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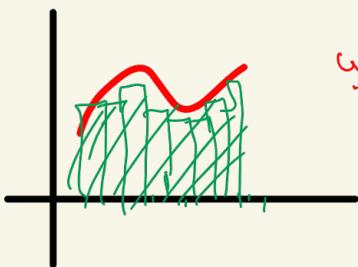
Jan 5, 2026

Part 1: Integrals

Part 2: Application of Integrals

Part 3: Sequences and Series

## Definite Integrals



$$y = f(x)$$

If  $f(x) > 0$

$$\int_a^b f(x) dx = \text{area below the graph}$$

Conceptual Interpretations =  
continuously accumulation  
of quantity.

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Indefinite Integrals  $\int f(x) dx$

essentially antiderivatives

$$x^2 \rightarrow \text{antider. } \frac{x^3}{3} \quad \text{so}$$

the most general antider.

is  $\frac{x^3}{3} + C$ .

$$\int f(x) dx = F(x) + C \text{ means}$$

that most general antider.

of  $F(x)$

$$F'(x) = f(x) \quad \int_0^1 x^2 dx = \frac{1}{3}$$

$$\int x^2 dx = \frac{x^3}{3} + C$$

# FTC: Fundamental Theory of Calculus

$$\text{FTC: } \int_0^1 x^2 dx = \left[ \frac{x^3}{3} \right]_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}$$

this always works

January 7, 2026

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$\int (f'(x)g(x) + f(x)g'(x)) dx = f(x)g(x) + C$$

$$\int f'(x)g(x) dx + \int f(x)g'(x) = f(x)g(x) + C$$

$$\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x) dx$$

$$\int x^2 dx - 7 = \frac{x^3}{3} + (C - 7) = \int x^2 dx$$

1)  
 $\int x \cdot \cos x dx$

Integration by parts

Solution:

$$\int x \cdot \boxed{\cos x} dx = \int x \cdot \boxed{(\sin x)'} dx =$$

$$= x \cdot \sin x - \int x' \sin x dx = x \cdot \sin x -$$

$$- \int \sin x dx = x \cdot \sin x + \cos x + C$$

2)

$$\int 2^x \cdot x dx = \int \left( \frac{2^x}{\ln 2} \right)' \cdot x dx =$$

$$= \frac{2^x}{\ln 2} \cdot x - \int \frac{2^x}{\ln 2} \boxed{\begin{matrix} 1 \\ \uparrow \\ x \end{matrix}} dx =$$

$$= \frac{2^x \cdot x}{\ln 2} - \frac{1}{\ln 2} \int 2^x dx \xrightarrow{\frac{2^x}{\ln 2}}$$

$$= \boxed{\frac{2^x \cdot x}{\ln 2} - \frac{2^x}{(\ln 2)^2}}$$

$$3) \int x^2 e^x dx = \int x^2 \cdot (e^x)' dx =$$

$$x^2 e^x - \int (x^2)' \cdot e^x dx = x^2 e^x - 2 \int$$

$$-2 \int x \cdot e^x dx = x^2 e^x - 2 \int x \cdot (e^x)' dx$$

$$= x^2 \cdot e^x - 2(x \cdot e^x - \int \boxed{x'} e^x dx) =$$

$$= x^2 \cdot e^x - 2 \cdot x \cdot e^x + 2 \boxed{\int e^x dx} e^x$$

$$= \boxed{x^2 e^x - 2x \cdot e^x + 2 \cdot e^x + C}$$

$$4) \int \ln x dx = \int \boxed{1} \cdot \ln x dx =$$

$$= \int x' \ln x dx = x \cdot \ln x - \int x (\ln x)' dx$$

$$= x \cdot \ln x - \int x \cdot \frac{1}{x} dx = x \cdot \ln x - \int 1 dx$$

$$= \boxed{x \cdot \ln x - x + C}$$

5)  $\int \cos^2 x \, dx = \int \cos x \cdot (\sin x)' \, dx$

$$= \cos x \cdot \sin x - \int (\cos x)' \sin x \, dx$$

$$= \cos x \cdot \sin x + \boxed{\int \sin^2 x \, dx}$$

$$= \cos x \cdot \sin x + \int (1 - \cos^2 x) \, dx$$

$$= \cos x \cdot \sin x + \int 1 \, dx - \int \cos^2 x \, dx$$

$$= \cos x \cdot \sin x + x - \int \cos^2 x \, dx$$

Solve for  $\int \cos^2 x \, dx$ :

$$2 \int \cos^2 x \, dx = \frac{1}{2} (\cos x \sin x + x) + C$$

$$u = x^2$$

$$u = g(x)$$

$$\frac{du}{dx} = 2x$$

$$g'(x)$$

$$du = 2x \, dx$$

$$g'(x) \, dx$$

$$\int u \underbrace{f(x)}_{dv} \underbrace{g'(x)}_{du} \, dx = \underbrace{u f(x)}_{u} \underbrace{g(x)}_{v} - \int v \underbrace{g(x)}_{u} \underbrace{f'(x)}_{du} \, dx$$

$$\int u \, dv = uv - \int v \cdot du$$

1)

$$\int u \underbrace{x \cdot \cos x}_{dv} \, dx :$$

$$u = x \quad v = \sin x$$

$$dv = \cos x \cdot dx$$

$$du = dx$$

$$\int \underbrace{x \cdot \cos x \, dx}_{u \, dv} = \int u \, dv = uv - \int v \, du$$

$$= x \cdot \sin x - \int \sin x \, dx + C = x \cdot \sin x - \cos x$$


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January 12, 2026

## Substitution Rule (aka "u-sub")

$$\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx$$

$\int f(x) g(x) \, dx \rightarrow$  integration by parts

$$f(x) g(x) - \int f'(x) g(x) \, dx$$

\* often serves the purpose

$\int f(g(x)) \, dx \times \rightarrow$  no such thing

$\int f(g(x)) g'(x) \, dx \rightarrow$  there is a formula  
for this, substitution rule

$$\int f(g(x))g'(x) dx = \int \underbrace{F'(g(x))g'(x)}_{(F(g(x))')} dx$$

$$= \int (F(g(x))' dx = F(g(x)) + C \quad \text{Let } u = g(x)$$

$$= F(u) + C = \int f(u) du \quad \checkmark$$

$$\text{let } u = x^2$$

$$\frac{du}{dx} = 2x \quad du = 2x dx$$

$$u = g(x)$$

$$du = g'(x) dx$$

$$\frac{du}{dx} = g'(x)$$

$$\int \underbrace{f(g(x))}_{u} \underbrace{g'(x)}_{du} dx = \int f(u) du$$

# Practice

1.

$$\int x \cdot \cos(x^2) dx$$

let  $u = x^2$

$$du = 2x dx \rightarrow \frac{du}{2}$$

$$\int x \cdot \cos(x^2) dx = \int \cos(u) \cdot \boxed{\frac{du}{2}} =$$

$$\frac{1}{2} \int \cos(u) du = \frac{1}{2} \sin(u) + C =$$

$$\frac{1}{2} \sin(x^2) + C$$

2.

$$\int x^2 \sqrt[3]{4x^3 + 3} dx$$

$$\text{let } u = 4x^3 + 3$$

$$du = 12x^2 dx \quad | \quad x^2 dx = \frac{du}{12}$$

$$\int x^2 \sqrt[3]{4x^3 + 3} dx = \int \sqrt[3]{u} \frac{du}{12} = \frac{1}{2} \int \sqrt[3]{u} du$$

$$\begin{aligned}
 &= \frac{1}{12} \int u^{\frac{1}{3}} du = \frac{1}{12} \cdot \frac{u^{\frac{4}{3}}}{\frac{4}{3}} + C = \\
 &= \frac{u^{\frac{4}{3}}}{16} + C = \boxed{\frac{(4x+3)^{\frac{4}{3}}}{16} + C}
 \end{aligned}$$

3.

$$\int \sin(10x) dx \quad \begin{aligned}
 &\text{let } u = 10x \\
 &du = 10 dx \\
 &dx = \frac{du}{10}
 \end{aligned}$$

$$\int \sin(10x) dx = \int \sin(u) \frac{du}{10} = \frac{1}{10} \int \sin(u) du$$

$$= -\frac{1}{10} \cos(u) + C = \boxed{-\frac{1}{10} \cos(10x) + C}$$

Useful shortent:

If  $\int f(x) dx = F(x) + C$ , then

$$\cdot \int f(x+a) dx = F(x+a) + C$$

$$\int \cos(x+\pi) dx = \sin(x+\pi) + C$$

$$\int f(a \cdot x) dx = \frac{F(a \cdot x)}{a} + C$$

$$\int \cos(\pi \cdot x) dx = \frac{\sin(\pi \cdot x)}{\pi} + C$$

$$5. \int \frac{x^3}{1+x^4} dx$$

Solution:

$$\text{Let } u = 1+x^4$$

$$du = 4x^3 dx \quad x^3 dx = \frac{du}{4}$$

$$\int \frac{x^3}{1+x^4} dx = \int \frac{du/4}{u} = \frac{1}{4} \int \frac{1}{u} du =$$

$$= \frac{1}{4} \ln|u| + C = \boxed{\frac{1}{4} \ln|1+x^4| + C}$$

$$6. \int \frac{x}{1+x^4} dx$$

$$\int \frac{x}{1+x^4} dx = \int \frac{x}{1+(x^2)^2} dx$$

Let  $u = x^2$

$$du = 2x dx \quad x dx = \frac{du}{2}$$

$$\int \frac{1}{1+x^4} x dx = \int \frac{1}{1+u^2} \frac{du}{2} =$$

$$= \frac{1}{2} \int \frac{1}{1+u^2} du = \frac{1}{2} \tan^{-1}(u) + C$$

$$= \boxed{\frac{1}{2} \tan^{-1}(x^2) + C}$$

$$7. \int \frac{1}{x^2 + 14x + 130} dx$$

Solution:  $x^2 + 14x + 130 = ( )^2 + C$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$x^2 + 14x + 130 = (x^2 + 2 \cdot x \cdot 7 + 7^2) - 49 + 130$$

$$= (x+7)^2 + 81 \rightarrow \text{completing the square}$$

$$\int \frac{1}{(x+7)^2 + 81} dx$$

Let  $u = x+7$

$$du = dx$$

$$\int \frac{1}{u^2 + 81} du = \int \frac{1}{u^2 + g^2} du =$$

$$= \frac{1}{g} \tan^{-1}\left(\frac{u}{g}\right) + C = \boxed{\frac{1}{9} \tan^{-1}\left(\frac{x+7}{9}\right) + C}$$

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$$\int \frac{\sin x \cdot \cos x}{1 + \sin^2 x} dx$$

Let  $u = \sin x \quad du = \cos x \ dx$

$$\int \frac{\sin x \cos x}{1 + \sin^2 x} dx = \int \frac{u}{1+u^2} du$$

$$\text{Let } v = 1+u^2$$

$$dv = 2u \, du$$

$$u \cdot du = \frac{dv}{2}$$

$$\int \frac{1}{v} \frac{dv}{2} = \frac{1}{2} \int \frac{1}{v} dv = \frac{1}{2} \ln|v| + C$$

$$\boxed{\frac{1}{2} \ln(1 + \sin^2 x) + C}$$

Alternative:

$$\text{Let } u = 1 + \sin^2 x \quad du = 2 \sin x \cdot \cos x \, dx$$

$$\int \frac{\sin x \cdot \cos x}{1 + \sin^2 x} = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + C$$

$$= \frac{1}{2} \ln(1 + \sin^2 x) + C$$

$$12. \int x^5 \cdot \cos(x^3) dx = \int x^2 \cdot x^3 \cdot \cos(x^3) dx$$

Solution:

$$\text{Let } u = x^3$$

$$du = 3x^2 dx$$

$$\int \underbrace{x^2 x^3 \cos(x^3)}_{u \cdot \cos u} dx = \int u \cdot \cos(u) \frac{du}{3} =$$

$$= \frac{1}{3} \int u \cdot \cos u du = \frac{1}{3} \int u \cdot (\sin' u) du =$$

$$= \frac{1}{3} [u \cdot \sin(u) - \int u' \sin u du] = \frac{1}{3} ($$

$$= \frac{1}{3} (u \cdot \sin u - \int \sin u du) = \frac{1}{3} (u \cdot \sin u$$

$$+ \cos u) + C = \frac{1}{3} (x^3 \cdot \sin x^3 + \cos(x^3)) + C$$

January 14, 2026

## Trigonometric Integrals

1. Integrals of form  $\int \sin^m x \cos^n x dx$
2. Integrals of form  $\int \tan^m x \sec^n x dx$
3. Integrals of form  $\int \sin(mx) \sin(nx) dx$   
... etc

$$1 \int \sin^m x \cdot \cos^n x dx$$

$$\int \cos^3 x dx = \int (1 - \sin^2 x) \cos x dx =$$

$$\cos^3 x = \cos x \cdot \cos^2 x = \cos x (1 - \sin^2 x)$$

$$\int \underbrace{(1 - \sin^2 x)}_{1-u^2} \underbrace{\cos x dx}_{du} = \int (1-u^2) du$$

$$1-u^2 \quad \text{let } u = \sin x$$

$$du = \cos x$$

$$\int \sin^m x \cdot \cos^n x \, dx$$

Case 1:  $n$  is odd

$$n = 2k + 1$$

$$\int \sin^m x \cdot \cos^{2k+1} x \, dx = \int \sin^m x \cdot (\cos x)^{2k} \cos x \, dx$$

$$= \int \sin^m x \cdot (1 - \sin^2 x)^k \cos x \, dx$$

case 2:  $m$  is odd ( $m = 2k + 1$ )

$$\int \sin^{2k+1} x \cdot \cos^n x \, dx = \int \sin x \cdot \sin x \cdot \cos^n x \, dx$$

$$= \int (1 - \cos^2 x)^k \cdot \cos^n x \cdot \sin x \, dx \quad u = \cos x$$

Case 3  $m \geq n$  odd, do any

Ex.

$$\int \sin^6 x \cdot \cos^5 x \, dx = \int \sin^6 x \cdot \cos^4 x \cdot \cos x \, dx$$

$$= \int \sin^6 x \cdot (\cos^2 x)^2 \cos x \, dx \quad \text{let } u = \sin x$$

$$du = \cos x \, dx$$

$$= \int \underbrace{\sin^6 x \cdot (1 - \sin^2 x)^2}_{u^6 \cdot (1-u^2)^2} \cos x \, dx$$

$$= \int u^6 (1-u)^2 \, du = \int u^6 (1-2u^2+u^4) \, du$$

$$= \int (u^6 - 2u^8 + u^{10}) \, du = \frac{u^7}{7} - 2 \frac{u^9}{9} + \frac{u^{11}}{11} + C$$

$$= \boxed{\frac{\sin^7 x}{7} - 2 \frac{\sin^9 x}{9} + \frac{\sin^{11} x}{11} + C}$$

$$\text{Ex. } \int \sin^4 x \cdot \cos^6 x \, dx = \int (\sin^2 x)^2 (\cos^2 x)^3 \, dx$$

$$= \int \left( \frac{1 - \cos(2x)}{2} \right)^2 \left( \frac{1 + \cos(2x)}{2} \right)^3 \, dx$$

Ex.

$$\int \cos^2 x \, dx$$

Solution:

$$\int \cos^2 x \, dx = \int \frac{1+\cos(2x)}{2} \, dx =$$

$$\frac{1}{2} \left( x + \int \cos(2x) \, dx \right) = \boxed{\frac{x}{2} + \frac{1}{2} \cdot \frac{\sin(2x)}{2} + C}$$

$$\int \cos(2x) = \frac{\sin(2x)}{2}$$

$$\text{Ex. } \int \cos^2 x \cdot \sin^2 x \, dx = \int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2}$$

$$= \frac{1}{4} \int (1-\cos(2x))(1+\cos(2x)) \, dx =$$

$$= \frac{1}{4} \int (1-\cos^2(2x)) \, dx = \frac{1}{4} x - \frac{1}{4} \int \cos^2(2x)$$

$$= \frac{x}{4} - \frac{1}{4} \int \frac{1+\cos(4x)}{2} \, dx$$

$$= \frac{x}{4} - \frac{x}{4} - \frac{1}{8} \int \cos(4x) dx$$

$$= \frac{x}{8} - \frac{1}{8} \cdot \frac{\sin(4x)}{4} + C = \boxed{\frac{x}{8} - \frac{\sin(4x)}{32} + C}$$

odd                    even

$$\int \tan^m x \cdot \sec^n x dx$$

1. If  $n$  is even,  $n=2k$

$$\int \tan^m x \cdot \sec^{2k} x dx = \int \tan^m x \cdot \underbrace{\sec x \cdot \sec x}_{(\sec^2 x)^{k-1}} dx$$

$$= \int \tan^m x \underbrace{(1+\tan^2 x)^{k-1}}_{du} \underbrace{\sec^2 x}_{\sec x} dx$$

in terms of  $u$

2. If  $m=2k+1$

$$\int \tan^{10} x \cdot \sec^6 x dx = \int \tan^{10} x \cdot \sec^4 x \cdot \sec^2 x$$

$$= \int \tan^{10} x (\sec^2 x)^2 \cdot \sec^2 x dx$$

$$= \int \tan^{10} x \cdot (1 + \tan^2 x)^2 \cdot \sec^2 x \, dx$$

$$\text{let } u = \tan x$$

$$du = \sec^2 x \, dx$$

$$= \int u^{10} (1+u^2)^2 \, dx = \int u^{10} (1+2u^2+u^4) \, du$$

$$= \int (u^{10} + 2u^{12} + u^{14}) \, du = \frac{u^{11}}{11} + 2 \frac{u^{13}}{13} + \frac{u^{15}}{15} + C$$

$$= \boxed{\frac{\tan^{11} x}{11} + \frac{2 \tan^{13}}{13} + \frac{\tan^{15}}{15} + C}$$

$$\int \cot^3 x \cdot \csc^3 x \, dx$$

Solution:

$$\int \cot^2 x \cdot \csc^2 x (\cot x \cdot \csc x) \, dx =$$

$$\int (\csc^2 x - 1) \csc^2 x \cot x \cdot \csc x \, dx$$

$$\text{let } u = \csc x$$

$$du = -\cot x \csc x \, dx$$

$$= - \int (u^2 - 1) u^2 \, du = - \int (u^4 - u^2) \, du =$$

$$= \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \boxed{\frac{\csc^3 x}{3} - \frac{\csc^5 x}{5} + C}$$

$$1 \quad \int \cos(7x) \cdot \cos(4x) \, dx =$$

$$= \int \frac{1}{2} (\cos(7-4x) + \cos(7x+4x)) \, dx$$

$$= \frac{1}{2} \int (\cos(3x) + \cos(11x)) \, dx$$

$$= \frac{1}{2} \left( \frac{\sin(3x)}{3} + \frac{\sin(11x)}{11} \right) + C$$

# Tutorial

January 16, 2026

$$1. \int x^2 \cdot \tan(x^3 + 1) dx$$

Solution

$$\text{Let } u = x^3 + 1$$

$$du = 3x^2 dx \rightarrow \frac{du}{3} = x^2 dx$$

$$\int \underbrace{x^2 \cdot \tan(x^3 + 1)}_{\text{circled}} dx$$

$$= \int \frac{du}{3} \cdot \tan(u) = \frac{1}{3} \int \tan(u) du =$$

$$= \frac{1}{3} \ln |\sec(u)| + C = \boxed{\frac{1}{3} \ln |\sec(x^3 + 1)| + C}$$

$$2. \int \frac{x^9}{4+x^{10}} dx$$

$$\text{Let } u = 4+x^{10}$$

$$du = 10x^9$$

$$x^9 = \frac{du}{10}$$

$$\int \frac{x^9}{4+x^{10}} = \frac{1}{10} \int \frac{1}{u} \cdot du = \frac{1}{10} \ln|u| + C$$

$$= \boxed{\frac{1}{10} \ln|4+x^{10}| + C}$$

$$3. \int \frac{x^4}{4+x^{10}} dx = \int \frac{x^4}{4+(x^5)^2} dx$$

$$\text{Let } u = x^5$$

$$du = 5x^4 dx \quad x^4 dx = \frac{du}{5}$$

$$\begin{aligned}
 &= \frac{1}{5} \int \frac{1}{4+u^2} du = \frac{1}{5} \int \frac{1}{2^2+u^2} du = \\
 &= \frac{1}{5} \cdot \frac{1}{2} \tan^{-1}\left(\frac{u}{2}\right) + C = \\
 &= \frac{1}{5} \cdot \frac{1}{2} \cdot \tan^{-1}\left(\frac{x^5}{2}\right) + C
 \end{aligned}$$

$$4. \int \frac{1}{\sqrt{21+4x-x^2}}$$

Solution

$$\begin{aligned}
 21+4x-x^2 &= 21-(x^2-4x) = \\
 &= 21 - \underbrace{(x^2-2 \cdot x \cdot 2 + 2^2)}_{(x-2)^2} - 2 = 25 - (x-2)^2
 \end{aligned}$$

$$\int \frac{1}{\sqrt{25-(x-2)^2}} dx$$

$$\begin{aligned}
 \text{Let } u &= x-2 \\
 du &= dx
 \end{aligned}$$

$$\begin{aligned}
 &= \int \frac{1}{\sqrt{25 - u^2}} du \quad \int \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \\
 &= \int \frac{1}{\sqrt{5^2 - u^2}} du = \sin^{-1}\left(\frac{u}{5}\right) + C \\
 &= \boxed{\sin^{-1}\left(\frac{x-2}{5}\right) + C}
 \end{aligned}$$

6.  $\int x^5 \sqrt{4+x^2} dx$

Solution

$$\text{Let } u = 4+x^2$$

$$du = 2x dx$$

$$\boxed{x dx = du/2}$$

$$\int x^5 \sqrt{4+x^2} dx = \int \underbrace{x \cdot (x^2)^2}_{u^2} \sqrt{4+x^2} \underbrace{dx}_{\frac{du}{2}}$$

$$= \frac{1}{2} \int (u-4)^2 \sqrt{u} \ du \rightarrow \text{brute force}$$

$$= \frac{1}{2} \int (u^2 - 8u + 16) u^{\frac{1}{2}} \ du =$$

$$= \frac{1}{2} \int (u^{\frac{5}{2}} - 8u^{\frac{3}{2}} + 16u^{\frac{1}{2}}) \ du =$$

$$= \frac{1}{2} \left( \frac{u^{\frac{7}{2}}}{\frac{7}{2}} - 8 \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{16u^{\frac{3}{2}}}{\frac{3}{2}} \right) + C =$$

$$= \boxed{\frac{(x^2+4)^{\frac{7}{2}}}{7} - 8 \frac{(x^2+4)^{\frac{5}{2}}}{5} + 16 \frac{(x^2+4)^{\frac{3}{2}}}{3} + C}$$

$$7. \int \frac{1}{1+e^{-x}} dx = \int \frac{1}{1+\frac{1}{e^x}} dx =$$

$$= \int \frac{e^x}{e^x+1} dx =$$

$$\text{let } u = e^x + 1 \\ du = e^x dx$$

$$= \int \frac{1}{u} du = \ln|u| + C = \boxed{\ln(e^x+1) + C}$$

## TRIG INTEGRALS

9.  $\int \tan^3 x \cdot \sec^1 x \ dx =$

$\overset{3}{\circlearrowleft}$   
odd ✓

$$= \int \tan^2 x \cdot \tan x \cdot \sec x \ dx =$$

$$= \int (\sec^2 - 1) \tan x \cdot \sec x \ dx$$

Let  $u = \sec x$

$$du = \tan x \cdot \sec x \ dx$$

$$= \int (u^2 - 1) du = \frac{u^3}{3} - u + C =$$

$$\boxed{\frac{\sec^3}{3} - \sec x + C}$$

$$\begin{aligned}
 \int \cos^4 x \, dx &= \int (\cos^2 x)^2 \, dx \\
 &= \int \left( \frac{1 + \cos(2x)}{2} \right)^2 \, dx \\
 &= \frac{1}{4} \int (1 + \cos(2x))^2 \, dx = \frac{1}{4} \int 1 + 2 \cdot \cos(2x) + \cos^2(2x) \, dx \\
 &= \frac{x}{4} + \frac{1}{2} \int \cos(2x) \, dx + \frac{1}{4} \int \cos^2(2x) \, dx \\
 &= \frac{x}{4} + \frac{1}{2} \frac{\sin(2x)}{2} + \frac{1}{4} \int \frac{1 + \cos(4x)}{2} \, dx \\
 &= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{1}{8} \int 1 + \cos(4x) \, dx = \\
 &= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{x}{8} + \frac{1}{8} \frac{\sin(4x)}{4} + C \\
 &= \boxed{\frac{3x}{8} + \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C}
 \end{aligned}$$

January 19, 2026

## Trigonometric Substitution

$$\int x^2 \sqrt{16 - x^2} dx = \int 16 \sin^2 \theta \sqrt{16 - 16 \sin^2 \theta} \cdot 4 \cos \theta d\theta$$

$\Downarrow$

$$= 4 \sqrt{1 - \sin^2 \theta} = 4 \sqrt{\cos^2 \theta}$$

$$x = 4 \sin \theta$$

$$= 4 \cos \theta$$

$$dx = 4 \cos \theta d\theta$$

$$= 256 \int \underline{\sin^2 \theta \cos^2 \theta} d\theta$$

trig integral

1. For  $\sqrt{a^2 - x^2}$

$x = a \cdot \sin \theta$ , choose  $\theta$  such that  
this happens

Is there even such  $\theta$ ?

$$a^2 - x^2 \geq 0 \quad x^2 \leq a^2 \quad -a \leq x \leq a$$

yes, because  $-a \leq x \leq a$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = a \sqrt{1 - \sin^2 \theta}$$
$$= a \sqrt{\cos^2 \theta} = a |\cos \theta| = a \cdot \cos \theta$$

2.

For  $\sqrt{a^2 + x^2}$  no restrictions

$$\text{let } x = a \cdot \tan \theta$$

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2 \tan^2 \theta} = a \cdot \sqrt{\sec^2 \theta}$$
$$= a |\sec \theta| = a \cdot \sec \theta$$

3.

$$\sqrt{x^2 - a^2}$$

$$x^2 - a^2 \geq 0$$

$$x^2 \geq a^2$$

$$\underline{x \geq a}$$

$$\text{or } \underline{x \leq -a}$$

Ex 1.

$$\int x \sqrt{16-x^2} dx = \int x \sqrt{4^2 - x^2} dx$$

let  $x = 4 \sin \theta \quad (-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$

$$dx = 4 \cos \theta d\theta$$

Then  $\sqrt{16-x^2} = \sqrt{16-16 \sin^2 \theta} =$

$$= \sqrt{16(1-\sin^2 \theta)} = 4 \sqrt{\cos^2 \theta} = 4 |\cos \theta|$$

$$= 4 \cos \theta$$

because  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\int x \sqrt{16-x^2} dx = 4 \int 4 \sin \theta \cdot 4 \cos \theta \cdot 4 \cos \theta$$

$$= 64 \int \sin \theta \cos^2 \theta d\theta$$

let  $u = \cos \theta$

$$du = -\sin \theta d\theta$$

$$= 64 \int \cos^2 \theta \underbrace{\frac{\sin \theta}{-\cos \theta}}_{-d\theta} d\theta$$

$$= -64 \int u^2 du = \boxed{-\frac{64}{3} u^3 + C}$$

$$= -\frac{64}{3} \cdot \cos^3 \underline{\theta} + C$$

$$\sin \theta = \frac{x}{4}$$

$$\theta = \sin^{-1} \left( \frac{x}{4} \right)$$

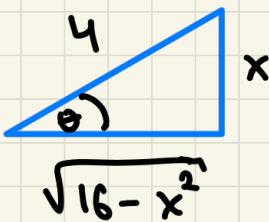
because  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$= \boxed{-\frac{64}{3} \cos^3 \left( \sin^{-1} \left( \frac{x}{4} \right) \right) + C}$$

If  $x = 4 \sin \theta$ ,  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

compute  $\cos \theta$  in terms of  $x$

$$\sin \theta = \frac{x}{4}$$



$$\cos \theta = \frac{\sqrt{16 - x^2}}{4}$$

**Conclusion**

$$\begin{aligned} \int x \sqrt{16 - x^2} dx &= -\frac{64}{3} \cos^3 \theta + C \\ &= -\frac{64}{3} \left( \frac{\sqrt{16 - x^2}}{4} \right)^3 + C \\ &= \boxed{-\frac{(\sqrt{16 - x^2})^3}{3} + C} \end{aligned}$$

2.

$$\int \frac{x^3}{\sqrt{x^2 + 9}} dx$$

$x^2 + 3^2$

Solution

$$\text{let } x = 3 \tan \theta$$

$$dx = 3 \sec^2 \theta d\theta$$

$$\text{Then } \sqrt{x^2 + 9} = \sqrt{9 \tan^2 \theta + 9} = 3 \sqrt{\tan^2 \theta + 1}$$

$$= 3 \sqrt{\sec^2 \theta} = 3 |\sec \theta| = 3 \sec \theta$$

$$\int \frac{x^3}{\sqrt{x^2 + 9}} = \int \frac{27 \tan^3 \theta}{3 \sec \theta} \cdot 3 \sec^2 \theta d\theta$$

$$= 27 \int \tan^3 \theta \sec \theta d\theta$$

$$= 27 \int \tan^3 \theta \cdot \sec \theta \, d\theta$$

$$= 27 \int \tan^2 \theta \cdot \tan \theta \cdot \sec \theta \, d\theta$$

$$= 27 \int (\sec^2 - 1) \tan \theta \cdot \sec \theta \, d\theta$$

$$\text{let } u = \sec \theta$$

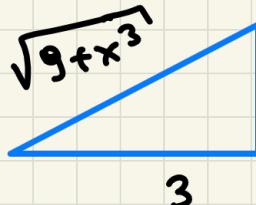
$$du = \tan \theta \sec \theta \, d\theta$$

$$= 27 \int (u^2 - 1) \, du = 27 \left( \frac{u^3}{3} - u \right) + C$$

$$= 9u^3 - 27u + C$$

$$= 9 \sec^3 \theta - 27 \sec \theta + C$$

$$x = 3 \tan \theta$$



$$\tan \theta = \frac{x}{3}$$

$$\sec \theta = \frac{\sqrt{9+x^2}}{3}$$

$$\int \frac{x^3}{\sqrt{x^2+9}} dx = 9 \sec^3 \theta - 27 \sec \theta + C$$

$$= 9 \left( \frac{\sqrt{9+x^2}}{3} \right)^3 - 27 \frac{\sqrt{9+x^2}}{3} + C$$

$$= \left( \frac{\sqrt{9+x^2}}{3} \right)^3 - 9 \sqrt{9+x^2} + C$$

$$3. \int \frac{\sqrt{x^2-1}}{x^6} dx$$

Solution

$$x = \sec \theta$$

$$dx = \tan \theta \sec \theta d\theta$$

$$\sqrt{x^2-1} = \sqrt{\sec^2 \theta - 1} = \sqrt{\tan^2 \theta} = \tan \theta$$

Then  $\int \frac{\sqrt{x^2-1}}{x^6} dx = \int \frac{\tan \theta}{\sec^6 \theta} \tan \theta \sec \theta d\theta$

$$= \int \frac{\tan^2 \theta}{\sec^5 \theta} d\theta = \int \frac{\frac{\sin^2 \theta}{\cos^2 \theta}}{\frac{1}{\cos^5 \theta}} d\theta =$$

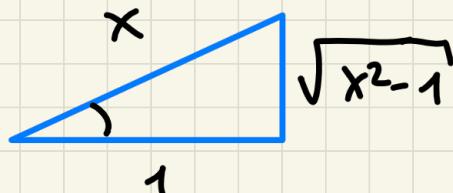
$$= \int \cos^3 \theta \cdot \sin^2 \theta d\theta = \int \cos^2 \theta \cdot \sin^2 \theta \cos \theta d\theta$$

$$= \int (1 - \sin^2 \theta) \sin^2 \theta \cos \theta d\theta =$$

$$= \int (1 - u^2) u^2 du = \int (u^2 - u^4) du =$$

$$\frac{u^3}{3} - \frac{u^5}{5} + C = \boxed{\frac{\sin^3 \theta}{3} - \frac{\sin^5 \theta}{5} + C}$$

$$\sec \theta = \frac{x}{1}$$



$$\sin \theta = \frac{\sqrt{x^2 - 1}}{x}$$

Then:

$$\int \frac{\sqrt{x^2-1}}{x^6} dx = \frac{\sin^3 \theta}{3} - \frac{\sin^5 \theta}{5} + C$$

$$= \boxed{\frac{1}{3} \left( \frac{\sqrt{x^2+1}}{x} \right)^3 - \frac{1}{5} \left( \frac{\sqrt{x^2-1}}{x} \right)^5 + C}$$

January 21, 2026

## Integrals of Rational Functions using Partial Fraction Decomposition

Polynomials:  $P(x) = a_0 + a_1 x + \dots + a_n x^n$

Rational functions:  $f(x) = \frac{P(x)}{Q(x)}$

where  $P(x)$ ,  $Q(x)$  are polynomials

$f(x) = \frac{P(x)}{Q(x)}$  is a "proper" rational function if  $\deg P(x) < \deg Q(x)$

Ex

$\frac{2x+7}{3x^2-2x+2}$  is proper

$\frac{2}{4x+2x+7} \rightarrow$  bigger than  
 $\frac{4x+2x+7}{3x^2-2x+2}$  is not proper

$$\int \frac{P(x)}{Q(x)} dx$$

Step # 0 : boil down to a proper rational function

Division with remainder for nums:  
If  $a, b > 0$  integers, there exist  $q, r$  integers such that  $a = bq + r$   $0 \leq r < b$

If  $P(x)$  and  $Q(x)$  are polynomials  
there exist  $S(x)$  and  $R(x)$   
such that:

$$P(x) = Q(x) S(x) + R(x)$$

$$\int \frac{P(x)}{Q(x)} dx = \int \frac{Q(x)S(x) + R(x)}{Q(x)} dx =$$

proper rat. funct.

$$\int \left( S(x) + \frac{R(x)}{Q(x)} \right) dx = \int S(x) dx + \int \frac{R(x)}{Q(x)} dx$$

Ex.

$$\int \frac{2x^3 - 5x^2 + 5x + 3}{x-1} dx$$

Solution.

$$\begin{array}{r} \cdot 2x^2 - 3x + 2 \\ x-1 / \underline{- 2x^3 - 5x^2 + 5x + 3} \\ \underline{2x^3 - 2x^2} \\ - \quad \underline{- 3x^2 + 5x + 3} \\ \underline{- 3x^2 + 3x} \\ \hline - \quad \underline{\underline{2x + 3}} \\ - \quad \underline{\underline{2x - 2}} \\ \hline \end{array}$$

$$\begin{array}{l} P(x) \\ 2x^3 - 5x^2 + 5x + 3 = \\ Q(x) \quad S(x) \quad R(x) \\ = (x-1)(2x^2 - 3x + 2) + 5 \end{array}$$

$$= \int \frac{(x-1)(2x^2 - 3x + 2) + 5}{x-1} dx =$$

$$= \int \left( 2x^2 - 3x + 2 + \frac{5}{x-1} \right) dx =$$

$$= \boxed{\frac{2x^3}{3} - \frac{3x^2}{2} + 2x + 5 \ln|x-1| + C}$$

Raised Goal :  $\int \frac{P(x)}{Q(x)} dx$  if  $\frac{P(x)}{Q(x)}$  proper polyn.

Case 1 :  $Q(x)$  is a product of linear functions

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\dots(a_kx+b_k)$$

There exists constants  $A_1, A_2, A_3, \dots, A_k$ ,

$\dots A_k$  such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \dots + \frac{A_k}{a_kx+b_k}$$

Ex.  $\frac{P(x)}{Q(x)} = \frac{\text{smth of degree zero or less}}{(3x-7)(5x+2)(2x+1)} =$

$$\frac{A}{3x-7} + \frac{B}{5x+2} + \frac{C}{2x+1}$$

Ex 1

$$\int \frac{x}{x^2+x-2} dx$$

Sol.

$$\text{Factor } x^2 + x - 2 = (x+2)(x-1)$$

The form of the PFD is:

$$\frac{x}{x^2 + x - 2} = \frac{A}{x+2} + \frac{B}{x-1}$$

Find A, B

$$\frac{x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} \quad | \cdot (x+2)(x-1)$$

$$x = A(x-1) + B(x+2)$$

$$x = Ax - A + Bx + 2B$$

$$x = (A+B)x + (2B-A)$$

$$\underline{1 \cdot x + 0} = x = \underline{(A+B)x} + \underline{(2B-A)}$$

$$\begin{cases} A+B = 1 \\ 2B-A = 0 \end{cases} \quad A = 2B$$

$$2B+B = 1 \quad 3B=1 \quad B = \frac{1}{3}$$

$$A = \frac{2}{3}$$

The P.F.D is :

$$\frac{x}{x^2-x-2} = \frac{2/3}{x+2} + \frac{1/3}{x-1}$$

$$\int \frac{x}{x^2+x-2} dx = \int \left( \frac{2/3}{x+2} + \frac{1/3}{x-1} \right) dx$$

$$= \frac{2}{3} \int \frac{1}{x+2} dx + \frac{1}{3} \int \frac{1}{x-1} dx$$

$$= \boxed{\frac{2}{3} \ln|x+2| + \frac{1}{3} \ln|x-1| + C}$$

Shortcut to find A and B

$$x = A(x-1) + B(x+2)$$

$$\text{take } x=1 \quad 1 = A \cdot 0 + B \cdot 3 \Rightarrow B = \frac{1}{3}$$

$$\text{take } x=-2 \quad -2 = A(-2-1) + B \cdot 0 \Rightarrow A = -\frac{2}{3}$$

!!! okay but doesn't always work !!!

Case 2.

Example:

$$\frac{P(x)}{Q(x)} = \frac{\dots\dots}{(5x+1)(2x+2)^3(x-3)^2}$$

$$\frac{A}{(5x+1)} + \frac{B}{(2x+7)} + \frac{C}{(2x+7)^2} + \frac{D}{(2x+7)^3} + \frac{E}{x-3} + \frac{F}{(x-3)^2}$$

$$(Q_i x + b_i)^r$$

$$\frac{B_1}{a_i x + b_i} + \frac{B_2}{(a_i x + b_i)^2} + \dots + \frac{B_r}{(a_i x + b_i)^r}$$

Case 3p

$$\frac{P(x)}{Q(x)} = \dots \frac{\dots}{(5x+1)(2x+7)^3(x-3)^2(x^4+2x+7)(2x^2+3x+19)^3}$$

$$\frac{A}{(5x+1)} + \frac{B}{(2x+7)} + \frac{C}{(2x+7)^2} + \frac{D}{(2x+7)^3} + \frac{E}{x-3} + \frac{F}{(x-3)^2}$$

$$+ \frac{6x+11}{x^2+2x+7} + \frac{Ix+J}{2x^2+3x+19} + \frac{Kx+L}{(2x^2+3x+19)^2} + \frac{Mx+N}{(2x^2+3x+19)^3}$$

$$Ex \int \frac{x+1}{x^4+x^2} dx$$

Sol.

$$\text{Factor } x^4 + x^2 = \underline{x^2} (\underline{x^2 + 1})$$

Form of the P.F.D is

$$\frac{x+1}{x^4+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} \quad | \quad x^2(x^2+1)$$

Find A, B, C, D

$$x+1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$$

$$x+1 = A(x^3+x) + B(x^2+1) + (Cx^3+Dx^2)$$

$$x+1 = \underline{Ax^3} + \underline{Ax} + \underline{Bx^2} + \underline{B} + \underline{Cx^3} + \underline{Dx^2}$$

$$x+1 = (A+C)x^3 + (B+D)x^2 + Ax + B$$

$$0 \cdot x^3 + 0 \cdot x^2 + 1 \cdot x + 1 = \underline{(A+C)x^3} + \underline{(B+D)x^2} + \underline{Ax} + \underline{B}$$

$$\begin{cases} A+C=0 \\ B+D=0 \\ A=1 \\ B=1 \end{cases} \quad \begin{matrix} C=-1 \\ D=-1 \end{matrix}$$

$$\frac{x+1}{x^4+x^2} = \frac{1}{x} + \frac{1}{x^2} - \frac{x+1}{x^2+1}$$

$$\int \frac{1}{x} + \frac{1}{x^2} - \frac{x+1}{x^2+1} dx =$$

$$= \ln|x| - \frac{1}{x} - \int \frac{x+1}{x^2+1}$$

$$= \int \frac{x}{x^2+1} dx + \underbrace{\int \frac{1}{x^2+1}}_{\tan^{-1} x}$$

$$\text{let } u = x^2+1$$

$$du = 2x dx$$

$$\frac{1}{2} \int \frac{1}{u} du + \tan^{-1} x + C = \frac{1}{2} \ln(x^2+1) + \tan^{-1}$$

$$= \boxed{\ln|x| - \frac{1}{x} - \frac{1}{2} \ln(x^2+1) - \tan^{-1} x + C}$$