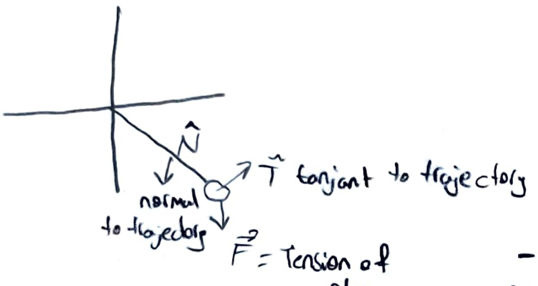
Multi-Variable Calculus {18.02}

Dot Product:

$$\vec{A}\cdot\vec{B} = \sum a_i b_i = |\vec{A}|\,|\vec{B}|\cos\theta \rightarrow \cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}$$

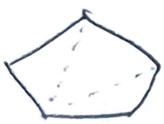


- component of F along $\hat{T}$ is what causes pendulum to swing

- component along $\hat{N}$ is responsible for the tension of the string

---

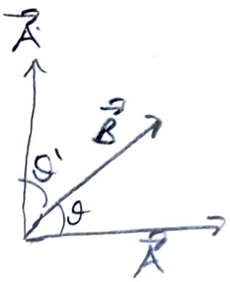
Area:



- easier: area of triangle

$$\text{Area} = \frac{1}{2}|\vec{A}|\,|\vec{B}|\sin\theta$$

We could find cos and use $\sin^2\theta + \cos^2\theta = 1$.



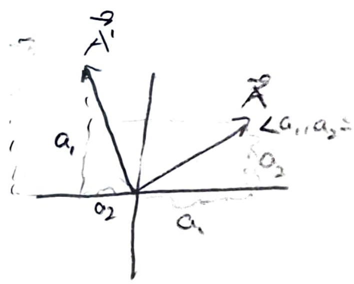
$\vec{A}' = \vec{A}$ rotated $90°$

$\theta' = \frac{\pi}{2} - \theta$

$\cos(\theta') = \sin\theta$

$\vec{A} = \langle a_1, a_2\rangle$

$\vec{A}' = \langle a_2, a_1\rangle$



$$|\vec{A}|\,|\vec{B}|\sin\theta = |\vec{A}'|\,|\vec{B}|\cos\theta' = \vec{A}'\cdot\vec{B}$$

$= \langle a_1, a_2\rangle \cdot \langle b_1, b_2\rangle$

$= a_1 b_2 - a_2 b_1$

$\Rightarrow \det(\vec{A},\vec{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \Rightarrow$ det of $\vec{A}$ and $\vec{B}$ = area of

Author: Abdullah Said Töngel

Cross Product

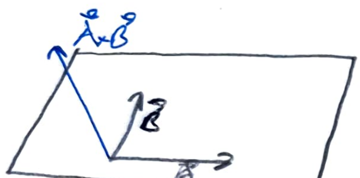
2 vectors in 3-space

Def: $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{k}$

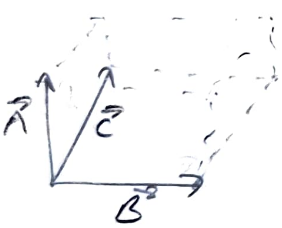
is a vector

Theorem:

* $|\vec{A}\times\vec{B}|$ = area of parallelogram
* dir $(\vec{A}\times\vec{B}) \perp$ to the plane of the parallelogram.



-use right-hand rule for right handed coordinate systems.



Volume = area · height
$= |\vec{B}\times\vec{C}| \cdot (\vec{A}\cdot\vec{n})$

$\left(\vec{A}\cdot\frac{(\vec{B}\times\vec{C})}{|\vec{B}\times\vec{C}|}\right)$

$= \vec{A}\cdot(\vec{B}\times\vec{C}) = \det(\vec{A},\vec{B},\vec{C})$

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

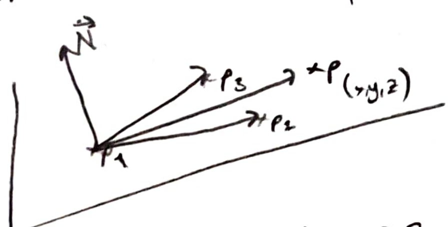
in particular

$\vec{A}\times\vec{A} = 0$

Application:

equation of plane P_1, P_2, P_3

= condition on (x,y,z) telling us whether P is in the plane?



Other sol'n: P is in the plane $\iff \vec{P_1P} \perp \vec{N}$
$\iff \vec{P_1P}\times\vec{N} = 0$

$\det(\vec{P_1P}, \vec{P_1P_2}, \vec{P_1P_3}) = 0$

How to find that

$\vec{N} \perp$ plane : take cross product of $\vec{P_1P_2}\times\vec{P_1P_3}$

So $\vec{P_1P} \cdot \vec{N} = 0 = \vec{P_1P} \cdot (\vec{P_1P_2} \times \vec{P_1P_3}) = 0 = \det.$

$AB(x) = A(Bx)$, Note: $AB \neq BA$

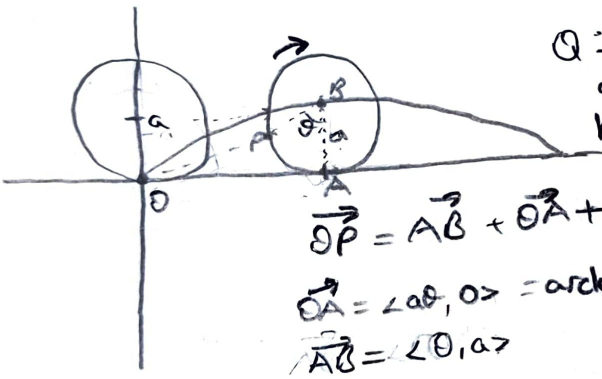
In the plane rotation by $90°$ counterclockwise

$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $R\hat{\imath} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  , $R\hat{\jmath} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

R found from $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -y \\ x \end{bmatrix}$

Cycloid:

wheel of radius a rolling on the floor = x-axis , P = a point on rim of wheel start at 0 , what happens?



Q: position $(x(\theta), y(\theta))$ of point P? as a function of the angle θ which the wheel has rotated.

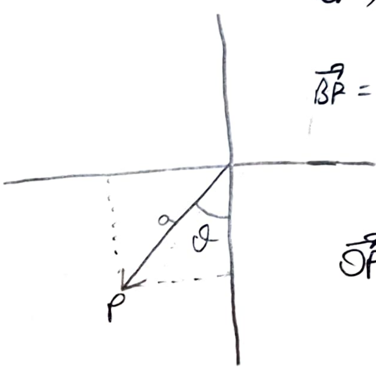
$\vec{OP} = \vec{AB} + \vec{OA} + \vec{BP}$

$\vec{OA} = \langle a\theta, 0 \rangle$ = arclength from A to P.

$\vec{AB} = \langle 0, a \rangle$

$\vec{BP}$, $\begin{cases} * |\vec{BP}| = a \\ * \text{ angle } \theta \text{ with vertical} \end{cases}$

$\vec{BP} = \langle -a\sin\theta, -a\cos\theta \rangle$



$\vec{OP} = \langle a\theta - a\sin\theta, a - a\cos\theta \rangle$

Q: what happens near the bottom point?

A: take length unit = radius : $a = 1$

$\begin{cases} x(\theta) = \theta - \sin\theta \\ y(\theta) = 1 - \cos\theta \end{cases}$

$\sin\theta \approx \theta$, $\cos\theta \approx 1$

$\underline{0 \, \& \, \theta \approx 0}$

We need better approximation.

Use tailor approx. for $t << 1$

$f(t) \approx f(0) + t f'(0) + \frac{t^2}{2} f''(0) + \frac{t^3}{3} f'''(0) + \dots$

$\sin\theta \approx \theta - \frac{\theta^3}{6}$

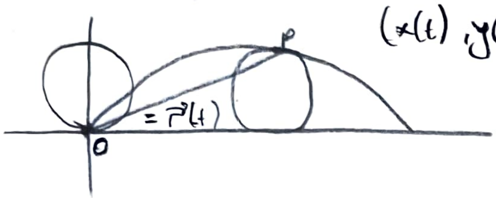
$\cos\theta \approx 1 - \frac{\theta^2}{2}$

$x(\theta) \simeq \theta - (\theta - \frac{\theta^3}{6}) \simeq \frac{\theta^3}{6}$ So $|x| << |y|$

$y(\theta) \simeq 1 - (1 - \frac{\theta^2}{2}) \simeq \frac{\theta^2}{2}$ $\frac{y}{x} \approx \frac{\theta^2}{2} \cdot \frac{6}{\theta^3} = \frac{3}{\theta} \to \infty$ as $\theta \to 0$

slope at origin is ∞

Parametric Equations:



$(x(t), y(t), z(t))$ position of a moving point.

Position vector $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Example: cycloid (wheel radius 1, at unit speed)

$\vec{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$

Velocity vector: $\vec{v} = \frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$

$\vec{v} = \langle 1 - \cos t, \sin t \rangle$

Speed $= |\vec{v}| = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2(1-\cos t)}$

$\vec{a} = \frac{d\vec{v}}{dt} = \langle \sin t, \cos t \rangle$

Arclength:

S = distance travelled along trajectory

S vs. t.

$\left| \frac{ds}{dt} \right| = \text{speed} = |\vec{v}|$

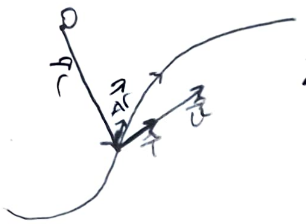
Example: length of an arch of cycloids:

$$\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$$

unit tangent vector

$\hat{T} = \frac{\vec{v}}{|\vec{v}|}$

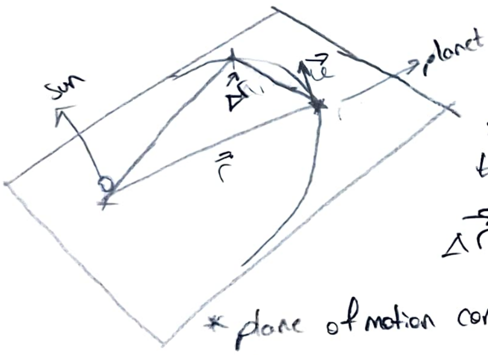
$\vec{v} = \frac{d\vec{r}}{dt} = \underbrace{\frac{d\vec{r}}{ds}}_{\hat{T}} \underbrace{\frac{ds}{dt}}_{|\vec{v}|} = \hat{T}\frac{ds}{dt}$, velocity has $\begin{cases} \text{direction: tangent to trajectory } \hat{T} \\ \text{length : speed } \frac{ds}{dt} \end{cases}$



$\vec{\Delta r} \approx \hat{T} \cdot \Delta s$, $\frac{\Delta s}{\Delta t} \approx$ speed , $\frac{\Delta \vec{r}}{\Delta t} \approx \hat{T}\frac{\Delta s}{\Delta t}$ gives $\frac{d\vec{r}}{dt} = \hat{T}\frac{ds}{dt}$

Example: Kepler's Second Law:

Motion of planets is in a plane and the area is swept out by the line from sun to planet at a constant rate



area swept in time Δt. $\approx \frac{1}{2}|\vec{r} \times \Delta\vec{r}| \approx \frac{1}{2}|\vec{r} \times \vec{v}|\Delta t$

$\Delta\vec{r} \approx \vec{v}\Delta t$

* plane of motion contains $\vec{r}$ and $\vec{v}$. Direction of $\vec{r} \times \vec{v}$ is normal to the plane of motion.

Kepler's Second Law $\Leftrightarrow \vec{r} \times \vec{v} =$ constant vector

$\Leftrightarrow \frac{d}{dt}(\vec{r} \times \vec{v}) = 0 = \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r}\,\frac{d\vec{v}}{dt} = 0 = \vec{v} \times \vec{v} + \vec{r} \times \vec{a} = 0$

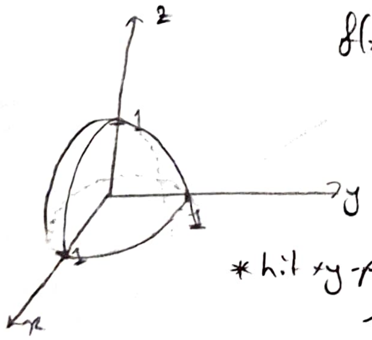
$\vec{v} \times \vec{v} = 0$

$\Leftrightarrow \vec{r} \times \vec{a} = 0$

$\Leftrightarrow \vec{a} \parallel \vec{r}$

Kepler's 2nd Law $\Rightarrow \vec{a} \parallel \vec{r} \Leftrightarrow$ gravitational force $\parallel \vec{r}$

Calculus of Several Variables



$f(x,y) = 1 - x^2 - y^2$

in yz-plane: $x = 0$, $z = 1 - y^2$

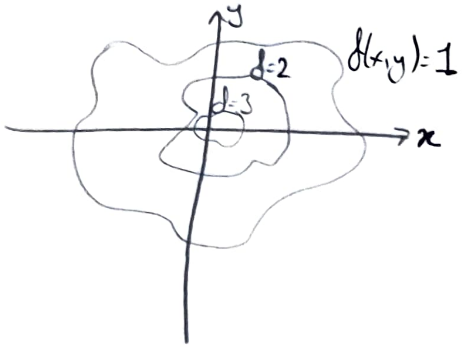
in xz-plane: $y = 0$, $z = 1 - x^2$

* hit xy-plane when $z = 0$

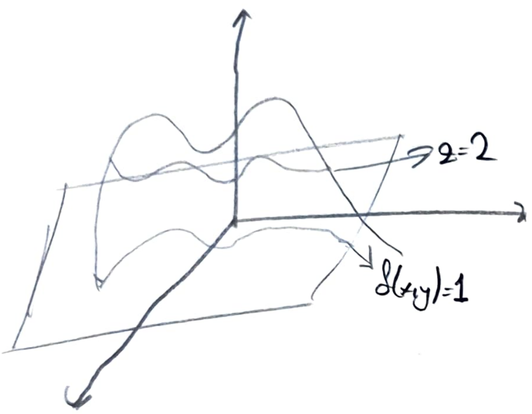
$1 - x^2 - y^2 = 0$

$x^2 + y^2 = 1 \rightarrow$ unit circle

Contour Plot: Shows all the points where $f(x,y)$ = some fixed constants, chosen at regular interval.



$\Longleftrightarrow$ we slice the graph by horizontal plates $z=c$



$\Rightarrow$ Level Curve of f.

function of 1-variable $f(x)$: $f'(x) = \dfrac{df}{dx} = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Approximation: $x_0 \rightsquigarrow f(x_0)$

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0)$$

Question: How can we do it for $f(x,y)$?

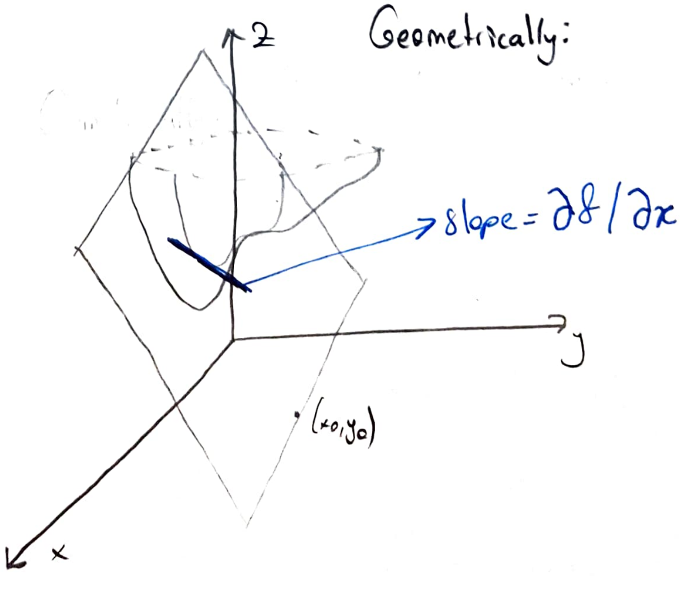
Partial Derivatives:

$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x}$$

"partial"

$$\frac{\partial f}{\partial y}(x_0,y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0,y_0)}{\Delta y}$$

Geometrically:



$$\frac{\partial f}{\partial x} = f_x$$

treat y as constant, x as variable

$f(x,y) \rightsquigarrow \frac{\partial f}{\partial x} = f_x \ , \ \frac{\partial f}{\partial y} = f_y$

If we change;

$x \rightsquigarrow x+\Delta x$

$y \rightsquigarrow y+\Delta y$ $\quad z = f(x,y)$: then $\Delta z \approx f_x \Delta x + f_y \Delta y$

f_x and f_y are slopes of 2 tangent lines

If $\frac{\partial f}{\partial x}(x_0,y_0) = a \Rightarrow L_1 = \begin{cases} z = z_0 + a(x-x_0) \\ y = y_0 \end{cases}$

If $\frac{\partial f}{\partial y}(x_0,y_0) = b \Rightarrow L_2 = \begin{cases} z = z_0 + b(y-y_0) \\ x = x_0 \end{cases}$

L_1, L_2 are both tangent to the graph $z=f(x,y)$. Together they determine a plane

$$z = z_0 + a(x-x_0) + b(y-y_0)$$

Approximation Formula: Graph of f is close to its tangent plane

Def'n (x_0,y_0) is a critical point if $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$.

Possibilities:

* Local Min. , * Local Max. , * Saddle

Depends on which side you're looking, it can be min or max.

Example: $x^2 - 2xy + 3y^2 + 2x - 2y$

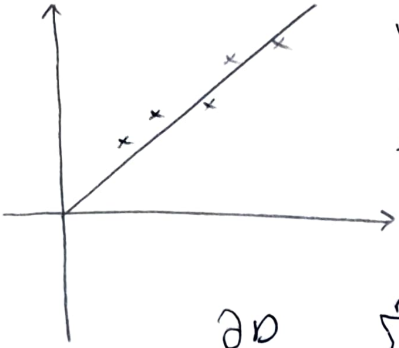
$x^2 - 2xy + y^2 = (x-y)^2 \Rightarrow (x-y+1)^2$

$\Rightarrow (x-y+1)^2 + 2y^2 - 1$ { It's square -1 so its min. value $= -1$ }

Least-Squares Interpolation

Given experimental data

$(x_1,y_1), (x_2,y_2), \dots (x_n,y_n)$, find "best" fit line $y=ax+b$



We'll try to minimize total square deviation.

Deviation for each data point : $y_i - (ax_i+b)$

Minimize :

$$D(a,b) = \sum_{i=1}^{n} \left[y_i - (ax_i+b)\right]^2$$

$$\frac{\partial D}{\partial a} = \sum_{i=1}^{n} 2\left[y_i-(ax_i+b)\right]\cdot(-x_i) = 0$$

$$\frac{\partial D}{\partial b} = \sum_{i=1}^{n} 2\left(y_i-(ax_i+b)\right)\cdot(-1) = 0$$

$$\Leftrightarrow \begin{cases} \sum_{i=1}^{n}(x_i^2 a + x_i b - x_i y_i) = 0 \Rightarrow \left(\sum_{i=1}^{n} x_i^2\right)a + \left(\sum_{i=1}^{n} x_i\right)b = \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n}(x_i a + b - y_i) = 0 \Rightarrow \left(\sum_{i=1}^{n} x_i\right)a + nb = \sum_{i=1}^{n} y_i \end{cases}$$

Best exponential fit : $y = ce^{ax} \Leftrightarrow \ln(y) = \ln(c) + ax$

Second-Derivative Test

First consider $w = ax^2 + bxy + cy^2$

Example: $w = x^2 + 2xy + 3y^2 = (x+y)^2 + 2y^2$

In general, if $a \neq 0$:

$$w = a\left(x^2 + \frac{b}{a}xy\right) + cy^2 = a\left(x + \frac{b}{2a}y\right)^2 + \left(c - \frac{b^2}{4a}\right)y^2$$

$$= ax^2 + \cancel{a}2x\frac{b}{2\cancel{a}}y + \cancel{\frac{\cancel{a}b^2}{4a^{\cancel{2}}}y^2} + cy^2 \cancel{- \frac{b^2}{4a}y^2}$$ $\Rightarrow$ just wanted show that its true

$$w = \frac{1}{4a}\left[4a^2\left(x + \frac{b}{2a}y\right)^2 + (4ac - b^2)y^2\right]$$

3 cases: 1-) $4ac - b^2 < 0 \Rightarrow$ 1 term ≥ 0, the other $\leq 0 \Rightarrow$ saddle point

2-) $4ac - b^2 = 0 \Rightarrow$ all values of y becomes critical since $ky = 0 \Rightarrow$ degenerate

3-) $4ac - b^2 > 0 \Rightarrow w = \frac{1}{4a}\left[+(\dots)^2 + (\dots)^2\right] \geq 0$

$\rightarrow$ if $a > 0$ then $w > 0$; minimum 0.

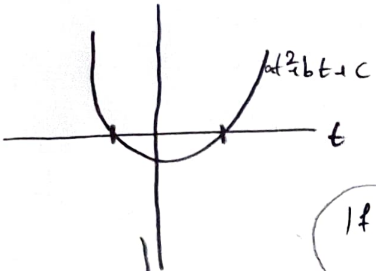
$\rightarrow$ if $a < 0$ then $w < 0$; we get maximum 0.

b^2-4ac : quadratic formula?

$$w = ax^2 + bxy + cy^2 = \underbrace{y^2}_{\geq 0}\underbrace{\left[a\left(\frac{x}{y}\right)^2 + b\left(\frac{x}{y}\right) + c\right]}$$

if $b^2-4ac>0$

then this takes $+,-$ values



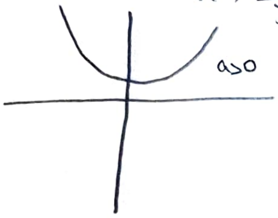
$\Rightarrow w$ takes positive and negative values

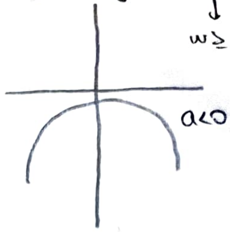
$\Rightarrow$ saddle

If $b^2-4ac<0$ then, $\frac{w}{y^2}$ is always positive or always negative

$\downarrow$ $w\geq 0$, min

$\downarrow$ $w\leq 0$, max.





In general we want to look second-derivatives.

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}, \quad \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

At a critical point (x_0,y_0) of f, Let $A = f_{xx}(x_0,y_0)$, $B = f_{xy}(x_0,y_0)$, $C = f_{yy}(x_0,y_0)$

If $AC-B^2>0$ and $A>0$, local minimum.

" $AC-B^2>0$ and $A<0$, " maximum.

" $AC-B^2<0 \Rightarrow$ saddle

" $AC-B^2=0 \Rightarrow$ can't conclude

Verify in special case $w = ax^2+bxy+cy^2$

$w_x = 2ax+by$, $w_{xx}=2a$, $w_{xy}=b$

$w_y = bx+2cy$, $w_{yy}=2c$, $w_{yx}=b$

so $\left.\begin{array}{l} A=2a \\ B=2c \\ C=b \end{array}\right\}$ $AC-B^2 = 4ac-b^2$

quadratic approximation:

$$\Delta f \approx f_x(x-x_0) + f_y(y-y_0) + \frac{1}{2}f_{xx}(x-x_0)^2 + \frac{1}{2}f_{yy}(y-y_0)^2 + f_{xy}(x-x_0)(y-y_0)$$

we can do it for n-degree. if we want to continue writing

$\Rightarrow$ so the general case reduces to the quadratic case

Note: In the degenerate case what actually happens depends only the higher order derivatives

Differentials:

Implicit differentiation

$y = f(x)$

$dy = f'(x)dx$

Example:

$y = \sin^{-1}(x)$

$x = \sin(y) \rightsquigarrow dx = \cos(y)dy \Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$

Total Differential:

$df = f_x dx + f_y dy + f_z dz$, $f(x,y,z)$ Note: df is not the same thing as Δf

What can we do with that?

1-) Encode how change in x, y, z, affect f

2-) placeholder for small variations $\Delta x, \Delta y, \Delta z$, to get approx. formula

$$\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$$

3-) Divide by something like dt to get an infinitesimal rate of change. when $x = x(t), y = y(t), z = z(t)$

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

so, why this is valid?

1st Attempt: $df = f_x dx + f_y dy + f_z dz$

$dx = x'(t)dt,\ dy = y'(t)dt,\ dz = z'(t)dt$

$df = f_x \cdot x'(t)dt + f_y \cdot y'(t)dt + f_z \cdot z'(t)dt$

Divide by $dt \Rightarrow$ get chain rule.

A better way: $\frac{\Delta f}{\Delta t} \approx \frac{f_x \Delta x + f_y \Delta y + f_z \Delta z}{\Delta t}$

when $\Delta t \to 0$

$\Rightarrow \frac{\Delta f}{\Delta t} \to \frac{df}{dt}$, $\frac{\Delta x}{\Delta t} \to \frac{dx}{dt}$, ...

so $\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$

$\approx$ becomes $=$ in limit
since error will be infinitely small.

Example:

$w = z^2 y + x$, $x = t,\ y = e^t,\ z = \sin t$

$$\frac{dw}{dt} = 1\frac{dx}{dt} + z^2 \frac{dy}{dt} + 2zy \frac{dz}{dt}$$

$$= \frac{dx}{dt} + \sin^2 t \frac{dy}{dt} + 2\sin t \cdot e^t \frac{dz}{dt}$$

$$= 1 + \sin^2 t \cdot e^t + \sin t \cdot e^t \cdot \cos t$$

Substitute: $w = z^2 y + x$

$w(t) = \sin^2 t e^t + t$

$$\frac{dw}{dt} = 2\cos t \sin t \cdot e^t + e^t \cdot \sin^2 t + 1$$

Application: justify product, quotient rules.

$f = uv$, $u = u(t)$, $v = v(t)$

$$\frac{d(uv)}{dt} = f_u \frac{du}{dt} + f_v \frac{dv}{dt} = v\frac{du}{dt} + u\frac{dv}{dt}$$

$g = \frac{u}{v}$, $u = u(t)$, $v = v(t)$

$$\frac{d(u/v)}{dt} = \frac{1}{v}\frac{du}{dt} + \frac{-u}{v^2}\frac{dv}{dt}$$

$w = f(x,y)$, $x = x(u,v)$, $y = y(u,v)$ Question: $\frac{\partial w}{\partial u}$, $\frac{\partial w}{\partial v}$ in terms of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$, x_u, x_v, y_u, y_v

$w(u,v) = f(x(u,v), y(u,v))$

$$dw = f_x dx + f_y dy = f_x(x_u du + x_v dv) + f_y(y_u du + y_v dv)$$

$$= \underbrace{(f_x x_u + f_y y_u)}_{\partial f/\partial u} du + \underbrace{(f_x x_v + f_y + y_v)}_{\partial f/\partial v} dv$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

Gradient Vector:

$$\frac{dw}{dt} = w_x\frac{dx}{dt} + w_y\frac{dy}{dt} + w_z\frac{dz}{dt} = \nabla w \cdot \frac{d\vec{r}}{dt}$$

$\nabla w = \langle w_x, w_y, w_z \rangle \Rightarrow$ gradient vector.

Gradient of w at some point (x,y,z)

$$\frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

Theorem: $\nabla w \perp$ level surface $\{w = \text{constant}\}$

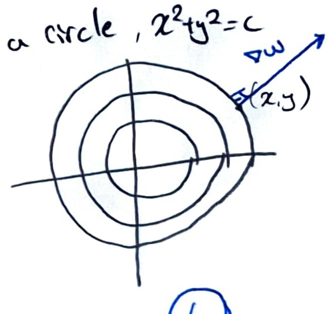
Exp. $w = a_1x + a_2y + a_3z$, $\nabla w = \langle a_1, a_2, a_3 \rangle$

level surface: $a_1x + a_2y + a_3z = C$.

plane with normal $\langle a_1, a_2, a_3 \rangle$

Exp. $w = x^2 + y^2$, $w = c$ is a circle, $x^2 + y^2 = c$

$\nabla w = \langle 2x, 2y \rangle$



Proof: Let $w = f(x,y,z)$ be a function of 3 variables. By saying it $\perp$ to the surface we meant that it is $\perp$ to the tangent to any curve that lies on the surface and goes through P. P is any point $P(x_0, y_0, z_0)$ at surface level.

Let
$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

We let $g(t) = f(x(t), y(t), z(t))$. Since curve is on the lvl surface we have: $g(t) = c$. Differentiating this eq. respect to t:

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}\Big|_P \frac{dx}{dt}\Big|_0 + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

In vector form we have

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = 0$$

$$\Leftrightarrow \nabla f_P \cdot r'(t) = 0$$

Since the dot product is zero, we have shown that gradient $\perp$ lvl surface.

Note: We can define the tangent plane as the gradient vector equals to normal vector of plane.

Example: find the tangent plane to surface $x^2 + y^2 - z^2 = 4$ at $\langle 2,1,1 \rangle$

gradient $= \nabla w$

$= \nabla w = \langle 2x, 2y, -2z \rangle = \langle 4, 2, -2 \rangle$

So eq is $4x + 2y - 2z = 8$, $\boxed{2x + y - z = 4}$

Another way to do?

first find the gradient: $\nabla w = \langle 2x, 2y, -2z \rangle$

Evaluate the gradient at given point: $\langle 4, 2, -2 \rangle$

Write the eq. of tangent plane as

$$w_x(x_0,y_0,z_0)(x-x_0) + w_y(x_0,y_0,z_0)(y-y_0) + w_z(x_0,y_0,z_0)(z-z_0) = 0$$

Using $(x_0, y_0, z_0) = (2,1,1)$ and $\nabla w(2,1,1) = (4,2,-2)$

$$4(x-2) + 2(y-1) - 2(z-1) = 0$$

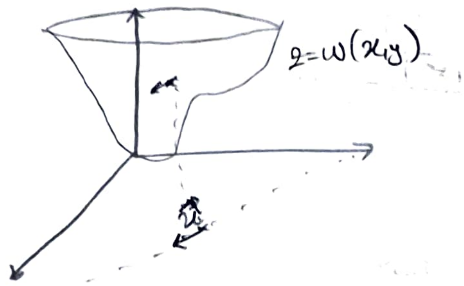
from that we get

$$2x + y - z = 4$$

Directional Derivatives:

$w = w(x,y) \rightsquigarrow$ know $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \Rightarrow$ derivatives are in dir. of $\hat{i}$ or $\hat{j}$ for 2-d space or so o

What if we move in direction of $\hat{u}$ = unit vector



Straight line trajectory

$\vec{r}(s)$, $\frac{d\vec{r}}{ds} = \hat{u}$

arclength
distance along line

If $\hat{u} = \langle a, b \rangle$

$$\begin{cases} x(s) = x_0 + as \\ y(s) = y_0 + bs \end{cases} \Rightarrow \text{plug into } \frac{dw}{ds}$$

Def'n:

$\left.\frac{dw}{ds}\right|_{\hat{u}} \Rightarrow$ directional derivative in direction of $\hat{u}$.

$\left.\frac{dw}{ds}\right|_{\hat{u}}$ = slope of slice of graph by a vertical plane $\parallel \hat{u}$.

Chain Rule implies: $\frac{dw}{ds} = \nabla w \cdot \frac{d\vec{r}}{ds} = \nabla w \cdot \hat{u}$

$\left.\frac{dw}{ds}\right|_{\hat{u}} = \nabla w \cdot \hat{u} = |\nabla w|\,|\hat{u}| \cos\theta$

$\Rightarrow$ max. when $\cos(\theta) = 1$

$\Rightarrow \hat{u} = \text{dir}(\nabla w)$

So: dir. of ∇w = dir. of fastest increase of w

$|\nabla w| = \left.\frac{dw}{ds}\right|_{\hat{u} = \text{dir}(\nabla w)}$

$\Rightarrow$ min. value of $\frac{dw}{ds}$, when $\cos(\theta) = -1$, $\hat{u}$ is in dir. of $(-\nabla w)$

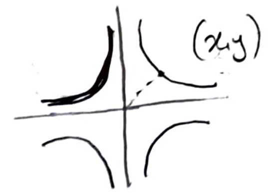
$\Rightarrow \left.\frac{dw}{ds}\right|_{\hat{u}} = 0$ when $\cos(\theta) = 0$ $\quad \hat{u} \perp \nabla w \Leftrightarrow \hat{u}$ tangent to the level

Lagrange Multipliers:

Let f be a function $f(x,y,z)$ where x,y,z are not independent.

$g(x,y,z)=c$

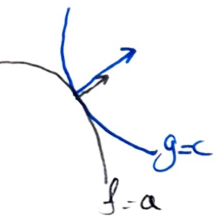
Example: point closest to origin on hyperbola $xy=c$



Minimize $f(x,y)=x^2+y^2$

subject to constraint $\underbrace{xy}_{g(x,y)}=c$

Observe: at the minimum level curve of f is tangent to hyperbola $g=c$



when this happens $\nabla f \parallel \nabla g$

so $\nabla f = \lambda \nabla g$

That creates system of equations:

$\nabla f = \lambda \nabla g \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases}$, $f=x^2+y^2$, $g=xy$

constraint $\quad g=c$

$2x = \lambda y \Rightarrow 2x - \lambda y = 0$
$2y = \lambda x \Rightarrow \lambda x - 2y = 0$ $\Rightarrow \begin{bmatrix} 2 & -\lambda \\ \lambda & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$xy = c \Rightarrow xy = c$

when det $=0$: $-4+\lambda^2=0 \Rightarrow \lambda = \pm 2$

λ becomes the Lagrange Multiplier.

when $\lambda = 2$:

$2x-2y=0$

$x=y$

$(\sqrt{c}, \sqrt{c})$

or

$(-\sqrt{c}, -\sqrt{c})$

when $\lambda=-2$

$2x+2y=0$

$x=-y$

$(\sqrt{c}, -\sqrt{c})$

or

$(-\sqrt{c}, \sqrt{c})$

$\Downarrow$

if $c\leq 0$

if $c>0$ then no sol'n

Why is this method valid?

At a constrained min or max in any direction along level $g=c$ the rate of change of f must be zero.

For any $\hat{u}$ tangent to $g=c$, we must have

$\left.\dfrac{df}{ds}\right|_{\hat{u}} = 0 \Rightarrow \nabla f \cdot \hat{u}$

So any such $\hat{u}$ is $\perp$ ∇f

So $\nabla f \perp$ level set of g

know $\nabla g \perp$ level " " g

so $\nabla f \parallel \nabla g$

Warning: Method can't tell you whether a sol'n is min/max
need to check by comparing values.

Non-Independent Variables

$f(x,y,z)$, $g(x,y,z)=c$ then $z=z(x,y)$

$\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$

Example: $x^2+yz+z^3=8$ at $(2,3,1)$

Differential of it: $2x\,dx+y\,dz+z\,dy+3z^2dz = 2x\,dx+z\,dy+(y+3z^2)dz=0$ at $(2,3,1)$

$4dx+dy+6dz=0$

If we see $z=z(x,y)$

$dz=-\frac{1}{6}(4dx+dy)$

$\dfrac{\partial z}{\partial x}=-\dfrac{4}{6}=-\dfrac{2}{3}$, $\dfrac{\partial z}{\partial y}=-\dfrac{1}{6}$

In general $g(x,y,z)=c$, then

$dg=g_x dx+g_y dy+g_z dz=0$

Solve for dz:

$dz=\dfrac{-g_x dx-g_y dy}{g_z}$

, So for $\dfrac{\partial z}{\partial x}$, set $y=$ constant $\Rightarrow dy=0$

$dz=\dfrac{-g_x}{g_z}dx$, $\dfrac{\partial z}{\partial x}=\dfrac{-g_x}{g_z}$

Let's take a simple example

$f(x,y)=x+y$

$x=u$

$y=u+v$

$\dfrac{\partial f}{\partial x}=1$, $\dfrac{\partial f}{\partial u}=2 \Rightarrow x=u$ but $\dfrac{\partial f}{\partial x}\neq\dfrac{\partial f}{\partial u}$

change $x=x$ keep y constant

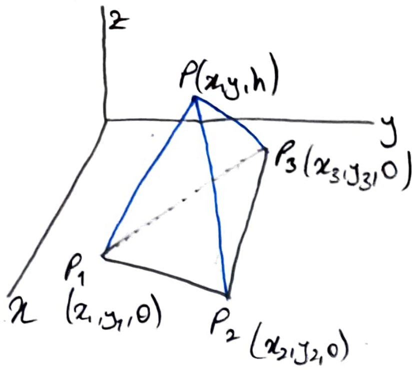
change $u=x$ but keep v constant, $v=y-x$

So we need a clear notation.

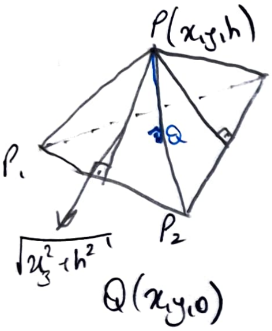
$\left(\dfrac{\partial f}{\partial x}\right)_y$ = keep $y=c$, $\left(\dfrac{\partial f}{\partial u}\right)_v$ = keep $v=c$

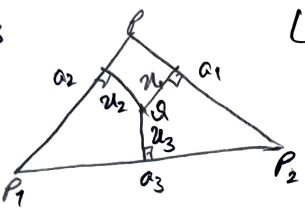
$\left(\dfrac{\partial f}{\partial x}\right)_y \neq \left(\dfrac{\partial f}{\partial x}\right)_v = \left(\dfrac{\partial f}{\partial u}\right)_v$

Advanced Example: Want to build a pyramid with given triangular base and volume with minimum surface area.



$vol = \frac{1}{3} \cdot \text{Area(base)}\ \underbrace{\text{height}}_{\text{fixed} = h}$





Let x_1, x_2, x_3 distances from Q to sides.

heights of faces: $\sqrt{x_1^2+h^2}$, $\sqrt{x_2^2+h^2}$, $\sqrt{x_3^2+h^2}$

Side area $= \frac{1}{2}a_1\sqrt{x_1^2+h^2} + \frac{1}{2}a_2\sqrt{x_2^2+h^2} + \frac{1}{2}a_3\sqrt{x_3^2+h^2} = f(x_1,x_2,x_3)$

Cut base into 3 pieces $= \overbrace{\text{Area(base)}}^{g} = \frac{1}{2}a_1x_1 + \frac{1}{2}a_2x_2 + \frac{1}{2}a_3x_3$

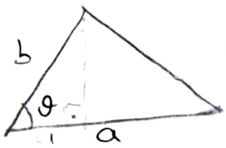
$\nabla f = \lambda \nabla g$:
$$\frac{\partial f}{\partial x_1} = \cancel{\tfrac{1}{2}a_1}\frac{x_1}{\sqrt{x_1^2+h^2}} = \lambda \cancel{\tfrac{1}{2}a_1}$$
$$\frac{\partial f}{\partial x_2} = \cancel{\tfrac{1}{2}a_2}\frac{x_2}{\sqrt{x_2^2+h^2}} = \lambda \cancel{\tfrac{1}{2}a_2}$$
$$\frac{\partial f}{\partial x_3} = \cancel{\tfrac{1}{2}a_3}\frac{x_3}{\sqrt{x_3^2+h^2}} = \lambda \cancel{\tfrac{1}{2}a_3}$$

$$\lambda = \frac{x_1}{\sqrt{x_1^2+h^2}} = \frac{x_2}{\sqrt{x_2^2+h^2}} = \frac{x_3}{\sqrt{x_3^2+h^2}}$$

$\Rightarrow x_1 = x_2 = x_3$

Point Q must be incenter.

Example: area of triangle



$A = \frac{1}{2} ab\sin\theta$.

$\underbrace{a = b\cdot\cos\theta}_{\text{constant}}$

rate of change of A with respect to θ

1-) treat a, b, θ as independent: $\frac{\partial A}{\partial \theta} = \left(\frac{\partial A}{\partial \theta}\right)_{a,b}$

$$\frac{\partial A}{\partial \theta} = \frac{1}{2} ab\cos\theta$$

2-) keep a constant.
b will change: $b = b(a,\theta) = \left(\frac{a}{\cos\theta}\right)$

θ changes, a constant: $a = b\cos\theta \Rightarrow$ b changes. How does A change?

3-) keep b constant
$a = (b,\theta)$ changes $\left(\frac{\partial A}{\partial \theta}\right)_b$

Compute $\left(\frac{\partial A}{\partial \theta}\right)_b = ?$

Method 0: Solve for b and substitute.

$a = b\cos\theta \Rightarrow b = \frac{a}{\cos\theta} = a\sec\theta$.

$$A = \frac{1}{2}ab\sin\theta = \frac{1}{2}a\cdot\frac{a}{\cos\theta}\cdot\sin\theta = \frac{1}{2}a^2\tan\theta$$

So $\left(\frac{\partial A}{\partial \theta}\right)_a = \frac{1}{2}a^2\sec^2\theta$

Normally, we have two systematic ways.

1-) Differentials
- keep a fixed: $da = 0$
- constraint $a = b\cos\theta$

$$0 = da = \cos\theta\cdot db - b\cdot\sin\theta\cdot d\theta$$
$$\cos\theta\, db = b\cdot\sin\theta\, d\theta$$
$$db = \frac{b\cdot\sin\theta\, d\theta}{\cos\theta} = b\cdot\tan\theta\, d\theta$$

$A = \frac{1}{2}ab\sin\theta$

$$dA = \frac{1}{2}b\sin\theta\, da + \frac{1}{2}a\sin\theta\, db + \frac{1}{2}ab\cos\theta\, d\theta.$$

So $dA = \frac{1}{2}a\sin\theta(b\cdot\tan\theta\, d\theta) + \frac{1}{2}ab\cos\theta\, d\theta$

$$= \frac{1}{2}ab\,\underbrace{(\sin\theta\tan\theta + \cos\theta)}_{\sec\theta}\,d\theta$$

So $\left(\frac{\partial A}{\partial \theta}\right)_a = \frac{1}{2}ab\sec\theta$

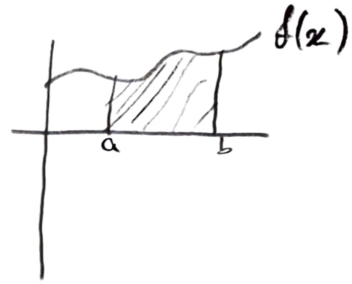
Summary:
1-) write dA in terms of $da, db, d\theta$
2-) $a = $ const $\Rightarrow$ set $dA = 0$
3-) differentiate constraints $\Rightarrow$ solve for db in terms of $d\theta$, plug into dA, get Answer.

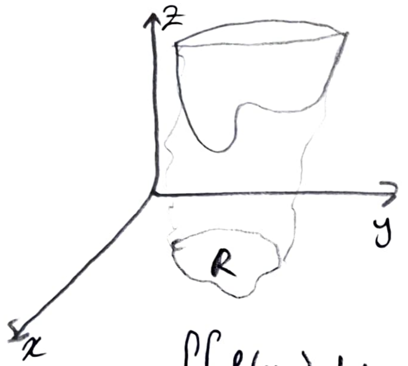
2-) Chain Rule: $\left(\frac{\partial}{\partial\theta}\right)_a$ in formula for A.

$$\left(\frac{\partial A}{\partial \theta}\right)_a = A_\theta\underbrace{\left(\frac{\partial \theta}{\partial \theta}\right)_a}_{1} + A_a\underbrace{\left(\frac{\partial a}{\partial \theta}\right)_a}_{0} + A_b\underbrace{\left(\frac{\partial b}{\partial \theta}\right)_a}_{\text{use constraint}}$$

Double Integrals:

Remember that $\int_a^b f(x)\,dx$ = area below the graph of f. over $[a,b]$

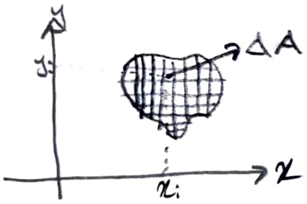




$\Rightarrow$ Double integral equalls to the volume below the graph $z=f(x,y)$ for a region R in xy-plane

$$\iint_R f(x,y)\,dA$$

Def: Cut R into small pieces of of Area ΔA_i

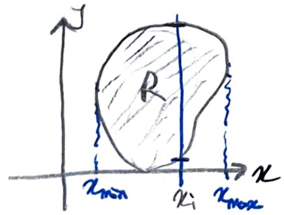


$\sum_i f(x_i, y_i)\cdot \Delta A_i$, Take the limit as $\Delta A_i \to 0$, get $\iint$

To compute: $\iint_R f(x,y)\,dA$: take slices , Let $S(x)$ = area of slice by plane // yz plane

Then volume $\int_{x_{min}}^{x_{max}} S(x)\,dx$

$$S(x) = \int_{y_{min}(x)}^{y_{max}} f(x,y)\,dy$$



$$\Rightarrow \iint f(x,y)\,dA = \int_{x_{min}}^{x_{max}} \left[\int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\,dy \right] dx$$

{ Iterated Integral. }

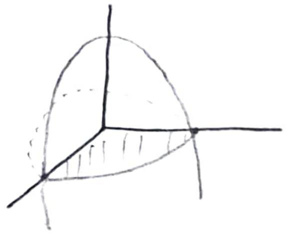
Example 1:

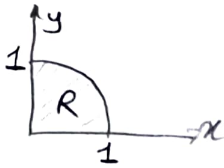
$z = 1 - x^2 - y^2$, region $0 \leq x \leq 1$, $0 \leq y \leq 1$

$$\int_0^1 \int_0^1 (1-x^2-y^2)(dy)(dx) \Rightarrow y - x^2 y - \frac{y^3}{3} \Big|_0^1 = 1 - x^2 - \frac{1}{3} = \frac{2}{3} - x^2$$

$$\Rightarrow \int_0^1 \frac{2}{3} - x^2\,dx = \frac{2}{3}x - \frac{x^3}{3} \Big|_0^1 = \frac{1}{3} + cx + d$$

Example-2: $1-x^2-y^2=z$,





R = quarter-disk

$x^2+y^2\le 1$

$x\ge 0,\ y\ge 0$

For given x, the range of y is from $y=0$ to $\sqrt{1-x^2}$

$$\iint_R f(x,y)\,dy\,dx = \int_0^1\int_0^{\sqrt{1-x^2}} 1-x^2-y^2\,dy\,dx$$

$$\Rightarrow \int_0^{\sqrt{1-x^2}} 1-x^2-y^2 dy = \left[y-x^2y-\frac{y^3}{3}\right]_0^{\sqrt{1-x^2}} = \sqrt{1-x^2}-x^2\sqrt{1-x^2}-\frac{\sqrt{1-x^2}^{\,3}}{3} = (1-x^2)\sqrt{1-x^2}-\frac{(1-x^2)^{3/2}}{3}$$

$$=\frac{2}{3}(1-x^2)^{3/2} \Rightarrow \int_0^1 \frac{2}{3}(1-x^2)^{3/2}dx = \int_0^{\pi/2}\frac{2}{3}(\cos^3\theta).\cos\theta\, d\theta = \frac{2}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta = \frac{2}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$\begin{Bmatrix} x=\sin\theta \\ (1-x^2)^{1/2}=\cos\theta \\ dx=\cos\theta . d\theta \end{Bmatrix}$$

$$=\frac{2}{3}\int_0^{\pi/2}\frac{(1+2\cos 2\theta+\cos^2 2\theta)}{4}d\theta = \frac{2}{3}\int_0^{\pi/2}\frac{\left(1+2\cos 2\theta+\left(\frac{1+\cos 4\theta}{2}\right)\right)}{4}d\theta = \frac{2}{3}\left(\frac{\theta+2.\sin 2\theta+\frac{\theta+\sin 4\theta}{2}}{4}\right)\Bigg|_0^{\pi/2}$$

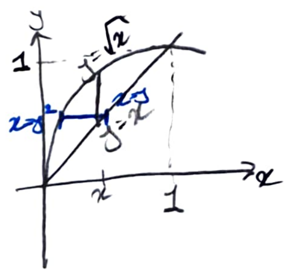
$$=\frac{\frac{\pi}{2}+\cancel{2\sin\pi}+\frac{\frac{\pi}{2}+\sin 2\pi}{2}}{4}-(0+2\sin 0+0+\sin 0) = \frac{\pi}{8}+\frac{\pi}{16}+0 = \left(\frac{3\pi}{16}\right)\frac{2}{3} = \frac{6\pi}{48} = \frac{\pi}{8}\ \{\text{Bruh}\}$$

Calculation of it will be easier in polar coordinates.

Exchanging order of integration will not change the sol'n but we must be "VERY" carefull about the boundaries.

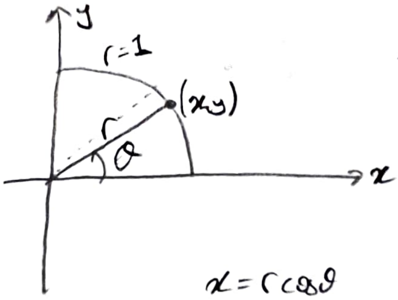
Exp. 1:
$$\int_1^2\int_2^3 dx\,dy = \int_2^3\int_1^2 dy\,dx$$

Exp. 2:
$$\int_0^1\int_x^{\sqrt{x}}\frac{e^y}{y}dy\,dx = \int_0^1\int_{y^2}^{y}\frac{e^y}{y}dx\,dy$$



Inner: $\left[\frac{e^y}{y}x\right]_{y^2}^{y} = e^y - e^y y$

$$\int_0^1 e^y - ye^y dy = (e^y+e^y-ye^y)\Big|_0^1 = 2e-e-2 = e-2$$



$1-x^2-y^2=f$

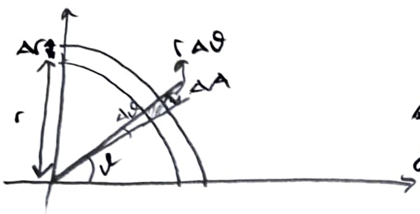
$x^2+y^2\leq 1$

$y, x \geq 0$

$x = r\cos\theta$

$y = r\sin\theta$

$$\int_0^{\pi/2}\int_0^1 (1-x^2-y^2)\, r\,dr\,d\theta$$



$\Delta A \approx \Delta r \cdot r\Delta\theta$

$dA = r\,dr\,d\theta$

$$f = 1-x^2-y^2 = 1-(x^2+y^2) = 1-r^2 \rightarrow \int_0^{\pi/2}\int_0^1 (1-r^2)\, r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 d\theta = \int_0^{\pi/2} \frac{1}{4}\,d\theta = \frac{1}{4}\cdot\frac{\pi}{2} = \frac{\pi}{8}$$

{Much Easier}

Applications of Double-Integrals:

1-) Find area of f in region R:

$$\text{Area}(R) = \iint_R f\,dA$$

, Mass of a flat object with density δ = mass per unit area.

$\Delta m = \delta \cdot \Delta A$

$$\text{Mass} = \iint_R \delta\,dA$$

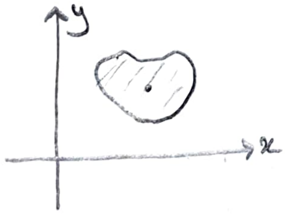
2-) Average value of f in R

$$\text{Avg of } f = \bar{f} = \frac{1}{\text{Area}(R)}\iint_R f\,dA$$

, weighted avg. of f. with density δ.

$$\frac{1}{\text{Mass}(R)}\iint_R f\,\delta\,dA$$

2a-) Center of mass of a (plane) object (with density δ)



$(\bar{x}, \bar{y})$

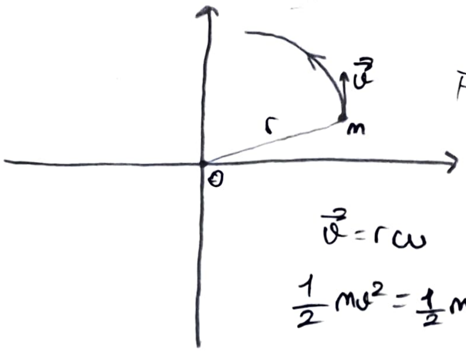
$$\bar{x} = \frac{1}{\text{Mass}(R)}\iint x\,\delta\,dA, \quad \bar{y} = \frac{1}{\text{Mass}(R)}\iint y\,\delta\,dA$$

3-) Moment of inertia

Mass = how hard it is to impart a translation motion to a solid

moment of inertia = rotation motion about that axis.
about an axis

Idea of inertia: Kinetic Energy of a point mass $= \frac{1}{2} m v^2$



For a mass m at distance r and angular velocity ω.

$\vec{v} = r\omega$

$\frac{1}{2} m v^2 = \frac{1}{2} m r^2 \omega^2$

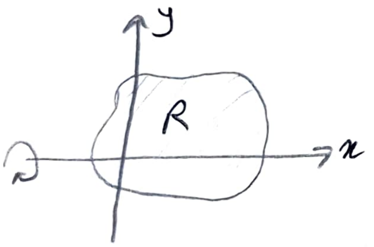
$\{$Moment of inertia $= mr^2\}$

For a solid unit density δ: $\Delta m \approx \delta \Delta A$, has moment of inertia:

$\Delta m\, r^2 = r^2 \delta \Delta A$

$\iint_R r^2 \delta\, dA \Rightarrow$ Moment of inertia about the origin $= I_0$

Rotational KE is $= \frac{1}{2} I_0 \omega^2$

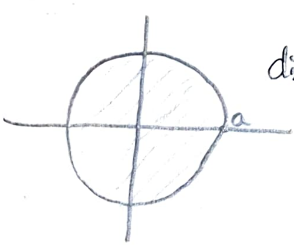


distance to x-axis $= |y|$

$(\quad " \quad)^2 = y^2$

$I_x = \iint y^2 \delta\, dA$

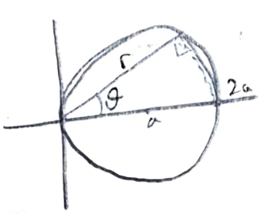
Example:



disk of radius a $(\delta = 1)$

$I_0 = \iint r^2\, dA = \int_0^{2\pi}\int_0^{a} r^2 \cdot r\, dr\, d\theta = 2\pi \frac{a^4}{4} = \frac{\pi a^4}{2}$

About a point on circumference



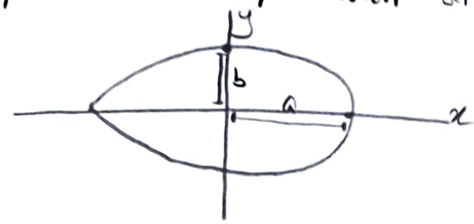
$r = 2a\cos\theta$

$I_0 = \iint r^2 r\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{2a\cos\theta} r^2 r\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\pi/2}\left(\left[\frac{r^4}{4}\right]_0^{2a\cos\theta}\right) d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(2a\cos\theta)^4}{4}\, d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 \cdot a^4 \cos^4\theta\, d\theta = \frac{3}{2}\pi a^4$

Changing Variables in Double Integrals:

Exp. 1: area of ellipse with semiaxises a, b



$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

$$\iint_{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2<1} dx\,dy \underset{\text{Set}}{=} \iint_{u^2+v^2<1} ab\,du\,dv$$

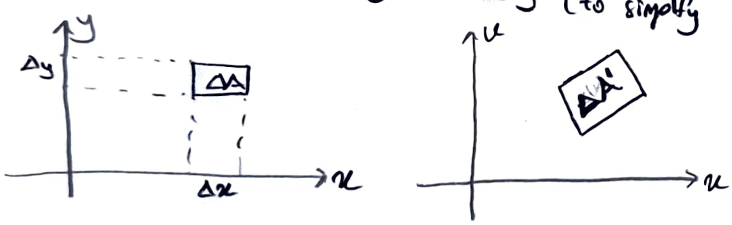
Set

$$\left.\begin{array}{l} \frac{x}{a}=u \Rightarrow du=\frac{1}{a}dx \\ \frac{y}{b}=v \Rightarrow dv=\frac{1}{b}dy \end{array}\right\} du\,dv=\frac{1}{ab}dx\,dy \Rightarrow dx\,dy=ab\,du\,dv$$

$$\to \frac{1}{ab}\iint_{u^2+v^2<1} du\,dv = ab.\ \text{area(unit disk)} = \pi.ab$$

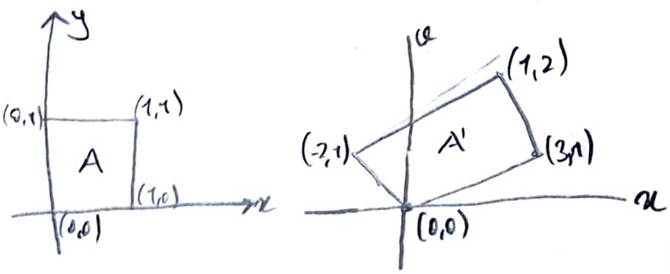
In general: We need to find scalar factor ($dx\,dy$ vs. $du\,dv$)

Example 2: Say $u=3x-2y$, $v=x+y$ (to simplify integrant of the bounds)



Relation between $dA=dx\,dy$ and $dA'=du\,dv$

Area scaling factor for this case does not depend on the choice of rectangle.



det A = 1

det A' = 5

Area(A') = 5 . Area(A)

So $dA'=5dA$

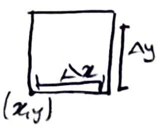
$du\,dv = 5\,dx\,dy$

So $\iint \cdots dx\,dy = \iint \cdots \frac{1}{5}\,du\,dv$

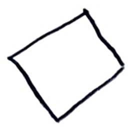
In general case: $u = u(x,y)$, $v = v(x,y)$

$$\Delta u \approx u_x \Delta x + u_y \Delta y$$
$$\Delta v \approx v_x \Delta x + v_y \Delta y \Rightarrow \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

Small xy-rectangle



$\Rightarrow$ small uv-parallelogram



Same argument $\Rightarrow$

$$\langle \Delta x, 0 \rangle \to \langle \Delta u, \Delta v \rangle \approx \langle u_x \Delta x, v_x \Delta x \rangle$$
$$\langle 0, \Delta y \rangle \to \langle \Delta u, \Delta v \rangle \approx \langle u_y \Delta y, v_y \Delta y \rangle$$
$\Big\}$ area' = det(Matrix) $\cdot \Delta x \Delta y$

Jacobian: $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$, Then $du\,dv = |J|\,dx\,dy = \left|\dfrac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$

Exp.1: polar coordinates

$x = r\cos\theta$
$y = r\sin\theta$, $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \Rightarrow \det = r\cos^2\theta + r\sin^2\theta = r$

So $dx\,dy = |r|\,dr\,d\theta = r\,dr\,d\theta$

We can compute u,v in terms of x,y and can compute x,y in terms of u,v. So;

Remark:
$$\left(\frac{\partial(u,v)}{\partial(x,y)}\right) \cdot \left(\frac{\partial(x,y)}{\partial(u,v)}\right) = 1$$
, So we can compute the easiest one

Exp.2:

Compute $\displaystyle\int_0^1\!\!\int_0^1 x^2 y\,dx\,dy$ by changing to $u = x$, $v = xy$

1-) Area element:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x \quad , \text{so } du\,dv = x\,dx\,dy$$
$x \geq 0$

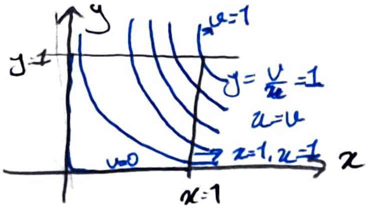
2-) Integrant in terms of u,v

$$x^2 y\,dx\,dy = x^2 y \frac{1}{x} du\,dv = xy\,du\,dv = v\,du\,dv$$

$$\iint_{??} v\,du\,dv$$

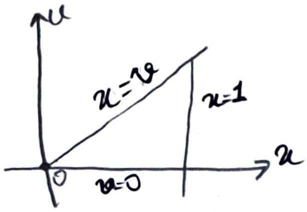
12

8-) Bounds



$$\int_0^1 \int_v^1 u \, du \, dv$$

Or you can draw xv



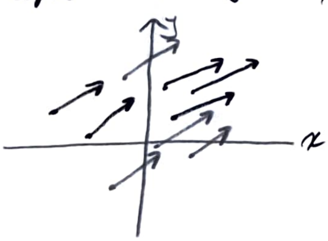
$$\int_0^1 \int_v^1 v \, dx \, dv$$

Vector Fields:

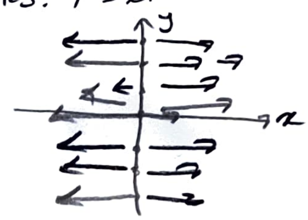
$\vec{F} = M\hat{i} + N\hat{j}$, M and N are functions of x, y. At each point (x, y), $\vec{F}$ a vector that depends on (x, y)

Examples: velocity in a fluid for field $\vec{F}$.

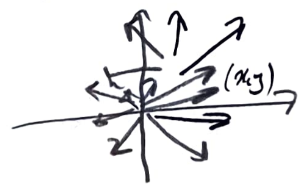
Examples: $\vec{F} = 2\hat{i} + \hat{j}$



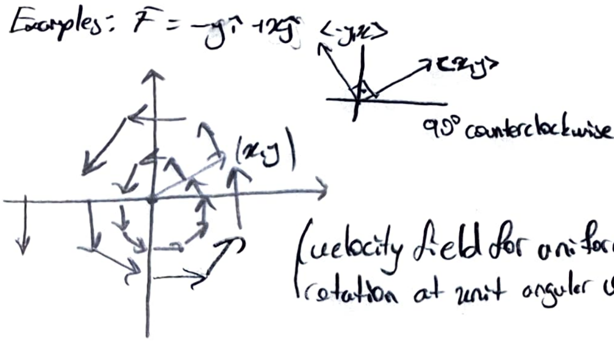
, Examples: $F = x\hat{i}$



Examples: $\vec{F} = x\hat{i} + y\hat{j}$



Examples: $F = -y\hat{i} + x\hat{j}$

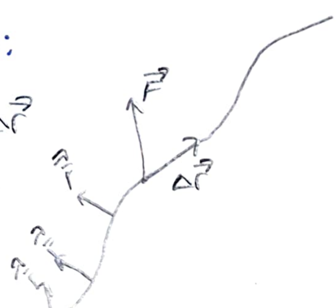


(velocity field for uniform rotation at unit angular velocity)

Work and Line Integrals:

$W = (\text{force})(\text{distance})\ \vec{F} \cdot \Delta\vec{r}$

Along a trajectory C, work adds up to



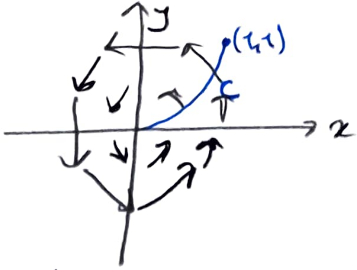
$$W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \lim_{\Delta r_i \to 0} \sum_i \vec{F}_i \cdot \Delta\vec{r}_i = \sum_i \vec{F}_i \cdot \left(\frac{\Delta\vec{r}_i}{\Delta t} \Delta t\right) \overset{\lim_{\Delta t \to 0}}{\Rightarrow} \int_{t_1}^{t_2} \vec{F} \cdot \frac{d\vec{r}}{dt} \, dt$$

velocity vector $\frac{d\vec{r}}{dt}$

Exp. 1: $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$

$C: x = t \quad 0 \le t \le 1$
$\quad y = t^2$



$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt} \cdot dt = \int_0^1 \langle -t^2, t \rangle \cdot \langle 1, 2t \rangle \, dt = \int_0^1 (-t^2 + 2t^2)\, dt = \int_0^1 t^2 dt = \frac{1}{3}$$

$\vec{F} = \langle -y, x \rangle \underset{\substack{x=t \\ y=t^2}}{=} \langle -t^2, t \rangle \quad , \frac{dx}{dt} = 1$

$\frac{dy}{dt} = 2t$

Another way to solve:

$\left.\begin{array}{l} \vec{F} = \langle M, N \rangle \\ d\vec{r} = \langle dx, dy \rangle \end{array}\right\} \vec{F} \cdot d\vec{r} = M dx + N dy$

$\int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy$: Express x, y in terms of a single variable and then substitute.

Let's make the example at above using this.

$$\int_C \vec{F} \cdot d\vec{r} = \int_C -y dx + x dy = \int_C -t^2 dt + t\, 2t dt = \int_0^1 t^2 dt = \frac{1}{3}$$

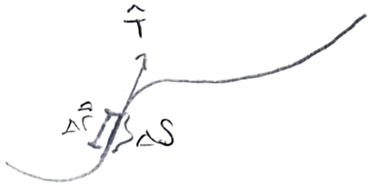
$dx = dt$
$dy = 2t dt$

Note: $\int_C \vec{F} \cdot d\vec{r}$ depends on the line C but not on parameterization.

eg. could do $\begin{cases} x = \sin\theta \\ y = \sin^2\theta \end{cases} \quad 0 \le \theta \le \pi/2$ (not practical). You should chose the elegant way

Geometric Approach:

$d\vec{r} = \langle dx, dy \rangle = \hat{T} ds$



Note: $\underbrace{\frac{d\vec{r}}{dt}}_{\text{velocity}} = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle = \hat{T} \cdot \underbrace{\frac{ds}{dt}}_{\text{speed}}$

So $\int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy = \int_C \vec{F} \cdot \hat{T}\, ds$

when $\vec{F} \perp \hat{T}$, $\int_C \vec{F} \cdot \hat{T} ds = 0$

Example.) C: circle of radius a at origin, counterclockwise

$\vec{F} = -y\hat{\imath} + x\hat{\jmath}$

Now $\vec{F} \parallel \hat{T}$, $\vec{F} \cdot \hat{T} = |\vec{F}| = a$

$$\int_C \vec{F} \cdot \hat{T} ds = \int_C a ds = a \int_C ds = a . \underbrace{\text{arclength}(C)}_{2\pi a} = 2\pi a^2$$

or from computing

$x = a\cos\theta$
$y = a\sin\theta$
$$\int_0^{2\pi} -y\,dx + x\,dy = \int_0^{2\pi} -(a\sin\theta)(-a\sin\theta\,d\theta) + (a\cos\theta)(a\cos\theta\,d\theta) = \int_0^{2\pi} a^2\underbrace{(\sin^2\theta + \cos^2\theta)}_{1}\,d\theta = 2\pi a^2$$

Example:

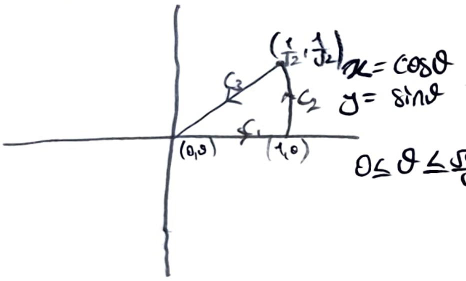
$\vec{F} = \langle x, y\rangle$, $\int_C \vec{F}\cdot d\vec{r}$, $C = C_1 + C_2 + C_3$; enclosing sector of unit disk $0 \le \theta \le \frac{\pi}{4}$

$$\int_{C_i} y\,dx + x\,dy$$

1-) x-axis: from $(0,0)$ to $(1,0)$: $y=0$, $dy=0$

$$\int_{C_1} 0\,dx + 0 = 0$$

2-) C_2: portion of unit circle



$0 \le \theta \le \frac{\pi}{4}$

$dx = -\sin\theta\,d\theta$
$dy = \cos\theta\,d\theta$, $\int_{C_2} y\,dx + x\,dy = \int_0^{\pi/4} \sin\theta\cdot(-\sin\theta\,d\theta) + \cos\theta(\cos\theta\,d\theta)$

$$= \int_0^{\pi/4} \cos^2\theta - \sin^2\theta\,d\theta = \int_0^{\pi/4} \cos 2\theta\,d\theta = \frac{1}{2}\sin 2\theta\Big|_0^{\pi/4} = \frac{1}{2}(\sin\tfrac{\pi}{2} - \sin 0)$$
$$= \frac{1}{2}$$

3-) $\int_{C_3} y\,dx + x\,dy$

could do $x = \frac{1}{\sqrt2} - \frac{1}{\sqrt2}t$ $\quad 0 \le t \le 1$
$y = \frac{1}{\sqrt2} - \frac{1}{\sqrt2}t$

we can do slightly better: $x = t$, $y = t$ $\quad t$ from 0 to $\frac{1}{\sqrt2}$ gives us $(-C_3)$ means C_3 backwards.

$\int_{-C_3} = -\int_{C_3}$, $dx = dt$, $dy = dt$, $\int_{-C_3} y\,dx + x\,dy = \int_0^{1/\sqrt2} t\,dt + t\,dt = t^2\Big|_0^{1/\sqrt2} = \frac{1}{2}$

$$\int_{C_3} y\,dx + x\,dy = -\frac{1}{2}$$

Total work $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + \frac{1}{2} - \frac{1}{2} = 0$

Special Case:

$\vec{F} = \nabla f$, $f(x,y)$ is called potential. Then we can simplify the evaluation of

$\int_C \vec{F} \cdot d\vec{r}$

Fundamental Theorem of Calculus for Line Integrals:

$$\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)$$

$$\int_C f_x\,dx + f_y\,dy = \int_C df = f(P_1) - f(P_0)$$

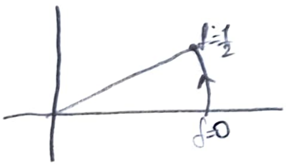
Proof:

$$\int_C \nabla f \cdot d\vec{r} = \int_C f_x\,dx + f_y\,dy \qquad C: x = x(t), \; dx = x'(t) \qquad t_0 \le 0 \le t_1$$
$$y = y(t), \; dy = y'(t)$$

$$\int_C \nabla f\, d\vec{r} = \int_C \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt} \right) dt = \int_C \frac{df}{dt} dt = \int_{t_0}^{t_1} \frac{df}{dt} dt = \Big[f(x(t), y(t)) \Big]_{t_0}^{t_1} = f(P_1) - f(P_0).$$

Example: $\vec{F} = \langle y, x \rangle$, $f(x,y) = xy$

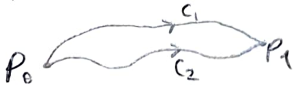
$$S_0 : \int_{C_2} \vec{F} \cdot d\vec{r} = f\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) - f(1,0) = \tfrac{1}{2}$$



Warning: The things that we made above only implies if $\vec{F}$ is a gradient field.

Consequences of Fundamental Theorem

* Path-independence



$$\int_{C_1} \nabla f\, d\vec{r} = \int_{C_2} \nabla f\, d\vec{r}$$

* $\vec{F} = \nabla f$ is conservative

C closed curve $\Rightarrow \int_C \vec{F} \cdot d\vec{r} = 0$

Equavilent Properties

1-) $\vec{F}$ is conservative : $\int_C \vec{F} \cdot d\vec{r} = 0$ along all closed curves C.

2-) $\int_C \vec{F} d\vec{r}$ is path independent

3-) $\vec{F}$ is a gradient field if $\vec{F} = \langle f_x, f_y \rangle$

4-) $M\,dx + N\,dy$ is an exact differential $= df$

Testing whether $\vec{F} = \langle M, N \rangle$ is gradient field?

Say if $\vec{F} = \nabla f$: $M = f_x$, $N = f_y$ then $f_{yx} = f_{xy} \Rightarrow M_y = N_x$

Conversely : if $\vec{F} = \langle M, N \rangle$ is differentiable everywhere and $M_y = N_x$ then $\vec{F}$ is a gradient field

Example : $\vec{F} = \underbrace{-y}_{M}\hat{i} + \underbrace{x}_{N}\hat{j}$

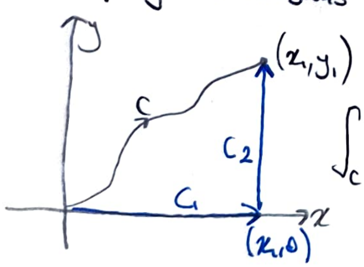
$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$, F is not a gradient field.

Example : $\vec{F} = (4x^2 + axy)\hat{i} + (3y^2 + 4x^2)\hat{j}$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= ax \\ \frac{\partial N}{\partial x} &= 8x \end{aligned}\right\} a = 8$$

Let's find its potential.

1-) Computing Line Integrals



$$\int_C \vec{F} \cdot d\vec{r} = f(x_1, y_1) - f(0,0) \Rightarrow f(x_1, y_1) = \int_C \vec{F} \cdot d\vec{r} + \underbrace{f(0,0)}_{\text{constant}}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (4x^2 + 8xy)\,dx + (3y^2 + 4x^2)\,dy$$

$\vec{F} = \langle 4x^2 + 8xy, 3y^2 + 4x^2 \rangle$

on C_1 : x goes from 0 to x_1 $\Rightarrow \int_0^{x_1} (4x^2)\,dx = \frac{4}{3} x_1^3$
$y = 0$, $dy = 0$

on C_2 : y goes from 0 to y_1 $\Rightarrow \int_0^{y_1} (3y^2 + 4x_1^2)\,dy$
$x = x_1$ $dx = 0$

$$= \left[y^3 + 4x_1^2 y \right]_0^{y_1} = y_1^3 + 4x_1^2 y_1$$

So $f(x,y) = \frac{4}{3}x^3 + y^3 + 4x^2 y + c$.

But we could also find it with another way.

2-) Antiderivatives

Want to solve $\begin{cases} f_x = 6x^2+8xy \ (1) \\ f_y = 3y^2+4x^2 \ (2) \end{cases}$

$(1) \Rightarrow f = \frac{6}{3}x^3 + 4x^2y + g(y)$

$f_y = 4x^2 + g'(y)$ – match this with eq. (2)

$4x^2 + g'(y) = 4x^2 + 3y^2 \Rightarrow g'(y) = 3y^2 \Rightarrow g(y) = y^3 + C$

$f = \frac{6}{3}x^3 + 4x^2y + y^3 + C$

Definition: $\operatorname{curl}(\vec{F}) = N_x - M_y$

$\vec{F} = \langle M, N \rangle$ is a gradient field in a region of the plane

$\Rightarrow$ conservative:

$$\oint_C \vec{F}\cdot d\vec{r} = 0 \text{ for closed } C$$

If $\vec{F}$ defined in entire plane

$N_x = M_y$

Green's Theorem:

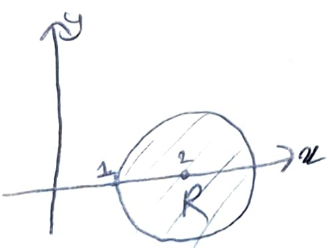
If C is a closed curve, enclosing a region R, counterclockwise, $\vec{F}$ vector field defined & differentiable in R

then
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl} \vec{F} \, dA$$

$$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA$$

Example:

Let C circle of radius 1 centered $(2,0)$



$$\oint \underbrace{ye^{-x}}_{M}\,dx + \underbrace{\left(\tfrac{1}{2}x - e^{-x}\right)}_{N} dy$$

directly:

$x = 2 + \cos\theta$, $dx = -\sin\theta\, d\theta$

$y = \sin\theta$, $dy = \cos\theta\, d\theta$, looks a bit long, a bit ...

So use Green's theorem

$$\iint_R \operatorname{curl} \vec{F}\, dA = \iint_R (N_x - M_y)\, dA = \iint_R \left(\tfrac{1}{2} + e^{-x}\right) - \left(e^{-x}\right) dA = \iint_R \left(\tfrac{1}{2}\right) dA = \underbrace{\text{Area}(R)}_{\pi} \cdot \underbrace{\bar{x}}_{2} = 2\pi$$

$$\bar{x} = \frac{1}{\text{Area}} \iint x \, dA$$

$$\left(\frac{1}{\text{mass}} \iint x \,\delta\, dA\right)$$

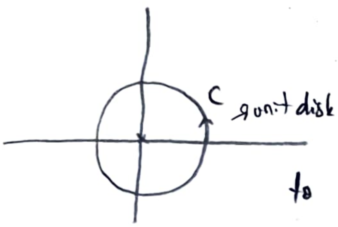
$$\oint \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl} \vec{F}\, dA = 0 \text{ if } \operatorname{curl} \vec{F} = 0$$

This proves : if curl $\vec{F}=0$ everywhere in R then $\oint \vec{F}\cdot d\vec{r} = 0$

Consequences: if $\vec{F}$ defined everywhere in the plane, curl $\vec{F}=0$ then, $\vec{F}$ is conservative.

Proof: 

$$\oint \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl} \vec{F}\, dA = \iint 0\, dA = 0$$



Vector field is not defined on origin, so you can't apply Green's thr. to the vector field, when C encloses the origin.

Proof of Green's Theorem:

$$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA$$

Observe: $\oint_C M\,dx = \iint_R -M_y\, dA$ (where $N=0$)

Then we can do the same for N and summing them will give the general case

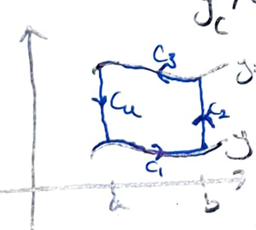
$$\oint_{C_1} M\,dx = \iint_{R_1} -M_y\, dA \quad , \quad \oint_{C_2} M\,dx = \iint_{R_2} -M_y\, dA$$

$$\oint_C M\,dx = \oint_{C_1} + \oint_{C_2} = \int_R -M\,dA$$

Cut R into vertically simple pieces, $a < x < b$, $f_1(x) < y < f_2(x)$

Main Step:

$\oint_C M\,dx = \iint_R -M_y\, dA$ if R vertically simple, C = boundary of R counterclockwise



$$\int_{C_1} M\,dx = \int_a^b M(x, f_1(x))\,dx \quad , \quad \int_{C_2} M(x,y)\,dx = 0 \quad , \quad \int_{C_3} = \int_b^a M(x, f_2(x))\,dx$$

$x=b$, $dx=0$

$$\int_{C_4} = 0 \; ; \; \oint_C M\,dx = \int_a^b M(x, f_1(x))\,dx - \int_a^b M(x, f_2(x))\,dx \quad , \quad \iint_R -M_y\, dA = \int_a^b \int_{f_1(x)}^{f_2(x)} -\frac{\partial M}{\partial y}\, dy\, dx$$

Inner: $\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\, dy = M(x, f_2(x)) - M(x, f_1(x))$

So

$$\iint_R \frac{\partial M}{\partial y}\, dy\, dx = \int_a^b M(x, f_2(x)) - M(x, f_1(x)) = \oint_C M\, dx.$$

We can do the same for N and sum them up to get to final form of Green's Theorem.
End of Proof.

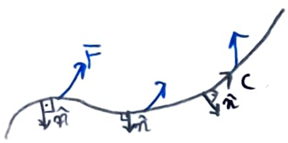
Flux:

c plane curve, $\vec{F}$ vector field

Flux of $\vec{F}$ across C

$$\oint_C \vec{F} \hat{n}\, ds$$

$\hat{n}$ = normal unit vector to C, 90° clockwise from $\hat{T}$

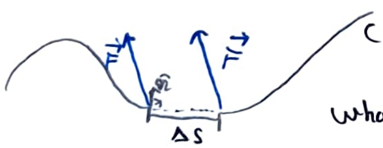


Break C into small pieces ΔS, Flux $= \lim_{\Delta S \to 0} \left(\sum \vec{F} \hat{n} \Delta S \right)$

Work: $\int_C \vec{F} d\vec{r} = \int_C \vec{F} \cdot \hat{T}\, ds$

Interpretation: For $\vec{F}$ a velocity field, Flux measures how much fluid passes through c per unit time

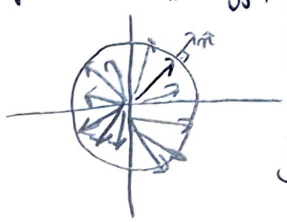
what passes through a portion of C in unit time



Area = base . height
$= \Delta S \ (\vec{F} \cdot \hat{n})$

what flows across C left-to-right is counted positively, right-to-left negatively.

Exp.1: $\vec{F} = x\hat{i} + y\hat{j}$, C circle of radius a, counterclockwise



$\vec{F} \parallel \hat{n}$, $\vec{F} \cdot \hat{n} = |\vec{F}||\hat{n}| = a$

$$\oint_C \vec{F} \hat{n}\, ds = \oint_C a\, ds = a . \text{length} = 2\pi a . a = 2\pi a^2$$

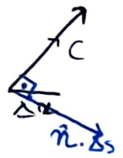
Exp.2: $\vec{F} = \langle -y, x \rangle$, Same c

$\vec{F}$ tangent to C, $\vec{F} \cdot \hat{n} = 0$, Flux = 0

How do we do calculation using components.

Remember : $d\vec{r} = \vec{T}ds = \langle dx, dy \rangle$

$\hat{n}$ is $\vec{T}$ rotated $90°$ clockwise , $\hat{n} = \langle dy, -dx \rangle$



$\vec{T}\Delta s = \Delta\vec{r} = \langle \Delta x, \Delta y \rangle$

Say $\vec{F} = \langle P, Q \rangle$ then

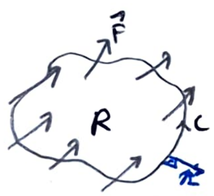
$$\int_C \vec{F}\cdot\hat{n}\,ds = \int_C \langle P, Q \rangle \cdot \langle dy, -dx \rangle = \int_C -Q\,dx + P\,dy$$

Green's theorem for Flux:

If C is a curve that encloses a region counter clockwise and $\vec{F}$ defined in R, then

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \underbrace{\operatorname{div}\vec{F}}_{\text{divergence of } \vec{F}}\,dA \quad , \quad \operatorname{div}\langle P, Q \rangle = P_x + Q_y$$

{Flux out of R through C}



Proof:

$$\oint_C -Q\,dx + P\,dy = \iint_R (P_x + Q_y)\,dA$$ From the previous proof, that's also true.

Example: $\vec{F} = x\hat{i} + y\hat{j}$, C = circle of radius a.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 1 + 1 = 2 \quad , \quad \oint \vec{F}\cdot\hat{n}\,ds = \iint_R 2\,dA = 2 \cdot \text{Area}(R) = 2\pi a^2$$

Interpretation of $\operatorname{div}\vec{F}$:

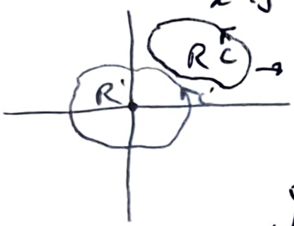
1-) measures how much the flow expanding

2-) "Source rate" = amount of fluid added to the system per unit time and area

We've seen Green's thm: $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \text{curl}\,\vec{F}\,dA$, $\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \text{div}\,\vec{F}\,dA$

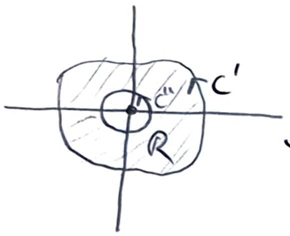
Only works if $\vec{F}$ and derivatives are defined everywhere in region R.

Example: $\vec{F} = \frac{-y\hat{i} + x\hat{j}}{x^2+y^2}$; $\vec{F}$ not def'n at origin, curl $\vec{F} = 0$ everywhere else

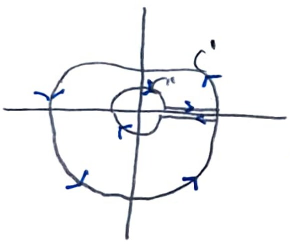


$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\,dA = 0$

$\oint_{C'} \vec{F}\cdot\hat{n}\,ds = ?$, can't use Green directly



$\oint_{C'} \vec{F}\cdot d\vec{r} - \oint_{C''} \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\,dA = 0$ (in our case curl$=0$)



Total Line Integral

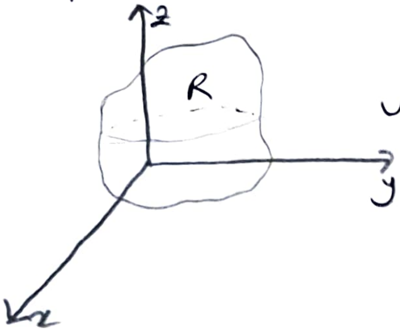
$= \oint_{C'} - \int_{C''}$

Definition: a connected region in the plane is simply connected if the interior of an closed curve in R is also contained in R

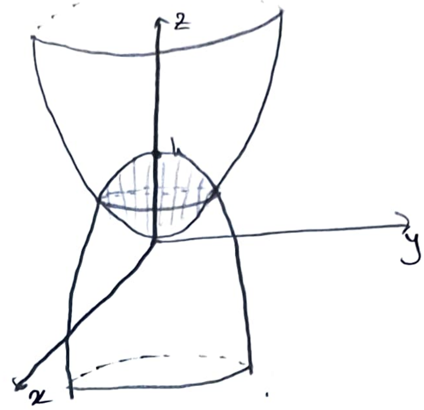
If domain where $\vec{F}$ is defined (+differentiable) is simply connected, then can always apply Green's thm.

If curl $\vec{F} = 0$ and the domain of def'n is simply connected then $\vec{F}$ is concervative.

Triple Integrals:



$$\iiint_R f \underbrace{dV}_{\text{volume}} \Rightarrow dV = dz\,dy\,dx \text{ (order can change for the problem)}$$

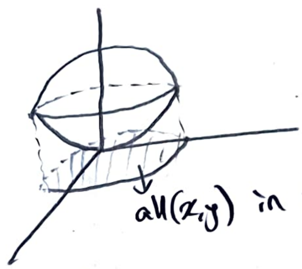


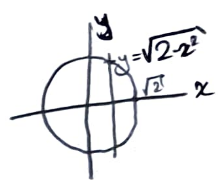
Example: Region between:

$z = x^2+y^2$, $z = 4-x^2-y^2$

$$V = \iiint 1\,dV$$

$$\int\int\int_{x^2+y^2}^{4-x^2+y^2} dz\,dy\,dx$$





$$\int_{\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{-\sqrt{2-x^2}}\int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx$$

wherever

$z_{bottom} < z_{top}$

$x^2+y^2 < 4-x^2-y^2$

$2x^2+2y^2 < 4$

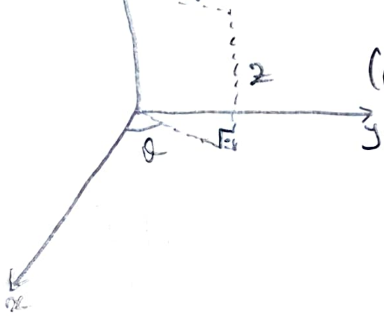
$x^2+y^2 < 2$: disk of radius $\sqrt{2}$

Better Way: Use polar coordinates instead xy.

Inner: $\displaystyle\int_{x^2+y^2}^{4-x^2-y^2} dz = \left[z\right]_{x^2+y^2}^{4-x^2-y^2} = 4-2x^2+2y^2$

$$\int_{-\sqrt{2}}^{\sqrt{2}}\int_{\sqrt{2-x^2}}^{\sqrt{2-x^2}} 4-2x^2-2y^2\,dy\,dx = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{4-r^2} dz\,r\,dr\,d\theta \text{ (easier to evaluate)}$$

{This is called cylindrical coordinates}



(r,θ,z)

$x = r\cos\theta$

$y = r\sin\theta$

Applications: – Mass: density $\delta = \frac{\Delta m}{\Delta V}$, $\underset{\text{mass element}}{dm} = \delta \, dV$ $\iiint_R \delta \, dV = \text{Mass}$

– Avg. Value of $f(x,y,z)$ in R:

$$\bar{f} = \frac{1}{\text{Vol}(R)} \iiint_R f \, dV \quad \text{or with weighted avg.} \quad \frac{1}{\text{Vol}(R)} \iiint_R \delta f \, dV.$$

– Center of mass: $(\bar{x}, \bar{y}, \bar{z})$

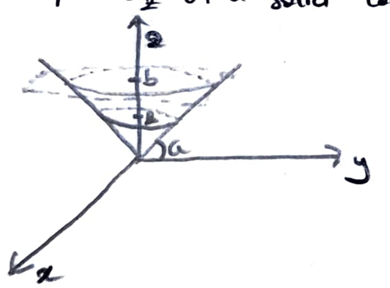
$$\bar{x} = \frac{1}{\text{Mass}} \iiint_R x \, \delta \, dV \quad \ldots$$

– Moment of Inertia: Respect to an axis

$$\iiint_R (\text{distance to axis})^2 \, \delta \, dV$$

$$I_z = \iiint_R r^2 \delta \, dV = \iiint_R (x^2+y^2) \delta \, dV, \quad I_x = \iiint_R (y^2+z^2) \delta \, dV, \quad I_y = \iiint_R (x^2+z^2) \delta \, dV$$

Example: I_z of a solid cone between $z = ar$, $z = b$ and $\delta = 1$

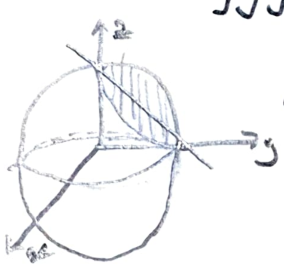


$$I_z = \int \int \int r^2 r \, dr \, d\theta \, dz$$

Eqn of cone: $z = ar \Rightarrow r = z/a$

$$I_z = \int_0^b \int_0^{2\pi} \int_0^{z/a} r^2 r \, dr \, d\theta \, dz = \frac{\pi b^5}{10a^4}$$

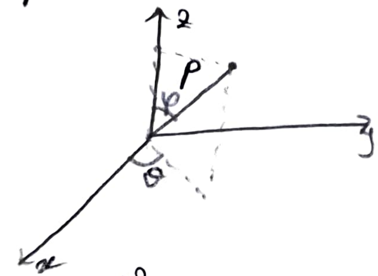
Example 3: Setup $\iiint_R$ for region $z \geq 1-y$ inside unit ball centered at origin



$x^2+y^2+z^2 \leq 1$

$$\int_0^1 \int_{-\sqrt{2y-2y^2}}^{\sqrt{2y^2}} \int_{1-y}^{\sqrt{1-x^2-y^2}} dz \, dx \, dy$$

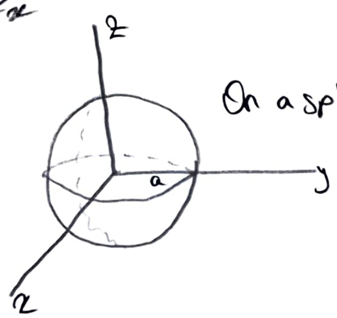
Spherical Coordinates:



ρ = rho = distance from origin

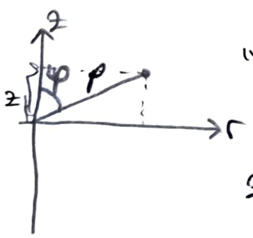
$\varphi = \phi$ = phi = angle down from +z axis

$0 \leq \varphi \leq \pi$, θ = angle counterclockwise from x-axis



On a sphere $\rho = a$, $\varphi \leftrightarrow$ latitude

$\theta \leftrightarrow$ longitude



"polar coordinates in rz-plane"

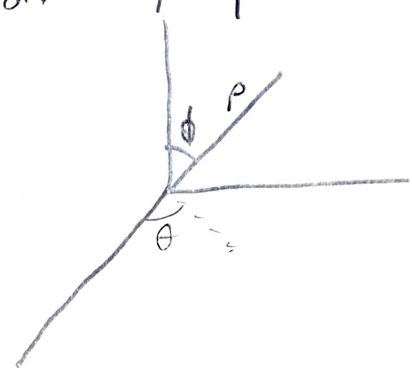
$z = \rho \cdot \cos\phi$, $r = \rho \sin\phi$; $x = r\cos\theta = \rho \sin\phi \cos\theta$

$y = r\sin\theta = \rho \sin\phi \sin\theta$

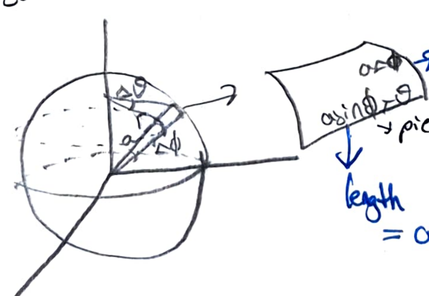
$\rho = \sqrt{r^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$ $z = \rho\cos\phi$

Triple Integral in spherical coordinates

$dV = \dots d\rho \, d\varphi \, d\theta$, $\Delta V \approx \quad \Delta\rho \, \Delta\varphi \, \Delta\theta$



Surface area on sphere of radius a?



$\rightarrow$ length $= a\Delta\phi$

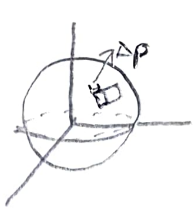
$\rightarrow$ piece of circle of radius r.

$r = a\sin\phi$

length $= a\sin\phi\,\Delta\theta$

$\Delta S \approx (a\sin\phi\,\Delta\theta)(a\,\Delta\phi) = a^2 \sin\phi\,\Delta\theta\,\Delta\phi$

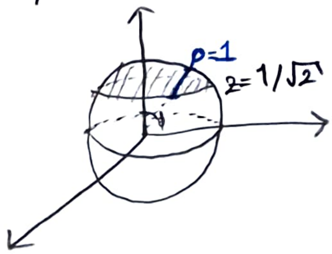
$dS = a^2 \sin\phi \, d\theta \, d\phi$



$\Delta V \approx \Delta\rho \cdot \Delta S = \rho^2 \sin\phi \, \Delta\rho \, \Delta\theta \, \Delta\phi$

$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

Example: Vol of portion of unit sphere above $z = 1/\sqrt{2}$

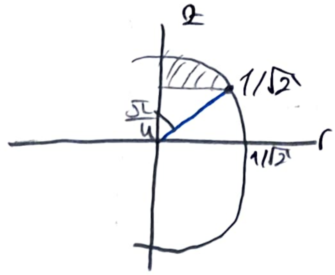


$$\int\int\int_{\frac{1}{\sqrt{2}}\sec\phi}^{1} 1\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$z = 1/\sqrt{2}$

$\rho\cos\phi = 1/\sqrt{2}$, ,

$\rho = \frac{1}{\sqrt{2}} \cdot \sec\phi$



here , $\rho = 1$ (sphere)

and $\rho\cos\phi = \frac{1}{\sqrt{2}}$

so $\cos\phi = \frac{1}{\sqrt{2}}$

$\phi = \pi/4$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_{\frac{1}{\sqrt{2}}\sec\varphi}^{1} 1\,\rho^2 \sin\phi \, d\rho \, d\theta \, d\varphi = \frac{2\pi}{3} - \frac{5\pi}{6\sqrt{2}}$$

Applications:

Gravitational Attraction

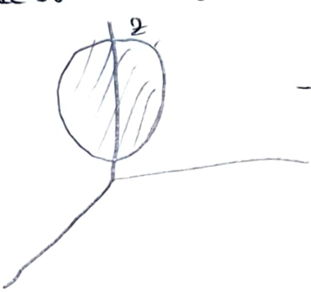
Gravitational Force Extracted by ΔM at (x,y,z) on a mass m at origin

$$|\vec{F}| = G.\frac{m\Delta M}{\rho^2} \quad , \quad \text{dir } \vec{F} = \frac{\langle x,y,z\rangle}{\rho} \quad , \quad \vec{F} = \frac{G.m\Delta M}{\rho^2} \cdot \frac{\langle x,y,z\rangle}{\rho}$$

So integrating

$\Delta M \approx \delta\Delta V \Rightarrow \iiint \frac{G.m \langle x,y,z\rangle}{\rho^3} \delta \, dV = \vec{F}$

-place solid so z axis is an axis of symmetry



Then $\vec{F} = \langle 0, 0, F\rangle$

z-component:

$$Gm\iiint \frac{z}{\rho^3}\delta \, dV = Gm\iiint \frac{\rho\cos\varphi}{\rho^3}\delta\rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= Gm\iiint \sin\varphi\cos\varphi \, \delta \, d\rho \, d\theta \, d\varphi \quad \{ z \text{ component of } \vec{F} \}$$

Vector Fields in Space:

At every point, $\vec{F} = \langle P, Q, R \rangle$; P, Q, R are functions of x, y, z

Examples: Force fields

(eg gravitational attraction of a solid mass M at $(0,0,0)$ on a mass m at (x, y, z)

$\vec{F}$ directed towards origin, magnitude $\sim c/\rho^2$

$$\vec{F} = \frac{-c \langle x, y, z \rangle}{\rho^3}$$

Velocity fields, $\vec{v}$

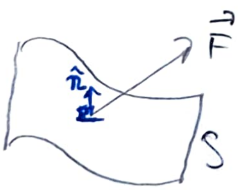
gradient fields, $u = u(x,y,z)$, $\nabla u = \langle u_x, u_y, u_z \rangle$

Flux: In 2d we were taking a line integral, now it's 3d and we will take surface integral

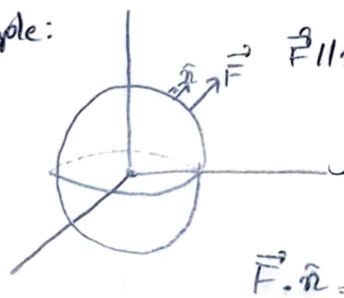
$\int \vec{F} \hat{n} \, ds \Rightarrow 2d$

$$\iint_S \vec{F} \hat{n} \, \underbrace{dS}_{\text{surface area element}} \Rightarrow 3D \quad , \quad d\vec{S} = \hat{n} \cdot dS$$

$\vec{F}$, vector field ; S, surface in space



Example:



$\vec{F} \parallel \hat{n}$ sphere with radius a.

$$\iint_S \vec{F} \, d\vec{S} = \iint_S \vec{F} \hat{n} \, dS \quad , \quad \hat{n} = \frac{1}{a} \langle x, y, z \rangle$$

$$\vec{F} \cdot \hat{n} = |\vec{F}| = a$$

$$\iint_S a \, dS = a \iint dS \; , \; \begin{matrix} = \text{area}(S) \\ = 4\pi a^2 \end{matrix} \Rightarrow 4\pi a^3$$

Example: Sphere radius a

$\vec{H} = z\hat{k}$, $\vec{H}\cdot\hat{n} = \langle 0,0,z\rangle = \frac{z^2}{a}$

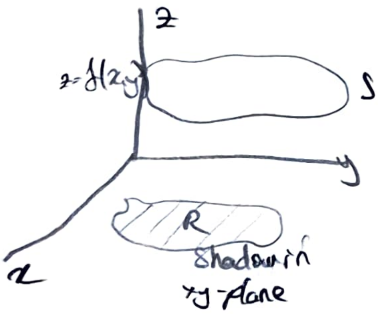
$$\iint_S \vec{H}\hat{n}\,dS = \iint_S \frac{z^2}{a}\,dS$$

$\Delta S \approx (a\sin\varphi\,\Delta\theta)(a\,\Delta\varphi) = a^2\sin\varphi\,\Delta\theta\,\Delta\varphi \Rightarrow dS = a^2\sin\varphi\,\Delta\theta\,\Delta\varphi$

$z = a\cos\varphi$

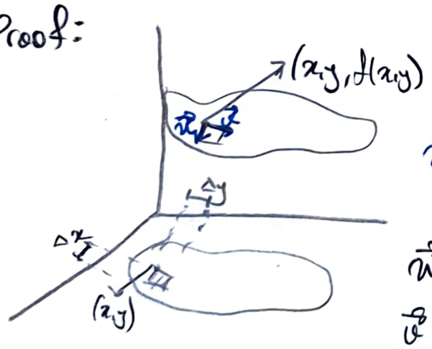
$$\iint_S \frac{a^2\cos^2\varphi}{a}\,a^2\sin\varphi\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{z^2}{a}\,dS = 2\pi a^3\left(-\frac{1}{3}\cos^3\varphi\right)_0^{\pi} = \frac{4}{3}\pi a^3$$

Def'n: If S is graph of $z = f(x,y)$



$\hat{n}\,dS = \pm\langle -f_x, -f_y, 1\rangle\,dx\,dy$

Proof:



$\vec{u}\times\vec{v} = \Delta S\,\hat{n}$

$\vec{u}$: from $(x,y,f(x,y))$ to $(x+\Delta x, y, f(x+\Delta x,y)) \simeq f(x,y) + \Delta x f_x$

$\vec{v}$: from $(x,y,f(x,y))$ to $(x, y+\Delta y, f(x,y+\Delta y)) \simeq f(x,y) + \Delta y f_y$

$\vec{u} \simeq \langle \Delta x, 0, f_x\Delta x\rangle = \langle 1, 0, f_x\rangle\cdot\Delta x$

$\vec{v} \simeq \langle 0, y, f_y\Delta y\rangle = \langle 0, 1, f_y\rangle\cdot\Delta y$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}\Delta x\,\Delta y = \langle -f_x, -f_y, 1\rangle\,\Delta x\,\Delta y = \Delta S\,\hat{n}$$

So $\hat{n}\,dS = \pm\langle -f_x, -f_y, 1\rangle\,dx\,dy$, +- because we choose the dir. of normal vector.

Divergence Theorem:

If S is a closed surface enclosing a region D, oriented with $\hat{n}$ outwards and $\vec{F}$ defined and differentiable everywhere in D, then

$$\oiint_S \vec{F} \cdot dS = \iiint_D \operatorname{div} \vec{F} \, dV \quad \text{where } \operatorname{div}(P\hat{i} + Q\hat{j} + R\hat{k}) = P_x + Q_y + R_z$$

Before proof;

∇ "del" notation.

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle, \quad \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$

$$\nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \Rightarrow \text{divergence.}$$

{To understand the rememberence of it.}

~~Proof of Divergence Theorem:~~

Proof of Divergence Theorem:

$\oiint_S \langle 0,0,R\rangle \cdot \hat{n}\,dS = \iiint_D R_z\,dV$, then we're gonna do it for each component and their sum will be general case.

If region D is vertically simple

$$\iiint_D R_z\,dV = \iint_U \int_{z_1(x,y)}^{z_2(x,y)} R_z\,dz\,dx\,dy = \iint_U \left[R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))\right] dx\,dy$$

$\oiint_{S=\text{bottom}+\text{top}+\text{sides}} \langle 0,0,R\rangle \cdot \hat{n}\,dS$, Top: $\hat{n}\,dS = \langle -\frac{\partial z_2}{\partial x}, -\frac{\partial z_2}{\partial y}, 1\rangle\,dx\,dy$

$$\langle 0,0,R\rangle \cdot \hat{n}\,dS = R\,dx\,dy$$

$$\oiint_{top} \langle 0,0,R\rangle \cdot \hat{n}\,dS = \oiint_{top} R\,dx\,dy = \iint_U R(x,y,z_2(x,y))\,dx\,dy$$

Bottom:

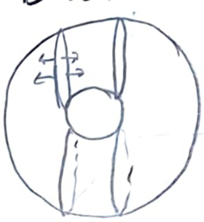
$\hat{n}\,dS = \langle \frac{\partial z_1}{\partial x}, +\frac{\partial z_1}{\partial y}, -1\rangle\,dx\,dy$, $\langle 0,0,R\rangle \cdot \hat{n}\,dS = -R\,dx\,dy$

$$\iint_{bottom} -R\,dx\,dy = \iint_U -R(x,y,z_1(x,y))$$

Sides are vertical so Flux in sides $=0$.

So; $\iiint_D R_z\,dV = \oiint_{top+bottom+sides} \langle 0,0,R\rangle \cdot \hat{n}\,dS$

If D is not vertically simple then we'll cut it into vertically simple pieces



End of proof.

Diffusion Equation:

u = concentration at a given point $= u(x,y,z,t)$

$$\frac{\partial u}{\partial t} = k\nabla^2 u = k\,\nabla\cdot\nabla u = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$, also called heat equation

Fluid flows from high concentration to low concentration.

$\vec{F}$ is directed along $-\nabla u$

In fact : $\vec{F} = -k\nabla u$

Flux out of D through S:

$$\oint\!\!\oint_S \vec{F}\cdot\hat{n}\,dS = \text{amount of fluid through } S \text{ per unit time} = \frac{d}{dt}\iiint_D u\,dV$$

$$\iiint_D \operatorname{div}\vec{F}\,dV = \oint\!\!\oint \vec{F}\cdot\hat{n}\,dS = -\iiint_D u\,dV = -\iiint_D \frac{\partial u}{\partial t}\,dV$$

$$\frac{d}{dt}\left(\left(\sum_i u(x_i,y_i,z_i,t)\Delta V_i\right)\right) = \sum_i \frac{\partial u}{\partial t}(x_i,y_i,z_i,t)\,\Delta V_i$$

For any region D

$$\frac{1}{\text{vol}(D)}\iiint \operatorname{div}\vec{F}\,dV = \frac{1}{\text{vol}(D)}\iiint_D -\frac{\partial u}{\partial t}\,dV$$

avg. of $(\operatorname{div}\vec{F})$ in D = avg. of $\left(\frac{-\partial u}{\partial t}\right)$ in D, so $\operatorname{div}\vec{F} = -\frac{\partial u}{\partial t}$

Line Integrals In Space:

Vector field: $\vec{F} = \langle P\hat{i}, Q\hat{j}, R\hat{k} \rangle$, Curve C in space : Work $= \int_C \vec{F}\cdot d\vec{r}$

$d\vec{r} = \langle dx, dy, dz \rangle$

Test For Gradient Fields:

$\vec{F} = \langle P, Q, R \rangle \overset{?}{=} \langle f_x, f_y, f_z \rangle$, If so then;

$P_y = f_{xy} = f_{yx} = Q_x$, $P_z = f_{zx} = f_{xz} = R_x$, $Q_z = f_{zy} = f_{yz} = R_y$.

=> criterion:

$\vec{F} = \langle P, Q, R \rangle$ (defined in a simply connected region)

is a gradient field $\iff \{P_y = Q_x , P_z = R_x , Q_z = R_y\}$

($P\,dx + Q\,dy + R\,dz$ is exact ; ($= df$))

Example: For which a and b is

$axy\,dx + (x^2 + z^3)dy + (byz^2 - 4z^3)\,dz$ exact?

$P_y = ax = 2x = Q_x \Rightarrow a = 2$

$P_z = 0 = 0 = R_x$

$Q_z = 3z^2 = bz^2 = R_y \Rightarrow b = 3$

For $a=2$ and $b=3$ this is exact.

So what's the potential.

1 $f(x_1, y_1, z_1) - f(0,0,0)$

$\int_C \vec{F}\cdot d\vec{r}$

2 Antiderivatives:

Want:

$f_x = 2xy$, $f_y = x^2 + z^3$, $f_z = 3yz^2 - 4z^3$

- $f_x = 2xy \overset{\int dx}{\Rightarrow} f = x^2y + g(y,z)$

$\downarrow \partial/\partial y$

- $f_y = x^2 + z^3 = x^2 + g'_y$, so $g'_y = z^3$

$\xrightarrow{\int dy} g = yz^3 + h(z)$

- $f_z = 3yz^2 - 4z^3 = 3yz^2 + h'$

by $\frac{\partial}{\partial z}$ of f: $f = x^2y + g = x^2y + yz^3 + h(z)$

so $h'(z) = -4z^3$, $h = -z^4 + C$.

$f = x^2y + yz^3 - z^4 + C$

Curl In 3D:

if $\vec{F} = \langle P\hat{i}, Q\hat{j}, R\hat{k} \rangle$

$\text{curl}\,\vec{F} = (R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k}$

If f is defined in a simply connected region

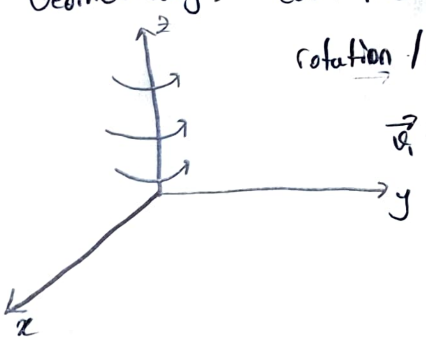
$\vec{F}$ = conservative $\Leftrightarrow \text{curl}\,\vec{F} = 0$.

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$, we've seen $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

$\nabla \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \text{div}\,\vec{F}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial}{\partial y}R - \frac{\partial}{\partial z}Q\right)\hat{i} - \left(\frac{\partial}{\partial x}R - \frac{\partial}{\partial z}P\right)\hat{j} + \left(\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P\right)\hat{k} = \text{curl}\,\vec{F}$$

Geometrically: "curl measures rotation component in a velocity field"



rotation / z-axis at angular velocity ω

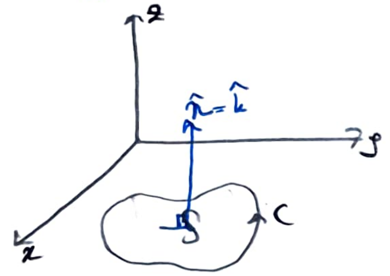
$\vec{v} = \langle -\omega y, \omega x, 0 \rangle$; $\text{curl}\,\vec{v} = 2\omega\hat{k}$

STOKE'S THM:

If C is a closed curve in space and S is any surface bounded by C and $\vec{F}$ defined everywhere on S.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot \hat{n}\, dS$$

Comparing Stoke's with Green's Thm:



S: portion of xy-plane, bounded by a curve c (counterclockwise)

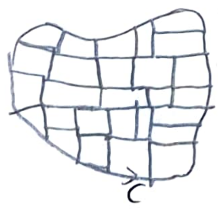
$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy \overset{\text{Stokes}}{=} \iint_S (\nabla\times\vec{F})\cdot\underset{\hat{k}}{\cancel{\hat{n}}}\, dS \quad , \quad (\nabla\times\vec{F})\cdot\hat{k} = z\text{-component of curl } \vec{F}$$

$$\vec{F} = \langle P, Q, R\rangle \quad \Rightarrow \quad = \iint_S (Q_x - P_y)\, dx\, dy$$

Green's Thm. is special case of Stoke's in xy-plane.

* why Stoke's is true?

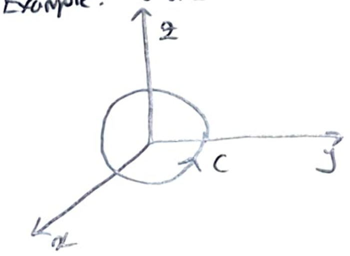
- We know it for C, S in xy-plane.
- Also for C, S in any plane (using that curl, work, flux make sense independently of the coord. system.)
- Given any S: decompose it into tiny, almost flat pieces



Sum of (work around each piece) = work along c

Sum of flux through each piece = flux through S

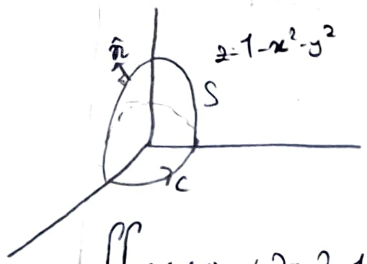
Example: work $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ around unit circle in xy-plane counterclockwise.



directly: $\oint_C \vec{F}\cdot d\vec{r} = \oint_C z\,dx + x\,dy + y\,dz = \oint_0^{2\pi} \cos t(\cos t\, dt)$

$z = 0$
$x = \cos t$
$y = \sin t$

$$= \int_0^{2\pi} \cos^2 t\, dt = \pi$$



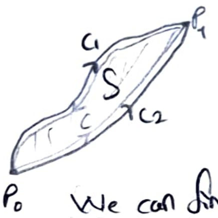
$\iint_S (\nabla\times\vec F)\hat n\,dS$, $\nabla\times F = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = 1\hat\imath - (-1)\hat\jmath + 1\hat k = \langle 1,1,1\rangle$

$\hat n\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy$
$= \langle 2x, 2y, 1\rangle\,dx\,dy$

$\iint_S \langle 1,1,1\rangle\cdot\langle 2x,2y,1\rangle\,dx\,dy = \iint_S (2x+2y+1)\,dx\,dy$ = swithch to polar coord. $= \pi$

Theorem: If $\vec F$ is defined in a simply connected region and curl $\vec F = 0$ then F is a gradient field and $\int_C \vec F\cdot d\vec r$ is path-independent.

Proof: Assume curl $\vec F = 0$

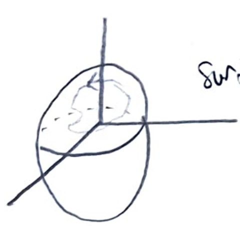


$$\int_{C_1}\vec F\cdot d\vec r - \int_{C_2}\vec F\cdot d\vec r = \int_C \vec F\cdot d\vec r \overset{\text{Stokes}}{=} \iint_S (\text{curl}\,\vec F)\cdot d\vec S = 0$$

$C = C_1 - C_2$

We can find region S because region is simply connected

Remark: Topology classifies surfaces in plane.

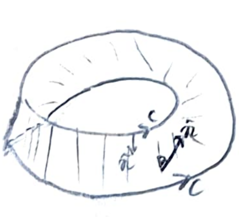


Surface of sphere is simply connekted.

Surface of a torus; Not.

2 "independent" loops that don't bound.

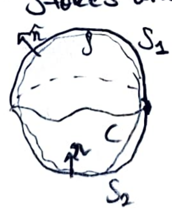
- Orientability:



non-orientabile surface

Flux cannot be defined.

Stokes and Surface Independence:



Stokes: $\oint_C \vec F\cdot d\vec r = \iint_{S_1}(\nabla\times\vec F)\cdot\hat n\,dS = \iint_{S_2}(\nabla\times\vec F)\hat n\,dS$

why same?

$$\iint_{S_1}(\nabla\times\vec F)\hat n\,dS - \iint_{S_2}(\nabla\times\vec F)\hat n\,dS = \iint_S(\nabla\times\vec F)\hat n\,dS$$

by div. thm. $= \iiint_D \text{div}(\nabla\times F)\,dV = 0$

$\text{div}(\nabla\times F) = 0$ always!

$\vec F = \langle P, Q, R\rangle$:

$\nabla\times\vec F = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$

$\text{div}(\nabla\times\vec F) =$

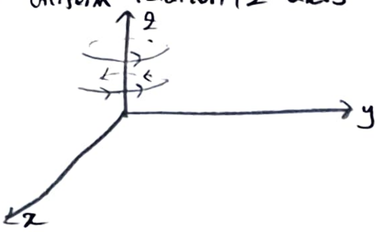
$= (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x + P_y)_z$

$= \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}} = 0$

$\nabla\cdot(\nabla\times\vec F) = 0$, Note: For real vectors $u\cdot(u\times v) = 0$

Curl: recall : $\vec{v}$ velocity field $\Rightarrow$ curl $\vec{v} = 2\cdot$ angular velocity vector. (rotation part of the motion [illegible]

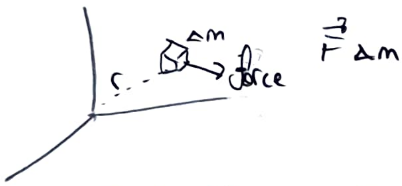
E.g. uniform rotation / z-axis



$$\vec{v} = w(-y\hat{i} + x\hat{j})$$
$$\nabla \times \vec{v} = 2w\hat{k}$$

For a force field ?

Torque : $\vec{\tau} = \vec{r} \times \underbrace{\vec{F}\Delta m}_{\text{force on mass } \Delta m}$.



$\vec{F}\Delta m$

For traslation : $\dfrac{\text{Force}}{\text{mass}} = \text{Acceleration} = \dfrac{d}{dt}(\text{velocity})$

$\downarrow$ curl $\qquad \downarrow$ curl $\qquad \downarrow$ curl

For rotation : $\dfrac{\text{Torque}}{\text{moment of Inertia}} = \text{Angular acceleration} = \dfrac{d}{dt}(\text{angular velocity})$

Consequence: If force $\vec{F}$ derives from a potential

$\Rightarrow$ Curl $\vec{F} = 0 \Rightarrow \vec{F}$ does not generate any rotation motion.

Maxwell's Equations:

$\vec{F} = q\vec{E}$, $\vec{F} = q\vec{v} \times \vec{B}$

Gauss-Coulomb Law: $\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}$ $\rho \to$ electric charge density. $\varepsilon_0 \to$ constant

$$\oiint \vec{E}\cdot dS = \iiint_D \operatorname{div}\vec{E}\, dV = \frac{1}{\varepsilon_0}\iiint_D \rho\, dV = \frac{Q}{\varepsilon_0}$$ $Q \to$ electric charges in D.

Faraday's Law: $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

$$\oint \vec{E}\, d\vec{r} = \iint_S (\nabla \times \vec{E})\, d\vec{S} = \iint_S \left(-\frac{\partial \vec{B}}{\partial t}\right) d\vec{S}$$

{End of Course}