STATISTICS FOR MANAGEMENT (IDS 570)

HOMEWORK 6 SOLUTION

Problem 1

(a) Using $P(X = i) = C_i^n \cdot p^i \cdot (1-p)^{n-i}$, we have that $P(X < 8) = P(X \leq 7) = \sum_{i=0}^{7} C_i^n \cdot 0.6^i \cdot 0.4^{n-i} = 0.213$.

(b) To find $P(6 < X < 12)$ take $P(X \leq 11) - P(X \leq 6) = \sum_{i=0}^{11} C_i^n \cdot 0.2^i \cdot 0.8n^{n-i} - \sum_{i=0}^{6} C_i^n \cdot 0.2^i \cdot 0.8^{n-i} = 0.9999 - 0.9133 = 0.0866$.

(c) $\mu_X = E(X) = p = 0.3$ and $\sigma_X^2 = Var[X] = p \cdot (1-p) = 0.3 \cdot 0.7 = 0.21$.

(d) $P(X > 5) = 1 - P(X \leq 5) = 1 - \sum_{i=0}^{5} \frac{e^{-4.6} \cdot 4.6^i}{i!} = 1 - 0.6857 = 0.3143$.

Problem 2

We can draw a Payoff table:

	success	failure
invest	$76,000	$-14,000
don't invest	$ 0	$ 0

(a) $E[X] = 0.35 \cdot 76,000 + 0.65 \cdot (-14,000) = 17,500$. Since the expectation is positive, it is worth it.

(b) $p \cdot 76,000 + (1-p) \cdot (-14,000) \geq 0$ implies $p \geq 0.1556$.

Problem 3

(a) First recall that the sum of two Poisson distributions with means λ_1 and λ_2 respectively is a Poisson distribution with mean $\lambda_1 + \lambda_2$. Therefore, the number of clients the coffee shop gets in an hour is distributed according to a Poisson distribution with mean 48. In addition, the expected number of clients in 30 min is given by scaling the mean accordingly: it is $48 \cdot \frac{30}{60} = 24$.

(b) Since Poisson is a memoryless distribution, the desired probability is $P(Y = 0) = \frac{e^{-\lambda} \cdot \lambda^0}{0!}$: The mean for 20 minutes will be $\frac{20}{60} \cdot 18 = 6$.

$P(Y = 0) = \frac{e^{-6} \cdot 6^0}{0!} = 0.0025$.

(c) $P(Z < 10) = P(Z \leq 9) = \sum_{i=0}^{9} \frac{e^{-\lambda} \cdot \lambda^i}{i!}$. The mean for 2 hours will be $\frac{120}{60} \cdot 30 = 60$.

Therefore, $P(Z \leq 9) = \sum_{i=0}^{9} \frac{e^{-60} \cdot 60^i}{i!}$

Problem 4

(a) Because $f(x)$ is the pdf of a random variable, (1) $f(x) \geq 0$ and (2) $\int_{-\infty}^{\infty} f(x)dx = 1$. Using (2) we get that

$$1 = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{0} 0\,\mathrm{d}x + \int_{0}^{1} kx^3\,\mathrm{d}x + \int_{1}^{\infty} 0\,\mathrm{d}x = k\frac{x^4}{4}\Bigg]_{x=0}^{x=1} = k\left(\frac{1}{4} - \frac{0}{4}\right) = \frac{k}{4},$$

and so $k = 4$. Notice that with $k = 4$, $f(x) \geq 0$ everywhere, and so (1) is satisfied.

(b) Since $f(x) = 0$ for all $x \leq 0$ we can infer that $F(x) = 0$ for all $x \leq 0$ and since $f(x) = 0$ for all $x \geq 1$ we can infer that $F(x) = 1$ for $x \geq 1$. It remains to determine $F(x)$ for $0 \leq x \leq 1$.

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t = \int_{-\infty}^{0} 0\,\mathrm{d}t + \int_{0}^{x} 4t^3\,\mathrm{d}t = t^4\Bigg]_{t=0}^{t=x} = x^4.$$

So,

$$F(x) = \begin{cases} 0, & x < 0 \\ x^4, & 0 \leq x \leq 1 \\ 1, & x \geq 1 \end{cases}$$

(c) $P(\frac{1}{2} \leq X \leq 2) = F(2) - F(\frac{1}{2}) = 1 - \left(\frac{1}{2}\right)^4 = \frac{15}{16}$.

(d) We will compare $P(0 \leq X \leq \frac{1}{2})$ and $P(\frac{1}{2} \leq X \leq 1)$.

$P(0 \leq X \leq \frac{1}{2}) = F(\frac{1}{2}) - F(0) = \left(\frac{1}{2}\right)^4 - (0)^4 = \frac{1}{16}$, and

$P(\frac{1}{2} \leq X \leq 1) = F(1) - F(\frac{1}{2}) = (1)^4 - \left(\frac{1}{2}\right)^4 = 1 - \frac{1}{16} = \frac{15}{16}$.

Therefore it is 15 times more likely that X is between $\frac{1}{2}$ and 1 than between 0 and $\frac{1}{2}$.

(e) This question asks for the x such that $F(x) = 0.5$. From above, we know that $F(x) = x^4$. Setting this equal to 0.5 and solving for x yields $x = \sqrt[4]{0.5} = 0.8409$.

Problem 5

(a) For $f(x)$ to be a pdf, we need to find A that will make (1) $f(x) \geq 0$ and (2) $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$. We will start with the second condition and check (1).

$$1 = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{1} 0\,\mathrm{d}x + \int_{1}^{A} \frac{x}{4},\mathrm{d}x + \int_{A}^{\infty} 0,\mathrm{d}x = \frac{x^2}{8}\Bigg]_{x=1}^{x=A} = \frac{A^2 - 1}{8}.$$

From the equation above we get that $A = \pm 3$. Observe that with $A = -3$, $f(x)$ is not defined properly because the support of $f(x)$ would be in the interval $1 \leq x \leq -3$, which is not possible. And so we conclude that $A = 3$.

(b) Since $f(x) = 0$ for all $x \leq 1$ we can infer that $F(x) = 0$ for all $x \leq 1$ and since $f(x) = 0$ for all $x \geq 3$ we can infer that $F(x) = 1$ for $x \geq 3$. It remains to determine $F(x)$ for $1 \leq x \leq 3$.

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t = \int_{-\infty}^{1} 0\,\mathrm{d}t + \int_{1}^{x} \frac{t}{4}\,\mathrm{d}t = \frac{t^2}{8}\Bigg]_{t=1}^{t=x} = \frac{x^2 - 1}{8}.$$

So,

$$F(x) = \begin{cases} 0, & x < 1 \\ \dfrac{x^2-1}{8}, & 1 \leq x \leq 3 \\ 1, & x \geq 3 \end{cases}$$

(c)
$$E[X] = \int_{-\infty}^{\infty} xf(x)\,\mathrm{d}x = \int_1^3 x \cdot \frac{x}{4}\,\mathrm{d}x = \frac{x^3}{12}\Bigg]_{x=1}^{x=3} = \frac{13}{6}.$$

(d)
$$Var[X] = E[X^2] - (E[X])^2.$$

Using the above formula, we first calculate $E[X^2]$.

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x = \int_1^3 x \cdot \frac{x^3}{4}\,\mathrm{d}x = \frac{x^4}{16}\Bigg]_{x=1}^{x=3} = 5.$$

Therefore,

$$Var[X] = 5 - \left(\frac{13}{6}\right)^2 = \frac{11}{36}.$$

(e) This question asks for the number x such that $F(x) = 0.375 = \frac{3}{8}$. Therefore we want the x such that

$$\frac{x^2-1}{8} = \frac{3}{8}.$$

Solving for x yields $x \pm 2$. However, since -2 is not in the range of values for which the cdf is $\frac{x^2-1}{8}$, the desired value is $x = 2$.

Problem 6

(a) Let X_1, X_2, X_3, X_4, X_5 be the amount of time it takes Bob to get to school on Monday, Tuesday, Wednesday, Thursday, and Friday, respectively. Then,

$$X_1, X_3, X_5 \sim U[10, 16]$$

and

$$X_2, X_4 \sim U[18, 20].$$

In addition, let Y be the amount of time that it takes Bob to get to school during a week. Then $Y = X_1 + X_2 + X_3 + X_4 + X_5$.

We desire $E[Y] = E[X_1+X_2+X_3+X_4+X_5] = E[X_1]+E[X_2]+E[X_3]+E[X_4]+E[X_5]$. Since $X_1, X_3, X5 \sim U[10, 16]$, the expected value for each of these random variables is $\frac{10+16}{2} = 13$ and since $X_2, X_4 \sim U[18, 20]$, the expected value for each of these random variables is $\frac{18+20}{2} = 19$. Therefore,

$$E[Y] = 13 + 19 + 13 + 19 + 13 = 77.$$

(b) Since each of the X_i's are mutually independent,

$$Var[Y] = Var[X_1] + Var[X_2] + Var[X_3] + Var[X_4] + Var[X_5].$$

Given that $X_1, X_3, X_5 \sim U[10, 16]$ and $X_2, X_4 \sim U[18, 20]$ it follows that $Var[X_1] = Var[X_3] = Var[X_5] = \frac{(16-10)^2}{12} = 3$ and $Var[X_2] = Var[X_4] = \frac{(20-18)^2}{12} = \frac{1}{3}$. So

$$Var[Y] = 3 + \frac{1}{3} + 3 + \frac{1}{3} + 3 = 9.67.$$

(c) We desire $P(X_2 \geq 19 | X_1 < 15)$. However, since X_1 and X_2 are independent, $P(X_2 \geq 19 | X_1 < 15) = P(X_2 \geq 19)$, and since $X_2 \sim U[18, 20]$,

$$P(X_2 \geq 19) = \frac{1}{2}.$$

(d) We desire

$$P(X_3 \geq 14 | X_3 \geq 13) = \frac{P(X_3 \geq 14, X_3 \geq 13)}{P(X_3 \geq 13)} = \frac{P(X_3 \geq 14)}{P(X_3 \geq 13)} = \frac{\left(\frac{16-14}{16-10}\right)}{\left(\frac{16-13}{16-10}\right)} = \frac{2}{3}.$$