ENPM-667
Problem Set -2

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① a) determinat of $\begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 3 & -3 & 4 & -2 \\ -2 & 1 & -2 & 1 \end{vmatrix}$

Minors of the 4x4 matrix are multiplied by their Co-efficients and added together to get the determinant values

$$\Rightarrow 1\begin{vmatrix} 1 & -2 & 1 \\ -3 & 4 & -2 \\ 1 & -2 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & -2 & 1 \\ 3 & 4 & -2 \\ -2 & -2 & 1 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 & 1 \\ 3 & -3 & -2 \\ -2 & 1 & 1 \end{vmatrix}$$

$$- 3\begin{vmatrix} 0 & 1 & -2 \\ 3 & -3 & 4 \\ -2 & 1 & -2 \end{vmatrix}$$

$$\Rightarrow 1\left[1(4-4) + 2(-3+2) + 1(6-4)\right]$$

$$+2\left[0 - 1(3-4) + 1(3-6)\right]$$

$$-3\left[0 - 1(-6+8) - 2(3-6\right]$$

$$\Rightarrow 1(0-2+2)+2(+1+3)-3(-2+6)$$

$$\Rightarrow 0-4-12$$

$$\Rightarrow \boxed{-16} \rightarrow \text{Answer 1(a)}$$

b) $$\begin{vmatrix} gc & ge & a+ge & gb+ge \\ 0 & b & b & b \\ c & e & e & b+e \\ a & b & b+f & b+d \end{vmatrix}$$

$$\det = gc\begin{vmatrix} b & b & b \\ e & e & b+e \\ b & b+f & b+d \end{vmatrix} - ge\begin{vmatrix} 0 & b & b \\ c & e & b+e \\ a & b+f & b+d \end{vmatrix} + (a+ge)\begin{vmatrix} 0 & b & b \\ c & e & b+e \\ a & b & b+d \end{vmatrix}$$

$$-gb+ge\begin{vmatrix} 0 & b & b \\ c & e & e \\ a & b & b+f \end{vmatrix}$$

$$\Rightarrow gc\begin{vmatrix} b & b & b \\ e & e & b+e \\ b & b+f & b+d \end{vmatrix}$$

$$\Rightarrow gc\left[b(eb+ed-(b+f)(b+e)) - b(eb+ed-(b^2+eb)) \right.$$

$$\left. + b(eb+ef-eb) \right)$$

$$\Rightarrow gc\Big[\cancel{eb^2} + \cancel{edb} - \cancel{b^3} - \cancel{b^2e} - fb^2 - \cancel{efb}$$

$$- \cancel{cb^2} - \cancel{cdb} \quad \cancel{b^3} + \cancel{eb^2} + \cancel{eb^2} + \cancel{efb} - \cancel{eb^2}$$

$$\Rightarrow \quad (-fb^2) \Rightarrow -gcfb^2 \text{ — } (1)$$

$$\Rightarrow \quad ge\begin{vmatrix} 0 & b & b \\ c & c & b+e \\ a & b+f & b+d \end{vmatrix}$$

$$\Rightarrow ge\Big[\big(0 - b(cb + cd - ab - ae) + b(cb + fc - ac)\big)\Big]$$

$$\Rightarrow ge\Big[\big(-\cancel{cb^2} - cdb + ab^2 + \cancel{aeb} + \cancel{cb^2} + fbc - \cancel{abe}\big)\Big]$$

$$\Rightarrow ge\big[-cbd + ab^2 + fbc\big]$$

$$\Rightarrow -gecbd + ab^2ge + fbcge \rightarrow (2)$$

$$\Rightarrow (a+ge)\begin{vmatrix} 0 & b & b \\ c & e & b+e \\ a & b & b+d \end{vmatrix}$$

$$\Rightarrow (a+ge)\left[0 - b(cb+cd-ab-ea)+b(cb-ae)\right]$$

$$\Rightarrow (a+ge)\left[-\cancel{cb^2} - cbd + ab^2 + \cancel{eab} + \cancel{b^2c} - \cancel{aeb}\right]$$

$$\Rightarrow (a+ge)(ab^2 - cbd)$$

$$\Rightarrow a^2b^2 - abcd + geab^2 - bcdeg \longrightarrow 3$$

$$\Rightarrow -gb+ge\begin{vmatrix} 0 & b & b \\ c & e & e \\ a & b & b+f \end{vmatrix}$$

$$\Rightarrow -gb+ge((0 - b(cb+cf-ae)+b(bc-ae)))$$

$$\Rightarrow -gb+ge(-\cancel{b^2c} - bcf + \cancel{abe} + \cancel{b^2c} - \cancel{abe})$$

$$\Rightarrow -gb+ge(-bcf)$$

$$\Rightarrow +gb^2cf - gebcf \longrightarrow 4$$

$$\det = ① - ② + ③ - ④$$

$$\Rightarrow -\cancel{gcfb^2} + \cancel{gecbd} - \cancel{ab^2ge} - \cancel{fbcge} + a^2b^2 - abcd$$
$$+\cancel{geab^2} - \cancel{bcdeg} - \cancel{gb^2cf} + \cancel{gebcf}$$

$$\boxed{\det \Rightarrow a^2b^2 - abcd} \rightarrow \text{Answer}$$

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② given matrix

$$\begin{vmatrix} x & a & a & 1 \\ a & x & b & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0$$

lets do Row operations

Subtract all R_4 from all Rows

$$\Rightarrow \quad \begin{aligned} R_1 &\rightarrow R_1 - R_4 \\ R_2 &\rightarrow R_2 - R_4 \\ R_3 &\rightarrow R_3 - R_4 \end{aligned} \quad \Rightarrow \begin{vmatrix} x-a & a-b & a-c & 0 \\ 0 & x-b & b-c & 0 \\ 0 & 0 & x-c & 0 \\ a & b & c & 1 \end{vmatrix}$$

now det $\Rightarrow$

$$(x-a)\begin{vmatrix} x-b & b-c & 0 \\ 0 & x-c & 0 \\ b & c & 1 \end{vmatrix} - (a-b)\begin{vmatrix} 0 & b-c & 0 \\ 0 & x-c & 0 \\ a & c & 1 \end{vmatrix} + (a-c)\begin{vmatrix} 0 & x-b & 0 \\ 0 & x-c & 0 \\ a & b & 1 \end{vmatrix}$$

$$(x-a)\Big[(x-b)\big[(x-c)-0\big]-(b-c)\big[0\big]+0\Big]$$
$$-(a-b)\big[0\big]+a-c\big[0\big]=0$$

$$\Rightarrow (x-a)(x-b)(x-c)=0$$

from this equation we get three values of

$$x-a=0$$
$$x-b=0$$
$$x-c=0$$

So $\boxed{\begin{matrix} x=a \\ x=b \\ x=c \end{matrix}}$ for which the determinant of given matrix is zero

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(Or) $\begin{vmatrix} x & a & a & 1 \\ a & x & b & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix}$

Put $x=a \Rightarrow \begin{vmatrix} a & a & a & 1 \\ a & a & b & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix}$

$C_1 = C_4$

$R_1 \to C_1/a \Rightarrow \begin{vmatrix} 1 & a & a & 1 \\ 1 & a & b & 1 \\ 1 & b & x & 1 \\ 1 & b & c & 1 \end{vmatrix}$ if two columns are equal from Properties of det det $= 0$

So $x = a$

Now put $x = b$

$\begin{vmatrix} b & a & a & 1 \\ a & b & b & 1 \\ a & b & b & 1 \\ a & b & c & 1 \end{vmatrix}$ Now $R_2 = R_3$ so det $= 0$

$\Rightarrow x = b$ (2 rows are equal)

Now put $x = c$

$\begin{vmatrix} c & a & a & 1 \\ a & c & b & 1 \\ a & b & c & 1 \\ a & b & c & 1 \end{vmatrix}$ Now $R_3 = R_4$ so det $= 0$

So $x = c$ (2 rows are equal)

So for det $= 0$ $\boxed{x = a,\ x = b,\ x = c}$

(3) $A = \begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

to find eigen values we compute

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & \gamma_1 & 0 \\ \gamma_2 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 1-\lambda\left((1-\lambda)^2 - \gamma_1(\gamma_2 - \gamma_2\lambda)\right) = 0$$

$$\Rightarrow 1-\lambda(1-\lambda)^2 = \gamma_1\gamma_2(1-\lambda)$$

one of the solution can be $\lambda = 1, 1$

$$\Rightarrow (1-\lambda)^2 = \gamma_1\gamma_2$$

$$\Rightarrow \quad 1-\lambda = \pm\sqrt{\gamma_1\gamma_2}$$

$$\Rightarrow \quad \lambda = 1-\sqrt{\gamma_1\gamma_2}\,, \quad 1+\sqrt{\gamma_1\gamma_2}$$

So eigen values will be

$$\boxed{\lambda = 1,\ 1-\sqrt{\gamma_1\gamma_2}\,,\ 1+\sqrt{\gamma_1\gamma_2}}$$

1 has multiplicity 2.

To find eigen vectors $\Rightarrow [A-\lambda I]V = 0$

$$\left[\begin{vmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\right] V = 0$$

for $\lambda = 1$

$$\Rightarrow \begin{vmatrix} 0 & \gamma_1 & 0 \\ \gamma_2 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 0x_1 + \gamma_1 x_2 + 0x_3 = 0$$

$$x_1\gamma_2 + 0x_2 + 0x_3 = 0$$

$$0x_1 + 0x_2 + 0x_3 = 0$$

$\rightarrow$ apply Cramers rule

By cramer's rule

$$\frac{x_1}{\begin{vmatrix} \gamma_1 & 0 \\ 0 & 0 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & 0 \\ \gamma_2 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & \gamma_1 \\ \gamma_2 & 0 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-\gamma_1\gamma_2} = K$$

$\Rightarrow$ multiply by $-\gamma_1\gamma_2$ everywhere [K is arbitary constant]

$$\Rightarrow \frac{x_1}{0} = \frac{-x_2}{0} = \frac{x_3}{1} = K$$

So for $\lambda = 1$ eigen vector $= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} K$

Now put $\lambda = 1 + \sqrt{\gamma_1\gamma_2}$ in $A - \lambda I$

$$\Rightarrow \begin{vmatrix} 1-\lambda & \gamma_1 & 0 \\ \gamma_2 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} \Rightarrow \begin{vmatrix} -\sqrt{\gamma_1\gamma_2} & \gamma_1 & 0 \\ \gamma_2 & -\sqrt{\gamma_1\gamma_2} & 0 \\ 0 & 0 & -\sqrt{\gamma_1\gamma_2} \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} -\sqrt{\gamma_1\gamma_2} & \gamma_1 & 0 \\ \gamma_2 & -\sqrt{\gamma_1\gamma_2} & 0 \\ 0 & 0 & -\sqrt{\gamma_1\gamma_2} \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -\sqrt{\gamma_1 \gamma_2}\, x_1 + \gamma_1 x_2 + 0 x_3 = 0$$

$$\Rightarrow \quad \gamma_2 x_1 - \sqrt{\gamma_1 \gamma_2}\, x_2 + 0 x_3 = 0$$

$$\Rightarrow \quad 0 x_1 + 0 x_2 - \sqrt{\gamma_1 \gamma_2}\, x_3 = 0$$

$\rightarrow$ Cramer's rule

By clamer's rule

$$\frac{x_1}{\begin{vmatrix} \gamma_1 & 0 \\ 0 & -\sqrt{\gamma_1 \gamma_2} \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -\sqrt{\gamma_1 \gamma_2} & 0 \\ 0 & -\sqrt{\gamma_1 \gamma_2} \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -\sqrt{\gamma_1 \gamma_2} & \gamma_1 \\ 0 & 0 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{-\gamma_1 \sqrt{\gamma_1 \gamma_2}} = \frac{-x_2}{\gamma_1 \gamma_2} = \frac{x_3}{0}$$

$$\Rightarrow \quad \frac{x_1}{\frac{-\gamma_1 \sqrt{\gamma_1 \gamma_2}}{-\gamma_1 \gamma_2}} = \frac{x}{1} = \frac{x_3}{0}$$

$$\Rightarrow \quad \frac{x_1}{\frac{\gamma_1}{\sqrt{\gamma_1 \gamma_2}}} = \frac{x}{1} = \frac{x_3}{0}$$

$$\frac{x_1}{\sqrt{\frac{\gamma_1}{\gamma_2}}} = \frac{x_2}{1} = \frac{x_3}{0} = K_1$$

So eigen vector for eigen value $\lambda = 1 + \sqrt{\gamma_1 \gamma_2}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \sqrt{\gamma_1/\gamma_2} \\ 1 \\ 0 \end{bmatrix} K_1$$

Now put $\lambda = 1 - \sqrt{\gamma_1 \gamma_2}$

$$\Rightarrow \begin{vmatrix} 1-\lambda & \gamma_1 & 0 \\ \gamma_2 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sqrt{\gamma_1 \gamma_2} & \gamma_1 & 0 \\ \gamma_2 & \sqrt{\gamma_1 \gamma_2} & 0 \\ 0 & 0 & \sqrt{\gamma_1 \gamma_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we get

$$\sqrt{\gamma_1\gamma_2}\,x_1 + \gamma_1 x_2 + 0x_3 = 0$$
$$\gamma_2 x_1 + \sqrt{\gamma_1\gamma_2}\,x_2 + 0x_3 = 0$$
$$0x_1 + 0x_2 + \sqrt{\gamma_1\gamma_2}\,x_3 = 0$$

$\rightarrow$ apply Clamer's rule

$$\Rightarrow \frac{x_1}{\begin{vmatrix} \gamma_1 & 0 \\ 0 & \sqrt{\gamma_1\gamma_2} \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} \sqrt{\gamma_1\gamma_2} & 0 \\ 0 & \sqrt{\gamma_1\gamma_2} \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} \sqrt{\gamma_1\gamma_2} & \gamma_1 \\ 0 & 0 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{\gamma_1\sqrt{\gamma_1\gamma_2}} = \frac{x_2}{-\gamma_1\gamma_2} = \frac{x_3}{0}$$

$$\Rightarrow \frac{x_1}{\dfrac{\gamma_1\sqrt{\gamma_1\gamma_2}}{-\gamma_1\gamma_2}} = \frac{x_2}{1} = \frac{x_3}{0} = k_2$$

$$\Rightarrow \frac{x_1}{-\sqrt{\dfrac{\gamma_1}{\gamma_2}}} = \frac{x_2}{1} = \frac{x_3}{0}$$

So for eigen value $\lambda = 1 - \sqrt{\gamma_1\gamma_2}$

eigen vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\sqrt{\dfrac{\gamma_1}{\gamma_2}} \\ 1 \\ 0 \end{bmatrix} k_2$

For eigen $1, 1+\sqrt{\gamma_1\gamma_2}, 1-\sqrt{\gamma_1\gamma_2}$ the Coresponding Eigen vectors are

$$x_1 = K[0 \quad 0 \quad 1]^T \; ; \; x_2 = K_1\left[\sqrt{\frac{\gamma_1}{\gamma_2}} \quad 1 \quad 0\right]^T$$

$$x_3 = K_2\left[-\sqrt{\frac{\gamma_1}{\gamma_2}} \quad 1 \quad 0\right]$$

The normalized Eigen vector are

$$x_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad x_2 = \frac{1}{\sqrt{\frac{\gamma_1+\gamma_2}{\gamma_2}}}\begin{bmatrix} \sqrt{\gamma_1/\gamma_2} \\ 1 \\ 0 \end{bmatrix} \qquad x_3 = \frac{1}{\sqrt{\frac{\gamma_1+\gamma_2}{\gamma_2}}}\begin{bmatrix} -\sqrt{\frac{\gamma_1}{\gamma_2}} \\ 1 \\ 0 \end{bmatrix}$$

(b) Conditions for Eigen values to be real

The eigen values of hermitian matrix are always real numbers

$$A = \begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let if

$$\gamma_1 = a+ib \text{ then } \gamma_2 = a-ib$$

then $A = \begin{bmatrix} 1 & a+ib & 0 \\ a-ib & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 1 & a-ib & 0 \\ a+ib & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A conjugate

Transpose of $\bar{A} = \begin{bmatrix} 1 & a+ib & 0 \\ a-ib & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A^*$

which is equal to A

So $\Rightarrow$ $A^* = \bar{A}$

So A is a hermitian matrix, eigen values of hermitian matrix are real numbers

In conlusion eigen values are real when z_1 and z_2 are conjugate

$\boxed{\text{and } z_1 \times z_2 \in R.}$

ⓒ Condition for eigen vector to be orthogonal

$\Rightarrow$ Eigen vector corresponding to distinct eigen values of a hermitian matrix are orthogonal

$\rightarrow$ For the eigen vectors to be orthogonal the dot product of them should be equal to zero

$$\Rightarrow X_2 \cdot X_3 = 0 \quad \Rightarrow [X_2]^T [X_3] = 0$$

$$\Rightarrow \begin{bmatrix} \sqrt{\dfrac{z_1}{z_1+z_2}} & \sqrt{\dfrac{z_2}{z_1+z_2}} & 0 \end{bmatrix} \begin{bmatrix} -\sqrt{z_1/z_1+z_2} \\ \sqrt{z_1/z_1+z_2} \\ 0 \end{bmatrix}$$

$$\Rightarrow -\frac{z_1}{z_1+z_2} + \frac{z_2}{z_1+z_2} = 0 \quad \Rightarrow \boxed{z_1 = z_2}$$

Condition for eigen vectors to be tur

4) LU decomposition of $\begin{bmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

Lets find lower triangular matrix (L).

general form of $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ u_{21} & 1 & 0 & 0 \\ u_{31} & u_{32} & 1 & 0 \\ u_{41} & u_{42} & u_{43} & 1 \end{bmatrix}$

To get u lets do row operations to get matrix in echlon form

$\Rightarrow \quad R_2 = R_2 - \frac{R_1}{2}$

$\Rightarrow \begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{bmatrix}$ $\longrightarrow$ multiplification factor $-\frac{1}{2}$ is used so $u_{21} = +1/2$

$\Rightarrow \quad R_3 = R_3 - \frac{5R_1}{2}$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -\frac{9}{2} \\ 0 & \frac{21}{2} & -\frac{7}{2} & -\frac{17}{2} \\ 3 & -6 & -3 & 1 \end{bmatrix}$$

$\longrightarrow$ multiplication factor $-\frac{5}{2}$ is used

so $\mu_{31} = \frac{5}{2}$

$$R_4 = R_4 - \frac{3R_1}{2}$$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & \frac{11}{2} & -\frac{7}{2} & -\frac{9}{2} \\ 0 & \frac{21}{2} & -\frac{7}{2} & -\frac{17}{2} \\ 0 & -\frac{3}{2} & -\frac{9}{2} & -\frac{7}{2} \end{bmatrix}$$

$\longrightarrow$ $-\frac{3}{2}$ is used as multiplication factor

So $\mu_{41} = \frac{3}{2}$

$$R_3 = R_3 - \frac{21R_2}{11}$$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & \frac{11}{2} & -\frac{7}{2} & -\frac{9}{2} \\ 0 & 0 & \frac{35}{11} & \frac{1}{11} \\ 0 & -\frac{3}{2} & -\frac{9}{2} & -7/2 \end{bmatrix}$$

multiplication factor $-\frac{21}{11}$ is used

so $\mu_{32} = \frac{21}{11}$

$\Rightarrow$

$$R_4 = R_4 + \frac{3R_2}{11}$$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 0 & 35/11 & 1/11 \\ 0 & 0 & -60/11 & -52/11 \end{bmatrix}$$

$\longrightarrow + \frac{3}{11}$ is used as a multiplication factor so $\mu_{42} = -3/11$

$$R_4 = R_4 + \frac{12R_3}{7}$$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & \frac{11}{2} & -\frac{7}{2} & -\frac{9}{2} \\ 0 & 0 & \frac{35}{11} & \frac{1}{11} \\ 0 & 0 & 0 & -\frac{32}{7} \end{bmatrix}$$

$\longrightarrow 12/7$ is used as a multiplication factor so $\mu_{43} = -\frac{12}{7}$

finally ecolon form is obtained

so

$$L = \begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & \frac{11}{2} & -\frac{7}{2} & -\frac{9}{2} \\ 0 & 0 & \frac{35}{11} & \frac{1}{11} \\ 0 & 0 & 0 & -\frac{32}{7} \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & \frac{11}{2} & -\frac{7}{2} & -\frac{9}{2} \\ 0 & 0 & \frac{35}{11} & \frac{1}{11} \\ 0 & 0 & 0 & -\frac{32}{7} \end{bmatrix}$$

$Ax = b$

We know that $A = LU$

so $LUX = b$

let $UX = y$

so $Ly = b$ → lets try this

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{5}{2} & \frac{21}{11} & 1 & 0 \\ 3/2 & -\frac{3}{11} & -\frac{12}{7} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 8 \\ -5 \end{bmatrix}$$

$\Rightarrow \quad y_1 = -4 \longrightarrow (1)$

$\Rightarrow \quad \frac{y_1}{2} + y_2 = 1 \longrightarrow (2)$

$\Rightarrow \quad \frac{5}{2}y_1 + \frac{21}{11}y_2 + y_3 = 8 \longrightarrow (3)$

$\Rightarrow \quad \frac{3}{2}y_1 - \frac{3}{11}y_2 - \frac{12}{7}y_3 + y_4 \Rightarrow -5 \longrightarrow (4)$

using all these four equations

$$Y_2 = 1-\left(\frac{-4}{2}\right)$$

$$Y_2 = 3$$

$$\Rightarrow \quad -4\left(\frac{5}{2}\right) + \frac{21}{11}(3) + Y_3 = 8$$

$$\Rightarrow \quad -10 + \frac{63}{11} + Y_3 = 8$$

$$\Rightarrow \quad 18 - \frac{63}{11} = Y_3$$

$$Y_3 = \frac{135}{11}$$

$$\Rightarrow \quad \frac{3}{2}(-4) - \frac{3}{11}(3) - \frac{12}{7}\left(\frac{135}{11}\right) + Y_4 = -5$$

$$\Rightarrow \quad -6 - \frac{9}{11} - \frac{1620}{77} + Y_4 = -5$$

$$\Rightarrow \quad Y_4 = 160/7$$

$$Y_4 \Rightarrow$$

Now put Y values in $UX = Y$

$$\Rightarrow \begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -\frac{7}{2} & -\frac{9}{2} \\ 0 & 0 & \frac{35}{11} & \frac{1}{11} \\ 0 & 0 & 0 & -\frac{32}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 135/7 \\ \frac{160}{7} \end{bmatrix}$$

$$2x_1 - 3x_2 + x_3 + 3x_4 = -4$$

$$\frac{11}{2}x_2 - \frac{7}{2}x_3 - \frac{9}{2}x_4 = 3$$

$$\frac{35}{11}x_3 + \frac{1}{11}x_4 = \frac{135}{11}$$

$$-\frac{32}{7}x_4 = \frac{160}{7}$$

from these equations

$$\boxed{x_4 = -5} \text{ put}$$

$$\frac{35}{11}(x_3) + \frac{1}{11}(-5) = \frac{135}{11}$$

$$35x_3 = 140$$

$$\boxed{x_3 = 4} \text{ put}$$

$$\frac{11}{2}(x_2) - \frac{7}{2}(4) - \frac{9}{2}(-5) = 3$$

$$11x_2 - 28 + 45 = 6$$

$$\boxed{x_2 = -1} \text{ put}$$

$$2x_1 - 3(-1) + 4 + 3(-5) = -4$$

$$\Rightarrow 2x_1 + 3 + 4 - 15 = -4$$

$$\Rightarrow 2x_1 = 4$$

$$\boxed{x_1 = 2}$$

So

$$\boxed{\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ -5 \end{bmatrix}}$$

4(i) solution.

(ii) $b = (-10 \;\; 0 \;\; -3 \;\; -24)^T$

$Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{5}{2} & \frac{21}{11} & 1 & 0 \\ 3/2 & -\frac{3}{11} & -\frac{12}{7} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ -3 \\ -24 \end{bmatrix}$$

$\Rightarrow \boxed{y_1 = -10}$

$\Rightarrow \frac{y_1}{2} + y_2 = 0$

$\Rightarrow \frac{5}{2} y_1 + \frac{21}{11} y_2 + y_3 = -3$

$\Rightarrow \frac{3}{2} y_1 - \frac{3}{11} y_2 - \frac{12}{7} y_3 + y_4 = -24$

Solving these equations.

$\Rightarrow -\frac{10}{2} + y_2 = 0$

$\Rightarrow \boxed{y_2 = 5}$

$\Rightarrow \frac{5}{2}(-10) + \frac{21}{11}(5) + y_3 = -3$

$\Rightarrow -25 + \frac{105}{11} + y_3 = -3$

$\Rightarrow \frac{+170}{11} - 3 = y_3$

$\Rightarrow \boxed{+137/11} = y_3$ //

$$\Rightarrow \frac{3}{2}(-10) - \frac{3}{11}(5) - \frac{12}{7}\left(\frac{137}{11}\right) + y_4 = -24$$

$$\Rightarrow -15 - \frac{15}{11} - \frac{1644}{77} + y_4 = -24$$

$$y_4 = \frac{96}{7}$$

$$\begin{bmatrix} -1 & 1 & 4 & -3 \end{bmatrix}$$

Now apply $UX = Y$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & \frac{11}{2} & \frac{-7}{2} & \frac{-9}{2} \\ 0 & 0 & \frac{35}{11} & \frac{1}{11} \\ 0 & 0 & 0 & \frac{-32}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \\ 137/11 \\ \frac{96}{7} \end{bmatrix}$$

$$2x_1 - 3x_2 + x_3 + 3x_4 = -10$$

$$\frac{11}{2}x_2 - \frac{7}{2}x_3 - \frac{9}{2}x_4 = 5$$

$$\frac{35}{11}x_3 + \frac{x_4}{11} = \frac{137}{11}$$

$$\frac{-32}{7}x_4 = \frac{96}{7}$$

Solving these equations

$$\boxed{x_4 = -3}$$

$$35x_3 - 3 = 137 \Rightarrow 35x_3 = 140$$

$$\Rightarrow \boxed{x_3 = 4}$$

$$\Rightarrow \frac{11}{2}x_2 - \frac{7(4)}{2} - \frac{9}{2}(-3) = 5$$

$$\Rightarrow 11x_2 - 28 + 27 = 10$$

$$\Rightarrow 11x_2 - 1 = 10$$

$$\Rightarrow \boxed{x_2 = 1}$$

$$\Rightarrow 2(x_1) - 3(1) + (4) + 3(-3) = -10$$

$$\Rightarrow 2x_1 = -2$$

$$\boxed{x_1 = -1}$$

So when $b = \begin{bmatrix} -10 \\ 0 \\ -3 \\ -24 \end{bmatrix} \longrightarrow \boxed{x = \begin{bmatrix} -1 \\ 1 \\ 4 \\ -3 \end{bmatrix}}$

4(ii) solution

Now lets find the det by $|L| \times |U|$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{5}{2} & \frac{21}{11} & 1 & 0 \\ 3/2 & \frac{-3}{11} & \frac{-12}{7} & 1 \end{vmatrix} \times \begin{vmatrix} 2 & -3 & 1 & 3 \\ 0 & \frac{11}{2} & -\frac{7}{2} & -\frac{9}{2} \\ 0 & 0 & \frac{35}{11} & \frac{1}{11} \\ 0 & 0 & 0 & \frac{-32}{7} \end{vmatrix}$$

$\det(L) \Rightarrow$

$$\Rightarrow 1 \times \begin{vmatrix} 1 & 0 & 0 \\ \frac{21}{11} & 1 & 0 \\ -\frac{3}{11} & -\frac{12}{7} & 1 \end{vmatrix} - 0 + 0 - 0$$

$$\Rightarrow 1[1] = 1 \,//$$

$\det(U)$

$$\Rightarrow 0 - 0 + 0 - 2\begin{vmatrix} 0 & 0 & \frac{11}{2} \\ 0 & \frac{35}{11} & -\frac{7}{11} \\ -\frac{32}{7} & \frac{1}{11} & -\frac{9}{2} \end{vmatrix}$$

$$\Rightarrow -2\left(\frac{11}{2}\left(-\left(-\frac{35}{11} \times \frac{32}{7}\right)\right)\right)$$

$$\Rightarrow -160 \,//$$

$$\det(L) \times \det(U) = -160 \times 1$$

$$\Rightarrow -160$$

Now lets find $\det(A)$

$$\Rightarrow \begin{vmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{vmatrix}$$

$$\Rightarrow 2\begin{vmatrix} 4 & -3 & -3 \\ 3 & -1 & -1 \\ -6 & -3 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & -3 & -3 \\ 5 & -1 & -1 \\ 3 & -3 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 4 & -3 \\ 5 & 3 & -1 \\ 3 & -6 & 1 \end{vmatrix}$$

$$- 3\begin{vmatrix} 1 & 4 & -3 \\ 5 & 3 & -1 \\ 3 & -6 & -3 \end{vmatrix}$$

$$\Rightarrow 2\left[4(-4)+3(-3)-3(-15)\right]+3\left[1(-4)+3(8)-3(-12)\right]$$

$$+1\left[-3-4(8)-3(-39)\right]-3\left[1(-15)-4(-12)-3(-39)\right]$$

$$\Rightarrow 2[20]+3[56]+[82]-3[150]$$

$$\Rightarrow 40+168+82-450$$

$$\Rightarrow -160$$

So we got $\det(L) \times \det(U) = \det(A)$

Hence proved.

Problem-5

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

finding eigen values.

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{vmatrix}$$

$$\Rightarrow (-1-\lambda)\left[(-1-\lambda)(-1-\lambda) - 4)\right] - 2\left[2(-1-\lambda) - 4\right]$$
$$+ 2\left[4 - (-2-2\lambda)\right] = 0$$

$$\Rightarrow (-1-\lambda)\left[1+\lambda+\lambda+\lambda^2-4\right] - 2\left[-2-2\lambda-4\right]$$
$$+ 2\left[4+2+2\lambda\right] = 0$$

$$\Rightarrow (-1-\lambda)\left[1+2\lambda+\lambda^2-4\right] - 2\left[-6-2\lambda\right]$$
$$+ 2\left[6+2\lambda\right] = 0$$

$$\Rightarrow -1-2\lambda-\lambda^2+4-\lambda-2\lambda^2-\lambda^3+4\lambda+12+4\lambda$$
$$+ 12+4\lambda = 0$$

$$-\lambda^3 - 3\lambda^2 + 9\lambda + 27 = 0$$

$$\Rightarrow \lambda^3 + 3\lambda^2 - 9\lambda - 27 = 0$$

$$\Rightarrow (\lambda+3)^2(\lambda-3) = 0$$

$$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-3) = 0$$

$$\lambda = -3, -3, 3$$

To find eigen vectors $(A - \lambda I)X = 0$

$$\begin{bmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Put $\lambda = 3$

$$\Rightarrow \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}$$

Reduce this into echelon form

$R_2 \to 2R_2 + R_1$

$R_3 \to 2R_3 + R_1$

$$\Rightarrow \begin{bmatrix} -4 & 2 & 2 \\ 0 & -6 & 6 \\ 0 & 6 & -6 \end{bmatrix}$$

$R_3 \to R_3 + R_2 \Rightarrow \begin{bmatrix} -4 & 2 & 2 \\ 0 & -6 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

echelon form

$$\begin{bmatrix} -4 & 2 & 2 \\ 0 & -6 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

let $z = k_1$

$\Rightarrow -6y + 6z = 0$

$\Rightarrow -6y + 6k_1 = 0 \qquad y = k_1$

$\Rightarrow -4x + 2y + 2z = 0$

$-4x + 2k_1 + 2k_1 = 0 \Rightarrow x = k_1$

So eigen vector for $\lambda = 3$ is $k_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

for $\lambda = -3$

$[A - \lambda I] \Rightarrow \begin{vmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{vmatrix}$

$R_3 \to R_3 - R_1$
$R_2 \to R_2 - R_1$ $\Rightarrow \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

echelon form

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

if $z = k_3$ & $y = k_2$

then $2x + 2k_2 + 2k_3 = 0$

$$x = -k_2 - k_3$$

So eigen vectors are $\Rightarrow \begin{bmatrix} -k_2 - k_3 \\ k_2 \\ k_3 \end{bmatrix}$

$$\Rightarrow k_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So $\vec{x}_2 = k_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\quad \vec{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} k_3$

So eigen vectors are $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \vec{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$$\vec{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\rightarrow$ eigen vectors are independent to each and (from matrix) $A = A^T$ and Symetric, hence the given matrix is diagonizable $\rightarrow$ 5(i) Answer

S = Vector matrix $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$S^{-1} = \frac{\operatorname{adj}(S)}{|S|}$$

$$\det|S| = 1(1) + (1) - 1(-1) = 3$$

and $\text{Cofactor}(S) \Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$(\text{Cofactor}(S))^T = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$S^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Diagonalizable matrix $= S^{-1} \cdot A \cdot S$

$$\Rightarrow \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}}$$

6) given that

$$(A+UV^T)^{-1} = A^{-1} - A^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1}$$

and A, $A+UV^T$ and $I+V^TA^{-1}U$ are - non singular matrices.

$\Rightarrow$ Multiply the equation with $(A+UV^T)$

Therefore

$$I = (A+UV^T)\left[A^{-1} - A^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1}\right]$$

$$\Rightarrow A \cdot A^{-1} - AA^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1} + UV^TA^{-1}$$
$$- UV^TA^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1}$$

$$\Rightarrow I + UV^TA^{-1} - IU(I+V^TA^{-1}U)^{-1}V^TA^{-1} -$$
$$UV^TA^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1}$$

$$\Rightarrow I + UV^TA^{-1} - (I+V^TA^{-1}U)^{-1}V^TA^{-1}\left[I \cdot U + UV^TA^{-1}U\right]$$

$$\Rightarrow I + UV^TA^{-1} - \left[(I+V^TA^{-1}U)^{-1}V^TA^{-1}\right]\left[U(I+UV^TA^{-1})\right]$$

$$\Rightarrow I + UV^TA^{-1} - UV^TA^{-1}$$

$$\Rightarrow I$$

So LHS = RHS //

So $(A+U^T)^{-1} = A^{-1} - A^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1}$