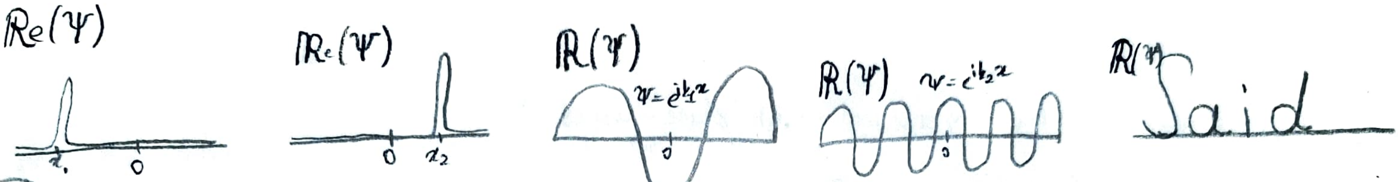
Quantum Physics 1

How do we configure a system or a quantum particle, we need to know its position and momentum.

$\{\vec{x}, \vec{p}\}$, $\vec{E}(\vec{x},\vec{p})$, $\vec{L}(\vec{x},\vec{p})$

In Real world, uncertainty of position and momentum, $\Delta x \Delta p \gtrsim \hbar/2$

(I) Specifying a state by (x,p) clearly will not work. The configuration of a quantum object is specified by a "wavefunction" denoted as $\Psi(x) \Rightarrow \mathbb{C}$ function

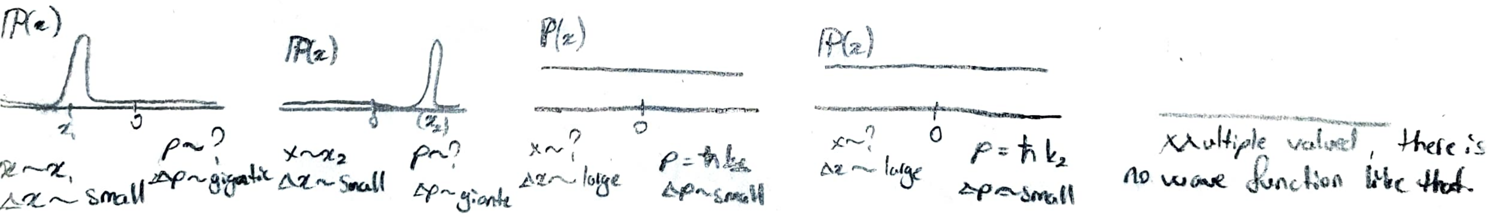


(II) $\mathbb{P}(x) = |\Psi(x)|^2$ determines probability density that the object is state $\Psi(x)$ will be found at x.

$$\mathbb{P}(x, x+dx) = \mathbb{P}(x)\,dx = |\Psi(x)|^2 dx \quad , \quad \int_{All} dx\, \mathbb{P}(x) = 1 = \int_{All} |\Psi(x)|^2 dx$$

Dim of wavefunction $= 1/\sqrt{L}$ since $\dim(dx) = L$, so $[\Psi(x)] = 1/\sqrt{L}$

$|e^{i\alpha}|^2 = 1$, $|B|^2 = B^* B$

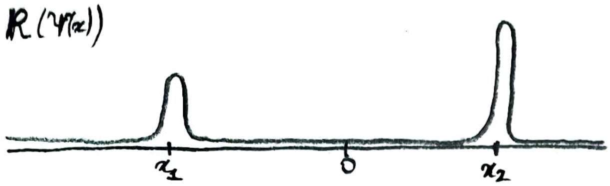


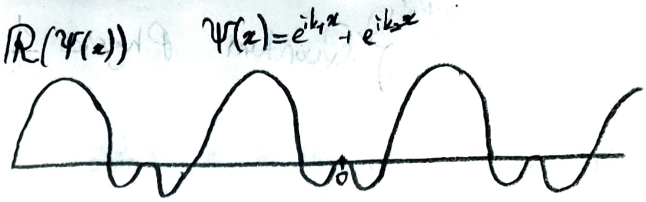
de Broglie Relation: $E \sim \hbar\omega = h\nu$, $P = \hbar k = h\lambda \Rightarrow \omega = 2\pi\nu \Rightarrow k = 2\pi/\lambda$

(III) Given Two Possible States of Quantum System Corresponding to Two Wavefunctions $\Psi_1(x)$ and $\Psi_2(x)$, the system can also be in a superposition of these,

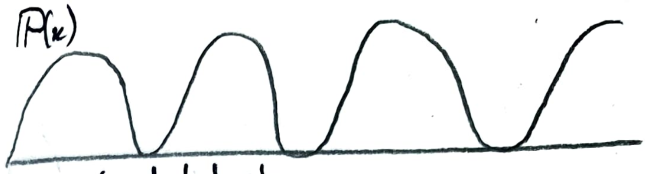
$$\Psi(x) = \alpha\, \Psi_1(x) + \beta\, \Psi_2(x) \qquad \alpha, \beta \in \mathbb{C} \text{ , Subject to normalization.}$$

$$\mathbb{P}(x) = \frac{|\Psi(x)|^2}{\int_{All} dx\, |\Psi(x)|^2}$$, alterna way to think about probability distribution.





$x \sim$ in between on avg
$\Delta x \sim (x_1 - x_2)$



$x \sim$ not much but okay.
$\Delta x \sim$ huge

$$\mathbb{P}(x) = |\alpha \Psi_1 + \beta \Psi_2|^2 = |\alpha|^2|\Psi_1|^2 + |\beta|^2|\Psi_2|^2 + \alpha^* \Psi_1^* \beta \Psi_2 + \alpha \Psi_1 \beta^* \Psi_2^* = \mathbb{P}_1 + \mathbb{P}_2 + \text{interference terms}$$

Thm: Any $f(x)$ can be built by superposing enough plane-waves e^{ikx}

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \hat{f}(k) e^{ikx} \quad , \lambda = 2\pi/k$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) e^{-ikx}$$

Any $\Psi(x)$ can be expressed as a superposition

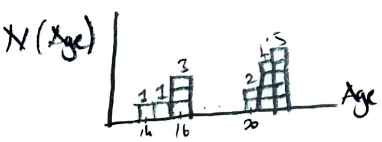
$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \tilde{\Psi}(k) e^{ikx}$$ of state with definite momentum. $p = \hbar k$.

of more equally $\equiv$ easily interpretable WF's.

Reasonable $\Psi(x)$ can be expressed as superposition

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int dk\, \tilde{\Psi}(k) e^{ikx} \qquad e^{ikx} = \cos(kx) + i \sin(kx)$$

$$\Psi(x) = \int dx_0\, \Psi(x_0)\, \delta(x - x_0)$$



$\mathbb{P}(a) = \dfrac{N(a)}{N_{total}}$, $\left(\sum_a \mathbb{P}(a) = 1\right)$, avg age $= 19.4$

$$\langle a \rangle = \frac{\sum_a N(a) \cdot a}{N_{total}} = \text{avg age.} = \sum_a a\, \mathbb{P}(a) \quad , \text{avg } a^2 \, , \langle a^2 \rangle = \sum_a a^2\, \mathbb{P}(a)$$

$\langle f(a) \rangle = \sum_a f(a)\, \mathbb{P}(a)$, $\langle a \rangle^2 \neq a^2$ always.

dev $\Rightarrow (a - \langle a \rangle)$, Note $\langle a - \langle a \rangle \rangle = \langle a \rangle - \langle a \rangle = 0$

std. dev $\langle (a - \langle a \rangle)^2 \rangle = \langle a^2 - 2a\langle a \rangle + \langle a \rangle^2 \rangle = \langle a^2 \rangle - 2\langle a \rangle^2 + \langle a \rangle^2 = \langle a^2 \rangle - \langle a \rangle^2 = \Delta a^2$

$\Delta a_{\mathbb{P}(a)} =$ uncertainity in a given $\mathbb{P}(a)$

Abdullah Said Tongel

Now for continues Variables

$\langle x \rangle = \int_{-\infty}^{\infty} dx\, \mathbb{P}(x) x \quad , \langle f(x) \rangle = \int_{-\infty}^{\infty} dx\, \mathbb{P}(x) f(x) \quad , \Delta x = \langle x^2 \rangle - \langle x \rangle^2$

$\langle x \rangle = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 x \quad , f(x) = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 f(x) \quad , \langle x \rangle = \langle x \rangle_\psi = \langle \psi | x | \psi \rangle$

Well, we should do with momentum too.

$\langle p \rangle = \int_{-\infty}^{\infty} dp\, \mathbb{P}(p) p \overset{?}{=} \int_{-\infty}^{\infty} dp\, |\tilde{\psi}(p)|^2 p$, is it true?

$\langle p \rangle = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 \cancel{p(x)}$, We can't say that we're in a definite position and definite momentum

that being in a

$\delta(x - x_0) = \frac{1}{\sqrt{2\pi}} \int dk\, e^{ikx}\, e^{ikx}$, Being in a state with definite position means

Superposition with arbitrary momentum.

So we want some good def'n of p given that $\psi(x)$.

Hint 1-) $k = 2\pi / \lambda \leftrightarrow p = \hbar k \quad , e^{ikx} \leftrightarrow p = \hbar k$

Note: $\partial_x e^{ikx} = ik e^{ikx}$

$\quad \hbar \partial_x e^{ikx} = \hbar k e^{ikx} = p e^{ikx}$, Check the units. $\frac{\hbar}{2} \geq \Delta x \Delta p \quad , [k] = \frac{1}{L}$, good.

$\frac{\hbar}{i} \partial_x \overset{"?"}{=} p$

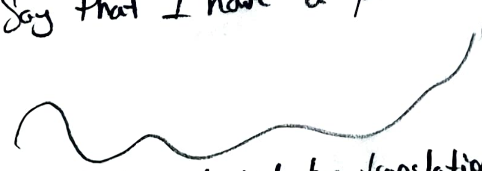
Noether's Theorem:

To every symmetry is associated a conserved quantity transition

Transition	$x \to x + L$	$\dot{p} = 0$
Time Transition	$t \to t + T$	$\dot{E} = 0$
Rotational	$\vec{x} \to \mathbb{R} x$	$\dot{\vec{L}} = 0$

Say that I have a particle which moving in some potential

$u(x) \quad | \quad \dot{p} = -\nabla u$



Constant potential has translation in

Symmetry, So Force $= 0$, $\dot{p} = 0$

$T_L f(x) = f(x - L)$

$T_L f(x) = f(x - L) = f(x) - L \partial_x f(x) + \frac{L^2}{2} \partial_x^2 f(x) - \frac{L^3}{6} \partial_x^3 f(x) \ldots$

$= (1 - L \partial_x + \frac{L^2}{2} \partial_x^2 - \frac{L^3}{6} \partial_x^3 + \ldots) \cdot f(x)$

$= e^{-L \partial_x} f(x)$

- Translation in x are generated by ∂_x
- Noether's Thm: Translation in $x \leftrightarrow$ momentum

$\Rightarrow$ Not totally shoking that, in QM;

$\partial_x \sim p$, $\partial_t \sim E$, $\partial_\theta \sim L_z$

Thing that $\hat{p} = \frac{\hbar}{i} \partial_x$, is the "fact" of Quantum Mechanics.

$\psi(x) = e^{ikx}$ $\quad \hbar k = \hat{p}$

$\hat{p}\psi_k(x) = \hbar k \psi_k(x)$, $\psi_s = \frac{1}{\sqrt{2}} e^{ik_1 x} + \frac{1}{\sqrt{2}} e^{-ik_2 x}$

$p \leftrightarrow \frac{\hbar}{i}\frac{\partial}{\partial x}$, $\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x)\psi(x) \frac{\hbar}{i}\frac{\partial}{\partial x} dx$

$\psi(x) = N e^{ikx} \rightarrow$ has $p = \hbar k$

$\hat{p}\psi = \frac{\hbar}{i}\frac{\partial}{\partial x}\psi = \hbar k \psi$

Operators and Measurable Quantities:

Operators act on objects. Vectors are objects and ops. are matrices, an example.

Ex $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ object, $M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, Mv is an object.

In quantum mechanics;
 objects: Complex functions of x.
 ops: Act on functions : $\hat{O}\ f(x) \rightarrow$ another function of x.

Some operators:

$1: f(x) \rightarrow f(x)$, $\frac{\partial}{\partial x} : f(x) \rightarrow \frac{\partial f(x)}{\partial x}$, $\hat{x} : f(x) \rightarrow x f(x)$, $\hat{S}_q : f(x) \rightarrow (f(x))^2$

$\hat{\mathbb{P}}_{42} : f(x) \rightarrow 42$, $\hat{\delta}_a \equiv \delta(x-a)$; $\hat{\delta}_a f(x) = \delta(x-a) f(x)$, $\hat{O}_{h(x)} : f(x) \rightarrow h(x) f(x)$

Linear Op: $\hat{O}(a f(x) + b g(x)) \rightarrow a \hat{O} f(x) + b \hat{O} g(x)$

Linear Operators can have "eigenfunctions":

$$\hat{A} \underbrace{f_a(x)}_{\text{eigenfunction}} = \underbrace{a}_{\text{eigen value}} f_a(x)$$

$a \in \mathbb{C}$, may take infinite values.

$\hat{A} = \hat{p}$ has eigenfunction e^{ikx} and eigen value $\hbar k$; $\hat{p} e^{ikx} = \hbar k e^{ikx}$

$\hat{p}, \hat{x}$, Energy Operation $= \frac{\hat{p}^2}{2m} + V(\hat{x}) = \hat{E} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(\hat{x})$

$$\hat{E}\psi(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(\hat{x})\psi(x)$$

A. Said Tongel

IV For each observable, we have an accosiated operator;

momentum $\to \hat{p}$ position $\to \hat{x}$ energy $\to \hat{E}$, $\hat{A} \equiv \hat{p}, \hat{x}, \hat{E}, \ldots$

$\langle \hat{A} \rangle = \int \Psi^*(x)\, \Psi(x)\, \hat{A}\, dx$ uncertanity of $(\Delta A)_\Psi = \sqrt{\langle \hat{A}^2 \rangle - \langle A \rangle^2}$

Comutation?

$(\hat{p}\hat{x}) f(x) \equiv \hat{p}(\hat{x} f(x)) = \hat{p}(x f(x)) = \frac{\hbar}{i} \frac{\partial}{\partial x}(x f(x)) = \left(\frac{\hbar}{i}\right) f(x) + \frac{\hbar}{i} x \frac{\partial f}{\partial x}$

$(\hat{x}\hat{p}) f(x) \equiv \hat{x}(\hat{p} f(x)) = \hat{x}(p f(x)) = (\hbar/i)\, x \frac{\partial f}{\partial x}$

$(\hat{x}\hat{p} - \hat{p}\hat{x}) f(x) = -\frac{\hbar}{i} f(x) = i\hbar f(x)$

$[\hat{x}, \hat{p}] f(x) = i\hbar f(x)$, $[\hat{x}, \hat{p}] = i\hbar \mathbb{1}$

V Upon measuring an observable A accosiated with $\hat{A}$

1-) The measured values must be one of the eigenvalues of $\hat{A}$

2-) After measurement, system collapses into Ψ_a , $\hat{A}\Psi_a = a\Psi_a$

Position of $\Psi(x)$ at x_0.

$\Psi \sim \delta(x - x_0)$, $\hat{x}\, \delta(x-x_0) = x\, \delta(x-x_0) = x_0\, \delta(x-x_0)$

VI Given an observable $\hat{A}$ and its eigenfunctions $\phi_a(x)$, normalize them by

$\int \Psi_a^*(x)\, \Psi_b(x)\, dx = \delta_{ab}$

1-) Can expand $\Psi(x)$ as

$$\Psi(x) = \sum_a c_a \Psi_a(x)$$

2-) Probability of measuring $\hat{A}$ and getting a is

$$|c_a|^2 = P_\Psi(a_0)$$

Ex: $\hat{x}$, $\{\delta(x - x_0), \forall x_0\}$

$$\Psi(x) = \int \underbrace{\Psi(x_0)}_{\text{"}c_a\text{"}}\, \underbrace{\delta(x - x_0)}_{\Psi_a}\, \underbrace{dx_0}_{\Sigma}$$

Schrodinger Equation:

$E = \hbar\omega$, $\vec{p} = \hbar\vec{k}$ $(p = \hbar k)$; Planck , Einstein , Coupton.

$(E, \vec{p}) = \hbar(\omega, \vec{k})$; $p^\mu = (E, \vec{p})$, $x^\mu = (t, \vec{x})$

$\Psi(x,t) = e^{i(kx - \omega t)}$, $p = \hbar k$

$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x} \leftrightarrow p = \hbar k$, $\hat{E} = i\hbar\frac{\partial}{\partial t} \leftrightarrow E = \hbar\omega$

$i\hbar\frac{\partial}{\partial t}\Psi = i\hbar(-i\omega)\Psi = \hbar\omega\Psi$

For this Ψ $\frac{\hbar}{i}\frac{\partial}{\partial x}\Psi(x,t) = p\,\Psi(x,t)$, $i\hbar\frac{\partial}{\partial t}\Psi(x,t) = E\,\Psi(x,t)$

$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \hat{E}\,\Psi(x,t)$, $\hat{E} = \frac{\hat{p}^2}{2m} + V(\hat{x}) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x)\,\Psi(x,t) \Big\}$$ Schrodinger's Equation.

We know that any wavefunction can be expressed as the superposition of two wavefunctions.
We also know that Schrodinger's Equation is linear. So we can say:
If we know the state of a system at any time, we can also know all of the states of the system at any time.

So we can say that the system is deterministic.

$\hat{A}\,\varphi_a(x) = a\,\varphi_a(x)$

But if we measure the system at some point, the superposition of states will collapse to some specific state a, $\hat{A} = a \Rightarrow \Psi(x,t) = \varphi_a(x)$.
System starts to evolve according to Schrodinger Equation but with new initial condition $\varphi_a(x)$

So we say that in Quantum mechanics, we have "2" different definitions of time Evolution.

1-) Schrodinger Equation, "Deterministic"

2-) Collapse, Probabilities collapse to some non-deterministic state.

And the probability distribution determined by the which wavefunction you have

$\mathbb{P}(\hat{A} = a) = |c_a|^2$, $c_a = (\varphi_a|\Psi)$,

$$(\varphi_a|\Psi) = \int_{-\infty}^{\infty}\varphi_a^*(x)\,\Psi(x)\,dx$$

How is that possible?

A. Said Tengel

Copenhag Interpretation says that; "as long as the math fits the data just shut up and calculate". That is not acceptable so let us find an explanation.

Theory of Decoherence:

It basically says that, that the reason you have this problem between Schrodinger's deterministic time and collapse is caused by the measuring the quantum objects with macroscopic systems when you try to measure quantum "objects" with a system made by classical dynamics, quantum effects get dominated (gets washed out) by the interaction with laws of classical physic. You get rid of the interference effects. Dealing with that is hard because if you want to treat the system according to Schrodinger Evolution you have to work trajectories and motion of dynamics of every single particle in the whole system.

So here is the question;

If you take a system where you have just one quantum object and other enormously high numbered particles act with classical dynamics, If you take whole system does Schrodinger Evo. collapses for this single quantum microsystem?

And the answer is yes, we're gonna see that later on.

Solving Schrodinger's Equation

For solving Schrodinger's equation we need a specific energy operator $\hat{E}$.

$$\hat{E} = \frac{p^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \quad , \quad \hat{p} = \frac{\hbar}{i}\partial_x$$

So

$$i\hbar\,\partial_t \Psi(x,t) = -\frac{\hbar^2}{2m}\partial_x^2 \Psi(x,t) + \frac{m\omega^2}{2}x^2\Psi(x,t)$$

For a given specific $\hat{E}$, there are many ways to solve equation.

a-) Brute Force: Solving roughly.

b-) Extreme Cleverness: Elegant use of math, use structures of differential eq.

c-) Numerically, by computer.

1-) Start with Eigenfunctions:

$[\hbar] = [E][t] = [p][x]$

$$\Psi(x,0) = \varphi_E(x) \quad , \quad \hat{E}\varphi_E(x) = E\varphi_E(x)$$

$$i\hbar\,\partial_t \Psi(x,0) = E\Psi(x,0) \Rightarrow \partial_t \Psi(x,0) = \frac{-iE}{\hbar}\Psi(x,0) = \frac{-iE}{\hbar}\varphi_E(x) \Rightarrow \Psi(x,t) = e^{-i\frac{Et}{\hbar}}\varphi_E(x) \quad , \quad \omega = E/\hbar$$

$\Psi(x,t) = e^{-i\omega t}\varphi_E(x) \equiv$ "Stationary State"

$\mathbb{P}(x,t) = |\Psi(x,t)|^2 = |\varphi_E(x)|^2$: "Means Independent in Time"

$$\langle x \rangle_t = \int \mathbb{P}(x,t)\,x\,dx = \int |\varphi_E(x)|^2 x\,dx \quad , \quad \text{if } \Psi(x,0) = \sum_n c_n \varphi_n(x)$$

$$\Rightarrow \Psi(x,t) = \sum_n c_n e^{-i\omega_n t}\varphi_n(x)$$

Linearity of Schrodinger Equation: $i\partial_t \Psi(x,t) = \frac{\hat{E}}{i\hbar}\Psi(x,t)$

Suppose;

$\Psi_n(x,t) = e^{-i\omega t}\varphi_{E_n}(x)$; $E_n = \hbar\omega_n$, solves S.E.

Then;

$\Rightarrow \Psi(x,t) = \sum_n c_n \Psi_n(x,t)$, $\Delta\omega\,\Delta t \gtrsim 1$, $\hbar(\Delta\omega\,\Delta t) \gtrsim \hbar \Rightarrow \Delta E\,\Delta t \gtrsim \hbar$, there is always some accuracy level on the measurements. We can't make measurement with zero error.

For a given $\hat{E} \rightarrow \{\varphi_{E_n}(x)\}$ with eigenvalue "E".

Any wavefunction $\Psi(x,0) = \sum_n c_n \varphi_{E_n}(x)$, can be expanded as the superposition of eigenfunctions. So we can take an arbitrary condition and solve Schrodinger Equation.

$\hat{p} \rightarrow \{\frac{1}{\sqrt{2\pi}} e^{ikx}\} \rightarrow p = \hbar k$, $f(x) = \frac{1}{\sqrt{2\pi}}\int e^{ikx} \tilde{f}(k)\,dk$

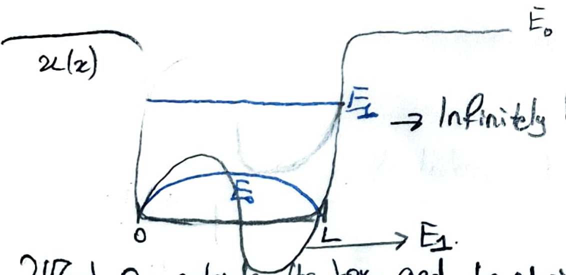
$\hat{x} \rightarrow \{\hat{x}\,\delta(x-y) = y\,\delta(x-y)\}$, $f(x) = \int \delta(x-y) f(y)\,dy$

Example:

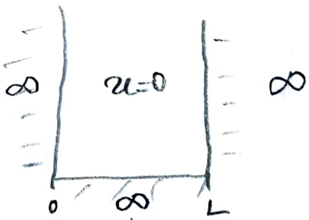
$\hat{E} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\partial_x^2$, $\hat{E}\varphi_E(x) = E\varphi_E(x) \Leftrightarrow -\frac{\hbar^2}{2m}\varphi_E''(x) = E\varphi_E(x)$

$\varphi_E''(x) + \frac{2mE}{\hbar^2}\varphi_E(x) = 0$, $\frac{2mE}{\hbar^2} = k^2 \Rightarrow \varphi_E(x) = Ae^{ikx} + Be^{-ikx}$, $E = \frac{\hbar^2 k^2}{2m}$.

Example:



$\rightarrow$ Infinitely high and square.



$\Psi(x) = 0$, outside the box, and boundaries: $x = 0, L$.

Inside $\varphi_E = \alpha\cos(kx) + \beta\sin(kx)$

$x = 0 \Rightarrow \alpha = 0$

$x = L \Rightarrow kL = (n+1)\pi \sim k_n = \frac{(n+1)\pi}{L}$

$\Rightarrow \varphi_E = A_n \sin(k_n x)$

$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi(n+1)^2}{2m} \Rightarrow E_1 = 4E_0$

The allowed energy levels are always bigger than zero, you can't have zero energy.

A. Said Tengel

$$(f\,|\,g)=\int_{-\infty}^{\infty} f^*(x)\,g(x)\,dx \quad ,\ 1$$

$$\hat{x}\,\varphi_{x_0}(x)=x_0\,\varphi_{x_0}(x)\ ,\ (\varphi_{x_0}(x)\,|\,\varphi_{x_1}(x))=\delta(x_0-x_1)$$
$$\hat{p}\,\varphi_k(x)=\hbar k\,\varphi_k(x)\ ,\ (\varphi_k(x)\,|\,\varphi_{k'}(x))=\delta(k-k')$$
$$\hat{E}\,\varphi_n=E_n\,\varphi_n\ ,\ (\varphi_n\,|\,\varphi_m)=\varphi_{nm}.$$

2

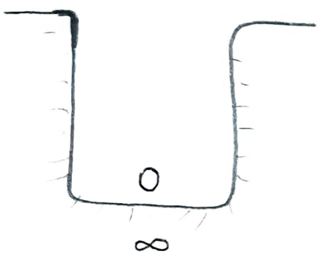
$$i\hbar\,\partial_t\,\Psi(x,t)=\hat{E}\,\Psi(x,t)$$
$$\Psi(x,0)=\varphi_n(x)\Rightarrow\Psi(x,t)=\varphi_n(x)\,e^{-i\frac{E_n t}{\hbar}}$$
$\omega_n=E_n/\hbar$.

$$\Psi(x)=\int_{-\infty}^{\infty}\delta(x-x_0)\,\Psi(x_0)\,dx_0$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}\,\tilde{\Psi}(k)\,dk$$
$$=\sum\!\!\!\!\!\!\int_n \varphi_n(x)\,C_n$$

3

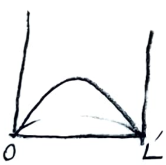
Energy lvl's can be discrete or continues.

$$\Psi(x,t)=\sum\!\!\!\!\!\!\int_n \varphi_n(x)\,e^{-i\omega_n t}\,C_n\ .\Rightarrow\text{more convenient.}$$

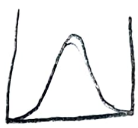


$$\hat{E}=\frac{\hat{p}^2}{2m}+\mathcal{U}(x)\ ,\ \Psi(x,t)=\sum_n C_n e^{i\frac{\hbar k_n^2}{2m}t}.$$
$$\varphi_n(x)=\sqrt{\frac{2}{L}}\sin(k_n x),\ k_n=\frac{(n+1)\pi}{L}$$
$$E_n=\frac{\hbar^2 k_n^2}{2m}$$

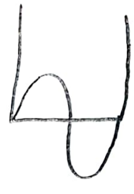
$\mathrm{Re}\,\Psi_{x_0}$



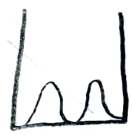
$P(x_0)$



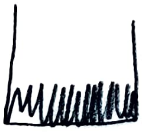
$\mathrm{Re}(\Psi_1)$



$P(\Psi_1)(x)$



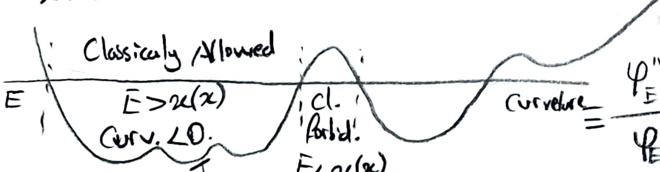
$\mathrm{Re}\,\Psi_{x_{101}}$



$P(x_{101})$

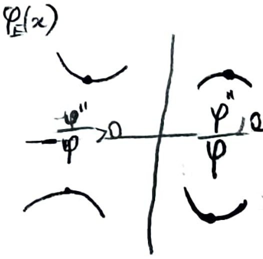
Qualitative Behaviour:

$\mathcal{U}(x)$



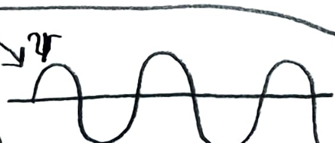
$$-\frac{\hbar^2}{2m}\varphi_E''(x)+\mathcal{U}(x)\,\varphi_E(x)=E\,\varphi_E(x)$$

$$\text{curvature}=\frac{\varphi_E''(x)}{\varphi_E(x)}=-\frac{2m}{\hbar^2}\left(E-\mathcal{U}(x)\right)$$



If $\dfrac{\varphi''}{\varphi}=K^2\Rightarrow$ Sol'n: $Ae^{kx}+Be^{-kx}$

If $\dfrac{\varphi''}{\varphi}=-K^2\Rightarrow$ Sol'n: $Ae^{ikx}+Be^{-ikx}$

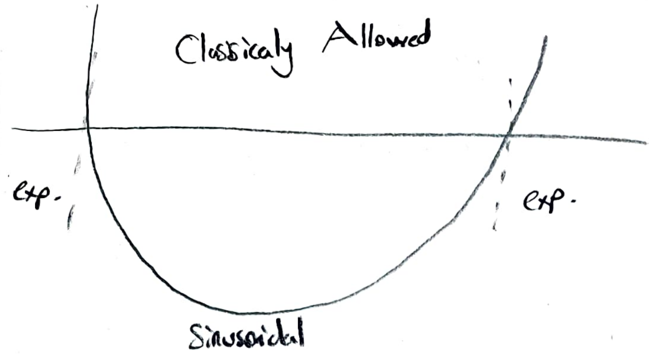


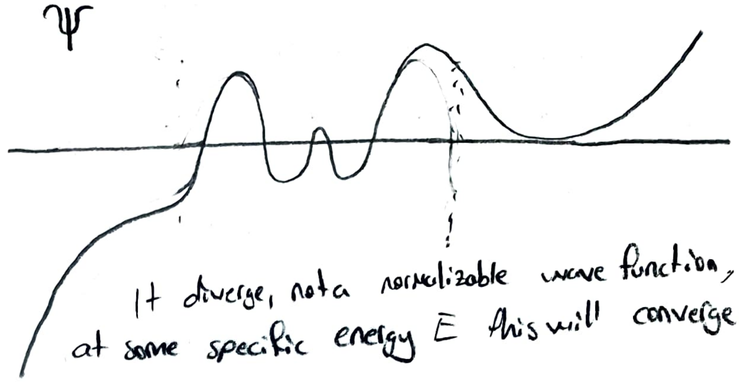
classicaly allowed region, sinusoidal

growing or shrinking exponentials on classically forbidden region

5

Exp.





Schrodinger eqn. energy eigenstate

$$\frac{\hbar^2}{2m}\partial_x^2 \Psi(x) + u(x)\Psi(x) = E\,\Psi(x)$$

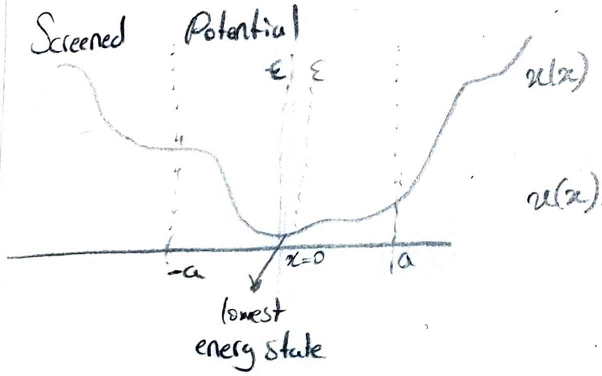
, if $u(x)$ is a smooth potential then

$$\left.\begin{array}{l}\Psi(x_0)=0\\ \Psi'(x_0)=0\end{array}\right\} \Rightarrow \Psi(x)\equiv 0.$$

Bound State $\Psi(x)\to 0$ as $|x|\to\infty$

For bound states in 1-d, no degeneracy is possible.

Screened Potential

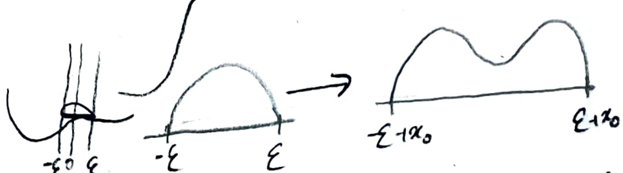


$u(x)$ $\quad \begin{array}{c} u(x)\to\infty \\ |x|\to\infty \end{array}$ infinite # of nodes

$$u(x)=\begin{cases} u(x), & |x|<a \\ \infty, & |x|>a \end{cases}$$

Let's take $a=\varepsilon \to 0$.

If we scratch this interval, the wave function will start oscillating.



but from property of $u(x)$, wave function can't cross and pass through the axis since wave function can't vanish,

We say that ground state has "no nodes".

A. Said Töngel

Now look excited states ; $\psi_1, \psi_2, \psi_3 \dots$

Since we're working on 1 dimensional space, degeneracy is not possible.
Which means ψ_1 can not have same or lower energy compared to ground state ψ_0.

So we say, for $\psi_0, \psi_1, \psi_2, \dots, \psi_n$; ψ_n has n nodes
$E_n > E_{n'}$ for $n > n'$.

Harmonic Oscillator:

$$\hat{E} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 x^2 \; ; \; [\omega] = 1/T, \quad \frac{1}{2} m\omega^2 x^2 = u(x).$$

Energy quantity $[E] = [\hbar\omega]$, Lengths "a" $\frac{\hbar^2}{ma^2} = E = m\omega^2 a^2 \Rightarrow a^2 = \hbar / m\omega$

$$-\frac{\hbar^2}{2m} \partial_x^2 \psi(x) + \frac{1}{2} m\omega^2 x^2 \psi(x) = \hat{E}\, \psi(x)$$, we need to make it dimensionless so we can solve.

Multiply by $2/\hbar\omega$.

$$-\underbrace{\underbrace{\frac{\hbar}{2m\omega}}_{L^2} \partial_x^2}_{\text{no unit}} \psi(x) + \underbrace{\underbrace{\frac{m\omega}{\hbar}}_{1/L^2} x^2}_{\text{unitless}} \psi(x) = \underbrace{\frac{2E}{\hbar\omega}}_{\mathcal{E}} \psi(x) = \underset{\text{dimensionless.}}{\mathcal{E}}\, \psi(x)$$, apply $x = au$.

$$\frac{d}{dx} = \frac{1}{a}\frac{d}{du}, \quad \frac{d^2}{dx^2} = \frac{1}{a^2}\frac{d^2}{du^2} \Rightarrow \frac{\hbar}{2m\omega} \frac{1}{a^2} \frac{d^2}{du^2} + \frac{m\omega}{\hbar} x^2 \psi = \mathcal{E}\psi$$

$$\Rightarrow \boxed{-\frac{d^2\psi}{du^2} + u^2\psi = \mathcal{E}\psi \quad \text{I}} \Rightarrow \partial_u^2 \psi + u^2\psi - \mathcal{E}\psi = 0$$

when $u \to \infty$, $\psi'' = +u^2\psi$. try $\psi = u^k e^{\alpha u^2/2}$

$$\psi' = \alpha u\, u^k e^{\alpha u^2/2} + k u^{k-1} e^{\alpha u^2/2}$$

$$\psi'' = (\alpha u)^2 u^k e^{\alpha u^2/2} + 2(\alpha u)(k) u^{k-1} e^{\alpha u^2/2} + k(k-1) u^{k-2} e^{\alpha u^2/2}$$

$$= \alpha^2 u^2 u^k e^{\alpha u^2/2} \underbrace{\left[1 + \frac{2k+1}{\alpha}\frac{1}{u^2} + \frac{k(k-1)}{\alpha^2}\frac{1}{u^4}\right]}_{K} = \alpha^2 u^2 \psi K. \text{ if } \alpha^2 = 1.$$

So as $|u| \to \infty$

$$\psi(u) \to A u^k e^{-u^2/2} + \underbrace{B u^k e^{u^2/2}}_{\text{we'll want this part to vanish.}}$$

Write $\psi(x)=h(x)e^{-x^2/2}$, $\underbrace{\psi(x)=\frac{\psi(x)}{e^{-x^2/2}}}_{h(x)}e^{-x^2/2}$ $\Rightarrow$ we're hoping to get a simplier differential equation. $\Rightarrow$ Plug into I.

$$\frac{-d^2\psi}{dx^2}+x^2\psi=\mathcal{E}\psi \Rightarrow \frac{-d^2h(x)e^{-x^2/2}}{dx^2}+x^2h(x)e^{-x^2/2}=\mathcal{E}h(x)e^{-x^2/2}$$

$$\Rightarrow \boxed{\frac{d^2h(x)}{dx^2}2x\frac{dh(x)}{dx}+(\mathcal{E}-1)h(x)=0. \quad \text{II}}$$

Let's take power series.

$h(x)=a_0+a_1x+a_2x^2+a_3x^3$, see the behaviour of it.

$$h(x)=\sum_{j=0}^{\infty}a_jx^j \Rightarrow \frac{dh}{dx}=\sum_{j=0}^{\infty}ja_jx^{j-1}$$

$$\Rightarrow \frac{d^2h}{dx^2}-2x\frac{dh}{dx}+(\mathcal{E}-1)h(x)=0=\sum_{j=0}^{\infty}\left[(j+2)(j+1)a_{j+2}-2ja_j+(\mathcal{E}-1)a_j\right]x^j$$

$$=\sum_{j=0}^{\infty}\left[(j+2)(j+1)a_{j+2}-2(2j+1-\mathcal{E})a_j\right]x^j=0, \quad a_{j+2}=\frac{(2j+1-\mathcal{E})}{(j+2)(j+1)}a_j$$

a_0, a_1 determines the whole function since its 2nd order ODE.

Solutions fixed by given a_0, a_1 $(h(0), h'(0))$

From a_0's get evens; from a_1 get odd solutions.

For $j\gg$, $a_{j+2}\approx\frac{2}{j}a_j$, this behaviour is bad.

Let's see how $e^{x^2}=\sum_{n=0}^{\infty}\frac{1}{n!}(x^2)^n=\sum_{j=0,2,4}\frac{1}{(j/2)!}x^j$; $c_j=\frac{1}{(j/2)!}$, $\frac{c_{j+2}}{c_j}=\frac{(j/2)!}{\left(\frac{j+2}{2}\right)!}=\frac{1}{\frac{j}{2}+1}\approx\frac{2}{j}$

Our series will diverge like e^{x^2}, if we want it to terminate

$2j+1=\mathcal{E}$, means $\mathcal{E}$ must be an odd integer, makes $a_{j+2}=0$.

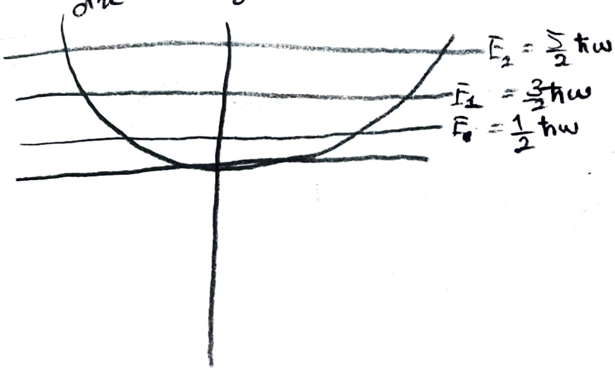
$h(x)=a_jx^j+a_{j-2}x^{j-2}\dots$, call $j=n$, $2n+1-\mathcal{E}=0$, $h(x)=a_nx^n+a_{n-2}x^{n-2}+\dots$

$$\mathcal{E}_n=\frac{E_n}{(\hbar\omega/2)}=2n+1, \quad E_n=\frac{\hbar\omega}{2}(2n+1)=\hbar\omega\left(n+\frac{1}{2}\right)$$

$$h_n(x)=H_n(x)=2^nx^n+\mathcal{O}(x^{n-2}), \quad \frac{d^2H_n}{dx^2}-2x\frac{dH_n}{dx}+2nH_n=0.$$

$$e^{-z^2+2zx}=\sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(x)$$

n-th degree polynomial means n nodes exists.



$$\psi_n=H_n(x)e^{-x^2/2}$$

$$\psi_n(x)=H_n\left(\frac{x}{a}\right)e^{-x^2/2a^2}.$$

A. Said Tüngel

Operator Methods for Harmonic Oscillators

$$\hat{E} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \quad ; \quad [\hbar m\omega] = [p]^2, \left[\frac{\hbar}{m\omega}\right] = [x]^2 ; \quad x_0 = \sqrt{\frac{2\hbar}{m\omega}}, \quad p_0 = \sqrt{2\hbar\omega m}$$

$$\hat{E} = \hbar\omega\left[\frac{\hat{p}^2}{p_0^2} + \frac{\hat{x}^2}{x_0^2}\right], \quad x_0 p_0 = 2\hbar . \quad \left(\frac{\hat{x}}{x_0} - \frac{i\hat{p}}{p_0}\right)\left(\frac{\hat{x}}{x_0} + \frac{i\hat{p}}{p_0}\right) = \frac{\hat{x}^2}{x_0^2} + \frac{\hat{p}^2}{p_0^2} + \frac{i}{2\hbar}[\hat{x},\hat{p}]$$

$$\Rightarrow \frac{\hat{x}^2}{x_0^2} + \frac{\hat{p}^2}{p_0^2} - \frac{1}{2} . \Rightarrow \hat{E} = \hbar\omega\left[\frac{\hat{p}^2}{p_0^2} + \frac{\hat{x}^2}{x_0^2}\right] = \hbar\omega\left(\left(\frac{\hat{x}}{x_0} + \frac{i\hat{p}}{p_0}\right)\left(\frac{\hat{x}}{x_0} - \frac{i\hat{p}}{p_0}\right) + \frac{1}{2}\right)$$

Def: $\hat{a} = \left(\frac{\hat{x}}{x_0} + \frac{i\hat{p}}{p_0}\right)$, $\hat{a}^{+} = \left(\frac{\hat{x}}{x_0} - \frac{i\hat{p}}{p_0}\right)$, $E = \hbar\omega\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right)$

Given any θ, a Linear Operator, we can build θ^{+}

$$\int f^*(\hat{\theta}^{+} g)\,dx \equiv \int (\hat{\theta} f)^* g\,dx, \quad \hat{\theta}^{+} = \text{Hermitian Adjoint of } \hat{\theta}.$$

Ex.1:

$c \in \mathbb{C}$, c^{+} : $\int f^*(c^{+} g)\,dx \equiv \int (cf)^* g\,dx = \int f^*(c^* g)\,dx$. true for all f, g.

So $\Rightarrow c^{+} = c^*$.

Ex.2: what is ∂_x^{+}

$$\int f^*(\partial_x^{+} g)\,dx = \int (\partial_x f)^* g\,dx . = \int (\partial_x f^*) g\,dx = \underbrace{-\int f^* \partial_x g\,dx}_{\text{From integration by parts.}} = \int f^*(-\partial_x g)\,dx$$

$\partial_x^{+} = -\partial_x$

"We assumed that surface terms vanish."

Ex.3: $\hat{x}^{+} =$ $\qquad \hat{x} f(x) = x f(x)$

$$\int f^*(\hat{x}^{+} g)\,dx = \int (\hat{x} f)^* g\,dx = \int x f^* g\,dx = \int f^*(xg)\,dx, \quad \hat{x}^{+} = \hat{x}.$$

Def: $\hat{\theta}^{+} = \hat{\theta}$ called Hermitian.

Fact: Any operator which equals to its adjoint has Real eigenvalues only.

Ex.4: $\hat{p}^{+} = \hat{p}$ since $\hat{p} = \frac{\hbar}{i}\partial_x$

Physical Fact: If all Observables are Real

Then in Quantum Mechanics $\exists$ All Operators $\leftrightarrow$ Observables must be Real.

So we can say, the operator we defined as $\hat{a}$ is not Hermitian and NOT associated with an observable.

$\hat{E} = \hat{E}^{\dagger}$, $[\hat{a}, \hat{a}^{\dagger}] = \left[\left(\frac{\hat{x}}{x_0} + i\frac{\hat{p}}{p_0}\right), \left(\frac{\hat{x}}{x_0} - i\frac{\hat{p}}{p_0}\right)\right] = \frac{i}{x_0 p_0}[x,p] + \frac{i}{x_0 p_0}[p,x]$

$\Rightarrow \frac{-i}{2\hbar}[x,p] + \frac{i}{2\hbar}[p,x] = \frac{-i}{2\hbar} i\hbar + \frac{i}{2\hbar} (-i\hbar) = \frac{1}{2} + \frac{1}{2} = 1.$

Therefore $[\hat{a}, \hat{a}^{\dagger}] = 1$, $[\hat{a}^{\dagger}, \hat{a}] = -1$.

$[\hat{E}, \hat{a}] = \hbar\omega [a^{\dagger}a, \hat{a}] = \hbar\omega(\hat{a}^{\dagger}\hat{a}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{a}) = \hbar\omega(\hat{a}^{\dagger}\hat{a} - \hat{a}\hat{a}^{\dagger})\hat{a} = -\hbar\omega\hat{a}$

$\Rightarrow [\hat{E}, \hat{a}] = -\hbar\omega\hat{a} \Rightarrow [\hat{E}, \hat{a}^{\dagger}] = \hbar\omega\hat{a}^{\dagger}$

Suppose $\hat{E}\varphi_E = E\varphi_E$, consider $\psi = \hat{a}\varphi_E$ $\{ABf(x) = ([A,B] + BA)f(x)\}$

$\hat{E}\psi = \hat{E}\hat{a}\varphi_E = (\hat{E}\hat{a} - \hat{a}\hat{E} + \hat{a}\hat{E})\varphi_E = ([\hat{E},\hat{a}] + \hat{a}\hat{E})\varphi_E = (-\hbar\omega\hat{a} + E\hat{a})\varphi_E.$

$= (E - \hbar\omega)\hat{a}\varphi_E = (E - \hbar\omega)\psi \Rightarrow$ eigenvalue decreased by $\hbar\omega$.

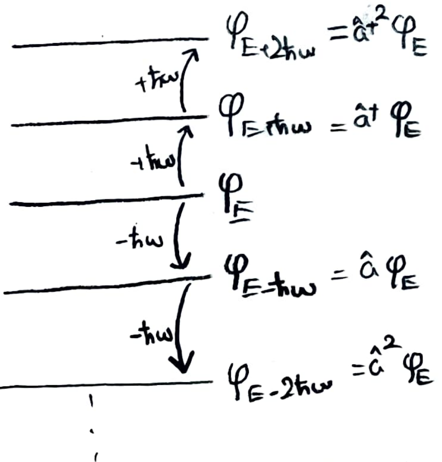
And, consider $\psi = \hat{a}^{\dagger}\varphi_E$

$\hat{E}\psi = \hat{E}\hat{a}^{\dagger}\varphi_E = [\hat{E}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{E} + \hat{a}^{\dagger}\hat{E}]\varphi_E = ([E, \hat{a}^{\dagger}] + \hat{a}^{\dagger}\hat{E})\varphi_E = (\hbar\omega\hat{a}^{\dagger} + E\hat{a}^{\dagger})\varphi_E$

$= (E + \hbar\omega)\hat{a}^{\dagger}\varphi_E = (E + \hbar\omega)\psi.$

$\hat{E}\varphi_E = E\varphi_E \Rightarrow \begin{cases} \hat{a}\varphi_E \text{ has energy value } E - \hbar\omega, \ \psi \propto \varphi_{E-\hbar\omega} \Rightarrow \text{called Lowering Operator} \\ \hat{a}^{\dagger}\varphi_E \text{ has energy value } E + \hbar\omega, \ \psi \propto \varphi_{E+\hbar\omega} \Rightarrow \text{called Raising Operator} \end{cases}$

$\hat{a}\varphi_E = \varphi_{E-\hbar\omega}, \ \hat{a}^{\dagger}\varphi_E = \varphi_{E+\hbar\omega}$



The energy states evenly spaced because of the commutator, when we first computed the commutator of E with $\hat{a}, \hat{a}^{\dagger}$ it was determined that it will leave a pattern like that.
They're called Landau Levels.
This ladder extends to infinitely up.
Let's show the min. value

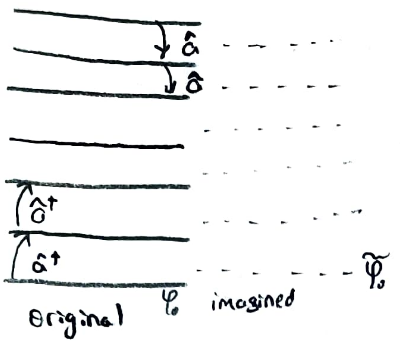
$$\langle E \rangle = \int \varphi_E^* \hat{E} \varphi_E \, dx = \int \underbrace{|\tilde{\psi}(p)|^2 \frac{p^2}{2m}}_{\text{positive}} dp + \int \underbrace{|\psi(x)|^2 \frac{m\omega^2}{2} x^2}_{\text{positive}} dx$$

So E must have a "ground state".

minimum E , $\hat{a}\varphi_E = 0 \Rightarrow$ non-normalizable $\{$ But, can't we create different intersection ladders? $\}$

A. Said Tüngel

Imagine it is not all of them, if there is another state there must be a states of ladder.



Can there be two states with two different energies a_0

Can there be two different states with annihilated by a_0

$\hat{E}\varphi_0 = \hbar\omega\left(a^{+}a + \frac{1}{2}\right)\varphi_0$, $\hat{a}$ also called annihilation operator

$= \frac{1}{2}\hbar\omega\,\varphi_0$ So any annihilated ground state must have

"Some" energy, means the imagined ladder is degenerate.

we can not have degenaricies in 1-D.

Last time we showed the Node Theorem and said that is the all states must exists.
which means we missed no states and

$$\hat{a}\varphi_0(x) = 0 = \left(\frac{\hat{x}}{x_0} + i\frac{\hat{p}}{p_0}\right)\varphi_0(x) = \left(\frac{\hat{x}}{x_0} + i\frac{\hbar}{i}\partial_x \frac{1}{p_0}\right)\varphi_0 = \left(\partial_x + \frac{p_0}{\hbar x_0}x\right)\varphi_0(x)$$

"0" function, not zero energy.

$\Rightarrow$ Its solution is Gaussian $\Rightarrow \left(\partial_x + \frac{2}{x_0^2}x\right)\varphi_0(x) \Rightarrow \varphi_0 = c.e^{-x^2/x_0^2}$, $\varphi_1 = N\left(\partial_x - \frac{2}{x_0^2}x\right)\varphi_0$

$$\varphi_k = N\left(\partial_x - \frac{2}{x_0^2}x\right)^k \varphi_0$$

$\hat{A}, \hat{B}$; is it possible for there to be function φ_{ab} simultaneously.

$$\hat{A}\varphi_{ab} = a\,\varphi_{ab} , \quad \hat{B}\varphi_{ab} = b\,\varphi_{ab}$$

Take $[A,B]\varphi_{ab} = (AB - BA)\varphi_{ab} = AB\varphi_{ab} - BA\varphi_{ab} = ab\,\varphi_{ab} - ba\,\varphi_{ab} = (ab - ba)\varphi_{ab} = 0.$

$[A,B]$ must be zero.

Let's say.

$[A,B] = C.\mathbb{1}$. The state of being in a definite value of a,b simultaneousl doesnot exists

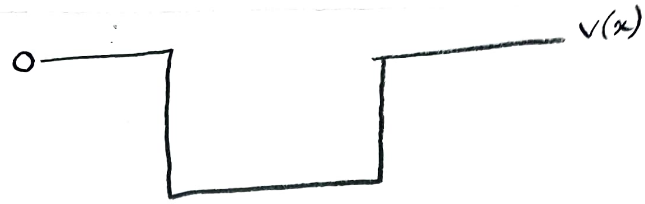
Because being in a state with definite value of a would be an eigenfunction of a, and its also true for the b case but there are no common states or common eigenfunctions because commutator never kills the state

$$[x,p] = i\hbar\mathbb{1} , \quad \Delta_\psi A\,\Delta_\psi B \geq \frac{1}{2}\left|\langle\psi|[A,B]|\psi\rangle\right|$$

$$\langle\psi|i\hbar\mathbb{1}|\psi\rangle = i\hbar$$

$\Delta_\psi x\,\Delta_\psi p \geq \frac{\hbar}{2}$, uncertainty does not depends on wave function.

Finite Potential Well

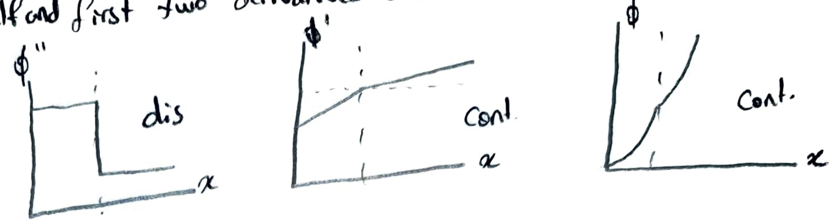


$$\hat{E}\,\varphi_E(x) = -\frac{\hbar^2}{2m}\varphi_E''(x) + V(x)\varphi(x)$$

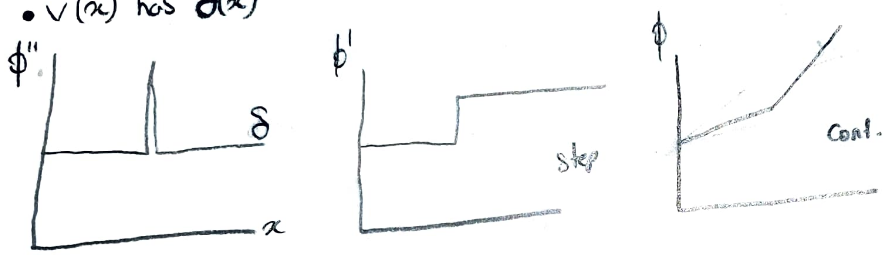
$$\varphi_E''(x) = \frac{2m}{\hbar^2}\left(V(x) - E\right)\varphi_E(x)$$, there should be discrete energy levels.

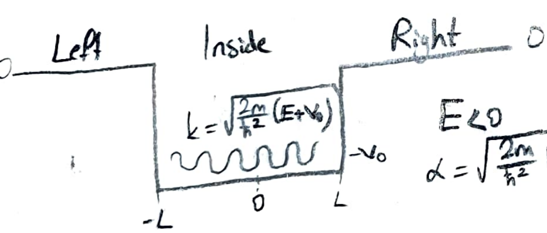
Continuity of $\varphi_E(x)$

- $V(x)$ continuous, means functions itself and first two derivatives are also continues.
- $V(x)$ has step-discontinuity
  - $\varphi(x)$, $\varphi'(x)$ continues but
  - $\varphi''(x)$ is discontinues



- $V(x)$ has $\delta(x)$





$E < 0$

$\alpha = \sqrt{\frac{2m}{\hbar^2}|E|}$

General Solution

$$\phi_E(x) = \begin{cases} A\cos(kx) + B\sin(kx) & \Rightarrow \text{Inside}. \\ Ce^{\alpha x} + De^{-\alpha x} & \Rightarrow \text{Left}. \\ Ge^{\alpha x} + Fe^{-\alpha x} & \Rightarrow \text{Right}. \end{cases}$$

If $E > V(x)$

$\phi'' = -k^2\phi \Rightarrow$ osc.

If $E < V(x)$

$\phi'' = \alpha^2\phi \Rightarrow$ exp.

$$k^2 = \frac{2m}{\hbar^2}(E - V(x))$$

$$\alpha^2 = \frac{2m}{\hbar^2}(V(x) - E)$$

Normalizable:

$\phi(x \to -\infty) = 0 \Leftrightarrow D = 0$

$\phi(x \to \infty) = 0 \Leftrightarrow G = 0$

parity, Sym. (even)	$B = 0$, $C = F$
antisym (odd)	$A = 0$, $C = -F$

For Even:

Right B.C. $\{x = L\}$

$\phi = Ce^{-\alpha L} = A\cos(kL) \Rightarrow C = A\cos(kL)e^{\alpha L}$

$\phi' = -\alpha Ce^{-\alpha L} = -kA\sin(kL) \Rightarrow C = +A\frac{k}{\alpha}\sin(kL)e^{\alpha L}$

$$-\frac{\phi'}{\phi} = k\tan(kL) = \alpha \Rightarrow kL\tan(kL) = L\alpha = \text{Condition on } E.$$

$$k^2 = \frac{2m}{\hbar^2}(V_0 + E), \quad \alpha^2 = \frac{2m}{\hbar^2}(-E)$$

A. Said Tongel

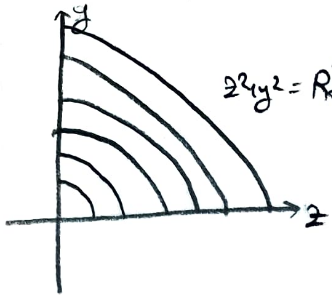
Graphical Solution:

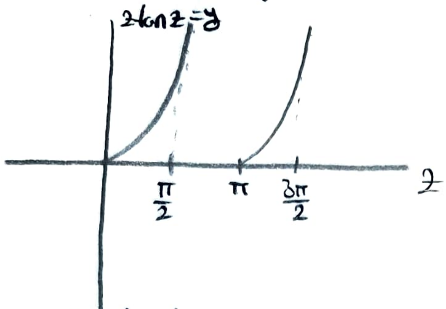
$kL = z$, $\alpha L = y$

$z^2 + y^2 = \frac{2mL^2}{\hbar^2} V_0 \equiv R_0^2$

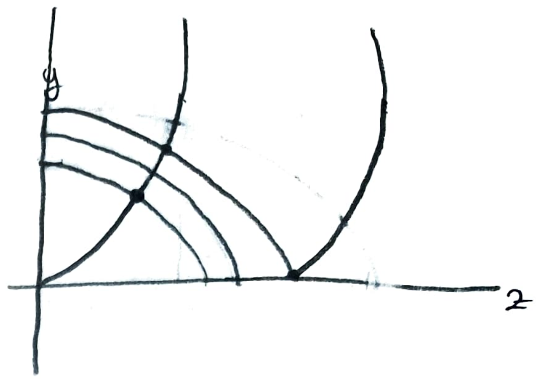
$z\tan z = y$, $z^2+y^2 = R_0^2$

We want solutions simultaneously for this equations.



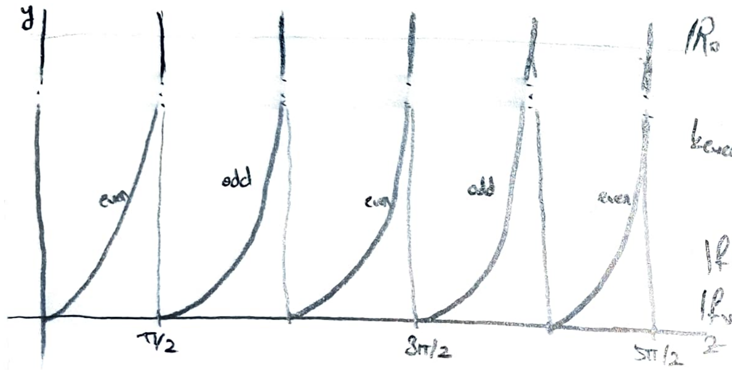


$\Longrightarrow$



In 1-D we always have at least 1 bound state.

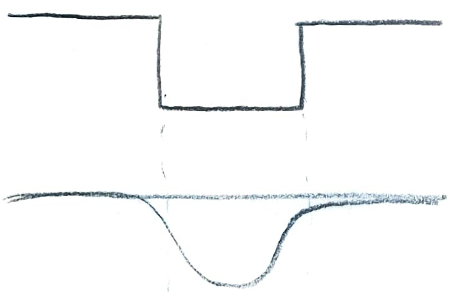
If we take "well" arbitrarly deep



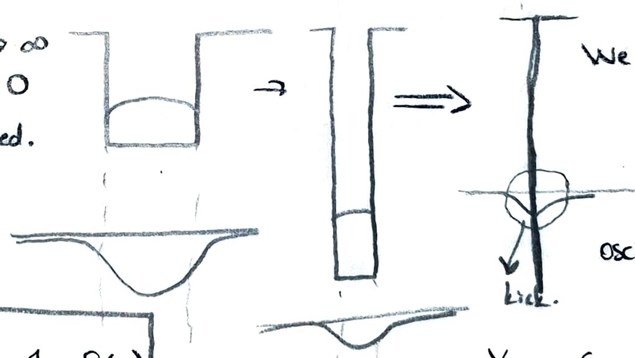
R_0 $\quad z = \frac{2n+1}{2}\pi$,

$k_{even} = \frac{2n+1}{2L}\pi$, $k_{odd} = \frac{n+1}{L}\pi$

If we make well deeper, we get more states and If we make width more narrow we get less states.



as $V_0 \to \infty$
$L \to 0$
Area fixed.

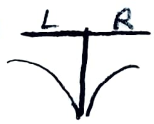


We get a δ function.

oscillates more rapidly

$\frac{V_0}{\varepsilon}$, ε is width.

Ex: $V = -V_0\,\delta(x)$



$\delta(\alpha x) = \frac{1}{|\alpha|}\delta(x)$

$\phi_E(x) = \begin{cases} A e^{\alpha x} + \cancel{B e^{-\alpha x}} & x<0 \\ \cancel{C e^{\alpha x}} + D e^{-\alpha x} & x>0 \end{cases}$

$A = D$ since sym.

$\phi'' = -\frac{2m}{\hbar^2}\left(V_0\,\delta(x) + E\right)\phi$

$\phi'' = \delta$

$\phi' = $ step

$\phi = $ cont.

$\int_{-\varepsilon}^{\varepsilon} \phi'' dx = \phi'(\varepsilon) - \phi'(-\varepsilon) = -\frac{2m}{\hbar^2} V_0\,\phi(0) + O(\varepsilon)$

$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\phi'' dx \Rightarrow \Delta\phi'(0) = -\frac{2m}{\hbar^2}V_0\,\phi(0) \Rightarrow$ This is only solvable if $\alpha = \frac{mV_0}{\hbar^2}$.

$E = -\frac{\hbar^2\alpha^2}{2m}$, Delta function has "1" even bound state

For odd ones, $A = -D$ and wavefunction $\phi(0) = 0$

This becomes zero for it and discontinuity is also zero

Means there is no potential.

Which implies no odd bound state

"Every isolated Delta function has 1 bound state".

Scattering in 1-D Space:

$\vec{m}$

$$\phi_E = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)} \Rightarrow E = \frac{\hbar^2 k^2}{2m} \quad \text{to solve Schrodinger Eq.} \quad \omega = \frac{\hbar k^2}{2m}$$

$\underbrace{x = \frac{\omega t}{k}}_{\text{moving along } +x \text{ direction}}$ $\quad$ $\underbrace{x = \frac{-\omega t}{k}}_{\text{moving along } -x \text{ direction}}$ $\quad$ ϕ_E is not normalizable

$\phi_k = \frac{1}{\sqrt{2\pi}} e^{ikx}$ such that $(\phi_k | \phi_{k'}) = \delta(k-k')$

Evolution in time of a well localized wavepacket.

Ex: Consider a free particle in a min. uncertainty Ψ.

$$\Psi(x,0) = \frac{1}{\sqrt{a\sqrt{\pi}}} e^{-x^2/2a^2} \rightarrow \text{what is } \Psi(x,t)?$$

$$\Psi(x,0) = \sum_n c_n \varphi_n(x) \Rightarrow \Psi(x,t) = \sum_n c_n \varphi_n(x) e^{-i\frac{E_n t}{\hbar}} \quad , \quad \hat{E}\varphi_n = E_n \varphi_n$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (e^{ikx}) \tilde{\Psi}(k)\, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \frac{\sqrt{a}}{\sqrt[4]{\pi}} e^{-k^2 a^2/2}\, dk.$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{a}{\sqrt{\pi}}} \int_{-\infty}^{\infty} e^{i(kx-\omega_k t)} e^{-k^2 a^2/2}\, dk = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{a}{\sqrt{\pi}}} \int_{-\infty}^{\infty} e^{i(kx - \frac{\hbar k^2}{2m} t)} e^{-k^2 a^2/2}\, dk$$

$$= \sqrt{\frac{a}{\sqrt{\pi}}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} e^{-k^2/2 (a^2 + \frac{i\hbar}{m} t)}\, dk$$

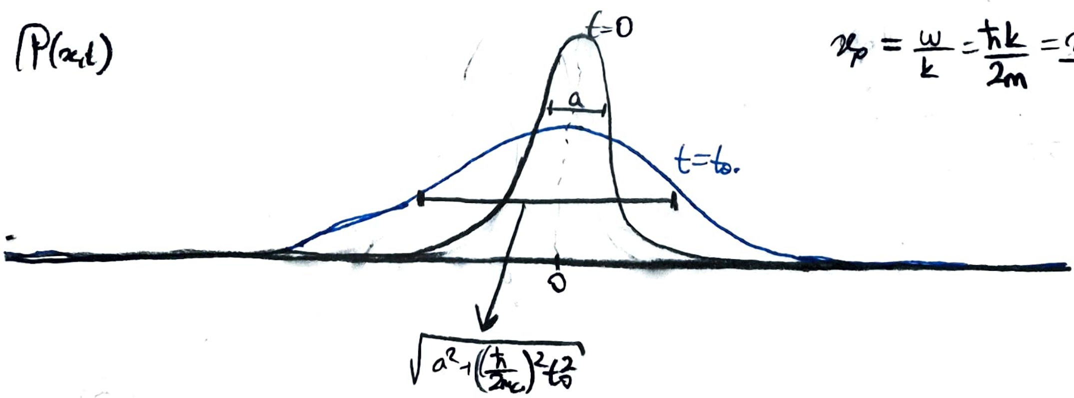
$$\Psi(x,t) = \frac{\sqrt{a}}{\sqrt[4]{\pi}} \frac{1}{\sqrt{a^2 + \frac{i\hbar}{m} t}} e^{-x^2/2 \cdot (a^2 + \frac{i\hbar}{m} t)}$$

$$P(x,t) = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{a^2 + (\frac{\hbar}{2ma})^2 t^2}} e^{-x^2/2 (a^2 + (\frac{\hbar}{2ma})^2 t^2)}$$

Probability at any point as time goes on is decreasing. Width is getting increased. $v = \frac{\hbar}{2ma}$.

$$v_p = \frac{\omega}{k} = \frac{\hbar k}{2m} = \frac{v_c}{2} ; \quad v_g = \frac{\partial \omega}{\partial k} = \frac{\hbar k}{m} = v_c.$$

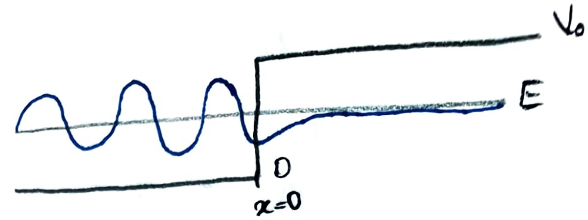
$P(x,t)$



A. Said Töngel

Ex: Potential Step:

$$\phi_E(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & L \\ \cancel{Ce^{\alpha x}} + De^{-\alpha x} & R \end{cases}$$



ϕ, ϕ' must be continues. at $x=0$.

ϕ: $A+B=D$, ϕ': $ik(A-B)=-\alpha D$.

$\Rightarrow D = \frac{2k}{k+i\alpha}$, $B = \frac{k-i\alpha}{k+i\alpha} = e^{i\phi}$

$\omega = E/\hbar$

$$\phi_E(x,t) = \begin{cases} Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)} & L \\ De^{-\alpha x-\omega t i} & R \end{cases}$$

$Ae^{ikx-\omega t i}$ } Right Moving Contribution

$Be^{-ikx-i\omega t}$ } Left Moving Contribution

$De^{-\alpha x-i\omega t}$ } Exponentialy falling, rotating phase

Suppose we have a wave function such

$\int |\psi|^2 dx = N \neq 1$. (1) . Then dP;

(2) $dP = \frac{1}{N} |\psi|^2 dx$ Since $\int dP = \frac{1}{N}\int |\psi|^2 dx = \frac{1}{N} . N = 1$ (3)

N is called normalization factor as long as $\int |\psi|^2 dx < \infty$ we can find a N and normalize the wavefunction. This functions called normalizable or square integrable functions.

By adjusting the coeff. of ψ we can make it normalized. Assuming (1) we define ψ' as

$$\psi' = \frac{1}{\sqrt{N}} \psi \quad (4)$$

is properly normalized. So

$$\int |\psi'|^2 dx = \frac{1}{N}\int |\psi|^2 dx = 1. \quad (5)$$

Now let's look at ψ as a probability amplitude. Suppose ψ is a normalized wavefunction at initial time t_0.

$$\int_{-\infty}^{\infty} \psi^*(x,t_0)\, \psi(x,t_0)\, dx = 1. \quad (6)$$

Since Schrodinger Equation determine ψ for all times, Do we have

$$\int_{-\infty}^{\infty} \psi^*(x,t)\, \psi(x,t)\, dx = 1 \; ? \quad (7)$$

First Define the Probability Density, $\rho(x,t)$

$$\rho(x,t) \equiv \Psi^*(x,t)\,\Psi(x,t) = |\Psi(x,t)|^2 \qquad (8)$$

Define also $N(t)$ as the integral of probability density throughout space

$$N(t) \equiv \int \rho(x,t)\,dx \qquad (9)$$

Since wavefunction initially normalized, $N(t_0)=1$.

To say that $N(t)=1$, we should show that

$$\frac{dN(t)}{dt} = 0 \qquad (10)$$

We call this conservation of probability. Let's check on Schrodinger Equation

$$\frac{dN(t)}{dt} = \int_{-\infty}^{\infty} \frac{\partial \rho(x,t)}{\partial t}\,dx = \int_{-\infty}^{\infty}\left(\frac{\partial \Psi^*}{\partial t}\Psi(x,t) + \Psi^*(x,t)\frac{\partial \Psi(x,t)}{\partial t}\right)dx \qquad (11)$$

Take its complex conjugate,

$$i\hbar\frac{\partial \Psi}{\partial t} = \hat{H}\Psi \Rightarrow \frac{\partial \Psi}{\partial t} = -\frac{i}{\hbar}\hat{H}\Psi \qquad (12)$$

$$-i\hbar\frac{\partial \Psi^*}{\partial t} = (\hat{H}\Psi)^* \Rightarrow \frac{\partial \Psi^*}{\partial t} = \frac{i}{\hbar}(\hat{H}\Psi)^* \qquad (13)$$

Now use (12) and (13) in (11),

$$\frac{dN(t)}{dt} = \int_{-\infty}^{\infty}\left(\frac{i}{\hbar}(\hat{H}\Psi)^*\Psi - \frac{i}{\hbar}\Psi^*(\hat{H}\Psi)\right)dx$$

$$= \frac{i}{\hbar}\left[\int_{-\infty}^{\infty}(\hat{H}\Psi)^*\Psi\,dx - \int_{-\infty}^{\infty}\Psi^*(\hat{H}\Psi)\,dx\right] \qquad (14)$$

To show that time derivative of $N(t)$ vanishes

$$\int_{-\infty}^{\infty}(\hat{H}\Psi)^*\Psi\,dx = \int_{-\infty}^{\infty}\Psi^*(\hat{H}\Psi)\,dx \qquad (15)$$

must be equal. If $\hat{H}$ is a Hermitian operator the condition will be satisfied. $\hat{H}$ is Hermitian if

$$\int_{-\infty}^{\infty}(\hat{H}\Psi_1)^*\Psi_2\,dx = \int_{-\infty}^{\infty}\Psi_1^*(\hat{H}\Psi_2)\,dx \qquad (16)$$

By defining Hermitian conjugate $\hat{H}^{\dagger}$

$$\int_{-\infty}^{\infty}\Psi_1^*(\hat{H}\Psi_2)\,dx = \int_{-\infty}^{\infty}(\hat{H}^{\dagger}\Psi_1)^*\Psi_2\,dx \qquad (17)$$

$\hat{H}$ is Hermitian if $\hat{H} = \hat{H}^{\dagger}$.

A. Said Tüngel

Now take a look to the integrand of equation (14).

$$\frac{\partial \rho}{\partial t} = \frac{i}{\hbar}\left[(\hat{H}\Psi)^*\Psi - \Psi^*(\hat{H}\Psi)\right]$$

$$= \frac{i}{\hbar}\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi^*}{\partial x^2}\Psi - \frac{\partial^2\Psi}{\partial x^2}\Psi^*\right) + V(x,t)\Psi^*\Psi - \Psi^* V(x,t)\Psi\right] \quad (18)$$

$$= \frac{\hbar}{2im}\left(\frac{\partial^2\Psi^*}{\partial x^2} - \Psi^*\frac{\partial^2\Psi}{\partial x^2}\right) \quad (19)$$

To show that the integral is zero it must be a total derivative.

$$\Rightarrow \frac{\partial}{\partial x}\left[\frac{\hbar}{2im}\left(\frac{\partial\Psi^*}{\partial x}\Psi - \Psi^*\frac{\partial\Psi}{\partial x}\right)\right] \quad (20)$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}J(x,t) \quad \text{where } J(x,t) = \frac{\hbar}{2im_e}\left(\Psi^*\partial_x\Psi - \Psi\partial_x\Psi^*\right)$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}J(x,t)$$ $J(x,t)$ is called current associated probability ρ.

→ Conservation equation.

Potential Step Problem;

Now if we come back to our

$$\Psi = \begin{cases} \Psi_I + \Psi_R & L \\ \Psi_T & R \end{cases} \qquad \left.\begin{aligned} T &= \left|\frac{J_T}{J_I}\right| = 0 \\ R &= \left|\frac{J_R}{J_I}\right| = 1 \end{aligned}\right\}$$

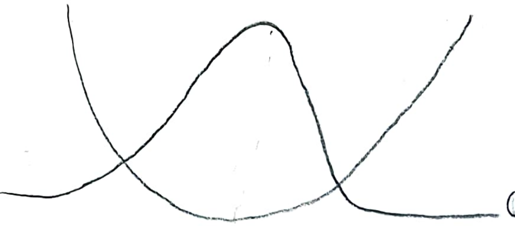
$$J_I = \frac{\hbar k}{m}|A|^2 \qquad J_R = \frac{\hbar k}{m}|B|^2$$

$$J_T = 0.$$

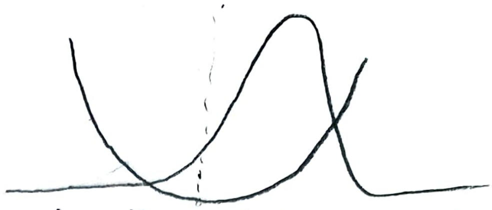
Def:

$$\left.\begin{aligned} T &\equiv \left|\frac{J_T}{J_I}\right| \Rightarrow \text{Transmition Probability} \\ R &\equiv \left|\frac{J_R}{J_I}\right| \Rightarrow \text{Reflection Probability} \end{aligned}\right\}$$

Coherent States: The ground state of a Harmonic Oscillator, Gaussian.



Displace the State a bit $\longrightarrow$



$\varphi_0 \sim N e^{-x^2/2\sigma^2}$

If we take a state φ_0 and translate it by x_0 and act with annihilation operator

-Normally if we act on φ_0 with $\hat{a}$ -

$\hat{a}\,\varphi_0(x) = 0$, since φ_0 is ground state but

$\hat{a}\,\varphi_0(x-x_0) = c\,\varphi_0(x)$ it turns out that displaced Gaussian States are eigenstates of the annihilation operator

Which are "Coherent States".

Coherent States mostly behaves classicaly even created by quantum mechanical wavefunction.

Ex. Scattering:

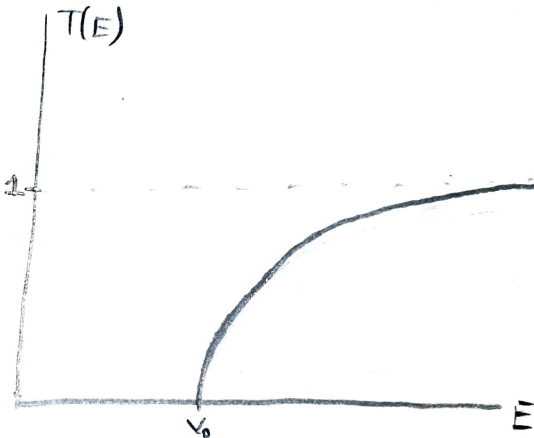
$\rightarrow$

$E > V_0$

$\frac{\hbar^2 k_1^2}{2m} = E$, $\frac{\hbar^2 k_2^2}{2m} = E - V_0$

$$\phi_E = \begin{cases} \overset{In}{A e^{ik_1 x}} + \overset{Out}{B e^{-ik_1 x}} \\ \overset{Out}{C e^{ik_2 x}} + \overset{In}{D e^{-ik_2 x}} \end{cases}$$

$\left.\begin{matrix} L \\ R \end{matrix}\right\} e^{-i\omega t}$



Incident From $L \Rightarrow D=0,\ A\neq 0$

" " $R \Rightarrow A=0,\ D\neq 0$

$\boxed{D=0}$ $C = \frac{2k_1}{k_1+k_2} A$, $B = \frac{k_1-k_2}{k_1+k_2} A$

$$\Rightarrow R = \left|\frac{k_1-k_2}{k_1+k_2}\right|^2 = \left|\frac{1-\sqrt{1-V_0/E}}{1+\sqrt{1-V_0/E}}\right|^2$$

$$T = \frac{4k_1k_2}{|k_1+k_2|^2} = \frac{4\sqrt{1-V_0/E}}{|1+\sqrt{1-V_0/E}|^2}$$

$R + T = 1$

$\boxed{A=0}$ $C = \frac{k_1-k_2}{k_1+k_2} D$, $B = \frac{2k_1}{k_1+k_2} D$

But transmition was also this.

So we can say transmition and reflection amplitude are the same.

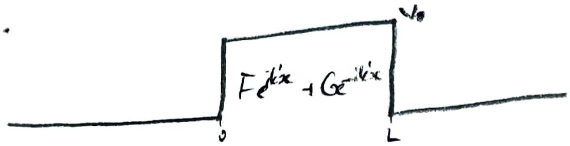
In classical mechanics when an object travels from low to high potential it would slow down and when traveling from high to low it gets faster.

But in Quantum Mechanics this thing is symmetric whether its moving from high to low or low to high.

From the detailed shape of the graph of transmission as a function of Energy, we can deduce what the potential and using phase shift.

A. Said Töngel

Ex.



$$\frac{\hbar^2 k^2}{2m_e} = E, \quad \frac{\hbar^2 k'^2}{2m_e} = E - V_0$$

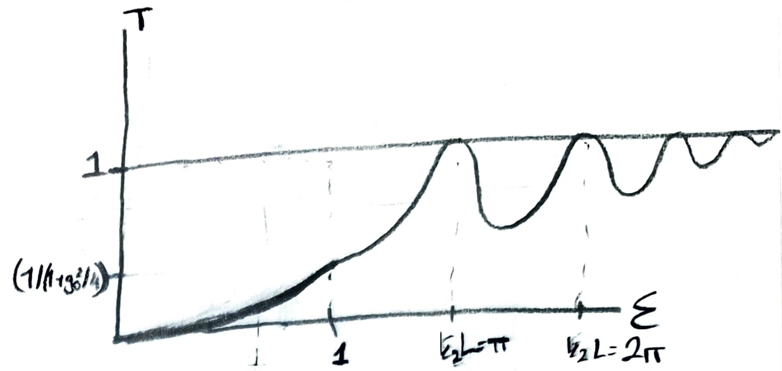
$$g_0^2 = \frac{2m_e L^2 V_0}{\hbar^2}, \quad \mathcal{E} = \frac{E}{V_0}.$$

$D = 0$, $E > V_0$

$$T = \left|\frac{C}{A}\right|^2 = \frac{4k^2k'^2}{4k^2k'^2\cos^2(k'L) + (k^2+k'^2)\sin^2(k'L)}$$

$$T_{\substack{E>V_0 \\ \mathcal{E}\geq 1}} = \frac{1}{1 + \frac{1}{4\mathcal{E}(\mathcal{E}-1)}\sin^2(g_0\sqrt{\mathcal{E}-1})}$$

$$\Rightarrow \sim \frac{1}{1 + \frac{g_0^2}{4}} \quad \text{as } \mathcal{E} \to 1.$$



$$T_{\substack{E<V_0 \\ \mathcal{E}<1}} = \frac{1}{1 + \frac{1}{4\mathcal{E}(1-\mathcal{E})}\sinh^2(g_0\sqrt{1-\mathcal{E}})}$$

This transmission across classicaly disallowed barier is called "Quantum Tunelling".

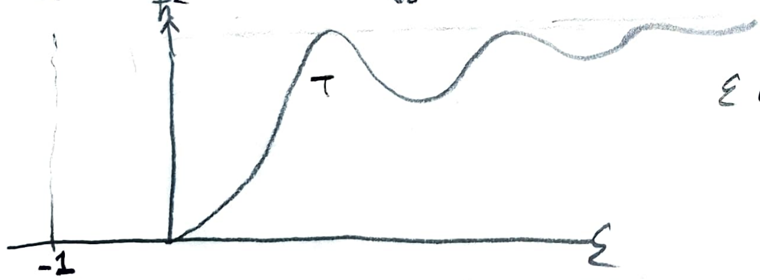
For $L \gg 1$, $T \sim e^{-2\alpha L}$

α is proportional with E.

Ex.

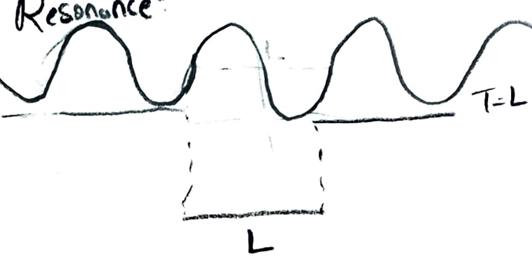
$$\Rightarrow T_{\mathcal{E}>0} = \frac{1}{1 + \frac{1}{4\mathcal{E}(\mathcal{E}+1)}\sin^2(g_0\sqrt{\mathcal{E}+1})}$$

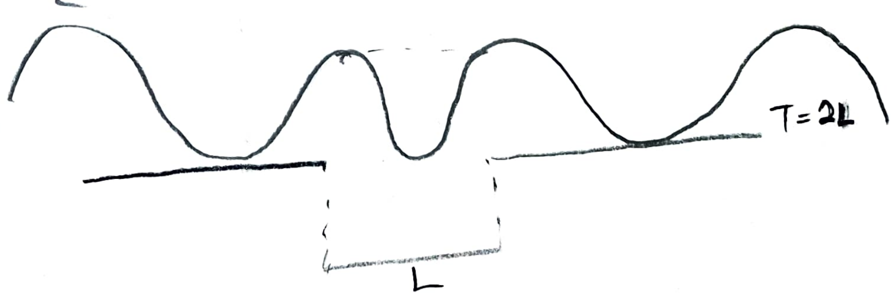
$$g_0^2 = \frac{2m_e L^2}{\hbar^2} V_0, \quad \mathcal{E} = \frac{E}{V_0}$$

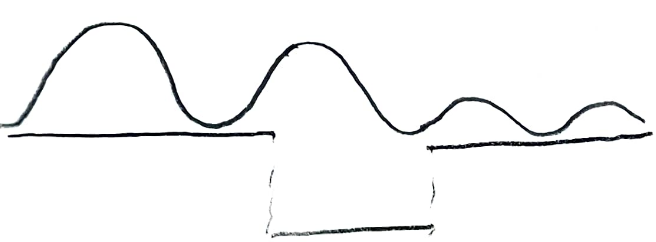


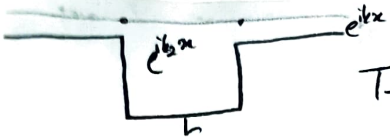
$\mathcal{E}$ must be greater than -1.

Resonance:





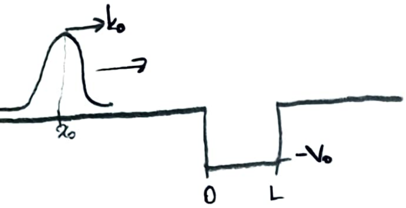




$T_{LR} \neq T_L T_R$, probabilities don't add up wavefunctions do.

$$t_{LR} = t_L e^{ik_2L} t_R + t e^{ik_2L} r e^{ik_2L} r e^{ik_2L} t + \dots$$

$$\Rightarrow t_{LR} = t_L e^{ik_2L} t_R \left(\frac{1}{1-|r|^2 e^{2ik_2L}} \right) = \frac{1}{e^{ik_2L} - \frac{2i}{T_R}\sin(k_2L)} \Rightarrow T_{LR} = \frac{1}{1 + \frac{1}{4E(E+1)}\sin^2(g_0\sqrt{E+1})}$$



$$\Psi(x,0) = N e^{-(x-x_0)^2/2\sigma^2} \cdot e^{ik_0x}, \quad \tilde{f}(k) = \tilde{N} e^{-\sigma^2(k-k_0)^2/2} e^{-ikx_0}$$

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int e^{ikx} \tilde{f}(k)\, dk, \quad \Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int \tilde{f}(k) e^{ikx - wti} dk \Bigg\} \frac{\hbar^2k^2}{2m} = \hbar w$$

$$E = \frac{\hbar^2 k^2}{2m} \qquad \phi_k = \frac{1}{\sqrt{2\pi}} \left[e^{ikx}\theta(-x) + r e^{-ikx}\theta(-x) + t e^{ikx}\theta(x) \right]$$

, r = reflection amplitude
t = transmittion amplitude

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int \tilde{f}(k)\phi_k\, dk = \frac{1}{\sqrt{2\pi}} \int \left[\tilde{f}(k) e^{ikx}\theta(-x) + \tilde{f}(k) r e^{-ikx}\theta(-x) + \tilde{f}(k) \cdot t e^{ikx}\theta(x) \right] dk.$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int \left[\tilde{f}(k) e^{i(kx-wt)}\theta(-x) + \tilde{f}(k) r e^{-i(kx+wt)}\theta(-x) + \tilde{f}(k) t e^{i(kx-wt)}\theta(x) \right] dk$$

$$\theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

at $t=0$:

$$\int \left(\tilde{f}(k) e^{ikx}\theta(-x) \right) dk. \Rightarrow G(x_0, k_0, 0)\,\theta(-x)$$. That gives us our Gaussian back.

$$\frac{d}{dk}\left(kx - w(k)t\right)\Big|_{k_0} = x - \frac{\hbar k}{m}t = 0 \Rightarrow x = \frac{\hbar k}{m}t = v_0 t$$, where v_0 is classical velocity

$$\Big\{ e^{i(kx-wt)}\theta(-x) \equiv \text{Term 1}, \; r e^{-i(kx+wt)}\theta(-x) \equiv \text{Term 2}, \; t e^{i(kx-wt)}\theta(x) \equiv \text{Term 3}. \Big\}$$

As time goes forward Term 1 will gone and get replaced by term 2 and 3.

Focus on 3rd term.

$$t = \sqrt{T} e^{-i\varphi}, \quad \theta(x) \frac{1}{\sqrt{2\pi}} \int \tilde{f}(k) e^{+i(kx - wt - \varphi)} \cdot \sqrt{T}\, dk$$

$$\Rightarrow \frac{d}{dk}\left(kx - wt - \varphi\right)\Big|_{k_0} = 0 \qquad , \frac{d\varphi}{dk} = \frac{dw}{dk}\frac{d\varphi}{dw} = v_0 \hbar \frac{d\varphi}{dE}$$

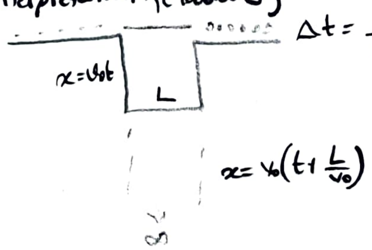
$$= x - v_0 t - \frac{d\varphi}{dk}$$

$$= x - v_0\left(t + \hbar\frac{\partial\varphi}{\partial E}\right) = 0$$

$$x = v_0\left(t + \hbar\frac{\partial\varphi}{\partial E}\right)$$

Interpretation (classicaly)

$$\Delta t = \frac{L}{V_0}$$



$\Rightarrow$ Compare with Q.R.

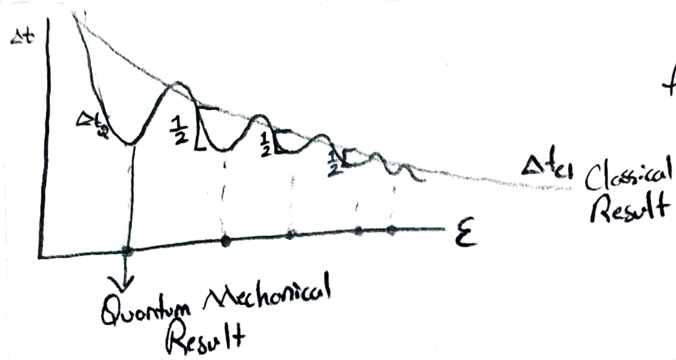
$$\Delta_{cl} = \frac{L}{V_0} \overset{?}{=} \hbar\frac{\partial\varphi}{\partial E}$$

$$\varphi = k_2L - \arctan\left(\frac{k_1^2 + k_2^2}{2k_1k_2}\tan(k_2L)\right)$$

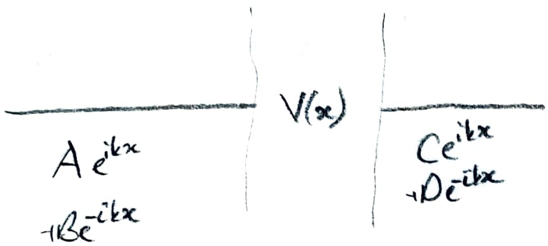
• Near resonance $k_2L = n\pi$

$$\Rightarrow \hbar\frac{\partial\varphi}{\partial E} = \frac{L}{2v_0}\left(1 + \frac{E}{V_0}\right)$$

A. Said Tongel



Quantum Mechanicaly its taking more time to pass through because there is reflection.



$$\begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix}$$

The "S-scattering-" Matrix

1-) Stuff does not disappear

$$|A|^2 + |D|^2 = |B|^2 + |C|^2 \ , \ (A^* \ D^*)\begin{pmatrix} A \\ D \end{pmatrix} = (B^* \ C^*)\begin{pmatrix} B \\ C \end{pmatrix} = (A^* \ D^*) S^\dagger S \begin{pmatrix} A \\ D \end{pmatrix}$$

$S^\dagger S = \mathbb{1}$, S is unitary matrix.

Consequence of that, eigenvalues of S are pure phases; $S_1 = e^{i\varphi_1}$, $S_2 = e^{i\varphi_2}$

$|S_{11}| = |S_{22}|$, $|S_{12}| = |S_{21}|$, $|S_{12}|^2 + |S_{11}|^2 = 1$, $S_{11}S_{12}^* + S_{21}S_{22}^* = 0$

Case, $\boxed{D=0}$

$A=1$, for normalization. $\Rightarrow \begin{cases} \frac{B}{A} = S_{11} = r_{\leftarrow} \\ \frac{C}{A} = S_{21} = t_{\rightarrow} \end{cases}$ $|r_{\rightarrow}| = |r_{\leftarrow}|$, $|S_{12}|^2 + |S_{11}|^2 = T + R = 1$.

Suppose our system is time reversel invarient $t \to -t$, ψ is a sol'n then ψ^* is also sol'n.

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} \\ Ce^{ikx} + De^{-ikx} \end{cases} , \ \psi^* = \begin{cases} A^*e^{-ikx} + B^*e^{ikx} \\ C^*e^{-ikx} + D^*e^{ikx} \end{cases} , \ \begin{pmatrix} A^* \\ D^* \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} B^* \\ C^* \end{pmatrix} \Rightarrow S^* S = \mathbb{1}$$

$\Rightarrow S = S^T \Rightarrow S_{21} = S_{12}$

We can determine the bound states from the information gathered from scattering.

$\begin{pmatrix} B \\ C \end{pmatrix} = S_E \begin{pmatrix} A \\ D \end{pmatrix}$, Take $E<0$, $k \to i\alpha$ $\quad S_E \begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq 0$ unless S_E is divergence. $S \sim \frac{c}{0}$.

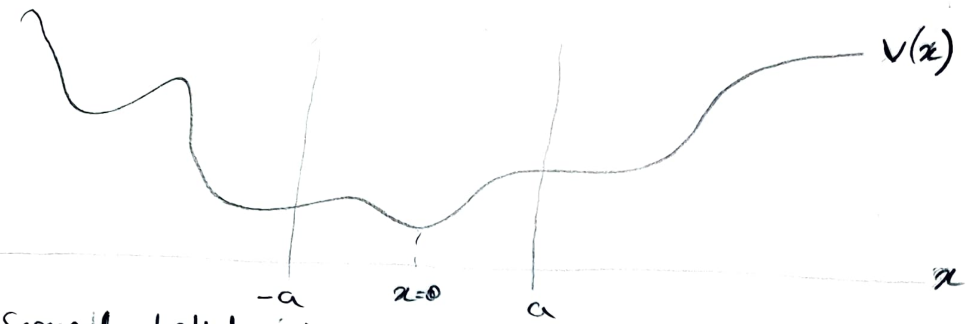
$$\psi = \begin{cases} Ae^{-\alpha x} + Be^{\alpha x} \\ Ce^{-\alpha x} + De^{\alpha x} \end{cases} , \ A, D \text{ must be zero.} \Rightarrow \begin{pmatrix} A \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

For finite square well: $S_{21} = t_{LR} = \dfrac{2k_1 k_2 e^{ik_1 L}}{2k_1 k_2 \cos(k_2 L) - i(k_1^2 + k_2^2)\sin(k_2 L)}$

When S_{21} has a pole, $\dfrac{k_2 L}{2}\tan\left(\dfrac{k_2 L}{2}\right) = \dfrac{k_1 L}{2}$ $\Rightarrow$ condition for the bound states of a square well.

Node Theorem {Repeated}:

If $\psi_1, \psi_2, \psi_3, \ldots$ are energy eigenstates of a 1-D potential where $E_1 < E_2 < E_3 < \ldots$ then $\psi_n(x)$ has exactly $n-1$ nodes

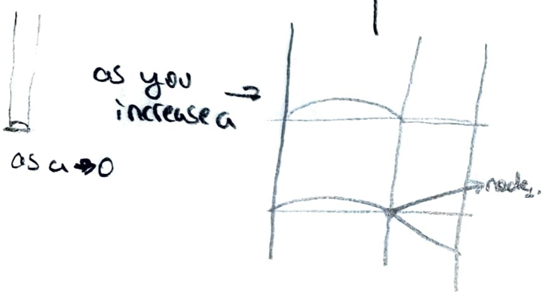


Consider that $\psi(x_0) = \psi'(x_0) = 0$ Not possible.

Screened potential $V_a(x)$

$$V(x) = \begin{cases} v(x) & |x| < a \\ \infty & |x| > a \end{cases}$$

Bound states of $V_a(x)$ as $a \to \infty$ are the bound states of $v(x)$



$[\hat{E}, \hat{A}] = +\hbar\omega\hat{A} \Rightarrow E_n$ evenly spaced by $\hbar\omega$ $\quad \hat{A}\phi_E = \phi_{E+\hbar\omega}$

$[\hat{E}, \hat{B}] = 0 \Rightarrow$

1-) $\phi_{E,B}$ are simultaneous eigenfunctions

2-) $\hat{B}\phi_E = \tilde{\phi}_E$.

$[\hat{E}, \hat{u}] = 0$, $\hat{u}^\dagger\hat{u} = \mathbb{1} \Rightarrow$

1-) Both simultaneous eigenfunctions

2-) $u\phi_E = \tilde{\phi}_E$

3-) $u = e^{i\hat{A}}$

$\hat{T}_L = e^{-L\partial_x} \Rightarrow$ Translation by L.

$\hat{B}_q = e^{-q\partial_p} \Rightarrow$ Boost by q

$\hat{u}_t = e^{-i\frac{t}{\hbar}\hat{E}} \Rightarrow$ Time translation.

A. Said Tüngel

Quantum Mechanics in 3-D:

$\hat{x}, \hat{y}, \hat{z}, \hat{p}_x, \hat{p}_y, \hat{p}_z \mid [\hat{x}, \hat{p}_x] = i\hbar, \; [\hat{x}, \hat{y}] = 0, \; [\hat{p}_x, \hat{p}_y] = 0, \; [\hat{p}_y, \hat{y}] = -i\hbar$

$\Delta x \Delta p \geq \frac{\hbar}{2}$

No sim.
eg.func.

$$\hat{E} = \frac{\hat{p}^2}{2m} + u(\hat{x}, \hat{y}, \hat{z}) \qquad , \hat{p} = -i\hbar \vec{\nabla}$$

SE.

$$i\hbar \partial_t \Psi(x,y,z,t) = i\hbar \partial_t \Psi(\vec{r}, t) = \left[-\frac{\hbar^2}{2m} \vec{\nabla}^2 + u(\vec{r}) \right] \Psi(\vec{r}, t)$$

$\dim(\Psi(\vec{r}, t)) = 1 / L^{3/2}$.

Ex. 1: Free particle in 3-D * Cartesian coords.

$$E \phi_E = \frac{-\hbar^2}{2m} (\partial_x^2 + \partial_y^2 + \partial_z^2) \phi_E$$

$$\phi_E(x,y,z) = \phi_x(x) \phi_y(y) \phi_z(z)$$

$E\phi_E = \frac{-\hbar^2}{2m} (\phi_x'' \phi_y(y) \phi_z(z) + \phi_x(x) \phi_y''(y) \phi_z(z) + \phi_x(x) \phi_y(y) \phi_z''(z))$, divide both sides by ϕ_E.

$$E = \frac{-\hbar^2}{2m} \left(\frac{\phi_x''}{\phi_x} + \frac{\phi_y''}{\phi_y} + \frac{\phi_z''}{\phi_z} \right) \Rightarrow \frac{-2m}{\hbar^2} E = f(x) + g(y) + h(z)$$

$\Rightarrow$ This implies that

$$\frac{\phi_x''}{\phi_x} = -\varepsilon_x, \; \frac{\phi_y''}{\phi_y} = \varepsilon_y, \; \frac{\phi_z''}{\phi_z} = -\varepsilon_z \Rightarrow \frac{+2m}{\hbar^2} E = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\phi''(x) + \varepsilon_x \phi(x) = 0$$

$\Rightarrow \phi_x = Ae^{ik_x x} + Be^{-ik_x x} \Rightarrow K_x^2 = \varepsilon_x$ shows that ε's are positive.

$$E = \frac{\hbar^2}{2m} (K_x^2 + K_y^2 + K_z^2) = \frac{\hbar^2}{2m} \vec{K}^2$$

$\phi_E = \underbrace{N e^{ik_x x} e^{ik_y y} e^{ik_z z}}_{\text{Some good basis}} = N \cdot e^{i\vec{k} \cdot \vec{r}}, \quad E = \frac{\hbar^2 \vec{k}^2}{2m}$

Gen Sol'n is

$$\phi_E = Ae^{ik_x x} \phi_y \phi_z + Be^{-ik_x x} \phi_y \phi_z$$

Suppose we take K_1 such that $\frac{\hbar^2 K_1^2}{2m} = E = \frac{\hbar^2 K_2^2}{2m}$ where $K_1 \neq K_2$

In 1-D $(K, -K)$ had same energy.

In 3-D K_1, K_2 /same magnitude → same energy means degenerate

System is symmetric so degeneracies occurs

Symmetry ⟺ degeneracy

3-D Harmonic Oscillator

$$V(\vec{r}) = \frac{m\omega^2}{2}(x^2+y^2+z^2) = \frac{m\omega^2}{2}|\vec{r}|^2$$

$$E\phi_E = \left[\frac{-\hbar^2}{2m}\left[\partial_x^2 + \partial_y^2 + \partial_z^2\right] + \frac{m\omega^2}{2}(x^2+y^2+z^2)\right]\phi_E$$

$$= \left[\left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{m\omega^2}{2}x^2\right) + ('' y) + ('' z)\right]\phi_E$$

$$E_x\phi_x = \left(\frac{-\hbar^2}{2m}\partial_x^2 + \frac{m\omega^2}{2}x^2\right)\phi_x \ , \quad \phi_E = \phi_{n_x}(x)\phi_{n_y}(y)\phi_{n_z}(z) \ , \quad E = E_{n_x} + E_{n_y} + E_{n_z}$$

$\phi_E = \phi_\ell(x)\,\phi_m(y)\,\phi_n(z)$

$E = \hbar\omega_0\left(\ell + m + n + \frac{3}{2}\right)$

} Basis of solutions of the energy eigenfunctions for 3-D Harmonic Oscillator

Degen.

$\frac{7}{2}$ — $(2,0,0)(0,2,0)(0,0,2)(1,1,0)(0,1,1)(1,0,1)$

$\frac{5}{2}$ — $(1,0,0)(0,1,0)(0,0,1)$

$\frac{3}{2}\hbar\omega_0$ — $(0,0,0)$

$d_4 = 15$
$d_3 = 10$
$d_2 = 6$
$d_1 = 3$
$d_0 = 1$

$d_n = \frac{(n+1)(n+2)}{2}$

* There can be degeneracies among bound states.

Imagine that potential was not symmetric.

Exp.

$$\left('' \omega_0 x\right) + \left('' \omega_0 y\right) + \left('' \omega_0 z\right) \to \left('' \omega_0 x\right) + \left('' \omega_1 y\right) + \left('' \omega_2 z\right) \text{ where } \omega_0 \neq \omega_1 \neq \omega_2 \text{ or } \omega_0 \neq 1 \text{ or } \omega_0 \neq \omega_2$$

$$E = \hbar\omega_0\left(\ell+\tfrac{1}{2}\right) + \hbar\omega_1\left(m+\tfrac{1}{2}\right) + \hbar\omega_2\left(n+\tfrac{1}{2}\right)$$

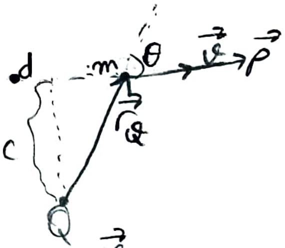
we don't get degeneracy, so this implies that

"when you have symmetry there is degeneracy, where you don't have symmetry you don't get degeneracy

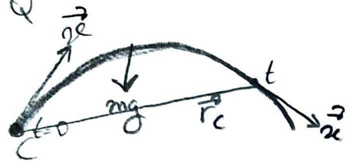
Note: Getting a degenerate case without symmetry may happen with rational ℓ, m, n but its unlikely to face.
Generally getting a case like that is impossible but could happen

A. Said Tongel

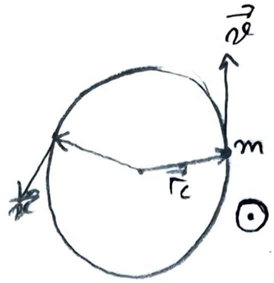
Classical Angular Momentum:



$\vec{L}_Q = \vec{r}_Q \times \vec{p} = (\vec{r}_Q \times \vec{v})\, m.$ $\quad |L_Q| = m v\, r_Q \sin\theta \quad \vec{L}_d = 0.$

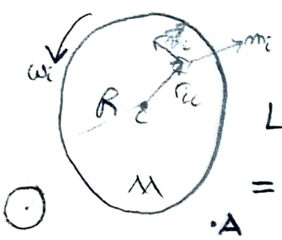


$t=0,\ \vec{L}_c = 0$
$t=t,\ \vec{L}_c \neq 0$



$|\vec{L}_c| = m|\vec{r}_c \times \vec{v}| = m r v$

$\vec{L}_Q = \vec{r}_Q \times \vec{p} \qquad \frac{d\vec{L}}{dt}\Big|_Q = \underbrace{\cancel{\frac{dr}{dt} \times \vec{p}}}_{0} + \vec{r}_Q \times \underbrace{\frac{dp}{dt}}_{F}\Big|_Q = \vec{r}_Q \times \vec{F} = \tau \Rightarrow$ Torque.



$L_{ci} = m_i r_{ic} v_i \quad , v = \omega R,\ v_i = \omega r_{ic}$

$= m_i r_{ic}^2 \omega \Rightarrow L_{disk} = \omega \underbrace{\sum_i m_i r_{ic}^2}_{I} = I_c \omega$, I_c = moment of inertia respect to c.

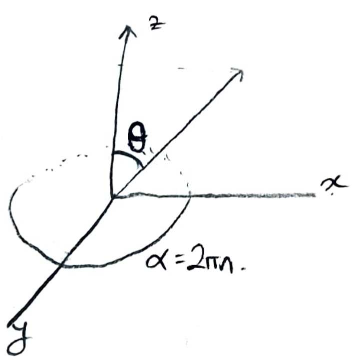
what about respect to a point A?

Even if you calculate the angular momentum respect to any point, as long as there is a rotation about center of mass.- the angular momentum will be conserved. This is called;
"Spin Angular Momentum"

$\vec{L} = \vec{r} \times \vec{p} = (x,y,z) \times (p_x, p_y, p_z) = \hat{i}(y p_z - z p_y) + \hat{j}(x p_z - z p_x) + \hat{k}(x p_y - y p_x)$ } Classicaly

In QM.

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}} = \begin{vmatrix} y p_z - z p_y \\ z p_x - x p_z \\ x p_y - y p_x \end{vmatrix} = \begin{vmatrix} \hat{L}_x \\ \hat{L}_y \\ \hat{L}_z \end{vmatrix} , \quad \hat{L}_z = -i\hbar \partial_\alpha$$



$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\partial_\theta \sin\theta\, \partial_\theta + \frac{1}{\sin^2\theta}\partial_\alpha\right)$$

$$L_z \phi_m = -i\hbar \partial_\theta \phi_m = \hbar m \phi_m \Rightarrow \phi_m = c e^{im\alpha} : m \in \mathbb{Z}$$

$[L_x, L_y] = [yP_z - zP_y, zP_x - xP_z] = [yP_z, zP_x] - [yP_z, xP_z] + [zP_y, zP_x] + [zP_y, xP_z]$

$= yP_x[P_z, z] - yx\underbrace{[P_z, P_z]}_{0} + \underbrace{P_y P_x[z, z]}_{0} + xP_y[z, P_z] = -i\hbar yP_x + i\hbar xP_y = i\hbar(xP_y - yP_x) = i\hbar L_z$

$\Rightarrow [L_x, L_y] = i\hbar L_z$ | $[\hat{L}^2, L_x] = 0$ | We can find simultaneous eigenfunctions of $\hat{L}^2$ with L_x, L_y, L_z. So we can say;

$[L_y, L_z] = i\hbar L_x$ | $[\hat{L}^2, L_y] = 0$

$[L_z, L_x] = i\hbar L_y$ | $[\hat{L}^2, L_z] = 0$

$[\hat{L}^2, L_x]$ or $[\hat{L}^2, L_y]$ or $[\hat{L}^2, L_z]$ gives complete set of commuting observables for angular momentum.

We want to find eigenfunctions of $\{L^2, L_z\}$

$L_+ \equiv L_x + iL_y$ | $[\hat{L}^2, L_\pm] = 0$, $[L_z, L_+] = \hbar L_+$

$L_- \equiv L_x - iL_y$ | $[L_z, L_-] = -\hbar L_-$

Definition.

Let Y_{lm} be eigenfunctions of $L^2 L_z$.

$L^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}$, $L_z Y_{lm} = \hbar m Y_{lm}$

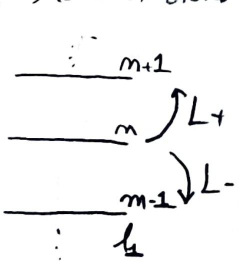
① $L_\pm$ leave l alone

$L^2 L_+ Y_{lm} = L_+ L^2 Y_{lm} = \hbar^2 l(l+1) L_+ Y_{lm}$

② $L_\pm$ raise/lower m by 1.

$L_z L_+ Y_{lm} = ([L_z, L_+] + L_+ L_z) Y_{lm} = (\hbar L_+ + \hbar m L_+) Y_{lm} = \hbar(m+1) L_+ Y_{lm}$.

$\Rightarrow$ ladder of states



Is this tower infinite? $\Rightarrow$ No.

$L^2 = L_x^2 + L_y^2 + L_z^2$

$\langle\rangle_{Y_{lm}} \Rightarrow \hbar^2 l(l+1) = \langle L_x^2\rangle + \langle L_y^2\rangle + \hbar^2 m^2$

$\Rightarrow \geq \hbar^2 m^2 \Rightarrow m_+, m_-$.

Suppose $L_+ Y_{lm_+} = 0 \Rightarrow (L_+ Y_{lm} | L_+ Y_{lm}) = 0 = (Y_{lm} | L_- L_+ Y_{lm}) = (Y_{lm} | [L_x^2 + L_y^2 + i[L_x, L_y]] Y_{lm})$

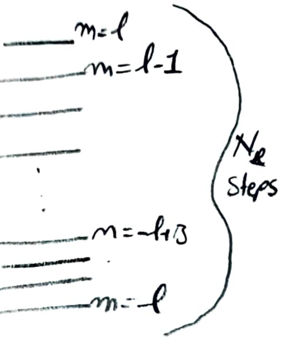
$= (Y_{lm} | [L^2 - L_z^2 - \hbar L_z] Y_{lm}) = (\hbar^2 l(l+1) - m^2\hbar^2 - m\hbar^2) = 0$.

$\Rightarrow \hbar^2(l(l+1) - m_+(m_+ + 1)) = 0$

$\Rightarrow m_+ = +l \Rightarrow m_- = -l$.

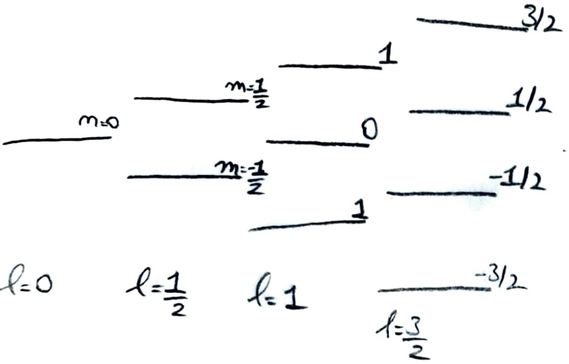
A. Said Tongel

Which means m spans values from $-l$ to l in unit steps.



$$2l = N_l - 1 \quad \rightarrow N_l = \mathbb{Z}_+$$

$$l = \frac{N_l - 1}{2} \rightarrow l \text{ is an integer or half integer.}$$



for all defined l.

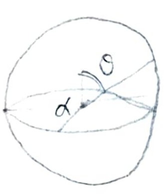
For the state $l, m=l$

$\langle L^2 \rangle = \hbar^2 l(l+1)$

$\langle L_z^2 \rangle = \hbar^2 l^2$

$\Rightarrow \langle L_x^2 \rangle + \langle L_y^2 \rangle = \hbar^2 l$, $\langle L_{x,y}^2 \rangle = \frac{1}{2}\hbar^2 l.$

$Y_{lm}(\theta,\alpha)$; $L_z = \frac{\hbar}{i}\partial_\alpha$, $L_\pm = \hbar e^{\pm i\alpha}(\partial_\theta \pm i\cot\theta\, \partial_\alpha)$



$$\hbar m Y_{lm} = \frac{\hbar}{i}\partial_\alpha Y_{lm} \Rightarrow im Y_{lm} = \partial_\alpha Y_{lm} \Rightarrow Y_{lm}(\theta,\alpha) = e^{im\alpha} \cdot P_{lm}(\theta)$$

$$Y_{lm}(\theta,2\pi) = e^{im2\pi} P_{lm}(\theta) = \begin{cases} Y_{lm}(\theta,0), & m \Rightarrow \text{integer} \\ -Y_{lm}(\theta,0), & m \Rightarrow \text{half-integer} \end{cases}$$

---

m int, $Y_{lm}(\theta,0) = Y_{lm}(\theta,0)$

m half-int, $Y_{lm}(\theta,0) = -Y_{lm}(\theta,0) = 0 \Rightarrow Y_{lm}(\theta,0) = 0$ if m is half integer.

which means half integers can not describe a particle state.

$L_+ Y_{ll} = 0.$

$$\Rightarrow \hbar e^{i\alpha}(\partial_\theta + i\cot\theta\, \partial_\alpha) e^{il\alpha} P(\theta) = 0 = (\partial_\theta - l\cot\theta) P_{ll}(\theta)$$

$$\Rightarrow P_{ll}(\theta) \propto \sin^l(\theta) \Rightarrow Y_{ll} = N_{ll} e^{il\alpha}\sin^l(\theta)$$

$Y_{l\,l-1} \propto L_- Y_{ll}$

Ex: $l=0$

$Y_{00} = \frac{1}{\sqrt{4\pi}}$

$l=1$

$Y_{11} = c\, e^{i\alpha}\sin(\theta)$

$Y_{10} = b \cdot \cos(\theta)$

$Y_{1-1} = a_1 e^{-i\alpha}\sin(\theta)$

$l=2$

$Y_{22} = c \cdot e^{i2\alpha}\sin^2\theta$

$\vdots$

$Y_{20} = a \cdot (3\cos^2\theta - 1)$

$Y_{2-2} = b \cdot e^{-i2\alpha}\sin^2\theta.$

Laplacian in 3D {Spherical Coordinates}

$$\vec{\nabla}^2 = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\left[\partial_\theta^2 + \cot\theta\,\partial_\theta + \frac{1}{\sin^2\theta}\partial_\alpha^2\right] \underset{\downarrow}{=} \left(\frac{1}{r}\partial_r r\right)^2 + \frac{1}{r^2}\left(-\frac{1}{\hbar^2}\hat{L}^2\right)$$

From product rule.

$$\hat{E} = \frac{\hat{p}^2}{2m} + \mathcal{U}(r), \quad \hat{p}^2 = -\hbar^2\nabla^2 = \frac{-\hbar^2}{r}\partial_r^2 r + \frac{\hat{L}^2}{r^2}$$

$$\hat{E} = \frac{-\hbar^2}{2mr}\partial_r^2 r + \frac{1}{2mr^2}\hat{L}^2 + \mathcal{U}(r)$$

$[\hat{E}, \hat{L}^2] = 0$. So we can find common eigenbasis of functions which are both $\hat{E}$'s and $\hat{L}^2$'s.

Seperation of Variables:

$\hat{E}\phi_E = E\phi_E \quad \phi_E(\vec{r}) = \varphi(r)\,Y_{lm}(\theta,\alpha)$, l must be integer - we're talking about rotational angular momentum -

$$\hat{E}\phi_E = \left(\frac{-\hbar^2}{2mr}\partial_r^2 r + \frac{\hbar^2 l(l+1)}{2mr^2} + \mathcal{U}(r)\right)\phi_E, \quad \frac{\hbar^2 l(l+1)}{2mr^2} + \mathcal{U}(r) = V_{\text{effective}}(r)$$

we reduced angular dependence so $\phi_E = \varphi(r)$

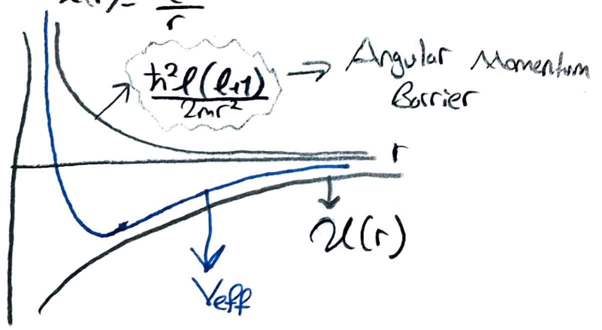
$$E\varphi(r) = \left(\frac{-\hbar^2}{2m}\frac{1}{r}\partial^2 r + V_{\text{eff}}(r)\right)\varphi(r)$$

$$\varphi(r) = \frac{1}{r}u(r): \quad \frac{1}{r}\partial_r^2 r\,\varphi(r) = \frac{1}{r}\partial^2 u(r), \quad V_{\text{eff}}\varphi = \frac{1}{r}V_{\text{eff}}\,u(r)$$

$$\Rightarrow E\,u(r) = \left(\frac{-\hbar^2}{2m}\partial_r^2 + V_{\text{eff}}(r)\right)u(r)$$

Example:

$\mathcal{U}(r) = \frac{-e^2}{r}$



A. Said Tüngel

General Facts for Centeral Potentials.

As $r \to 0$, what must be true for $u(r)$?

if $u(r) \to c$ as $r \to 0 \Rightarrow \phi = \frac{c}{r}$, $\vec{\nabla}^2 \frac{1}{r} \sim \frac{1}{4} \partial_r^2$ is not possible since its diverges and become $\vec{\nabla}^2 \frac{1}{r} = \delta(r)$.

So $u(r) \to 0$ as $r \to 0 \Rightarrow \phi$ becomes constant

- E only depends on l, but not on m

for each $m \in \{l, l-1, \dots, -l\}$

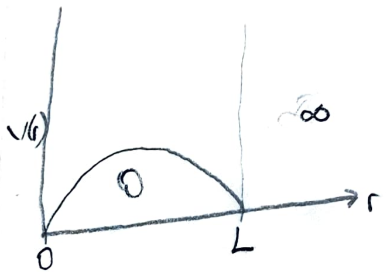
For each m's energy is the same E_l.

$d(E_l) = 2l+1 \equiv$ Degeneracy of E_l. Because of Rotational Symmetry

$[\hat{E}, L_x] = 0$, $[\hat{E}, L_y] = 0$, $[\hat{E}, L_z] = 0$

So we can find a common eigenbasis of E with $L_{x,y,z}$. $[E, L_\pm] = 0$.

Example:



$$\hat{E}u = \frac{\hbar^2}{2m}\left(-\partial_r^2 + \frac{l(l+1)}{r^2}\right)u.$$

$$E_0 = \frac{\hbar^2 k^2}{2m}$$

Special case $l=0$: $E u_0 = \frac{-\hbar^2}{2m} \partial_r^2 u_0 = \frac{-\hbar^2}{2m} u_0''$

$\Rightarrow u_0 = A\cos(kr) + B\sin(kr)$

Boundary: $u(0) = 0 \Rightarrow A = 0$

$u(L) = 0 \Rightarrow kL = n\pi$

$$\Rightarrow \phi_{E_0} = \gamma_{00} \frac{1}{r} \sin\left(\frac{n\pi}{L} r\right) = N \frac{1}{r} \sin\left(\frac{\pi n}{L} r\right)$$

Coulomb Potential: $V(r) = \frac{-e^2}{r}$

$$\frac{-\hbar^2}{2m} u''(r) + \left(\frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2} - \frac{e^2}{r}\right) u(r) = E_\ell u(r)$$

① $[e^2] = E.L = \frac{p^2 L}{2m}$, $[\hbar] = p.L$, $[m] = m$

$\Rightarrow r_0 = \frac{\hbar^2}{2me^2} \Rightarrow E_0 = \frac{e^2}{r_0} = \frac{2me^4}{\hbar^2} = 4\underbrace{R}_{13.6 eV} \rightarrow R = $ Rydberg's Constant.

② $r = r_0 \rho \quad E = -E_0 \mathcal{E}$

$$\left(-\partial_\rho^2 + \frac{\ell(\ell+1)}{\rho^2} - \frac{1}{\rho} + \mathcal{E}\right) u = 0$$

③ Asymptotic Analysis

$$\left.\begin{array}{l} \cdot \rho \to \infty \to u \sim e^{-\sqrt{\mathcal{E}}\rho} \\ \cdot \rho \to 0 \to u \sim \rho^{(\ell+1)} \end{array}\right\} \Rightarrow u = \rho^{(\ell+1)} e^{-\sqrt{\mathcal{E}}\rho} v(\rho) = e^{(-r/2r_0 n)} \left(\frac{r}{r_0}\right)^{\ell+1} v_{n\ell}\left(\frac{r}{r_0}\right)$$

$\Rightarrow \rho v'' + 2(1+\ell-\sqrt{\mathcal{E}}\rho)v' + (1-2\sqrt{\mathcal{E}}(\ell+1))v = 0$

$\Rightarrow v = \sum_{j=0}^{\infty} a_j \rho^j \rightarrow a_{j+1} = \frac{2\sqrt{\mathcal{E}}(j+\ell+1)-1}{(j+1)(j+2\ell+2)} a_j$, To terminate;

$a_{j_{max}+1} = 0$ must be

$\Rightarrow \mathcal{E} = \frac{1}{4n^2}$, $n = j_{max} + \ell + 1$.

$\Rightarrow E_{n,\ell,m} = \frac{-E_0}{4n^2}$, independent of n, ℓ, m

Sepctrum of Light of Hydrogen $\sim \frac{E_0}{4n^2} \rightarrow$ Rydberg Relation

When $j \gg$,

$a_{j+1} \approx a_j \frac{2\sqrt{\mathcal{E}}}{(j+1)} \Rightarrow a_j = a_0 \frac{(2\sqrt{\mathcal{E}})^j}{j!}$ = Expention coeffs. of exponential

Then v becomes $= e^{2\sqrt{\mathcal{E}}\, r/r_0} \Rightarrow$ This destroys our asymptotic analysis so at some point it must terminet if

$$2\sqrt{\mathcal{E}}(j_{max}+\ell+1) = 1. \Rightarrow \mathcal{E} = \frac{1}{4(j_{max}+\ell+1)^2}$$

Ψ_{ℓ,m,j_m}, $E_{\ell,m,j_m} = \frac{-E_0}{4\underbrace{(j_m+\ell+1)^2}_{n}}$; $n > \ell$, n is an integer

$d(E_n) = \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{\ell} 1 = \sum_{\ell=0}^{n-1} 2\ell+1 = n^2$, there is a hugh degeneracy.

A. Said Türgel

$\Rightarrow$ Sym $\Leftrightarrow$ Conserved Quantity $[\hat{E},\hat{\vec{A}}]=0$.

This quantity $\vec{A}=\vec{p}\times\vec{L}-m_e e^2\hat{r}$ $\Rightarrow$ This showed that there is some conserved quantity but about sym?

There are two explanations.

- $\delta_G x = f(\vec{x},\vec{p})$
- If you work with Coulomb problem in 3D = freeparticle in 4D, constrained to S^3.

There is a $SO(4)$ = special orthogonal transformation in 4D.

* Ang. Momentum in 3D = $SO(3)=3 = L_x, L_y, L_z$
" " " 4D = $SO(4)=6$ $\Rightarrow$ if you map this in 3D coulomb problem, becomes.

$\Rightarrow SO(4)=L_x, L_y, L_z, 3\hat{A}$'s.

There is an enhanced symmetry group and there is a conserved quantity which related to a operator $\hat{A}$ which commutes with $\hat{E}$.

$i\hbar\frac{d}{dt}\langle\hat{B}\rangle_{\psi(t)}=\langle[\hat{E},\hat{B}]\rangle$, $[\hat{E},\hat{B}]=0\Rightarrow\frac{d}{dt}\langle\hat{B}\rangle=0$

Classicaly to being a conserved quantity means its time independent.
Quantum mechanicaly it means that commutes with energy $\hat{E}$.
$L_x, L_y, L_z \Rightarrow [\hat{E},\hat{L}_{x,y,z}]=0$, so their time der. of their expectation values are both zero.

Let's try to break/split degeneracy $\Leftrightarrow$ Break Symmetry.
change $\hat{E}$, preserve $[\hat{E},\hat{L}]=0$.

Ex. 1: Relativistic corrections to kinetic energy

$$KE=\sqrt{m_e^2c^4+p^2c^2}-m_ec^2=m_ec^2\left(\sqrt{1+\frac{p^2}{m_e^2c^2}}-1\right)$$ $\Rightarrow$ if we taylor expand this.

$$\Rightarrow \frac{\vec{p}^2}{2m_e}-\frac{p^4}{8m_e^3c^2}$$ $\approx$ it is an approximation, $\Rightarrow E_{\tilde{H}_y}=E_c-\frac{p^4}{8m_e^3c^2}$, $E_{nlm}=\frac{-E_0}{n^2}$

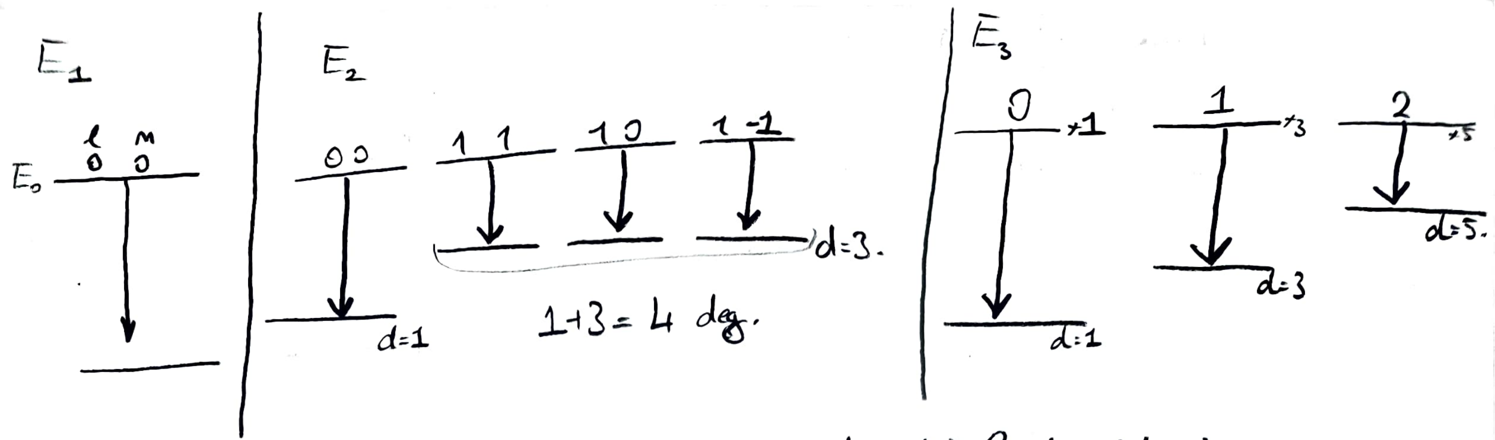
We have to solve the eigenvalue eq. again, also energy eigenvalue has a 4th derivative.
So let's see what we can do.

- Dimensional analysis, try to break degeneracy.
- Use the expectation value of p^4

$$\Rightarrow E_{nlm}^{\tilde{H}_y}=\frac{-E_0}{n^2}-\frac{E_0^2}{n^4m_ec^2}\left(\frac{4n}{l+\frac{1}{2}}-3\right)$$ $\rightarrow$ E depends on l and n, Yay.

$\Rightarrow [E_{\tilde{H}_y},\hat{A}]\neq 0$, $[\hat{E},\hat{L}]=0$ still.

Let's see degeneracy explicitly.



What must be do to break deg. caused by ang. momentum $\iff$ Break rotational sym.

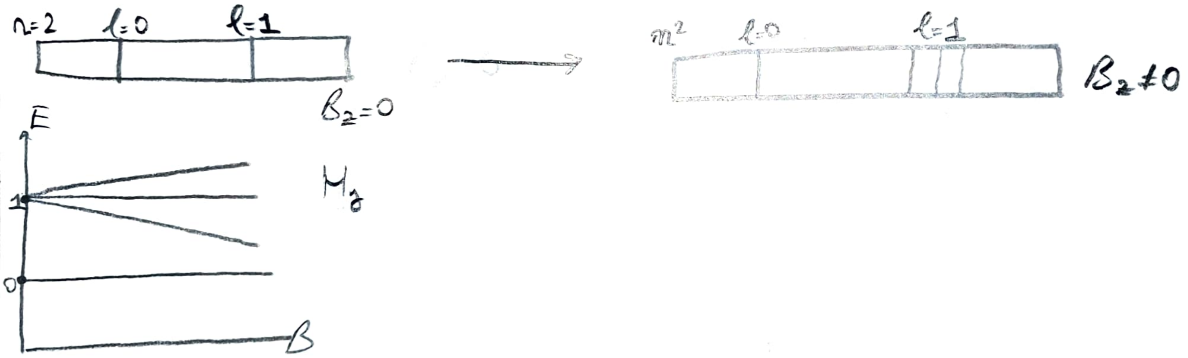
Zeeman: using magnetic field, tried to get a spectrum.

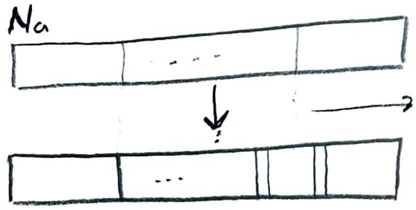
$$E = E_c - \vec{\mu}\cdot\vec{B} \qquad \vec{\mu} = \frac{IA}{c} = \frac{-e}{2m_e c}\vec{L}$$

$\Rightarrow E_c - \mu_z B_z = E_c + \frac{e}{2m_e c} L_z \Rightarrow$ Since L_z is dep. of m

Our E is dep. of m

$$E^2_{nlm} = \frac{-E_0}{n^2} + \frac{e\hbar}{2m_e c} B_z m \Rightarrow E^2_{nlm} = E^{H_y}_{nlm} + \frac{e\hbar}{2m_e c} B_z m$$





$\rightarrow l = \frac{3}{2}$, what the?

Zeeman found that.

Also referred as {Anomalous Zeeman} effect.

We solved that the particular sol'ns was Y_{lm}'s, but it must be zero for half-integers.

Coulomb: $d(E) = n^2$ | But they observed that $d(n) = 2n^2$

If you have 4e you can put them in different or same states right.

But there are twice as many groupings.

Pauli said: 1 No two e^- can live in the same quantum state

2 There are twice as many states as you think in H.

{I posit the existence of an additional quantum number which the e can have which takes one of two values}

A: Said Tangel .

Goudsmit and Uhlenbeck tried to explain this.

1-) There is Zeeman effect.

2-) There is also some wrong total angular momentum, there are twice as many states for one in coulomb potential.

They supposed that the total angular momentum was not right

They " " e can have some intrinsic angular momentum

$e: J = \vec{L} + \vec{S}$, which has the property $S = \frac{1}{2}$, $m_s = \pm 1/2$.

If you have some integer rot. ang. mom. and S, what is the total ang-mom.?

Some half-int number, this can explain anomalous Zeeman effect.

$\hat{E}^{\dagger} = \hat{E}$, $i\hbar \partial_t \Psi = \hat{E} \Psi \Rightarrow \Psi(t) = \underbrace{e^{-\frac{i}{\hbar}\hat{E}t}}_{\text{unitary}} \Psi(0)$ $\boxed{Q}$

if $\hat{E}$ is not Hermitian, energy eigenvalues are no longer real and probability would no longer be conserved.

{Assume that $\hat{E}$ is t independent}.

Suppose we have a stationary state with $\hat{E} \in \mathbb{C}$, $\hat{E} = E_R + i\Gamma$ $\boxed{Q}$

$$\phi_{(t)} = e^{-\frac{i}{\hbar}\hat{E}t}\phi_2(0) = e^{-i\frac{E_R t}{\hbar}} e^{-i\Gamma}\phi(0)$$

When we take norm square of it shows that,

Having loss of of probability is having a complex eigenvalue

$$\Psi_{n\ell m} = N \frac{1}{r} \underbrace{R_{n\ell}(r)}_{u(r)} Y_{\ell m}(\theta, \alpha) = N \frac{1}{r} R_{n\ell}(r) P_\ell(\cos\theta) e^{im\alpha}, \Rightarrow \text{Stationary State.}$$

$$\frac{d}{dt}\langle \vec{r} \rangle = 0 \quad \rho(\vec{r}) = |\Psi_{n\ell m}(\vec{r})|^2$$

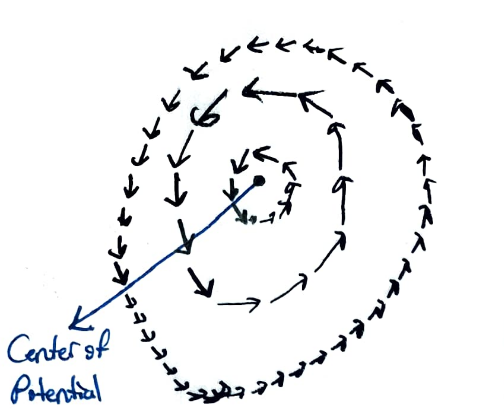
$$\partial_t \rho(\vec{r}) = -\vec{\nabla}\cdot\vec{J}, \quad \vec{J} = \frac{\hbar}{m_e} \mathrm{Im}(\Psi^* \vec{\nabla} \Psi)$$

In St. State $\Psi_{n\ell m}$, $\partial_t \rho(\vec{r}) = 0 \Rightarrow \vec{\nabla}\cdot J = 0$

What's current? $\left(\nabla_r = \partial_r, \ \nabla_\theta = \frac{1}{r}\partial_\theta, \ \nabla_\alpha = \frac{1}{r\sin\theta}\partial_\alpha\right)$

$$J_r = 0, \ J_\theta = 0, \ J_\alpha = \frac{\hbar}{m_e} \mathrm{Im}\left(\Psi^* \frac{1}{r\sin\theta} im \Psi\right) = \frac{1}{m_e} |\Psi|^2 \frac{1}{r\sin\theta} m\hbar$$

$$\Rightarrow \vec{J} = \frac{|\Psi|^2}{m_e r \sin\theta} \hbar m \, \hat{e}_\alpha \cdot \Rightarrow \text{Probability Density Rotating.}$$



$\vec{I} = -e\vec{J} \Rightarrow \vec{\mu} = \mu_B m \hat{z}$ where $\vec{\mu}_B = \frac{e\hbar}{2m_e}$.

What happens when the state has a real eigenfunction?

$(\Psi_{nlm} + \Psi_{nl-m}) = \Psi_{nl}$, for a free particle.

Current will be $\equiv$ zero.

Its some arbitrary superposition of having opposite ang. mom.

So expectation value of mom. is zero

If we're interested in to find states with def. E and def. momentum, it is not going to be a real function on some dir.-

"End of Coulomb Potential"

Multiple Particles:

$$CM: (x_1, p_1, x_2, p_2) \;, \; QM: \Psi(\underset{\text{1st p.}}{x_1}, \underset{\text{2nd p.}}{x_2}) \;, \; \hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2$$

$$\mathbb{P}(x_1, x_2) = |\Psi(x_1, x_2)|^2 \;; \; [\hat{x}_1, \hat{x}_2] = 0, [\hat{x}_1, \hat{p}_1] = i\hbar, [\hat{x}_1, \hat{p}_2] = 0.$$

Ex: Two free particles, $V(a,b) = 0$

$$\hat{E} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \;, \; \Psi(a,b) = K(a)\phi(b)$$

$$\frac{p_a^2}{2m}K(a) = \epsilon_a K(a) \Rightarrow K(a) = ce^{ik_a a} \;, \; \frac{p_b^2}{2m_2}\phi(b) = e^{ik_b b}.$$

$$\Psi_E(a,b) = e^{i(k_a a + k_b b)} \;, \; E = \frac{\hbar^2}{2m}(k_a^2 + k_b^2)$$

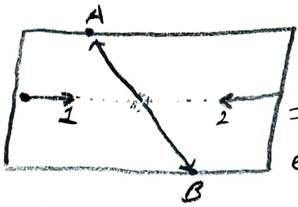
Ex. $\hat{E} = \frac{p_a^2}{2m_1} + \frac{p_b^2}{2m_2} + V(|a-b|)$; a = proton, b = electron

$$\underbrace{R}_{\text{Center of mass position.}} = \left(\frac{1}{m_a + m_b}\right)(m_a a + m_b b) \;, \; r = a - b$$

$$\hat{E} = -\frac{\hbar^2}{2m_t}\nabla_R^2 - \frac{\hbar^2}{2\mu}\nabla_r^2 + V(r) \;, \; \mu = \frac{m_a m_b}{m_a + m_b} \simeq m_b$$

A. Said Tongel

Identical Particles:



which particle ended up where?

$\Rightarrow \mathbb{P}(A,B) = \mathbb{P}(B,A)$, because particles are identical. We can't know which ended up where.

$\hat{P}_{12} : 1 \leftrightarrow 2 : \mathbb{P}(A,B) = \hat{\mathbb{P}}(B,A)$, $\hat{P}\Psi(A,B) = \Psi(B,A) \Rightarrow$ it does not have to be invariant.

$\Psi(B,A) = e^{i\theta_{AB}}\Psi(A,B)$

$\hat{P}\hat{P}\Psi(A,B) = e^{2i\theta_{AB}}\Psi(A,B) \Rightarrow (e^{i\theta_{AB}})^2 = 1 \Rightarrow e^{i\theta_{AB}} = \pm 1$

$\hat{P}(A,B) = \pm\Psi(A,B) \Rightarrow \hat{P}^2\Psi = \Psi \Rightarrow$ eigenvalues of $\hat{P}$ are ± 1

If are our particles our distinguishable, then

$\Psi_D(A,B) = \chi(A)\phi(B)$, $\mathbb{P}_D(A,B) \neq \mathbb{P}_D(B,A)$, $|\Psi_D(A,B)|^2 = |\chi(A)\phi(B)|^2$

If we have indistinguishabe particles;

$|\Psi(A,B)|^2 = |\Psi(B,A)|^2 \Rightarrow \Psi_{\pm} = \frac{1}{\sqrt{2}}(\chi(A)\phi(B) \pm \chi(B)\phi(A))$

$\Rightarrow \hat{P}\Psi_{\pm}(A,B) = \pm\Psi(A,B)$ → + means sym. → − means anti-sym. {for combinations of states}

If you have identical particles, then

$\hat{E}\hat{P} = \hat{P}\hat{E} \Rightarrow [\hat{E},\hat{P}] = 0 \Rightarrow \langle\hat{P}\rangle$ doesn't change in time

Initially

$P\Psi_A = +\Psi(0) \Rightarrow P\Psi(t) = +\Psi(t)$

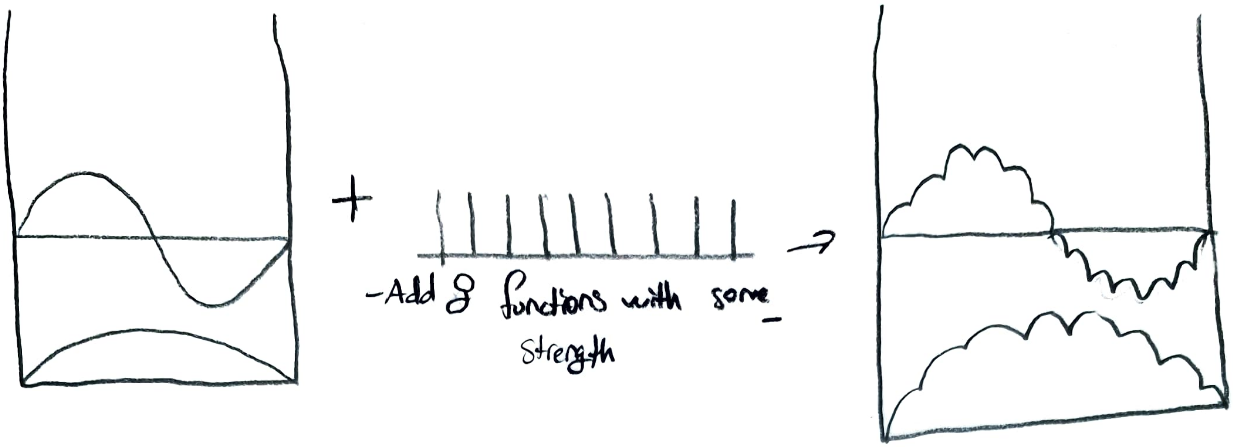
1. Distinguisable Particles
2. Identical, $P\psi = +\psi \Rightarrow$ Called Bosons
3. ", $P\psi = -\psi \Rightarrow$ Called Fermions

$\Psi_F(A,B) = -\Psi(B,A) \Rightarrow \Psi(A,A) = -\Psi(A,A) = 0$ } Pauli Principle.

$\Psi_{F/B} = \frac{1}{\sqrt{2}}(\chi(A)\phi(B) \mp \chi(B)\phi(A))$

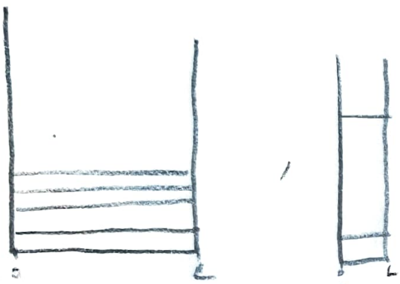
$\Psi_B(A,A) = \sqrt{2}\,\chi(A)\phi(A)$

$\Psi_{\pm}(A,B) = \pm\Psi(B,A)$ → Bosons → Fermions



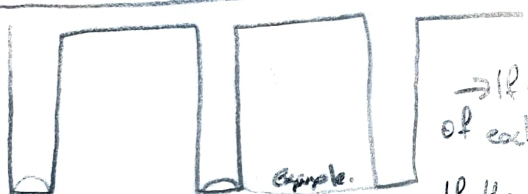
$\Rightarrow$  - Eventually will look like this

$\boxed{Q}$



$E_n \simeq \frac{n^2}{L^2}$ $\quad$ $E_n \simeq \frac{n^2}{L^2}$

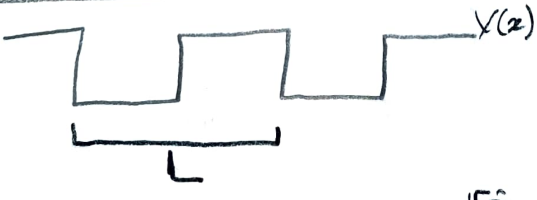
when we have a narrow box the energy differences between states are gigantic compared to a wider box.



$\Rightarrow$ If the height of the well is infinitely high then the states of each well is some linear combinations of  . If the height of the barrier is large but finite, from the energy convention we can determine its' specific energy.

$\boxed{Q}$

Periodic Potentials:



$V(x+L) = V(x)$

$\hat{T}_L V(x) = V(x)$

$\hat{T}_L(\hat{V}(x) f(x)) = \hat{V}(x)\, T_L f(x)$

$V(x+L) f(x+L) = \hat{V}(x) f(x+L)$

$[\hat{T}_L, \hat{V}(x)] = 0$

$\hat{T}_L = e^{i\hat{p}L/\hbar} \Rightarrow [\hat{T}_L, \hat{p}^2] = 0$

$\Rightarrow [\hat{E}, \hat{T}_L] = 0, \ [\hat{E}, \hat{p}] \neq 0$

Note: $\hat{T}_L$ is unitary

$\hat{T}_L \hat{T}_L^{\dagger} = \mathbb{1}$

$\hat{T}_L^{\dagger} = e^{\frac{-i\hat{p}L}{\hbar}}.$

A. Saad Töngel

$\hat{E}\phi_{E\alpha} = E\phi_{E\alpha}, \quad \hat{T}_L \phi_{E\alpha} = e^{i\alpha}\phi_{E\alpha}$

Let $u(x) = e^{-iqx}\phi_\alpha$

$\hat{T}_L u(x) = e^{iq(x+L)}\phi_\alpha(x+L)$

$= e^{iqL} e^{i\alpha} e^{iqx}\phi_\alpha(x)$

$= e^{i(\alpha - qL)} u(x) \Rightarrow q = \frac{\alpha}{L} \Rightarrow u(x) \to u(x)$

"periodic"

$\Rightarrow \phi_{E,q} = e^{iqx} u(x)$; $e^{i\alpha} = e^{iqL}$, $u(x)$ is periodic.

$q \simeq q + \frac{2\pi}{L}$, $e^{i\alpha} \to e^{i\alpha}$

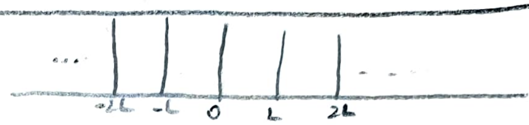
(1-) $\phi_{E,q}(x)$ is not periodic by L unless $q=0$.

(2-) $\mathbb{P}(x+L) = \mathbb{P}(x) = |u(x)|^2$ is periodic

To normalize probability we need wave packets. $\Rightarrow$ all $\phi_{E,q}$ are extended, not localized.

(3-) $\phi_{E,q}(x)$ is not a momentum eigenstate $\Leftrightarrow q\hbar$ is not "the momentum"; crystal momentum.

Example: $V(x) = \sum_{-\infty}^{\infty} \frac{\hbar^2}{2mL} \underbrace{g_0}_{\text{dimensionless strength}} \delta(x - nL)$



between $0 < x < L$, $\phi_{E,q} = A e^{ikx} + B e^{-ikx}$ $\qquad \frac{\hbar^2 k^2}{2m} = E$

" $\quad L < x < 2L$, $\phi_{E,q} = e^{iqL}(A e^{ikx} + B e^{-ikx})$

B.C at δ's.

- $\phi(0^+) = \phi(0^-) \equiv \phi(L^-) e^{-iqL}$

$\Rightarrow A + B = A e^{i(k-q)L} + B e^{-i(k+q)L}$ (1)

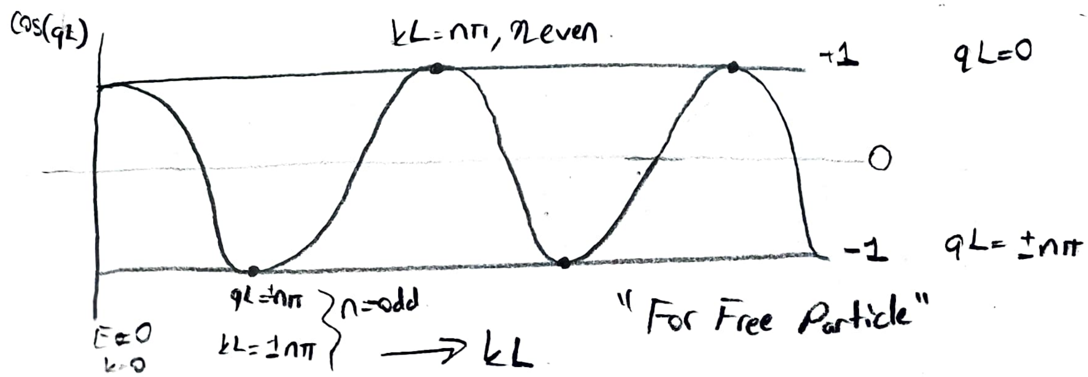
- $\phi'(0^+) - \phi'(0^-) = \frac{g_0}{L} \phi(0)$

$\Rightarrow ik(A-B) - ik(A e^{i(k-q)L} - B e^{-i(k+q)L}) = \frac{g_0}{L}(A+B)$ (2)

$\Rightarrow A = B.(kq)_1$, $A = B(kq)_2$. if you set this two equations equal to each other;

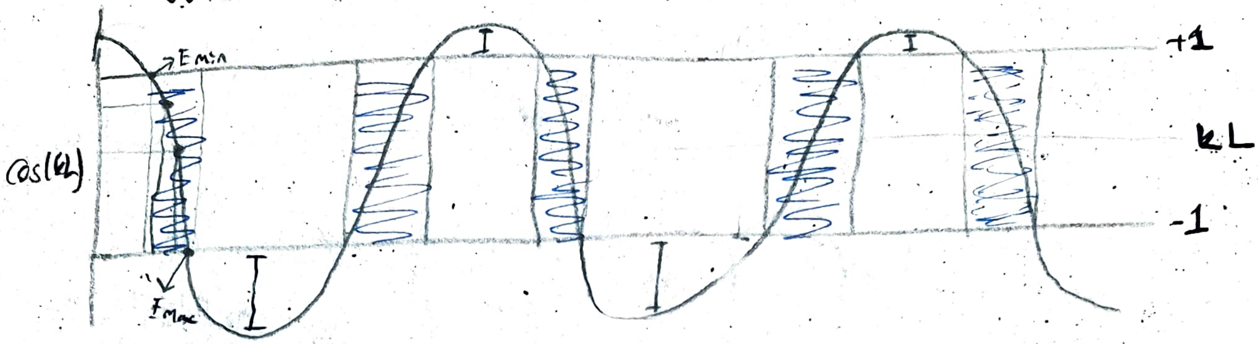
$\Rightarrow \cos(qL) = \cos(kL) + \frac{g_0}{2kL} \sin(kL)$, k determines E.

$\Rightarrow$ There exists states for any (q, E) satisfies the given last equation.



$g_0 = 0 \Rightarrow \cos(qL) = \cos(kL)$
$\Rightarrow k = q \Rightarrow E = \frac{\hbar^2 q^2}{2L}$

$g_0 \neq 0$.

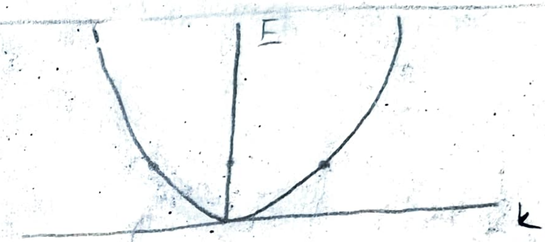


$\exists$ E, T_L energy eigenstate for any (E, q) such that there exists a solution.

$\forall$ E in the colored region corresponds to a state.

There are continuous band of allowed and continuous bands of disallowed energies.

Free Particle: $\hat{E} = \frac{\hat{p}^2}{2m} \Rightarrow [\hat{E}, \hat{p}] = 0$



$\exists$ basis $\hat{E}\phi_{Ek} = E\phi_{Ek}$, $\hat{p}\phi_{Ek} = \hbar k\, \phi_{Ek}$

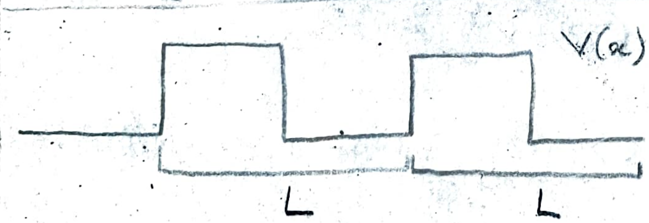
$\Rightarrow$ Solve Eq. $\Rightarrow \phi_{Ek} = \frac{1}{\sqrt{2\pi}} e^{ikx}$, $E_k = \frac{\hbar^2 k^2}{2m} = \omega_k \hbar$

Note: all $\phi_E = e^{i(kx - \omega_k t)}$ are extended

$\Rightarrow$ Build wavepackets $\psi = \int f(k) e^{i(kx - \omega_k t)} dk$ → peaked at k_0.

$v_g = \frac{\partial \omega}{\partial k}(k_0) \stackrel{fp}{=} \frac{\hbar k}{m} = \frac{\langle p \rangle}{m}$

$m_* = \frac{\langle p \rangle}{v_g}$



$\hat{E}_{g_0} = \frac{\hat{p}^2}{2m} + g_0 V(x)$

$\Rightarrow [\hat{E}, \hat{p}] \neq 0$ but $[\hat{E}, \hat{T}_L] = 0 \Rightarrow \exists$ common eigenbasis of $\hat{E}, \hat{T}_L$.

$\phi_q(x) = e^{iqx} u(x)$, $\hat{T}_L \phi_q(x) = e^{iqL} \phi_q(x)$ → periodic function, $u(x+L) = u(x)$

$\{ T_L f(x) = f(x+L) \}$

$\Rightarrow \hat{E}$ eigenfunctions can be organized as

$$\hat{E}\phi_{Eq} = E\phi_{E,q}, \quad \hat{T}_L \phi_{Eq} = e^{iqL} \phi_{Eq}$$

A. Said Töngel

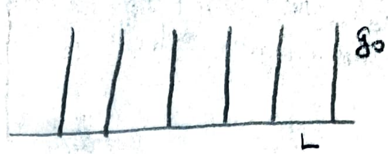
=) Solve Equation =) Relationships between E, q. For $g_0 = 0$ f.p, we find

$$E_q = \frac{\hbar^2 q^2}{2m}, \quad \phi_{qE} = \frac{1}{\sqrt{2\pi}} e^{iqx} \cdot c. \qquad \boxed{\Psi = \int e^{i(qx - w_q t)} u(x) f(q)\, dq.}$$

$$v_g = \frac{\partial w}{\partial q}(q_0), \quad m_* = \frac{\langle p \rangle}{v_g}$$

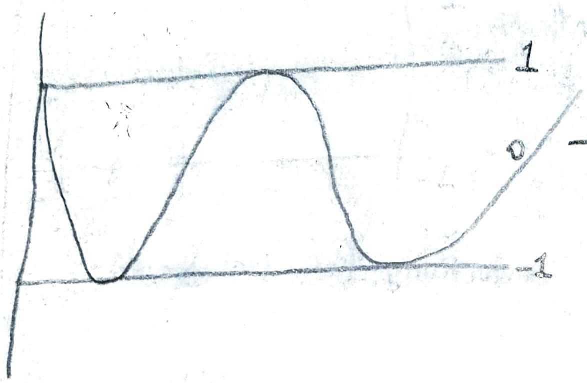
When $g_0 \neq 0$.

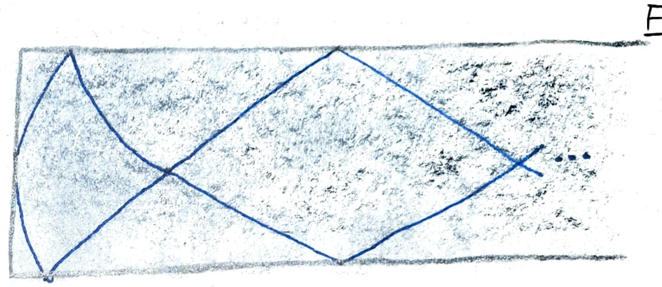
$$\phi_{Eq} = e^{iq(x)} u(x) \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$



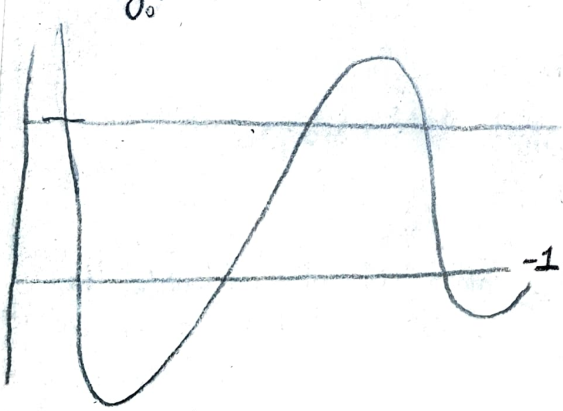
$$\cos(qL) = \cos(kL) + \frac{g_0}{2kL} \sin(kL)$$

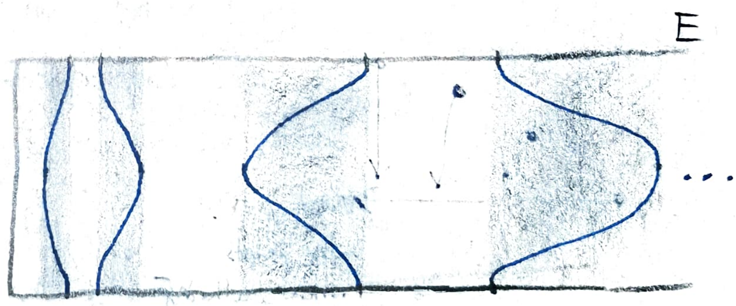
When $g_0 = 0$,





when $g_0 \neq 0$,





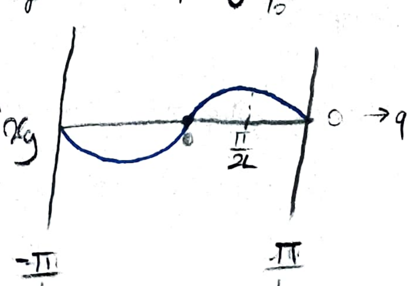
* Lessons from $g_0 \neq 0$.

1.) E eigenvalues are restricted to lie within bands.

2.) E eigenstates are all extended.

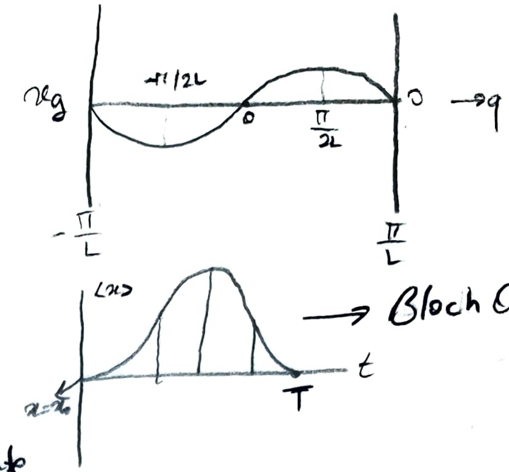
=) build wave packets $\Psi = \int_{q_0} f(q)\, e^{i(qx - w_k t)} u_q(x)\, dq$.

3.) $v_g = \frac{\partial w}{\partial q}(q_0)$, v_g



4-) $\langle F \rangle = \frac{d}{dt} \langle \hbar q \rangle$

For a wavepacket localized around q_0.



→ Bloch Oscillation.

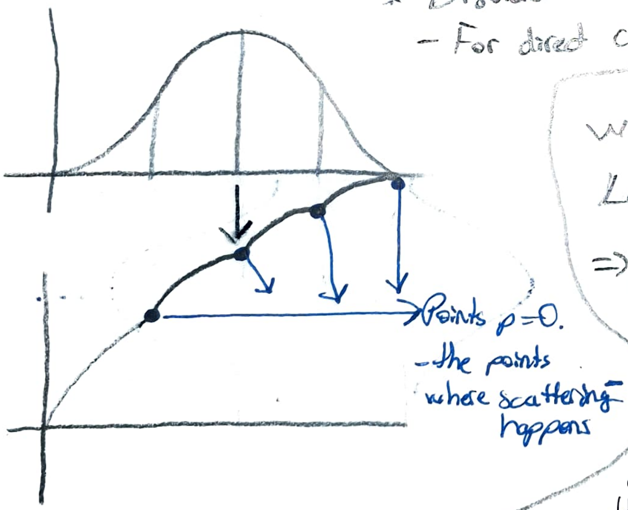
* How does the direction of force and acceleration can be opposite?

It accelerates back and forth even the dir. of force is same - in a periodic lattice, like copper..

But copper condutes. e^- don't oscillate back and forth. Let us explain

- The mass of ions are not infinite so they wiggle, as a consequence they're not perfectly periodic. That gives us bunch of thing.
  - The eigenfunctions we found are not exact eigenfunctions since we need to deal with wiggling of lattice
  - e^- can move along and kick one of these ions and scatter some of its momentum into ions. This changes the structure of lattice. We get a chain of momentum scattering and they move up.

* Disorder is essential in conductivity for real solids.
  - For direct currents..



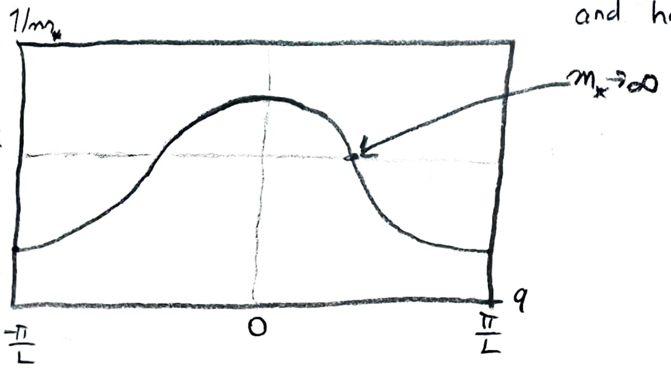
What about the force and dir of acc.?

Let's calculate the mass

$$\Rightarrow \frac{1}{m_*} = \frac{1}{\hbar^2} \frac{\partial^2 E}{\partial q^2}$$

* The effective mass is used to decribe the motion particles in a medium, Instead of using actual mass we describe an adjusted mass that accounts for the interactions in the medium.

$m_* \to \infty$ means the particle is localized at that point and has no motion.



A- Said Tongel

$\Psi(x)$ Such that $\langle\hat{x}\rangle_\Psi = x_0$, $\langle\hat{p}\rangle_\Psi = p_0$ $\boxed{Q}$

$\tilde{\Psi}(x) = e^{ikx}\Psi(x) \Rightarrow \langle\hat{x}\rangle_{\tilde{\Psi}} = x_0$, $\langle\hat{p}\rangle_{\tilde{\Psi}} = p_0 + \hbar k$.

$[\hat{x},\hat{p}] = i\hbar$, $T_L = e^{-\frac{i}{\hbar}pL}$, α governs the spatial variation of phase of Ψ.

$$\frac{d}{dt}\langle\hat{p}\rangle = \langle\frac{\partial V(x)}{\partial x}\rangle = \langle F\rangle$$

Crystal Momentum: $V(x+L) = V(x) \Rightarrow [\hat{E},\hat{T}_L] = 0$

$\Rightarrow \phi_{E,q} = e^{iqx}u(x)$, q governs the phase as a function of x.

$\Rightarrow \frac{d}{dt}\langle\hbar q\rangle = \langle F\rangle \rightarrow$ sharply peaked at q_0, Ψ. $\left\{\begin{array}{l} p\phi_{E,q} \neq p_0\phi_{E,q} \\ \hbar q \text{ is not eigenvalue of } \hat{p}. \end{array}\right.$

eig. value of T_L on $u(x) = e^{iqL}$.

- Momentum is strictly conserved if there is no force and if there is a force that it is just constanly increase
- For crystal momentum it's not increasing constantly it's periodic, it increases then decrease to its intial case under some period even if you don't change the force.

How can we know that $u(x)$ is Real? $\boxed{Q}$

① $T_L\phi_{E,q} = e^{i\alpha}\phi_{Eq} \Rightarrow \phi_{E,q}\cdot e^{-iqx} = u \equiv u_q(x)$, $q_L = \alpha \Rightarrow T_L u_q(x) = u_q(x) = u_q(x+L)$

$\Rightarrow \phi_{E,q} = e^{iqx}u_q(x) \rightarrow$ periodic.

② $q \sim q + \frac{2\pi}{L}$, $\phi_{q,E} = e^{iqx}u(x) \equiv e^{ikx} \rightarrow$ constant and $q = k$ however q is periodic by $\frac{2\pi}{L}$ but any k is allowed for a free particle

$\Rightarrow e^{ikx} = e^{iqx}\cdot\underbrace{e^{i(k-q)x}}_{u(x)}$ $u(x)$ is not Real for free particle.

If we want $u(x)$ to be Real, q can not be periodic.

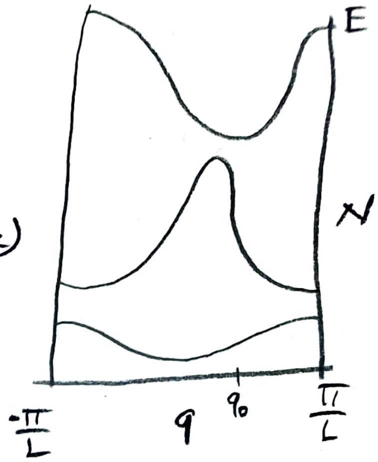
$q = q_0 + \frac{N\pi}{L}$

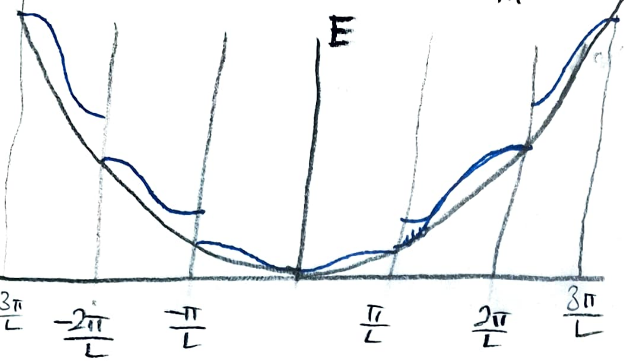
$\phi_{q_0 N} = e^{iq_0 x}e^{i\frac{N\pi}{L}x}u(x)$

Suppose u Real, $q \in \mathbb{R}$ (not periodic)

$$\phi_q = e^{iqx} u(x)$$

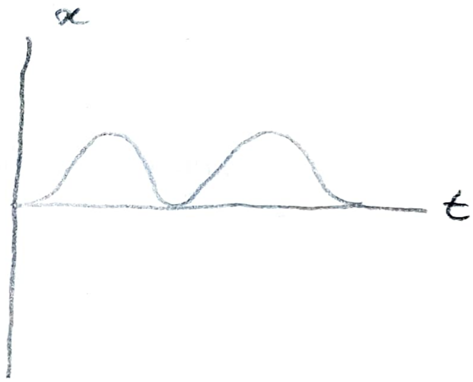
$$J_x = \frac{\hbar}{2m} \mathrm{Im}(\phi^* \partial_x \phi) = \frac{\hbar q}{m} u^2(x) = \left(\frac{\hbar q_0}{m} + \frac{n\pi\hbar}{mL}\right) u^2(x)$$



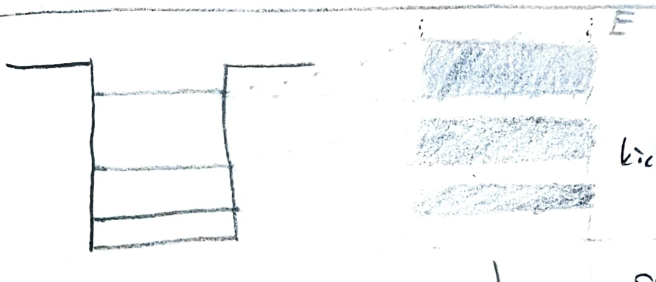


- Some values of E allowed, E gives inf. about k.
- q gives inf. about E.

For our periodic lattice, our particle oscillates back and forth, our materials—copper etc.— are not perfect. We need a perfect crystal to build this system and test it.

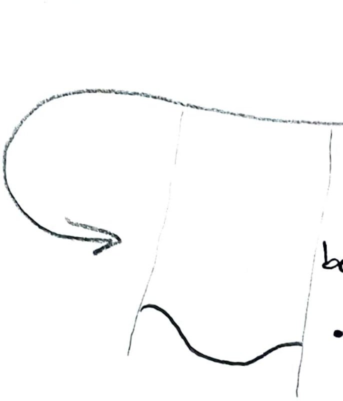


Now try to understand our -real- materials.



If you put e's into a system it will eventually kick back to ground state

SS: Stationary state.



$\langle x \rangle_{SS} = 0$, superposition states will correspond to different $\langle x \rangle$.
In order to induce a current we must put e's into superposition of being into higher energy states.

- Think this square-wells as atoms -hydrogen-.

In order to system be neutral, there must e^- for every well.
There are n states for n wells and to neutralize our systems we must put e's into those states which are in the first band.
If we want to induce a current, the next allowed state is in "next band".

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In order to take an e^- from the first band and put in 2nd band we have to put in a minimum energy ΔE.

So $\hbar\omega \geq \Delta E$.

$]\Delta E$ • Crystals are transparent unless at sufficiently high frequencies.

—Diamond and copper, they are both crystals.

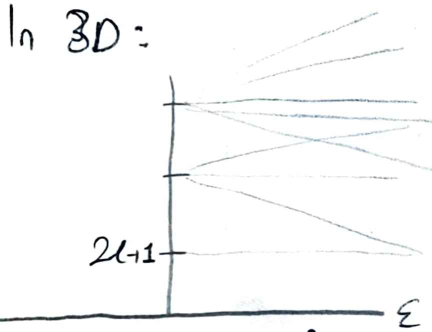
Diamond is tranparent for visible light - since copper is not transpardnt for visible light -

$\Delta E_D \gg \Delta E_C$.

In 1-D crystal if you have each band come from an allowed energy state, each well comes with integer number of e^-, you will always have filled bands and then a gap, you're stuck.

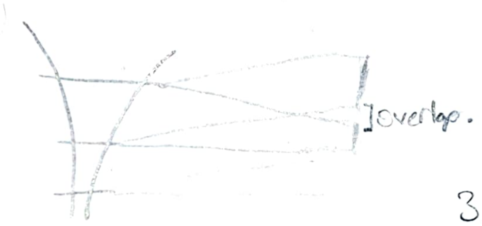
That property comes from "spin" which we'll talk later.

In 3D:



$$E = \frac{L^2}{2I} + \varepsilon L_z$$

In 3D, there are crossed states

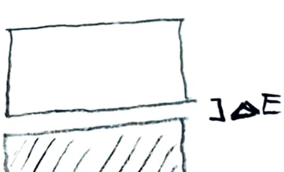


Let's say 1st band is filled

Now most of 2nd band and part of 3rd band avaliable, then much of 3rd and bit of 2nd band.

The E needed to excite will be so small, acting like a "classical conductor," because there is an unfilled band.

There were filled bonds which made them "Insulators".



P(of electron to be excited through empty gap)

$\sim e^{-\frac{\Delta E}{k_B T}}$

if this gap ΔE is small it makes us system a "semiconductor"

Spin:

$$[S_x, S_y] = i\hbar S_z \,, \quad [S_y, S_z] = i\hbar S_x$$

$$s = 1/2 \quad 3/2 \quad 5/2 \quad \ldots$$

$$S_\pm = S_x \pm i S_y$$

$$S^2 = \hbar^2 s(s+1) \quad , \; s \in \frac{2n+1}{2}$$

$$S_z = \hbar m_s \quad , \; -s \leq m_s \leq s$$

$S = 1/2$

$$\Psi(x) = \Psi_{+\frac{1}{2}}(x) + \Psi_{-\frac{1}{2}}(x) \quad \Big| \quad \mathbb{P}(x, \pm\hbar/2) = |\Psi_{\pm}(x)|^2$$

$$\Psi(x) = \begin{pmatrix} \Psi_\uparrow(x) \\ \Psi_\downarrow(x) \end{pmatrix} \qquad \Psi^\dagger = (\Psi_\uparrow^*, \; \Psi_\downarrow^*)$$

Norm: $$(\Psi | \Psi) = 1 = \int (|\Psi_\uparrow|^2 + |\Psi_\downarrow|^2)\, dx$$

$$\Psi(x) = \Psi_\uparrow(x)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \Psi_\downarrow(x)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \,, \quad |\Psi(x)\rangle = |\Psi_\uparrow(x)|\uparrow\rangle + |\Psi_\downarrow(x)|\downarrow\rangle$$

$$S_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle \,, \quad S_z|\downarrow\rangle = \frac{-\hbar}{2}|\downarrow\rangle, \quad S^2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3\hbar^2}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \ldots$$

$$S_z\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad S_z\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad S^2|\uparrow\rangle = \frac{3\hbar^2}{4}|\uparrow\rangle$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad S^2 = \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_+\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar\begin{pmatrix} 1 \\ 0 \end{pmatrix} \,, \quad S_+\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (0) \,, \quad S_-\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \,, \quad S_-\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} & S_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \,, \quad S_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ & S_x = \frac{S_+ + S_-}{2} \,, \quad S_y = \frac{S_+ - S_-}{2i} \end{aligned}\right\} \Rightarrow \begin{aligned} & S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ & S_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{aligned}$$

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$\underbrace{S_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{\sigma^x}$, $\underbrace{S_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}}_{\sigma^y}$, $\underbrace{S_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{\sigma^z}$

$$[S_x, S_y] \overset{?}{=} i\hbar S_z = \left(\frac{\hbar}{2}\right)^2 \left[\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}\right] = \frac{\hbar^2}{4}\left(\begin{pmatrix}i&0\\0&-i\end{pmatrix} - \begin{pmatrix}-i&0\\0&i\end{pmatrix}\right)$$

$$= \frac{\hbar^2}{4}\begin{pmatrix}2i&0\\0&-2i\end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix}i&0\\0&-i\end{pmatrix} = (i\hbar)\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = i\hbar S_z$$

S_x : Eigen values ?

$$S_x\begin{pmatrix}1\\1\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\1\end{pmatrix} = \uparrow_z + \downarrow_z$$

$$S_x\begin{pmatrix}1\\-1\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}1\\-1\end{pmatrix} = \uparrow$$

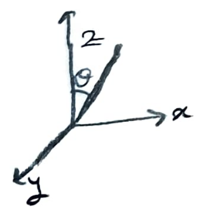
$$\uparrow_x = \frac{1}{\sqrt{2}}(\uparrow_z + \downarrow_z)$$

$$\downarrow_x = \frac{1}{\sqrt{2}}(\uparrow_z - \downarrow_z)$$

$\mathbb{P}(\uparrow_z, \downarrow_x) = \mathbb{P}(\uparrow_z)$ given that we measured $\downarrow_x$.

$= \left|\frac{1}{\sqrt{2}}\right|^2 = 1/2$

S_y : $\uparrow_y = \frac{1}{\sqrt{2}}(\uparrow_z + i\downarrow_z)$, $\downarrow_y = \frac{1}{\sqrt{2}}(\uparrow_z - i\downarrow_z)$

$\hat{S}_\theta \equiv$ 

$$\Rightarrow \begin{aligned}\uparrow_\theta &= \cos\left(\frac{\theta}{2}\right)\uparrow_z + \sin\left(\frac{\theta}{2}\right)\downarrow_z \\ \downarrow_\theta &= \sin\left(\frac{\theta}{2}\right)\uparrow_z - \cos\left(\frac{\theta}{2}\right)\downarrow_z.\end{aligned}$$

$\hat{z}$

$B_z \simeq B_0 + \beta z$

$$\Psi_e = a\uparrow_z + b\downarrow_z$$

$\vec{z}$

$$E = -\mu_0 \cdot \vec{S} \cdot \vec{B}_z = \begin{cases} 0 & L \\ -\mu_0 \cdot S_z(B_0 + \beta z) & \text{In Region} \\ 0 & R \end{cases}$$

$$-\frac{\hbar}{2}\mu_0 B_z\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}C(z)&0\\0&-C(z)\end{pmatrix}$$

$E\uparrow_z = C(z)\uparrow_z$, $E\downarrow_z = -C(z)\downarrow_z$

$E_{\uparrow_z} = +C(z)$, $E_{\downarrow_z} = -C(z)$

$$\Rightarrow \Psi(t) = a e^{+i\frac{E_\uparrow t}{\hbar}}\uparrow_z + b e^{-i\frac{E_\downarrow t}{\hbar}}\downarrow_z$$

$$= a e^{i\frac{\mu_0 B_0 t}{2}} e^{i\frac{\mu_0 \beta t z}{2}}\uparrow_z + b e^{-i\frac{\mu_0 B_0 t}{2}} e^{-i\frac{\mu_0 \beta t z}{2}}\downarrow_z$$

$$\downarrow_\theta = \cos\left(\frac{\theta}{2}\right)\downarrow_0 + i\sin\left(\frac{\theta}{2}\right)\uparrow_0$$

Entanglement:

$\Psi(0) = \frac{1}{\sqrt{2}}(\uparrow_z + \downarrow_z) = \uparrow_x$, $E = -\mu_b B_z S_z \Rightarrow E_1 = 1\hbar\omega$, $\omega = \frac{\mu_b B_z}{2}$

$\Rightarrow \Psi(t) = \frac{1}{\sqrt{2}}\left(e^{i\omega t}\uparrow_z + e^{-i\omega t}\downarrow_z\right)$

$\omega t = \frac{\pi}{2} \Rightarrow \Psi_{after} = \frac{i}{\sqrt{2}}(\uparrow_z - \downarrow_z) = i\downarrow_x$

we have

1-) The system can be put in $\uparrow_z, \downarrow_z$

2-) Ability to evolve states from $\Psi_1 \to \Psi_2$ w/ $\vec{B}$ fields.

This will allows us to do Quantum Computing.

Quantum Computing:

N bits $\{0,1\}$, N binary numbers

N Qbits $\{|0\rangle, |1\rangle\}$, $2^N \in \mathbb{C}$ numbers

$\Psi = \alpha\,|000\ldots0\rangle + \beta\,|10000\ldots\rangle$

$\vdots$

2^N possibility

Systems: N Qbits $\underset{\uparrow}{1}$ $\underset{\downarrow}{0}$

In: $\Psi_{in}^{(N)} \xrightarrow[\text{implement Algorithm}]{\text{Evolve w/ } \hat{E}} \Psi_{out}^{(N)}$

- In/out are superposition of values we measure.
- Measure $\Rightarrow$ Interference
  $\Rightarrow$ Probabilistic outcomes
- Orchestrate the Interference to get definite outcome
- Focus on checkable problems.

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Qbit = a 2-state system $|0\rangle$ $|1\rangle$

$\psi = \alpha|0\rangle + \beta|1\rangle$, $\mathbb{P}(0) = |\alpha|^2$

2-Qbits =

$\psi = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \Gamma|11\rangle$

$\mathbb{P}(A=0) = \alpha^2 + \beta^2$, $\mathbb{P}(A=0, B=1) = |\beta|^2$

Seperable States

$|\psi\rangle = |\psi_1\rangle|\psi_2\rangle$

eg = $(a|0\rangle + b|1\rangle)(c|0\rangle - d|1\rangle) = ac|00\rangle - ad|01\rangle + bc|10\rangle - bd|11\rangle$

Measure $A=0$: $|\psi\rangle = |0\rangle \propto (c|0\rangle - d|1\rangle)$

Generic States are Not Seperable.

$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \Rightarrow$ called $\boxed{\text{Entangled}}$

Suppose $B=1$. $\Rightarrow \mathbb{P}(A=0) = 0$, $\mathbb{P}(A=1) = 1$

How to Compute:

Sch. Evo w/our tuned $\hat{E}$.

Ex 1: Gate NOT: $\underset{\binom{1}{0}}{|0\rangle} \to \underset{\binom{0}{1}}{|1\rangle}$, $\underset{\binom{0}{1}}{|1\rangle} \to \underset{\binom{1}{0}}{|0\rangle}$.

$\mathcal{U}_{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -ie^{\frac{\pi}{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i\left(\cos(\pi/2) + i\sin(\pi/2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$

$\mathcal{U}$ for B_x for $t\omega = \frac{\pi}{2}$

Ex 2: Turn on B_y for a time t.

$$|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$U_H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

CNOT: Apply Not to the second qubit if and only if first qubit is 1.

00 → 00
01 → 01
10 → 11
11 → 10

what can't you do with this operations?

Sch. Evo. ⇒ Linear, unitary

No - Cloning

Suppose that $|x\rangle|y\rangle \to |$constant,

$$U|x\rangle|y\rangle \to |x\rangle|x\rangle, \ \forall |y\rangle$$
$$U|z\rangle|y\rangle \to |z\rangle|z\rangle, \ \forall |y\rangle$$

Because U is unitary it preserves inner product since inner product implies

$$\langle x|z\rangle = (\langle x|z\rangle)^2$$

$$\langle x|z\rangle \times \langle y|y\rangle = \langle x|z\rangle \times \langle x|z\rangle$$

Assuming $\langle y|y\rangle = 1$, this reduce to

$$\langle x|z\rangle = (\langle x|z\rangle)^2$$

This is true if $\underbrace{\langle x|z\rangle = 0}_{\text{orthogonal}}$ or $\underbrace{\langle x|z\rangle = 1}_{\text{Same state}}$

Cloning an arbitrary unknown quantum state is impossible

You also can't lose information since universe preserves info.
{ The information of state y, that we tried to copy into it a state }

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Now Let's see what we can do.

- You can entangle.

$|0\rangle|0\rangle$

$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \times (|0\rangle)$

$= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

By using Hadamard and CNOT operations to a two state system, we entangled the states.

Deutsch-Jozsa Alghoritm:

Suppose that I have a function which I want to evaluate but it cost \$ 10^6 and I want to know whether

$f(0) = f(1) \Rightarrow$ Classicaly requires 2 evaluations which cost \$ 2.10^6.

We'll build a setup which will require less resources.

$f(0) \oplus f(1) = \begin{cases} 0 & \text{if same} \\ 1 & \text{if not} \end{cases}$

Setup: $\mathcal{U}_f : |x\rangle|y\rangle \to |x\rangle|f(x) \oplus y\rangle$, $\oplus$ = add mod two, $|x\rangle \to |f(x)\rangle$

Algorithm In: $\Psi = |0\rangle|1\rangle$.

1-) Aplly Hadamard on both qubits.

$$\Psi \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{2}\Big((|0\rangle(|0\rangle - |1\rangle)) + (|1\rangle)(|0\rangle - |1\rangle)\Big)$$

2-) Apply $\mathcal{U}_f$ operator.

$$\frac{1}{2}\Big[|0\rangle \times (|f(0) \oplus 0\rangle - |f(0) \oplus 1\rangle) + |1\rangle(|f(1) \oplus 0\rangle - |f(1) \oplus 1\rangle)\Big]$$

$\Rightarrow f(0) = 0 \Rightarrow (0 \ -1)$

$\Rightarrow f(0) = 1 \Rightarrow (1 - 0) = -(0 - 1)$

$\Big\} = (-1)^{f(0)}(|0\rangle - |1\rangle)$

$$\Rightarrow \frac{1}{2}\Big[(-1)^{f(0)}|0\rangle \times (|0\rangle - |1\rangle) + (-1)^{f(1)}|1\rangle \times (|0\rangle - |1\rangle) = \frac{1}{2}(-1)^{f(0)}\Big[|0\rangle + (-1)^{f(0) \oplus f(1)}|1\rangle\Big](|0\rangle - |1\rangle)$$

3-) Don't care about 2nd qubit for now

If $f(0)=f(1) \Rightarrow |0\rangle + |1\rangle$

If $f(0) \neq f(1) \Rightarrow |0\rangle - |1\rangle$

4-) Hadamard the first qubit.

$$\Psi_{out} = \frac{1}{2}\left(1+(-1)^{f(0)\oplus f(1)}|0\rangle\right) + \frac{1}{2}\left(1-(-1)^{f(0)\oplus f(1)}|1\rangle\right)$$

5-) Measure first qubit.

$\Rightarrow 0 \Rightarrow$ same

$\Rightarrow 1 \Rightarrow$ different.

But still it cost us \$ 10^6.

Let's see the part of Jozsa

N Bit DJ problem for every x_1 measurement, 2^N, classicaly

N Qbits " " it requires "1" Quantum Operation

EPR:

Let's say that I have an entangled state $\frac{1}{\sqrt{2}}(\uparrow\uparrow + \downarrow\downarrow)$

when we measure α, we know the state of β.

Now there are 2 possibilities

1-) It was already determined that both would be up or down

2-) xor there is some nonlocality in universe such that a distant measurement on α has effects on the state of β.

Abdullah Said Tongel

There are some answers to this problem.

1-) The quantum mechanics is insufficient and there is some hidden variable that we haven't observed which makes role on determining the particles spin from the creation.

It looks probabilistic but actually there is some hidden fact which controlling the probability distribution.

2-) That is some upseting fact but we're not going to throw it out just because there is some nasty problem that we need to face. QM models works like a champ in other problems.

Now let's look at the problem with a different aspect.

Let's measure the spin of α along z and measure β along x direction. Let's measure about S_z and S_x on both particles.

BUT know we have bot information about S_z and S_x but S_x and S_z DO NOT. At the very beggining of spin we said that spin along x and z both COMUTE. but know we throw

Einstein said "QM is incomplete" but Bell came with an idea.

Bell used classical statistics to show that Argument 1 is trash.

$$N(A,\bar{B}) + N(B,\bar{C}) \geq N(A,\bar{C})$$

$$\rightarrow N(A,\bar{B},C) + N(\bar{A},B,\bar{C}) \geq 0$$

First, we will measure α at 0 and β at ϑ.

Then, we will measure α at ϑ and β at 2ϑ

Finally, " " " α at 0 and β at 2ϑ

$$\left.\begin{aligned} &A: \uparrow_0,\ \bar{A}: \downarrow_0,\ B: \uparrow_\vartheta,\ \bar{B}: \downarrow_\vartheta,\ C: \uparrow_{2\vartheta},\ \bar{C}: \downarrow_{2\vartheta} \\ &\mathbb{P}(\uparrow_0, \downarrow_\vartheta) + \mathbb{P}(\uparrow_\vartheta, \downarrow_{2\vartheta}) \geq \mathbb{P}(\uparrow_0, \downarrow_{2\vartheta}) \end{aligned}\right\} \downarrow_\vartheta = \cos(\vartheta/2)\downarrow_0 + \sin\left(\tfrac{\vartheta}{2}\right)\uparrow_0$$

$$\sin^2\left(\frac{\vartheta}{2}\right) + \sin^2\left(\frac{\vartheta}{2}\right) \geq \sin^2(\vartheta) \text{ , at small } \vartheta;$$

$$\Rightarrow \left(\frac{\vartheta}{2}\right)^2 + \left(\frac{\vartheta}{2}\right)^2 \ngeq \vartheta^2 \Rightarrow \text{Violation of the Bell's Inequality}$$

We observed this result which is an empirical proof that Arg. 1 is not possible.