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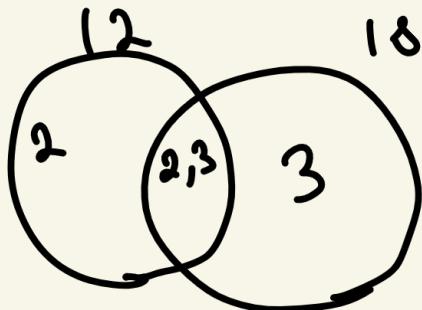
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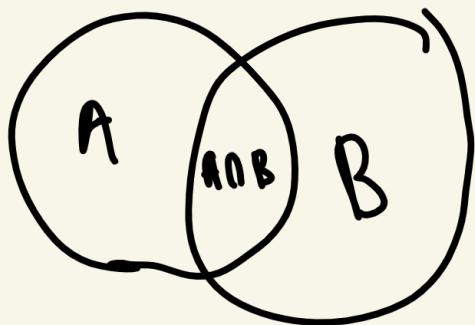


office hours  
Erie 3125

January 5, 2026



$$\frac{ab}{\gcd(a,b)} = \text{lcm}(a,b)$$



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proposition 1:

For all  $x \in \mathbb{R}$ ,  $x \cdot 0 = 0$

and  $0 \cdot x = 0$

proof: Let  $x \in \mathbb{R}$

$$x+0=x$$

$$1+0=1 \quad \text{by identity}$$

$$0+0=0$$

$$x(1+0) = x \cdot 1$$

$$x(1+0) = x \quad \text{by identity}$$

$$x \cdot 1 + x \cdot 0 = x \quad \text{by distributivity}$$

$$x + x \cdot 0 = x \quad \text{by identity}$$

$$-x + (x+x \cdot 0) = -x+x$$

$$-x+x+x \cdot 0 = -x+x \quad \text{Associativ.}$$

$$0 + x \cdot 0 = 0 \quad \text{invertibility}$$

$$x \cdot 0 = 0 \quad \text{identity}$$

Also,  $0 \cdot x = 0$  for all

$x \in \mathbb{R}$ ,  $x \cdot 0 = 0$  and  $0 \cdot x = 0$

January 7, 2026

Proposition 2: For all  $x, y \in \mathbb{R}$

$$(-x)y = -(xy)$$

proof: Let  $x, y \in \mathbb{R}$

Universal Generalization

To prove "for all  $x \in \mathbb{R}, P(x)"$

Let  $x \in \mathbb{R}$

\* Demonstrate  $P(x)$

Therefore, for all  $x \in \mathbb{R}, P(x)$

$$-\square + \square = 0$$

$$-(xy) + xy = 0 \quad \text{by invertibility}$$

$$\underbrace{-(xy)}_{\text{keep}} + xy + (-x) \cdot y = 0 + \underbrace{(-x) \cdot y}_{\text{keep}}$$

$$-(xy) + xy + (-xy) = 0 + (-x) \cdot y$$

$$-(xy) + xy + (-1)(xy) = 0 + (-x) \cdot y$$

$$1x = x \quad -(1x) = -x \quad (-1)x = -x$$

$$-(xy) + (x + (-x))y = (-x)y \quad \begin{matrix} \text{Distr.} \\ \text{and Identities} \end{matrix}$$

$$-(xy) + 0 \cdot y = (-x)y \quad \text{Invertibility}$$

$$-(xy) + 0 = (-x)y \quad \text{Prop 1}$$

$$-(xy) = (-x)y \text{ by identity}$$

Therefore, for all  $x, y \in \mathbb{R}$   $(-x)y = -xy$

Proposition 3:

$$\text{For all } x, y \in \mathbb{R}, (-x)(-y) = xy$$

Proof: Let  $x, y \in \mathbb{R}$

$$-x + x = 0 \quad \text{Invertibility}$$

$$(-x + x)(-y) = 0 \cdot (-y)$$

$$(-x)(-y) + x(-y) = 0 \cdot (-y) \text{ dist.}$$

$$(-x)(-y) + x(-y) = 0 \quad \text{Prop 1}$$

$$(-x)(-y) + x(-y) + xy = 0 + xy$$

$$(-x)(-y) + x(-y) + xy = xy \quad (\text{Ident.})$$

$$(-x)(-y) + x(-y+y) = xy \quad \text{dist.}$$

$$(-x)(-y) + x \cdot 0 = xy \quad \text{invertib.}$$

$$(-x)(-y) + 0 = xy \quad \text{Prop 1}$$

$$(-x)(-y) = xy \quad \text{Identity}$$

For all  $x, y \in \mathbb{R}$   $(-x)(-y) = xy$

January 9, 2026

My dog, is yellow  
subject

$X$  is yellow  $\nearrow$  open sentence

Examples:  $y = 2x + 1$   
 $x < 3$

Each variable has an allowable set of values called the

"universe of discourse" for variable

Example: X is wearing Y

Quantified Statements:

Two of my cats are orange

predicate: X is orange

Universe of discourse for X:

The set of all my cats

Universal Quantified Statements

"All my cats are orange"

predicate: X is orange

universe: C = set of my cats

Notation:  $\forall x \in C, x \text{ is orange}$

Read: "for all values of X in C,  
X is orange"

# Existential Quantified Statements

"Some of my cats are orange"

Notation:  $\exists x \in C, X \text{ is orange}$

Read: "for at least one value  
of  $X$  in  $C$ ,  $X$  is orange"

"there is a value of  $X$   
in  $C$  where  $X$  is orange"

"there exists an  $X$  in  $C$   
for which  $X$  is orange"

$\mathbb{N}$  natural numbers  $\{1, 2, 3, 4, \dots\}$

$\mathbb{Z}$  integers  $\{\dots -4, -3, \dots 0, 1, 2, 3, \dots\}$

$\mathbb{Q}$  rational numbers All fractions & integers

$\mathbb{R}$  real num's

combination

$\mathbb{C}$  complex num's of real / imagin. numbers

1.  $\forall x \in \mathbb{N}, \boxed{0 \leq x}$  True

2.  $\forall x \in \mathbb{R}, \boxed{0 < x^2}$  False

3.  $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \boxed{x < y}$  True

4.  $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, \boxed{x < y}$  False

5.  $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \boxed{y = 2x}$  True

6.  $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \boxed{x = 2y}$  False  
 $5 = 2y ?$

7.  $\forall x \in \mathbb{Z}$ , If  $\exists y \in \mathbb{Z}, x = 2y$

If its even then

then  $\exists q \in \mathbb{Z}, x = 2q$  True

8.  $\forall x \in \mathbb{R}$ , If  $\forall a \in (0, \infty), x \leq a$

then  $x \leq 0$

If its  $x \geq 0$ , but never less or equal to zero

~~True~~

9.  $\forall x \in \mathbb{R}$  If  $\forall a \in \mathbb{R}, a \cdot x \leq 0$ ,

$\{0\}$

then  $\forall b \in \mathbb{R}, 0 \leq b \cdot x$ . True

January 12, 2026

Proposition:  $0 < 1$

Proof by contradiction:

To prove a proposition  $P$

Assume  $\neg P$  (negation of  $P$ )



derive a contrad.  $Q$  and  $\neg Q$

Assume :  $0 \neq 1$

Since  $0 \neq 1$ , we have  $1 > 0$

by trichotomy

$-1 + 1 < -1 + 0$  by monotonicity

$$0 < -1$$

then:

$0(-1) < (-1)(-1)$  by monotonicity

$0 < 1$  by Prop 1 & Prop 3

Now,  $0 < 1$  and  $0 \neq 1$ . This is contradiction

Therefore,  $0 < 1$

Proposition:  $1+1 \neq 1$

Proof:

Assume  $1+1 = 1$

$$1+1(-1) = 1 + (-1)$$

$$1 = 0$$

But,  $1 \neq 0$  This is contradiction

Therefore,  $1+1 \neq 1$

Proposition:  $1+1 \neq 0$

Proof:

Assume  $1+1 = 0$

We know  $0 < 1$

then  $1+0 < 1+1$  by monotonic.

So,  $1 < 0$

this is a contradiction, since  $0 < 1$

Therefore,  $1+1 \neq 0$

Definition:

$$2 = 1+1 \quad 3 = 2+1 \quad 4 = 3+1 \quad 5 = 4+1$$

Example:  $2 < 4$

Proof:

$$0 < 1$$

then  $1+0 < 1+1$

so  $1 < 2$

by transitivity,  $0 < 2$

then  $1+0 < 2+1$

so  $1 < 3$

then  $1+1 < 3+1$ , so  $2 < 4$

January 14, 2026

Proposition:

$\forall x, y \in \mathbb{R}, \text{if } x < y, \text{ then } -y < -x$

## Direct Proof

To prove If  $P$ , then  $Q$

Assume  $P$

\* Demonstrate  $Q$

Therefore, If  $P$ , then  $Q$

## Proof

Let  $x, y \in \mathbb{R}$

Assume  $x < y$

by monot

$$-x + x + (-y) < -x + y + (-y)$$

$$0 + (-y) < -x + 0$$

$$-y < -x \quad \therefore$$

Therefore, If  $x < y$ , then  $-y < -x$

## Monotonicity (Negative Multiplication)

$\forall x, y, z \in \mathbb{R}$ , If  $x < y$  and  $z < 0$ ,

then  $yz < xz$

Proof

Let  $x, y, z \in \mathbb{R}$

target  
 $yz < xz$

Assume  $x < y$  and  $z < 0$

Since  $z < 0$ , we have  $0 < -z$   
 $-z + z < -z + 0$

Then  $x(-z) < y(-z)$

So,  $-xz < -yz$

$$xz + (-xz) + yz < xz + (-yz) + yz$$

$$0 + yz < xz + 0$$

$$yz < xz$$

Therefore, if  $x < y$  and  $z < 0$ ,

QED      then  $yz < xz$

$\forall x, y \in \mathbb{R}$ , if  $x < y$ , then  $x+2 < y+3$

Proof Let  $x, y \in \mathbb{R}$

Assume  $x < y$

$$\begin{array}{c} | \\ x+2 < y+3 \end{array}$$

$x+2 < y+2$  by monotonicity

Since  $2 < 3$ ,  $y+2 < y+3$  by monot.

$x+2 < y+2$  &  $y+2 < y+3$

By transitivity,  $x+2 < y+3$

Using transitivity to Prove  $A < B$

①  $A < C$

②  $C < B$

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$A < B$

January 16, 2026

## Rules of Negation

1.  $\neg(\forall x \in U, P(x))$  is  $\exists x \in U, \neg P(x)$
2.  $\neg(\exists x \in U, P(x))$  is  $\forall x \notin U, \neg P(x)$
3.  $\neg(P \text{ and } Q)$  is  $\neg P \text{ or } \neg Q$
4.  $\neg(P \text{ or } Q)$  is  $\neg P \text{ and } \neg Q$
5.  $\neg(\text{If } P, \text{then } Q)$  is  $P \text{ and } \neg Q$

$\forall x \in \mathbb{R}, \text{if } \boxed{\forall a \in (0, \infty), a \leq x}, \text{ then } \boxed{x \leq 0}$

Negation

$\exists x \in \mathbb{R}, \boxed{\forall a \in (0, \infty), a \leq x} \text{ and } \boxed{0 < x}$

1.  $\forall x \in \mathbb{Z}$ , if  $\exists a \in \mathbb{Z}, x = 2a+1$ , then  
 $\exists b \in \mathbb{Z}, x = 3b$  ✓

$\exists x \in \mathbb{Z}, \exists a \in \mathbb{Z}, x = 2a+1$  and  
 $\forall b \in \mathbb{Z}, x \neq 3b$  ✓

2.  $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x < y$

$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x \geq y$  ✓

3.  $\forall x, y \in \mathbb{R}$ , if  $0 < x < 1$  and  
 $x+y=1$ , then  $0 < y < 1$

$\exists x, y \in \mathbb{R}, 0 < x < 1$  and  $x+y=1$   
and  $y \leq 0$  or  $1 \leq y$  ✓

1.  $A < C > A < B$   
2.  $C < B$

1.  $\forall x, y \in \mathbb{R}$ , If  $0 < y < x$ , then  $y < 2x$

**Structure** Let  $x, y \in \mathbb{R}$

Assume  $0 < y < x$

$0 < y$  and  $y < x$

then  $0 < x$        $x+0 < x+x$

so  $x < 2x$

$y < x < 2x$

so  $y < 2x$

2.  $\forall x, y \in \mathbb{R}$ , If  $x < 2 < y$ , then  $x+2 < y^2$

$x < 2$      $2 < y$

$2 \cdot 2 < 2 \cdot y$  so  $4 < 2y$

Since  $0 < 2 < y$   
we have  $0 < y$        $2 < y$  so  $2(y) < y(y)$

$2y < y^2$

then  $4 < y^2$

$x < 2$  so  $x+2 < 4$

Now,  $x+2 < 4$

January 19, 2026

## Proving Statements with Existential Qualifiers

### Universal Generalization

To prove  $\forall x \in \mathbb{R}, P(x)$

Let  $x \in \mathbb{R}$

\* Show  $P(x)$

Therefore,  $\forall x \in \mathbb{R}, P(x)$

### Existential Generalization

To prove  $\exists x \in \mathbb{R}, P(x)$

Let  $x = \boxed{\quad}$  some specific value

\* Demonstrate  $P(x)$

for that specific value

Therefore,  $\exists x \in \mathbb{R}, P(x)$

## Example

$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \exists z \in \mathbb{R}, x < y$

proof:

let  $x \in \mathbb{R}$

let  $y = x+1$

Since  $0 < 1, x+0 < x+1$

then  $x < y$

Therefore,  $\exists y \in \mathbb{R}, x < y$

## Example

$\forall x, y \in \mathbb{R}, \text{If } x < y, \text{ then } \exists z \in \mathbb{R} \underline{x < z < y}$

Proof:

let  $x, y \in \mathbb{R}$

Assume  $x < y$

let  $z = \frac{x+y}{2}$

Since  $x < y$

$$\text{then } x+x < y+x$$

$$2x < y+x$$

$$\frac{1}{2}(2x) < \frac{1}{2}(y+x)$$

$$\text{So, } x < z$$

Also, since  $x < y$ ,  $x+y < y+y$

then  $x+y < 2y$

so  $\frac{-1}{2}(x+y) < \frac{-1}{2}2y$

Now  $\frac{x+y}{2} < y$ , so  $z < y$

Therefore,  $\exists z \in \mathbb{R}$ ,  $x < z < y$

## Example

$\forall x \in \mathbb{R}$ , if  $2 < x$ , then

$\exists a \in \mathbb{R}$ ,  $1 < a$  and  $1+a < x$

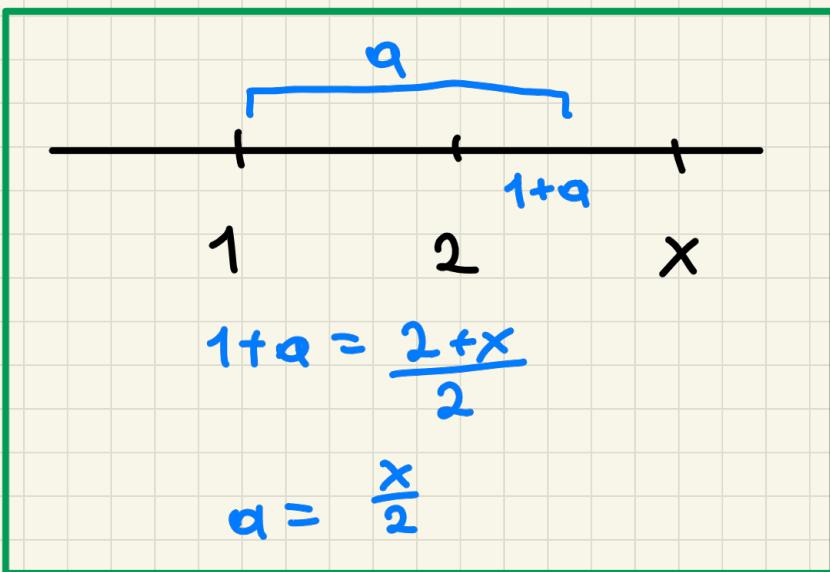
## Proof

Let  $x \in \mathbb{R}$

Assume  $2 < x$

$1 < a$

Let  $a = x/2$



Since  $2 < x$ ,  $\frac{1}{2}(2) < \frac{x}{2}$

so,  $1 < a$

Then,  $1 + a < a + a$

so,  $1 + a < 2a$

$1 + a < 2\left(\frac{x}{2}\right)$  so  $1 + a < x$

Therefore  $1 < a$  and  $1 + a < x$

so,  $\exists a \in \mathbb{R}$ ,  $1 < a$  and  $1 + a < x$

## Example

$\forall x, y \in \mathbb{R}$ , if  $0 < x < 1$  and  
 $0 < y < 1$ ,

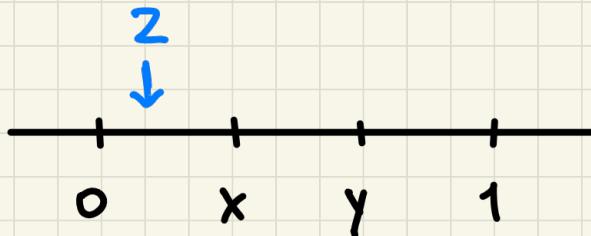
then  $\exists z \in \mathbb{R}$ ,  $0 < z < x$   
and  $0 < z < y$

## Proof

let  $x, y \in \mathbb{R}$

Assume  $0 < x < 1$  and  $0 < y < 1$

let  $z = x \cdot y$



TBC

January 21, 2026

$\wedge$  "and"

$\vee$  "or"

$$\mathbb{B} = \{T, F\}$$

$\wedge$	T	F
T	T	F
F	F	F

$\vee$	T	F
T	T	F
F	T	F

Inclusive OR - includes the possibility of both statements being true

$$\neg \text{ "not"} \quad \neg F = T \quad \neg T = F$$

$x \Rightarrow y$  statement or binary operation

In COMP 1000  $x \Rightarrow y$  is the same as  $\neg x \vee y$  but not MATH 1020

$$\text{In IR } x(y+z) = xy + xz$$

in  $\mathbb{B}$  boolean

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$x \wedge (x \vee y) = x$$

$$x \vee (x \wedge y) = x$$

$$x \Rightarrow y \text{ on } \mathbb{B}$$

behaves analogous to  $x \leq y$  on  $\mathbb{R}$

$$x \wedge T = x \quad x \vee F = x$$

$$x \vee T = T \quad x \wedge F = F$$

January 23, 2026

Identity:

$$x \vee F = x$$

$$x \wedge T = x$$

## Annihilation:

$$x \wedge F = F$$

$$x \vee T = T$$

## Complementation:

$$x \wedge \neg x = F \quad \neg \boxed{\neg x} \wedge \boxed{\neg x} = F$$

$$x \vee \neg x = T \quad \neg \boxed{\neg x} \vee \boxed{\neg x} = T$$

## Proposition (Involution)

$$\forall x \in B, \neg \neg x = x$$

Proof:

$$\text{let } x \in B$$

$$x \vee F = x \quad (\text{Identity})$$

$$x \vee (\neg \neg x \wedge \neg x) = x \quad (\text{Complementation})$$

$$(x \vee \neg \neg x) \wedge (x \vee \neg x) = x \quad (\text{Distributivity})$$

$$(x \vee \neg \neg x) \wedge \underline{T} = x \quad (\text{Complementation})$$

$$(x \vee \neg \neg x) \wedge \underline{\neg \neg x \vee \neg x} = x$$

$$\neg \neg x \vee (x \wedge \neg x) = x$$

$$\neg\neg x \vee F = x$$

So,  $\neg\neg x = x$  By identity

①  $\forall x, y \in B, (x \wedge \neg y) \vee (y \wedge \neg x) = (x \vee y) \wedge (\neg x \vee \neg y)$

②  $\forall x, y \in B, \text{ if } x \wedge \neg y = x, \text{ then}$

$$\neg x \wedge y = y$$

①

let  $x, y \in B$

$$(x \wedge \neg y) \vee (\underline{y \wedge \neg x}) = (x \vee y) \wedge (\neg x \vee \neg y)$$

$$(x \wedge y) \vee \underline{(x \wedge \neg x)} \vee \underline{(\neg y \wedge y)} \vee \underline{(\neg y \wedge \neg x)}$$

$$(x \wedge y) \vee (\neg y \wedge \neg x) = (x \vee y) \wedge (\neg x \vee \neg y)$$

$$\neg(x \wedge y) \wedge \neg(\neg y \wedge \neg x) \quad \text{DM}$$

$$(\neg x \vee \neg y) \wedge (y \vee x)$$

$$(y \vee x) \wedge (\neg x \vee \neg y) = (x \vee y) \wedge (\neg x \vee \neg y)$$

②

let  $x, y \in \mathbb{B}$

assume  $x \wedge \neg y = x$  .....  $\neg x \wedge y = y$

$$\neg(x \wedge \neg y) = \neg x$$

$$\neg x \vee \neg \neg y = \neg x$$

$$\neg x \vee y = \neg x \quad | \wedge y$$

$$(\neg x \vee y) \wedge y = \neg x \wedge y \quad \text{Absorption}$$

$$\underline{y = \neg x \wedge y}$$

Axioms of a binary operation

Existence:  $\forall x, y \in \mathbb{R}, x + y \in \mathbb{R}$

Uniqueness:  $\forall a, b, x \in \mathbb{R}, \text{ If } a = b, \text{ then } a + x = b + x$

Existence:  $\forall x, y \in \mathbb{B}, x \wedge y \in \mathbb{B}$

Uniqueness:  $\forall a, b, x \in \mathbb{B}, \text{ If } a = b, \text{ then } a \wedge x = b \wedge x$

③

$\forall x, y, a \in \mathbb{B}, \text{ If } x \Rightarrow y \text{ and } a \Rightarrow b, \text{ then } x \wedge a \Rightarrow y \wedge b$

let  $x, y, a \in \text{IB}$

Assume  $x \Rightarrow y$  and  $a \Rightarrow b$

$$x \wedge a \Rightarrow y \wedge a$$

$$x \wedge a \Rightarrow \boxed{y \wedge a}$$

$$\boxed{y \wedge a} \Rightarrow y \wedge b$$

$$x \wedge y \Rightarrow y \wedge b$$

Since  $a \Rightarrow$ ,  $y \wedge a \Rightarrow y \wedge b$

By transitivity,  $x \wedge a \Rightarrow y \wedge b$

January 26, 2026

Proposition: For all  $x, y \in \text{IB}$ , the following statements are equal

(1)  $x \Rightarrow y$

(2)  $x \wedge y = x$

(3)  $x \vee y = y$

(4)  $\neg x \vee y = T$

Proof : let  $x, y \in IB$

(1)  $\Rightarrow$  (2)

Assume  $x \Rightarrow y \dots x \wedge y = x$

$x \wedge y \Rightarrow x$  by consistency

Since  $x \Rightarrow y, x \wedge x \Rightarrow x \wedge y$

Then  $x \Rightarrow x \wedge y$

$x \wedge y = x$  by antisymmetry

Therefore, IF (1) then (2)

---

(2)  $\Rightarrow$  (3)

Assume  $x \wedge y = x \dots xy = y$

Then  $(x \wedge y) \vee y = xy$

By Absorption,  $y = xy$

Therefore, If (2) then (3)

---

(3)  $\Rightarrow$  (4)

Assume  $x \vee y = y$  .....  $\neg x \vee y = T$

$$\overline{\neg x \vee x \vee y} = \neg x \vee y \quad \vee \neg x$$

$$T \vee y = \neg x \vee y$$

Then,  $T = \neg x \vee y$   
If (3) then (4)

---

(4)  $\Rightarrow$  (1)

Assume  $\neg x \vee y = T$  ....  $x \Rightarrow y$

$$x \wedge (\neg x \vee y) = x \wedge T \quad \wedge x$$

$$(x \wedge \neg x) \vee (x \wedge y) = x$$

$$F \vee (x \wedge y) = x$$

$$x \wedge y = x$$

Also,  $x \wedge y \Rightarrow y$  by consistency

Then  $x \Rightarrow y$

Therefore, If  $(y)$  then  $(x)$  January 30, 2026

## Proving Conditional Statements

Meet - join - formulas

To prove "If  $A$ , then  $B$ "

Direct

Proof

$$A \Rightarrow B$$

$$A \wedge B = A$$

Assume

$$A \vee B = B$$

Contraposition

$\downarrow$   
Proposition (Contraposition)  
& demon<sup>n</sup>

$\forall x, y \in IB, x \Rightarrow y$  if and only if

$$\neg y \Rightarrow \neg x$$

Proof :

let  $x, y \in IB$

Assume  $x \Rightarrow y \dots \neg y \Rightarrow \neg x$

Then  $x \wedge y = x$  meet formula

So,  $\neg y \vee \neg x = \neg x$

$A \quad B$

$B$

join formula

Therefore,  $\neg y \Rightarrow \neg B$

---

Conversely, assume  $\neg x \Rightarrow \neg y$

Then,  $\neg y \wedge \neg x = \neg y$  meet

So,  $\neg\neg y \wedge \neg\neg x = \neg\neg y$

$y \vee x = y$  Involution

Then,  $x \vee y = y$

$x \Rightarrow y$  join

January 30, 2026

# Proving Conditional Statements

Direct Proof

Assume A



\* Demonstrate B

Contraposition

Assume  $\neg B$



\* Demonstrate  $\neg A$

Therefore, If A  
then B

Therefore, If A  
then B

①  $\forall x, y \in \mathbb{R}$ , If  $y^2 \leq x+2$ , then  
 $2 \leq x$  or  $y \leq 2$

②  $\forall x, y, z \in \mathbb{R}$ , If  $xz + yz \leq x^2 + y^2$   
then  $x \leq 0$  or  $y \leq x$  or  $z \leq y$

③  $\forall x, y \in \mathbb{R}$ , If  $\forall a \in (-\infty, x-a)$ ,  
then  $y \leq x$

Proof:

$$x+2 < y^2$$

1. Assume  $x < 2$  and  $2 < y \dots$

Since  $x < 2$ ,  $x+2 < 4$

Since  $0 < 2 < y$ , we

know  $2 < y$

$$\text{Now, } 2 \cdot y < y \cdot y = 2y < \underline{y^2}$$

Since,  $2 < y$ ,  $x+2 < 2y$

2. Assume  $0 < x < y < z$   $\dots x^2 + y^2 < xz + yz$

Then  $x < z$ , so  $x^2 < xz$

Then,  $x^2 + yz < xz + yz$

Also,  $y < z$  and  $0 < y$ , so  $y^2 < yz$

Then,  $x^2 + y^2 < \boxed{x^2 + yz}$

By transitivity,  $x^2 + y^2 < xz + yz$

③ proof: Let  $x, y \in \mathbb{R}$

Assume  $x < y \dots \exists a \in (-\infty, 0), x < y$

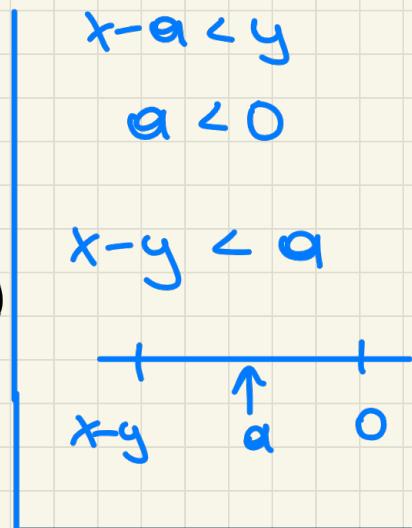
Let  $a = \frac{x-y}{2}$   
 $2a = x-y$

Since  $x < y \quad x-y < 0$

Then  $\frac{x-y}{2} < 0, \text{ so } a \in (-\infty, 0)$

Since  $a < 0, 2a < a$

So,  $x-y < a$



Then  $x-a < y$

February 2, 2026

# Instantiations (gathering information from your assumption)

## Example

$$\forall x \in \mathbb{R}, ax \leq x \text{ then } x = 0$$

Proof:

Let  $x \in \mathbb{R}$

Assume  $\forall a \in \mathbb{R}, ax \leq x$

$$\square \cdot x \leq x$$

Since  $0 \in \mathbb{R}, 0 \cdot x \leq x$

Then  $0 \leq x$

Since  $2 \in \mathbb{R}, 2 \cdot x \leq x$

Then  $2x - x \leq 0, \underline{x \leq 0}$

Thus,  $x = 0$

## Example

$\forall x \in \mathbb{R}$ , If  $\forall a \in (1+x, \infty)$ ,  
 $2x \leq 2+a$ , then  $x \leq 4$

Proof:

Let  $x \in \mathbb{R}$

Assume  $\forall a \in (1+x, \infty)$ ,  $2x \leq 2+a$   $\dots x \leq 4$

$$2x \leq 2 + \square$$

Since  $1+x < 2+x$ , we know  
that  $2+x \in (1+x, \infty)$

Universal  
Instantiation Then  $2x \leq 2 + (2+x)$

$$2x \leq 4+x$$

$$2x - x \leq 4, \quad \underline{x \leq 4}$$

Then  $x \leq 4$

## Example

$\forall x \in \mathbb{R}$ , if  $\exists a \in \mathbb{R}$ ,  $a \neq 1$   
and  $ax = x$ , then  $x = 0$

## Proof

Let  $x \in \mathbb{R}$

Assume  $\exists a \in \mathbb{R}$ ,  $a \neq 1$  and  $ax = x$  ....  $x = 0$

Existential Instantiation

Choose  $\alpha \in \mathbb{R}$  with  $\alpha \neq 1$  and  $\alpha x = x$

$$\alpha x - x = 0$$

$$(\alpha - 1)x = 0$$

Since  $\alpha \neq 1$ , we know  $\alpha - 1 \neq 0$

$$(\alpha - 1)^{-1} (\alpha - 1)x = (\alpha - 1)^{-1} 0$$

$$\underline{x = 0}$$

## Example

$\forall x \in \mathbb{R}$ , if  $\exists b \in (0, \infty), \forall a \in (0, \infty)$   
 $bx < a$ , then  $\forall c \in (0, \infty)$ ,  $x < c$

Proof

Let  $x \in \mathbb{R}$

Assume  $\exists b \in (0, \infty), \forall a \in (0, \infty)$   $bx < a$   
Let  $c \in (0, \infty)$

Choose  $\beta \in (0, \infty)$ , where  $\forall a \in (0, \infty)$ ,  $\beta x < a$

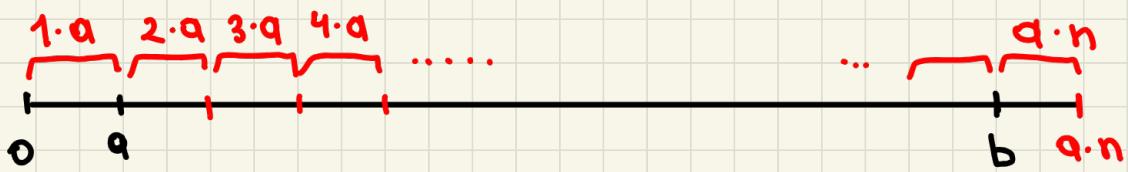
Since  $\beta \in (0, \infty)$  and  $c \in (0, \infty)$ ,  
we know  $\beta c \in (0, \infty)$

Then  $\beta x < \beta c$ , then  $x < c$

February 4, 2026

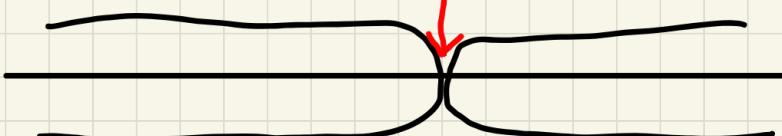
Theorem (The Archimedean Property)

$\forall a, b \in \mathbb{R}$ , if  $0 < a$ , then  $\exists n \in \mathbb{N}, b < na$



$$A = \{x \in \mathbb{Q} \mid 0 < x, x^2 < 2\}$$

$$B = \{x \in \mathbb{Q} \mid 0 < x, 2 < x^2\}$$



$$c = \sqrt{2}$$

Dedekind Cut

Proof:

Let  $a, b \in \mathbb{R}$

Assume  $0 < a$

$\forall n \in \mathbb{N}, a_n \leq b$

reachable  
Let  $A = \{a_n \mid n \in \mathbb{N}\}$

unreachable  
Let  $B = \{y \in \mathbb{R}, \forall n \in \mathbb{N}, a_n \leq y\}$

Since  $a \in A, b \in B, A$  and  $B$  are not empty

Also, for any  $y \in B$  and any  $x \in A$

$x = an$  for some  $n \in \mathbb{N}$ , and so  $an \leq y$

Then A and B form a Dedekind cut

Then there is a number

February 6, 2026

## The Archimedean Property

$\forall a, b \in \mathbb{R}$ , IF  $0 < a$ , then  $\exists n \in \mathbb{N}$ ,  $b < n$

Ex.

$\forall \epsilon \in \mathbb{R}$ , IF  $0 < \epsilon$ , then  $\exists n \in \mathbb{N}$ ,  $\frac{1}{n} < \epsilon$

Proof:

Let  $\epsilon \in \mathbb{R}$

Assume  $0 < \epsilon$

By the Archimedean Property

$\exists n \in \mathbb{N}$ ,  $1 < \epsilon n$

Choose  $n \in \mathbb{N}$  with  $1 < \epsilon n$

Then  $\frac{1}{n} < \epsilon$

Therefore,  $\exists n \in \mathbb{N}, \frac{1}{n} < \epsilon$

Ex.  $\forall x \in \mathbb{R}$ , If  $x < 1$ , then  $\exists n \in \mathbb{N}, x < \frac{n}{n+1}$

Proof:

Let  $x \in \mathbb{R}$

Assume  $x < 1 \Rightarrow 0 < 1-x$

By the Archimedean Property

$\exists n \in \mathbb{N}, x < (1-x)n$

Choose  $n \in \mathbb{N}$  with  $x < (1-x)n$

Scrap paper

$$x < \frac{n}{n+1}$$

$$(n+1)x < n$$

$$nx + x < n$$

$$x < n - nx$$

$$\boxed{x} < \boxed{1-x}n$$

$$x < n - xn$$

$$xn + x < n$$

$$x(n+1) < n$$

$$x < \frac{n}{n+1}$$

Therefore  $\exists n \in \mathbb{N}, x < \frac{n}{n+1}$

①  $\forall x \in \mathbb{R}$ , if  $5 < x$ , then  $\exists n \in \mathbb{N}, \frac{5n+7}{n+1} < x$

②  $\forall x, y \in \mathbb{R}$ , if  $x < y$ , then  $\exists n \in \mathbb{N}, (3+n)x < (n-3)y$

③  $\forall x \in \mathbb{R}$ , if  $\forall n \in \mathbb{N}, \frac{3n+2}{n+1} \leq x$ , then  $3 \leq x$

1) Let  $x \in \mathbb{R}$

Assume  $5 < x \Rightarrow 0 < x-5$

By the Archimedean Property

$\exists n \in \mathbb{N}, 7-x < (x-5)n$

Choose  $n \in \mathbb{N}$  with<sup>↑</sup>

$$7-x < xn - 5n$$

$$5n + 7 < xn + x$$

$$5n + 7 < x(n+1)$$

$$\frac{5n+7}{n+1} < x$$

$$\frac{5n+7}{n+1} < x$$

$$5n+7 < x(n+1)$$

$$5n+7 < nx+x$$

$$5n - nx + 7 < x$$

$$n(5-x) + 7 < x$$

$$x > n(5-x)$$

$$x > 5 + n + 7$$

2)

Let  $x, y \in \mathbb{R}$

Assume  $x < y$

By the Archimedean Property

$$\exists n \in \mathbb{N}, 3x + 3y < (y - x)n$$

Choose  $n \in \mathbb{N}$  with  $3x + 3y < (y - x)n$

$$3x + 3y < ny - nx$$

$$3x + xn < yn - 3y$$

$$\exists n \in \mathbb{N}, (3+n)x < (n-3)y$$

3)

Let  $x \in \mathbb{R}$

Assume  $x > 3$   
 $x - 3 > 0$

$$x - 2 < n(3 - x)$$

$$x - 2 < 3n - nx$$

$$3n + 2 > xn + x$$

$$3n - xn > x - 2$$

$$n(3 - x) > x - 2$$

$$n > (x - 2)/(3 - x)$$

$$x + nx < 3n + 2$$

$$x(1+n) < 3n + 2$$

$$x < \frac{3n+2}{1+n}$$