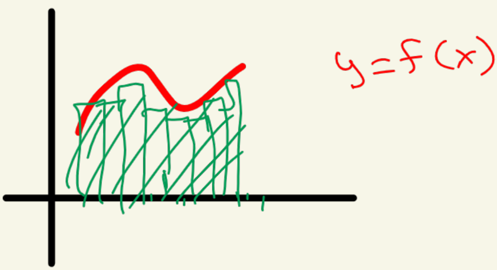
Jan 5, 2026

Part 1: Integrals

Part 2: Application of Integrals

Part 3: Sequences and Series

Definite Integrals



If $f(x) > 0$

$$\int_a^b f(x)\,dx = \text{area below the graph}$$

Conseptual Intepretations = continuosly accumulation of quantity.

Indefinite Integrals $\int f(x)$

essentialy antiderivitives

$x^2 \to$ antider. $\frac{x^3}{3}$ so

the most general antider.

is $\frac{x^3}{3} + C$.

$\int f(x)\,dx = F(x) + C$ means

that most general antider.

of $f(x)$

$F'(x) = f(x)$ $\int_0^1 x^2\,dx = \frac{1}{3}$

$\int x^2\,dx = \frac{x^3}{x} + c$

FTC: Foundamental Theory of Calculus

FTC: $\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}$

this always works

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$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$\int (f'(x)\,g(x) + f(x)\,g'(x))\,dx = f(x)+g(x)+C$

$\int f'(x)+g(x)\,dx + \int f(x)\,g'(x) = f(x)g(x)+C$

$\int f(x)\,g'(x) = f(x)g(x) - \int f'(x)\,g(x)\,dx$

Integration by parts

$\int x^2\,dx - 7 = \frac{x^3}{3} + (C-7) = \int x^2\,dx$

1)

$\int x\cdot\cos x\,dx$ Integration by parts

Solution:

$$\int x \cdot \boxed{\cos x}\, dx = \int x \cdot \boxed{(\sin x)'}\, dx =$$

$$= x \cdot \sin x - \int x' \sin x \, dx = x \cdot \sin x -$$

$$- \int \sin x \, dx = x \cdot \sin x + \cos x + C$$

2)

$$\int 2^x \cdot x \, dx = \int \left(\frac{2^x}{\ln 2}\right)' \cdot x \, dx =$$

$$= \frac{2^x}{\ln 2} \cdot x - \int \frac{2^x}{\ln 2} \boxed{\overset{1}{x'}}\, dx =$$

$$= \frac{2^x \cdot x}{\ln 2} - \frac{1}{\ln 2} \int 2^x \, dx \nearrow \frac{2^x}{\ln 2}$$

$$= \boxed{\frac{2^x \cdot x}{\ln 2} - \frac{2^x}{(\ln 2)^2}}$$

3) $\int x^2 e^x \, dx = \int x^2 \cdot (e^x)' \, dx =$

$x^2 e^x - \int (x^2)' \cdot e^x \, dx = x^2 e^x - 2\int$

$-2\int \underbrace{x \cdot e^x} \, dx = x^2 e^x - 2 \int x \cdot (e^x)' \, dx$

$= x^2 \cdot e^x - 2\left(x \cdot e^x - \int \overset{\to 1}{\boxed{x'}} e x \, dx \right) =$

$= x^2 \cdot e^x - 2 \cdot x \cdot e^x + 2 \boxed{\int e^x \, dx}$ e^x

$= \boxed{x^2 e^x - 2x \cdot e^x + 2 \cdot e^x + C}$

4) $\int \ln x \, dx = \int \overset{x'}{\boxed{1}} \cdot \ln x \, dx =$

$= \int x' \ln x \, dx = x \cdot \ln x - \int x \, (\ln x)' \, dx$

$= x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx = x \cdot \ln x - \int 1 \, dx$

$$= \boxed{x \cdot \ln x - x + c}$$

5) $\int \cos^2 x \, dx = \int \cos x \cdot (\sin x)' \, dx$

$= \cos x \cdot \sin x - \int (\cos x)' \sin x \, dx$

$= \cos x \cdot \sin x + \boxed{\int \sin^2 x \, dx}$

$= \cos x \cdot \sin x + \int (1 - \cos^2 x) \, dx$

$= \cos x \cdot \sin x + \int 1 \, dx - \int \cos^2 x \, dx$

$= \cos x \cdot \sin x + x - \int \cos^2 x \, dx$

Solve for $\int \cos^2 x \, dx$:

$$\boxed{2\int \cos^2 x \, dx = \frac{1}{2}(\cos x \sin x + x) + C}$$

$u = x^2$ $\qquad\qquad$ $u = g(x)$

$\frac{du}{dx} = 2x$ $\qquad\qquad$ $g'(x)$

$du = 2x\,dx$ $\qquad\qquad$ $g'(x)\,dx$

$$\int \underbrace{f(x)}_{u}\,\underbrace{g'(x)}_{dv}\,dx = \underbrace{f(x)}_{u}\,\underbrace{g(x)}_{v} - \int \underbrace{g(x)}_{v}\,\underbrace{f'(x)}_{du}$$

$$\int u\,dv = uv - \int v \cdot du$$

1)

$$\int \underbrace{x}_{u} \cdot \underbrace{\cos x \, dx}_{dv} :$$

$u = x \qquad v = \sin x$

$dv = \cos x \cdot dx$

$du = dx$

$$\int \underbrace{x}_{u} \cdot \underbrace{\cos x \, dx}_{dv} = \int u \, dv = uv - \int v \, du$$

$$= x \cdot \sin x - \int \sin x \, dx = x \cdot \sin x - \cos x + C$$

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Substitution Rule (aka "u-sub")

$$\int (f(x) + g(x)) dx = \int f(x) \, dx + \int g(x) \, dx$$

$\int f(x) g(x) \, dx \rightarrow$ integration by parts

$f(x) g(x) - \int_{*} f'(x) g(x) \, dx$

* often serves the purpose

$\int f(g(x)) \, dx$ ✗ $\rightarrow$ no such thing

$\int f(g(x)) g'(x) \, dx \rightarrow$ there is a formula for this, subsitution rule

$$\int f(g(x))g'(x)\,dx = \int \underbrace{F'(g(x))g'(x)}_{(F(g(x))'}\,dx$$

$$= \int (F(g(x))'\,dx = F(g(x)) + C$$ Let $u = g(x)$

$$= F(u) + C = \int f(u)\,du$$ ✓

let $u = x^2$

$\frac{du}{dx} = 2x$ $\quad du = 2x\,dx$

$u = g(x)$

$du = g'(x)\,dx$

$\frac{du}{dx} = g'(x)$

$$\int f(\underbrace{g(x)}_{u})\underbrace{g'(x)}_{du}\,dx = \int f(u)\,du$$

Practice

1.

$$\int x \cdot \cos(x^2)\, dx$$

let $u = x^2$

$du = 2x\, dx \rightarrow \frac{du}{2}$

$$\int x \cdot \cos(x^2)\, dx = \int \cos(u) \cdot \boxed{\frac{du}{2}} =$$

$$\frac{1}{2} \int \cos(u)\, du = \frac{1}{2} \sin(u) + C =$$

$$\frac{1}{2} \sin(x^2) + C$$

2.

$$\int x^2 \sqrt[3]{4x^3 + 3}\, dx$$

let $u = 4x^3 + 3$

$du = 12x^2\, dx \mid x^2 dx = \frac{du}{12}$

$$\int x^2 \sqrt[3]{4x^3 + 3} = \int \sqrt[3]{u}\, \frac{du}{12} = \frac{1}{2} \int \sqrt[3]{u}\, du$$

$$= \frac{1}{12}\int u^{\frac{1}{3}}\,du = \frac{1}{12}\,\frac{u^{\frac{4}{3}}}{4/3} + C =$$

$$= \frac{u^{\frac{4}{3}}}{16} + C = \boxed{\frac{(4x^3+3)^{\frac{4}{3}}}{16} + C}$$

3.

$$\int \sin(10x)\,dx$$

let $u = 10x$

$du = 10\,dx$

$dx = \frac{du}{10}$

$$\int \sin(10x)\,dx = \int \sin(u)\,\frac{du}{10} = \frac{1}{10}\int \sin(u)\,du$$

$$= -\frac{1}{10}\cos(u) + C = \boxed{-\frac{1}{10}\cos(10x) + C}$$

Useful shortcut :

If $\int f(x)\,dx = F(x) + C$, then

- $\int f(x+a)\,dx = F(x+a) + C$

$$\int \cos(x+7)\,dx = \sin(x+7) + C$$

$$\int f(a\cdot x)\,dx = \frac{F(a\cdot x)}{a} + C$$

$$\int \cos(7\cdot x)\,dx = \frac{\sin(7x)}{7} + C$$

5. $\displaystyle\int \frac{x^3}{1+x^4}\,dx$

Solution:

Let $u = 1+x^4$

$du = 4x^3\,dx \quad x^3dx = \frac{du}{4}$

$$\int \frac{x^3}{1+x^4}\,dx = \int \frac{du/4}{u} = \frac{1}{4}\int \frac{1}{u}\,du =$$

$$= \frac{1}{4}\ln|x| + C = \boxed{\frac{1}{4}\ln|1+x^4| + C}$$

6. $\displaystyle\int \frac{x}{1+x^4}\,dx$

$$\int \frac{x}{1+x^4}\,dx = \int \frac{x}{1+(x^2)^2}\,dx$$

Let $u = x^2$

$du = 2x\,dx \qquad x\,dx = \frac{du}{2}$

$$\int \frac{1}{1+x^4}\,x\,dx = \int \frac{1}{1+u^2}\,\frac{du}{2} =$$

$$= \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}(u) + c$$

$$= \boxed{\frac{1}{2}\tan^{-1}(x^2) + c}$$

7. $\displaystyle\int \frac{1}{x^2+14x+130}\,dx$

Solution: $x^2+14x+130 = (\qquad)^2 + c$

$(a+b)^2 = a^2+2ab+b^2$

$$x^2+14x+130=(x^2+2\cdot x\cdot 7+7^2)-7^2+130$$

$$=(x+7)^2+81 \rightarrow \text{completing the square}$$

$$\int \frac{1}{(x+7)^2+81}\,dx$$

Let $u=x+7$

$du=dx$

$$\int \frac{1}{u^2+81}\,du=\int \frac{1}{u^2+9^2}\,du=$$

$$=\frac{1}{9}\tan^{-1}\left(\frac{u}{9}\right)+C=\boxed{\frac{1}{9}\tan^{-1}\left(\frac{x+7}{9}\right)+C}$$

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$$\int \frac{\sin x\cdot\cos x}{1+\sin^2 x}\,dx$$

Let $u=\sin x$ $\quad du=\cos x\,dx$

$$\int \frac{\sin x \cos x}{1+\sin^2 x}\, d = \int \frac{u}{1+u^2}\, du$$

Let $v = 1+u^2$

$dv = 2u \; du$

$u \cdot du = \frac{dv}{2}$

$$\int \frac{1}{v} \frac{dv}{2} = \frac{1}{2}\int \frac{1}{v}\, dv = \frac{1}{2}\ln|v| + c$$

$$\boxed{\frac{1}{2}\ln(1+\sin^2 x) + C}$$

Alternative:

Let $u = 1+\sin^2 x$ $\quad du = 2\sin \cdot \cos x$

$$\int \frac{\sin x \cdot \cos x}{1+\sin^2 x} = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| + c$$

$$= \frac{1}{2}\ln(1+\sin^2 x) + C$$

12. $\int x^5 \cdot \cos(x^3) = \int x^2 \cdot x^3 \cdot \cos(x^3)\, dx$

Solution: Let $u = x^3$

$du = 3x^2\, dx$

$$\int x^2 \underbrace{x^3 \cos(x^3)}_{u \cdot \cos u}\, dx = \int u \cdot \cos(u) \frac{du}{3} =$$

$$= \frac{1}{3} \int u \cdot \cos u \quad du = \frac{1}{3} \int u \cdot (\sin' u)\, du =$$

$$= \frac{1}{3} \left[u \cdot \sin(u) - \int u' \sin u \, du \right] = \frac{1}{3} ($$

$$= \frac{1}{3} \left(u \cdot \sin u - \int \sin u \, du \right) = \frac{1}{3} (u \cdot \sin u$$

$$+ \cos u) + C = \frac{1}{3} (x^3 \cdot \sin x^3 + \cos(x^3) + C)$$

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Trigonometric Integrals

1. Integrals of form $\int \sin^m x \cos^n x \, dx$
2. Integrals of form $\int \tan^m x \sec^n x \, dx$
3. Integrals of form $\int \sin(mx) \sin(nx) \, dx$ .... etc

1 $\int \sin^m x \cdot \cos^n x \, dx$

$$\int \cos^3 x \, dx = \int (1-\sin^2 x) \cos x \, dx =$$

$$\cos^3 x = \cos x \cdot \cos^2 x = \cos x (1-\sin^2 x)$$

$$\int \underbrace{(1-\sin^2 x)}_{1-u^2} \underbrace{\cos x \, dx}_{du} = \int (1-u^2) \, du$$

let $u = \sin x$

$du = \cos x$

$$\int \sin^m x \cdot \cos^n x \, dx$$

Case 1: n is odd

$$n = 2k+1$$

$$\int \sin^m x \cdot \cos^{2k+1} x \, dx = \int \sin^m x \cdot (\cos^2 x)^k \cdot \cos x$$

$$= \int \sin^m x \cdot (1-\sin^2 x)^k \cos x \, dx$$

case 2: m is odd $(m = 2k+1)$

$$\int \sin^{2k+1} x \cdot \cos^n dx = \int \sin^{2k} x \cdot \sin x \cdot \cos^n x$$

$$= \int (1-\cos^2 x)^k \cdot \cos^n x \cdot \sin x \, dx \qquad u = \cos x$$

Case 3 m & n odd, do any

Ex.

$$\int \sin^6 x \cdot \cos^5 x \, dx = \int \sin^6 x \cdot \cos^4 x \cdot \cos x \, dx$$

$$= \int \sin^6 x \cdot (\cos^2 x)^2 \cos x \, dx$$

let $u = \sin x$

$du = \cos x \, dx$

$$= \int \underbrace{\sin^6 x \cdot (1-\sin^2 x)^2}_{u^6 \cdot (1-u^2)^2} \underbrace{\cos x}_{du} \, dx$$

$$= \int u^6 (1-u)^2 \, du = \int u^6 (1-2u^2+u^4) \, du$$

$$= \int (u^6 - 2u^8 + u^{10}) \, du = \frac{u^7}{7} - 2\frac{u^9}{9} + \frac{u^{11}}{11} + C$$

$$= \boxed{\frac{\sin^7 x}{7} - \frac{2\sin^9 x}{9} + \frac{\sin^{11}}{11} + C}$$

Ex. $\int \sin^4 x \cdot \cos^6 x \, dx = \int (\sin^2 x)^2 (\cos^2 x)^3 \, dx$

$$= \int \left(\frac{1-\cos(2x)}{2}\right)^2 \left(\frac{1+\cos(2x)}{2}\right)^3 dx$$

Ex.

$$\int \cos^2 x \, dx$$

Solution:

$$\int \cos^2 x \, dx = \int \frac{1+\cos(2x)}{2} \, dx =$$

$$\frac{1}{2}\left(x + \int \cos(2x)\, dx\right) = \boxed{\frac{x}{2} + \frac{1}{2} \cdot \frac{\sin(2x)}{2} + C}$$

$$\int \cos(2x) = \frac{\sin(2x)}{2}$$

Ex. $$\int \cos^2 x \cdot \sin^2 x \, dx = \int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2}$$

$$= \frac{1}{4} \int (1-\cos(2x))(1+\cos(2x)) \, dx =$$

$$= \frac{1}{4} \int (1-\cos^2(2x)) \, dx = \frac{1}{4} x - \frac{1}{4} \int \cos^2(2x)$$

$$= \frac{x}{4} - \frac{1}{4} \int \frac{1+\cos(4x)}{2} \, dx$$

$$= \frac{x}{4} - \frac{x}{4} - \frac{1}{8}\int \cos(4x)\, dx$$

$$= \frac{x}{8} - \frac{1}{8}\cdot\frac{\sin(4x)}{4} + C = \boxed{\frac{x}{8} - \frac{\sin(4x)}{32} + C}$$

odd even

$$\int \tan^{m} x \cdot \sec^{n} x \; dx$$

1. If n is even, $n = 2k$

$$\int \tan^{m} x \cdot \sec^{2k} x\, dx = \int \tan^{m} x \cdot \underbrace{\sec^{2k-2} x}_{(\sec^2 x)^{k-1}} \cdot \sec^2 x$$

$$= \int \underbrace{\tan^{m} x\,(1+\tan^2 x)^{k-1}}_{\text{interms of } u} \underbrace{\sec^2 x}_{du} \; dx$$

2. If $m = 2k+1$

$$\int \tan^{10} x \cdot \sec^{6} x\; dx = \int \tan^{10} x \cdot \sec^4 x \cdot \sec^2 x$$

$$= \int \tan^{10} x\,(\sec^2 x)^2 \cdot \sec^2 x \; dx$$

$$= \int \tan^{10} x \, (1+\tan^2 x)^2 \cdot \sec^2 x \;\; dx$$

let $u = \tan x$

$du = \sec^2 x \;\; dx$

$$= \int u^{10} (1+u^2)^2 \; dx \;\; = \int u^{10} (1+2u^2+u^4) du$$

$$= \int (u^{10} + 2u^{12} + u^{4}) \, du \;\; = \frac{u^{11}}{11} + 2\frac{u^{13}}{13} + \frac{u^{15}}{15} + c$$

$$= \boxed{\frac{\tan^{11} x}{11} + \frac{2\tan^{13}}{13} + \frac{\tan^{15}}{15} + C}$$

$$\int \cot^3 x \cdot \csc^3 x \;\; dx$$

Solution:

$$\int \cot^2 x \cdot \csc^2 x \, (\cot x \cdot \csc x) \;\; dx =$$

$$\int (\csc^2 x - 1)\csc^2 x \quad \cot x \cdot \csc x \, dx$$

let $u = \csc x$

$du = -\cot x \csc x \, dx$

$$= -\int (u^2 - 1)\, u^2 \, du = -\int (u^4 - u^2)^2 \, du =$$

$$= \int (u^2 - u^4)\, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \boxed{\frac{\csc^3 x}{3} - \frac{\csc^5 x}{5} + C}$$

1 $\int \cos(7x) \cdot \cos(4x) \, dx =$

$$= \int \frac{1}{2}\left(\cos(7 - 4x) + \cos(7x + 4x)\right) dx$$

$$= \frac{1}{2}\int \left(\cos(3x) + \cos(11x)\right) dx$$

$$= \frac{1}{2}\left(\frac{\sin(3x)}{3} + \frac{\sin(11x)}{11}\right) + C$$