① $dF(x,y) = \left(\frac{1}{x^2+2} + \frac{\alpha}{y}\right) dx + (xy^{\beta}+1)\, dy$

given that it is a exact differential

So It is in the form of

$dF = M dx + N dy$

$M = \frac{1}{x^2+2} + \frac{\alpha}{y}$

$N = xy^{\beta} + 1$

for exact differential $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow \frac{\partial}{\partial y}\left(\frac{1}{x^2+2} + \frac{\alpha}{y}\right) = \frac{\partial}{\partial x}(xy^{\beta}+1)$

$\Rightarrow \alpha \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = y^{\beta} \frac{\partial}{\partial x} x$

$\Rightarrow -\frac{\alpha}{y^2} = y^{\beta}$

$\Rightarrow -\alpha y^{-2} = y^{\beta}$

compare bases and exponents

$\Rightarrow \alpha = -1$

$\beta = -2$

Sub $\Rightarrow \alpha, \beta$ in equation

$$dF(x,y) = \left(\frac{1}{x^2+2} - \frac{1}{y}\right)dx + \left(xy^{-2}+1\right)dy$$

for exact differential Solution will be

$$U(x,y) = \int A(x,y)\,dx + F(y)$$

$$\Rightarrow \int\left(\frac{1}{(x^2+2)} - \frac{1}{y}\right)dx + F(y) = C_1$$

$$\Rightarrow \frac{\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)}{\sqrt{2}} - \frac{x}{y} + F(y) = C_1$$

To get $F(y)$ $\int U(x,y) = B(x,y)$

wrt y

$$\Rightarrow \frac{d}{dy}\left[\frac{\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)}{\sqrt{2}}\right] - \frac{d}{dy}\left[\frac{x}{y}\right] + F(y) = xy^{-2}+1$$

$$\Rightarrow 0 + \frac{x}{y^2} + \frac{d}{dy}F(y) = \frac{x}{y^2} + 1$$

$$\Rightarrow \frac{d}{dy}F(y) = 1$$

$\Rightarrow$ on integrating $F(y) = y + C_2$

Sub F(y) in the solution

$$\Rightarrow \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{x}{y} + y + C_2 = C_1$$

$$\Rightarrow \boxed{\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{x}{y} + y = C}$$

is the solution

(2) Given that $R\frac{dq}{dt} + \frac{q}{c} = v(t)$

where $v(t) = V_0 \sin(\omega t)$

$\Rightarrow$ It can be rearranged as

$$R\,dq + \frac{q}{c}\,dt = v(t)\,dt$$

Since $\frac{\partial M}{\partial t} \neq \frac{\partial N}{\partial q}$ this is not a exact differential

So to make it exact differential multiply the equation with integration factor

$$IF = \exp \int \frac{1}{A}\left(\frac{\partial N}{\partial q} - \frac{\partial M}{\partial t}\right) dt$$

$$\Rightarrow \exp \int \frac{1}{R}\left(\frac{1}{c} - 0\right) dt$$

$$\Rightarrow \exp \int \frac{1}{Rc}\, dt$$

$$\Rightarrow e^{t/Rc} \longrightarrow I\cdot F$$

now multiply on both sides of equation with $I\cdot F$ to make it exact.

$$\Rightarrow \underbrace{e^{t/CR}\,dq + \frac{e^{t/CR}}{CR}\,q\,dt}_{} = \frac{V_0 \sin(\omega t)}{R} \times e^{t/CR}\,dt$$

This is in $f'(uv)$ form

$$\Rightarrow \int \frac{d}{dt}\left(e^{t/Rc}\,q\right) = \int \frac{V_0 \sin(\omega t)}{R}\, e^{t/Rc}\,dt$$

$$\Rightarrow q\,e^{t/Rc} = \frac{V_0}{R} \int \sin(\omega t)\, e^{t/Rc}\,dt$$

$$\left[\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bt - b \cos bt\right] + C \right]$$

applying this

$$\Rightarrow q\,e^{t/Rc} = \frac{V_0}{R}\left[\frac{e^{t/RC}}{\left(\frac{1}{Rc}\right)^2 + \omega^2}\left[\frac{1}{Rc}\sin(\omega t) - \omega \cos(\omega t)\right] + c \right]$$

$$\Rightarrow q\,e^{t/Rc} = \frac{V_0}{R}\left[\frac{Rc^2\, e^{t/Rc}}{1+(\omega Rc)^2}\left[\frac{\sin(\omega t) - (\omega Rc)\cos(\omega t)}{Rc}\right] + c \right]$$

$$q e^{t/RC} \Rightarrow \frac{V_0 C e^{t/RC}}{1+(\omega RC)^2}\left[\sin(\omega t) - (\omega RC)\cos(\omega t)\right] + C_1$$

given that when $t = 0$ $q = 0$

So applying this

$$0 = \frac{V_0 C(1)}{(1+\omega RC)^2}\left[0 - (\omega RC)\cos(0)\right] + C_1$$

$$\Rightarrow \quad 0 = \frac{V_0 C}{(1+\omega RC)^2}\left[(-\omega RC) + C_1\right]$$

$$C_1 = \frac{V_0 \omega RC^2}{1+(\omega RC)^2}$$

Put C_1 in equation

$$q = \frac{V_0 C}{1+(\omega RC)^2}\left[\sin(\omega t) - (\omega RC)\cos\omega t\right] + \frac{V_0 C^2 \omega R e^{-t/RC}}{1+(\omega RC)^2}$$

$$\boxed{q = \frac{V_0 C}{1+(\omega RC)^2}\left[\sin(\omega t) - (\omega RC)\cos\omega t + \omega RC e^{-t/RC}\right]}$$

↳ Solution

(3) given $\frac{dy}{dx} = \frac{-2x^2+y^2+x}{xy}$

on rearranging it

$\Rightarrow (xy)\,dy = (-2x^2+y^2+x)\,dx$

$\Rightarrow (2x^2+y^2+x)\,dx + (xy)\,dy = 0$

$M = 2x^2+y^2+x$

$N = xy$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = y$

So $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So equation is not exact differential

I.F is required to convert to exact

$I.F = e^{\int f(x)\,dx}$

$f(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$

$\Rightarrow \frac{1}{xy}(2y - y) = \frac{1}{x}$

$$IF = e^{\int \frac{1}{x} dx}$$

$$= e^{\log x}$$

$$\Rightarrow \boxed{x \longrightarrow I \cdot F}$$

So multiply equation with I·F on both sides

$$\Rightarrow x\left(2x^2 + y^2 + x\right) dx + x^2 y\, dy = 0$$

$$\Rightarrow \text{Solution } \int M\, dx + F(y) = C$$

$$U_{(x,y)} = \int \left(2x^3 + xy^2 + x^2\right) x^2 + F(y) = C$$

$$U_{(x,y)} \Rightarrow \frac{2x^4}{4} + \frac{x^2 y^2}{2} + \frac{x^3}{3} + F(y) = C$$

$$\Rightarrow F_y = \frac{\partial}{\partial y}(U)_{(x,y)} = N(x,y)$$

$$\Rightarrow F_y \Rightarrow 0 + x^2 y + \frac{\partial}{\partial y} F(y) = x^2 y$$

$$\Rightarrow \frac{d}{dy} F(y) = 0$$

$$\Rightarrow F(y) = C$$

now sub $F(y)$ in solution

$$\Rightarrow \frac{x^4}{2} + \frac{x^3}{3} + \frac{x^2 y^2}{2} + c_1 = c$$

solution $= \boxed{\frac{x^4}{2} + \frac{x^3}{3} + \frac{x^2 y^2}{2} = c}$ → Solution

4a given $\frac{\partial^2 f}{\partial t^2} + 2\frac{df}{dt} + 5f = 0$

and boundary conditions

$f(0) = 1 \quad f'(0) = 0$

To roots rewrite as

$$\lambda^2 + 2\lambda + 5 = 0$$

roots of quadratic equation is given

as $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

so $\frac{-2 \pm \sqrt{4 - 4(20)}}{2} = \frac{-2 \pm 4i}{2}$

$\Rightarrow -1 \pm 2(i)$

roots are Complex with $\alpha = -1$
$\beta = 2$

So Complement Function C.F is given by

$$C.F = c_1 e^{(\alpha + i\beta)t} + c_2 e^{(\alpha - i\beta)t}$$

on rearranging we get

$$\Rightarrow e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right)$$

Put α, β values

$$\Rightarrow e^{-t}\left(c_1 \cos(2t) + c_2 \sin(2t)\right)$$

and we have $f(0) = 1$

$$\Rightarrow f(0) = e^{0}(c_1 + 0) = 1$$

$$\boxed{c_1 = 1} \rightarrow (1)$$

$f'(0) = 0$

$$\Rightarrow e^{-t}\left(c_1\left(2(-\sin(2t))\right) + c_2\left(2\cos(2t)\right)\right)$$

$$+ -e^{-t}\left(c_1 \cos(2t) + c_2 \sin(2t)\right)$$

$$f'(0) = -e^{0}(c_1 + 0) + e^{0}(2c_2) = 0$$

$\Rightarrow -c_1 + 2c_2 = 0 \rightarrow$ (2)

Put (1) in (2)

$\Rightarrow -1 + 2c_2 = 0$

$$\boxed{c_2 = \frac{1}{2}}$$

So equation is

$$CF = e^{-t}\left(\cos(2t) + \frac{1}{2}(\sin 2t)\right)$$

Since the equation is homogeneous

Solution = C.F

$$\boxed{\text{So } e^{-t}\left(\cos(2t) + \frac{1}{2}(\sin 2t)\right)}$$

4b) $$\frac{d^2f}{dt^2} + 2\frac{df}{dt} + 5f = e^{-t}(\cos(3t))$$

$\Rightarrow$ from 4a we get

$$f(t) \Rightarrow \underbrace{e^{-t}(c_1\cos(2t) + c_2\sin(2t))}_{C.F}$$

By method of undetermined coefficients

Assume $y_p = e^{-t}\left[a\cos(3t) + b\sin(2t)\right]$

Now $f'(t) = ae^{-t}\left[-3\sin(3t)\right] + a\cos(3t)(-e^{-t})$
$+ e^{-t} b(2\cos(2t)) + b\sin 2t(-e^{-t})$

$$f'(t) = ae^{-t}\left[(-3\sin(3t)) - \cos(3t)\right] + be^{-t}\left[2\cos(2t) - \sin 2t\right]$$

and

$$f''(t) = -ae^{-t}\left[(-3\sin 3t) - \cos(3t)\right] + ae^{-t}\left[9\cos(3t) + 3\sin(3t)\right] - be^{-t}\left[2\cos(2t) - \sin(2t)\right] + be^{-t}\left[-4\sin(2t) - 2\cos(2t)\right]$$

Put these f, f', f'' in $f^2 + f' + f = e^{-t}\cos(3t)$

we get

$$e^{-t}(\cos(3t)(-5a) + \sin(2t)(-5b)] = e^{-t}\cos(3t)$$

Comparing Coefficients

$-5a = 1 \Rightarrow a = -1/5$

$-5b = 0 = b = 0$

so putting these a and b values in $e^{-t}(a\cos(3t) + b\sin(2t))$

we get

$$PI = e^{-t}\left(\frac{-1}{5}\cos(3t)\right)$$

//

Now solution will be $\Rightarrow$ CF + PI

$$\Rightarrow e^{-t}\left(c_1\cos(2t) + c_2\sin(2t)\right) - \frac{1}{5}e^{-t}\cos(3t)$$

Now we have $f(0) = 0$

$$f(0) = e^{-0}\left(c_1\cos(0) + c_2\sin(0)\right) - \frac{1}{5}e^{0}\left(\cos 0\right) = 0$$

$$\Rightarrow (c_1 + 0) - \frac{1}{5} = 0$$

$$\Rightarrow \boxed{c_1 = \frac{1}{5}}$$

$\Rightarrow f'(0) = 0 \rightarrow$ apply this we get c_2

$$\Rightarrow f'(t) = -e^{-t}\left(c_1\cos(2t) + c_2\sin(2t)\right) + e^{-t}\left(-2c_1\sin 2 + 2c_2\cos(2t)\right) + \frac{e^{-t}\cos 3t}{5} + \frac{3e^{-t}\sin 3t}{5}$$

$$f'(0) = -c_1 + 2c_2 + \frac{1}{5} = 0$$

$$\Rightarrow \boxed{c_2 = 0}$$

So put these c_1, c_2 in equations

Solution will be

$$e^{-t}\left(\frac{\cos(2t)}{5} - e^{-t}\frac{\cos(3t)}{5}\right) \Rightarrow \boxed{\frac{e^{-t}}{5}\left[\cos(2t) - \cos(3t)\right]}$$

Solution

5. Method of laplace transforms

$$y''(t) + y(t) = \sin(2t)$$

given boundary conditions $y(0) = 2$

$y'(0) = 1$

So applying laplace

$$s^2\bar{y}(s) - s\,y(0) - y'(0) + \bar{y}(s) = \frac{2}{s^2+2^2}$$

$$\Rightarrow s^2\bar{y}(s) - s(2) - 1 + S\bar{y} = \frac{2}{s^2+2^2}$$

$$\Rightarrow \bar{y}(s)(s^2+1) - 2s - 1 = \frac{2}{s^2+2^2}$$

$$\Rightarrow \bar{y}(s) = \frac{2}{(s^2+4)(s^2+1)} + \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

We need to convert the term into partial fraction

$$\frac{2}{(s^2+4)(s^2+1)} = \frac{A}{(s^2+4)} + \frac{B}{(s^2+1)}$$

$$\frac{2}{(s^2+4)(s^2+1)} = \frac{As^2+A+Bs^2+4B}{(s^2+4)\quad(s^2+1)}$$

$$\Rightarrow 2 = (A+B)s^2 + A+4B$$

Comparing coeft of s^2 $\quad A+B=0$

Comparing constants $\quad A+4B=2$

from above equations

$$A=-B,$$

$$-B+4B=2$$

$$3B=2 \Rightarrow \boxed{B=\frac{2}{3},\ A=-2/3}$$

Sub A & B in Partial fractions

$$\Rightarrow -\frac{2}{3}\left(\frac{1}{s^2+2^2}\right)+\frac{2}{3}\left(\frac{1}{s^2+1}\right)+\frac{2s}{(s^2+4)}+\frac{1}{(s^2+1)}$$

applying inverse laplace for the above

$$\Rightarrow y(t) = -\frac{2}{3}\left(\frac{1}{2}\sin 2t\right)+\frac{2}{3}(\sin t)+2\cos t+\sin t$$

$$\Rightarrow \boxed{y(t) = -\frac{1}{3}(\sin(2t)) + \frac{5}{3}\sin(t)) + 2\cos(t)}$$

Solution

⑥ Method of undetermined coefficients

$$\frac{d^3y}{dx^3} + \frac{3d^2y}{dx^3} + \frac{3dy}{dx} + y = 30e^{-x}$$

$\Rightarrow$ It can be written as

$\Rightarrow \lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

This nothing but $(\lambda+1)^3 = 0$

so $(\lambda+1)(\lambda+1)(\lambda+1) = 0$

roots of the equation is $\lambda = -1, -1, -1$

roots are repeated

C.F for this is

$$y(x) = \left(C_1 + C_2 x \; --- \; C_K x^{K-1}\right) e^{\lambda_1 x}$$

$$\boxed{y(x) = \left(C_1 + C_2 x + C_3 x^2\right) e^{-x}} \rightarrow CF$$

PI of $30e^{-x}$

let $y = a_0 e^{-x} x^3$ (assuming)

$y' = a_0\left(e^{-x}(3x^2) + (x^3(e^{-x}(-1))\right)$

$y' \Rightarrow a_0\left(3x^2 e^{-x} - x^3 e^{-x}\right)$

Now $y'' = a_0\left[\left[3x^2 e^{-x}(-1) + e^{-x} 6x\right] - \left[-x^3 e^{-x} + e^{-x} 3x^2\right]\right]$

$$\Rightarrow a_0\left(-3x^2 e^{-x} + 6xe^{-x} + e^{-x}x^3 - 3x^2 e^{-x}\right)$$

$$\Rightarrow a_0\left(6xe^{-x} + x^3 e^{-x} - 6x^2 e^{-x}\right)$$

$$y''' \Rightarrow a_0\left(6\left(e^{-x} - e^{-x}x\right) + x^3(-e^{-x}) + e^{-x}(3x^2) + 6x^2e^{-x} - 12xe^{-x}\right]$$

$$\Rightarrow a_0\left(6e^{-x} - 6e^{-x}x - e^{-x}x^3 + 3x^2e^{-x} + 6x^2e^{-x} - 12xe^{-x}\right]$$

$$\Rightarrow a_0\left(-x^3e^{-x} + 9e^{-x}x^2 - 18e^{-x}x + 6e^{-x}\right)$$

Substitute y, y', y'', y''' in $y''' + 3y'' + 3y' + y = 30e^{-x}$

$$\Rightarrow a_0 e^{-x}\left(-x^3 + 9x^2 - 18x + 6 + 3(6x + x^3 - 6x^2) + 3(3x^2 - x^3) + x^3\right) = 30e^{-x}$$

$$\Rightarrow a_0 e^{-x}\left(-\cancel{x^3} + \cancel{9x^2} - \cancel{18x} + 6 + \cancel{18x} + \cancel{3x^3} - \cancel{18x^2} + \cancel{9x^2} - \cancel{3x^3} + \cancel{x^3}\right) = 30e^{-x}$$

$\Rightarrow$ So we get $a_0 e^{-x}(6) = 30e^{-x}$

$$\Rightarrow a(6) = 30$$

$\Rightarrow a = 5$ So $PI = 5e^{-x}x^3$

Solution = CF+PI $\boxed{y = e^{-x}\left[c_1 + c_2 x + c_3 x^2 + 5x^3\right]}$

7a) Method of variation of Parameters

$y'' - y = x^n$

CF $\Rightarrow$ $D^2 - 1 = 0$

$(D+1)(D-1) = 0$

$D = 1 \quad D = -1$

So root are real and distinct.

$\Rightarrow$ C.F $= y = c_1 e^x + c_2 e^{-x}$

$y_1 = e^x \quad y_2 = e^{-x} \quad X = x^n$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^0 - e^0$$

$\Rightarrow -2$

We know $PI = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

$$\Rightarrow -e^x \int \frac{e^{-x} x^n}{-2} dx + e^{-x} \int \frac{e^x x^n}{-2} dx$$

$$\Rightarrow \frac{e^{-x}}{-2} \int e^{-x} x^n dx + \frac{-e^{-x}}{2} \int e^x x^n dx$$

we know $\int e^{x} x^{n} = e^{x}\left[\sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} x^{k}\right]$

So $\Rightarrow -\frac{1}{2}\left[\sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} x^{k}\right] - \frac{1}{2}\left[\sum_{k=0}^{n} \frac{n!}{k!} x^{k}\right]$

Complete solution $=$ CF + PI

$$\Rightarrow \boxed{c_1 e^{-x} + c_2 e^{-x} - \frac{1}{2}\left[\sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} x^{k}\right] - \frac{1}{2}\left[\sum_{k=0}^{n} \frac{n!}{k!} x^{k}\right]}$$

7b) $y'' + y = \tan x$

$\Rightarrow \lambda^2 + 1 = 0$

$\Rightarrow \lambda^2 = -1$

$\Rightarrow \lambda = \pm i$

roots are imaginary

So C.F $\Rightarrow y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

here $\alpha = 0 \quad \beta = 1$

So $y = e^{0}(c_1 \cos x + c_2 \sin x)$

$$y = C_1 \cos x + C_2 \sin x$$

$$y_1 = \cos x \qquad y_2 = \sin x \qquad X = \tan x$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W = 1$$

$$PI = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$\Rightarrow -y_1 \int \frac{\sin x \tan x}{1} dx + \sin x \int \frac{\cos x \tan x}{1} dx$$

$$\Rightarrow -\cos x \int \frac{\sin^2 x}{\cos x} dx + \sin x \int \sin x \, dx$$

$$\Rightarrow -\cos x \int \frac{(1-\cos^2 x)}{\cos x} + \sin x \int \sin x \, dx$$

$$\Rightarrow -\cos x \int \sec x - \cos x + \sin x \int \sin x \, dx$$

$$\Rightarrow -\cos x \left[\ln(\sec x - \tan x) - \sin t \right]$$

$$- \sin x \cos x$$

$$y_p = \cos x \cancel{\sin x} - \cos x \left(\ln |\sec x + \tan x) \right| - \sin x \cancel{\cos x}$$

$$y_p = -\cos x \,\left| \ln (\sec x + \tan x) \right|$$

$$y = CF + PI$$

$$= \boxed{C_1 \cos x + C_2 \sin x - \cos x \left(\ln (\sec x + \tan x) \right)}$$

Solution.