Modeling

Homework- 2

Sai Dinesh Gelam

UID:120167140

1.1) Composition of transforms

Given operations:

Rotation by angle ϕ about the world Z-axis is represented by $R_{Z,\emptyset}$

Translation by y along the current Y-axis is represented by $T_{Y,y}$

Rotation by angle θ about the current Z-axis is represented by $R_{Z,\theta}$

Rotation by angle φ about the world X-axis is represented by $R_{X,\varphi}$

There should be two types of transformations needed to be made in order to attain the required position.

a) Rotation by world axis
b) Rotation by current axis

The only difference between these two rotations in computation is the order of multiplication of matrices.

Pre-multiplication --Rotation by World axis

Post-multiplication -- Rotation by Current axis

Now lets define rotational matrix of each operation

$$R_{Z,\emptyset} = \begin{bmatrix} \cos(\emptyset) & -\sin(\emptyset) & 0 & 0 \\ \sin(\emptyset) & \cos(\emptyset) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{Y,y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{Z,\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{X,\varphi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) & 0 \\ 0 & \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

First lets Rotate by angle ϕ about the **world Z-axis**: $R_{Z,\emptyset} * I$ (I is a identity matrix)

Now translate along **current Y axis** by y units: $R_{Z,\emptyset} * T_{Y,y}$ (since it is operation by current axis it should be post multiplied).

Rotation by angle θ about the **current Z-axis**: $R_{Z,\emptyset} * T_{Y,y} * R_{Z,\theta}$ (since it is operation by current axis it should be post multiplied).

Rotation by angle φ about the **world X-axis:** $R_{X,\varphi} * R_{Z,\emptyset} * T_{Y,y} * R_{Z,\theta}$ (since it is operation by world axis it should be pre multiplied).

So, the final production matrix is $R_{X,\varphi} * R_{Z,\emptyset} * T_{Y,y} * R_{Z,\theta}$

Substituting matrices we get

$R_{X,\varphi} * R_{Z,\emptyset} * T_{Y,y} * R_{Z,\theta}$ =

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) & 0 \\ 0 & \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\emptyset) & -\sin(\emptyset) & 0 & 0 \\ \sin(\emptyset) & \cos(\emptyset) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

On multiplying all the Matrices, we get final matrixes as

$$\begin{bmatrix} -\sin(\phi)\sin(\theta) + \cos(\phi)\cos(\theta) & -\sin(\phi)\cos(\theta) - \sin(\theta)\cos(\phi) & 0 & -y\sin(\phi) \\ \sin(\phi)\cos(\psi)\cos(\theta) + \sin(\theta)\cos(\phi)\cos(\psi) & -\sin(\phi)\sin(\theta)\cos(\psi) + \cos(\phi)\cos(\psi)\cos(\theta) & -\sin(\psi) & y\cos(\phi)\cos(\psi) \\ \sin(\phi)\sin(\psi)\cos(\theta) + \sin(\psi)\sin(\theta)\cos(\phi) & -\sin(\phi)\sin(\psi)\sin(\theta) + \sin(\psi)\cos(\phi)\cos(\theta) & \cos(\psi) & y\sin(\psi)\cos(\phi) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.2)Modeling beyond rigid transformations

Given senerio:

F-f : Fixed frame

E-f : Earth Frame

S: Sun’s frame

a : Major axis length of elipse

b : Minor axis length of elipse

θ : Rotation angle of earth around F-ez

α : Revolution angle of earth around S

ω_e: Angular velocity of Rotation

ω_e: Angular velocity of Revolution

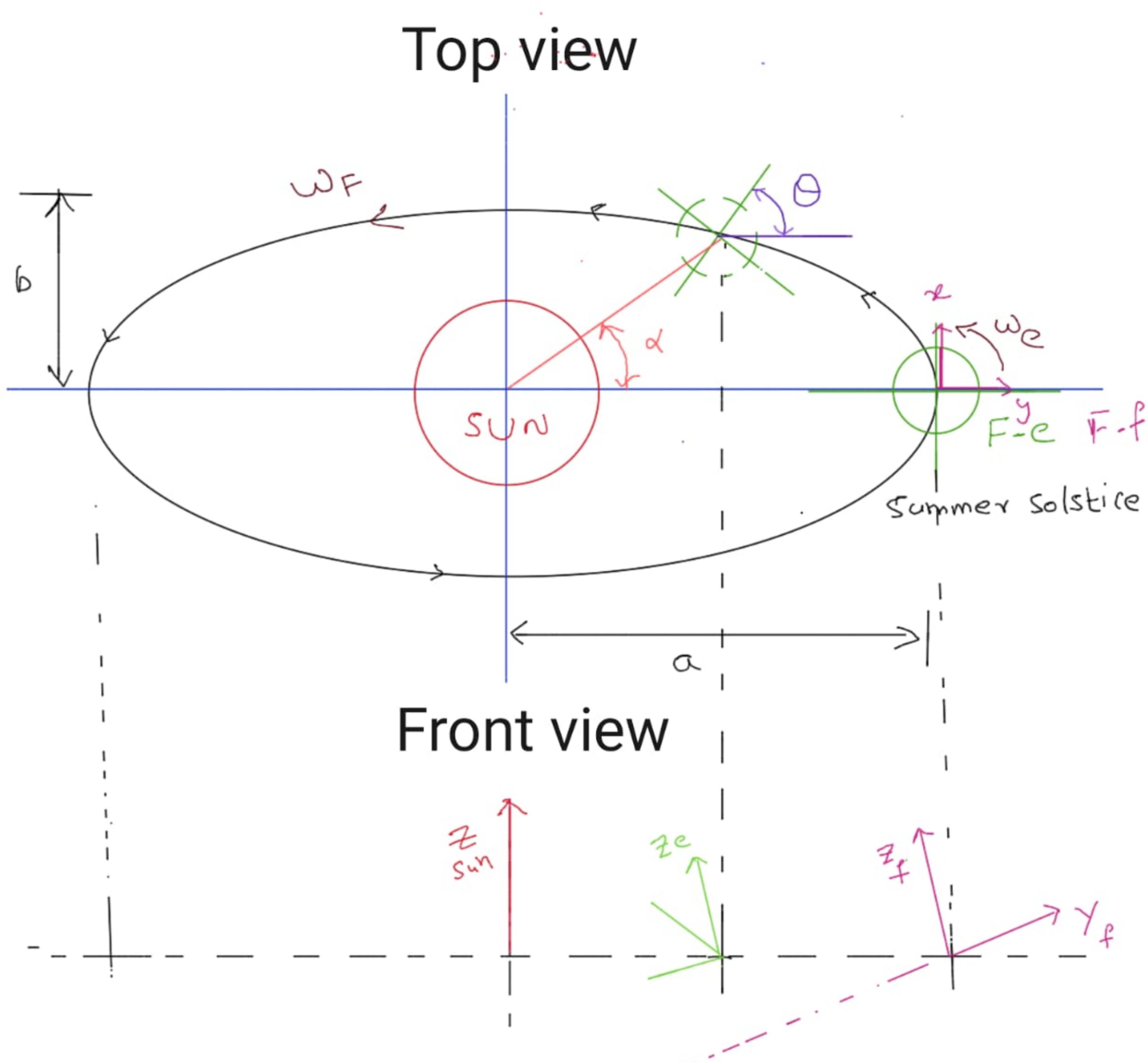


Figure 1 The figure shows the orientation of axes of fixed, earth and sun at t=0 and during motion

Since it is given that at t=0 the earth axes coincide with fixed frame, it can be concluded that Sun axes (Sz) is at the angle of 23.5^{θ} with respective F-fz and it is in the distance of "-a "along current axis(after rotation along Sx.

Transformation(rotation) of 23.5 of sun axes from

F-f about Sx → $R_{SX} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(23.5) & -\sin(23.5) & 0 \\ 0 & \sin(23.5) & \cos(23.5) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Transformation (translation) of Sun axes along current Sy → $T_{SY} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Transformation (Translation) of earth axes about Sun's axes

using polar co-ordinate of ellipse $T_{Ex,y}$ → = $\begin{bmatrix} 1 & 0 & 0 & bsin\alpha \\ 0 & 1 & 0 & acos\alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Now Earth's z-axis(F-eZ) should be aligned -23.5^{θ}wrt to Sun's Z

So, transform (Rotate) -23.5 about (F-e X) → $R_{EX} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-23.5) & -\sin(-23.5) & 0 \\ 0 & \sin(-23.5) & \cos(-23.5) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Now Rotate Earth about its own axis (F-ez) by rotation angle θ → $R_{EZ} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Lets consider transformation from Fixed frame to sun Frame = ${}^{F}_{S}H$ ➔ $R_{SX} * T_{SY}$

Lets consider transformation from Sun frame to Earth frame = ${}^{S}_{E}H$ ➔ $T_{Ex,y} * R_{EX}$

So therefore to get transformation from fixed frame to earth frame we need to multiply the above two

$${}^{F}_{E}H = {}^{F}_{S}H * {}^{S}_{E}H$$

So ${}^{F}_{E}H = R_{SX} * T_{SY} * T_{Ex,y} * R_{EX}$

Now include rotation of earth about its own axis = ${}_{E}^{F}H * R_{EZ}$

So to get the final orientation of earth axes with respective fixed frame will be given by the equation

$$=R_{SX} * T_{SY} * T_{Ex,y} * R_{EX} * R_{EZ}$$

On substituting these matrices, we get ➔

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(23.5) & -\sin(23.5) & 0 \\ 0 & \sin(23.5) & \cos(23.5) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & bsin\alpha \\ 0 & 1 & 0 & acos\alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-23.5) & -\sin(-23.5) & 0 \\ 0 & \sin(-23.5) & \cos(-23.5) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

On multiplying all the matrices, we get ➔

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & bsin(\alpha) \\ \sin(\theta) & \cos(\theta) & 0 & acos(\alpha)\cos(\theta) - acos(\theta) \\ 0 & 0 & 1 & asin(\theta)\cos(\alpha) - asin(\theta) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where θ is defined as $\theta = \frac{360*t}{24*3600}$ t is in seconds

And α is defined as $\alpha=\frac{360*t}{365*24*3600}$ t is in seconds

a is semi major axes of ellipse = 152.1 * 10^6 km

b is semi minor axes of ellipse = 141.7 *10^6 km

verification:

When t=0 the transformation should result in reference frame orientation

At t=0 ➔ $\theta = 0$; α = 0

So we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notes : when t≠0 we get some component of z because the plane of revolution of earth is inclined to reference frame.

1.3) Transform Estimation

The given cube can be transformed to required orientation (EFGH should be on ZX plane and F should be on Z axis, G should be on X axis) by some set of transformations (rotations and then translations)

These transformations are as follows.

a) *Rotate the cube 90 deg clockwise about world Y.*

➔ $R_{Y,90} = \begin{bmatrix} cos\,(-90) & 0 & sin\,(-90) & 0 \\ 0 & 1 & 0 & 0 \\ -sin\,(-90) & 0 & cos(-90) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

b) Rotate the cube 90 deg clockwise about world Z.

➔ $R_{Z,90} = \begin{bmatrix} \cos(90) & -\sin(90) & 0 & 0 \\ \sin(90) & \cos(90) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c) Rotate the cube 45 deg clockwise about world Y.

➔ $R_{Y,45} = \begin{bmatrix} cos\,(-90) & 0 & sin\,(-90) & 0 \\ 0 & 1 & 0 & 0 \\ -sin\,(-90) & 0 & cos(-90) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

d) Translate the cube by 1m along positive Y

➔ $T_{y,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

e) Translate the cube $\frac{1}{\sqrt{2}}$ along positive Z

➔ $T_{z,\frac{1}{\sqrt{2}}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

f) Translate the cube by $\sqrt{2}$ along positive X

➔ $T_{x,\sqrt{2}} = \begin{bmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

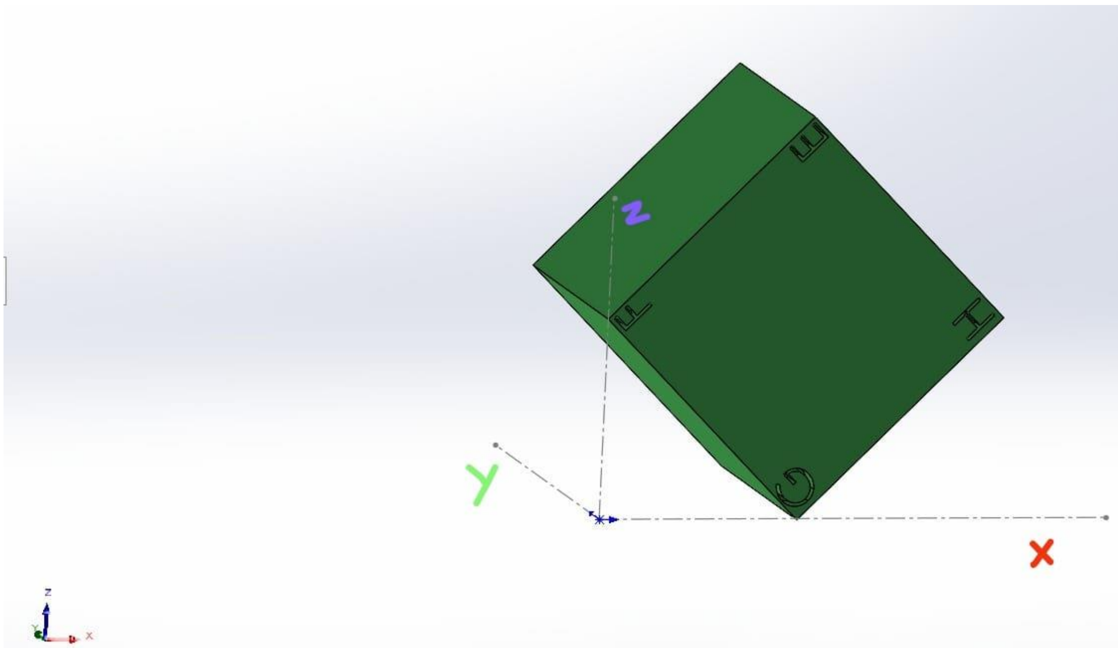
Now as the transformations all are done using fixed axis so we need to pre-multiply of all the rotation matrix.

➔ $T_{x,\sqrt{2}} * T_{z,\frac{1}{\sqrt{2}}} * \ T_{y,1} * R_{Y,45} * \ R_{Z,90} * R_{Y,90}$

On multiplying all we get homogeneous matrix of

$$\boldsymbol{H} = \begin{bmatrix} -0.707 & 0.707 & 0 & 1.1414 \\ 0 & 0 & 1 & 1 \\ 0.707 & -0.707 & 0 & 0.707 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The final orientation of the cube in the required position is shown the picture below



From this final matrix we can deduce Euler angles by comparing with the actual YZY combination matrix

$$\begin{bmatrix} -\sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi)\cos(\theta) & -\sin(\theta)\cos(\psi) & \sin(\phi)\cos(\psi)\cos(\theta) + \sin(\psi)\cos(\phi) \\ \sin(\theta)\cos(\phi) & \cos(\theta) & \sin(\phi)\sin(\theta) \\ -\sin(\phi)\cos(\psi) - \sin(\psi)\cos(\phi)\cos(\theta) & \sin(\psi)\sin(\theta) & -\sin(\phi)\sin(\psi)\cos(\theta) + \cos(\phi)\cos(\psi) \end{bmatrix}$$

$$\theta = \cos^{-1}(r_{22})$$

$$\theta = \cos^{-1}(0)$$

$$\theta = 90 \text{ deg}$$

Similarly, for ϕ we get

$$\phi = \tan^{-1}\frac{r_{23}}{r_{21}}$$

$$\phi = \tan^{-1}(1)$$

$$\phi = 90 \text{ deg}$$

Similarly, for ψ we get

$$\psi = \tan^{-1}\frac{r_{32}}{r_{12}}$$

$$\psi = \tan^{-1}\left(\frac{\left(\frac{1}{\sqrt{2}}\right)}{\left(\frac{1}{\sqrt{2}}\right)}\right)$$

$$\psi = 45°$$

So, Euler angles YZY for the transformation are $\theta = 90°; \phi = 90°; \psi = 45°$

1.4) Trajectory Optimization

Given that the required angle rotations are ψg = 45∘, θg = 25∘, ϕg = 16∘ which are consecutive rotations about the global X, Y, Z axes.
Max angular velocity about any axes is ω_{max}= 2 deg/sec

So now lets define the rotation matrices for each rotation

$$R_{\emptyset Z} = \begin{bmatrix} \cos\emptyset & -\sin\emptyset & 0 \\ \sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{\theta Y} = \begin{bmatrix} \cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{\theta X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & sin\varphi & \cos\varphi \end{bmatrix}$$

Now, given that the order of rotation is X, Y, Z about global frame so we have to pre-multiply the matrices

So final rotation matrix R is given by

R ➔ $R_{\emptyset Z}*R_{\theta Y}*R_{\theta X}$

$$R \Rightarrow \begin{bmatrix} \cos\emptyset & -\sin\emptyset & 0 \\ \sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & sin\varphi & \cos\varphi \end{bmatrix}$$

On substituting values of φ = 45∘, θ = 25∘, $\emptyset$ = 16∘ we get

$$\Rightarrow \begin{bmatrix} \cos 16 & -\sin 16 & 0 \\ \sin 16 & \cos 16 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 25 & 0 & sin25 \\ 0 & 1 & 0 \\ -sin25 & 0 & \cos 25 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45 & -\sin 45 \\ 0 & sin45 & \cos 45 \end{bmatrix}$$

R ➔ $\begin{bmatrix} 0.870 & -0.0922 & 0.481 \\ 0.249 & 0.761 & -0.597 \\ -0.422 & 0.640 & 0.640 \end{bmatrix}$ this will be the final rotation matrix

Now, using **Axis angle representation method,** let's find the angle α and vector k which gives the equivalent rotation for the above-mentioned R.

in this method we know that α = $cos^{-1}(\frac{trace\ (R)-1}{2})$

and trace = r_{11} + r_{22}+r_{33}

trace = 0.870+0.761+0.640 =2.27

α = $cos^{-1}(\frac{2.27-1}{2})$

α = 50.58 deg

Now, to find the rotation axis vector we have the formula`

$$\hat{k} = \frac{1}{2*\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

$$\hat{k} = \frac{1}{2*\sin(50.58)}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

$$\hat{k} = \begin{bmatrix} 0.803 \\ 0.587 \\ 0.102 \end{bmatrix}$$

Rotation about $\hat{k}$ by α with w_k angular velocity can be resolved into w_x, w_y, w_z

$$w_k k = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

For K_x= 45 deg which is highest angle is to be rotated by max angular velocity so w_x $=$ 2 deg/ sec

$$w_k = \frac{w_x}{k_x}$$

$$w_k = \frac{2}{0.803} = 2.49 \text{ deg/sec}$$

So, using find w_k we will find w_y, w_z

$$w_y = 2.49 * 0.586 = 1.46 \text{ deg/s}$$

$$w_z = 2.49 * 0.102 = 0.25 \text{ deg/s}$$

Now the shortest time possible will be.

$$Tmin = \frac{\alpha}{w_k}$$

$$Tmin = \frac{50.42}{2.49}$$

$$Tmin = 20.25\ sec$$

Now we have all the required angular velocities $\begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} 2 \\ 1.46 \\ 0.25 \end{bmatrix}$ deg/sec

On plotting w_x, w_y, w_z with respective to time and incrementing the angle $\varphi, \theta, \emptyset$ with time we get the following graphs