ENPM 662
Final exam

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Work Space

Work space is all available space for the effector to reach

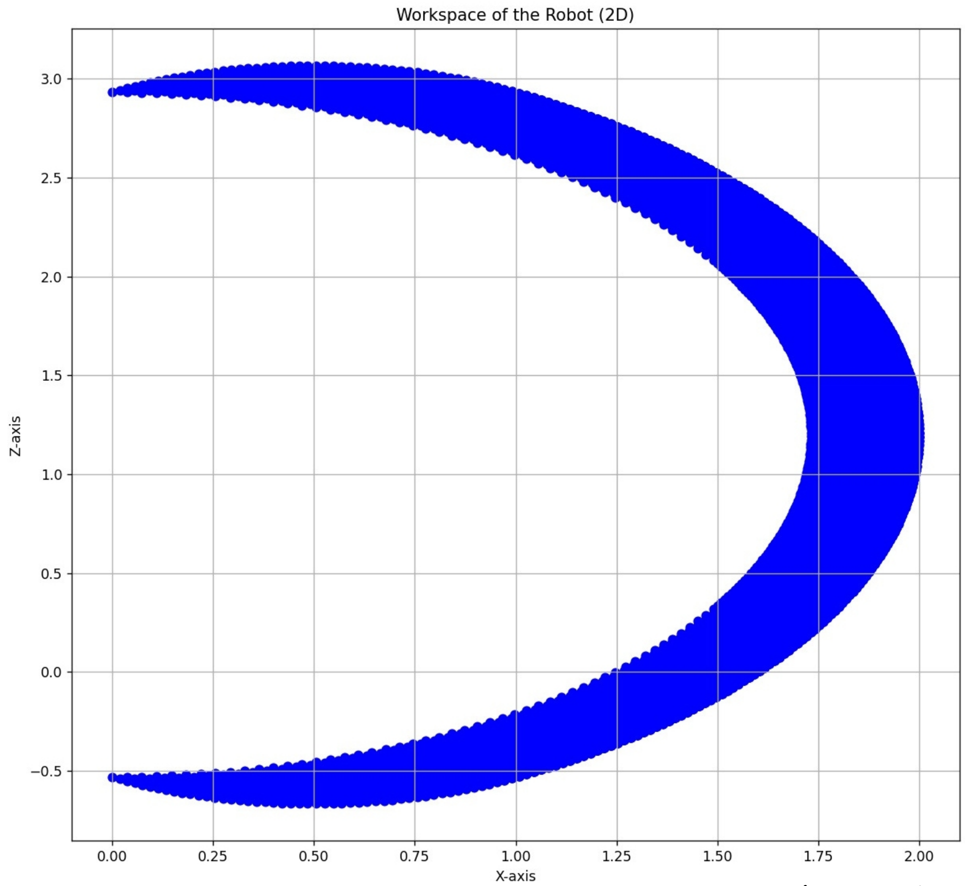
To find this we can run a code which plots all the possible points

For simplicity and to view the unreachble points in the space, I just considered 2-D plots (z-x plane) excluding link 1 rotation so ($\theta_1 = 0$)

To plot this we can run two nested for loops from ($\theta_2 = -60$ to $+60$) and $\theta_3 = -60$ to 60. and printing all the effector positions by finding x, z cordinates from final transforms.

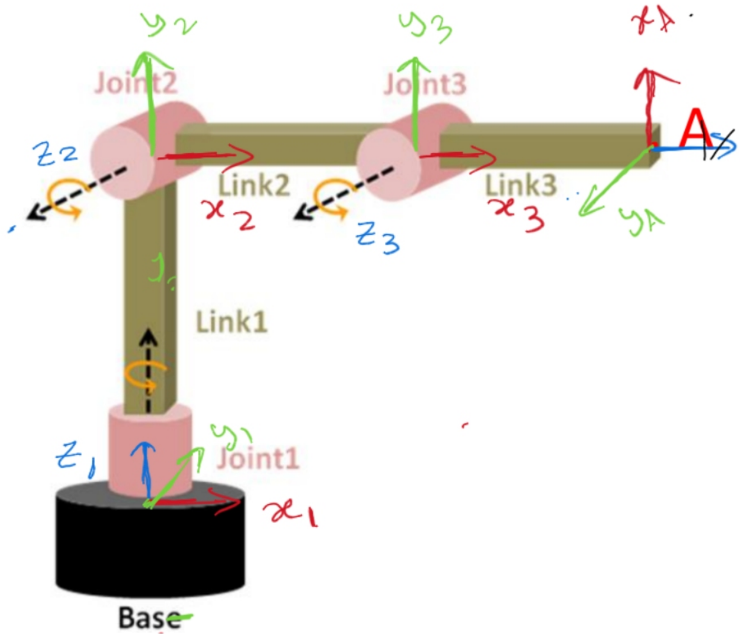
```python
# Joint limits
theta2_limits = np.linspace(-np.pi/3, np.pi/3, 100)
theta3_limits = np.linspace(-np.pi/3, np.pi/3, 100)
end_effector_positions = []
for theta2 in theta2_limits:
    for theta3 in theta3_limits:
        # Forward kinematics equations (adjust based on your robot's kinematics)
        x = cos(theta2) + cos(theta2 + theta3)
        z = sin(theta2) + sin(theta2 + theta3)+1.2
        # print(theta2)

        end_effector_positions.append([x, z])
        # print(x)
```

on implementing the above "for loops" we get the following workspace plot



This plot is only in 1^{st} quadrant this plot can be rotated along z-axis to get full plot (3D) workspace

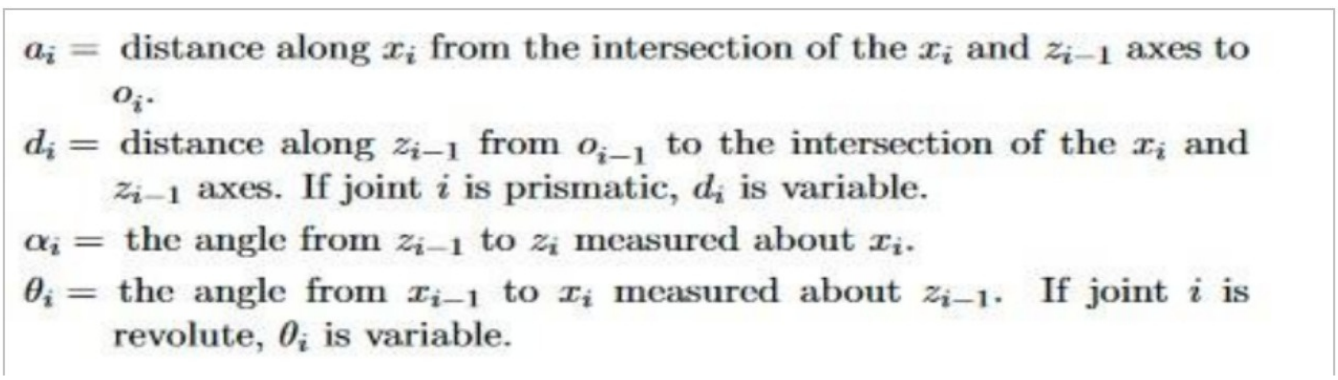
Frame assignment



DH Table

links	a	α_i	d_i	θ_i
1-2	0	90	1.2	θ_i
2-3	1	0	0	θ_i
3-A	1	0	0	θ_i

For this I followed the following Spong Convection



a_i = distance along x_i from the intersection of the x_i and z_{i-1} axes to o_i.

d_i = distance along z_{i-1} from o_{i-1} to the intersection of the x_i and z_{i-1} axes. If joint i is prismatic, d_i is variable.

α_i = the angle from z_{i-1} to z_i measured about x_i.

θ_i = the angle from x_{i-1} to x_i measured about z_{i-1}. If joint i is revolute, θ_i is variable.

Transformation matrix are given by the following matrix function

$$T = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cos \propto_i & \sin\theta_i \sin \propto_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cos \propto_i & -\cos\theta_i \sin \propto_i & a_i \sin\theta_i \\ 0 & \sin \propto_i & \cos \propto_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Final transformation is given by

$$T_A = T_1^2 \times T_2^3 \times T_3^A$$

at home position $(\theta_i = 0)$ we get the final TA as follows

$$\begin{bmatrix} 0 & 0 & 1.0 & 0 \\ 1.0 & 0 & 0 & 2.0 \\ 0 & 1.0 & 0 & 1.2 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

To find Jacobian matrix I followed method one,

$$J_v = [J_{v_1} \text{ --- } J_{v_n}]$$

$$J_\omega = [J_{\omega_1} \text{ --- } J_{\omega_n}]$$

$$J_{v_i} = R_{i-1} \times (o_n - o_{i-1})$$

$$J_{w_i} = R_{i-1} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So $$J_i = \begin{bmatrix} z_{i-1} \times (o_n - o_{i-1}) \\ z_{i-1} \end{bmatrix}$$

$$J = \begin{bmatrix} R_0^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (o_6^0 - o_0^0) & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (o_6^0 - o_1^0) & R_2^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (o_6^0 - o_2^0) \\ R_0^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & R_2^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

J_{at} home position $(\theta_i = 0)$ $\Rightarrow$

$$\begin{bmatrix} 0 & 0 & 0 \\ 2.0 & 0 & 0 \\ 0 & 2.0 & 1.0 \\ 0 & 0 & 0 \\ 0 & -1.0 & -1.0 \\ 1.0 & 0 & 0 \end{bmatrix}$$

J_{at} $(\theta_1 = 0,\ \theta_2 = 45,\ \theta_3 = -45)$ $\Rightarrow$

$$\begin{bmatrix} 0 & -0.707106781186548 & 0 \\ 1.70710678118655 & 0 & 0 \\ 0 & 1.70710678118655 & 1.0 \\ 0 & 0 & 0 \\ 0 & -1.0 & -1.0 \\ 1.0 & 0 & 0 \end{bmatrix}$$

Now defining velocity matrix

$$\varepsilon = \begin{bmatrix} V_n^0 \\ \omega_n^0 \end{bmatrix}$$

$V_n \Rightarrow$ linear velocities

$\omega_n \rightarrow$ angular velocities

$\omega_n = 0$ (In our case)

$$V_n = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

In our case $\dot{x} = 0$, $\dot{y} = 0$ $\dot{z} = \frac{0.707}{20}$ m/s

$$\text{So } \varepsilon = \begin{bmatrix} 0 \\ 0 \\ \dot{z} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\varepsilon = J \cdot \dot{q}$$

$$\dot{q} = J^{-1} \varepsilon$$

This $\dot{q}$ will be used in inverse kinematics

Dynamics

The Lagrangian $\Rightarrow L$

$$L(q,\dot{q}) = K(q,\dot{q}) - P(q)$$

$K \rightarrow$ Kinetic energy matrix

$P \rightarrow$ Potential energy matrix

lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}}\right) - \frac{\partial L(q,\dot{q})}{\partial q} = \tau$$

on solving these we get

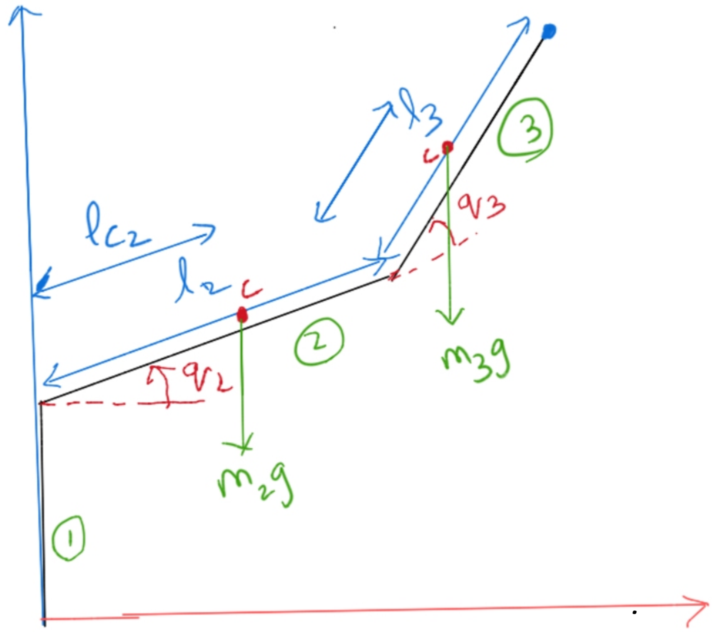
$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau + J^T F$$

$M \rightarrow$ mass matrix

$C \rightarrow$ centripetal and coriolis matrix

$g \rightarrow$ gravity matrix

Consider the following oreintation of the given robotic arm



where

$l_{c_i} \rightarrow$ distance from joint to center of mass of link

$q_i \rightarrow$ joint angles

$m_i \rightarrow$ mass of the link

$l_i \rightarrow$ length of the link

$I_{c_i} \rightarrow$ mass moment of Inertia

finding overall kinetic energy of the overall manupulator

$$K = \sum_{i=1}^{n} \frac{1}{2} m_i V_{c_i}^T V_{c_i} + \frac{1}{2} \omega_i^T I_{c_i} \omega_i$$

$V_c \rightarrow$ linear velocity matrix of cg of link

$\omega_i \rightarrow$ angular velocity matrix of link about joint

Note

Since the 1^{st} Link of our manupulator is not having any involvement in altering Kinetic energies (In drawing line) it is not been considered in calculating total Kinetic energies

So from geometery method we get positions of x_{c_2}, y_{c_2} and x_{c_3}, y_{c_3} as follows

(let arm is moving in $x-y$ plane)

$$x_{c_2} = l_{c_2} \cos(q_2)$$
$$y_{c_2} = l_{c_2} \cos(q_2)$$

$$x_{c_3} = l_2 \cos q_2 + l_{c_3} \cos(q_2 + q_3)$$
$$y_{c_3} = l_2 \sin q_2 + l_{c_3} \sin(q_2 + q_3)$$

we Know that $V_x = \frac{dx}{dt}$ $V_y = \frac{dy}{dt}$

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} \Rightarrow \frac{dx}{d\theta}\dot{\theta}$$

Simillarly for $\frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt} = \frac{dy}{d\theta}\dot{\theta}$

$$V_{c_2} = \begin{bmatrix} \dot{x}_{c_2} \\ \dot{y}_{c_2} \end{bmatrix} = \begin{bmatrix} -\dot{q}_2 l_{c_2} \sin q_2 \\ \dot{q}_2 l_{c_2} \cos q_2 \end{bmatrix}$$

$$V_{c_2}^T = \left[-\dot{q}_2 l_{c_2} \sin q_2 \,,\; \dot{q}_2 l_{c_2} \cos q_2 \right]$$

$$V_{c_2}^T V_{c_2} = \left[-\dot{q}_2 l_{c_2} \sin q_2 \quad \dot{q}_2 l_{c_2} \cos q_2 \right] \begin{bmatrix} -\dot{q}_2 l_{c_2} \sin q_2 \\ \dot{q}_2 l_{c_2} \cos q_2 \end{bmatrix}$$

on simplification

$$\boxed{V_{c_2}^T V_{c_2} = \dot{q}_2^2 l_{c_2}^2}$$

$$V_{c_3} = \begin{bmatrix} \dot{x}_{c_3} \\ \dot{y}_{c_3} \end{bmatrix} = \begin{bmatrix} -\dot{q}_2 l_2 \sin q_2 - (\dot{q}_2 + \dot{q}_3) l_{c_3} \sin(q_2 + q_3) \\ \dot{q}_2 l_2 \cos q_2 + (\dot{q}_2 + \dot{q}_3) l_{c_3} \cos(q_2 + q_3) \end{bmatrix}$$

finding $V_{c_3}^T V_{c_3}$

$$V_{c_3}^T V_{c_3} = \dot{q}_2^2 l_2^2 + (\dot{q}_2 + \dot{q}_3)^2 l_{c_3}^2$$

$$+ 2\dot{q}_2(\dot{q}_2 + \dot{q}_3) l_2 l_{c_2} (\sin q_2 \sin(q_2 + q_3)$$

$$+ \cos q_2 \cos(q_2 + q_3))$$

$$K_v = \frac{1}{2}\sum_{i=1}^{n} m_i V_{c_i}^T V_{c_i}$$
(linear)

$$\Rightarrow \frac{1}{2} m_2 \dot{q}_2^2 l_{c_2}^2 + \frac{1}{2} m_3 (\dot{q}_2^2 l_2^2 + (\dot{q}_2 + \dot{q}_3)^2 l_{c_3}^2$$
$$+ 2\dot{q}_2(\dot{q}_2 + \dot{q}_3) l_2 l_{c_3} \cos q_3$$

$$\Rightarrow \frac{1}{2}[\dot{q}_2 \; \dot{q}_3] \begin{bmatrix} m_2 l_{c_2}^2 + m_3(l_2^2 + l_{c_3}^2 + 2 l_2 l_{c_3} \cos q_3) & \cdot\, m_3(l_{c_3}^2 + l_2 \cdot l_{c_3} \cos q_2) \\ m_3 (l_{c_3}^2 + l_2 l_{c_3} \cos q_3 & m_3 l_{c_3}^2 \end{bmatrix} \begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$K_\omega = \frac{1}{2}\sum_{i=1}^{n} \omega_i^T I_{c_i} \omega_i$$
(angular)

$$\Rightarrow \frac{1}{2}\dot{q}_2^2 I_2 + \frac{1}{2}(\dot{q}_2 + q_3)^2 I_3$$

$$\Rightarrow \frac{1}{2}[\dot{q}_2 \; \dot{q}_3] \begin{bmatrix} I_2 + I_3 & I_3 \\ I_3 & I_3 \end{bmatrix} \begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$K(q,\dot{q}) = K_1 + K_2 \Rightarrow \frac{1}{2} q^T M(q)(q)$$

On comparing we get $M(q)$

Since link 1 doesn't contribute to kinetic energy the following will be $\dot{q}$ matrix

$$\dot{q} \Rightarrow \begin{bmatrix} 0 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$K(q,\dot{q}) = \frac{1}{2}\dot{q}^T \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & m_2 l_{c_2}^2 + m_2\left(l_2^2 + l_{c_3}^2 + 2 l_2 l_{c_3}\cos q_3\right) + I_2 + I_3 & m_3\left(l_{c_3}^2 + l_2 l_{c_3}\cos q_2\right) + I_3 \\ 0 & m_3(l_{c_3}^2 + l_2 l_{c_3}\cos q_3 + I_3 & m_3 l_{c_3}^2 + I_3 \end{bmatrix}}_{M(q)} \dot{q}$$

Now

C is given :-

$$C(q,\dot{q})\dot{q} \Rightarrow \dot{M}(q)(\dot{q}) - \frac{1}{2}\frac{\partial}{\partial q}\left(\dot{q}^T M(q)(\dot{q})\right)$$

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial q}\Big(m_2\dot{q}_2^2 l_{c_2}^2 + m_2(\dot{q}_2^2 l_2^2 + (\dot{q}_2 + \dot{q}_3) l_{c_3}^2 + 2\dot{q}_2(\dot{q}_2 + \dot{q}_3) l_2 l_{c_3}\cos q_3 + \dot{q}_2^2 I_2 + (\dot{q}_2 + \dot{q}_3)^2 I_3$$

on solving the above equation we get

C matrix as follows

$$C(q,\dot{q}) \Rightarrow \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -m_2 l_2 l_{c_3}\dot{q}_3 \sin q_3 & -m_3 l_3 l_{c_3}(\dot{q}_2 + \dot{q}_3)\sin q_3 \\ 0 & m_2 l_2 l_{c_2}\dot{q}_2 \sin q_2 & 0 \end{bmatrix}}_{C(q,\dot{q})} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

Now calculating g matrix

```
PE = g*((m*z1/2)+(m*(z1+((z2-z1)/2)))+(m*(z2+((z3-z2)/2))))
```

$$g(q) = \frac{\partial P(q)}{\partial q} \Rightarrow \begin{bmatrix} \frac{\partial p(q)}{\partial q_1} \\ \frac{\partial p(q)}{\partial q_2} \\ \frac{\partial p(q)}{\partial q_3} \end{bmatrix}$$

```
G =  Matrix([[diff(PE, th0)],
      [diff(PE, th1)],
      [diff(PE, th2)],
      ])
```

```
M_q = Matrix([[0, 0, 0],
              [0, m*lc*lc + m*(l*l+l*l+(2*l*l*sp.cos(q2)))+(2.0*I), m*((lc*lc)+l*l*sp.cos(q2))+I],
              [0, m*((lc*lc)+(l*lc*sp.cos(q2)))+I, (m*lc*lc)+I]])

C_q = Matrix([[0, 0, 0],
            [0, -(m*l*lc*q3_dot*sp.sin(q3)), -(m*l*lc*(q2_dot+q3_dot)*sp.sin(q3))],
            [0, (m*l*lc*q2_dot*sp.sin(q3)), 0]])
```

So Joint torques can be calculated as follows

$$\tau = M(q)(\ddot{q}) + c(q,\dot{q})\dot{q} + g(q) - J^T(F)$$

where $F = \begin{bmatrix} -10 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$

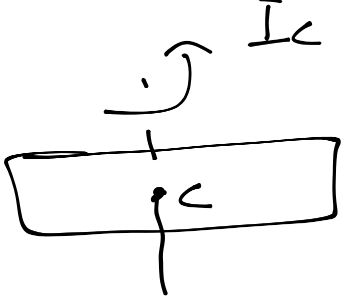
Some calculations

$$\text{Mass} = \text{Volume} \times \text{density}$$

$$\Rightarrow (0.09^2 - 0.08^2) \times 1 \times 7600$$

$$m \Rightarrow 12.92 \text{ kg}$$

Mass moment of inertia



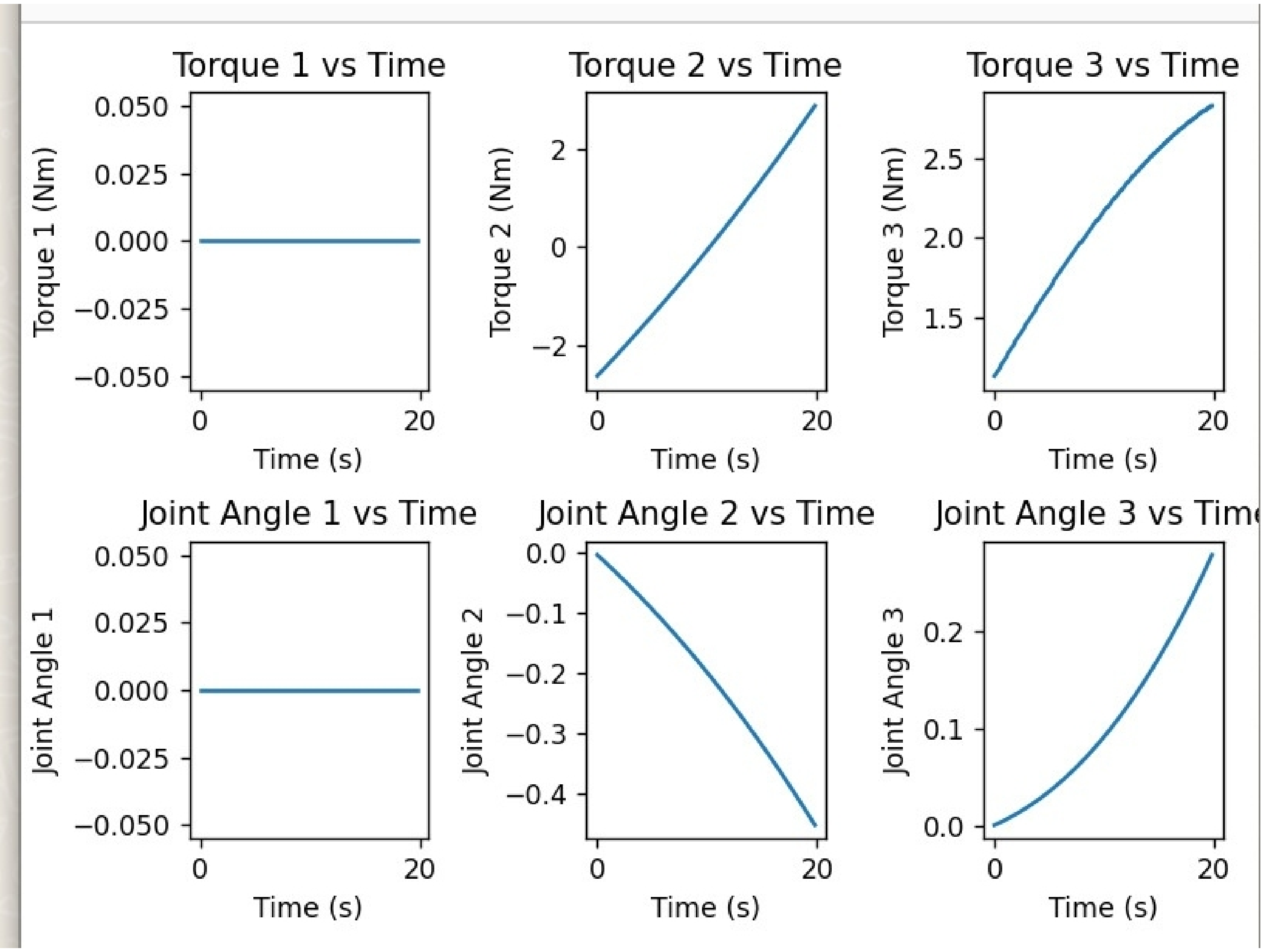
$$I_c = \frac{ml^2}{12}$$

of Bar

$$\Rightarrow \frac{12.92 \times (1)^2}{12} =$$

$$\Rightarrow 1.0766 \text{ kg m}^2$$

Output of Joint angles and torques



Note:- angles are started from 0 but not 45° because as initial (home position) to draw line is set 45° position by adding +45 and -45 to θ_2 to θ_3 respectively. Since the home position is changed to 45° so angles will start ploting from zero (relavitely)

Question 2

1) To show any rotation matrix is orthogonal with determinant equal to one

For simplicity Lets consider 2×2 matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

A Square matrix A is said to orthogonal if the following satisfies

$$A^T A = I$$

$$A^I = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^I A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos\theta^2 + \sin^2\theta & -(\cos\theta)(\sin\theta) + \cos\theta\sin\theta \\ -((\cos\theta)(\sin\theta)) + (\sin\theta\cos\theta) & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$(\cos\theta^2 + \sin\theta^2 = 1)$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow 1$$

Now finding $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

$$\Rightarrow \cos\theta^2 + \sin\theta^2 = 1$$

So, $|A| = 1 \rightarrow (2)$

Simillarly, (1) & (2) are true for any Rotation matrix of any order

2) To show that length of a free vector is not changed by rotation, $\|v\| = \|Rv\|$

let v be the vector under Rotation

R be the Rotation matrix

magnitude of v is given as $\|v\|$

$$\|v\| = \sqrt{v \cdot v}$$

After Rotation by "R" let v becomes Rv

So length of $Rv = \sqrt{Rv \cdot Rv}$

we know that R is a orthogonal matrix and Preserves dot product

i.e $Rv \cdot Rv = v \cdot v$

So $\sqrt{Rv \cdot Rv} = \sqrt{v \cdot v} = \|v\|$

So, we get

$$\|V\| = \|RV\|$$

To Prove it take a Rotation matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos\theta(V_1) - \sin\theta(V_2) \\ \sin\theta(V_1) + \cos\theta(V_2) \end{bmatrix}$$

let $\theta = 0$

$$\Rightarrow \begin{bmatrix} V_1 - 0 \\ 0 + V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Rightarrow \|V\| \Rightarrow \sqrt{V_1^2 + V_2^2}$$

let $\theta = 90^\circ$

$$\Rightarrow \begin{bmatrix} 0 - V_2 \\ V_1 \end{bmatrix} = \begin{bmatrix} -V_2 \\ V_1 \end{bmatrix} \Rightarrow \|V\| = \sqrt{V_1^2 + V_2^2}$$

So from above we conclude that Rotation matrix doesnot change the vector magnitude. of a free vector

(3) To show that distance between points is not changed by rotation that is

$\|P_1 - P_2\| = \|RP_1 - RP_2\|$

let RP_1, RP_2 are two points in space arrived from rotation of space of points P_1 & P_2

$$\|RP_1 - RP_2\|^2 = (RP_1 - RP_2)(RP_1 - RP_2)$$

$$\Rightarrow (RP_1)(RP_1) - 2(RP_1)(RP_2) + (RP_2)(RP_2)$$

we know $a \cdot b = a^T b$

So applying this

$$\Rightarrow (RP_1)^T(RP_1) - 2(RP_1)^T(RP_2) + (RP_2)^T(RP_2)$$

$$\Rightarrow P_1^T R^T R P_1 - 2P_1^T R^T R P_2 + P_2^T R^T R P_2$$

Since R is orthogonal matrix $R^T R = I$

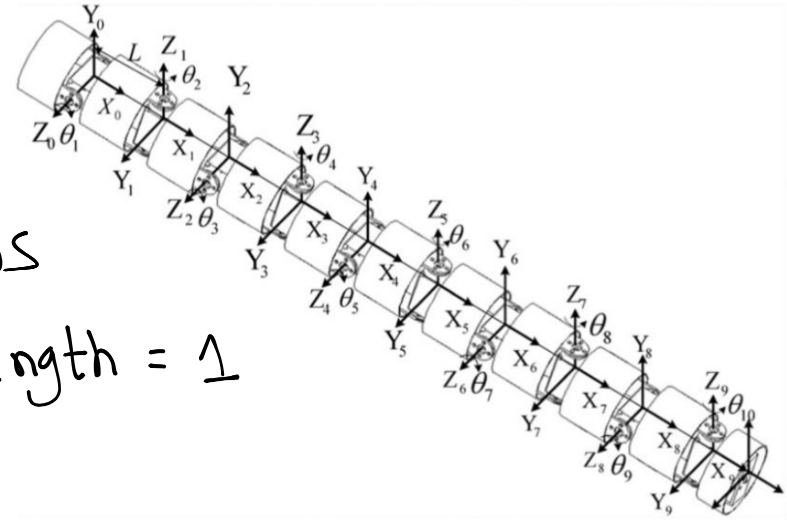
$$\Rightarrow P_1^T (I) P_1 - 2P_1^T (I) P_2 + P_2^T (I) P_2$$

$$\Rightarrow P^T P_1 - 2P_1^T P_2 + P_2^T P_2$$

$$\Rightarrow \|P_1 - P_2\|^2$$

This shows distance doesn't change by Rotation

Question 3



Possible D-H will be as follows

assume:- link length = 1

Link	a_i(in m)	α_i	d_i(in m)	θ_i
0-1	1	90	0	θ_1
1-2	1	-90	0	θ_2
2-3	1	90	0	θ_3
3-4	1	-90	0	θ_4
4-5	1	90	0	θ_5
5-6	1	-90	0	θ_6
6-7	1	90	0	θ_7
7-8	1	-90	0	θ_8
8-9	1	90	0	θ_9
9-10	1	-90	0	θ_{10}

To get a snake like movement I am assuming every Joint should follow a "Sine" trignometery but with a distinct Phase angles

When a body is said to be in SHM (simple harmonic motion) the motion is governed by the following equation

$$y = A \sin(\omega t + \phi)$$

$\omega \rightarrow$ angular frequency
$\phi \rightarrow$ phase difference among particles
$A \rightarrow$ linear or Angular amplitudes
$t \rightarrow$ time

we can use different phase angle shift for all the Simiarl joints ($\alpha = 90$) and other for joints ($\alpha = -90$)
so the implimentation will be

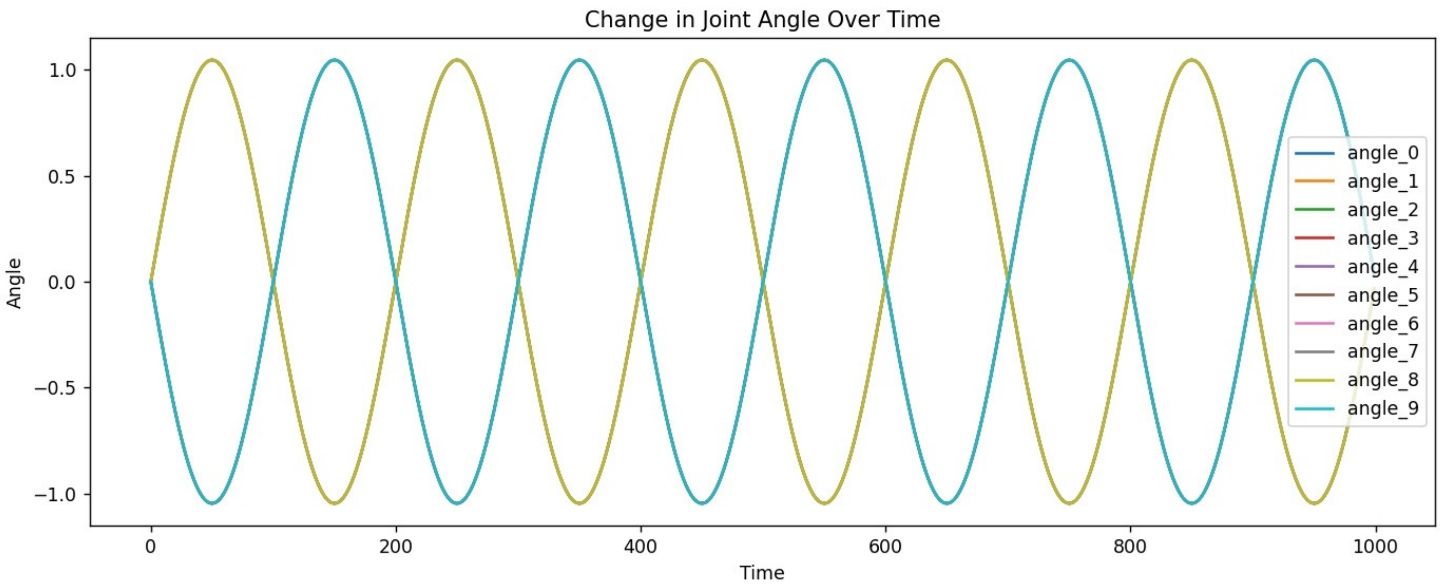
$$\theta_x = A_x \sin(\omega_x t + \phi_x) \Rightarrow \text{for } a = -90$$

$$\theta_y = A_y \sin(\omega_y t + \phi x) \Rightarrow \text{for } a \Rightarrow 90$$

The following code logic is used to get the angles from 1 to 10 joints inorder to get Snake motion (sine trajectory)

```python
# Iterating to calculate joint angles over time
for time_step in range(1000):
    for joint_index in range(10):
        if joint_index % 2 == 0:
            theta_1 = generate_sine_wave(amplitude_x, frequency_x * time_step, joint_index * phase_diff_x)
            angle_data[time_step, joint_index] = theta_1
        else:
            theta_2 = generate_sine_wave(amplitude_y, frequency_y * time_step, joint_index * phase_diff_y -
            angle_data[time_step, joint_index] = theta_2
```

Output



All the angles with same phase difference will coincide each other hence resulting in only 2 plots

References :

https:www.outube.com/watch?v=QN-Awth50aA
https://www.ivlabs.in/uploads/1/3/0/6/130645069/rebis.pdf