

Problem Set # 5

Due Date: Thursday Dec 14 @ 11:59pm EST in ELMS

Instructions:

- Please write clearly and show all your work. Credit will be given for all correct steps. Points will be deducted for any missing steps.
- Please write your name clearly on the assignment.
- Problems 1 and 2 are from a previous final exam.
- No extensions will be provided for this problem set.

Problem 1: Let $\mathbf{A} \in \mathfrak{R}^{n \times n}$ and $\mathbf{B} \in \mathfrak{R}^{m \times m}$. Show that $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{X}(0)e^{\mathbf{B}t}$ is the solution to the following equation

$$\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{X}(t)\mathbf{B}$$

Problem 2: A (Euclidean) ball, $\mathbf{B}(x_c, r)$, in \mathfrak{R}^n is given by

$$\mathbf{B}(x_c, r) = \left\{ x \in \mathfrak{R}^n \mid \|x - x_c\| \leq r \right\}$$

where $r > 0$ and $\|\cdot\|$ denotes the Euclidean norm, $\|x\| = \sqrt{x^T x}$. The vector $x_c \in \mathfrak{R}^n$ is the center of the ball and r is the radius of the ball. **Prove that $\mathbf{B}(x_c, r)$ is a Convex set.**

Problem 3: Consider the following state-space equation

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Consider the following cost function:

$$J = \int_0^\infty \left(\mathbf{x}^T \mathbf{Q} \mathbf{x} + u^2 \right) dt$$

where \mathbf{Q} is given by:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}, \quad \gamma > 0$$

Design an LQR controller for this system and provide the state feedback $u = \mathbf{K}\mathbf{x}$. You need to do all these calculations by hand and show all your steps. No credit will be given for simulation based results. Points will be deducted for any missing steps.

Problem 4: Consider the following state-space representation

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t)$$

- Write this system in the standard form of Uncontrollable Systems using the Similarity Transformation discussed in class.
- Write the controllable part of this system as a state equation.

Show all your work. No credit will be given for simulation based results. Points will be deducted for any missing steps.

Problem 5: Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \mathbf{x}$$

Investigate the stability of this system using the following Lyapunov equation:

$$\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} = -\mathbf{Q}, \quad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Show all your work. No credit will be given for simulation based results.