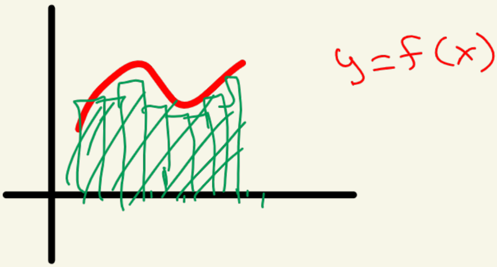
Jan 5, 2026

Part 1: Integrals

Part 2: Application of Integrals

Part 3: Sequences and Series

Definite Integrals



If $f(x) > 0$

$$\int_a^b f(x)\,dx = \text{area below the graph}$$

Conseptual Intepretations = continuosly accumulation of quantity.

Indefinite Integrals $\int f(x)$

essentialy antiderivitives

$x^2 \to$ antider. $\frac{x^3}{3}$ so

the most general antider.

is $\frac{x^3}{3} + C$.

$\int f(x)\,dx = F(x) + C$ means

that most general antider.

of $f(x)$

$F'(x) = f(x)$ $\quad \int_0^1 x^2\,dx = \frac{1}{3}$

$\int x^2\,dx = \frac{x^3}{x} + c$

FTC: Foundamental Theory of Calculus

FTC: $\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}$

this always works

January 7, 2026

$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$\int (f'(x)g(x) + f(x)g'(x))\,dx = f(x)+g(x)+C$

$\int f'(x)+g(x)\,dx + \int f(x)g'(x) = f(x)g(x)+C$

$\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x)\,dx$

Integration by parts

$\int x^2\,dx - 7 = \frac{x^3}{3} + (C-7) = \int x^2\,dx$

1)

$\int x \cdot \cos x\,dx$ Integration by parts

Solution:

$$\int x \cdot \boxed{\cos x}\, dx = \int x \cdot \boxed{(\sin x)'}\, dx =$$

$$= x \cdot \sin x - \int x' \sin x \, dx = x \cdot \sin x -$$

$$- \int \sin x \, dx = x \cdot \sin x + \cos x + C$$

2)

$$\int 2^x \cdot x \, dx = \int \left(\frac{2^x}{\ln 2}\right)' \cdot x \, dx =$$

$$= \frac{2^x}{\ln 2} \cdot x - \int \frac{2^x}{\ln 2} \boxed{\overset{1}{\underset{x'}{\uparrow}}} \, dx =$$

$$= \frac{2^x \cdot x}{\ln 2} - \frac{1}{\ln 2} \int 2^x \, dx \nearrow \frac{2^x}{\ln 2}$$

$$= \boxed{\frac{2^x \cdot x}{\ln 2} - \frac{2^x}{(\ln 2)^2}}$$

3) $\int x^2 e^x \, dx = \int x^2 \cdot (e^x)' \, dx =$

$x^2 e^x - \int (x^2)' \cdot e^x \, dx = x^2 e^x - 2\int$

$-2\int \underbrace{x \cdot e^x} \, dx = x^2 e^x - 2\int x \cdot (e^x)' \, dx$

$= x^2 \cdot e^x - 2\left(x \cdot e^x - \int \overset{1}{\boxed{x'}} e^x \, dx \right) =$

$= x^2 \cdot e^x - 2 \cdot x \cdot e^x + 2 \boxed{\int e^x \, dx} \; e^x$

$= \boxed{x^2 e^x - 2x \cdot e^x + 2 \cdot e^x + C}$

4) $\int \ln x \, dx = \int \overset{x'}{\boxed{1}} \cdot \ln x \, dx =$

$= \int x' \ln x \, dx = x \cdot \ln x - \int x \, (\ln x)' \, dx$

$= x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx = x \cdot \ln x - \int 1 \, dx$

$= \boxed{x \cdot \ln x - x + C}$

5) $\int \cos^2 x \, dx = \int \cos x \cdot (\sin x)' \, dx$

$= \cos x \cdot \sin x - \int (\cos x)' \sin x \, dx$

$= \cos x \cdot \sin x + \boxed{\int \sin^2 x \, dx}$

$= \cos x \cdot \sin x + \int (1 - \cos^2 x) \, dx$

$= \cos x \cdot \sin x + \int 1 \, dx - \int \cos^2 x \, dx$

$= \cos x \cdot \sin x + x - \int \cos^2 x \, dx$

Solve for $\int \cos^2 x \, dx$:

$$\boxed{2\int \cos^2 x \, dx = \frac{1}{2}(\cos x \sin x + x) + C}$$

$u = x^2$ $\qquad$ $u = g(x)$

$\frac{du}{dx} = 2x$ $\qquad$ $g'(x)$

$du = 2x\,dx$ $\qquad$ $g'(x)\,dx$

$$\int \underbrace{f(x)}_{u}\underbrace{g'(x)}_{dv}\,dx = \underbrace{f(x)}_{u}\underbrace{g(x)}_{v} - \int \underbrace{g(x)}_{v}\underbrace{f'(x)}_{du}$$

$$\int u\,dv = uv - \int v \cdot du$$

1)

$$\int \underbrace{x}_{u} \cdot \underbrace{\cos x \; dx}_{dv} :$$

$u = x$ $\qquad$ $v = \sin x$

$dv = \cos x \cdot dx$

$du = dx$

$$\int \underbrace{x}_{u} \cdot \underbrace{\cos x \, dx}_{dv} = \int u \, dv = uv - \int v \, du$$

$$= x \cdot \sin x - \int \sin x \, dx = x \cdot \sin x - \cos x + C$$

January 12, 2026

Substitution Rule (aka "u-sub")

$$\int (f(x) + g(x))dx = \int f(x) \, dx + \int g(x) \, dx$$

$\int f(x) g(x) \, dx \rightarrow$ integration by parts

$$f(x) g(x) - \int_* f'(x) g(x) \, dx$$

* often serves the purpose

$\int f(g(x)) \, dx$ $\times$ $\rightarrow$ no such thing

$\int f(g(x)) g'(x) \, dx \rightarrow$ there is a formula for this, subsitution rule

$$\int f(g(x))g'(x)\,dx = \int \underbrace{F'(g(x))g'(x)}_{(F(g(x))'}\,dx$$

$$= \int (F(g(x))'\,dx = F(g(x)) + C \quad \text{Let } u = g(x)$$

$$= F(u) + C = \int f(u)\,du \checkmark$$

$$\text{let } u = x^2$$

$$\frac{du}{dx} = 2x \qquad du = 2x\,dx$$

$$u = g(x)$$

$$du = g'(x)\,dx$$

$$\frac{du}{dx} = g'(x)$$

$$\int f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)}_{du}\,dx = \int f(u)\,du$$

Practice

1.

$$\int x \cdot \cos(x^2)\ dx$$

let $u = x^2$

$du = 2x\ dx \rightarrow \frac{du}{2}$

$$\int x \cdot \cos(x^2)\,dx = \int \cos(u) \cdot \boxed{\frac{du}{2}} =$$

$$\frac{1}{2}\int \cos(u)\ du = \frac{1}{2}\sin(u) + C =$$

$$\frac{1}{2}\sin(x^2) + C$$

2.

$$\int x^2 \sqrt[3]{4x^3+3}\ dx$$

let $u = 4x^3+3$

$du = 12x^2\ dx \mid x^2 dx = \frac{du}{12}$

$$\int x^2 \sqrt[3]{4x^3+3} = \int \sqrt[3]{u}\ \frac{du}{12} = \frac{1}{2}\int \sqrt[3]{u}\ du$$

$$= \frac{1}{12}\int u^{\frac{1}{3}}\,du = \frac{1}{12}\,\frac{u^{\frac{4}{3}}}{4/3} + C =$$

$$= \frac{u^{\frac{4}{3}}}{16} + C = \boxed{\frac{(4x^3+3)^{\frac{4}{3}}}{16} + C}$$

3.

$$\int \sin(10x)\,dx$$

let $u = 10x$

$du = 10\,dx$

$dx = \frac{du}{10}$

$$\int \sin(10x)\,dx = \int \sin(u)\frac{du}{10} = \frac{1}{10}\int \sin(u)\,du$$

$$= -\frac{1}{10}\cos(u) + C = \boxed{-\frac{1}{10}\cos(10x) + C}$$

Useful shortent :

If $\int f(x)\,dx = F(x) + C$, then

- $\int f(x+a)\,dx = F(x+a) + C$

$$\int \cos(x+7)\,dx = \sin(x+7) + C$$

$$\int f(a\cdot x)\,dx = \frac{F(a\cdot x)}{a} + C$$

$$\int \cos(7\cdot x)\,dx = \frac{\sin(7x)}{7} + C$$

5. $\int \frac{x^3}{1+x^4}\,dx$

Solution:

Let $u = 1+x^4$

$du = 4x^3\,dx \quad x^3 dx = \frac{du}{4}$

$$\int \frac{x^3}{1+x^4}\,dx = \int \frac{du/4}{u} = \frac{1}{4}\int \frac{1}{u}\,du =$$

$$= \frac{1}{4}\ln|x| + C = \boxed{\frac{1}{4}\ln|1+x^4| + C}$$

6. $\int \frac{x}{1+x^4}\,dx$

$$\int \frac{x}{1+x^4}\,dx = \int \frac{x}{1+(x^2)^2}\,dx$$

Let $u = x^2$

$du = 2x\,dx \qquad x\,dx = \frac{du}{2}$

$$\int \frac{1}{1+x^4}\,x\,dx = \int \frac{1}{1+u^2}\,\frac{du}{2} =$$

$$= \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}(u)+c$$

$$= \boxed{\frac{1}{2}\tan^{-1}(x^2)+c}$$

7. $\displaystyle\int \frac{1}{x^2+14x+130}\,dx$

Solution: $x^2+14x+130 = (\quad)^2+c$

$(a+b)^2 = a^2+2ab+b^2$

$$x^2+14x+130=(x^2+2\cdot x\cdot 7+7^2)-7^2+130$$

$$=(x+7)^2+81 \rightarrow$$ completing the square

$$\int \frac{1}{(x+7)^2+81}\,dx$$

Let $u=x+7$

$du=dx$

$$\int \frac{1}{u^2+81}\,du=\int \frac{1}{u^2+9^2}\,du=$$

$$=\frac{1}{9}\tan^{-1}\left(\frac{u}{9}\right)+C=\boxed{\frac{1}{9}\tan^{-1}\left(\frac{x+7}{9}\right)+C}$$

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$$\int \frac{\sin x\cdot\cos x}{1+\sin^2 x}\,dx$$

Let $u=\sin x$ $\quad du=\cos x\,dx$

$$\int \frac{\sin x \cos x}{1+\sin^2 x}\, d = \int \frac{u}{1+u^2}\, du$$

Let $v = 1+u^2$

$dv = 2u \; du$

$u \cdot du = \frac{dv}{2}$

$$\int \frac{1}{v}\,\frac{dv}{2} = \frac{1}{2}\int \frac{1}{v}\, dv = \frac{1}{2}\ln|v| + c$$

$$\boxed{\frac{1}{2}\ln(1+\sin^2 x) + C}$$

Alternative:

Let $u = 1+\sin^2 x$ $\quad du = 2\sin \cdot \cos x$

$$\int \frac{\sin x \cdot \cos x}{1+\sin^2 x} = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| + c$$

$$= \frac{1}{2}\ln(1+\sin^2 x) + C$$

12. $\int x^5 \cdot \cos(x^3) = \int x^2 \cdot x^3 \cdot \cos(x^3)\, dx$

Solution: Let $u = x^3$

$du = 3x^2\, dx$

$$\int x^2 \underbrace{x^3 \cos(x^3)}_{u \cdot \cos u} dx = \int u \cdot \cos(u) \frac{du}{3} =$$

$$= \frac{1}{3} \int u \cdot \cos u \quad du = \frac{1}{3} \int u \cdot (\sin' u)\, du =$$

$$= \frac{1}{3} \left[u \cdot \sin(u) - \int u' \sin u\, du \right] = \frac{1}{3} ($$

$$= \frac{1}{3} \left(u \cdot \sin u - \int \sin u\, du \right) = \frac{1}{3} (u \cdot \sin u$$

$$+ \cos u) + C = \frac{1}{3} (x^3 \cdot \sin x^3 + \cos(x^3) + C)$$

January 14, 2026

Trigonometric Integrals

1. Integrals of form $\int \sin^m x \cos^n x \, dx$
2. Integrals of form $\int \tan^m x \sec^n x \, dx$
3. Integrals of form $\int \sin(mx) \sin(nx) \, dx$ .... etc

1 $\int \sin^m x \cdot \cos^n x \, dx$

$$\int \cos^3 x \, dx = \int (1-\sin^2 x) \cos x \, dx =$$

$$\cos^3 x = \cos x \cdot \cos^2 x = \cos x (1-\sin^2 x)$$

$$\int \underbrace{(1-\sin^2 x)}_{1-u^2} \underbrace{\cos x \, dx}_{du} = \int (1-u^2) \, du$$

let $u = \sin x$

$du = \cos x$

$$\int \sin^m x \cdot \cos^n x \, dx$$

Case 1: n is odd

$n = 2k+1$

$$\int \sin^m x \cdot \cos^{2k+1} x \, dx = \int \sin^m x \cdot (\cos^2 x)^k \cdot \cos x$$

$$= \int \sin^m x \cdot (1-\sin^2 x)^k \cos x \, dx$$

case 2: m is odd $(m = 2k+1)$

$$\int \sin^{2k+1} x \cdot \cos^n dx = \int \sin^{2k} x \cdot \sin x \cdot \cos^n x$$

$$= \int (1-\cos^2 x)^k \cdot \cos^n x \cdot \sin x \, dx$$

$u = \cos x$

Case 3 m & n odd, do any

Ex.

$$\int \sin^6 x \cdot \cos^5 x \, dx = \int \sin^6 x \cdot \cos^4 x \cdot \cos x \, dx$$

$$= \int \sin^6 x \cdot (\cos^2 x)^2 \cos x \, dx$$

let $u = \sin x$

$du = \cos x \, dx$

$$= \int \underbrace{\sin^6 x \cdot (1 - \sin^2 x)^2}_{u^6 \cdot (1 - u^2)^2} \underbrace{\cos x}_{du} \, dx$$

$$= \int u^6 (1-u)^2 \, du = \int u^6 (1 - 2u^2 + u^4) \, du$$

$$= \int (u^6 - 2u^8 + u^{10}) \, du = \frac{u^7}{7} - 2\frac{u^9}{9} + \frac{u^{11}}{11} + C$$

$$= \boxed{\frac{\sin^7 x}{7} - \frac{2\sin^9 x}{9} + \frac{\sin^{11}}{11} + C}$$

Ex. $\int \sin^4 x \cdot \cos^6 x \, dx = \int (\sin^2 x)^2 (\cos^2 x)^3 dx$

$$= \int \left(\frac{1 - \cos(2x)}{2} \right)^2 \left(\frac{1 + \cos(2x)}{2} \right)^3 dx$$

Ex.

$$\int \cos^2 x \, dx$$

Solution:

$$\int \cos^2 x \, dx = \int \frac{1+\cos(2x)}{2} \, dx =$$

$$\frac{1}{2}\left(x + \int \cos(2x)\, dx\right) = \boxed{\frac{x}{2} + \frac{1}{2} \cdot \frac{\sin(2x)}{2} + C}$$

$$\int \cos(2x) = \frac{\sin(2x)}{2}$$

Ex. $$\int \cos^2 x \cdot \sin^2 x \, dx = \int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2}$$

$$= \frac{1}{4}\int (1-\cos(2x))(1+\cos(2x)) \, dx =$$

$$= \frac{1}{4}\int (1-\cos^2(2x)) \, dx = \frac{1}{4}x - \frac{1}{4}\int \cos^2(2x)$$

$$= \frac{x}{4} - \frac{1}{4}\int \frac{1+\cos(4x)}{2} \, dx$$

$$= \frac{x}{4} - \frac{x}{4} - \frac{1}{8}\int \cos(4x)\,dx$$

$$= \frac{x}{8} - \frac{1}{8}\cdot\frac{\sin(4x)}{4} + C = \boxed{\frac{x}{8} - \frac{\sin(4x)}{32} + C}$$

$$\int \tan^{\overset{\text{odd}}{m}} x \cdot \sec^{\overset{\text{even}}{n}} x \; dx$$

1. If n is even, $n = 2k$

$$\int \tan^m x \cdot \sec^{2k} x\,dx = \int \tan^m x \cdot \underbrace{\sec^{2k-2} x}_{(\sec^2 x)^{k-1}} \cdot \sec^2 x$$

$$= \int \underbrace{\tan^m x\,(1+\tan^2 x)^{k-1}}_{\text{interms of } u} \underbrace{\sec^2 x}_{du} \; dx$$

2. If $m = 2k+1$

$$\int \tan^{10} x \cdot \sec^6 x\,dx = \int \tan^{10} x \cdot \sec^4 x \cdot \sec^2 x$$

$$= \int \tan^{10} x\,(\sec^2 x)^2 \cdot \sec^2 x \; dx$$

$$= \int \tan^{10} x \,(1+\tan^2 x)^2 \cdot \sec^2 x \; dx$$

let $u = \tan x$

$du = \sec^2 x \; dx$

$$= \int u^{10}(1+u^2)^2 \; dx = \int u^{10}(1+2u^2+u^4)\,du$$

$$= \int (u^{10}+2u^{12}+u^{14})\,du = \frac{u^{11}}{11} + 2\,\frac{u^{13}}{13} + \frac{u^{15}}{15} + C$$

$$= \boxed{\frac{\tan^{11} x}{11} + \frac{2\tan^{13}}{13} + \frac{\tan^{15}}{15} + C}$$

$$\int \cot^3 x \cdot \csc^3 x \; dx$$

Solution:

$$\int \cot^2 x \cdot \csc^2 x\,(\cot x \cdot \csc x)\; dx =$$

$$\int (\csc^2 x - 1)\csc^2 x \quad \cot x \cdot \csc x \, dx$$

let $u = \csc x$

$du = -\cot x \csc x \, dx$

$$= -\int (u^2-1)\, u^2 \, du = -\int (u^4 - u^2)^2 \, du =$$

$$= \int (u^2 - u^4)\, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \boxed{\frac{\csc^3 x}{3} - \frac{\csc^5 x}{5} + C}$$

1 $\displaystyle\int \cos(7x)\cdot\cos(4x)\, dx =$

$$= \int \frac{1}{2}\left(\cos(7-4x) + \cos(7x+4x)\right) dx$$

$$= \frac{1}{2}\int \left(\cos(3x) + \cos(11x)\right) dx$$

$$= \boxed{\frac{1}{2}\left(\frac{\sin(3x)}{3} + \frac{\sin(11x)}{11}\right) + C}$$

Tutorial

$\boxed{\text{January } 16, 2026}$

1. $\displaystyle\int x^2 \cdot \tan(x^3+1)\,dx$

Solution

Let $u = x^3 + 1$

$du = 3x^2\,dx \rightarrow \frac{du}{3} = x^2\,dx$

$$\int x^2 \cdot \tan(x^3+1)\,dx$$

$$= \int \frac{du}{3} \cdot \tan(u) = \frac{1}{3}\int \tan(u)\,du =$$

$$= \frac{1}{3}\ln|\sec(u)| + C = \boxed{\frac{1}{3}\ln|\sec(x^3+1)| + C}$$

2. $\int \frac{x^9}{4+x^{10}}\,dx$

Let $u = 4+x^{10}$

$du = 10x^9$

$x^9 = \frac{du}{10}$

$$\int \frac{x^9}{4+x^{10}} = \frac{1}{10}\int \frac{1}{u}\cdot du = \frac{1}{10}\ln|u| + C$$

$$= \boxed{\frac{1}{10}\ln|4+x^{10}| + C}$$

3. $\int \frac{x^4}{4+x^{10}}\,dx = \int \frac{x^4}{4+(x^5)^2}\,dx$

Let $u = x^5$

$du = 5x^4\,dx \qquad x^4\,dx = \frac{du}{5}$

$$= \frac{1}{5}\int \frac{1}{4+u^2}\,du = \frac{1}{5}\int \frac{1}{2^2+u^2}\,du =$$

$$= \frac{1}{5}\cdot\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C =$$

$$= \frac{1}{5}\cdot\frac{1}{2}\cdot\tan^{-1}\left(\frac{x^5}{2}\right) + C$$

4. $$\int \frac{1}{\sqrt{21+4x-x^2}}$$

Solution

$$21+4x-x^2 = 21-(x^2-4x) =$$

$$= 21 - (\underbrace{x^2 - 2\cdot x\cdot 2 + 2^2}_{(x-2)^2} - 2^2 = 25-(x-2)^2$$

$$\int \frac{1}{\sqrt{25-(x-2)^2}}\,dx$$

Let $u = x-2$

$du = dx$

$$=\int \frac{1}{\sqrt{25-u^2}}\,du \qquad \int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+C$$

$$=\int \frac{1}{\sqrt{5^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{5}\right)+C$$

$$= \boxed{\sin^{-1}\left(\frac{x-2}{5}\right)+C}$$

6. $\int x^5\sqrt{4+x^2}\,dx$

Solution

Let $u = 4+x^2$

$du = 2x\,dx$

$\boxed{x\,dx = du/2}$

$$\int x^5 \underbrace{\sqrt{4+x^2}}_{u^2}\,dx = \int x\cdot(x^2)^2\sqrt{4+x^2}\,dx \quad \frac{du}{2}$$

$$= \frac{1}{2}\int (u-4)^2\sqrt{u}\, du$$ → brute force

$$= \frac{1}{2}\int (u^2 - 8u + 16)\, u^{\frac{1}{2}}\, du =$$

$$= \frac{1}{2}\int \left(u^{\frac{5}{2}} - 8u^{\frac{3}{2}} + 16u^{\frac{1}{2}}\right) du =$$

$$= \frac{1}{2}\left(\frac{u^{7/2}}{7/2} - 8\,\frac{u^{5/2}}{5/2} + \frac{16u^{3/2}}{3/2}\right) + C =$$

$$= \boxed{\frac{(x^2+4)^{7/2}}{7} - 8\,\frac{(x^2+4)^{5/2}}{5} + 16\,\frac{(x^2+4)^{3/2}}{3} + C}$$

7. $$\int \frac{1}{1+e^{-x}}\, dx = \int \frac{1}{1+\frac{1}{e^x}}\, dx =$$

$$= \int \frac{e^x}{e^x+1}\, dx =$$

let $u = e^x + 1$

$du = e^x\, dx$

$= \int \frac{1}{u}\, du = \ln|u| + C = \boxed{\ln(e^x+1)+C}$

TRIG INTEGRALS

9. $\int \tan^{3} x \cdot \sec^{1} x \; dx =$

odd ✓

$= \int \tan^2 x \cdot \tan x \cdot \sec x \; dx =$

$= \int (\sec^2 - 1) \tan x \cdot \sec x \; dx$

Let $u = \sec x$

$du = \tan x \cdot \sec x \; dx$

$= \int (u^2 - 1)\, du = \frac{u^3}{3} - u + C =$

$$\boxed{\frac{\sec^3}{3} - \sec x + C}$$

$$\int \cos^4 x \, dx = \int (\cos^2 x)^2 \, dx$$

$$= \int \left(\frac{1+\cos(2x)}{2} \right)^2 dx$$

$$= \frac{1}{4} \int (1+\cos(2x))^2 \, dx = \frac{1}{4} \int 1 + 2 \cdot \cos(2x) + \cos^2(2x)$$

$$= \frac{x}{4} + \frac{1}{2} \int \cos(2x) \, dx + \frac{1}{4} \int \cos^2(2x) \, dx$$

$$= \frac{x}{4} + \frac{1}{2} \frac{\sin(2x)}{2} + \frac{1}{4} \int \frac{1+\cos(4x)}{2} \, dx$$

$$= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{1}{8} \int 1 + \cos(4x) \, dx =$$

$$= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{x}{8} + \frac{1}{8} \frac{\sin(4x)}{4} + C$$

$$= \boxed{\frac{3x}{8} + \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C}$$

January 19, 2026

Triganametric Substitution

$$\int x^2\sqrt{16-x^2}\,dx = \int 16\sin^2\theta\sqrt{16-16\sin^2\theta}\cdot 4\cos\theta\, d\theta$$

$\sqrt{16-x^2} = \sqrt{4^2-x^2}$

$= 4\sqrt{1-\sin^2\theta} = 4\sqrt{\cos^2\theta}$

$= 4\cos\theta$

$x = 4\sin\theta$

$dx = 4\cos\theta\, d\theta$

$$= 256\int \underbrace{\sin^2\theta\cos^2\theta}_{\text{trig integral}}\, d\theta$$

1. For $\sqrt{a^2-x^2}$

$x = a\cdot\sin\theta$, choose θ such that this happens

Is there even such θ?

$a^2 - x^2 \geq 0 \qquad x^2 \leq a^2 \qquad -a \leq x \leq a$

yes, becouse $-a \leq x \leq a$

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = a\sqrt{1-\sin^2\theta}$$

$$= a\sqrt{\cos^2\theta} = a|\cos\theta| = a \cdot \cos\theta$$

2.

For $\underline{\sqrt{a^2+x^2}}$ no restrictions

let $x = a \cdot \tan\theta$

$$\sqrt{a^2+x^2} = \sqrt{a^2 + a^2\tan^2\theta} = a \cdot \sqrt{\sec^2\theta}$$

$$= a|\sec\theta| = a \cdot \sec\theta$$

3. $\sqrt{x^2-a^2}$ $\qquad x^2 - a^2 \geq 0$

$x^2 \geq a^2$

$\underline{x \geq a}$ or $\underline{x \leq -a}$

Ex 1.

$$\int x\sqrt{16-x^2}\,dx = \int x\sqrt{4^2-x^2}\,dx$$

let $x = 4\sin\theta \ \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$

$dx = 4\cos\theta \ d\theta$

Then $\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} =$

$= \sqrt{16(1-\sin^2\theta)} = 4\sqrt{\cos^2\theta} = 4|\cos\theta|$

$= 4\cos\theta$

because $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\int x\sqrt{16-x^2}\,dx = 4\int 4\sin\theta \cdot 4\cos\theta \cdot 4\cos\theta$$

$$= 64\int \sin\theta\cos^2\theta\,d\theta$$

let $u = \cos\theta$

$du = -\sin\theta\,d\theta$

$$= 64\int \overset{u^2}{\cos^2\theta}\,\underbrace{\sin\theta\,d\theta}_{-d\theta}$$

$$= -64\int u^2\,du = \boxed{\frac{-64}{3}u^3 + C}$$

$$= \frac{-64}{3}\cdot\cos^3\theta + C$$

$$\sin\theta = \frac{x}{4}$$

$$\theta = \sin^{-1}\left(\frac{x}{4}\right)$$

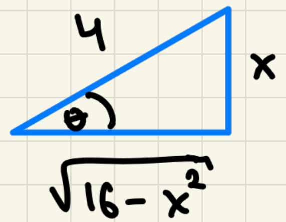
because $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= \boxed{\frac{-64}{3}\cos^3\left(\sin^{-1}\left(\frac{x}{4}\right)\right) + C}$$

If $x = 4\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

compute $\cos\theta$ in terms of x

$$\sin\theta = \frac{x}{4}$$



$$\cos\theta = \frac{\sqrt{16-x^2}}{4}$$

Conclusion

$$\int x\sqrt{16-x^2}\,dx = -\frac{64}{3}\cos^3\theta + C$$

$$= -\frac{64}{3}\left(\frac{\sqrt{16-x^2}}{4}\right)^3 + C$$

$$= \boxed{-\frac{\left(\sqrt{16-x^2}\right)^3}{3} + C}$$

2. $$\int \frac{x^3}{\sqrt{x^2+9}}\,dx$$

$x^2 + 3^2$

Solution

let $x = 3\tan\theta$

$dx = 3\sec^2\theta \, d\theta$

Then $\sqrt{x^2+9} = \sqrt{9\tan^2\theta+9} = 3\sqrt{\tan^2\theta+1}$

$= 3\sqrt{\sec^2\theta} = 3|\sec\theta| = 3\sec\theta$

$$\int \frac{x^3}{\sqrt{x^2+9}} = \int \frac{27\tan^3\theta}{3\sec\theta} \cdot 3\sec^2\theta \, d\theta$$

$$= 27\int \tan^3\theta \sec\theta \, d\theta$$

$$= 27\int \tan^3\theta \cdot \sec\theta \; d\theta$$

$$= 27\int \tan^2\theta \cdot \tan\theta \cdot \sec\theta \; d\theta$$

$$= 27\int (\sec^2 - 1)\tan\theta \cdot \sec\theta \; d\theta$$

let $u = \sec\theta$

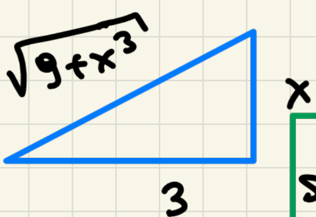
$du = \tan\theta \sec\theta \; d\theta$

$$= 27\int (u^2 - 1)\, du = 27\left(\frac{u^3}{3} - u\right) + C$$

$$= 9u^3 - 27u + C$$

$$= \boxed{9\sec^3\theta - 27\sec\theta + C}$$

$x = 3\tan\theta$



$\tan\theta = \frac{x}{3}$

$$\boxed{\sec\theta = \frac{\sqrt{9+x^2}}{3}}$$

$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = 9\sec^3\theta - 27\sec\theta + C$$

$$= 9\left(\frac{\sqrt{9+x^2}}{3}\right)^3 - 27\,\frac{\sqrt{9+x^2}}{3} + C$$

$$\boxed{= \left(\frac{\sqrt{9+x^2}}{3}\right)^3 - 9\sqrt{9+x^2} + C}$$

3. $\displaystyle\int \frac{\sqrt{x^2-1}}{x^6}\,dx$

Solution

$$x = \sec\theta$$

$$dx = \tan\theta\,\sec\theta\,d\theta$$

$$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$$

Then $\displaystyle\int \frac{\sqrt{x^2-1}}{x^6}\,dx = \int \frac{\tan\theta}{\sec^6\theta}\,\tan\theta\,\sec\theta$

$$= \int \frac{\tan^2\theta}{\sec^5\theta}\, d\theta = \int \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^5\theta}}\, d\theta =$$

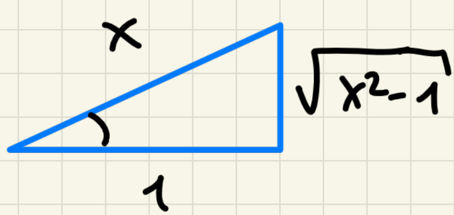
$$= \int \cos^3\theta \cdot \sin^2\theta \, d\theta = \int \cos^2\theta \cdot \sin^2\theta \cos\theta \cdot d\theta$$

$$= \int (1-\sin^2\theta)\sin^2\theta \cdot \cos\theta \, d\theta =$$

$$= \int (1-u^2)u^2 \, du = \int (u^2-u^4)\, du =$$

$$\frac{u^3}{3} - \frac{u^5}{5} + C = \boxed{\frac{\sin^3\theta}{3} - \frac{\sin^5\theta}{5} + C}$$

$$\sec\theta = \frac{x}{1}$$



$$\sin\theta = \frac{\sqrt{x^2-1}}{x}$$

Then:

$$\int \frac{\sqrt{x^2-1}}{x^6}\,dx = \frac{\sin^3\theta}{3} - \frac{\sin^5\theta}{5} + C$$

$$= \boxed{\frac{1}{3}\left(\frac{\sqrt{x^2+1}}{x}\right)^3 - \frac{1}{5}\left(\frac{\sqrt{x^2-1}}{x}\right)^5 + C}$$

January 21, 2026

Integrals of Rational Functions using Partial Fraction Decomposition

Polinomials: $P(x) = a_0 + a_1 x + \dots + a_n x^n$

Rational functions: $f(x) = \dfrac{P(x)}{Q(x)}$

where $P(x)$, $Q(x)$ are polynomials

$f(x) = \dfrac{P(x)}{Q(x)}$ is a "proper" rational function if $\deg P(x) < \deg Q(x)$

Ex

$\frac{2x+7}{3x^2-2x+2}$ is proper

2 → bigger than

$\frac{4x^2+2x+7}{3x^2-2x+2}$ is not proper

$$\int \frac{P(x)}{Q(x)}\,dx$$

Step # 0 : boil down to a proper rational function

Division with remainder for nums: If $a, b > 0$ integers, there exist q, r integers such that $a = bq + r$ $0 \leq r < b$

If $P(x)$ and $Q(x)$ are polynom. there exist $S(x)$ and $R(x)$ Such that:

$$P(x) = Q(x)\,S(x) + R(x)$$

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{Q(x)\,S(x) + R(x)}{Q(x)}\,dx =$$

$$\int \left(S(x) + \frac{R(x)}{Q(x)} \right) dx = \int S(x)\,dx + \int \overset{\text{proper rat. funct.}}{\frac{R(x)}{Q(x)}}\,dx$$

Ex. $\displaystyle\int \frac{2x^3 - 5x^2 + 5x + 3}{x-1}\,dx$

Solution.

$$\begin{array}{r|l}
 & 2x^2 - 3x + 2 \\
x-1 & 2x^3 - 5x^2 + 5x + 3 \\
- & 2x^3 - 2x^2 \\
\hline
- & -3x^2 + 5x + 3 \\
 & -3x^2 + 3x \\
\hline
- & 2x + 3 \\
 & 2x - 2 \\
\hline
 & 5
\end{array}$$

$$\overset{P(x)}{2x^3 - 5x^2 + 5x + 3} =$$

$$= \overset{Q(x)}{(x-1)}\,\overset{S(x)}{(2x^2 - 3x + 2)} + \overset{R(x)}{5}$$

$$= \int \frac{(x-1)(2x^2 - 3x + 2) + 5}{x-1}\,dx =$$

$$= \int \left(2x^2 - 3x + 2 + \frac{5}{x-1} \right) dx =$$

$$= \boxed{\frac{2x^3}{3} - \frac{3x^2}{2} + 2x + 5\ln|x-1| + C}$$

Raised Goal: $\int \frac{P(x)}{Q(x)}\,dx$ if $\frac{P(x)}{Q(x)}$ proper polyn.

Case 1: $Q(x)$ is a product of linear funct.

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\ldots(a_kx+b_k)$$

There exists constants $A_1, A_2, A_3, \ldots A_k$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \ldots + \frac{A_k}{a_kx+b_k}$$

Ex.

$$\frac{P(x)}{Q(x)} = \frac{\text{smth of degree zero or less}}{(3x-7)(5x+2)(2x+1)} =$$

$$\frac{A}{3x-7} + \frac{B}{5x+2} + \frac{C}{2x+1}$$

Ex 1

$$\int \frac{x}{x^2+x-2}\,dx$$

Sol. Factor $x^2+x-2=(x+2)(x-1)$

The form of the PFD is:

$$\frac{x}{x^2+x-2}=\frac{A}{x+2}+\frac{B}{x-1}$$

Find A, B $\quad \frac{x}{(x+2)(x-1)}=\frac{A}{x+2}+\frac{B}{x-1} \quad \Big| \cdot (x+2)(x-1)$

$$x=A(x-1)+B(x+2)$$

$$x=Ax-A+Bx+2B$$

$$x=(A+B)x+(2B-A)$$

$$\underline{1}\cdot x+\underline{0}=x=\underline{(A+B)}x+\underline{(2B-A)}$$

$$\begin{cases} A+B=1 \\ 2B-A=0 \end{cases} \qquad A=2B$$

$2B+B=1 \qquad 3B=1 \qquad B=\frac{1}{3}$

$A=\frac{2}{3}$

The P.F.D is :

$$\frac{x}{x^2-x-2} = \frac{2/3}{x+2} + \frac{1/3}{x-1}$$

$$\int \frac{x}{x^2+x-2}\,dx = \int \left(\frac{2/3}{x+2} + \frac{1/3}{x-1}\right) dx$$

$$= \frac{2}{3}\int \frac{1}{x+2}\,dx + \frac{1}{3}\int \frac{1}{x-1}\,dx$$

$$= \boxed{\frac{2}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + c}$$

Shortcut to find A and B

$x = A(x-1) + B(x+2)$

take $x = 1$ $\quad 1 = A\cdot 0 + B\cdot 3 \Rightarrow B = \frac{1}{3}$

take $x = -2$ $\quad -2 = A(-2-1) + B\cdot 0 = A = \frac{2}{3}$

!!! okay but doesn't always works !!!

Case 2.

Example:

$$\frac{P(x)}{Q(x)} = \frac{\cdots\cdots}{(5x+1)(2x+7)^3(x-3)^2}$$

$$\frac{A}{(5x+1)}+\frac{B}{(2x+7)}+\frac{C}{(2x+7)^2}+\frac{D}{(2x+7)^3}+\frac{E}{x-3}+\frac{F}{(x-3)^2}$$

$$(a_i x+b_i)^r$$

$$\frac{B_1}{a_i x+b_i}+\frac{B_2}{(a_i x+b_i)^2}+\cdots\cdots+\frac{B_r}{(a_i x+b_i)^r}$$

Case 3p

$$\frac{P(x)}{Q(x)}=\frac{\cdots\cdots}{(5x+1)(2x+7)^3(x-3)^2(x^2+2x+7)(2x^2+3x+19)^3}$$

$$\frac{A}{(5x+1)}+\frac{B}{(2x+7)}+\frac{C}{(2x+7)^2}+\frac{D}{(2x+7)^3}+\frac{E}{x-3}+\frac{F}{(x-3)^2}$$

$$+\frac{Gx+H}{x^2+2x+7}+\frac{Ix+J}{2x^2+3x+19}+\frac{Kx+L}{(2x^2+3x+19)^2}+\frac{Mx+N}{(2x^2+3x+19)^3}$$

Ex $\displaystyle\int\frac{x+1}{x^4+x^2}\,dx$

Sol.

Factor $x^4+x^2 = \underline{x^2}\,(\underline{x^2+1})$

Form of the P.F.D is

$$\frac{x+1}{x^4+x^2} = \underline{\frac{A}{x} + \frac{B}{x^2}} + \underline{\frac{Cx+D}{x^2+1}} \quad \Big|\; x^2(x^2+1)$$

Find A, B, C, D

$$x+1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$$

$$x+1 = A(x^3+x) + B(x^2+1) + (Cx^3+Dx^2)$$

$$x+1 = \underline{Ax^3} + \underline{Ax} + \underline{Bx^2} + \underline{B} + \underline{Cx^3} + \underline{Dx^2}$$

$$x+1 = (A+C)x^3 + (B+D)x^2 + Ax + B$$

$$\underbrace{0}\cdot x^3 + \underbrace{0}\cdot x^2 + \underbrace{1}\cdot x + \underbrace{1} = \underbrace{(A+C)}x^3 + \underbrace{(B+D)}x^2 + \underbrace{A}x + \underbrace{B}$$

$$\begin{cases} A+C = 0 \\ B+D = 0 \\ A = 1 \\ B = 1 \end{cases} \qquad \begin{matrix} C = -1 \\ D = -1 \end{matrix}$$

$$\frac{x+1}{x^4+x^2} = \frac{1}{x} + \frac{1}{x^2} - \frac{x+1}{x^2+1}$$

$$\int \frac{1}{x} + \frac{1}{x^2} - \frac{x+1}{x^2+1}\, dx =$$

$$= \ln|x| - \frac{1}{x} - \int \frac{x+1}{x^2+1}$$

$$= \int \frac{x}{x^2+1}\, dx + \underbrace{\int \frac{1}{x^2+1}}_{\tan^{-1} x}$$

let $u = x^2+1$

$du = 2x\, dx$

$$\frac{1}{2}\int \frac{1}{u}\, du + \tan^{-1} x + C = \frac{1}{2}\ln(x^2+1) + \tan^{-1}$$

$$= \boxed{\int \ln|x| - \frac{1}{x} - \frac{1}{2}\ln(x^2+1) - \tan^{-1} x + C}$$

let $x = 5\sin\theta$

$dx = 5\cos\theta \, d\theta$

then $\sqrt{25-x^2} = \sqrt{25-25\sin^2\theta} = 5\sqrt{1-\sin^2\theta}$

$= 5|\cos\theta| = 5\cos\theta$

$\cos\theta \geq 0$ because $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\int \sqrt{25-x^2}\, dx = \int 5\cos\theta \cdot 5\cos\theta \, d\theta =$$

$$25\int \cos^2\theta \, d\theta = 25\int \frac{1+\cos(2\theta)}{2} d\theta$$

$$\frac{25}{2}\int (1+\cos(2\theta))\, d\theta = \frac{25}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$

IF $x = 5\sin\theta$ $\qquad \sin(2\theta) = ?$

$\sin\theta = \frac{x}{5} \Rightarrow \theta = \sin^{-1}\left(\frac{x}{5}\right)$

$\sin(2\theta) = 2\sin\theta \cdot \cos\theta$

$\sin^2\theta + \cos^2\theta = 1 \qquad \cos^2\theta = 1 - \sin^2\theta$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$$= \frac{25}{2}\left(\sin^{-1}\left(\frac{x}{5}\right) + \frac{x\sqrt{25-x^2}}{25}\right) + C$$

$$\boxed{= \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right) + \frac{x\sqrt{25-x^2}}{2} + C}$$

④

$$\frac{x^8+x+1}{(2x+4)^2(3x^2+2x+7)(x^2+1)^3} = \frac{A}{2x+4} + \frac{B}{(2x+4)^2}$$

$$+ \frac{Cx+D}{3x^2+2x+7} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

⑤

degree 12

$$\frac{x^{12}+x+1}{(2x+4)^2(3x^2+2x+7)(x^2+1)^3}$$

degree : $2+2+2\cdot 3 = 10$ $\qquad 12 > 10$

not a proper rational function

$$\frac{x^8+x+1}{(2x+4)^2(3x^2+2x-1)(x^2+1)^3} = \frac{A}{2x+4} + \frac{B}{(2x+4)^2}$$

$$+\frac{C}{3x-1}+\frac{D}{x+1}+\frac{Ex+F}{x^2+1}+\frac{Gx+H}{(x^2+1)^2}+\frac{Ix+J}{(x^2+1)^3}$$

(11) $$\int \frac{1}{e^{2x}+e^{x}}\,dx = \int \frac{e^{x}}{e^{x}(e^{2x}+e^{x})}\,dx$$

let $u = e^{x}$

$du = e^{x}\,dx$

$$= \int \frac{1}{u^3+u^2}\,du \rightarrow \text{rational function}$$

Factor: $u^3+u^2 = u^2(u+1)$ PFD

$$\frac{1}{u^2(u+1)} = \frac{A}{u}+\frac{B}{u^2}+\frac{C}{u+1} \quad \Big| \cdot u^2(u+1)$$

$$1 = Au(u+1)+B(u+1)+Cu^2$$

$$1 = Au^2+Au+Bu+B+Cu^2$$

$$1 = (A+C)u^2+(A+B)u+Cu^2$$

$$0\cdot u^2+0\cdot u+1 = (A+C)u^2+(A+B)u+B$$

$$\begin{cases} A+B=0 \\ A+C=0 \\ B=1 \end{cases} \qquad C=1 \qquad A=-1$$

The PFD is

$$\frac{1}{u^2(u+1)} = -\frac{1}{u} + \frac{1}{u^2} + \frac{1}{u+1}$$

$$\int \frac{1}{e^{2x}+e^x}\,dx = \int \frac{1}{u^3+u^2}\,du = \int \left(-\frac{1}{u} + \frac{1}{u^2} + \frac{1}{u+1} \right.$$

$$= -\ln|u| - \frac{1}{u} + \ln|u+1| + C$$

$$= -\ln|e^x| - \frac{1}{e^x} + \ln|e^x+1| + C$$

$$= \boxed{-x - \frac{1}{e^x} + \ln(e^x+1) + C}$$