I)x = 0$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} - \end{bmatrix} = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$, $x_2 =$ no 2nd independent eigenvalue.

Lecture. 22. Diagonalization and Powers of A

Suppose n independent eigen vectors of A. $A-\lambda I$ singular and $Ax=\lambda x$

Put them in columns of S.

$$A \cdot S = A\begin{bmatrix} x_1 x_2 \cdots x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 \cdots \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 \cdots x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = S\Lambda$$

diagonal eigen matrix Λ

$AS = S\Lambda$

$S^{-1}AS = \Lambda$

$\Rightarrow A = S\Lambda S^{-1}$

If $Ax = \lambda x$

$A^2x = \lambda Ax = \lambda^2 x$

$\rightarrow A^2 = S\Lambda S^{-1} S\Lambda S^{-1}$

$= S\Lambda^2 S^{-1}$

$A^k = S\Lambda^k S^{-1}$

Theorem:

$A^k \to 0$ as $k \to \infty$ if all $|\lambda_i| < 1$.

A is sure to have n independent eigen vectors and be diagonalizable

If all the λ's are different.

(no repeated λ's)

Repeated eigen values may or may not have n independent eigen vectors.

Equation $u_{k+1} = Au_k$
start with a given vector u_0.

$u_1 = Au_0$, $u_2 = A^2u_0$, $u_k = A^k u_0$

To really solve: write

$u_0 = c_1x_1 + c_2x_2 + \cdots + c_nx_n = Sc$

$A^{100}u_0 = A^{100}(c_1x_1 + \cdots + c_nx_n) = c_1\lambda_1^{100}x_1 + c_2\lambda_2^{100}x_2 + \cdots + c_n\lambda_n^{100}x_n = S\Lambda^{100}c = u_{100}$

Fibonacci Example: $0, 1, 1, 2, 3, 5, 8, 13, \ldots, F_{100} = ?$

$F_{k+2} = F_{k+1} + F_k$ → second order eq. not a system or first order. der.

$F_{k+1} = F_{k+1}$, $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$, $u_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} u_k$

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (-\lambda + \lambda^2) - 1 = \lambda^2 - \lambda - 1 = 0$

$\lambda_1 = \frac{1+\sqrt{5}}{2}$, $x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ $\qquad \lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$\lambda_2 = \frac{1-\sqrt{5}}{2}$, $x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$c_1x_1 + c_2x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\varphi = \frac{1+\sqrt{5}}{2}$, $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$.

Since A is diagonalizable, we can express A^n using eigenvalues.

$A^n = SD^nS^{-1}$, where D is diagonal matrix

$D^n = \begin{bmatrix} \varphi^{100} & 0 \\ 0 & \bar{\varphi}^{100} \end{bmatrix}$, Using the Fibonacci Sequences;

$F_k = \frac{\varphi^k - \bar{\varphi}^k}{\sqrt{5}} \Rightarrow F_{100} = \frac{\varphi^k - \bar{\varphi}^k}{\sqrt{5}}$.

Lecture 23. First Order Constant Coeff. Linear ODE's:

$\frac{dx_1}{dt} = -x_1 + 2x_2$, $\frac{dx_2}{dt} = x_1 - 2x_2$ $\rightarrow A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$: Singular matrix. , $\lambda = 0, -3$

$\lambda_1 = 0$, $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} c_1$, $Ax_1 = 0x_1$. , $\lambda_2 = -3$, $\vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} c_2$., $Ax_2 = -3x_2$.

Solution: $x(t) = c_1 \vec{x}_1 e^{\lambda_1 t} + c_2 \vec{x}_2 e^{\lambda_2 t} \approx c_1 \lambda_1^k \vec{x}_1 + c_2 \lambda_2^k \vec{x}_2$: $u_{k+1} = Au_k$.

Check: $\frac{dx}{dt} = Ax$. Plug in $e^{\lambda_1 t} x_1$, $\lambda_1 e^{\lambda_1 t} \vec{x}_1 = Ae^{\lambda_1 t} x_1 \Rightarrow \lambda_1 \vec{x}_1 = A\vec{x}_1$. Checked.

$c_1 e^{\lambda_1 t} \vec{x}_1 + c_2 e^{\lambda_2 t} \vec{x}_2 \Rightarrow c_1 1 \vec{x}_1 + c_2 e^{-3t} \vec{x}_2$. Use initial conditions to find c_1, c_2.

At $t = 0$.

$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, sol'n: $c_1 = c_2 = 1/3 \Rightarrow \frac{1}{3}\vec{x}_1 + \frac{1}{3}\vec{x}_2 \cdot e^{-3t}$ } SSS $\Rightarrow (1/3)\vec{x}_1$ as $t \to \infty$ since $e^{-3t} \to 0$.

1. Stability, $x(t) \to 0$ / need $e^{\lambda t} \to 0$ / Re part of $\lambda < 0$.
2. Steady-State Sol'n.
   $\lambda_1 = 0$ and other Re $\lambda < 0$
3. Blow up if any Re of $\lambda > 0$.

2x2 stability condition
Re $\lambda_1, \lambda_2 < 0$
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: trace $= a + d = \lambda_1 + \lambda_2 < 0$.
det > 0 $(\lambda_1 \cdot \lambda_2)$

$[\vec{x}_1 \ \vec{x}_2] = S$.

$\frac{dx}{dt} = Ax$, Set $x = Sv$.

$S\frac{dv}{dt} = ASv \Rightarrow \frac{dv}{dt} = S^{-1}ASv = \Lambda v$, $\frac{dv_1}{dt} = \lambda_1 v_1$; $v(t) = e^{\Lambda t} v(0)$, $x(t) = Se^{\Lambda t} S^{-1} x(0)$ } $e^{At} = Se^{\Lambda t} S^{-1}$

Matrix Exponential, e^{At}:

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots + \frac{(At)^n}{n!} + \dots$$

$$(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \dots \text{ if } |\lambda(At)| < 1.$$

$$e^{At} = SS^{-1} + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1} t^2}{2!} + \frac{S\Lambda^3 S^{-1} t^3}{3!} + \dots + \frac{S\Lambda^n S^{-1} t^n}{n!} = Se^{\Lambda t} S^{-1}$$

but S must be inverseable and A can be diagonalizable.

$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, $e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$, if $e^{At} \to 0$ then λ's must be negative.

Exp. $y'' + by' + ky = 0$

$u = \begin{bmatrix} y' \\ y \end{bmatrix}$, $u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$

Lecture 24. Markov Matrices, Fourier Series and Projections.

Conditions of a Markov Matrix
1-) All entries must be ≥ 0
2-) All columns must add up to 1.

Markov matrix generally associated with probability of something.

$$A = \begin{bmatrix} a & c & e \\ b & d & f \\ 1-a-b & 1-c-d & 1-e-f \end{bmatrix}$$

Identities of Markov Matrix:
1-) $\lambda = 1$, is an eigenvalue
2-) All either eigenvalue $|\lambda_i| < 1$.
3-) Eigenvalues of A is equal to Eigenvalues of A^T.

$$(A - 1I) = \underbrace{\begin{bmatrix} a-1 & c & e \\ b & d-1 & f \\ 1-a-b & 1-c-d & 1-e-f \end{bmatrix}}_{\text{all columns are add up to zero.}}, \text{ we want to see that } A-I \text{ is singular.}$$

If we add up rows $(1,1,1)$ equals to zero so its dependent because $(1,1,1)$ is in the $N(A^T)$.
Then $\vec{x}_1$ is in the $N(A)$.

$$\det(A - \lambda I) = 0. \quad N(A) = \left\{ \begin{bmatrix} \frac{-e}{a-1} \\ \frac{-f}{d-1} \\ 1 \end{bmatrix} \right\}$$
$$\Downarrow$$
$$\det(A^T - \lambda I) = 0.$$

$u_{k+1} = A u_k$, A is Markov

Steady State of Markov.

$A^k u_0 = c_1 \lambda_1^k \vec{x}_1 + c_2 \lambda_2^k \vec{x}_2 + \dots + c_n \lambda_n^k \vec{x}_n = u_k$, as $k \to \infty$
Solution tends to $c_i \lambda_i^k \vec{x}_i$ for $\lambda_i = 1$.
All other λ's tends to zero.
So sol'n becomes; $c_i \lambda_i^k \vec{x}_i = c_i \vec{x}_i$.

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=k+1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_k, \quad \begin{bmatrix} u_{call} \\ u_{mass} \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$$

$$\begin{bmatrix} u_{ca} \\ u_{m} \end{bmatrix}_\infty = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$$

$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}; \lambda = 1, \tfrac{7}{10}, \quad \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \overset{\vec{x}_1}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}, \quad .7 \begin{bmatrix} .2 & .2 \\ .1 & .1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \overset{\vec{x}_2}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \Rightarrow u_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1000}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{2000}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Projections with Orthonormal Basis

$q_1, \dots, q_n$

$\vec{V} = x_1 q_1 + x_2 q_2 + \dots + x_n q_n$; $Qx = V \Rightarrow x = Q^{-1} V = Q^T V \Rightarrow x_1 = q_1^T V$.

$q_1^T V = x_1 q_1^T q_1 + 0 \dots + 0$.

Fourier Series:

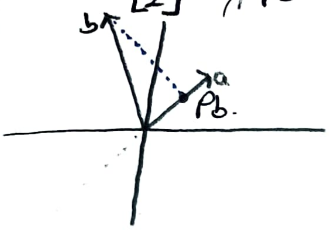
$f(x) = a_0 1 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \dots$

$v^T w = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$, $f^T g = \int_0^{2\pi} f(x) g(x) dx$, $a_1 \underbrace{\int_0^{2\pi} (\cos(x))^2 dx}_{=\pi} = \int_0^{2\pi} f(x) \cos(x) dx$

$$\Rightarrow a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(x) dx$$

Review - 2:

1-) $a=\begin{bmatrix}2\\1\\2\end{bmatrix}$; $P=A(A^TA)^{-1}A^T = \frac{aa^T}{a^Ta} = \frac{1}{9}\begin{bmatrix}2\\1\\2\end{bmatrix}\begin{bmatrix}2&1&2\end{bmatrix} = \frac{1}{9}\begin{bmatrix}4&2&4\\2&1&2\\4&2&4\end{bmatrix}$; $\lambda=0,0,1.$



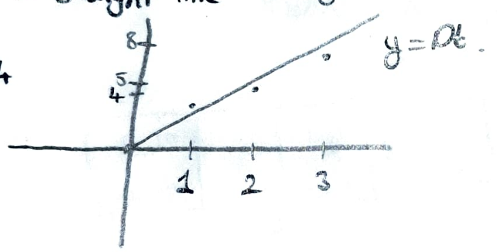
$Pa = \frac{aa^T}{a^Ta}\cdot a = a.$

$u_{k+1}=P.u_k$, $u_1 = Pu_0 = a\frac{a^Tu_0}{a^Ta} = a\frac{27}{9} = \overset{\hat{x}}{3}a. = \begin{bmatrix}6\\3\\6\end{bmatrix}$; $u_k = P^k u_0 = Pu_0 = \begin{bmatrix}6\\3\\6\end{bmatrix}$

$u_0=\begin{bmatrix}9\\9\\0\end{bmatrix}$, $u_0 = c_1x_1+c_2x_2+c_3x_3$; $A^k u_0 = c_1\lambda_1^k x_1 + c_2\lambda_2^k x_2 + c_3\lambda_3^k x_3.$

2-) Fiting a straight line throught the origin;

$t=1, y=4$
$2, 5.$
$3, 8.$



$1.D=4$
$2.D=5$
$3.D=8.$

$\underset{A}{\begin{bmatrix}1\\2\\3\end{bmatrix}}\underset{x}{D}=\underset{B}{\begin{bmatrix}4\\5\\8\end{bmatrix}}$; $A^TA\hat{D}=A^Tb.$

$14\hat{D}=38$

$\Rightarrow \hat{D} = 38/14.$

Projecting b onto column space of A (line)

$a_1=\begin{bmatrix}1\\2\\3\end{bmatrix}$, $a_2=\begin{bmatrix}1\\1\\1\end{bmatrix}$ → plane = colmn. space of A.

(orthogonal too) $B=\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{a_1^Ta_2}{a_1^Ta_1}a_1 . = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{6}{14}\begin{bmatrix}1\\2\\3\end{bmatrix}$

3-) 4×4; $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Under what conditions on λ's this matrix invertable?

a-) Invertiable $\Leftrightarrow$ no-zero eigenvalues.

b-) $\det(A^{-1}) = \left(\frac{1}{\lambda_1}\right)\left(\frac{1}{\lambda_2}\right)\left(\frac{1}{\lambda_3}\right)\left(\frac{1}{\lambda_4}\right)$

c-) what's the trace of $A+I = (\lambda_1+\lambda_2+\lambda_3+\lambda_4)+4.$

4-)

$A_4=\begin{bmatrix}1&1&0&0\\1&1&1&0\\0&1&1&1\\0&0&1&1\end{bmatrix}$ use cofactors to find

$D_n = 1\,D_{n-1} + -1\,D_{n-2} = D_{n-1} - D_{n-2}.$ } Solve this writing as a system. $D_1=1$, $D_2=0$.

$D_n=\det(A_n)$

$\begin{bmatrix}D_n\\D_{n-1}\end{bmatrix} = \begin{bmatrix}1&-1\\1&0\end{bmatrix}\begin{bmatrix}D_{n-1}\\D_{n-2}\end{bmatrix}$; $\lambda = \frac{1\pm\sqrt{-3}}{2} = \frac{1\pm3i}{2}$; $\Rightarrow e^{i\pi/3}, e^{-i\pi/3}.$

Lecture 25. Symmetric Matrices:

- Spectral Theorem for Real Symmetric Matrices:

Let $A \in \mathbb{R}^{n\times n}$ be real and symmetric. Then;

i-) The eigenvalues of A are real.

ii-) A is diagonalizable.

iii-) There is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

* A may be orthonormally diagonalized: $A = V\Lambda V^T$ where $V \in \mathbb{R}^{n\times n}$ is an orthonormal matrix of eigenvectors of A, and $\Lambda \in \mathbb{R}^{n\times n}$ is a real diagonal matrix of eigenvalues.

Proof:(i): Take an arbitrary eigenvalue λ of A with corresponding eigenvector $\vec{x}$.

Then

$$A\vec{x} = \lambda\vec{x}. \quad (1)$$

Taking the conjugate of both sides and using the fact that A is real so that $A = \bar{A}$, we get

$$A\overline{\vec{x}} = \bar{A}\overline{\vec{x}} = \overline{A\vec{x}} = \overline{\lambda\vec{x}} = \bar{\lambda}\overline{\vec{x}}. \quad (2)$$

Taking advantage of the fact that A is symmetric so that $A = A^T$, we take the transpose of both sides to get;

$$A\overline{\vec{x}} = \bar{\lambda}\overline{\vec{x}} \Rightarrow \overline{\vec{x}}^T A^T = \bar{\lambda}\overline{\vec{x}}^T \Rightarrow \overline{\vec{x}}^T A = \bar{\lambda}\overline{\vec{x}}^T$$

Then we multiply by $\vec{x}$ on both sides;

$$\overline{\vec{x}}^T A\vec{x} = \bar{\lambda}\overline{\vec{x}}^T\vec{x} \Rightarrow \lambda\overline{\vec{x}}^T\vec{x} = \bar{\lambda}\overline{\vec{x}}^T\vec{x}$$

$$0 = (\lambda - \bar{\lambda})\overline{\vec{x}}^T\vec{x}$$

using the fact that $(\vec{x}, \lambda)$ is an eigenvector-eigenvalue pair of A. Now $\vec{x}$ is non-zero if and only if $\vec{x}$ is nonzero.

$$\overline{\vec{x}}^T\vec{x} = \sum_{i=1}^{n} \bar{x}_i \cdot x_i = \sum_{i=1}^{n} |x_i|^2 \quad , \text{ so } \overline{\vec{x}}^T\vec{x} \text{ is non-zero if and only if } \vec{x} \text{ is nonzero, and thus that } \overline{\vec{x}}^T\vec{x} \text{ is nonzero,}$$

Since $\vec{x}$ is an eigenvector and thus nonzero, we know that $\overline{\vec{x}}^T\vec{x}$ is nonzero, and thus that $\lambda - \bar{\lambda} = 0$. Thus $\lambda = \bar{\lambda}$ so λ is real. Since λ is an arbitrary eigenvalue, all eigenvalues of A are real.

Proof{ii} and {iii} are explained at online notes.

Positive Definite Matricies:

Let $A \in \mathbb{R}^{n \times n}$ with positive entries and if the real number $x^T A x$ is positive for every non-zero column vector x.. All eigenvalues and all pivots are must be positive.

All subdeterminants of $n \times n$ matrix must be positive.

Lecture. 26. Complex Matricies:

$z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$, length of $z = \bar{z}^T z$, $z^H = \bar{z}^T$: Hermition.

$z \in \mathbb{C}^n$

In complex symmetric matricies $A^T \neq A$. $\bar{A}^T = A^H = A$.

Perpendicular

$q_1, q_2, \dots, q_n$. $q_i q_j = \begin{Bmatrix} 0 & i \neq j \\ 1 & i = j \end{Bmatrix}$, $Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$, $Q^T Q = I$ for $Q \in \mathbb{R}^{n \times n}$.

q's are unit length.

for $Q \in \mathbb{C}^{n \times n}$; $I = Q^H Q$ orthogonal also referred as unitary.

The most famous Complex Symmetric Matrix:

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix} \quad F_{ij} = \omega^{ij} \quad , \omega^n = 1, \omega = e^{i2\pi/n}. = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.$$

$ij = 0, \dots, n-1$.

For case $n=4$. $\omega^4 = 1$, $\omega = e^{2\pi i/4} = i$.

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & 1 \\ 1 & -i & -1 & -i \end{bmatrix}$$

, Since the col'mns are orthogonal we can find inverse easily. | Inner products must be zero.

If we devide the matrix with its length matrix becomes orthonormal. $\lg(F_4) = 2$

$\omega_{64} = e^{i2\pi/64}$, $\omega_{32} = e^{i2\pi/32} \Rightarrow (\omega_{64})^2 = \omega_{32}$

$$F_{64} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & \vdots & & 1 & 0 \\ 0 & 1 & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & & & \\ & \omega & & \\ & & \omega^2 & \\ & & & \ddots \ \omega^{31} \end{bmatrix} \Bigg\} \; 64^2 \to 2[32]^2 + 32 \to 2\left[[16]^2 + 16\right] + 32 \to \dots$$

A $2n$ sized Fourier Transform F times x which might require $(2n)^2 = 4n^2$ operations can instead be performed using two size n Fourier transforms plus two very simple matrix multiplication which require on the order of n multiplications. The matrix P picks out the even components $x_0, x_2, \dots$ of a vector first, then the odd ones. Thus we can do a Fourier transform of size 64 on a vector by seperating the vectors into odd and even components, performing a size 32 Fourier transform on each half of its components then recombining the two halves with multiplication by D. Instead n^2 operations we get

$\frac{1}{2} n \log_2 n$ operations.

Lecture 27.

Positive definite matricies:

- must be symmetric
- must be one of ; a-) all $\lambda_i > 0$, b-) all subdeterminents > 0, c-) all pivots > 0 or d-) $x^T A x > 0$ for all $x \neq 0$.

If a is singular for some $\lambda_i = 0$; some pivot position is also zero and also its some subdeterminant = zero, we call this matrix A "semidefinite".

Testing for $x^T A x > 0$ to see positive definite matrix.

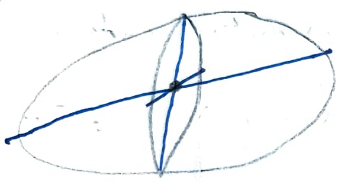
* if A is a positive definite $n \times n$ matrix, $x^T A x$ makes an $n+1$ dimensional paraboloid.

Example:

$$\underbrace{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}}_{A} \quad \text{dets} = 2, 3, 4 \quad \text{pivots}: 2, \tfrac{3}{2}, \tfrac{4}{3}, \quad \text{eigen v.}: 2 \pm \sqrt{2}, 2.$$

$$x^T A x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 - 2x_2x_3 > 0.$$

$$f = x^T A x.$$



$Q \Lambda Q^T = A.$ To find its minima, we need 2nd derivative test.

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \Rightarrow \text{this also must be positive definite. when 1st der.} = 0.$$

→ The length of the axises vary along eigenvalues $\lambda_1, \lambda_2, \lambda_3$, and they're in the dimention of the eigenvectors x_1, x_2, x_3

If $x^T A x > 0$ {except zero vector}, if A, B are pos. definite then $x^T(A+B)x > 0$; A+B is also
$x^T B x > 0$
definite positive matrix.

$$x^T V^T V x = (x^T V^T)(Vx) = (Vx)^T (Vx) = \|Vx\|^2 \geq 0.$$

Now V is an $m \times n$ matrix, rank of $V = n$.

Lecture 28. Similar Matricies and Jordan Form:

A and B are $n \times n$ matricies and similar under the condition

$B = M^{-1} A M$ for some invertable Matrix M.

$$\left.\begin{aligned} & Ax = \lambda x, \quad B = M^{-1} A M \\ & \underbrace{(M^{-1} A M)}_{B} M^{-1} x = \lambda M^{-1} x \\ & B M^{-1} x = \lambda M^{-1} x \end{aligned}\right\} \begin{aligned} & \text{Eigenvalues of } A = \text{eigenvalues of } B. \\ & \text{Eigenvectors of } B = M^{-1}(\text{eigenvectors of } A). \end{aligned}$$

If some $\lambda_i = \lambda_j$ matrix may not be diagonalizable.

$$\text{Exp.} \left.\begin{aligned} & \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\ & \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \end{aligned}\right\} \text{Same } \lambda\text{'s but 1st one is in a different matrix family.}$$

Jordan's Thm:

Every square matrix A is similar to a Jordan matrix J.

$$J = \begin{bmatrix} \boxed{J_1} & & \\ & \ddots & \\ & & \boxed{J_d} \end{bmatrix}$$

# of blocks = # of e.vectors. for all A.

Jordan block in form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix} \Rightarrow \text{has 1 eigenvector.}$$

Lecture 29. Singular Value Decomposition {SVD}

Row space of $\mathbb{R}^n \rightarrow$ colmn. space of $\mathbb{R}^m$

v_1, v_2 are orthonormal vectors in $\mathbb{R}^n$ and u_1, u_2 are orthonormals in $\mathbb{R}^m$.

Then

$$\sigma_1 u_1 = A v_1 \text{ and } \sigma_2 u_2 = A v_2 \; ; \quad A \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_r \end{bmatrix}$$

$$AV = U\Sigma \Rightarrow A = U\Sigma V^{-1} = U\Sigma V^T$$

$$\Rightarrow A^T A = V\Sigma \underbrace{U^T U}_{I} \Sigma V^T = V \Sigma^2 V^T$$

$$\Sigma^2 \approx A^T A$$

normalized by dividing its length.

Example 1: $\underbrace{\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}}_{A}$, $A^T A = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow 32 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow 18 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & +1/\sqrt{2} \end{bmatrix}}_{V^T}$$

Finding U

$$AA^T = U\Sigma V^T V \Sigma^T U^T = U \Sigma \Sigma^T U^T.$$

$$AA^T = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}; \; AA^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 32 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; AA^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 18 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{V^T}$$

$v_1, v_2, \ldots, v_r$ orthonormal basis for row space
$u_1, u_2, \ldots, u_r$ " " " column space
$v_{r+1}, \ldots, v_n$ " " " null space
$u_{r+1}, u_{r+2}, \ldots, u_n$ " " " $N(A^T)$.

and $\underline{A v_i = \sigma_i u_i}$.

Use this to determine sign of eigenvectors.

Linear Transforms. Lecture 30:

All transformations termed linearly if

1-) $T(x+y) = T(x) + T(y)$
2-) $T(cx) = c\,T(x)$

generally: $T(cx+dy) = c\,T(x) + d\,T(y)$.

Some Linear Transforms

- Projection: $\mathbb{R}^n \to \mathbb{R}^n$

- Rotation of Vectors: $\mathbb{R}^n \to \mathbb{R}^n$, 45° rotation ...

* Linear Transformations can be thinked as things that shifts the vector spaces into other vector spaces.

Transformations exists with only needing a coordinate axis: they're abstract in a way, but to write down a matrix we need coordinates. We can define any basis we want.

Say that $T(v) = Av$, How many vectors do we need to know the transformation of to define A?

$T: \mathbb{R}^3 \to \mathbb{R}^2$: If we know $T(v_1)$ and $T(v_2)$ we would know all $T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$. plane of v_1, v_2.

For an m-dimensional input: $T(v) = c_1 T(v_1) + \dots + c_m T(v_m)$.

If we used $v_1, \dots, v_m$ as basis then $c_1, \dots, c_m$ are our coordinates.

Construct matrix A that represents linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$.

- Choose basis $v_1, \dots, v_n$ for inputs in $\mathbb{R}^n$.
- Chose basis $w_1, \dots, w_m$ for outputs in $\mathbb{R}^m$

So to generalize a transformation to input bases $v_1, \dots, v_n$ and output bases $w_1, \dots, w_m$ expressed as some linear comb. of w.

1-) Apply transformation to every basisvector of v,

$$T(v_1) = c_{11}w_1 + c_{21}w_2 + \dots + c_{m1}w_m$$
$$T(v_2) = c_{12}w_1 + c_{22}w_2 + \dots + c_{m2}w_m \quad \text{and so on till } v_n.$$

2-) Since v is the input basis, $T(v) = d_1T(v_1) + d_2T(v_2) + \dots + d_nT(v_n)$.

3-) Assemble into matrix.

$$T(v) = \underbrace{\begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix}}_{T(v_1)} d_1 + \underbrace{\begin{bmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{m2} \end{bmatrix}}_{T(v_2)} d_2 + \dots \longrightarrow \underbrace{T(v)}_{\text{in w basis.}} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}}_{v}$$

Lecture 31 was all examples about Linear transformations and Lecture 32 was review of last 8 lectures.

Lecture 33 Right- Left and Pseudo Inverses:

Remember that; $AA^{-1} = A^{-1}A = I$ when $r=m=n \Rightarrow$ full rank, two-sided inverse.

Left Inverse, full column rank $r = n < m$ nullspace $= \{0\}$, independent columns.

$(A^TA)^{-1}$ exists and $\underbrace{(A^TA)^{-1}A^T}_{A^{-1}_{left}} A = I = \underset{n\times m}{A^{-1}_{left}} \cdot \underset{m\times n}{A} = I_{n\times n}$.

Right-Inverse, full row rank $= m < n$, $N(A^T) = \{0\}$. independent rows. ∞ solutions to $Ax = b$. $n-m$ free variables.

$A\underbrace{A^T(AA^T)^{-1}}_{A^{-1}_{right}} = I = AA^{-1}_{right} = I$

$AA^{-1}_{left} = A(A^TA)^{-1}A^T = P$ onto column space

$A^{-1}_{right}A = A^T(AA^T)^{-1}A = P$ onto row space.

If $\vec{x}, \vec{y}$ are vectors in row space and $\vec{x} \neq \vec{y}$ then column space $A\vec{x} \neq A\vec{y}$.

Proof: Suppose $Ax = Ay$, $A(x-y) = 0 \Rightarrow$ In null space and also row space.

$x \to Ax \Rightarrow x = A^+(Ax)$
$y \to Ay \Rightarrow y = A^+(Ay)$ } Called Pseudo Inverse.

How to find pseudo inverse A^+

1-) Start from SVD $= A = U\Sigma V^T$, $\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r & \\ & & & 0 \end{bmatrix}$; $m\times n$, rank r, $\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_r & \\ & & & 0 \end{bmatrix}$, $n\times m$, $\Sigma\Sigma^+ = \underset{m\times m}{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}}$

$\Sigma^+\Sigma = \underset{n\times n}{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}}$, You get projections onto row and column spaces.

$A^+ = V\Sigma^+U^T$.

Lecture. 34. Linear Algebra Review Lecture:

1-) Given

$Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has no solutions, $m=3$.

$Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has 1 solutions, $N(A)=\{0\}$. $r=n$

$\Big\}$ $m=3>n=r$. $\Big\}$ Find an A

$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

TF.

1-) $\det \overset{I}{A^TA} = \det AA^T$, X

2-) A^TA is invertible, ✓ since $r=n$.

3-) AA^T is pos. definite, X

$A^TA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\underset{n\times m}{A^T} y = c$ at least 1 sol'n for every c and in fact ∞ many solutions.

It has at least 1 sol'n because $r=n$, n independent rows.

$\dim N(A^T) = m-r > 0$ free variables so ∞ many sol'ns exists.

2-)

$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$,

-a) Solve $Ax = v_1 - v_2 + v_3$ then $x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

-b-) Suppose $v_1 - v_2 + v_3 = 0$. Then x is in $N(A)$ so sol'ns are never unique.

-c) If v_1, v_2, v_3 are orthonormal, what comb. of v_1 and v_2 is the closest point to v_3.

Then projection of v_3 onto v_1, v_2 plane is "zero".

3-)

$A = \begin{bmatrix} .2 & .4 & .3 \\ .4 & .2 & .3 \\ .4 & .4 & .4 \end{bmatrix}$, col1 + col2 = 2.col3., $x_k = A^k x(0) = c_1\underset{0}{\lambda_1^k}\vec{x}_1 + c_2\underset{1}{\lambda_2^k}\vec{x}_2 + c_3\underset{-.2}{\lambda_3^k}\vec{x}_3$.

$\lambda_i = 0, 1, -.2$.

$x(0) = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$

as $k\to\infty$, $x_\infty = c_2\vec{x}_2$, $\begin{bmatrix} -.8 & .4 & .3 \\ .4 & -.6 & .3 \\ .4 & .4 & -.6 \end{bmatrix} \underset{\text{what a shot!}}{\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $c_2 = 1$

$x_\infty = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$

4-) 2×2

1-) projection onto $a=\begin{bmatrix}4\\-3\end{bmatrix}$, $P=\frac{aa^T}{a^Ta}=\frac{\begin{bmatrix}4\\-3\end{bmatrix}\begin{bmatrix}4 & -3\end{bmatrix}}{\begin{bmatrix}4 & -3\end{bmatrix}\begin{bmatrix}4\\-3\end{bmatrix}}=\frac{\begin{bmatrix}16\\9\end{bmatrix}}{[169]}$

2-) $\lambda_1=0, \vec{x}_1=\begin{bmatrix}1\\2\end{bmatrix}, \lambda_2=3, \vec{x}_2=\begin{bmatrix}2\\1\end{bmatrix}$, $A=\begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}0&0\\0&3\end{bmatrix}\begin{bmatrix}1&2\\2&1\end{bmatrix}^{-1}$

3-) $A\neq B^TB$ for any B. Since B^TB is symmetric, any A which is not symmetric will do.

4-) A matrix that has orthogonal eigenvectors but not symmetric.
- can be a skew-symmetric, orthogonal matricies.

5-) $\begin{bmatrix}1&0\\1&1\\1&2\end{bmatrix}\begin{bmatrix}\hat{c}\\\hat{d}\end{bmatrix}=\underbrace{\begin{bmatrix}3\\4\\1\end{bmatrix}}_{b}$, $\begin{bmatrix}\hat{c}\\\hat{d}\end{bmatrix}=\begin{bmatrix}11/3\\-1\end{bmatrix}$

1-) What is P of b onto $C(A)$: $\frac{11}{3}\text{col}(1)-1(\text{col}(2))$.

2-) Find a b so least-square sol'n changes to zero.

$\{$ End of Linear Algebra Course from Gilbert Strang $\}$

02.03.2025.

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