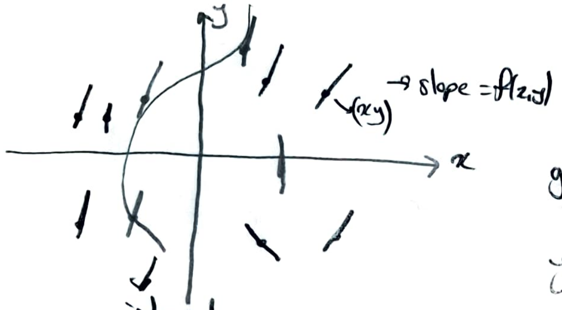
{ Differential Equations }

First Order ODE's (First Order Ordinary Differential Equations):

Analytic View:		Geometric View:
$y' = f(x,y)$	$\longleftrightarrow$	Direction Field
$y_1(x)$	$\longleftrightarrow$	Integral Curve



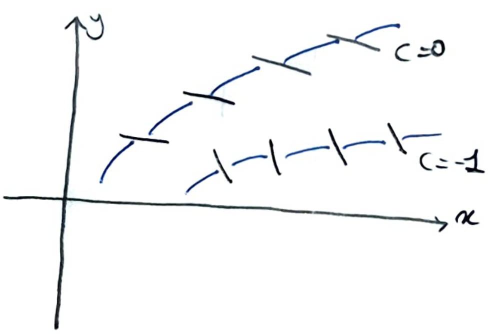
$y_1(x)$ sol'n to differential eq. if and only if graph of $y_1(x)$ is an integral curve

$y_1'(x) = f(x, y_1(x))$, slope of $y_1(x)$ = slope of the dir. field at that point $f(x, y_1(x))$

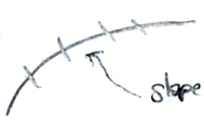
Same.

Drawing Direction Fields:

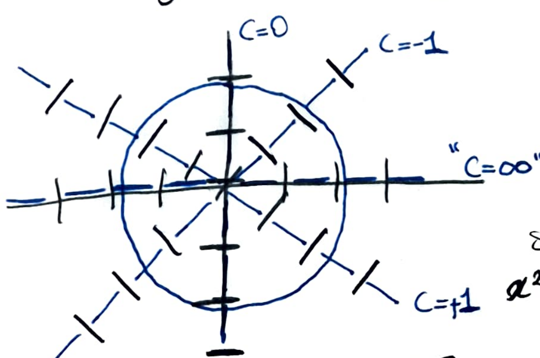
- Computer
  1-) Pick (x,y) [equally spaced]
  2-) Computes $f(x,y)$
  3-) Draws line element / slope $f(x,y)$



- Human
  1-) Pick slope = c
  2-) $f(x,y) = c$, plot this eqn., lvl curve of $f(x,y)$, called isoline
  3-)



Exp: $y' = \frac{-x}{y}$, $\frac{-x}{y} = c$, $y = \frac{-1}{c}x$

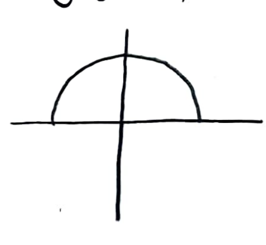


int. curves = circles.

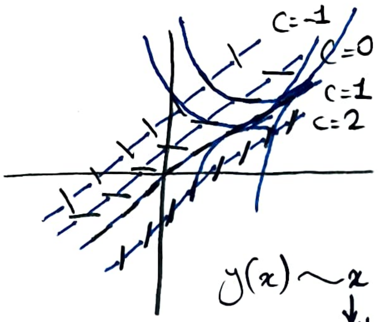
Solve ODE's by separation of variables.

$x^2+y^2 = c_1$

$y = y_1(x) = \pm\sqrt{c_1^2 - x^2}$

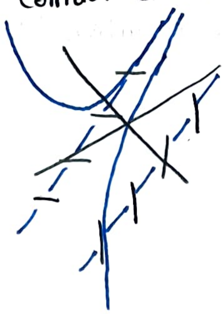


$y' = 1+x-y$, isocline : $y = 1+x-c$



$y(x) \sim x$
↓
sol'n.

Corridor between c=0 and c=2 isocline



Sol'n can't espace.

Rule 1-) Two integral curves can't cross at an angle, can't have two slopes at the same point. Two integral curves can't be tangent.

Existince and Uniquenes Thm says that, (x_0, y_0), $y' = f(x,y)$ has one and only one sol'n through the point (x_0, y_0) , Hyp: $f(x,y)$ should be continues near (x_0, y_0), $f_y(x,y)$ should be also near (x_0, y_0).

Exp. $xy' = y-1$

$$\frac{dy}{y-1} = \frac{dx}{x} \Rightarrow \ln|y-1| = \ln|x| + C_1 \Rightarrow y-1 = Cx \text{ , } y = Cx-1 \text{ , sol'n}$$

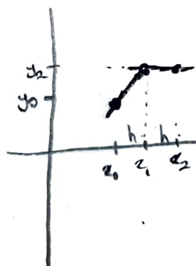
$$\frac{dy}{dx} = \frac{y-1}{x} \text{ , not defined when } x=0$$

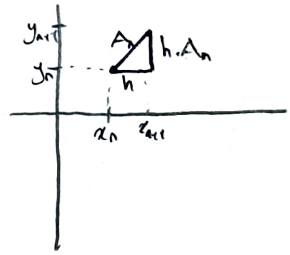
no existence

Numerical Sol'n:

$\left.\begin{array}{l} y' = f(x,y) \\ y(x_0) = y_0 \end{array}\right\}$ Initial Value Problems (IVP)

, Euler's Method





$y_{n+1} - y_n = hA_n$.

Equations :

$x_{n+1} = x_n + h$

$y_{n+1} = y_n + hA_n$

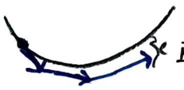
$A_n = f(x_n, y_n)$

Exp: $y' = x^2 - y^2$, $y(0) = 1$, $h = .1$

n	x_n	y_n	A_n	hA_n
0	0	1	-1	-.1
1	.1	.9	-.8	-.08
2	.2	.82		

$y(.2) \approx .82$

Euler too high or too low

$y_1(x)$  convex, E too low, $y''>0$

$y'' = 2x - 2yy'$

At $(0,1)$

$y(0)=1$

$y'(0)=-1$

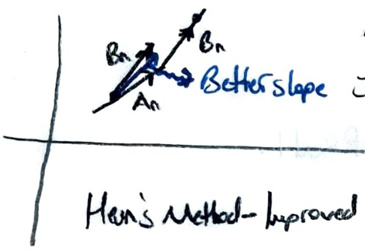
$y''(0)=2$, increasing, Convex, too low.

$y_1(x)$  concave, E too high, $y''<0$

Error e depends on step size.

$e \sim c_1 h \rightarrow$ Euler's first order method.

Better Method: Try to finding a better slope A_n.



$x_{n+1} = x_n + h$

Better slope $y_{n+1} = y_n + h\left(\frac{A_n + B_n}{2}\right)$, $\tilde{y}_{n+1} = y_n + hA_n$, $B_n = f(x_{n+1}, \tilde{y}_{n+1})$

Heun's Method - Improved Euler's Method - RK2

RK2: Second-Order Method $= e \sim c_2 h^2$

RK4: Standart Method, more accurate

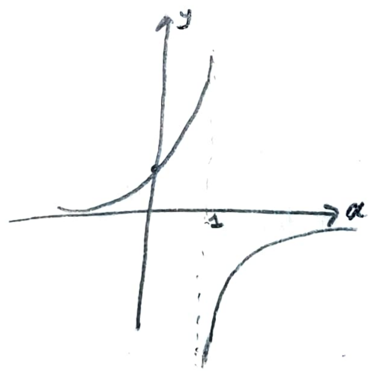
Runge-Kutta, 4th order

$\frac{A_n + 2B_n + 2C_n + D_n}{6} \rightarrow$

Pitfalls:

#1. You'll find

#2. $y' = y^2$, sol'ns: $y = \frac{1}{c-x}$ $\rightarrow$ singularity at $x=c$



$y(0)=1$

$y(2)=?$

First Order Linear Equation:

$a(x)y' + b(x)y = c(x)$, if $c=0$, this is homogenous.

Standard Linear Form: $y' + p(x)y = q(x)$

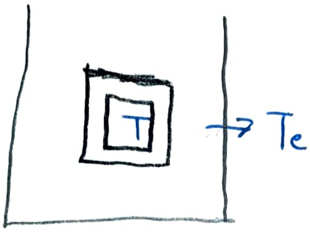
$y' = p(x)y + q(x)$ {1st Order Linear Form}

Models:

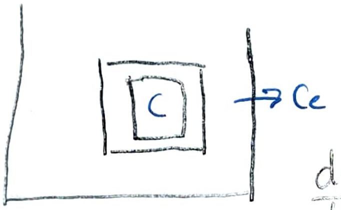
Temp – concentration

Mixing

Diffusion



$$\frac{dT}{dt} = k(T_e - T), \; k>0 \quad \{\text{Temp}\}$$



C, C_e = concentation of salt.

$$\frac{dC}{dt} = k(C_e - C)$$

In Standart Form:

$$\frac{dT}{dt} + kT = kT_e, \quad \frac{dC}{dt} + kC = kC_e$$

$y' + p(x)y = q(x) \rightsquigarrow$ Gonna solve by finding an integrating factor $u(x)$.

$uy' + puy = qu.$, if $u' = pu$, $\frac{du}{u} = \int p(x)dx = \ln u$

$u = e^{\int p(x)dx} \rightsquigarrow$ integrating factor.

Method:

0-) Standard Linear Form

1-) $e^{\int p(x)dx} = u$

2-) multiply both sides by u

3-) Integrate and find sol'ns.

Example: $xy' - y = x^3$

$y' - \frac{1}{x} y = x^2$

$e^{\int -\frac{1}{x}dx} = e^{(-\ln x)} = \frac{1}{x}$

$\underline{y' \cdot \frac{1}{x} - \frac{1}{x^2} y} = x \implies \frac{1}{x} y = \frac{x^2}{2} + C$

$\left(\frac{1}{x} y\right)'$

$\implies \boxed{y = \frac{x^3}{2} + Cx.}$

Substitution:

$y' = f(x,y)$, Scaling $x = \frac{x_1}{a}$, $y = \frac{y_1}{b}$; a, b constant values.

$\frac{dT}{dt} = k(M^4 - T^4)$, $T_1 = T/M$, dim (T_1) = dimensionless.

$\frac{dT_1}{dt} = kM^4(1 - T_1^4) = k_1(1 - T^4)$, One less constant., $k_1 = kM^3$ =) "lumping constant"

① Change Variables , ② Make Equation Dimensionless.

Direct: ③ Reduce Number or Simplify Constants.

New Variable = f(Old Variables) , $\int x\sqrt{1-x^2}\,dx$, $u = 1 - x^2$

$T_1 = T/M$

Inverse:

Old Variables = f(Old, New Var.) , $\int \sqrt{1-x^2}\,dx$, $u = \sin x, \cos x$.

$T = T_1 \cdot M$

Direct Substitution: {Bournoulli Equation}

$y' = p(x)y + q(x)y^n$ ($n \neq 0$, since we know how to do it , $n \neq 1$, linear comb.)

$\frac{y'}{y^n} = \frac{p(x)}{y^{n-1}} + q(x)$, $v = 1/y^{n-1} = y^{1-n}$, $v' = (1-n)y^{-n}y'$

$\frac{v'}{1-n} = p(x)v + q(x)$

Example:

$y' = \frac{y}{x} - y^2$, $\frac{y'}{y^2} = \frac{1}{xy} - 1$, $v = 1/y$, $v' = -\frac{1}{y^2} \cdot y'$

$-v' = \frac{v}{x} - 1$, $v' + \frac{v}{x} = 1$, int. factor, $1/x$ =) $e^{(\ln x)} = x$

$\underbrace{xv' + v}_{(xv)'} = x$ =) $xv = \frac{x^2}{2} + C$ =) $v = \frac{x}{2} + \frac{C}{x}$, $\frac{1}{y} = \frac{x}{2} + \frac{C}{x} = \frac{x^2 + 2C}{2x}$

$y = \frac{2x}{x^2 + 2C} = \frac{2x}{x^2 + C_1}$

Homogenous First Order Ordinary Differential Equations:

$y' = F(y/x)$

$y' = \frac{x^2y}{x^3+y^3} = \frac{y/x}{1+(y/x)^3}$, $xy' = \sqrt{x^2+y^2} \Rightarrow y' = \sqrt{1+(y/x)^2}$

Invariant under the operation "Zoom":

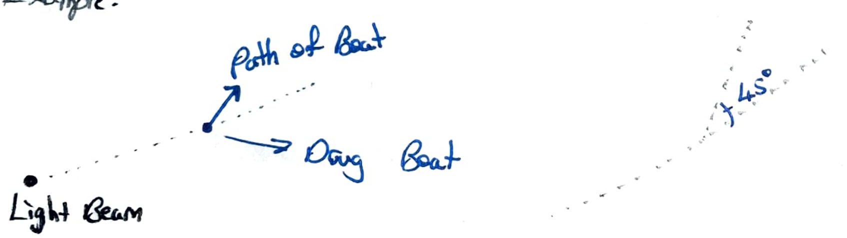
$x \to ax_1$, $y \to ay_1$

$\frac{dy_1}{dx_1} = F(y_1/x_1)$, $\frac{dy}{dx} = F(y/x) = F(y_1 a / x_1 a) = F(y_1/x_1)$

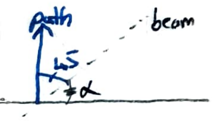
$y' = F(y/x)$, $y = zx$, $y' = z'x + z$, $z = y/x$

$z'x + z = F(z)$, $x\frac{dz}{dx} = F(z) - z$

Example:



What is the boats path?



$y = y(x)$, $\tan(\alpha) = y/x$

$y' = \tan(\alpha+45) = \frac{\tan\alpha + \tan 45°}{1 - \tan\alpha \tan 45°} = \frac{y/x + 1}{1 - y/x} = y+x / x-y$.

$z = y/x$

$y' = z'x + z = \frac{z+1}{1-z} \Rightarrow x\frac{dz}{dx} = \frac{z+1}{1-z} - z = \frac{1+z^2}{1-z}$, $\frac{1-z}{1+z^2}dz = \frac{dx}{x} = \ln x$

$\tan^{-1}(z) - \frac{1}{2}\ln(1+z^2) = \ln x + c$, $\tan^{-1} z = \ln\sqrt{1+z^2} + \ln x + c$

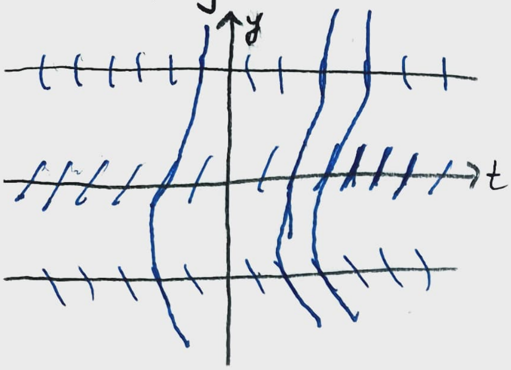
$\tan^{-1}(y/x) = \ln\sqrt{x^2+y^2} + c$

$\theta = \ln r + c$

$\underline{r = c_1 e^{\theta}}$

$\frac{dy}{dt} = f(y) \Rightarrow$ No t in RHS, makes it autonomous. (can solve by seperation of var.)

We're aiming to get qualitative information without solving the equation.



lines $y = y_0$, isoclines since $\frac{dy}{dt} = f(y_0)$

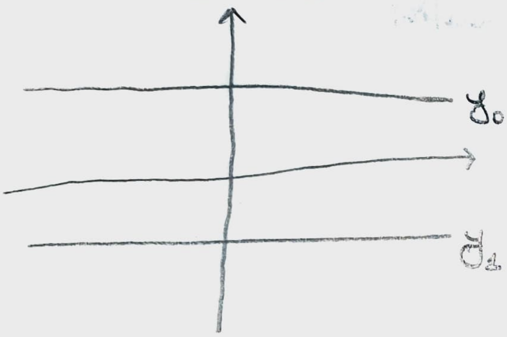
Critical point y_0, $f(y_0) = 0$

$y = y_0$ is a solution.

Proof:

$$\frac{dy_0}{dt} = f(y_0) = 0 \text{ (both sides = zero)}.$$

Suppose I have 2 constant horizontal lines as solutions.



We know that integral curves can't cross the curves, so y_0 and y_1 works as boundaries.

So;

1.) Find critical points.

2.) Graph $f(y) > 0$? < 0?

$\frac{dy}{dt} = f(y)$; if $f(y) > 0$, $y(t)$ increasing.
if $f(y) < 0$, $y(t)$ decreasing.

Example:

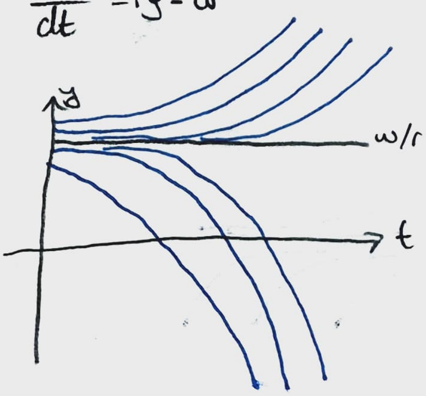
y = money in the bank

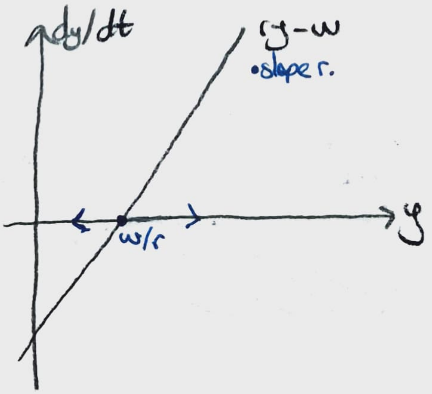
r = continues interest rate

ω = rate of embezzelement.

$$\frac{dy}{dt} = ry - w$$

1.) Critical Point, $y = w/r$.





Logistic Equations:

Population Behavier $y(t)$

$\frac{dy}{dt} = ky$, $k=c$, growth rate.

, logistic growth

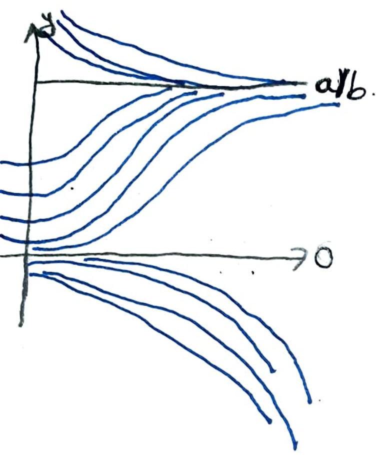
k declines as y increases , simplest choice ;

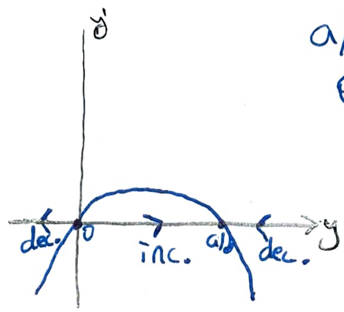
$k = a - by$

$\frac{dy}{dt} = ay - by^2$

Critical Points

$y(a-by) = 0$; $y=0$, a/b .

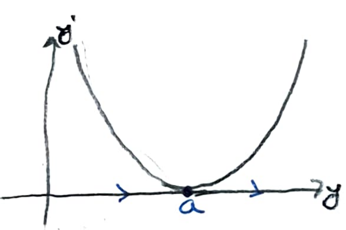


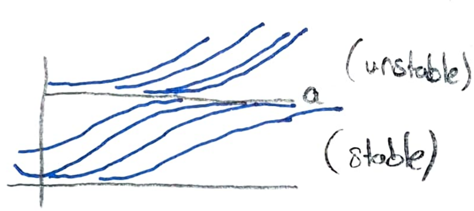


$a/b \Rightarrow$ Stable Solution

$0 \Rightarrow$ Unstable Solution

Suppose that;



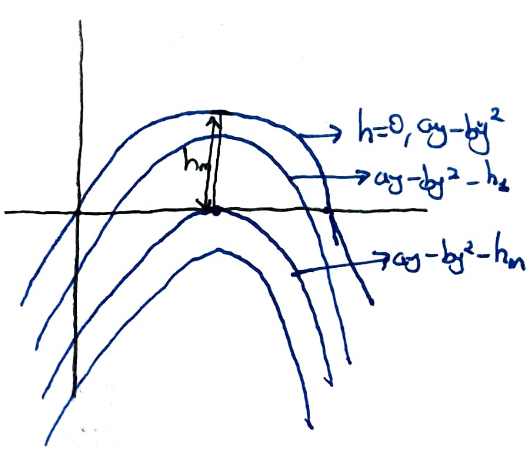


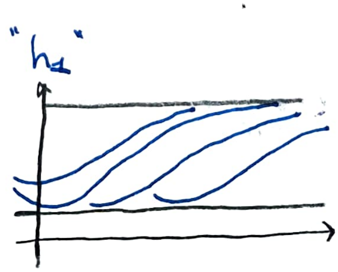
$a \Rightarrow$ Semi - Stable Critical Point.

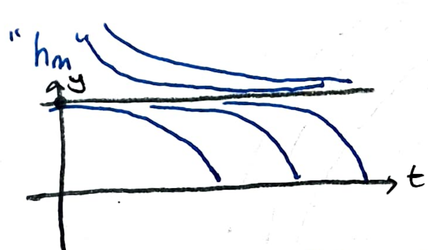
Logistic Equations with Harvesting.

Harvest rate at constant t.

$\frac{dy}{dt} = ay - by^2 - h$

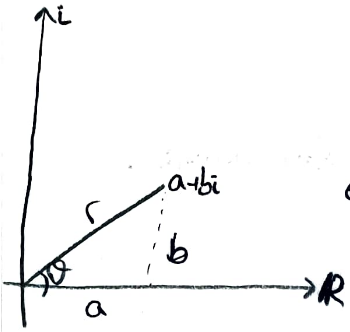






h_m = max. rate of harvesting.

Polar Representation of Complex Numbers



$$a+bi = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

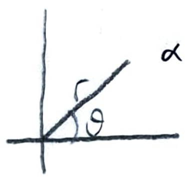
Exponentials

1-) Exp. law : $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$?

2-) Sol'n eqt. $\frac{dy}{dt} = ay$, $y(0)=1$. $\frac{d}{dt}e^{i\theta} = ie^{i\theta}$?

$$(\cos\theta_1 + i\sin\theta_2)(\cos\theta_2 + i\sin\theta_1) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1) = \cos(\theta_1+\theta_2) + i(\sin(\theta_1+\theta_2))$$

$$\frac{d}{dt}e^{it} = \frac{d}{dt}(\cos t + i\sin t) = -\sin t + i\cos t = i(\cos t + i\sin t) = ie^{it}, \quad y(0) = e^{i0} = \cos 0 + i\sin 0 = 1.$$



$\alpha = re^{i\theta}$, r = modules of α $|\alpha|$; θ = argument of α. $(\arg(\alpha))$

Example:

$\int e^x \cos x \, dx$, $\cos x$: Re part of e^{ix}

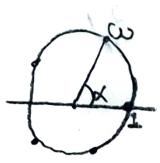
$$\underbrace{\int e^{-x} e^{ix} dx}_{Re} = \int e^{x(i-1)} dx = \frac{e^{x(i-1)}}{i-1} = \frac{1}{i-1} e^{-x}(\cos x + i\sin x) = \frac{-i-1}{2} e^{-x}(\cos x + i\sin x) = \frac{e^{-x}}{2}(-i-1)(\cos x + i\sin x)$$

$$\Rightarrow \frac{e^{-x}}{2}(-\cos x + \sin x)$$

Exp:

$\sqrt[5]{1}$

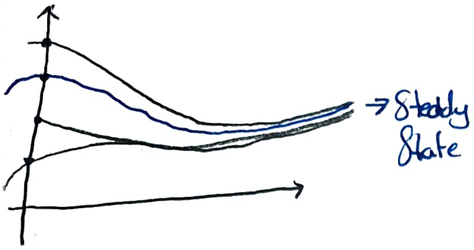
no answers as cx. #'s.



$\alpha = \frac{2\pi}{5}$

$\omega = e^{i\,2\pi/5}$ $(r=1)$, $\omega^5 = e^{(i\,2\pi/5)\cdot 5} = e^{i2\pi} = 1$, since 2π and 0 same angle.

Solve $y' + ky = k\,g_e(t)$, Sol'n: $y = \underbrace{e^{-kt}\int g(t)\, e^{kt}\, dt}_{\text{Steady state sol'n}} + \underbrace{ce^{-kt}}_{\substack{\downarrow \\ 0 \text{ as } t \to \infty \\ \text{transient}}} \to$ use $y(0)$



Other sol'ns must follow the SSS after a while.

Superposition Principle

$q_1(t) \to y_1(t)$ | $q_1 + q_2 \to y_1 + y_2$
$q_2(t) \to y_2(t)$ | $cq_1 \to cy_1$

Trigonometric Equations

$y' + ky = kq_e(t)$, $q_e = \cos(\omega t)$ ‖ $\cos(\omega t)$, find the response.

Hint: Complexify the problem (since easy to integrate exponentials.)

ω = angular frequency = # of complete oscillation in period 2π

$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$, $\tilde{y}' + k\tilde{y} = ke^{i\omega t}$; $\tilde{y} = y_1 + iy_2$

Find $\tilde{y}$, then y_1 will solve the orginal problem.

Solve: $e^{kt} \Rightarrow$ integrating factor

$(\tilde{y}e^{kt})' = ke^{(k+wi)t} \Rightarrow \tilde{y}e^{kt} = \dfrac{ke^{(k+i\omega)t}}{k+i\omega} \Rightarrow \tilde{y} = \dfrac{k}{k+i\omega}\dfrac{e^{(k+i\omega)t}}{e^{kt}} = \dfrac{k}{k+i\omega}e^{i\omega t} = \dfrac{1}{1+i(\omega/k)}e^{i\omega t}$.

Go Polar or Cartesian

> Remind: α complex number
>
> $\frac{1}{\alpha}\alpha = 1$, $|\frac{1}{\alpha}||\alpha| = 1$
>
> $\dfrac{1}{|\alpha|} = \left|\dfrac{1}{\alpha}\right|$
>
> $\arg(\frac{1}{\alpha}) + \arg(\alpha) = \arg(1) = 0$

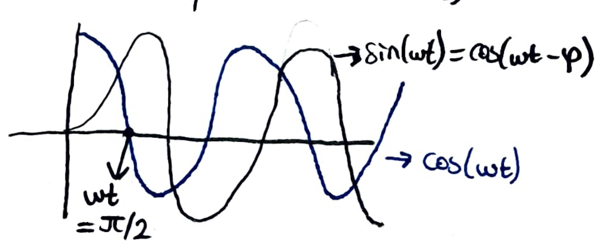
In Polar Form: $\dfrac{1}{1+i(\omega/k)}e^{i\omega t} = Ae = Ae^{-i\varphi}$, $\tilde{y} = Ae^{i\omega t - i\varphi}$

$= \dfrac{1}{\sqrt{1+(\omega/k)^2}}e^{i\omega t - i\varphi}$

$\arg(1 + i(\omega/k)) = \varphi$

$y_1 = \dfrac{1}{\sqrt{1+(\omega/k)^2}}\cos(\omega t - \varphi)$

$\varphi = \tan^{-1}(\omega/k)$ ‖ phase lag or delay of function.



Response:

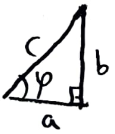
$\dfrac{1}{\sqrt{1+(\omega/k)^2}}\cos(\omega t - \varphi)$

If $k\uparrow$; $A\uparrow$, $\varphi\downarrow$

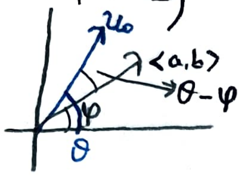
Going Cartesian:

$$y = \frac{1}{1+i(\omega/k)} e^{i\omega t} = \frac{1-i(\omega/k)}{1-(\omega/k)^2}\left(\cos(\omega t) - \frac{\omega}{k}\sin(\omega/t)\right),$$ to convert this

$$a\cos(\theta) + b(\sin(\theta)) = c \cdot \cos(\theta - \varphi)$$



Proof: {18.02}



$$\langle a,b\rangle \langle \cos\theta, \sin\theta\rangle = |\langle a,b\rangle| \cdot 1 \cdot \cos(\theta-\varphi)$$

Proof: {18.03}

$(a-bi)(\cos\theta + i\sin\theta)$ {Take Real Parts of It}

$$= \sqrt{a^2+b^2}\, e^{i(\theta-\varphi)} \Rightarrow \underbrace{\sqrt{a^2+b^2}}_{= c}\cos(\theta-\varphi) = a\cos\theta + b\sin\theta.$$

From that:

$$\Rightarrow \frac{1}{1-(\omega/k)^2}\sqrt{1+(\omega/k)^2} \cdot \cos(\omega t - \varphi), \quad \varphi = \tan^{-1}(\omega/k)$$

First Order Basic Linear Equations

1-) $y' + ky = kq(t)$
2-) $y' + ky = q(t)$
3-) $y' + p(t)y = q(t)$

$k>0$

If $k<0$, let's say for $y'+ky = q(t)$.

when $k>0$

$$\underbrace{e^{-kt}\int q(t)e^{kt}\,dt}_{\text{Important Part}} + \underbrace{ce^{-kt}}_{\Downarrow\ 0 \text{ as } k\to\infty}$$

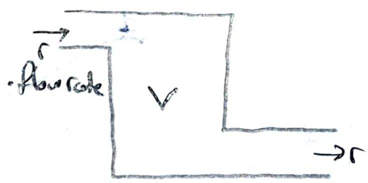
but when $k<0$;

$$\underbrace{e^{-kt}\int q(t)e^{kt}\,dt}_{\text{Some fixed function}} + \underbrace{ce^{-kt}}_{\Downarrow\ \infty \text{ or } -\infty \text{ "Important Part"}}$$

Mixing Problem:

$x(t)$: amount of salt in tank.

C_e: concentration of incoming salt.



$$\frac{dx}{dt} = \underset{\text{inflow}}{\text{rate of salt}} - \underset{\text{outflow}}{\text{rate of salt}} = r \cdot C_e - r\frac{x}{V}$$

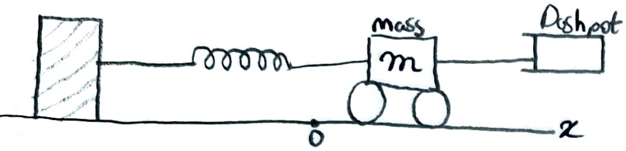
$$\frac{dx}{dt} + \frac{rx}{V} = r\cdot c_e(t) = \frac{dx}{dt} + rc = rc_e, \quad c(t) = \frac{x}{V}, \quad x = cV$$

$$\frac{dx}{dt} = V\frac{dc}{dt} \Rightarrow \frac{dc}{dt} + \frac{r}{V}c = \frac{r}{V}c_e, \quad k = r/V: \text{ basic parameter.}$$

Linear 2nd Order ODE's with Constant Coefficients

$y'' + Ay' + By = 0$

Assume: General Solution $y = c_1 y_1 + c_2 y_2$, where y_1 and y_2 are sol'ns.
Initial Conditions are satisfied by chosing c_1 and c_2
y_1 and y_2 are independent.



Equilibrom Position

$\underbrace{mx''}_{\text{Force}} = \underbrace{-kx}_{\text{Spring}} \underbrace{-cx'}_{\text{F. dashpot}}$

(Typical Model)

$mx'' + cx' + kx = 0$

$x'' + \frac{cx'}{m} + \frac{kx}{m} = 0$

Basic Method To Solve:

$y = e^{rt}$, $t =$ independent.

$r^2 e^{rt} + Are^{rt} + Be^{rt} = 0 \Rightarrow \boxed{r^2 + Ar + B = 0}$ Characteristic Equation of System.

Case 1-) roots, $r_1 \neq r_2$ (Real)

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Example:

$y'' + 4y' + 3y = 0$, $r^2 + 4r + 3 = 0 = (r+3)(r+1)$: $r = -3, -1$.

$y(0) = 1$

$y'(0) = 0$

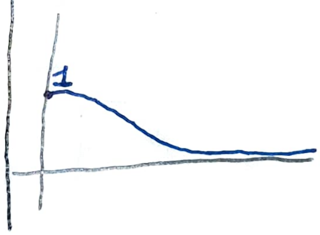
Sol'n: $y = c_1 e^{-3t} + c_2 e^{-t}$

$y' = -3c_1 e^{-3t} - c_2 e^{-t}$, $t = 0$

$-3c_1 - c_2 = 0$ | $c_1 = -1/2$

$1 = c_1 + c_2$ | $c_2 = 3/2$

$y = -\frac{1}{2} e^{-3t} + \frac{3}{2} e^{-t}$



Case 2-) Complex Roots, $r = a \pm bi$

$y = e^{(a \pm bi)t}$?

Sol'n: $y = e^{at + ibt}$ $= e^{at}(\cos(bt))$: Real Part
$e^{at}(\sin(bt))$: From Complex Part

So the sol'n becomes:

$y = e^{at}(c_1 \cos(bt) + c_2 \sin(bt))$

Thm: If you have a sol'n as $u + iv$ complex sol'n to $y'' + Ay' + By = 0$ the u and v are real sol'ns.

Proof: $(u + iv)'' + A(u + iv)' + B(u + iv) = 0$

$u'' + Au' + Bu + i(v'' + Av' + Bv) = 0$

$u'' + Au' + Bu = -i(v'' + Av' + Bv)$

$\underbrace{}_{\text{Real Part}} = \text{Complex Part} = \text{zero}$

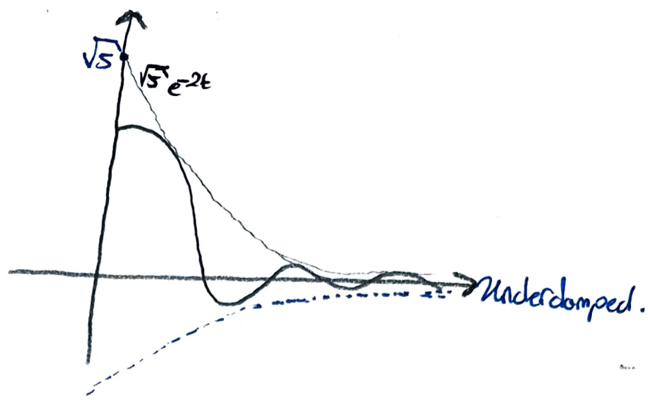
Example: $y''+4y'+5y=0$

$r^2+4r+5=0$

$r=\dfrac{-4\pm\sqrt{-4}}{2}=-2\pm i$

$e^{(-2+i)t} \Rightarrow e^{-2t}(c_1\cos t+c_2\sin t)=y,$

$y=e^{-2t}(\cos t+2\sin t)=\sqrt{5}\,e^{-2t}\cos(t-\varphi)$



Critical Damped: $r^2+Ar+B=0$, has 2 equal roots.

ODE must look like $y''+2ay'+a^2y=0$ Sol'n: $y=e^{-at}$

If you know "1" sol'n to the 2nd order ODE, you can find another sol'n as $y=y_1 x$.

$a^2)\ y=e^{-at}x$

$a\ |\ y'=-ae^{-at}x+e^{-at}x'$

$1\ |\ y''=a^2e^{-at}x-ae^{-at}x'-ae^{-at}x'+e^{-at}x''=a^2e^{-at}x-2ae^{-at}x'+e^{-at}x''$

$a^2e^{-at}x-2a^2e^{-at}x+2ae^{-at}x'+a^2e^{-at}x-2ae^{-at}x'+e^{-at}x''=e^{-at}x''=0$

$x=c_1t+c_2,\ y_1=e^{-at}t.$

$y=c_1y_1+c_2y_2$, $y=\underset{Cx}{C_1}\underset{Cx}{e^{(a+bi)t}}+\underset{Cx}{C_2}\underset{Cx}{e^{(a-bi)t}}$; $C_1,C_2\in\mathbb{C}$

Way 1-) Make imaginary part zero

$u+iv\to v=0$

change $i\to -i$

$y=\bar{C}_1e^{(a-bi)t}+\bar{C}_2e^{(a+bi)t}$, we want to see that first and second expression are the same.

$C_2=\bar{C}_1$, must be.

$y=(c+di)e^{(a-bi)t}+(c-di)e^{(a+bi)t}$ $\Rightarrow$ This is the general form of Solution in Science.

$e^{at}[c\underbrace{(e^{ibt}+e^{-ibt})}_{2\cos bt}+i d\underbrace{(e^{ibt}-e^{-ibt})}_{2di\sin(bt)}]$

Remind: $\cos a=\dfrac{e^{ia}+e^{-ia}}{2}$, $\sin a=\dfrac{e^{ia}-e^{-ia}}{2i}$

$\Rightarrow 2ce^{at}\cos(bt)-2de^{at}\sin(bt)$

$=e^{at}(2c\cos(bt)-2d\sin(bt))$

Oscillations:

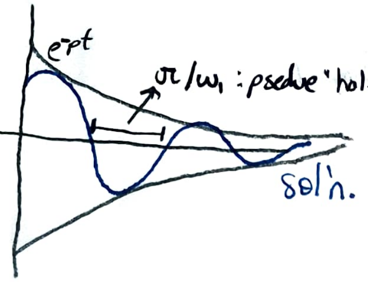
$y'' + 2py' + (\omega_0)^2 y = 0$ | when $p=0$ | $r = \pm i\omega_0^2$

$r^2 + 2pr + (\omega_0)^2 = 0$ | $y'' + \omega_0^2 y = 0$ | $y = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) = A\cos(\theta - \varphi)$ {Undamped}

$\omega_0 \downarrow$ circular frequency

$r = \pm p \pm \sqrt{p^2 - \omega_0^2}$

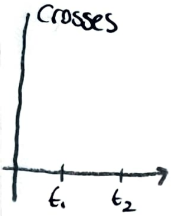
We got oscillations if $p^2 - \omega_0^2 < 0$, since we need complex roots $\Rightarrow |p| < |\omega_0|$



π/ω_1 : pseduo "half" period ω_1 : pseduo frequency

$$r = -p \pm \sqrt{-(\omega_0^2 - p^2)} = -p \pm \sqrt{-\omega_1^2} = -p \pm \omega_1 i.$$

$$\Rightarrow e^{-pt}(c_1\cos(\omega_1 t) + c_2\sin(\omega_1 t)) = e^{-pt} A\cos(\omega_1 t - \varphi)$$



$$t_2 = t_1 + \frac{2\pi}{\omega_1} \Rightarrow \omega_1 t_1 - \varphi = \pi/2 \ , \ \omega_1\left(t_1 + \frac{2\pi}{\omega_1}\right) - \varphi = \pi/2 + 2\pi$$

Depends on;

- p - only on ODE
- φ - } Initial
- A - } conditions
- ω_1 - only on ODE

1-) Why $y = c_1y_1 + c_2y_2$ is a solution?

2-) Why $c_1y_1 + c_2y_2$ is the all solution to the ODE?

Superposition principle says; y_1 and y_2 are sol'n to linear homogenous equation {can be higher order too} then $c_1y_1 + c_2y_2$ is a sol'n.

Proof: $y'' + py' + qy = 0$

$D^2y + pDy + qy = 0$, $\underbrace{(D^2 + pD + q)}_{\text{linear operator } L} y = 0$, $Ly = 0$, $L = D^2 + pD + q$

$x(x) \rightarrow \boxed{L} \rightarrow y(x)$, $L(x_1 + x_2) = L(x_1) + L(x_2)$, $L(c_1x_1) = c_1 L(x_1)$

D is linear so L is linear.

$L(c_1y_1 + c_2y_2) = 0$

$L(c_1y_1) + L(c_2y_2) = c_1 \underbrace{L(y_1)}_{0} + c_2 \underbrace{L(y_2)}_{0}$, First Question Answered.

Initial Value Problem:

Thm: $\{c_1y_1 + c_2y_2\}$ is enough to satisfy any initial condition

Proof:

$y(x_0) = a$; $y = c_1y_1 + c_2y_2$ | $c_1y_1(x_0) + c_2y_2(x_0) = y(x_0) = a$ } c_1, c_2 are variables

$y'(x_0) = b$; $y' = c_1y_1' + c_2y_2'$ | $c_1y_1'(x_0) + c_2y_2'(x_0) = y'(x_0) = b$ } plug in $x = x_0$.

Solvable for c_1, c_2 if $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}_{x_0} \neq 0$; Wronskian, $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

If y_1 and y_2 dependant; $W(y_1, y_2) = 0$, determinant will be zero.

If y_1 and y_2 are sol'n to the ODE, either $W(y_1, y_2) \equiv 0$, for all x or $W(y_1, y_2)$ is never zero for all x.

$C_1 y_1 + C_2 y_2 = C_1' u_1 + C_2' u_2$; $u_1 = \bar{C}_1 y_1 + \bar{C}_2 y_2$
$u_2 = \bar{\bar{C}}_1 y_1 + \bar{\bar{C}}_2 y_2$

Find normalized Solutions

$\bar{Y}_1, \bar{Y}_2$ $\quad \bar{Y}_1 : \bar{Y}_1(0) = 1$, $\bar{Y}_1'(0) = 0$, $\bar{Y}_2 : \bar{Y}_2(0) = 0$, $\bar{Y}_2'(0) = 1$

Example:

$y'' + y = 0$
$y_1 = \cos x = \bar{Y}_1$
$y_2 = \sin x = \bar{Y}_2$

$y'' - y = 0$
$y_1 = e^x$
$y_2 = e^{-x}$

gen. sol'n:
$C_1 e^{+x} + C_2 e^{-x} = y$
$C_1 e^{x} - C_2 e^{-x} = y'$

$\bar{Y}_1 : C_1 + C_2 = 1$
$C_1 - C_2 = 0$
$C_1 = C_2 = \frac{1}{2}$

$\bar{Y}_1 = \underbrace{\frac{e^x + e^{-x}}{2}}_{\cosh(x)}$, $\bar{Y}_2 = \underbrace{\frac{e^x - e^{-x}}{2}}_{\sinh(x)}$

Solution to Initial Value Problem : ODE + $y(0) = a = y_0$
$y'(0) = b = y_0'$ $\quad y = y_0 \bar{Y}_1 + y_0' \bar{Y}_2$

Existence and Uniqueness Theorem:

$y'' + py' + qy = 0$; p, q are continues for all x.

There is one and only one solution which satisfies given initial conditions.

$y(x_0) = A$
$y'(x_0) = B$.

We want all solutions to the ODE. So we claim that; $C_1 \bar{Y}_1 + C_2 \bar{Y}_2$ are all solutions.

Proof of Q2: Given sol'n $u(x)$ and $u(x_0) = u_0$, $u'(x_0) = u_0'$ then $u_0 \bar{Y}_1 + u_0' \bar{Y}_2$ satisfies its initial conditions.

And uniqness and existence says that there is only "1" solution to the ODE so the solution we've found $u = u_0 \bar{Y}_1 + u_0' \bar{Y}_2$ is the only and all solutions to the ODE.

2nd Order Inhomogenous ODE's

$y'' + p(x) y' + q(x) y = f(x)$, $f(x)$ referred as input, signal, driving term, forcing term etc.

Solution: $y(x)$ reffered as output, response.

$y'' + p(x) y' + q(x) y = 0 \Rightarrow$ associated, homogenous eq., reduced eq.

Thm: $Ly = f(x)$, L is a linear operator.
Sol'n to the equation has the form

$y_p + y_c$, $y_c = C_1 y_1 + C_2 y_2$ and y_p is a particular solution to $Ly = f(x)$

Proof: 1 $y_p + c_1y_1 + c_2y_2$ are sol'ns

$$L(y_p + c_1y_1 + c_2y_2) = \underbrace{L(y_p)}_{f(x)} + \underbrace{L(c_1y_1 + c_2y_2)}_{0}$$

Since y_p is a sol'n Therefore it satisfies $Ly = f(x)$

2 There're no solutions expect the ones above.

$x(x) = $ sol'n, $L(x) = f(x)$

$L(y_p) = f(x)$

$-$

$L(x - y_p) = 0$, therefore $x - y_p = c_1y_1 + c_2y_2$ x was not a new sol'n. It was one of the old

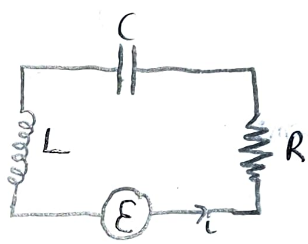
$$x = y_p + \tilde{c}_1y_1 + \tilde{c}_2y_2$$

Examples: Consider them as models.

$mx'' + bx' + kx = f(t)$ force system, $f(t) \neq 0$

spring - mass - dashpot passive system, $f(t) = 0$

$$mx'' = \underbrace{f(t)}_{\text{External force}} - \underbrace{bx' - kx}_{\text{force}}$$



Sum of Voltage Drops = 0

$Li' + Ri + q_c/C = E(t)$

$Li'' + Ri' + \frac{i}{C} = E'(t)$

$q_c' = i$.

$y' + ky = q(t)$, Sol'n was: $e^{-kt}\int q(t)e^{kt} + ce^{-kt}$, $k > 0$: $y = \underset{y_p}{\text{steady-state}} + \underset{y_c \to 0 \text{ as } t \to \infty}{\text{transient}}$

$y'' + Ay' + By = f(t)$, $y = y_p + c_1y_1 + c_2y_2$, when $c_1y_1 + c_2y_2 \to 0$

Under what circumstanced this thing as $t \to \infty$ for all c_1, c_2. becomes true?

If this is so, ODE is stable.

$$y = \underbrace{y_p}_{SSS} + \underbrace{c_1y_1 + c_2y_2}_{\text{transient}}$$

Char. roots	Sol'ns	{homogenous} Stab. conditions
$r_1 \neq r_2$	$c_1e^{r_1t} + c_2e^{r_2t}$	$r_1, r_2 < 0$
$r_1 = r_2$	$(c_1 + c_2t)e^{r_1t}$	$r_1 < 0$
$r = a \pm bi$	$e^{at}(c_1\cos(bt) + c_2\sin(bt))$	$a < 0$

ODE is stable if all characteristic roots have negative real parts.

$$y'' + Ay' + By = \underset{0}{\overset{\shortparallel}{f(t)}}$$

$y''+Ay'+By=f(x)$ find a particular solution. y_p. (general sol'n $y=y_p+y_c$)

Important $f(x)$: $e^{\alpha x}$ $\begin{cases}\sin(x)\\ \cos(x)\end{cases}$, $\begin{cases}e^{ax}\sin\omega x\\ e^{ax}\cos\omega x\end{cases}$, all special cases of $e^{(a+i\omega)x}=e^{\alpha x}$

($\alpha \in \mathbb{C}$)

$y''+Ay'+By=f(x)$

$\underbrace{(D^2+AD+B)}_{p(D)}y=f(x)$

$p(D)e^{\alpha x}=p(\alpha)e^{\alpha x}$

Proof: $(D^2+AD+B)e^{\alpha x}$

$=D^2e^{\alpha x}+ADe^{\alpha x}+Be^{\alpha x}$

Also true for $P(D)$.

Exponential Input Thm: for $e^{\alpha x}$

$$y_p=\frac{e^{\alpha x}}{p(\alpha)}$$

Proof: $p(D)y_p=e^{\alpha x}$

$$\frac{p(D)e^{\alpha x}}{p(\alpha)}=\frac{p(\alpha)e^{\alpha x}}{p(\alpha)x}=e^{\alpha x}$$ for $p(\alpha)\neq 0$

Exp: $y''-y'+2y=10e^{-x}\sin x$

Find particular sol'n. $\downarrow$ imaginary part of $\tilde{y}_p=\dfrac{10e^{-(1+i)x}}{(-1+i)^2-(-1+i)+2}$

Imaginary Part of it = Sol'n to original problem.

$(D^2-D+2)\tilde{y}=10e^{-(1+i)x}$

$$\tilde{y}_p=\frac{10e^{-(1+i)x}}{3-3i}=\frac{10}{3}\frac{e^{-(1+i)x}}{1-i}=\frac{10}{3}\frac{(1+i)}{2}e^{-x}(\cos x+i\sin x)$$

$$y_p=\operatorname{Im}(\tilde{y}_p)=\frac{5}{3}e^{-x}(\cos x+\sin x)=\frac{5}{3}e^{-x}\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$$

If $p(\alpha)=0$. $[\alpha\to a\in\mathbb{C}]$

Exponential Shift Rule

$$p(D)e^{ax}u(x)=e^{ax}p(D+a)u(x)$$

Proof: $D(e^{ax}u)=e^{ax}Du+ae^{ax}u=e^{ax}(Du+au)=e^{ax}(D+a)u$.

Using the notation I for Identity operator we can rewrite this as

$D(e^{ax}u)=e^{ax}(D+aI)u$

If we apply D second time.

$D^2(e^{ax}u)=e^{ax}(D+aI)^2u$

This generalizes to

$D^k(e^{ax}u)=e^{ax}(D+aI)^ku.$

$(D^2+AD+B)y = e^{ax} \quad (a \in \mathbb{C})$

$P(a)=0 \rightarrow a$ is simple root of $p(D)$

$y_p = \dfrac{xe^{ax}}{p'(a)}$

If a is double root:

$y_p = \dfrac{x^2 e^{ax}}{p''(a)}$

Proof: Simple root case

$p(D)=(D-b)(D-a)$ $\quad b \neq a.$ $\quad p(D)\dfrac{e^{ax}x}{p'(a)} = e^{ax}$ $\quad \left[y_p = \dfrac{e^{ax}x}{p'(a)} \right]$

$p'(D)=(D-b)+(D-a)$

$p'(a)= b-a$

$e^{ax}(D+a-b)\dfrac{\overbrace{(D)x}^{1}}{a-b} = e^{ax}\dfrac{(a-b)1}{a-b} = e^{ax}.$

Double Root Case

$p(D)=(D-a)(D-a)$ $\quad p(D)\dfrac{e^{ax}x^2}{p''(a)} = e^{ax} = e^{ax}\,p(D^2)\dfrac{x^2}{p''(a)} = e^{ax}\cdot\dfrac{2}{2} = e^{ax}.$

$p'(D) = (D-a)+(D-a)$

$p''(D) = 2$

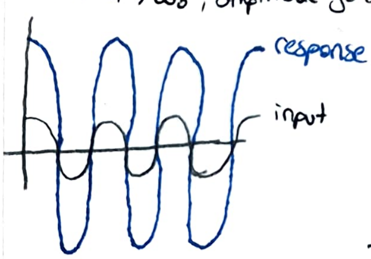
Resonance:

$y''+\omega_0^2 y = \cos\omega_1 t.$ $\quad \underset{\text{input}}{\omega_1} \neq \omega_0$, $(D^2+\omega_0^2)y = \omega_1 t$

$(D^2+\omega_0^2)\tilde{y} = e^{i\omega_1 t}$

$\tilde{y}_p = \dfrac{e^{i\omega_1 t}}{(i\omega_1)^2+\omega_0^2} = \dfrac{e^{i\omega_1 t}}{\omega_0^2-\omega_1^2}$, $\quad \mathrm{Re}(\tilde{y}_p) = y_p = \dfrac{\cos\omega_1 t}{\omega_0^2-\omega_1^2}$ = Response

when $\omega_1 \rightarrow \omega_0$, amplitude gets bigger



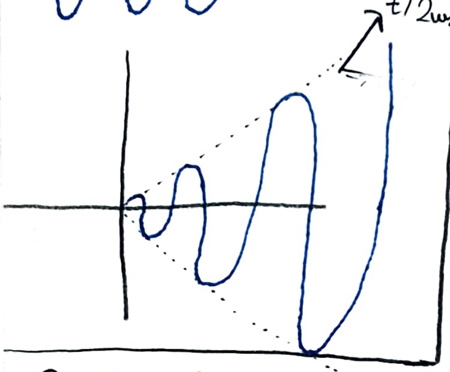
If $\omega_1 = \omega_0$

$(D^2+\omega_0^2)y = \cos\omega_0 t$

$(D^2+\omega_0^2)\tilde{y} = e^{i\omega t}$

$i\omega_0$ is simple root of $D^2+\omega_0^2$

$\tilde{y} = \dfrac{te^{i\omega t}}{2(i\omega_0)}$, $\quad \mathrm{Re}(\tilde{y}) = y = \dfrac{t(\sin(\omega_0 t))}{2\omega_0} = y_p$



$(D^2+\omega_0^2)y = \cos\omega_1 t$

$y_p = \dfrac{\cos\omega_1 t}{\omega_0^2-\omega_1^2} - \underbrace{\dfrac{\cos\omega_0 t}{\omega_0^2-\omega_1^2}}_{\text{(part of } y_c)}$, $\quad \lim\limits_{\omega_1\to\omega_0} \dfrac{\cos\omega_1 t - \cos\omega_0 t}{\omega_0^2 - \omega_1^2}$

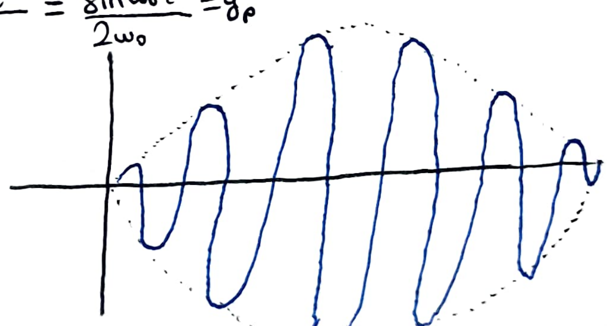
$= \lim\limits_{\omega_1\to\omega_0} \dfrac{-\sin\omega_1 t}{-2\omega_1} = \dfrac{\sin\omega_0 t}{2\omega_0} = y_p$

Geometric Meaning:

$\cos B - \cos A = 2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)$

$\dfrac{\cos\omega_1 t - \cos\omega_0 t}{\omega_1^2-\omega_0^2} = \dfrac{2\sin\left(\left(\frac{\omega_0-\omega_1}{2}\right)t\right)\underbrace{\sin\left(\left(\frac{\omega_0+\omega_1}{2}\right)t\right)}_{\sin(\omega_0 t)}}{\omega_1^2-\omega_0^2}$

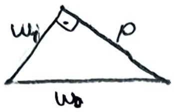
freq. small.



Damped Resonance

$x'' + 2px' + \omega_0^2 x = f(t)$

ω_0 = natural undamped frequency
ω_1 = psedue frequency, rotational damped frequency



$\omega_1^2 = \omega_0^2 - p^2$

$y'' + 2py' + \omega_0^2 y = \cos \omega t$

Problem: which ω gives maximum amplitude for the response, ω_r?

Let's say we know p and ω_0

answer: $\omega_r = \sqrt{\omega_0^2 - 2p^2}$

Fourier Series

$y'' + ay' + by = f(t)$, $f(t)$: exp, sint, cost, it was some special function but we'll see that if was not so special.

$f(t)$ is periodic and has period 2π.

$$f(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

Input	Response
$\sin(nt) \cdot b_n$	$\rightsquigarrow y_n^{(s)}(t) \cdot b_n$
$\cos(nt) a_n$	$\rightsquigarrow y_n^{(c)}(t)\, a_n$

$+$

$f(t) \rightsquigarrow \sum a_n y_n^{(c)}(t) + b_n y_n^{(s)}(t) + c_1$

Using superposition principle this is true since ODE is linear.

We're going to try to find fourier series for given $f(t)$ with period 2π.

$u(t), v(t)$ fns on $\mathbb{R}$, period 2π, Orthogonal on $[\pi, +\pi]$ if: $\int_{-\pi}^{\pi} u(t)\, v(t)\, dt = 0$

Theorem: $\begin{cases} \sin(nt), & n = 1, \dots, \infty \\ \cos(mt), & m = 0, \dots, \infty \end{cases}$ if any two distinct ones are orthogonal on $[-\pi, \pi]$

Include:

$\int_{-\pi}^{\pi} \sin^2(nt)\, dt =$ π

$\int_{-\pi}^{\pi} \cos^2(nt)\, dt =$

1-) Trig. Identities

2-) Complex Numbers

3-) Use ODE → that's the one we'll use.

Proof: $m \neq n$

Satisfy: $\sin(nt), \cos(nt)$, $x'' + n^2 x = 0$

x_n, v_m be any two of functions

$$\underbrace{\int_{-\pi}^{\pi} x_n'' v_m \, dt}_{\text{Unsymmetric}} = \underset{0}{x_n' v_m \Big|_{-\pi}^{\pi}} - \underbrace{\int_{-\pi}^{\pi} x_n' v_m' \, dt}_{\text{Symmetric}} \quad , \quad \int_{-\pi}^{\pi} x_n'' v_m dt = -n^2 \underbrace{\int_{-\pi}^{\pi} x_n v_m dt}_{\text{unsymmetric}} \quad , \quad \underbrace{-\int_{-\pi}^{\pi} x_n' v_m' dt}_{\text{symmetric}} = -n^2 \int_{-\pi}^{\pi} x_n v_m \, dt = \text{unsymmetric} = 0$$

$$\int_{-\pi}^{\pi} x_n v_m'' dt = \underset{0}{x_n v_m' \Big|_{-\pi}^{\pi}} - \underbrace{\int_{-\pi}^{\pi} x_n' v_m' dt}_{\text{Symmetric}} \quad , \quad \int_{-\pi}^{\pi} x_n v_m'' \, dt = -m^2 \underbrace{\int x_n v_m \, dt}_{\text{unsymmetric}} \quad , \quad \underbrace{-\int_{-\pi}^{\pi} x_n' v_m' dt}_{\text{Symmetric}} = \overbrace{-m^2 \int_{-\pi}^{\pi} x_n v_m \, dt}^{\text{unsymmetric}} = 0$$

$\int_{-\pi}^{\pi} x_n v_m \, dt = 0$ $(m \neq n)$. End of proof.

Given $f(t)$, 2π period and find a_n, b_n.

$$f(t) = \ldots \underbrace{a_k \cos(kt)}_{\text{any term}} + \cdots + \underbrace{a_n \cos(nt)}_{\text{wanted}} + \ldots$$

$$f(t)\cos(nt) = \ldots a_k \cos(kt)\cos(nt) + \cdots + a_n \cos^2(nt) \ldots$$

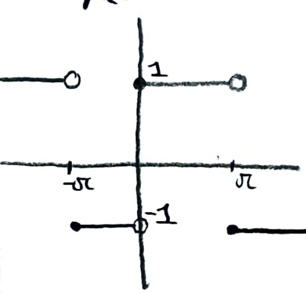
$$\int_{-\pi}^{\pi} f(t)\cos(nt) \, dt = \int_{-\pi}^{\pi} \ldots \underset{0}{a_k \cos(kt)\cos(nt)} + \cdots + \underset{a_n \pi}{a_n \cos^2(nt)} \ldots dt \quad , \quad \text{all terms} = 0 \text{ except } a_n \cos^2(nt) = \pi \cdot a_n$$

$$\int_{-\pi}^{\pi} f(t)\cos(nt) \, dt = a_n \pi \quad , \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos(nt) \, dt \quad , \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\sin(nt) \, dt \Big\} n = 1, 2, \ldots \infty$$

$$f(t) = c_0 + \cdots + a_n \cos(nt) + \ldots \quad , \quad \int_{-\pi}^{\pi} f(t) dt = 2\pi c_0 + \ldots \underset{0}{\int_{-\pi}^{\pi} a_n \cos(nt) dt} + \ldots \quad , \quad c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

$$\text{So; } f(t) = \overset{c_0}{\frac{a_0}{2}} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt). \Big\{ \text{General Form} \Big\}$$

Example:



$a_n = 0$

$$b_n = \underset{\frac{1-\cos n\pi}{n}}{-\int_{-\pi}^{0} \sin(nt) dt} + \underset{\frac{1-\cos n\pi}{n}}{\int_{0}^{\pi} \sin(nt) dt} = \frac{2}{n}(1 - \cos n\pi)$$

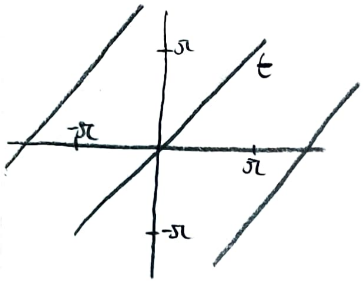
$\cos n\pi \rightarrow$ $-1, n = \text{odd}$; $1, n = \text{even}$ $\Big\} (-1)^n$

$b_n \Rightarrow \frac{4}{n} \{\text{odd}\}$

$0 \{\text{even}\}$

$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(nt)+b_n\sin(nt)$, $f(t)$ is periodic, has period 2π

$f(t)=g(t) \Rightarrow$ F.S of f = F.S of g



$f(t)$=even, $f(-t)=f(t)$: $f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(nt)$. (all b_n's are zero.)

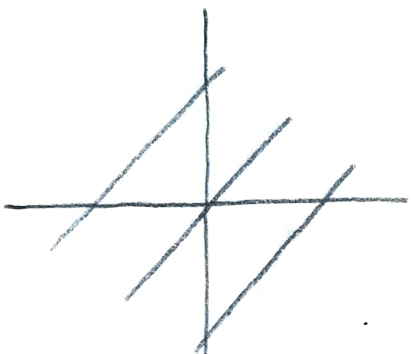
$f(-t)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(nt)-b_n\sin(nt)$

$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(nt)+b_n\sin(nt)$, $b_n=-b_n=0$ for all n.

$f(t)$=odd $\Rightarrow$ all $a_n=0$.

$f(t)$=even $\Rightarrow f(t)\cos(nt)$=even. , $a_n=\frac{2}{\pi}\int_0^{\pi} f(t)\cos(nt)dt$ ($b_n=0$, for $f(t)$=even)

$b_n=\frac{2}{\pi}\int_0^{\pi} \underbrace{f(t)}_{odd}\underbrace{\sin(nt)}_{odd}dt$, odd.odd = even function.



$f(t)$=odd.

$b_n=\frac{2}{\pi}\int_0^{\pi} t\sin(nt)dt=\frac{2}{\pi}\left[-\frac{t\cos(nt)}{n}\Big|_0^{\pi}\right]-\int_0^{\pi}-\frac{\cos nt}{n}dt$

$=\frac{2}{\pi}\left[\frac{-\pi}{n}(-1)^n+\underbrace{\frac{\sin(nt)}{n^2}\Big|_0^{\pi}}_{=0}\right] \Rightarrow b_n=\frac{2}{n}(-1)^{n+1}$

F.S for $f(t)=2\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin(nt) \Rightarrow 2(\sin t-\frac{1}{2}\sin 2t+\frac{1}{3}\sin 3t+\dots$

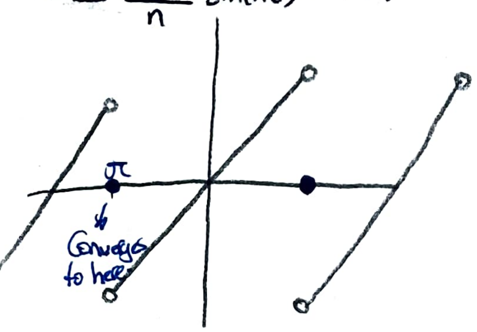
(Unlikely from Tailor Series the Fourier Series tries to get near to the original function under whole interval different from trying to find an approximation near $f(0)$, as the Tailor expention reffered.

Theorem:

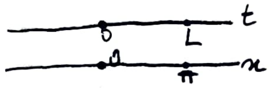
If f is continues at t_0, then $f(t_0)$ = sum of fourier series at t_0.

If f has jump discontinuity at t_0, then $f(t_0)$ converges to the average point (mid point) of jump.

$2\sum\frac{(-1)^{n+1}}{n}\sin(nt)=F(t)$



Extension #1 : Period is 2L.



$x = \frac{\pi}{L} t$

Use natural function : $\cos(n \frac{\pi}{L} t)$, $\sin(n \frac{\pi}{L} t)$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos\left(\frac{n\pi}{L} t\right) dt \ , \quad b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin\left(\frac{n\pi}{L} t\right) dt$$

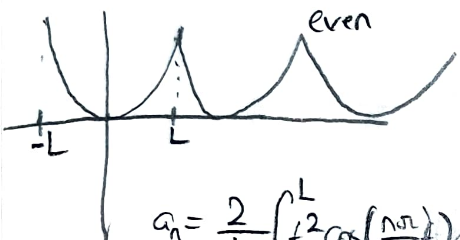
$f(t)$ even, period 2L , $a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi}{L}t\right)dt$.

$f(t)$ odd , period 2L, $b_n = \frac{2}{L}\int_0^L f(t) \sin\left(\frac{n\pi}{L}t\right)dt$

Extension #2: $f(t)$ defined on $[0, L]$

Make a periodic expension

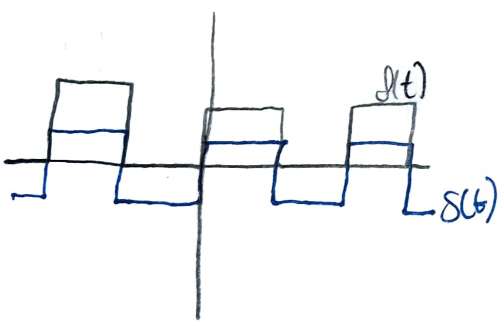
Let's talk for t^2.

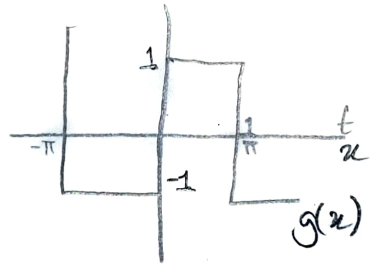


$$a_n = \frac{2}{L}\int_0^L t^2 \cos\left(\frac{n\pi}{L}t\right)dt$$

odd

$$b_n = \frac{2}{L}\int_0^L t^3 \sin\left(\frac{n\pi}{L}t\right) dt.$$





$g(x) = \frac{4}{\pi}\sum \frac{\sin(nx)}{n}$

$x = \pi t$.

$f(t) = s(t) + \frac{1}{2}$

$s(t) = \frac{1}{2} g(x)$

$f(t) = \frac{1}{2} + \frac{1}{2}\frac{4}{\pi}\sum_{odd} \frac{\sin(nx)}{n}$

$x'' + \omega_0^2 x = f(t)$, solve to find a particular sol'n, x_p.

Can find x_p if RHS: $\left\{\begin{array}{l}\cos \omega t \\ \sin \omega t\end{array}\right.$, $\left.\frac{\begin{array}{c}\cos(\omega t) \\ \sin(\omega t)\end{array}}{\omega_0^2 - \omega^2}\right\} = x_p$

If $f(t) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos \omega_n t + b_n \sin \omega_n t$, $\omega_n = n\pi/L$

$$x_p = \frac{a_0}{2\omega_0^2} \sum_1^{\infty} \frac{a_n \cos \omega_n t}{\omega_0^2 - \omega_n^2} + \frac{b_n \sin \omega_n t}{\omega_0^2 - \omega_n^2}$$

Input $f(t)$: $x'' + \omega_0^2 x = f(t)$

Responce: $\frac{1}{2\omega_0^2} + \frac{2}{\pi}\sum_{odd} \frac{\sin n\pi t}{(\omega_0^2 - (n\pi)^2)n}$

(2) Use differentiation of it. term by term.

Assume sol'n of form $x_p = c_0 + \sum c_n \sin(n\pi t)$, substitute into ODE.

$x_p'' = \sum_1^{\infty} -(n\pi)^2 c_n \sin(n\pi t)$; sum involving c_n's $\sin(n\pi t)$ = F.S. for $f(t)$.

$$\omega_0^2 c_0 + \sum c_n \left(\omega_0^2 - (n\pi)^2\right) \sin(n\pi t) = \frac{1}{2} + \sum_{odd} \frac{\sin(n\pi t)}{n}$$

$$c_0 = \frac{1}{2\omega_0^2} ; \quad \underset{odd}{c_n} = \frac{2}{\pi n} \frac{1}{\omega_0^2 - (n\pi)^2}$$

$$\sum_0^{\infty} a_n x^n = A(x) = \sum_0^{\infty} a(n) x^n$$

$a_n \rightsquigarrow A(x)$

$1 \rightsquigarrow \frac{1}{1-x}$, $|x|<1$. / if $x>1$ doesn't converge.

$\frac{1}{n!} \rightsquigarrow e^x$

I need continues analog

$t: 0 \leq t < \infty$ instead

$n = 0, 1, 2, \ldots$

$$\int_0^{\infty} a(t) x^t \, dt = A(x)$$, $x^t = e^{(\ln x)t}$, $0<x<1$ since we need it to converge : $\ln(x)<0$, $-s = \ln(x)$

$$\int_0^{\infty} f(t) e^{-st} \, dt = F(s) \Rightarrow$$ Laplace Transform.

$f(t) \to \boxed{\text{trans.}} \to F(s)$, $f(t) \to \boxed{\text{oper.}} \to g(t)$

$$\boxed{\begin{array}{l} \text{Linear Transform:} \\ \mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g) \\ \mathcal{L}(cf) = c\,\mathcal{L}(f) \end{array}}$$

$\mathcal{L}(f(t)) = F(s)$, $f(t) \rightsquigarrow F(s)$

$$\mathcal{L}(1) = \int_0^{\infty} e^{-st} dt = \lim_{R\to\infty} \int_0^R e^{-st} dt = \int_0^R e^{-st} dt$$

$$= \frac{e^{-st}}{-s}\Bigg|_0^R = \frac{e^{-sR} - 1}{-s} = \lim_{R\to\infty} \frac{e^{-sR}-1}{-s} = \frac{1}{s} \text{ if } s>0$$

$$\mathcal{L}(e^{at}) = \mathcal{L}(e^{at}\cdot 1) = \int_0^{\infty} e^{at} f(t) e^{-st} dt = \int_0^{\infty} e^{-(s-a)t} f(t)\, dt = F(s-a) \text{ if } s-a>0.$$

Can use also for $e^{(a+bi)t} \rightsquigarrow \frac{1}{s-(a+bi)}$

$\cos(at) = \frac{e^{iat} + e^{-iat}}{2}$, $\sin(at) = \frac{e^{+iat} - e^{-iat}}{2i}$

$$\mathcal{L}(\cos(at)) = \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right) = \frac{1}{2}\left(\frac{s+ia+s-ia}{s^2+a^2}\right) = \frac{1}{2}\left(\frac{2s}{s^2+a^2}\right) = \frac{s}{s^2+a^2} \quad s>0$$

$$\mathcal{L}(\sin(at)) = \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right) = \left(\frac{s+ia-(s-ia)}{s^2+a^2}\right)\frac{1}{2i} = \left(\frac{2ia}{s^2+a^2}\right)\frac{1}{2i} = \frac{a}{s^2+a^2} \quad s>0$$

We need to use partial fraction to find "Inverse" Laplace Transform.
Let's make an example! to remember.

$$\frac{1}{s(s+3)} = \frac{1/3}{s} + \frac{-1/3}{s+3}$$

$\mathcal{L}^{-1}$: $\frac{1}{3} - \frac{e^{-3t}}{3}$

$\mathcal{L}(t^n) = \int_0^\infty t^n e^{-st}\,dt$,

Remind: $(uv)' = u'v + uv'$

$\int u'v\,dx = \int (uv)'\,dx - \int uv'\,dx = uv - \int uv'\,dx = \int u'v\,dx.$

$$\int_0^\infty t^n e^{-st}\,dt = t^n \frac{e^{-st}}{-s}\Big]_0^\infty - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s}\,dt$$

, $\lim_{t\to\infty} \frac{t^n e^{-st}}{-s}$, $s>0$. $\Rightarrow \frac{1}{-s} \lim_{t\to\infty} \frac{t^n}{e^{st}} = 0$ by n app. of L'hopitals Rule.

$$\Rightarrow 0 - 0 + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st}\,dt = \frac{n}{s}\mathcal{L}(t^{n-1})$$

$$\mathcal{L}(t^n) = \frac{n}{s}\left(\mathcal{L}(t^{n-1})\right) = \frac{n(n-1)}{s^2}\left(\mathcal{L}(t^{n-2})\right) = \dots = \frac{n(n-1)\dots 1}{s^n}\mathcal{L}(t^0) = \frac{n!}{s^{n+1}}$$

Conditions for Laplace Transform:
$f(t)$ of "exponential type"

$|f(t)| \le c e^{kt}$ $c>0$ some constant and $k>0$ for all t.

Exp:

$|\sin t| \le 1 e^{0t}$, $t^n \le M e^t$ for some M all $t>0$

$\frac{t^n}{e^t} \le M$, $\frac{t^n}{e^t} \xrightarrow[t\to\infty]{} 0$

$\frac{1}{t}$: $\int_0^\infty \underbrace{e^{-st}}_{\approx 1}\frac{1}{t}\,dt$ like $\int_0^\infty \frac{dt}{t}$ $\Rightarrow$ doesn't converge Laplace Transform doesn't exist.

$e^{t^n} > e^{kt}$, $n>1$ $t^n >> kt$ as $t\to\infty$, e^{t^n} doesn't have Laplace Transform if $n>1$

$\boxed{\begin{matrix} y'' + Ay' + By = g(t) \\ y(0) = y_0,\ y'(0) = y_0' \end{matrix}}$ $\xrightarrow{\mathcal{L}}$ $\boxed{\text{Algebraic equation in } \bar{Y}(s)}$ $\xrightarrow[\bar{Y}]{\text{Solve for}}$ $\boxed{\bar{Y} = \frac{p(s)}{q(s)}}$ $\xrightarrow{\mathcal{L}^{-1}}$ $\boxed{y = y(t)}$

IVP. $y(t)$ $\bar{Y}(s)$

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big]_0^\infty - \int_0^\infty -s e^{-st} f(t)\,dt$$

, $\lim_{\substack{t\to\infty \\ s>0}} \frac{f(t)}{e^{st}} = 0$ since $f(t)$ is exp. type. $s>k$.

$= 0 - \underbrace{f(0)}_{\text{we have IVP. because of this.}} + s\,F(s)$

$f'(t) \rightsquigarrow sF(s) - f(0)$, $f''(t) \rightsquigarrow s\,\mathcal{L}(f'(t)) - f'(0) = s[sF(s) - f(0)] - f'(0)$

$f''(t) = [f'(t)]'$, $f''(t) \rightsquigarrow s^2 F(s) - s f(0) - f'(0)$, for 2nd order can be done for n-th order.

$y'' - y = e^{-t}$; $y(0)=1$, $y'(0)=0$

$\Big\downarrow \mathcal{L}$

$s^2 Y - s - Y = \frac{1}{s+1}$, $(s^2-1)Y = \frac{1}{s+1} + s = \frac{s^2+s+1}{s+1}$, $Y = \frac{s^2+s+1}{(s+1)^2(s-1)} = \frac{-1/2}{(s+1)^2} + \frac{C=1/4}{(s+1)} + \frac{3/4}{(s-1)}$

$\xrightarrow{\mathcal{L}^{-1}} -\frac{1}{2}te^{-t} + \frac{1}{4}e^{t} + \frac{3}{4}e^{t}$ $t \rightsquigarrow \frac{1}{s^2}$, $te^{-t} \rightsquigarrow \frac{1}{(s+1)^2}$ put $s=0$, $-1 = -\frac{1}{2} + C - \frac{3}{4} \Rightarrow C = 1/4$

Convolution: $f(t) * g(t)$

$$\left.\begin{aligned} F(s) &= \int_0^\infty f(t)\, e^{-st}\, dt \\ G(s) &= \int_0^\infty g(t)\, e^{-st}\, dt \end{aligned}\right\} F(s)G(s) = \int_0^\infty e^{-st} (f*g)\, dt$$

$$\left.\begin{aligned} F(x) &= \sum_0^\infty a_n x^n \ ;\ n \to t,\ x \to e^{-s} \\ G(x) &= \sum_0^\infty b_n x^n \ ;\ n \to t,\ x \to e^{-s} \end{aligned}\right\} F(x)G(x) = \sum_0^\infty c_n x^n$$

is there a formula for c_n in terms of a_n and b_n.

$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$; $f*g = g*f$ since $FG = GF$.

Exp:

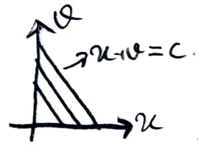
$t^2 * t = \int_0^t u^2 \cdot (t-u)\, du = \left.\frac{u^3 t}{3} - \frac{u^4}{4}\right]_0^t = \frac{t^4}{3} - \frac{t^4}{4} = \frac{t^4}{12}$

$\frac{2!}{s^3} \cdot \frac{1}{s^2} = \frac{2}{s^5}$, $\mathcal{L}^{-1}\left(\frac{2}{s^5}\right) = \frac{1}{12}t^4$ since $\mathcal{L}^{-1}\left(\frac{4!}{s^5}\right) = t^4$

Exp:

$f(t) * 1 = \int_0^t f(u)\, 1\, du = \int_0^t f(u)\, du$. $\Rightarrow$ Prove It.

$F(s)G(s) = \int_0^\infty e^{-su} f(u)\, du \cdot \int_0^\infty e^{-sv} g(v)\, dv = \int_0^\infty\!\!\int_0^\infty e^{-s(u+v)} f(u) g(v)\, du\, dv$



, $t = u+v$

$= \int_0^\infty\!\!\int_0^t e^{-st} f(u)\, g(t-u)\, du\, dt$, $du\, dv = \overbrace{\left(\frac{\partial(u,v)}{\partial(u,t)}\right)}^{1} du\, dt$,

1st) u varies, t is fixed

$u+v=C$.

if $u\uparrow$, $v\downarrow$; if $u\downarrow$, $v\uparrow$

$u = u$
$v = t-u$ $J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$

Radioactive waste: $f(t)$ = dump-rate, t = in years

Problem: start $t=0$. At time t how much radioactive waste is in pile?

Δt, t_i, t_{i+1} amount dumped $[t_i, t_{i+1}]$ $\approx f(t_i)\Delta t$

$\underbrace{\underset{\text{initial amount}}{A_0} e^{-kt}}_{\text{amt. left at time } t.}$ assume $k = C$.

$u_1\ u_2\ u_3\ u_4\ u_5 \ldots u_n$, 0, Δu, t, u

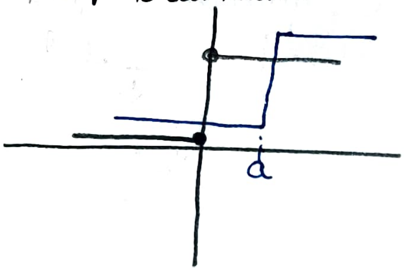
amount dumped in $[u_i, u_{i+1}] \approx f(u_i)\Delta u$ length of time u had on the pile.

By time t, it has decayed to $\approx \underbrace{f(u_i)\Delta u}_{\text{starting amount } u_i} \cdot e^{-k(t-u_i)}$

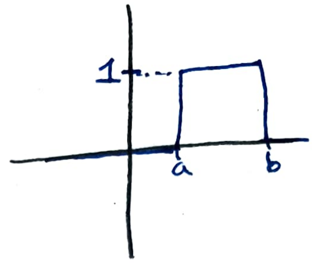
Total amount left in time $t \approx \sum_{i=1}^n f(u_i) e^{-k(t-u_i)} \Delta u$

let $\Delta u \to 0$ $\int_0^t f(u) e^{-k(t-u)}\, du = f(t) * e^{-kt}$.

Jump Discontinuities

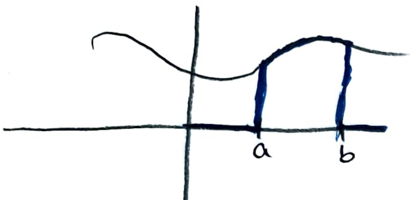


$u(t)$ "unit step"
$u(0)$ undefined
$u_a(t) = u(t-a)$



$u_{ab}(t)$: "unit box"

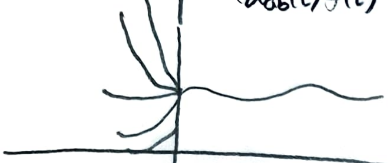
$u_{ab}(t) = u_a(t) - u_b(t)$
$= u(t-a) - u(t-b)$



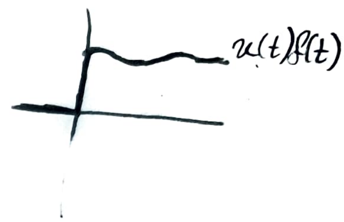
$u_{ab}(t) f(t)$

$\mathcal{L}(u(t)) = \int_0^\infty e^{-st} u(t)\,dt = 1/s \;; s>0 \;; \mathcal{L}(1) = 1/s \;, s>0$

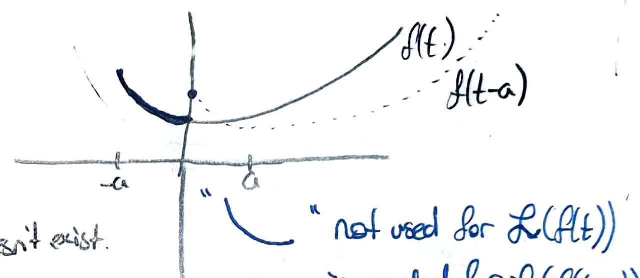
$\mathcal{L}^{-1}\left(\frac{1}{s}\right) =$ we can find multiple functions, since laplace transform doesn't not care about for $t<0$.



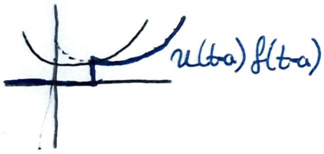
have some $F(s)$



$f(t) \overset{\mathcal{L}}{\rightsquigarrow} F(s) \;, \; F(s) \overset{\mathcal{L}^{-1}}{\rightsquigarrow} u(t) f(t)$



$\mathcal{L}(f(t-a))$ in terms of $\mathcal{L}(f(t))$ - doesn't exist.



$\left.\begin{array}{l} , u(t-a) f(t-a) \rightsquigarrow e^{-as} F(s). \\ u(t-a) f(t) \rightsquigarrow e^{-as} \mathcal{L}(f(t+a)) \end{array}\right\}$ t-axis translation.

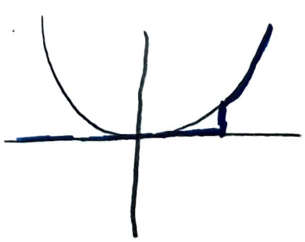
$\int_0^\infty e^{-st} u(t-a) f(t-a)\,dt \;, t_1 = t-a$

$\int_{-a}^\infty e^{-s(t_1+a)} u(t_1) f(t_1)\,dt_1 = e^{-as} \int_{-a}^\infty e^{-st_1} \underset{0}{u(t_1)} f(t_1)\,dt_1 = e^{-as} \int_0^\infty e^{-st_1} f(t_1)\,dt$, since $u(t_1) = 0$ for $t_1 < 0$ and $u(t_1) = 1$ for $t_1 > 0$.

$u(t-a) f(t-a) \rightsquigarrow e^{-as} \mathcal{L}(f(t))$
$u(t-a) f(t)$
$u(t-a) f(t-a+a) \rightsquigarrow e^{-as} \mathcal{L}(f(t+a))$

Exp: $u_{ab}(t) = u(t-a) - u(t-b) \overset{\mathcal{L}}{\rightsquigarrow} \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}$

Exp: $u(t-1) t^2 \overset{\mathcal{L}}{\rightsquigarrow} e^{-s} \mathcal{L}(t+1)^2 = e^{-s} \mathcal{L}(t^2 + 2t + 1) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) \longrightarrow$



Exp: $\mathcal{L}^{-1}\left(\frac{1 + e^{-\pi s}}{s^2+1}\right) : \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$

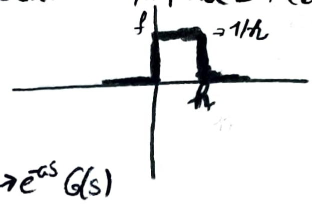
$\frac{1}{s^2+1} \overset{\mathcal{L}}{\rightsquigarrow} u(t)\sin(t) \;, \; \frac{e^{-\pi s}}{s^2+1} \overset{\mathcal{L}}{\rightsquigarrow} u(t-\pi)\sin(t-\pi)$

Answer: $u(t)\sin(t) + u(t-\pi)\sin(t-\pi) : f(t)$

$f(t) = \begin{cases} \sin(t), & 0 \le t \le \pi \\ \sin(t) + (-\sin(t)), & t > \pi \end{cases} , \quad f(t) = \begin{cases} \sin(t), & 0 \le t \le \pi \\ 0, & t > \pi \end{cases}$

For a unit impulse $f(t)$ - force $[a,b]$, $f(t) =$ constant F, impulse $= F(b-a)$

$\int_a^b f(t)dt$



$y''+y = \frac{1}{h} u_{0h}(t) \xrightarrow{\mathcal{L}}$ Remind: $u(t-a)g(t-a) \xrightarrow{\mathcal{L}} e^{-as}G(s)$

$$\frac{1}{h}\left[u(t)-u(t-h)\right] \xrightarrow{\mathcal{L}} \frac{1}{h}\left[\frac{1}{s}-\frac{e^{-hs}}{s}\right], \text{ let } h \to 0.$$

$\lim_{h\to 0} \frac{1-e^{-hs}}{hs} = \lim_{x\to 0} \frac{1-e^{-x}}{x} = \lim_{x\to 0}\frac{e^{-x}}{1} = 1.$ $\quad \frac{1}{h}u_{0h}(t) \xrightarrow{\mathcal{L}} \frac{1-e^{-hs}}{hs}$; as $h \to 0$

$\delta(t) \xrightarrow{\mathcal{L}} 1$

dirac-Delta Notation "not a function"

$\int_{-\infty}^{\infty} \delta(t)dt = 1.$

$f(t) * \delta(t) \xrightarrow[\mathcal{L}]{} F(s)\cdot 1$; $u(t) f(t) \xrightarrow{\mathcal{L}} F(s)$, $u'(t) = \delta(t)$

$y''+y = A\delta(t-t_0)$, kicked with $t=t_0$ with impulse A.

$y(0)=1$, $y'(0)=0$

$$s^2Y - s + Y = Ae^{-t_0 s} \Rightarrow Y = \frac{s}{s^2+1} + \frac{Ae^{-t_0 s}}{s^2+1}; \quad y = \cos(t) + u(t-t_0)A\sin(t-t_0), \quad t_0 = \pi/2$$

$$y = \begin{cases} \cos(t), & 0 \le t \le \pi/2 \\ (1-A)\cos(t), & t > \pi/2 \end{cases}$$

$y''+ay'+by = f(t)$; $y(0)=0$, $y'(0)=0$, $Y = F(s)\frac{1}{s^2+as+b}$, $y(t) = f(t) * w(t)$

$s^2Y + asY + bY = F(s)$

$\frac{1}{s^2+as+b} = W(s) \Rightarrow$ transfer function, $w(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+as+b}\right)$; $y(t) = f(t) * w(t) = \int_0^t (f(u)) w(t-u)du$

$w(t) =$ weight function of the system

$y''+ay'+by = \delta(t)$; $y(0)=y'(0)=0$, kick mass at $t=0$, unit impuls.

$\downarrow \mathcal{L}$

$s^2Y + asY + bY = 1$, $Y = \frac{1}{s^2+as+b} \xrightarrow{\mathcal{L}^{-1}} y(t) = w(t)$

Systems: {First-Orders}

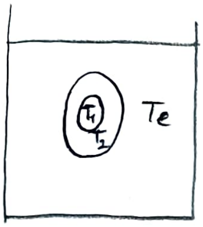
$x' = f(x,y;t)$ x, y dependent variables, t is independent.
$y' = f(x,y;t)$

Linear System

a, b, c, d => constants   a, b, c, d => functions of t.   Linear Homogenous $r_1, r_2 = 0$

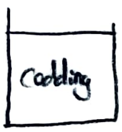
$$x' = ax + by + r_1(t)$$
$$y' = cx + dy + r_2(t)$$

$x(t_0) = x_0,\ y(t_0) = y_0$



$$\frac{dT_1}{dt} = k_1(T_2 - T_1),\quad T_1' = -k_1T_1 + k_1T_2$$

$$\frac{dT_2}{dt} = k_1(T_1 - T_2) + k_2(T_e - T_2),\quad T_2' = k_1T_1 - T_2(k_1+k_2) + k_2T_e$$



: $T_e = 100e^{-kt}$

: $T_e = 0$, Need to eliminate one of the dependent variable so we can work easily.

$$T_1' = -k_1T_1 + k_1T_2 \Rightarrow T_2 = \frac{T_1' + k_1T_1}{k_1}$$

$$\left(\frac{T_1' + k_1T_1}{\cancel{k_1}}\right)' = k_1^2T_1 - (k_1+k_2)\left(\frac{T_1' + k_1T_1}{\cancel{k_1}}\right) \Rightarrow T_1'' + k_1T_1' + \underbrace{(k_1+k_2)(T_1' + k_1T_1)}_{k_1T_1' + k_2T_1' + k_1^2T_1 + k_1k_2T_1} - k_1^2T_1 = 0$$

$$\Rightarrow T_1'' + (k_2 + 2k_1)T_1' + k_1k_2T_1 = 0 \Rightarrow r^2 + (k_2+2k_1)r + k_1k_2 = 0$$

$$\Delta = (k_2+2k_1)^2 - 4k_1k_2 = k_2^2 + 4k_1^2 + 4k_1k_2 - 4k_1k_2 = k_2^2 + 4k_1^2 \Rightarrow r_{1,2} = \frac{\overbrace{-(k_2+2k_1)}^{a} \pm \overbrace{\sqrt{k_2^2+4k_1^2}}^{b}}{2};\quad r_1 = \frac{a+b}{2},\ r_2 = \frac{a-b}{2}$$

$$T_1 = c_1e^{r_1t} + c_2e^{r_2t};\quad T_2 = \frac{1}{k_1}c_1r_1e^{r_1t} + \frac{1}{k_1}c_2r_2e^{r_2t} + T_1$$

$T_1(0) = x_1,\ T_2(0) = x_2;\quad x_1 = c_1 + c_2,\ x_2 = \left(c_1(r_1+k_1) + c_2(r_2+k_1)\right)/k_1.$

$x' = f(x,y)$
$y' = g(x,y)$
{autonomous}
Velocity field

Sol'ns:
$x = x(t)$
$\underbrace{y = y(t)}$
parametric curve.

$\left\{ \begin{array}{l} x'(t) \\ \underbrace{y'(t)} \end{array}\right.$
velocity of sol'n at time t.

Sol'n: curve parameterized with right velocity everywhere

$x' = -k_1 x + k_1 y$ $\quad x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
$y' = k_1 x - (k_1+k_2) y$ $\quad y = \frac{c_1 e^{r_1 t}}{k_1} + k_1 c_2 e^{r_2 t}$ } gen. sol'ns.

$$\begin{vmatrix} x \\ y \end{vmatrix}' = \begin{vmatrix} -k_1 & k_1 \\ k_1 & -(k_1+k_2) \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix}, \quad \begin{vmatrix} x \\ y \end{vmatrix} = c_1 \begin{vmatrix} e^{r_1} \\ e^{r_1}/k_1 \end{vmatrix} e^{t} + c_2 \begin{vmatrix} e^{r_2} \\ -k_1 e^{r_2} \end{vmatrix} e^{t}; \quad \left.\begin{matrix} x = a_1 e^{\lambda_1 t} \\ y = a_2 e^{\lambda_2 t} \end{matrix}\right\} \underbrace{r_1, r_2}_{\text{dep. of}}, \underbrace{k_1, k_2}_{\text{indep.}} \Rightarrow c.$$

$\Rightarrow$ trial solution $\begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} a_1 \\ a_2 \end{vmatrix} e^{\lambda t}$. substitute into system.

$$\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cancel{e^{\lambda t}} = \begin{vmatrix} -k_1 & k_1 \\ k_1 & -k_1-k_2 \end{vmatrix} \begin{vmatrix} a_1 \\ a_2 \end{vmatrix} \cancel{e^{\lambda t}} \Rightarrow \lambda a_1 = -k_1 a_1 + k_1 a_2 \; ; \; \lambda a_2 = k_1 a_1 - (k_1+k_2) a_2$$

$(-k_1 - \lambda) a_1 + k_1 a_2 = 0; k_1 a_1 + (-k_1 - k_2 - \lambda) a_2 = 0$. when does it have non-trivial sol'n?

$$\Leftrightarrow \underbrace{\begin{vmatrix} -k_1-\lambda & k_1 \\ k_1 & -k_1-k_2-\lambda \end{vmatrix}}_{A} = 0 \Rightarrow$$ k_1, k_2 were the conductivity constants for the system we created, known.

$$\det(A) = (k_1^2 + k_1 k_2 + k_1\lambda + k_1\lambda + k_2\lambda + \lambda^2) - k_1^2 = \overset{6}{k_1 k_2} + \overset{-4}{2k_1\lambda} + \overset{-2}{k_2\lambda} + \overset{1}{\lambda^2} = 0$$ for the given constants k_1, k_2.

{The same equation we've got from other method of solving except r must be replaced with λ}

Let's take $k_1 = 2$ and $k_2 = 3$ from that $\lambda = -1$ or -6.

$\lambda = -1$, find a_1, a_2;

$$\left.\begin{matrix} -a_1 + 2a_2 = 0 \\ 2a_1 - 4a_2 = 0 \end{matrix}\right\} \text{sol'n: } \begin{vmatrix} 2k \\ k \end{vmatrix}, \; k \neq 0$$

sol'n to system: $\begin{vmatrix} 2k \\ k \end{vmatrix} e^{-t}$

$\lambda = -6$, find a_1, a_2;

$$\left.\begin{matrix} 4a_1 + 2a_2 = 0 \\ 2a_1 + 1a_2 = 0 \end{matrix}\right\} \text{sol'n: } \begin{pmatrix} l \\ -2l \end{pmatrix}$$, sol'n to system: $\begin{pmatrix} l \\ -2l \end{pmatrix} e^{-t}$.

Superposition Principle:

$$\tilde{c}_1 \begin{pmatrix} 2k \\ k \end{pmatrix} e^{-t} + \tilde{c}_2 \begin{pmatrix} l \\ -2l \end{pmatrix} e^{-6t} \Rightarrow ; c_1/2 = \tilde{c}_1, \; c_2 = \tilde{c}_2$$

In general:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{B} \begin{pmatrix} x \\ y \end{pmatrix}; \quad \text{Trial: } \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t}; \quad \text{Substitute: } \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Homogenous system

$$\underbrace{\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}}_{A} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$ Solveable (non-trivially) $\Leftrightarrow \det(A) = 0$.

$$\det(A) = (a-\lambda)(d-\lambda) - (bc) = \lambda^2 - \underbrace{(a+d)}_{\substack{tr\,B \\ \text{(trace)}}}\lambda + \underbrace{ad - bc}_{\det(B)} = 0$$ $\Rightarrow$ characteristic equation of matrix.

roots λ_1, λ_2 are eigenvalues of B.

For each λ_i, find associated $\vec{\alpha}_i = \begin{pmatrix} a_{1i} \\ a_{2i} \end{pmatrix} \Rightarrow$ Called eigenvector associated to λ_i.

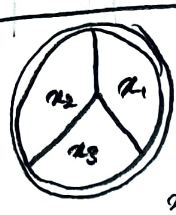
by solving system: $\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

gen. sol'n.

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} e^{\lambda_2 t}$$

$\begin{vmatrix} x \\ y \\ \vdots \end{vmatrix} = \vec{x}$; $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$, $\vec{x}' = A\vec{x}$ trial: $\vec{x} = \vec{\alpha} e^{\lambda t}$, $\vec{\alpha} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$\lambda \vec{\alpha} e^{\lambda t} = A \vec{\alpha} e^{\lambda t}$. $A\vec{\alpha} = \lambda \vec{\alpha} \Rightarrow (A - \lambda I)\vec{\alpha} = 0. \Rightarrow |A - \lambda I| = 0.$



x_i: temp in tank ; $x_i(t)$

$x_1' = k(x_3 - x_1) + k(x_2 - x_1) = -2kx_1 + kx_2 + kx_3$; $k = 1$.

$x_1' = -2x_1 + x_2 + x_3$, $x_2' = x_1 - 2x_2 + x_3$, $x_3' = x_1 + x_2 - 2x_3$.

$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = -(\lambda+2)^3 + 2 - 3(-2-\lambda) = 0. \Rightarrow (\lambda^3 + 3\lambda^2 \cdot 2 + 6 \cdot 2\lambda + 8) + 2 - 6 - 3\lambda = 0$

$\Rightarrow \lambda^3 + 6\lambda^2 + 9\lambda = 0 = (\lambda^2 + 3\lambda)(\lambda + 3)$: eigenvalues: $\lambda = 0$; $\lambda = -3$, {double}

$\lambda = 0$;

$\vec{\alpha}$: ? $A\vec{\alpha} = 0$, $A = \begin{vmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix}$, $\left.\begin{matrix} -2a_1 + a_2 + a_3 = 0 \\ a_1 + 2a_2 + a_3 = 0 \\ a_1 + 2a_2 - 2a_3 = 0 \end{matrix}\right\}$ does not have an inverse matrix.

$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{0t} = c$; constant sol'n. $a_1 = a_2 = a_3$.

$\lambda = -3$.

$\left.\begin{matrix} a_1 + a_2 + a_3 = 0 \\ a_1 + a_2 + a_3 = 0 \\ a_1 + a_2 + a_3 = 0 \end{matrix}\right\}$ $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-3t}$, $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-3t}$

gen. sol'n: $c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-3t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{0t} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

when $t \to \infty$, $\left(c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right) e^{-3t} \to 0$ and only constant sol'n remains which means system tends to the equilibrium state.

if λ is a repeated eigenvalue, you can find enough independent eigenvectors to make up the needed number of repeated solutions.

This kind of eigenvalue called "complete". Other case is called "defective".

- Complex eigenvalues
  - Calculate cx. eigenvalues
  - form sol'ns: $\vec{\alpha} e^{(a+bi)t}$ ← cx

take Re part, get two sol'ns.

$\begin{matrix} x' = x + 2y \\ y' = -x + y \end{matrix}$, $A = \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix}$

char. eq: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$. = eigenvalues.

system for eigenvector

$(1-i)a_1 + 2a_2 = 0$

$-a_1 + (-1-i)a_2 = 0$

$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{i-1}{2} \end{pmatrix}$: cx. sol'n: $\begin{pmatrix} 1 \\ \frac{i-1}{2} \end{pmatrix} e^{it}$

$\Rightarrow \left(\begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} i\right)(\cos(t) + i\sin(t))$

$= \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} \cos(t) - \begin{pmatrix} 0 \\ +1/2 \end{pmatrix} \sin(t) = \begin{pmatrix} x \\ y \end{pmatrix}$: $x = \cos(t)$, $y = -\frac{1}{2}(\cos(t) + \sin(t))$

{ellipses}

How to Sketch Solutions of $\begin{pmatrix}x\\y\end{pmatrix}' = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

$x' = -x + by$: Massachussets
$y' = cx - 3y$; N.H.

x } departures from normal
y } tourists adverfising budget

$x' = -x + 2y$
$y' = -3y$: $A = \begin{vmatrix}-1 & 2\\ 0 & -3\end{vmatrix} \Rightarrow \lambda^2 + 4\lambda + 3 = 0 \Rightarrow \lambda = -3, -1$. e-values.

$\lambda = -3$
$2a_1 + 2a_2 = 0 \quad \vec{\alpha} = \begin{pmatrix}1\\-1\end{pmatrix}$

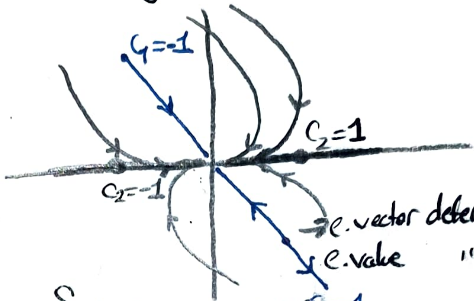
$\lambda = -1$
$0a_1 + 2a_2 = 0$
$\vec{\alpha} = \begin{pmatrix}1\\0\end{pmatrix}$

gen. sol'n: $\begin{pmatrix}x\\y\end{pmatrix} = c_1 \underline{\begin{pmatrix}1\\-1\end{pmatrix} e^{-3t}} + c_2 \underline{\begin{pmatrix}1\\0\end{pmatrix} e^{-t}}$.

$x = c_1 e^{-3t} + c_2 e^{-t}, \; y = -c_1 e^{-3t}$.

1. 4-easy sol'ns. } $c_1 = \pm 1, \; c_2 = 0$
2. Fill in. } $c_1 = 0, \; c_2 = \pm 1$

$x = e^{-3t}, \; y = -e^{-3t}$



→ e.vector determines the line and
↘ e.value " its direction.

{Stable–asymtotically}

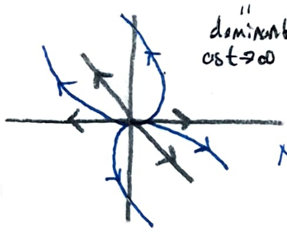
$c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-3t} + c_2\begin{pmatrix}1\\0\end{pmatrix}e^{-t} \to 0$ as $t \to \infty$

dominant as $t \to -\infty$ — dominant as $t \to \infty$

The fact that all the trajectory end up at origin called "sink"
The general way that this lines look decribed by word "node".

If A: $\vec{x} = c_1\begin{pmatrix}1\\-1\end{pmatrix}e^{+3t} + c_2\begin{pmatrix}1\\0\end{pmatrix}e^{t}$

dominant as $t \to \infty$ — dominant as $t \to -\infty$

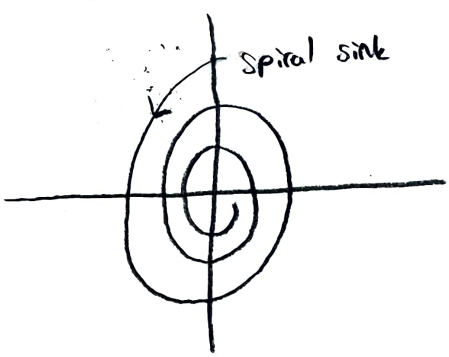


Nodal Source {source node}
(Unstable)

$x' = -x - y$
$y' = 2x - 3y$, $A = \begin{vmatrix}-1 & -1\\ 2 & -3\end{vmatrix} \Rightarrow \lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda = -2 \pm i$; $\vec{\alpha}$ cx. e-vector

$\begin{pmatrix}x\\y\end{pmatrix} = c_1\left[\begin{pmatrix}a_1\\a_2\end{pmatrix}\cos(t) + \begin{pmatrix}b_1\\b_2\end{pmatrix}\sin(t)\right]e^{-2t} + c_2\left[\begin{pmatrix}a_1\\a_2\end{pmatrix}\cos(t) - \begin{pmatrix}b_1\\b_2\end{pmatrix}\sin(t)\right]e^{-2t}$

as a curve bounded, period 2π; satisfies $Ax^2 + By^2 + Cxy = D$. $\Rightarrow$ hyperbolas, parabolas and ellipses since bounded



Theorem:

$\vec{x}' = A\vec{x}$. (A constant), 2×2.

$\boxed{A}$: Gen sol'n to $\vec{x}' = A\vec{x}$ is $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$ where $\vec{x}_1, \vec{x}_2$ are two linearly independent sol'n.

Easy to show that these are sol'ns. (from linearity and superposition principle)

Hard to show that those are the all sol'ns to the system.

$\boxed{B}$: Wronskian of sol'ns. $W(\vec{x}_1, \vec{x}_2)$: $\left|\vec{x}_1 \; \vec{x}_2\right| = \det(\vec{x}_1, \vec{x}_2)$

Thm: $W(t)$ → $\equiv 0$ (if $\vec{x}_1, \vec{x}_2$ are dependent)
→ never 0 (for any t value). if $(\vec{x}_1, \vec{x}_2)$ are linearly independent.

Fundamental matrix for $\vec{x}' = A\vec{x}$. $X := [\vec{x}_1 \; \vec{x}_2]$; $\vec{x}_1, \vec{x}_2$ independent solutions.

Properties:

1-) $|X| \neq 0$ for any t.

2-) $X' = AX \Longleftrightarrow [\vec{x}_1' \; \vec{x}_2'] = A[\vec{x}_1 \; \vec{x}_2] = [A\vec{x}_1 \; A\vec{x}_2] \Longleftrightarrow \vec{x}_1' = A\vec{x}_1, \; \vec{x}_2' = A\vec{x}_2$

Inhomogenous Systems:

$x' = ax + by + r_1(t)$
$y' = cx + dy + r_2(t)$

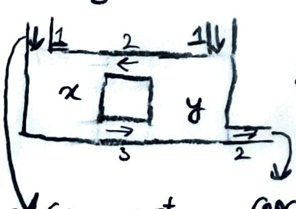
Thm. C: $\underbrace{\vec{x}_{gen}}_{\text{gen sol'n.}} = \underbrace{\vec{x}_c}_{\text{gen sol'n to } \vec{x}' = A\vec{x}} + \underbrace{\vec{x}_p}_{\text{particular sol'n.}}$ → find $\vec{x}_p$.

$\vec{x}' = A\vec{x} + r(t)$

At before we've struggled to find $\vec{x}_p$, from now on we'll see that its easy.

$x'' + x = \tan t$.

Mixing problem



Concn. $= 5e^{-t}$ Concn. $= 0$.

both tanks 1 liters, flow rates are liters per hour $\left[\text{Concentration} = \frac{x}{1}\right]$

x = amount salt tank 1.

y = " " " 2.

$\left.\begin{aligned} x' &= -3x + 2y + 5e^{-t} \\ y' &= 3x - 4y + 0. \end{aligned}\right\}$ $\vec{x}' = \begin{bmatrix} -3 & 2 \\ 3 & -4 \end{bmatrix}\vec{x} + \begin{pmatrix} 5e^{-t} \\ 0 \end{pmatrix}$

Method to solve $\vec{x}' = A\vec{x} + r(t)$, find $\vec{x}_p$. The method for solving called;

"Variation of Parameters."

Variation of Parameters

$\vec{x}' = A\vec{x} + r(t)$, $\vec{x}_{gen} = \vec{x}_c + \vec{x}_p$.

$\vec{x}_p = u_1(t)\vec{x}_1 + u_2(t)\vec{x}_2$, $\vec{x}_p = \mathbf{X}\vec{u} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ => substitute into that and see what u is.

$\vec{x}' = A\vec{x} + r(t)$, $\vec{x}_p' = A\vec{x}_p + r(t)$

$\cancel{\mathbf{X}_p'\vec{u}} + \mathbf{X}_p\vec{u}' = \cancel{A\mathbf{X}\vec{u}} + r(t) \Rightarrow \mathbf{X}_p\vec{u}' = r(t) \Rightarrow \vec{u}' = X^{-1}r(t)$ if $\det(X^{-1}) \neq 0$ since clmns are indep.

$\vec{u}' = \int \mathbf{X}^{-1} r(t)\,dt$ (integrate each individual entry)

Particular Solution: $\vec{x}_p = \mathbf{X}\int \mathbf{X}^{-1}\vec{r}\,dt$

$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$, $\vec{x} = \mathbf{X}\vec{c} = [\vec{x}_1\ \vec{x}_2]\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$[\mathbf{X}\vec{c}_1\ \mathbf{X}\vec{c}_2] = \mathbf{X}\underbrace{[\vec{c}_1\ \vec{c}_2]}_{C} = \mathbf{X}C$, $|C| \neq 0$.

$\vec{x}' = A\vec{x}$. The solution given by $\boxed{\text{formula}}$

1×1 case: $x' = ax$. sol'n is $x = ce^{at}$

$e^{at} = 1 + at + a^2t^2/2 + a^3t^3/3! + \cdots \Rightarrow \frac{de^{at}}{dt} = \left(a + a^2t + \frac{a^3t^2}{2!} + \cdots\right)$

$\Rightarrow a.e^{at}$.

A Fundamental Matrix for : $\vec{x}' = A\vec{x}$. is e^{At}. what the...

$e^{At} := I_2 + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots$ Is this a fundamental matrix for system? Satisfies $\mathbf{X}' = A\mathbf{X}$ ($\mathbf{X} = e^{At}$)

This is same calculation as for 1x1 case.

1-) $|\mathbf{X}(0)| = I_2 \neq 0$.

$x' = y$
$y' = x$, $\vec{x} = \overset{A}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}\vec{x}$; $A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = I_2$, $A^3 = I_2.A = A$, $A^4 = A.A = I_2$

$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}t + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{t^2}{2!} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}t^3/3! + \cdots$

$e^{At} = \begin{bmatrix} 1 + t^2/2! + t^4/4! + \cdots & t + t^3/3! + t^5/5! + \cdots \\ t + t^3/3! + t^5/5! + \cdots & 1 + t^2/2! + t^4/4! + \cdots \end{bmatrix} = \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix} = \begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}\frac{1}{2}$.

IVP: $\vec{x}' = A\vec{x}$; $\vec{x}(0) = \vec{x}_0$, find $\vec{x}(t)$

gen. sol'n: $\vec{x} = e^{At}.\vec{c}$

$\vec{x}(0) = e^{A0}.\vec{c}$, $\vec{x}_0 = I.\vec{c} \Rightarrow \vec{c} = \vec{x}_0$

$\boxed{\text{Solution: } \vec{x} = e^{At}\vec{x}_0}$

$e^{A+B} = e^Ae^B$; -A, B are square matricies - these statement is true under the cases.

1-) $A = cI$

2-) $B = -A$ $\longrightarrow$ $e^{A-A} = e^A.e^{-A}$
$I = e^A.e^{-A} = (e^A)^{-1} = e^{-A}$.

3-) $B = A^{-1}$

Calculation of e^{At}.

1-) Using series (too hard)

2-) $\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$

3-) $\underset{\text{F.M.}}{\mathbf{X}}.\mathbf{X}(0)^{-1}$ a-) F.M
b-) value at 0. $\mathbf{X}(0)\mathbf{X}(0)^{-1} = I$.

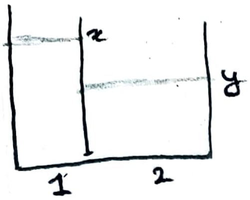
e^{At} has same identities

$\mathbf{X}.X(0)^{-1} = e^{At}$.

③ Decoupling: $u = ax + by$, $v = cx + dy$ } find u, v | in uv coordinates.

$u' = k_1 u$, $v' = k_2 v$.

Example:



flow rate through hole [cu.cm/s] $\propto$ area of hole $\times$ ht. difference (ht. diff $\propto$ pressure diff.)

$x' = c(y-x)$, $2y' = c(x-y)$ take $c=2$: $\begin{cases} x' = -2x+2y \\ y' = x-y \end{cases}$

$u = x+2y$ (total amount of water)
$v = x-y$ (pressure at the hole)

$u' = x' + 2y' = 0$
$v' = x' - y' = -3x+3y$

$\begin{cases} u' = 0 \\ v' = -3(x-y) = -3v \end{cases}$ decoupled;

Sol'n: $u = c_1$, $v = c_2 e^{-3t}$

In terms of xy

$x = \frac{1}{3}(u+2v) = \frac{1}{3}(c_1 + 2c_2 e^{-3t})$
$y = \frac{1}{3}(u-v) = \frac{1}{3}(c_1 - c_2 e^{-3t})$

$\vec{x} = \frac{1}{3}c_1\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{3}c_2\begin{pmatrix}2\\-1\end{pmatrix}e^{-3t}$

Solution with General Case:

$\begin{pmatrix}x\\y\end{pmatrix}' = \begin{bmatrix}-2 & 2\\ 1 & -1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} \Rightarrow \lambda^2 + 3\lambda = 0 \Rightarrow \lambda = 0, -3.$

$\lambda = 0$; $-2a_1 + 2b_1 = 0$. $\vec{\alpha}_1 = \begin{pmatrix}1\\1\end{pmatrix}$

$\lambda = -3$; $a_1 + 2b_1 = 0$ $\vec{\alpha}_2 = \begin{pmatrix}-2\\1\end{pmatrix}$

$E = \begin{bmatrix}1 & -2\\ 1 & 1\end{bmatrix}$, $D = E^{-1} = \begin{bmatrix}1 & 2\\ -1 & 1\end{bmatrix}\cdot\frac{1}{3}$

$\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}1 & 2\\ -1 & 1\end{pmatrix}\frac{1}{3}\begin{pmatrix}x\\y\end{pmatrix}$: $u = \frac{1}{3}(x+2y)$, $v = \frac{1}{3}(-x+y)$

$u' = 0$, $v' = -3v$.

For general Case Decoupling: e-values must be real and complete.

$\underbrace{\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}a_1 & a_2\\ b_1 & b_2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}}_{D\text{- decoupling mx.}}$. we need $\underbrace{\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a_1 & a_2\\ b_1 & b_2\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}}_{D^{-1} = E.} \Rightarrow E = (\vec{\alpha}_1, \vec{\alpha}_2)$

$\vec{\alpha}_1 \rightsquigarrow \begin{pmatrix}1\\0\end{pmatrix}$
$\vec{\alpha}_2 \rightsquigarrow \begin{pmatrix}0\\1\end{pmatrix}$
$xy \rightsquigarrow uv$

$xy = \begin{pmatrix}a_1 & a_2\\ b_1 & b_2\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}$; columns $\vec{\alpha}_1$, $\vec{\alpha}_2$

if $\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \rightsquigarrow \begin{pmatrix}a_1\\b_1\end{pmatrix}$
if $\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \rightsquigarrow \begin{pmatrix}a_2\\b_2\end{pmatrix}$

Substitute into System $\vec{x}' = A\vec{x}$ and see if its decoupled in the uv-plane.

$(A - \lambda_1 I)\vec{\alpha}_1 = \vec{0}$; we first defined λ_1 then $\vec{\alpha}_1$.

In Linear algebra, $A\vec{\alpha}_1 = \lambda_1\vec{\alpha}_1$ we first define $\vec{\alpha}_1$ then find the λ_1.

$$A.E = A[\vec{\alpha}_1\ \vec{\alpha}_2] = [A\vec{\alpha}_1\ A\vec{\alpha}_2] = [\lambda_1\vec{\alpha}_1\ \ \lambda_2\vec{\alpha}_2] = [\vec{\alpha}_1\vec{\alpha}_2]\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$$

$\Rightarrow E\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$, $\vec{u} = \begin{pmatrix}u\\v\end{pmatrix}$

Make Substitution: $\vec{x} = E\vec{u}$

$E\vec{u}' = AE\vec{u} = E\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}u.$
"multiply both sides by E^{-1}"

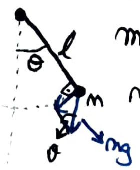
$\Rightarrow \vec{u}' = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\vec{u}.$

$\vec{u}' = \lambda_1 u$
$\vec{v}' = \lambda_2 v$

Sol'n:
$u = c_1 e^{\lambda_1 t}$
$v = c_2 e^{\lambda_2 t}$.

Then plug in uv into xy.

$x' = f(x,y)$
$y' = g(x,y)$ } autonomous, non-linear. Problem: Sketch its trajectories.



$m\vec{a} = \vec{F}$

$m\underbrace{l\theta''}_{a} = -mg\sin(\theta) - c\underbrace{l\theta'}_{v}$; $\theta'' + \frac{c}{m}\theta' + \frac{g}{l}\sin(\theta) = 0$, $\theta'' + c\theta' + k\sin\theta = 0$.

$\theta' = \omega$, $\omega' = -k\sin(\theta) - c\omega$.

1-) Find critical points of system.

(x_0, y_0): $f(x_0,y_0)=0$, $g(x_0,y_0)=0$ } $x = x_0$, $y = y_0$ for all time. } Solve Simultaneosly $f(x,y)=0$; $g(x,y)=0$ } Generally it's impossible to solve them. But we'll try.

Take $c=1$, $k=2$.

$\theta' = \omega$, $\omega' = -2\sin(\theta) - \omega$.

Crit Points: $\omega = 0$, $-2\sin(\theta) - \omega = 0$. } $\Rightarrow \theta = 0, \pm\pi, \pm 2\pi, \ldots$, $\omega = 0$.

Physically: $(0,0)$; Stable, $(\pi,0)$: Unstable } Physically, there are two crit points.

2-) For each cr. point (x_0,y_0), You linearelize system near (x_0,y_0). } Plot the trajectories of this linearized system near critical points.

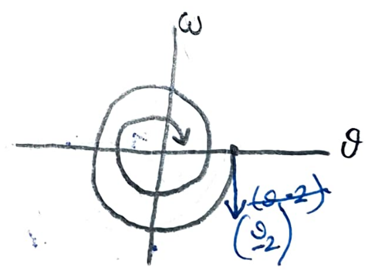
Lin'ze at $(0,0)$

$\theta' = \omega$

$\omega' = -2\theta - \omega$, since $\sin\theta \simeq \theta$ as $\theta \to 0$.

$\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$

$\lambda^2 + \lambda + 2 = 0 \Rightarrow \lambda = \dfrac{-1 \pm \sqrt{-7}}{2}$ } spiral (cx. root), sink (since $\lambda = -\frac{1}{2} + bi$)



Lin'ze at $(\pi,0)$

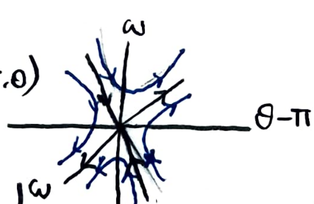
$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_0$ Calculated at (x_0,y_0) } This is the mx. of linearized system.

$J = \begin{pmatrix} 0 & 1 \\ -2\cos(\theta) & -1 \end{pmatrix}$

at $(0,0)$: $J_0 = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}$, at $(\pi,0)$: $J_0 = \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}$; $\lambda^2 + \lambda - 2 = 0 = (\lambda+2)(\lambda-1) \Rightarrow \lambda = 1, -2$.

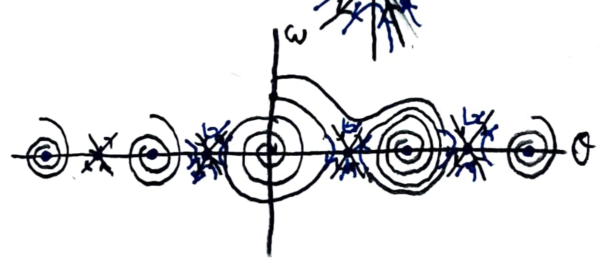
$\lambda = 1$; $-a_1 + a_2 = 0$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$.

$\lambda = -2$; $2a_1 + a_2 = 0$, $\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$. } At: $(\pi, 0)$



3-) Big Picture

Plot trajectories around each crit. points then add som.



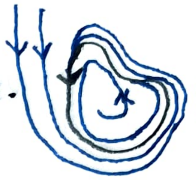
Behaviour of trajectories:

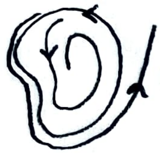
1-) Goes along to infinity

2-) End up at a critical point and sit there.

3-) It repeat itself in loop {closed trajectory} $\longrightarrow$  represents periodic behaviour of the system.

Ex: $x' = y$, $y' = -x$ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\lambda = \pm i$, $\vec{x} = c_1 \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$ 

But we're interested in limit cycles: closed trajectory, isolated, stable.  Stable.

 Unstable.

Existence Problem of Limit Cycles: In general we don't know much about them. {Poincaré-Bendixson}
Basically this is done by computer search drived for specific physical problem.

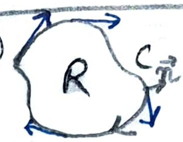
Non-Existence Theorems:

- Bendixson:

D is region of plane. Theorem says that calculate the divergence of vector field.
$\operatorname{div} \vec{F} = f_x + g_y$; f, g continues. Assume that $\operatorname{div} \vec{F} \neq 0$ in that region D.
$\Rightarrow$ So there're no closed trajectories in D.

Ex: $x' = x^3 + y^3$, $y' = 3x + y^3 + 2y$, $\operatorname{div} \vec{F} = 3x^2 + 3y^2 + 2 \Rightarrow$ always positive in the entire xy plane.

Indirect Argument {Proof}
Suppose a closed trajectory exists in D.

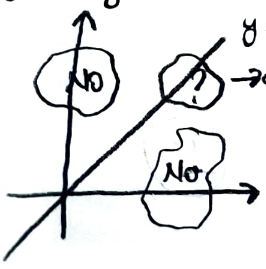
D 

$\oint_C \vec{F} \cdot \vec{n}\, ds \Rightarrow$ Flux of F across C.
$= 0$. since $\vec{F} \cdot \vec{n} = 0$.

Also can do by green's thm.

$\oint_C \vec{F} \cdot \vec{n}\, ds = \iint_R \operatorname{div} \vec{F}\, dA$, $\operatorname{div} \vec{F} >$ everywhere in R, $\operatorname{div} \vec{F} <$ " in R {since $\operatorname{div} \vec{F} \neq 0$ in R.}
$\oint_C = 0$; $\iint_R$ either > 0 or < 0 NOT 0. $\left\{ \oint_C \vec{F} \cdot \vec{n}\, ds \neq \iint_R \operatorname{div} \vec{F}\, dA \right.$, any explanation for this situation will be that a closed trajectory does not exists in D.

$x' = x^2 + y^2 + 1$, $y' = x^2 - y^2$; $\operatorname{div} \vec{F} = 2x - 2y = 0$ along the line $x = y$.



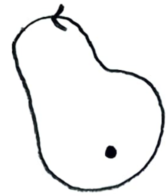
$\rightarrow \operatorname{div} \vec{F} = 0$, Criterion of Bendixson can't decide here. } we need to use "Critical Point" criterion.

Critical Point Criterion:

D - region

C closed trajectory of system in D.

$\Rightarrow$ inside that closed trajectory, there must be a critical point.



Thm: If D has no critical points, D has no closed trajectories in it.

Our last case;

$x' = x^2 + y^2 + 1$, $y' = x^2 - y^2$ } no crit points so no limit cycles

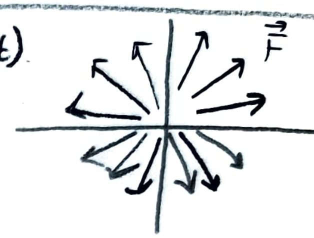
$$\begin{aligned} x' &= ax^2 + bxy + cy^2 + dx + ey + f \\ y' &= a'x^2 + b'xy + c'y^2 + d'x + e'y + f' \end{aligned}$$

$\Rightarrow$ How many limit cycles can a quadratic system have? The answer is unknown but researchers found a system with 4. We don't know how many are there.

$\dfrac{dx}{dt} = f(x,y)$, $\dfrac{dy}{dt} = g(x,y)$ } non-linear autonomous system

$\vec{F} = f\hat{i} + g\hat{j}$ Sol'ns: $x = x(t)$, $y = y(t)$

{How to eliminate t?}



$\dfrac{dy}{dx} = \dfrac{g(x,y)}{f(x,y)}$; $\vec{F}$ field becomes $\rightsquigarrow$ dir $\vec{F}$ field } Solution: $y = y(x)$: explicit | $h(x,y) = 0$: implicit

Ex.

$x' = y$, $y' = -x$ sol'n: $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$: Eliminate t.

$\dfrac{dy}{dx} = \dfrac{-x}{y} \rightsquigarrow y\,dy = -x\,dx \Rightarrow x^2 + y^2 = c$ Also can be done eliminating from solution.

Ex:

predator $x' = -ax + bxy$.
prey $y' = cy - dxy$.
$a, b, c, d > 0$.

Crit. Points: $x(-a + by) = 0$, $y(c - dx) = 0$ $\Rightarrow$ $y = 0 \Leftrightarrow x = 0$, $y = a/b \Leftrightarrow x = c/d$. | $(0,0)$, $(\frac{c}{d}, \frac{a}{b})$ {Voltera's Principle}

$(0,0)$, $\begin{bmatrix} -a & 0 \\ 0 & c \end{bmatrix}$: $\lambda = -a, c$. Saddle. {unstable}

$J = \begin{bmatrix} -a + by & bx \\ -dy & c - dx \end{bmatrix}$ at $(\frac{c}{d}, \frac{a}{b}) \to \begin{bmatrix} -a + b\frac{a}{b} & b\frac{c}{d} \\ -d\frac{a}{b} & c - d\frac{c}{d} \end{bmatrix} = \begin{bmatrix} 0 & bc/d \\ -da/b & 0 \end{bmatrix}$

linearized system;

$\Rightarrow x' = \dfrac{bc}{d} y$, $y' = -\dfrac{da}{b} x$ } linearized system is center. In the shape of elliptical closed trajectories.

} First eliminate t. Assume $a, b, c, d = 1$

$\dfrac{dy}{dx} = \dfrac{y(1-x)}{x(-1+y)} \Rightarrow \dfrac{y-1}{y} dy = \dfrac{1-x}{x} dx$

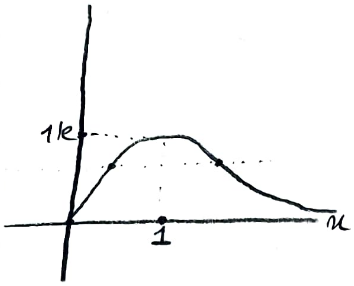
$\left(1 - \dfrac{1}{y}\right) dy = \left(\dfrac{1}{x} - 1\right) dx$

$y - \ln y = \ln x - x + c_1$.

$e^y \cdot \dfrac{1}{y} = x \dfrac{1}{e^x} \cdot c_2$

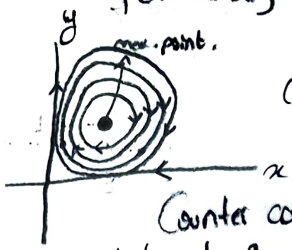
$\dfrac{x}{e^x} \cdot \dfrac{y}{e^y} = c$.

integral curves are graphs of this. Contour curves of $xe^{-x} \cdot ye^{-y} = c = h(x,y)$



xe^{-x}

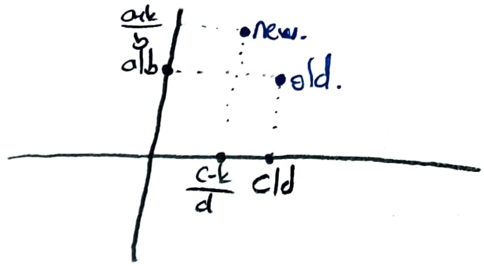
{clockwise} $h = xe^{-x} ye^{-y}$
$x=1$ $y=1$



Cont. curve of $h(x,y)$

Counter curves can intersect two times so its not spiral.

If we fish the preyors at rate k, the new center;



{End of Differential Equations Course}