STATISTICS FOR MANAGEMENT (IDS 570)

HOMEWORK 5 SOLUTION

Problem 1

(a) $E[X] = \sum_{i=1}^{6} \frac{1}{6} \cdot i = 3.5$.
(b) $Y = 6 - X$. Therefore, $E[Y] = E[6 - X] = 6 - E[X] = 2.5$ (by linearity of expectation).
(c) $Var[X] = \sum_{i=1}^{6} \frac{1}{6} \cdot (i - E[X])^2 = 2.92$
(d) $Var[Y] = Var[6 - X] = Var[X] = 2.92$.
(e) Using linearity of expectation, $E[Z] = E[X] - E[Y] = 1$.
(f) Note that $Z = X - (6 - X) = 2X - 6$. Therefore, $Var[Z] = 4 \cdot Var[X] = 11.68$.
(g) $Z > 0 \Rightarrow (2X - 6) > 0 \Rightarrow X > 3$. Thus $P(Z > 0) = P(X > 3) = \frac{1}{2}$.
(h) From part (a), we know $E[X] = 3.5$. Since X and Y are independent and have the same pdf, $E[X] = E[Y] = 3.5$. Therefore, $E[Z] = 0$ and $Var[Z] = Var[X] + Var[Y] = 5.83$.

Problem 2

(a) For $i = 1, \ldots, 6$,

$$P(X = i, Y = i) = \frac{1}{6} \cdot \frac{C_i^6}{2^6}.$$

Note that $C_0^6 = 1$, $C_1^6 = 6$, $C_2^6 = 15$, $C_3^6 = 20$, $C_4^6 = 15$, $C_5^6 = 6$, $C_6^6 = 1$. Thus,

$$\begin{aligned} P(\text{game results in a tie}) &= \sum_{i=1}^{6} P(X = i, Y = i) = \frac{1}{6} \sum_{i=1}^{6} \frac{C_i^6}{2^6} \\ &= \frac{1}{6} \cdot \frac{6 + 15 + 20 + 15 + 6 + 1}{64} = \frac{1}{6} \cdot \frac{63}{64}. \end{aligned}$$

(b)

$$\begin{aligned} P(X > Y) &= \sum_{i=1}^{5} P(Y = i, X > i) = \sum_{i=1}^{5} \frac{C_i^6}{2^6} \cdot \frac{6 - i}{6} = \frac{1}{2^6} \sum_{i=1}^{5} C_i^6 - \frac{1}{6 \cdot 2^6} \cdot \sum_{i=1}^{5} i \cdot C_i^6 \\ &= \frac{(1 + 6 + 15 + 20 + 15 + 6)}{64} - \frac{(0 \cdot 1 + 1 \cdot 6 + 2 \cdot 15 + 3 \cdot 20 + 4 \cdot 15 + 5 \cdot 6)}{6 \cdot 64} \\ &= \frac{63}{64} - \frac{31}{64} = \frac{1}{2}. \end{aligned}$$

(c) Let X denote your score and Y denote your friends score. Therefore,

$$P(\text{you win the game}) = \sum_{i=1}^{6} P(X = i, Y < i) = \sum_{i=1}^{6} \frac{1}{6} \cdot \frac{i - 1}{6} = \frac{5}{12}$$

You could also arrive at the answer by the following reasoning: since, the game is symmetric the probability that you win the game is equal to the probability that your friend wins the game. Therefore, $P(\text{you win the game}) = \frac{1}{2} \cdot (1 - P(\text{game is a tie}))$.

Problem 3

(a) Since, the table gives the joint probability distribution of X and Y , the sum of all entries should sum to one. Therefore, $p = (1 - 0.1 - 0.15 - 0.175 - 0.075 - 0.225) = 0.275$.

(b)

$$P(X = 1) = \sum_{i=1}^{3} P(X = 1, Y = i) = 0.5$$

$$P(X = 2) = \sum_{i=1}^{3} P(X = 2, Y = i) = 0.5$$

(c)

$$P(Y = 1) = \sum_{i=1}^{2} P(Y = 1, X = i) = 0.25$$

$$P(Y = 2) = \sum_{i=1}^{2} P(Y = 2, X = i) = 0.25$$

$$P(Y = 2) = \sum_{i=1}^{2} P(Y = 3, X = i) = 0.5$$

(d) $E[X] = 0.5 \cdot 1 + 0.5 \cdot 2 = 1.5$

(e) $E[Y] = 0.25 \cdot 1 + 0.25 \cdot 2 + 0.5 \cdot 3 = 2.25$

(f) $Cove(X, Y) = E[XY] - E[X]E[Y] = \sum_{i=1}^{2} \sum_{j=1}^{3} P(X = i, Y = j) \cdot ij - 1.5 \cdot 2.25 = 0$

(g) No, the two random variables are not independent.

$$P(X = 1 \text{and} Y = 1) = 0.1 \text{ and } P(X = 1) \cdot P(Y = 1) = 0.5 \cdot 0.25 = 0.125.$$

Thus,

$$P(X = 1 \text{ and } Y = 1) \neq P(X = 1) \cdot P(Y = 1)$$

which implies that the random variables X and Y are not independent.