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Trigonometry Summary

CSE 4303 / CSE 5365 Computer Graphics

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We require only relatively simple trigonometry for this class. However, you must understand and be comfortable with this subset.

Preliminaries

- Angles may be measured in either *radians* or *degrees*. There are 2π radians or 360 degrees in a circle. Radians are expressed simply as a number with or without the word 'radians' (or 'rad') following. Degrees are expressed as a number followed by the degree symbol $^\circ$ or the word 'degrees' (or 'deg').
  - Example: 5 or $5\,\mathrm{rad}$ means 5 radians. 30° or $30\,\mathrm{deg}$ means 30 degrees.
  - An angle measurement with no mark is in *radians*.
- An angle measurement θ in degrees, θ°, may be converted to radians thus,

$$\theta\,\mathrm{rad} = \frac{\theta^\circ \cdot \pi}{180^\circ}$$

- An angle measurement in radians, $\theta\,\mathrm{rad}$, may be converted to degrees thus,

$$\theta^\circ = \frac{\theta\,\mathrm{rad} \cdot 180^\circ}{\pi}$$

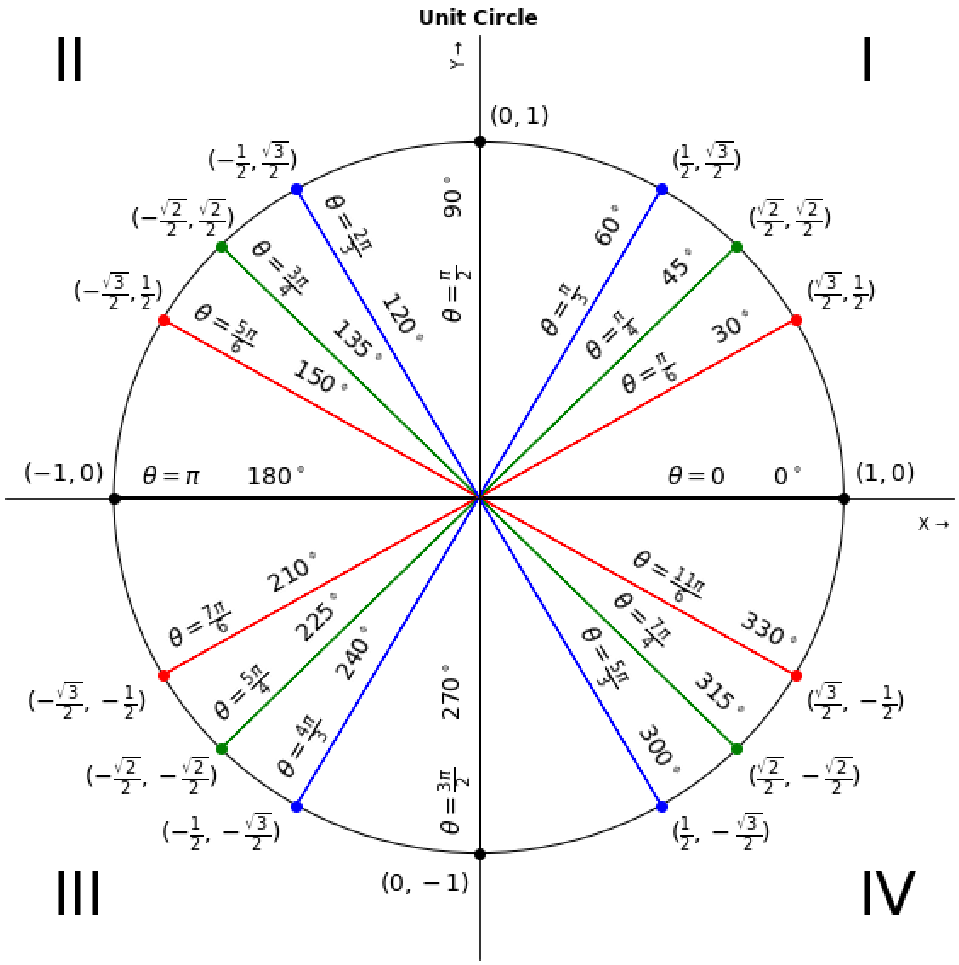
- Many kinds of calculations are easier when angles are expressed in radians. On the other hand, most persons are more familiar with degree measurement. Get comfortable with either usage.
- Many software packages use angles expressed in radians. Others use degrees. Be very careful that you know the standard for the software you use. (Some packages actually use radians for some operations and degrees for others.)
- Angles are measured from the positive side of the x axis with *counter-clockwise* being *positive* angle measurement. *Clockwise* is *negative* angle measurement.

Quadrants and the Unit Circle

- The *cartesian plane* is divided into four *quadrants* by the x and y axes. These quadrants are known as the *first* through the *fourth* quadrants and are labelled I, II, III, IV.

Quadrant	Coordinates
I	$x > 0, y > 0$
II	$x < 0, y > 0$
III	$x < 0, y < 0$
IV	$x > 0, y < 0$

- The coordinate axes themselves are not in any quadrant; they are the boundaries *between* quadrants.
- The *unit circle* is a circle with radius $= 1$ centered at the origin $= (0, 0)$.



Unit Circle
II
I
III
IV
Y →
X →
$(0, 1)$
$(1, 0)$
$(-1, 0)$
$(0, -1)$
$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
$(\frac{\sqrt{3}}{2}, \frac{1}{2})$
$(-\frac{1}{2}, \frac{\sqrt{3}}{2})$
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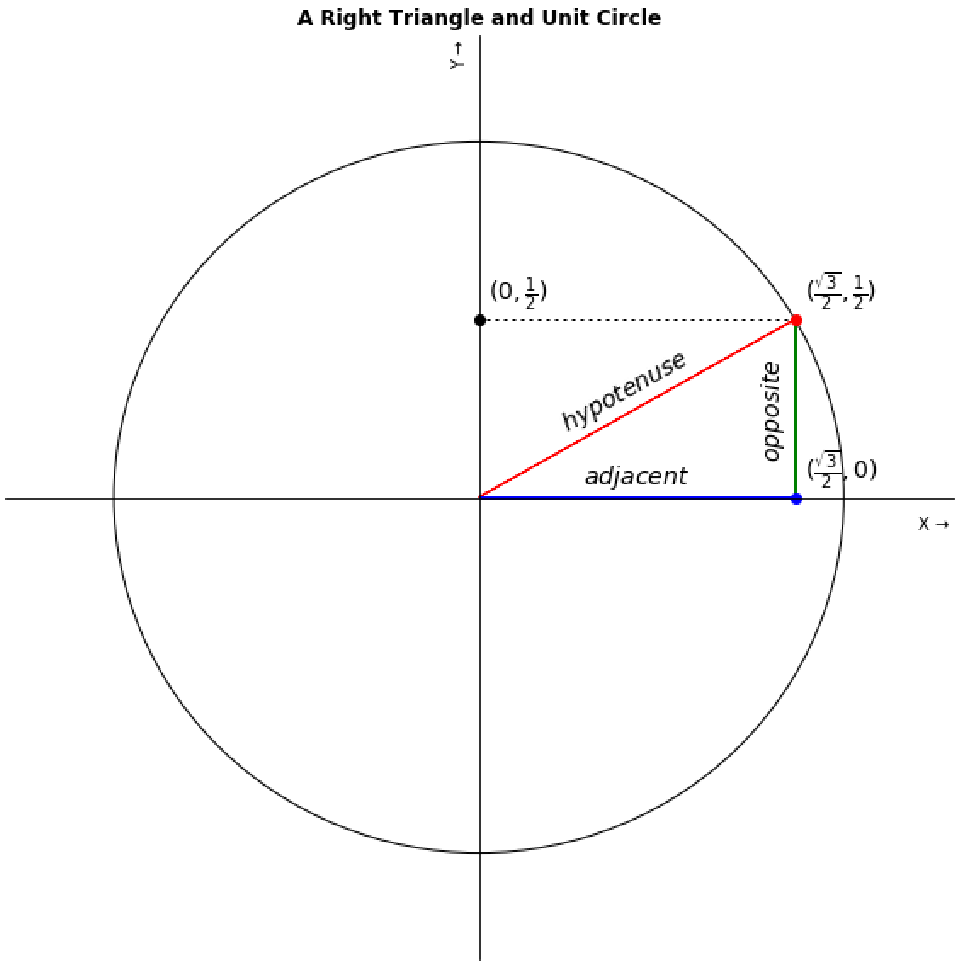
$(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$
$(\frac{\sqrt{3}}{2}, -\frac{1}{2})$
$\theta = 0$ 0°
$\theta = \frac{\pi}{6}$ 30°
$\theta = \frac{\pi}{4}$ 45°
$\theta = \frac{\pi}{3}$ 60°
$\theta = \frac{\pi}{2}$ 90°
$\theta = \frac{2\pi}{3}$ 120°
$\theta = \frac{3\pi}{4}$ 135°
$\theta = \frac{5\pi}{6}$ 150°
$\theta = \pi$ 180°
$\theta = \frac{7\pi}{6}$ 210°
$\theta = \frac{5\pi}{4}$ 225°
$\theta = \frac{4\pi}{3}$ 240°
$\theta = \frac{3\pi}{2}$ 270°
$\theta = \frac{5\pi}{3}$ 300°
$\theta = \frac{7\pi}{4}$ 315°
$\theta = \frac{11\pi}{6}$ 330°

- The angles shown in the *Unit Circle* plot fit simple patterns (which makes them easy to remember).
- While this diagram looks complicated and scary, one soon realizes that there are only *four* angles, each occuring four times in $\frac{\pi}{2}$ (90°) steps. If one learns the three angles in Quadrant I and the angle $\frac{\pi}{2}$ (90°), one knows all the rest. Just remember to use the proper $+$ and $-$ signs for the values associated with the angles in the other quadrants.
- When $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ radians (or $0^\circ, 90^\circ, 180^\circ, 270^\circ$), the angle meets the unit circle at $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$. Notice that the x, y coordinate pairs all have one 0 and one of ± 1.
- For $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (or $45^\circ, 135^\circ, 225^\circ, 315^\circ$), the angle meets the unit circle at $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$, $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$. Notice that the x, y coordinate pairs are all $(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$. This is $(\pm 0.707, \pm 0.707)$ when rounded to three digits. This number occurs in many places in CG. Learn to recognize it.
- For $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ (or $30^\circ, 150^\circ, 210^\circ, 330^\circ$), the angle meets the unit circle at $(\frac{\sqrt{3}}{2}, \frac{1}{2})$, $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$, $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$. Notice that the x, y coordinate pairs are all $(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})$. This is $(\pm 0.866, \pm 0.500)$ when rounded to three digits. The number $\frac{\sqrt{3}}{2} = 0.866\ldots$ occurs in many places in CG. Learn to recognize it.
- For $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ (or $60^\circ, 120^\circ, 240^\circ, 300^\circ$), the angle meets the unit circle at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$. Notice that the x, y coordinate pairs are all $(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$. This is $(\pm 0.500, \pm 0.866)$ when rounded to three digits.
- Notice that the only numbers that show up as coordinates are $0, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}$ $(\pm 0.707), \pm\frac{\sqrt{3}}{2}$ (± 0.866), and ± 1. It's not hard to learn these.
- Angle measurement is not unique. Since a circle has 2π radians (360°), angles that are a multiple of that value apart have the same properties. For example, $\frac{\pi}{4}$ and $\frac{9\pi}{4} = \frac{\pi}{4} + 2\pi$ $(45^\circ$ and $405^\circ = 45^\circ + 360^\circ)$ both meet the unit circle at $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

Trigonometric Functions

- The trigonometric functions are typically defined using a right triangle.

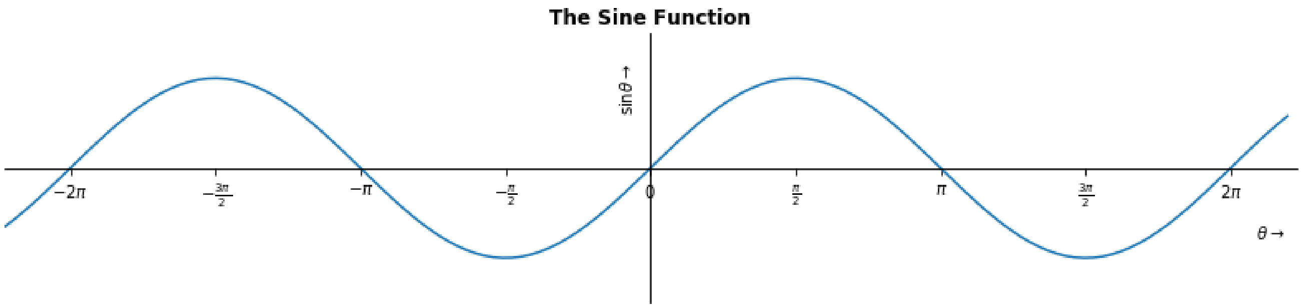


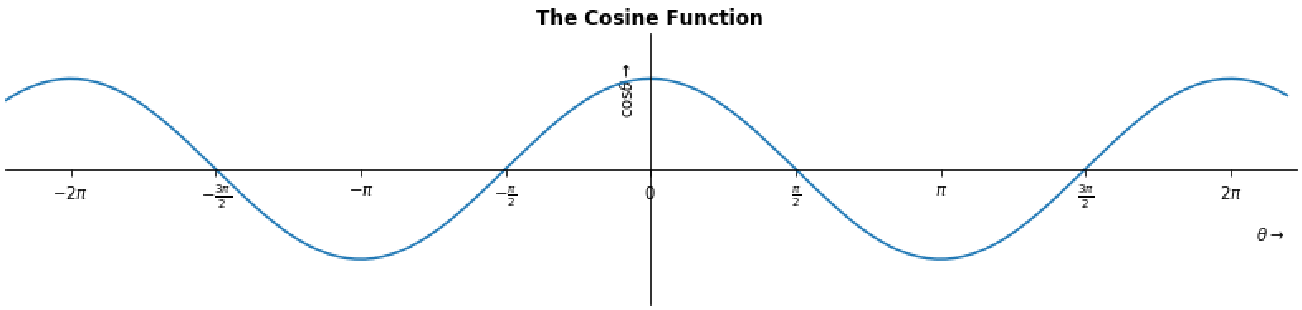
- In this example, the angle between the *adjacent* line and the *hypotenuse* line is $\frac{\pi}{6}\,(30^\circ)$.
- The *sine* of an angle is the y component of the spot where the *hypotenuse* of the right triangle touches the unit circle. In this case, the value of $\sin\frac{\pi}{6} = \frac{1}{2}$. This is also the length of the *opposite* side of the triangle.
- The *cosine* of an angle is the x component of the spot where the *hypoteneuse* of the right triangle touches the unit circle. In this case, the value of $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. This is also the length of the *adjacent* side of the triangle.
- If one has a right triangle that is *not* inscribed in the unit circle (that is, the length of the hypotenuse is $\neq 1$, one can make use of these definitions:

$$\sin\theta = \frac{opposite}{hypotenuse}$$

$$\cos\theta = \frac{adjacent}{hypotenuse}$$

- Since sin and cos are defined on the unit circle, we have $\sin^2\theta + \cos^2\theta = 1$, by the Pythagorean theorem, for *any* angle θ. This is useful for solving various trigonometry problems and for deriving handy trigonometric identities.
- There are other trigonometric functions, but they do not come up as much in CG as do sin and cos. If one needs these other functions, their definitions are not hard to find on-line.





- As one can see in the above graphs of the sin and cos functions, they are periodic, with a period of 2π.
- Sin starts at 0 for $\theta = 0$ moves up to $+1$ at $\frac{\pi}{2}$ then back to 0 at $\theta = \pi$. From there, it goes negative, reaching -1 at $\frac{3\pi}{2}$ then back to zero at $\theta = 2\pi$.
- Sin is what is known as an *odd* function because $\sin -x = -\sin x$. Sin is symmetric about the origin.
- Cos starts at $+1$ for $\theta = 0$ moves down to 0 at $\frac{\pi}{2}$ then continues down to -1 at $\theta = \pi$. From there, it goes swings back up, reaching 0 at $\frac{3\pi}{2}$ then peaks again at $+1$ for $\theta = 2\pi$.
- Cos is what is known as an *even* function because $\cos -x = \cos x$. Cos is symmetric about the y axis.