STATISTICS FOR MANAGEMENT (IDS 570)

HOMEWORK 4 SOLUTION

Problem 1

(a) Let B be the event that someone is happy with his/her boss and let H be the event that someone is happy with his/her hours of work. The data from the problem can be summarized as follows: $P(B) = 0.5, P(H) = 0.4, P(\overline{B} \cap \overline{H}) = 0.3$. Using this a table of probabilities can be filled in:

	H	$\overline{H}$	Prob.
B	0.2^5	0.3^4	**0.5**
$\overline{B}$	0.2^2	**0.3**	0.5^1
Prob.	**0.4**	0.6^3	**1**

The boldface numbers are the probabilities given by the data in the problem statement. Following the superscripts, we can fill in the remaining probabilities as follows. For ^{1}, $P(B) = 0.5$, and since $P(\overline{B}) = 1 - P(B)$ we have $P(\overline{B}) = 1 - 0.5 = 0.5$. For ^{2}, $P(\overline{B} \cap H) + P(\overline{B} \cap \overline{H}) = P(\overline{B})$, so that $P(\overline{B} \cap H) = P(\overline{B}) - P(\overline{B} \cap \overline{H}) = 0.5 - 0.3 = 0.2$. For ^{3}, $P(H) + P(\overline{H}) = 1$, so that $P(\overline{H}) = 1 - P(H) = 1 - 0.4 = 0.6$. For ^{4}, $P(B \cap \overline{H}) + P(\overline{B} \cap \overline{H}) = P(\overline{H})$, so that $P(B \cap \overline{H}) = P(\overline{H}) - P(\overline{B} \cap \overline{H}) = 0.6 - 0.3 = 0.3$. Finally, for ^{5}, $P(B \cap H) + P(B \cap \overline{H}) = P(B)$, so that $P(B \cap H) = P(B) - P(B \cap \overline{H}) = 0.5 - 0.3 = 0.2$.

(b) We desire $P(B|H) = \dfrac{P(B \cap H)}{P(H)} = \dfrac{0.2}{0.4} = 0.5$.

(c) One way of checking for independence is to check if $P(B)P(H) = P(B \cap H)$. Plugging in the values above yields $P(B)P(H) = 0.5 \cdot 0.4 = 0.2 = P(B \cap H)$, and therefore the events are independent.

Problem 2

(a) Let B be the event that we select a black ball and W be the event that we select a white ball. Additionally, let I be the event that we choose urn I and II be the event that we select urn II. Then the probability of selecting a black ball is $P(B) = P(B|I)P(I) + P(B|II)P(II) = \dfrac{2}{3} \cdot \dfrac{1}{2} + \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{7}{12}$.

(b) We desire $P(I|W)$. By Bayes' Theorem, $P(I|W) = \dfrac{P(W|I)P(I)}{P(W|I)P(I) + P(W|II)P(II)} = \dfrac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{1}{3} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}} = \dfrac{2}{5}$.

(c) In order for I and B to be independent, it must be the case that $P(I)P(B) = P(I \cap B)$. $P(I) = \dfrac{1}{2}$, $P(B) = \dfrac{7}{12}$ (from above) and hence $P(I)P(B) = \dfrac{7}{24}$. Furthermore, $P(B \cap I) = P(B|I)P(I) = \dfrac{2}{3} \cdot \dfrac{1}{2} = \dfrac{1}{3}$. Therefore, $P(B \cap I) = \dfrac{1}{3} \neq \dfrac{7}{24} = P(I)P(B)$ and so the events are not independent.

Problem 3

(a) Let Q be the event that a new worker meets the quota for sales and let T be the event that a new worker attends the training program. The desired probability is $P(Q) = P(Q|T)P(T) + P(Q|\overline{T})P(\overline{T}) = 0.8 \cdot 0.5 + 0.3 \cdot 0.5 = 0.55$.

(b) The desired probability is $P(T|Q)$ which by Bayes' Theorem equals $\dfrac{P(Q|T)P(T)}{P(Q|T)P(T) + P(Q|\overline{T})P(\overline{T})} = \dfrac{0.8 \cdot 0.5}{0.8 \cdot 0.5 + 0.3 \cdot 0.5} = \dfrac{8}{11}$.

(c) We proceed as in the first part and calculate $P(Q) = P(Q|T)P(T) + P(Q|\overline{T})P(\overline{T}) = 0.8 \cdot 0.9 + 0.3 \cdot 0.1 = 0.75$.

Problem 4

(a) Let T be the event that after testing a pipe the test comes back positive for damage and let D be the event that a new pipe is damaged after the winter season. The data in the problem can be summarized as follows: $P(D) = 0.5$, $P(T|D) = 0.9$, $P(T|\overline{D}) = 0.3$. Then by Bayes' Rule, $P(D|T) = \dfrac{P(T|D)P(D)}{P(T|D)P(D) + P(T|\overline{D})P(\overline{D})} = \dfrac{0.9 \cdot 0.5}{0.9 \cdot 0.5 + 0.3 \cdot 0.5} = 0.75$.

(b) Again by Bayes' Rule, $P(D|\overline{T}) = \dfrac{P(\overline{T}|D)P(D)}{P(\overline{T}|D)P(D) + P(\overline{T}|\overline{D})P(\overline{D})}$. Note that $P(\overline{T}|D) = 1 - P(T|D) = 0.1$ and $P(\overline{T}|\overline{D}) = 1 - P(T|\overline{D}) = 0.7$. Hence, $P(D|\overline{T}) = \dfrac{0.1 \cdot 0.5}{0.1 \cdot 0.5 + 0.7 \cdot 0.5} = 0.125$.

Problem 5

(a)

$$\begin{aligned}
P(A)P(\overline{B}) &= P(A)(1 - P(B)) \\
&= P(A) - P(A)P(B) \\
&= P(A) - P(A \cap B)(\text{because of the independence of } A \text{ and } B) \\
&= P(A \cap B) + P(A \cap \overline{B}) - P(A \cap B) \\
&= P(A \cap \overline{B})
\end{aligned}$$

and hence A and $\overline{B}$ are independent.

(b)

$$\begin{aligned}
P(\overline{A} \cap \overline{B}) &= P(\overline{A}) + P(\overline{B}) - P(\overline{A} \cup \overline{B}) \\
&= P(\overline{A}) + P(\overline{B}) - (1 - P(A \cap B)) \\
&= P(\overline{A}) + P(\overline{B}) + P(A \cap B) - 1 \\
&= (1 - P(A)) + (1 - P(B)) + P(A \cap B) - 1 \\
&= 1 - P(A) - P(B) + P(A \cap B) \\
&= 1 - P(A) - P(B) + P(A)P(B)(\text{because of the independence of } A \text{ and } B) \\
&= (1 - P(A))(1 - P(B)) \\
&= P(\overline{A})P(\overline{B})
\end{aligned}$$

and hence $\overline{A}$ and $\overline{B}$ are independent.

An alternative solution is to apply (a) to the pair (B, A) to get that B and $\overline{A}$ are independent and then to apply (a) again to the pair $(\overline{A}, B)$.

Problem 6

Let E be the event that we see a sum of 5 before a sum of 7. We want to compute $P(E)$. The easy way to solve this is by conditioning on the outcome of the first roll: Let F be the event that the first roll is a 5; let G be the event that the first roll is a 7; Let H be the event that the rolll is a sum other than 5 or 7. Then F, G, and H partition the sample space. Therefore, we have:

$$P(E) = P(E \mid F)P(F) + P(E \mid G)P(G) + P(E \mid H)P(H)$$

Now, we know that $P(F) = 4/36$, $P(G) = 6/36$, and $P(H) = 26/26$. Also, given that the first roll is a 5, the probability we get a 5 before a 7 is one: $P(E \mid F) = 1$. Similary, given that the first roll is a 7, the probability we get a 5 before a 7 is zero: $P(E \mid G) = 0$. Now, if the first roll is neither a 5 or a 7, we can think of the process starting all over again: the chance we get a 5 before a 7 is just like it was before we start rolling ($P(E)$). Therefore $P(E \mid H) = P(E)$. Thus,

$$P(E) = 1 \cdot (4/36) + 0 \cdot (6/36) + P(E) \cdot (26/36)$$

which gives us $P(E) = 4/36 + (26/36)P(E)$. Hence, $P(E) = 2/5 = 0.4$.

Problem 7

The probability of missing the target is $q = 1 - p = 0.7$. Hence, the probability that n missiles miss the target is $(o.7)^n$. Thus, we seek the smallest n for which $1 - (0.7)^n > 0.8$ or equivalently $(0.7)^n < 0.2$. Compute: $(0.7)^1 = 0.7$, $(0.7)^2 = 0.49$, $(0.7)^3 = 0.343$, $(0.7)^4 = 0.2401$, $(0.7)^5 = 0.16807$. Thus, at least $n = 5$ missiles should be fired.

Problem 8

a. We have $P(A \cap B) \geq 0$ and $P(E) > 0$; hence

$$P(A \mid E) = \frac{P(A \cap E)}{P(E)} \geq 0.$$

Thus (1) holds.

b. We have $S \cap E = E$; hence

$$P(S \mid E) = \frac{P(S \cap E)}{P(E)} = \frac{P(E)}{P(E)} = 1.$$

Thus (2) holds.

c. If A and B are disjoint events, then so are $A \cap E$ and $B \cap E$. Furthermore,

$$(A \cup B) \cap E = (A \cap E) \cup (B \cap E)$$

Hence,

$$P[(A \cup B) \cap E] = P[(A \cap E) \cup (B \cap E)] = P(A \cap E) + P(B \cap E).$$

Therefore,

$$P(A \cup B \mid E) = \frac{P[(A \cup B) \cap E]}{P(E)} = \frac{P(A \cap E) + P(B \cap E)]}{P(E)} = \frac{P(A \cap E)}{P(E)} + \frac{P(B \cap E)}{P(E)} = P(A \mid E) + P(B \mid E).$$

Thus (3) holds.