ENPM 667
Problem Set -5

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① The given state equation

$$\dot{x} = \underbrace{\frac{1}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}}_{A} x(t) + \underbrace{e^{t/2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} u(t)$$

Since B is in terms of (t) we can conclude that given system is a linear time varying system

So from controlability therom the system can be controlable, only when grammian matrix of controlability is invertable if and only if $n \times nm$ controllability matrix satisfies

$$\text{rank}\left(\left[B \vdots AB \vdots A^2B \vdots \cdots \vdots A^{n-1}B\right]\right) = n$$

here $n = 2$ (2x2 matrix)

So $B = \begin{bmatrix} e^{t/2} \\ e^{t/2} \end{bmatrix}$

$$AB = \begin{bmatrix} \frac{5}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{5}{12} \end{bmatrix}\begin{bmatrix} e^{t/2} \\ e^{t/2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{5}{12}e^{t/2} + \frac{1}{12}e^{t/12} \\ \frac{1}{12}e^{t/2} + \frac{5}{12}e^{t/12} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{t/2} \\ \frac{1}{2}e^{t/2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} e^{t/12} & \frac{1}{2} e^{t/12} \\ e^{t/12} & \frac{t}{2} e^{t/12} \end{bmatrix}$$

we can observe that column 2 is a multiple of column 1 (similar to Rows as well) which indicates dependences of columns and rows which gives the determinant zero

$$\left| \begin{bmatrix} B & AB \end{bmatrix} \right| = 0$$

So $\boxed{\text{rank} \begin{bmatrix} B & AB \end{bmatrix} < n}$

(not full rank)

Hence the given System is uncontrollable

(2) Given system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \gamma_1(t) & \gamma_2(t) \\ -\gamma_2(t) & \gamma_1(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The given equation of the system is in the form of

$$\dot{x}(t) = A x(t)$$

So we can find state transistion matrix by the equation

$$\phi(t_0, t_t) = P e^{D(t-t_0)} P^{-1} \quad \text{--- (1)}$$

finding eigen values of A

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} \gamma_1(t) - \lambda & \gamma_2(t) \\ -\gamma_2(t) & \gamma_1(t) - \lambda \end{vmatrix}$$

$$\Rightarrow (\gamma_1(t) - \lambda)^2 + \gamma_2^2(t) = 0$$

$$\Rightarrow \gamma_1^2 + \lambda^2 - 2\gamma_1\lambda + \gamma_2^2 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda(\gamma_1) + \gamma_1^2 + \gamma_2^2 = 0$$

$$\text{roots} = \; = \frac{2\gamma_1 \pm \sqrt{4\gamma_1^2 - 4(\gamma_1^2 + \gamma_1^2)}}{2}$$

$$\Rightarrow \frac{\cancel{2}\left(\gamma_1 \pm \sqrt{\cancel{\gamma_1^2} - (\cancel{\gamma_1^2} + \gamma_2^2)}\right)}{\cancel{2}}$$

$$\Rightarrow \gamma_1 \pm \sqrt{-\gamma_2^2}$$

$$\Rightarrow \gamma_1 \pm \gamma_2 i$$

$$\lambda_1 = \gamma_1 + \gamma_2 i \qquad \lambda_2 = \gamma_1 - \gamma i$$

eigen vector for λ_1, λ_2 will be following

$$\Rightarrow \begin{bmatrix} \gamma_1 - \lambda & \gamma_2 \\ -\gamma_2 & \gamma_1 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -i\gamma_2 & \gamma_2 \\ -\gamma_2 & -i\gamma_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\to \lambda_1 = \gamma_1 + \gamma_2 i$$

$$\Rightarrow \begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow -x_1 i + x_2 = 0$$

$$x_1 - x_2 i = 0$$

$$\Rightarrow X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

for $\lambda_2 = \sigma_1 - i\sigma_2$

$$\Rightarrow \begin{bmatrix} i\sigma_2 & \sigma_2 \\ -\sigma_2 & i\sigma_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \quad -ix_1 + x_2 = 0$$

$$-x_1 + ix_2 = 0$$

$$x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Now we get p matrix $\quad P = [x_1, X_2]$

$$P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \qquad |P| = -i - i = -2i$$

$$P^{-1} = \frac{\text{Adj}(P)}{\det(P)} = \frac{1}{-2i} \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

Now the state transistion matrix is given by

$$\phi(t_0\ t_1) = Pe^{D(t-t_0)}P^{-1}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{\int_{t_0}^{t_t} \gamma_1(t) + i\gamma_2(t)\, dt} & 0 \\ 0 & e^{\int_{t_0}^{t_t} \gamma_1(t) - i\gamma_2(t)\, dt} \end{bmatrix}$$

$$\times \quad \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

$\Rightarrow$ Let $c_1 = e^{\int_{t_0}^{t_t} \gamma_1(t) + i\gamma_2(t)\, dt}$

$$c_2 = e^{\int_{t_0}^{t_t} \gamma_1(t) - i\gamma_2(t)\, dt}$$

So $\Rightarrow$ $\frac{1}{2} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$

on multiplying we get

$$\Rightarrow \frac{1}{2} \begin{bmatrix} c_1 + c_2 & i(c_2 - c_1) \\ i(c_1 - c_2) & (c_1 + c_2) \end{bmatrix} = \phi(t_0, t_t)$$

state transistion matrix

Solution is given by

$$\boxed{x(t) = \phi(t_0, t_t)\, x(t_0)}$$

(3) Equations of motion :

$$J_1 \ddot{q}_1 = \tau$$

$$J_2 \ddot{q}_2 = \tau$$

So $J_1 \ddot{q}_1 = J_2 \ddot{q}_2$

State variables : q_1 q_2

$\dot{q}_1$ $\dot{q}_2$

So $x_1 = q_1$ $\dot{x}_1 = \dot{q}_1$

$x_2 = q_2$ $\dot{x}_2 = \dot{q}_2$

$$x = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}$$

State space equations

$\dot{x} = Ax + Bu$ Now we write state as following

$$\Rightarrow \dot{x} = \begin{bmatrix} \dot{q}_1 \\ \ddot{q}_1 \\ q_2 \\ \ddot{q}_2 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{4\times 4,\ A} \underbrace{\begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix}}_{4\times 1,\ x} + \underbrace{\begin{bmatrix} 0 \\ 1/J_1 \\ 0 \\ 1/J_2 \end{bmatrix}}_{B} \underbrace{(\tau)}_{u}$$

$\Rightarrow$ Controlability matrix $\Rightarrow$ $C \Rightarrow [B, AB, A^2B, A^3B]$

$$AB \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{J_1} \\ 0 \\ \frac{1}{J_2} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{J_1} \\ 0 \\ \frac{1}{J_2} \\ 0 \end{bmatrix}$$

$$A^2B \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/J_1 \\ 0 \\ 1/J_2 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Simillarly $A^3B = 0$

So $$C = \begin{bmatrix} 0 & 1/J_1 & 0 & 0 \\ 1/J_1 & 0 & 0 & 0 \\ 0 & 1/J_1 & 0 & 0 \\ 1/J_2 & 0 & 0 & 0 \end{bmatrix}$$

Since two columns of C are zero

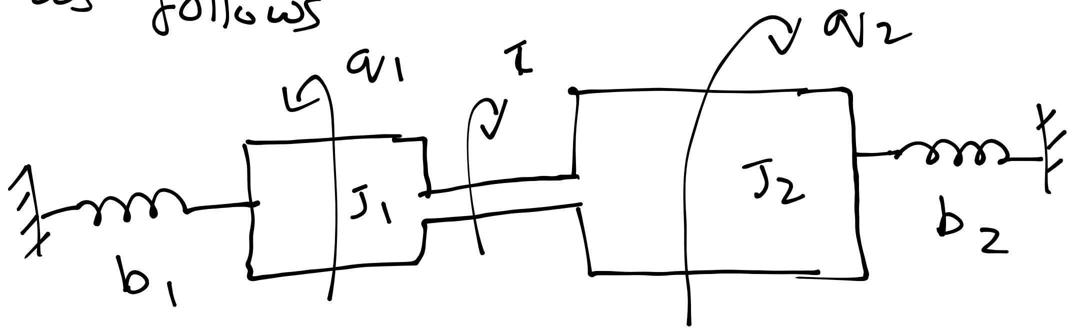
$|C| = 0$

So $\text{rank}\left[B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, A^3B\right] \neq n$

(not full rank)

So the system is <u>uncontrollable</u>

To make it controllable add supports external (dampers) to both the bodies as follows



Now the motion equations are

$$\tau = J_1\ddot{q}_1 + b_1\dot{q}_1$$

$$\tau = J_2\ddot{q}_2 + b_2\dot{q}_2$$

Now state variables are

$$\boxed{\begin{aligned} \ddot{q}_1 &= \frac{\tau - b_1\dot{q}_1}{J_1} \\ \ddot{q}_2 &= \frac{\tau - b_2\dot{q}_2}{J_2} \end{aligned}}$$

$$\Rightarrow \quad x = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} \quad \dot{x} = \begin{bmatrix} \dot{q}_1 \\ \ddot{q}_2 \\ \dot{q}_2 \\ \ddot{q}_2 \end{bmatrix}$$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-b_1}{J_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{-b_2}{J_2} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_3 \end{bmatrix}}_{J \to x} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{J_1} \\ 0 \\ \frac{1}{J_2} \end{bmatrix}}_{B} \underbrace{(\tau)}_{\bar{u}}$$

$$A_c = [B,\ AB\quad A^2B\quad A^3B]$$

$$\Rightarrow \quad AB \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-b_1}{J_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{-b_2}{J_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1/J_1 \\ 0 \\ \frac{1}{J_2} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{J_1} \\ \frac{-b_1}{J_1J_2} \\ \frac{1}{J_2} \\ \frac{-b_2}{J_1J_2} \end{bmatrix}$$

$$\Rightarrow \quad A^2B \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-b_1}{J_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{-b_2}{J_2} \end{bmatrix} \begin{bmatrix} \frac{1}{J_1} \\ \frac{-b_1}{J_1J_2} \\ \frac{1}{J_2} \\ \frac{-b_2}{J_1J_2} \end{bmatrix} = \begin{bmatrix} -b_1/J_1J_2 \\ b_1^2/J_1^2J_2 \\ \frac{1}{J_2} \\ \frac{+b_2^2}{J_1J_2^2} \end{bmatrix}$$

$$\Rightarrow A^3B \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{b_1}{J_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{b_2}{J_2} \end{bmatrix} \begin{bmatrix} -b_1/J_1J_2 \\ +b_1^2/J_1^2J_1 \\ \frac{1}{J_2} \\ \frac{b_2^2}{J_1J_2^2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} +b_1^2/J_1^2J_1 \\ -b_1^3/J_1^3J_1 \\ +b_2^2/J_1J_2^2 \\ -b_2^2/J_1J_2^2 \end{bmatrix}$$

Therefore B, AB, A^2B, $A^3B \neq 0$
(not null matrix)

So $A_c = [B, AB, A^2B, A^3B]$

$\det|A_c| \neq 0$

A_c is having full rank
hence the system is controllable
by adding external supports to hold
the system (dampers in this case)

(4) Given second order linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{B} u$$

$$AB = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1+6 \\ 1+4 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

controlability matrix $[B, AB]$

$$C \Rightarrow \begin{bmatrix} 1 & 7 \\ -2 & 5 \end{bmatrix}$$

$\det(C) \Rightarrow |C| = 5+14 = 19 \neq 0$

So the matrix C is having full rank (2)

So from the theory of controlability the system is controllable

So with a state feed back $u = Kx$

when we close the loop

$$A_c = A + BK$$

$$\Rightarrow \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

$$A_c = \begin{bmatrix} 1+K_1 & -3+K_2 \\ 1-2K_1 & -2-2K_2 \end{bmatrix}$$

Now lets find eigen values of A_c

$$\Rightarrow \begin{vmatrix} 1+K_1-\lambda & -3+K_2 \\ 1-2K_1 & -2-2K_2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1+K_1-\lambda)(-2-2K_2-\lambda) - (1-2K_1)(-3+K_2) = 0$$

$$\Rightarrow \lambda^2 + 2\lambda + 2K_2\lambda - \lambda - 2 - 2K_2 - K_1\lambda - 2K_1 - 2K_1K_2 - K_2 + 3 + 2K_1K_2 - 6K_1 = 0$$

$$\Rightarrow \lambda^2 + 2\lambda + 2K\lambda - \lambda - 2 - 2K_2 - K_1\lambda - 2K_1 - K_2 + 3 - 6K_1 = 0$$

$$\Rightarrow \lambda^2 + (1-K_1+2K_2)\lambda + (1-8K_1-3K_2) = 0$$

given that poles are $-2, 2$

Put these values in the above equation

$\Rightarrow$ we get

$$\lambda = -2 \Rightarrow 4 + (1-K_1+2K_2)(-2) + (1-8K_1-3K_2) = 0$$

$$\Rightarrow 4 - 2 + 2K_1 - 4K_2 + 1 - 8K_1 - 3K_2 = 0$$

$$\Rightarrow 3 - 6K_1 - 7K_2 = 0 \quad -(1)$$

$\lambda = 2$

$$\Rightarrow 4 + (1 - K_1 + 2K_2)(2) + (1 - 8K_1 - 3K_2) = 0$$

$$\Rightarrow 4 + (2 - 2K_1 + 4K_2) + (1 - 8K_1 - 3K_2) = 0$$

$$\Rightarrow 7 - 10K_1 + K_2 = 0 \quad - (2)$$

Solving (1) & (2)

We get $K_1 = +\frac{13}{19}$

$K_2 = -\frac{3}{19}$

So state feed back control is

$$u = K_1 x_1 + K_2 x_2$$

$$\boxed{u = +\frac{13}{19} x_1 - \frac{3}{19} x_2}$$

(5)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$AB \Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

So $[B, AB] = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ (columns are dependent)

$|C| = 0 \Rightarrow$ So uncontrolable

Now closed loop matrix is $A_c = A + BK$

$$A_c \Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [K_1 \ K_2]$$

$$\Rightarrow \begin{bmatrix} -1 & 0 \\ K_1 & K_2+2 \end{bmatrix}$$

finding eigen values of A_c $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 0 \\ K_1 & K_2+2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)(K_2+2-\lambda) = 0$$

So $\lambda = -1$ $\lambda = K_2+2$ $\} \rightarrow$ closed loop poles

Now one pole is at -1 & other is at k_2+2

So if $k_2 = -4$

$$\lambda_2 = -4+2 = -2$$

There $\lambda_1, \lambda_2 = -1, -2$ one pole can be placed at -2

But it is impossible place both the poles at -2

And since one of the pole is already in the left half plane, and other pole can also be placed on left half plane by selecting values of K_2 such that $(K_2 < -2)$

hence the system is "stablizable"

⑥ $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad AB = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$C = [B, AB] = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

$\det(C) = 0$ (dependable columns)

closed loop. matrix $A_c = A + BK$

$$A_c = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [K_1 \quad K_2]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ K_1 & K_2+2 \end{bmatrix}$$

eigen values of A_c

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ K_1 & K_2+2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(K_2+2-\lambda) = 0$$

so $\lambda_1 = +1$

$\lambda_2 = K_2 + 2$

So for $K_2 = -4$

$\lambda_2 = -4+2 = -2$

one pole can be placed at -2

however other poles is $\lambda_1 = 1$ which cannot be placed at -2

So It is imposible to place both the poles at -2

More over, since the λ_1 is $+1$ it is always on the positive (right) half so irrespective to K_1, K_2 values the system cannot be made stable

Hence the system is unstabilizable

⑦ LTI system can be framed as follows

$$\dot{x}(t) = Ax(t) + Bu(t) \quad - (1)$$

$$y(t) = Cx(t) + Du(t) \quad - (2)$$

input

output

lets apply laplace transforms on both the equations

$$\boxed{L(\dot{x}(t)) = SX(S)}$$

$$sX(s) = AX(s) + BU(s) \rightarrow (3)$$

$$Y(s) = CX(s) + DU(s) \rightarrow (4)$$

from (3) $\rightarrow$ $(SI - A)X(S) = BU(S)$

$$\Rightarrow X(S) = \frac{BU(S)}{(SI-A)}$$

$$\Rightarrow X(S) = (SI-A)^{-1}BU(S)$$

put this in (4)

we get

$$Y(S) = \left[C((SI-A)^{-1}BU(S)) + DU(S)\right]$$

$$Y(S) = \left[C(SI-A^{-1})B + D\right]U(S) \quad \rightarrow (5)$$

Now lets consider the system with input $\dot{u}(t)$ and output $\tilde{y}$

So state equations will be as follows

$$\Rightarrow \dot{x}(t) = Ax(t) + B\dot{u}(t) \rightarrow (6)$$

$$\tilde{y}(t) = Cx(t) + D\dot{u}(t) \rightarrow (7)$$

Applying laplace to 7 (zero intial condition)

$$L(\dot{x}(t)) = L(Ax(t)) + L(B\dot{u}(t))$$

$$\Rightarrow sx(s) = Ax(s) + BsU(s)$$

$$\Rightarrow sx(s) - Ax(s) = BsU(s)$$

$$\Rightarrow x(s)(sI - A) = BsU(s)$$

$$\Rightarrow X(s) = (sI-A)^{-1}Bs\,U(s)$$

put this in 7

$$\Rightarrow \tilde{y}(t) = \left[C(sI-A)^{-1}BsU(s) + DsU(s)\right]$$

$$\tilde{y}(t) \Rightarrow \left[C(sI-A)^{-1}B + D\right]sU(s)$$

$\rightarrow$ (8)

Compare 5 and 8 we get

$$\tilde{y}(s) = s\,y(s) \rightarrow (9)$$

Apply inverse Laplace in (9)

we get $\boxed{\tilde{y}(t) = \dot{y}(t)}$

hence for the input $\dot{u}(t)$ we get an output of $\dot{y}(t)$ for a LTI system with zero initial condition

(8) $$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix} x(t)$$

and $x(t_0) = x_0$

It is in the form of $\dot{x} = A(x)$

taking laplace on both sides

$$sX(s) - x(t_0) = AX(s)$$

$$sX(s) - A(X(s)) = x(t_0)$$

$$\Rightarrow (sI - A)X(s) = x(t_0)$$

$$\Rightarrow X(s) = (sI - A)^{-1} x(t_0)$$

(since $x(t_0) = x_0$)

$$\Rightarrow x(s) = (sI - A)^{-1} x_0$$

taking inverse laplace transform

$$x(t) = L^{-1}\{(SI-A)^{-1}\}x_0 \quad — ①$$

$$[SI-A] = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} S+1 & 0 & 0 \\ 0 & S+4 & -4 \\ 0 & 1 & S \end{bmatrix}$$

$$\det(SI-A) \Rightarrow (S+1)\big((S+4)(S)+4\big)$$

$$\Rightarrow (S+1)\underbrace{(S^2+4S+4)}_{(S+2)^2}$$

$$\Rightarrow (S+1)(S+2)^2$$

$$[(SI-A)]^{-1} = \frac{Adj\,[SI-A]}{|SI-A|}$$

$Adj\,[SI-A] \Rightarrow$

C_{11} (co factor matrix) $= \begin{vmatrix} S+4 & -4 \\ 1 & S \end{vmatrix} = (S+2)^2$

$$C_{12} = -\begin{vmatrix} 0 & -4 \\ 0 & s \end{vmatrix} = 0$$

$$C_{13} = \begin{vmatrix} 0 & s+4 \\ 0 & 1 \end{vmatrix} = 0$$

$$C_{21} = -\begin{vmatrix} 0 & 0 \\ 1 & s \end{vmatrix} = 0$$

$$C_{22} = \begin{vmatrix} s+1 & 0 \\ 0 & s \end{vmatrix} = s(s+1)$$

$$C_{23} = -\begin{vmatrix} s+1 & 0 \\ 0 & 1 \end{vmatrix} = -(s+1)$$

$$C_{31} = \begin{vmatrix} 0 & 0 \\ s+4 & -4 \end{vmatrix} = 0$$

$$C_{32} = -\begin{vmatrix} s+1 & 0 \\ 0 & -4 \end{vmatrix} = 4s+4$$

$$C_{33} = \begin{vmatrix} s+1 & 0 \\ 0 & s+4 \end{vmatrix} = (s+1)(s+4)$$

$$M \Rightarrow \begin{vmatrix} (s+2)^2 & 0 & 0 \\ 0 & s(s+1) & -(s+1) \\ 0 & 4s+4 & (s+1)(s+4) \end{vmatrix}$$

$$Adj \Rightarrow M^T$$

So $[SI - A]^{-1} \Rightarrow \dfrac{Adj}{|det|}$

$$\Rightarrow \frac{1}{(S+1)\ (S+2)^2} \begin{bmatrix} (S+2)^2 & 0 & 0 \\ 0 & S(S+1) & 4(S+1) \\ 0 & -1(S+1) & (S+4)(S+1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{S+1} & 0 & 0 \\ 0 & \frac{S}{(S+2)^2} & \frac{4}{(S+2)^2} \\ 0 & \frac{-1}{(S+2)^2} & \frac{S+4}{(S+2)^2} \end{bmatrix}$$

Now lets compute inverse Laplace for the above

$$L^{-1}\{[SI - A]^{-1}\} = \begin{bmatrix} L^{-1}\left[\frac{1}{S+1}\right] & 0 & 0 \\ 0 & L^{-1}\left[\frac{S}{(S+2)^2}\right] & L^{-1}\left[\frac{4}{(S+2)^2}\right] \\ 0 & L^{-1}\left[\frac{-1}{(S+2)^2}\right] & L^{-1}\left[\frac{S+4}{(S+2)^2}\right] \end{bmatrix}$$

$$L^{-1}\{[SI - A]^{-1}\} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t}(1-2t) & +4te^{-2t} \\ 0 & -e^{-2t}t & (1+2t)e^{-2t} \end{bmatrix}$$

from equation 1 we know that

$$L^{-1}\{(sI-A)^{-1}\} = x(t)$$

So using this we get

$$\boxed{x(t) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t}(1-2t) & +4te^{-2t} \\ 0 & -te^{-2t} & (1+2t)e^{-2t} \end{bmatrix} x_0.}$$