ENPM-667
Problem set - 3

Sai Dinesh Gelam
Uid : 120167140

① from spectral norm of a matrix we know that

a)
$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

Now lets consider that $y = cx$ where c is a arbitary constant and not equal to zero

So in terms of y

$$\|A\| = \max_{\|y\|=1} \|Ay\|$$

which is also equal to

$$\|A\| = \max_{\|y\|=1} \frac{\|Ay\|}{\|y\|}$$

Now put $y = cx$

$$\Rightarrow \max_{\|y\|=1} \frac{\|Acx\|}{\|cx\|}$$

from properties we know that $\|cx\| = |c| \, \|x\|$

when $\|y\| = 1$

$\|cx\| = 1$

c is constant $c \neq 0$

So $x \neq 0$

So

$$\|A\| = \max_{\|x\| \neq 0} \frac{\cancel{|c|}\,\|Ax\|}{\cancel{|c|}\,\|x\|}$$

Therefore we get the result

$$\boxed{\|A\| = \max_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|}}$$

(b) To prove that for any $n \times 1$ vector x.

$$\|Ax\| \leq \|A\|\,\|x\|$$

lets assume that for any vector x of $n \times 1$

$$\|Ax\| > \|A\|\,\|x\|$$

Now, $\|Ax\| \cdot \dfrac{1}{\|x\|} > \|A\|$

$$\Rightarrow \frac{\|Ax\|}{\|x\|} > \|A\| \longrightarrow (1)$$

By definition $\|A\| = \max\limits_{\|x\| \neq 0} \dfrac{\|Ax\|}{\|x\|}$

The max is $\geq \dfrac{\|Ax\|}{\|x\|}$ So the result (1) is wrong

So, the assumption

$$\|Ax\| > \|Ax\| \, \|x\|$$ is not ture

Therefore in conlusion we get

$$\boxed{\|Ax\| \leq \|Ax\| \, \|x\|}$$

(c) Using the conclusion above prove that for comfortable matrices A and B

$$\|AB\| \leq \|A\| \, \|B\|$$

from the defintion we know that

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

Now consider $\|AB\| = \max_{\|x\| \neq} \dfrac{\|ABx\|}{\|x\|}$

using the conclusion above

we know $\|Ax\| \leq \|A\| \, \|x\|$

So applying the same for AB

$$\max_{\|x\|=1} \|AB\| \leq \max_{x \neq 0} \frac{\|A\| \, \|Bx\|}{\|x\|}$$

$$\|AB\| \leq \max_{x \neq 0} \frac{\|A\| \, \|B\| \, \cancel{\|x\|}}{\cancel{\|x\|}}$$

$\Rightarrow$ So we conclude that

$$\boxed{\|AB\| \leq \|A\|\,\|B\|}$$

② given definition $\sigma(A) = \max\{\text{eigen values of } A\}$

Prove that $\sigma(A) \leq \|A\|$ for all $n \times n$ of A

from results of the above Problem

we know that $\|Ax\| \leq \|A\|\,\|x\| \rightarrow$ ①

from properties of matrix we know that

$$\underset{\substack{\downarrow \\ \text{eigen value}}}{AX = \lambda} X \rightarrow \text{eigen vector} \rightarrow ②$$

So applying ① on ②

we get $|\lambda|\,\|x\| \leq \|A\|\,\|x\|$

$\swarrow$ Since λ is constant

$$|\lambda|\cancel{\|x\|} \leq \|A\|\,\cancel{\|x\|}$$

$$|\lambda| \leq \|A\|$$

And Spectral radius $\sigma(A)$ will be given

as $\sigma(A) = \max\{|\lambda|\}$

$$\boxed{\therefore\ \sigma(A) \leq \|A\|}$$ hence Proved.

(3) Given that A is continuously differentiable $n \times n$ matrix and it is invertable at each t

we know that

$$A(t)A^{-1}(t) = I$$

differentiate both sides

$$\Rightarrow \frac{d}{dt}\left(A(t)A^{-1}(t)\right) = \frac{d}{dt}(I) \qquad \left[\because \frac{d}{dt}I = 0\right]$$

$$\Rightarrow \frac{d}{dt}A(t).A^{-1}(t) + A(t)\frac{d}{dt}A^{-1}(t) = 0$$

$$\Rightarrow \dot{A}(t).A^{-1}(t) + A(t)\dot{A}^{-1}(t) = 0$$

$$\Rightarrow \dot{A}(t).A^{-1}(t) = -A(t)\dot{A}^{-1}(t)$$

multiply on both sides with $A^{-1}(t)$

$$\Rightarrow -A^{-1}(t)\dot{A}(t)A^{-1}(t) = -\dot{A}^{-1}(t)A\cancel{(t)}A^{-1}\cancel{(t)}$$

$$\Rightarrow \dot{A}^{-1}(t) = -A^{-1}(t)\dot{A}(t)A^{-1}(t)$$

$$\Rightarrow \boxed{\frac{d}{dt}A^{-1}(t) = -A^{-1}(t)\dot{A}(t)A^{-1}(t)}$$

4)

given $\dot{x} = ax(t) + b(t)u(t)$

using laplace both sides

$$L\{\dot{x}\} = L\{ax(t)\} + L\{b(t)u(t)\}$$

$$L\{\dot{x}\} \rightarrow s.X(s) - x(0)$$

$$L\{ax(t)\} \rightarrow a.x(s)$$

$\Rightarrow$ putting these into equation

$$\Rightarrow s.X(s) - x(0) = a.x(s) + L\{b(t)u(t)\}$$

$$\Rightarrow s.x(s) - a.X(s) = x(0) + L\{b(t)u(t)\}$$

$$\Rightarrow (s-a)x(s) = x(0) + L\{b(t)u(t)\}$$

$$\Rightarrow X(s) = \frac{x(0)}{s-a} + \frac{L\{b(t)u(t)\}}{s-a}$$

consider $L\{b(t)u(t)\} = B(s)U(s)$

$$X(s) = \frac{x(0)}{s-a} + \frac{B(s)U(s)}{s-a}$$

Now lets apply inverse Laplace on both Sides.

$$L^{-1}[X(s)] = L^{-1}\left[\frac{x(0)}{s-a}\right] + L^{-1}\left[\frac{B(s)\,U(s)}{s-a}\right]$$

Now consider $B(s)\,U(s) = D(s)$

and $\frac{1}{(s-a)} = G(s)$

$$L^{-1}[X(s)] = L^{-1}\left[\frac{x(0)}{s-a}\right] + L^{-1}[D(s)\,G(s)]$$

$$x(t) = x_0 e^{at} + L^{-1}[D(s)\cdot G(s)]$$

since $L^{-1}[D(s)\cdot G(s)] = \int_0^t D(u)\, G(t-u)\, du$

$$\Rightarrow \int_0^t B(\tau)\,U(\tau)\, e^{a(t-\tau)}\, d\tau$$

So,

$$\boxed{x(t) = \int_0^t B(\tau)\,U(\tau)\cdot e^{a(t-\tau)}\, d\tau + x(0)\, e^{at}}$$

(5) $y^n(t) + a_{n-1} t^{-1} y^{(n-1)}(t) + a_{n-2} t^{-2} y^{(n-2)}(t) + \cdots + a_1 t^{-n+1} y^{(1)}(t) +$

$\cdots a_0 t^{-n} y(t) = 0$

where $y^n(t) = \dfrac{d^n y(t)}{dt^n}$

$\Rightarrow a_0 t^{-n} y(t) + a_1 t^{-n+1} y'(t) + \cdots\cdots a_{n-2} t^{-2} y^{(n-2)}(t) + a_{n-1} t^{-1} y^{(n-1)}(t)$

$+ y^n(t)$

The possible state space can be

$$x(t) = \left[t^{-n} y(t) \quad t^{-n+1} y'(t) \quad \cdots\cdots \quad t^{-1} y^{n-1}(t) \right]^T$$

$$x(t) = \left[x_1, x_2, \cdots\cdots x_n \right]^T$$

where $x_1 = t^{-n} y(t) \quad x_2 = t^{-n+1} y'(t) \quad \cdots\cdots$

$x_n = t^{-1} y^{n-1}(t)$

$\Rightarrow \dot{x}_1 \Rightarrow -n t^{-n-1} y(t) + t^{-n} y'(t)$

$\Rightarrow -n t^{-1} x_1 + t^{-1} x_2$

Similarly $\Rightarrow \dot{x}_2 = -(n-1) t^{-1} x_2 + t^{-1} x_3$

$x_n = -t^{-1} x_n + t^{-1} y^n(t)$

$y_n \Rightarrow -a_{n-1} t^{-1} y^{n-1}(t) \cdots\cdots\cdots - a_0 t^n y(t)$

$\Rightarrow -a_{n-1} x_n - a_{n-2} x_{n-1} \cdots\cdots - a_0 x_1$

$$\dot{x}(t) = \begin{bmatrix} t^{-1}(-n) & t^{-1} & 0 & 0 & \cdots & 0 \\ 0 & -(n-1)t^{-1} & t^{-1} & & & \\ 0 & 0 & \ddots & \ddots & & \\ \vdots & \vdots & & & \ddots & \\ -a_0 & -a_1 & \cdots & \cdots & \cdots & -a_{n+1} \end{bmatrix} x(t)$$

$$\dot{x}(t) = t^{-1} \underbrace{\begin{bmatrix} -n & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -(n-1) & 1 & & & & & \\ 0 & 0 & & \ddots & & & & \\ \vdots & \vdots & & & & \ddots & & \\ -a_0 & -a_1 & & & & & & -a_{n+1} \end{bmatrix}}_{A} x(t)$$

So, we get $\boxed{\dot{x}(t) = t^{-1} A x(t)}$

6) $$\frac{\partial}{\partial \tau}\Phi(t,\tau) = -\Phi(t,\tau)A(\tau)$$

using the Property

$$\Phi^{-1}(t,t_0) = \Phi(t_0,t)$$

and $$\Phi(t,t_0)\,\Phi(t_0,t) = I$$

using the above Property we can write that

$$\Phi(t,\tau)\,\Phi(\tau,t) = I$$

now differentiating on both sides

$$\Rightarrow \frac{\partial}{\partial \tau}\left[\Phi(t,\tau)\cdot\Phi(\tau,t)\right] = \frac{\partial}{\partial \tau}(I)$$

$$\Rightarrow \frac{\partial}{\partial \tau}\Phi(t,\tau)\,\Phi(\tau,t) + \Phi(t,\tau)\frac{\partial}{\partial \tau}\Phi(\tau,t) = 0$$

$\Rightarrow$ we know that for Transistion matrix

$$\frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\,\Phi(t,t_0)$$

$$\Rightarrow \frac{\partial}{\partial \tau}\Phi(t,\tau)\,\Phi(\tau,t) = -\Phi(t,\tau)\cdot A(\tau)\,\Phi(\tau,t)$$

$$\Rightarrow \frac{\partial}{\partial \tau}\Phi(t,\tau) = -\Phi(t,\tau)\cdot A(\tau)\,\underbrace{\Phi(\tau,t)\,\Phi^{-1}(\tau,t)}_{I}$$

So we get $$\boxed{\frac{\partial}{\partial \tau}\Phi(t,\tau) = -\Phi(t,\tau)A(\tau)}$$

(7) Given that $A = \begin{bmatrix} 1 & 0 \\ 1 & \eta(t) \end{bmatrix}$

where η is a bounded and continous function of t

$$\vec{\dot{X}} = A(t)x(t)$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & \eta(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t_2) \end{bmatrix}$$

after multiplication

we get $\dot{x}_1 = x_1(t)$

$$\Rightarrow \frac{dx_1}{dt} = x_1(t)$$

$$\Rightarrow dx_1 = x_1(t)\,dt \Rightarrow \frac{dx_1}{x_1(t)} = dt$$

on integrating we get

$$\Rightarrow \ln x_1(t) - \ln x_1(0) = t - t_0$$

$$\Rightarrow \ln\left(\frac{x_1(t)}{x_1(0)}\right) = t - t_0$$

$$\Rightarrow x_1(t) = e^{(t-t_0)} x_1(0) \longrightarrow (1)$$

Similarly $\dot{x}_2(t) = x_1(t) + x_2(t)\cdot\eta(t)$

$$\Rightarrow \frac{d}{dt}\dot{x}_2(t) - x_2(t)\eta(t) = x_1(t)$$

we know that for the form $\frac{dy}{dx} + y\cdot P(x) = Q$

Solution is given by

$$x_2(t)\cdot IF = \int (IF)Q(x)\,d(t)$$

So $If = e^{\int -\eta(t)dt}$

$$\Rightarrow x_2(t)e^{\int -\eta(t)dt} = \int x_1(t)e^{-\int \eta(t)dt}\,dt + C$$

$$\Rightarrow x_2(t)e^{\int -\eta(t)dt} = \int x_1(0)e^{t-t_0}\cdot e^{-\int \eta(t)dt}\,dt + C$$

$$\Rightarrow x_2(t)e^{-\int \eta(t)dt} = x_1(0)\int e^{t-t_0}e^{-\int \eta(t)dt}\,dt + C$$

Now lets assume $\Rightarrow$ $\boxed{\int e^{t-t_0}e^{-\int \eta(t)dt} = a(t)}$

and $\boxed{e^{-\int \eta(t)dt} = u(t)}$

So simplified equation will be

$$x_2(t)u(t) = x_1(0)a(t) + C$$

at $t = t_0$ $\quad x_2(t_0) = x_2(0)$

So, $c = x_2(0)\mu(0) - x_1(0)t(0)$

$$x_2(t)\mu(t) = x_1(0)a(t) + x_2(0)\mu(0) - x_1(0)t(0)$$

$$x_2 = \frac{x_1(0)a(t)}{\mu(t)} + \left(\frac{x_2(0)\mu(0) - x_1(0)a(0)}{\mu(t)}\right)$$

$$\Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{t-t_0} & 0 \\ \frac{a(t)}{\mu(t)} - \frac{a(0)}{\mu(t)} & \frac{x_2(0)\mu(0)}{\mu(t)} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

8) Exponential matrix of

$$A \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{bmatrix}$$

first lets diagonalize the matrix

finding eigen values.

$$|A - \lambda I| = 0$$

$$\Rightarrow \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 6 & 5 & -2-\lambda \end{bmatrix} = 0$$

$$-\lambda(+2\lambda + \lambda^2 - 5) - 1(-6) = 0$$

$$\Rightarrow -2\lambda^2 - \lambda^3 + 5\lambda + 6 = 0$$

$$\Rightarrow \lambda^3 + 2\lambda^2 - 5\lambda - 6 = 0$$

$$\Rightarrow -(\lambda - 2)(\lambda + 1)(\lambda + 3) = 0$$

so $\lambda = 2, -1, -3 \rightarrow$ these are the eigen values

put $\lambda = 2$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 6 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $-2x_1 + x_2 + 0x_3 = 0$

$0x_1 - 2x_2 + x_3 = 0$ → Apply cramer's rule

$6x_1 + 5x_2 - 4x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix}} = K$$

$$\Rightarrow \frac{x_1}{1} = \frac{-x_2}{-2} = \frac{x_3}{+4} = k$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} +1 \\ +2 \\ +4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/4 \\ 1/2 \\ 1 \end{bmatrix}$$

for $\lambda = -1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 6 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + x_2 + 0x_3 = 0$

$0x_1 + x_2 + x_3 = 0$ → Cramers rules

$6x_1 + 5x_2 - x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}} = K$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \rightarrow \text{eigen vector}$$

put $\lambda = -3$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 6 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 3x_1 + x_2 + 0x_3 &= 0 \\ 0x_1 + 3x_2 + x_3 &= 0 \end{aligned}\right] \rightarrow \text{Cramers rule}$$

$$6x_1 + 5x_2 + x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 3 & 1 \\ 0 & 3 \end{vmatrix}}$$

$$\frac{x}{1} = \frac{-x_2}{3} = \frac{x_3}{9}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/9 \\ -1/3 \\ 1 \end{bmatrix} \rightarrow \text{eigen vector}$$

So $P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -3 \\ 4 & 1 & 9 \end{bmatrix}$ $\Rightarrow$ on computing we get

$$P^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{4}{15} & \frac{1}{15} \\ 1 & \frac{-1}{6} & -\frac{1}{6} \\ -\frac{1}{5} & \frac{-1}{10} & \frac{1}{10} \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

So $A = PDP^{-1}$

Now, from exponential series

$$e^{xt} = 1 + xt + \frac{xt^2}{2!} + \cdots \frac{(xt)^n}{n!}$$

$$e^{At} = I + (At) + \frac{(At)^2}{2!} + \cdots \frac{(At)^n}{n!}$$

$$A^{nt} = PD^{nt}P^{-1}$$

$$e^{A} = Pe^{nt}P^{-1}$$

The exponential of a diagonal matrix is

$$e^{Dt} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}$$

So $e^{At} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -3 \\ 4 & 1 & 9 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} \frac{1}{5} & \frac{4}{15} & \frac{1}{15} \\ 1 & \frac{-1}{6} & \frac{-1}{6} \\ -\frac{1}{5} & \frac{-1}{10} & \frac{1}{10} \end{bmatrix}$

$$\Rightarrow e^{At} = \begin{bmatrix} e^{2t} & e^{-t} & e^{-3t} \\ 2e^{2t} & -e^{-t} & -3e^{-3t} \\ 4e^{2t} & e^{-t} & 9e^{-3t} \end{bmatrix} \begin{bmatrix} 1/5 & 4/15 & 1/15 \\ 1 & -1/6 & -1/6 \\ -1/5 & -1/10 & 1/10 \end{bmatrix}$$

$$\Rightarrow \frac{1}{30} \begin{bmatrix} 6e^{-3t}(e^{5t}+5e^{2t}-1) & e^{-3t}(8e^{5t}-5e^{2t}-3) & e^{-3t}(2e^{5t}-5e^{2t}+3) \\ 6e^{-3t}(2e^{5t}-5e^{2t}+3) & e^{-3t}(16e^{5t}+5e^{2t}+9) & e^{-3t}(4e^{5t}+5e^{2t}-9) \\ 6e^{-3t}(4e^{5t}+5e^{2t}-9) & e^{-3t}(32e^{5t}-5e^{2t}-27) & e^{-3t}(8e^{5t}-5e^{2t}+27) \end{bmatrix}$$

So,

$$e^{At} = \frac{e^{-3t}}{30} \underbrace{\begin{bmatrix} 6(e^{5t}+5e^{2t}-1) & 8e^{5t}-5e^{2t}-3 & 2e^{3t}-5e^{2t}+3 \\ 6(2e^{5t}-5e^{2t}+3) & 16e^{5t}+5e^{2t}+9 & 4e^{5t}+5e^{2t}-9 \\ 6(4e^{5t}+5e^{2t}-9) & 32e^{5t}-5e^{2t}-27 & 8e^{5t}-5e^{2t}+27 \end{bmatrix}}_{C}$$

$$\Rightarrow \boxed{e^{At} = \frac{e^{-3t}}{30}\begin{bmatrix} C \end{bmatrix}}$$