ENPM 662
Problem set -5

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(1) Show that $x(t) = e^{At}x(0)e^{Bt}$ is the solution to the following equation

$$\dot{x}(t) = Ax(t) + x(t)B$$

To prove this we can directly differentiate equation

$$x(t) = e^{At}x(0)e^{Bt}$$

$$\dot{x}(t) = \frac{d}{dt}\left[e^{At}x(0)e^{Bt}\right]$$

$$\Rightarrow x(0)\left[\frac{d}{dt}\left(e^{At}e^{Bt}\right)\right]$$

$$\Rightarrow x(0)\left[Ae^{At}e^{Bt} + Be^{Bt}e^{At}\right]$$

$$\dot{x}(t) \Rightarrow A\underbrace{e^{At}x(0)e^{Bt}}_{x(t)} + B\underbrace{e^{Bt}x(0)e^{At}}_{x(t)}$$

$$\dot{x}(t) = Ax(t) + Bx(t)$$

hence the differentiation of $x(t)$ leads to $\dot{x}(t) = Ax(t) + Bx(t)$

So, $x(t) = e^{At}x(0)e^{Bt}$ is the solution of $\dot{x}(t) = Ax(t) + x(t)B$

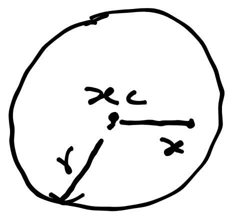
2) Euclidean ball $B(x_c, r)$ is given by

$$B(x_c, r) = \{ x \in R^n \mid \|x - x_c\| \leq r \}$$

$x_c \rightarrow$ Centre of the circle

$x \rightarrow$ random point

So considering any point on or in the circle



$$\|\bar{x} - \bar{x}_c\| \leq r$$

Now lets consider two arbitrary points x_1, x_2 in ball $B(x_c, r)$ so that

$$\|x_1 - x_c\| \leq r$$

$$\|x_2 - x_c\| \leq r$$

Now line segement joining those points can be written as

$$x(\theta) = (1-\theta)x_1 + \theta x_2$$

where $0 \leq \theta \leq 1$

Now if we show that this line segment lines in the ball itself then the ball is a convex set

So we have

$$\|x(\theta) - x_c\|$$

$$\Rightarrow \|(1-\theta)x_1 + \theta x_2 - x_c\|$$

from triangle inequality property of norms we get

$$\Rightarrow \|x(\theta) - x_c\| \le (1-\theta)\|x_1 - x_c\| + \theta\|x_2 - x_c\|$$

$$\|x_1 - x_c\| < r \qquad \|x_2 - x_c\| < r$$

$\Rightarrow$ So

$$\|x(\theta) - x_c\| \le (1-\theta) r + \theta r$$

$$\Rightarrow \|x(\theta) - x_c\| \le r - \cancel{r\theta} + \cancel{\theta r}$$

$$\Rightarrow \|x(\theta) - x_c\| \le r$$

line segement joint x_1 & x_2

$\Rightarrow$ from above it is Proved that line segment joining x_1 x_2 also lies in ball. It is proved that the set $B(x_c, r)$ is a convex set

4) $\dot{x}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t)$

$C = [B \vdots AB \vdots \dots A^2B]$

$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$A^2B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \end{bmatrix}$

column 2, 6, 4 are dependent to each other

So $\det(C) = 0$

$\text{rank}(C) = 2 \leq n$

$S = [V_1 \; V_2 \; V_3 \; \dots \; V_{n-r} \mid S_{n-n_r}]$

$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad S_{n-n_r} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$S^{-1} = \frac{Adj(S)}{|S|}$$

co. of matrix of $S = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

$$adj = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$S^{-1} = \frac{adj|S|}{det|S|} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Now $\hat{A} = S^{-1}AS = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$\hat{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{B} = S^{-1}B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad n_r \times n_r = 2$$

The standard form of uncontrollable system is given by $\dot{x} = \hat{A}x + \hat{B}u$

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u$$

Controllable part is given by

$$\boxed{\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u}$$

③ $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$

Cost function $J = \int_0^{\infty} (x^T Q x + u^2)\, dt$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}, \quad b > 0$$

here $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

we know that K (gain matrix) $= -R^{-1} B_K^T P$

where P is solution of Ricatting equation

$$A^T P + PA - PB_K R^{-1} B_K^T P = -Q$$

we also that general cost function is given by

$$J(K, \vec{x}(0)) = \int_0^{\infty} \vec{x}^T(t) Q \vec{x}(t) + \vec{U}_K^T(t) R\, U_k(t)\, dt$$

By comparing both the cost functions.

we get $R = 1$

i.e $R = \underline{I}$

So putting all the known values in Ricatti equation we get the following

and p be $\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ Since p is symetric matrix $P_{12} = P_{21}$

$$\Rightarrow A^T P + PA - PB_K R^{-1} B^T P = -Q$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P & P_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$- \underset{2\times 2}{\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = -Q$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ P_{11} & P_{12} \end{bmatrix} + \begin{bmatrix} 0 & P_{11} \\ 0 & P_{12} \end{bmatrix} - \begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & P_{11} \\ P_{11} & 2P_{12} \end{bmatrix} - \underset{2\times 1}{\begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix}} \underset{1\times 2}{\begin{bmatrix} P_{12} & P_{22} \end{bmatrix}} = -Q$$

$$\Rightarrow \begin{bmatrix} 0 & P_{11} \\ P_{11} & 2P_{12} \end{bmatrix} - \begin{bmatrix} P_{12}^2 & P_{12} P_{22} \\ P_{22} P_{12} & P_{22}^2 \end{bmatrix} = -Q$$

$$\Rightarrow \begin{bmatrix} -P_{12}^2 & P_{11} - P_{12}P_{22} \\ P_{11} - P_{12}P_{22} & 2P_{12} - P_{22}^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -\gamma \end{bmatrix}$$

on equating we get

$$\Rightarrow P_{11} - P_{12}P_{22} = 0$$

$$\Rightarrow -P_{12}^2 = -1$$

$$P_{12} = \pm 1$$

There are 2 cases $P_{12} = +1$ $P_{12} = -1$

$P_{12} = +1 \rightarrow$ case - $\underline{1}$

$$P_{11} - (1)P_{22} = 0$$

$$P_{11} = P_{22}$$

$$2P_{12} - P_{22}^2 = -\gamma$$

$$2 - P_{22}^2 = -\gamma$$

$$-\gamma - 2 = -P_{22}^2$$

$$\Rightarrow \sqrt{(2+\gamma)} = \pm P_{22}$$

$$P_{22} = \pm\sqrt{2+\gamma}$$

So $P_{1.1} = \begin{bmatrix} \sqrt{2+\gamma} & 1 \\ 1 & \sqrt{2+\gamma} \end{bmatrix}$

$P_{1.2} = \begin{bmatrix} -\sqrt{\gamma+2} & 1 \\ 1 & -\sqrt{\gamma+2} \end{bmatrix}$

Case 2

$P_{12} = -1$

$P_{11} = -P_{22}$

$-2P_{12} - P_{22}^2 = -\gamma$

$P_{22} = \pm\sqrt{\gamma-2}$

for $\gamma > 0$ we have a chance that P_{22} can be complex number since we know that matrix P can be only Positive symetric definite matrix case-2 will be not possible

So only considering case 1.1 and 1.2

case 1.1 lets find eigen values of P

$$\begin{vmatrix} \sqrt{\gamma+2}-\lambda & 1 \\ 1 & \sqrt{\gamma+2}-\lambda \end{vmatrix} = 0$$

$$\left(\sqrt{\gamma+2}-\lambda\right)^2 - 1 = 0$$

$\sqrt{\gamma+2} - \lambda = \pm 1$

$\boxed{\lambda = \pm 1 + \sqrt{\gamma+2}}$ $\gamma > 0$

always +ve

$$\begin{vmatrix} -\sqrt{q+2} - \lambda & 1 \\ 1 & -\sqrt{q+2} - \lambda \end{vmatrix} = 0 \longrightarrow \text{case } 1\cdot 2$$

$$\left(-\sqrt{q+2} - \lambda\right)^2 - 1 = 0$$

$$\Rightarrow \pm\left(\sqrt{q+2} + \lambda\right) - 1 = 0$$

$$\Rightarrow \pm\left(\sqrt{q+2} + \lambda\right) = 1$$

$$\lambda = 1 \quad \sqrt{q+2}$$ → can be positive or negetive

So case 2 cannot be choosen

$$P = \begin{bmatrix} \sqrt{q+2} & 1 \\ 1 & \sqrt{q+2} \end{bmatrix}$$

Now $K = -R^{-1} B^T P$

$$\Rightarrow -I \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{q+2} & 1 \\ 1 & \sqrt{q+2} \end{bmatrix}$$

$$\boxed{K = -\begin{bmatrix} 1 & \sqrt{q+2} \end{bmatrix}}$$

Putting back K in state feedback $u = Kx$

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & -\sqrt{\gamma+2} \end{bmatrix} X$$

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ -1 & -\sqrt{\gamma+2} \end{bmatrix} X$$

$$\boxed{\dot{X} = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{\gamma+2} \end{bmatrix} X}$$

(5) Investigate the Stability of the System

$$\dot{X} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} X \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

using Lyapunov equation

→ According to lyapunov stability therom, a system is said to be stable iff for a Symmetric +ve definite matrix Q there exists a Symmetric +ve definite P such that $A^T P + PA = -Q \rightarrow (1)$

let p be

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \qquad A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$$

⇒ Now putting P and A values in (1)

$$A^T P \Rightarrow \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} -3P_{11} - P_{12} & -3P_{12} - P_{22} \\ 2P_{11} - P_{12} & 2P_{12} - P_{22} \end{bmatrix}$$

$$PA \Rightarrow \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -3P_{11} - P_{12} & 2P_{11} - P_{12} \\ -3P_{12} - P_{22} & 2P_{12} - P_{22} \end{bmatrix}$$

$$A^T P + PA = \begin{bmatrix} -6P_{11} - 2P_{12} & 2P_{11} - 4P_{12} - P_{22} \\ 2P_{11} - 4P_{12} - P_{22} & 4P_{12} - 2P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

equating two we get following equations

$$-6P_{11} - 2P_{12} = -1 \rightarrow (1)$$

$$2P_{11} - 4P_{12} - P_{22} = 0 \rightarrow (2)$$

$$4P_{12} - 2P_{22} = -1 \rightarrow (3)$$

on solving these three equations we get

$$P_{11} = \frac{7}{40}$$

$$P_{12} = -\frac{1}{40} \qquad \text{so } P = \begin{bmatrix} 7/40 & -1/40 \\ -1/40 & \frac{18}{40} \end{bmatrix}$$

$$P_{22} = \frac{18}{40}$$

Now find eigen values of P

$$\frac{1}{40} \begin{vmatrix} 7-\lambda & -1 \\ -1 & 18-\lambda \end{vmatrix} = 0$$

$$\left((7-\lambda)(18-\lambda) - 1 \right) = 0$$

$$\Rightarrow 126 - 7\lambda - 18\lambda + \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 25\lambda + 125 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{25}{2} \pm \frac{\sqrt{(25)^2 - 4(1)(125)}}{2}$$

$$\lambda = \frac{25}{2} \pm \frac{5\sqrt{5}}{2}$$

$$\lambda_1 = \frac{25}{2} + \frac{5\sqrt{5}}{2} > 0$$

$$\lambda_2 = \frac{25}{2} - \frac{5\sqrt{5}}{2} > 0 .$$

Since eigne values are positive
P is positive definite symetric matrix
Since P exist the system is stable