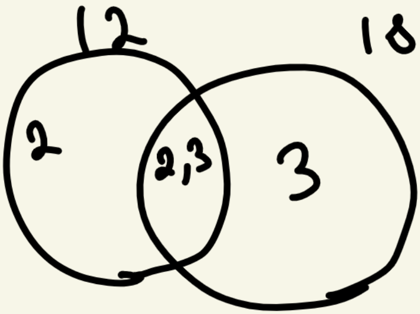
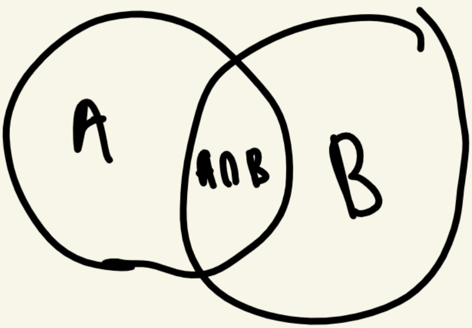
office hours
Erie 3125

January 5, 2026



$$\frac{ab}{\gcd(a,b)} = \operatorname{lcm}(a,b)$$



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proposition 1:

For all $x \in \mathbb{R}$, $x \cdot 0 = 0$

and $0 \cdot x = 0$

proof: Let $x \in \mathbb{R}$

$x + 0 = x$

$1 + 0 = \underline{1}$ by identity

$0 + 0 = 0$

$x(1+0) = x \cdot 1$

$x(1+0) = x$ by identity

$x \cdot 1 + x \cdot 0 = x$ by distributivity

$x + x \cdot 0 = x$ by identity

$-x + (x + x \cdot 0) = -x + x$

$-x + x + x \cdot 0 = -x + x$ Assosiativ.

$0 + x \cdot 0 = 0$ invertibility

$x \cdot 0 = 0$ identity

Also, $0 \cdot x = 0$ for all

$x \in \mathbb{R}$, $x \cdot 0 = 0$ and $0 \cdot x = 0$

January 7, 2026

Proposition 2: For all $x, y \in \mathbb{R}$

$$(-x)y = -(xy)$$

proof: Let $x, y \in \mathbb{R}$

Universal Generalization

To prove "For all $x \in R$, $P(x)$"

Let $x \in \mathbb{R}$

* Demonstrate $P(x)$

Therefore, For all $x \in \mathbb{R}$, $P(x)$

$$-\Box + \Box = 0$$

$-(xy) + xy = 0$ by invertibility

$$\underbrace{-(xy)}_{\text{keep}} + xy + (-x)\cdot y = 0 + \underbrace{(-x)\cdot y}_{\text{keep}}$$

$$-(xy) + xy + (-xy) = 0 + (-x)\cdot y$$

$$-(xy) + xy + (-1)(xy) = 0 + (-x)\cdot y$$

$1x = x \qquad -(1x) = -x \qquad (-1)x = -x$

$-(xy) + (x + (-x))y = (-x)y$ Distrib. and Ident.

$-(xy) + 0\cdot y = (-x)y$ Invertibility

$-(xy) + 0 = (-x)y$ Prop 1

$-(xy) = (-x)y$ by identity

Therefore, for all $x, y \in \mathbb{R}$ $(-x)y = -(xy)$

Proposition 3:

For all $x, y \in \mathbb{R}$, $(-x)(-y) = xy$

Proof: Let $x, y \in \mathbb{R}$

$-x + x = 0$ Invertibility

$(-x + x)(-y) = 0 \cdot (-y)$

$(-x)(-y) + x(-y) = 0 \cdot (-y)$ dist.

$(-x)(-y) + x(-y) = 0$ Prop 1

$(-x)(-y) + x(-y) + xy = 0 + xy$

$(-x)(-y) + x(-y) + xy = xy$ (Ident.)

$(-x)(-y) + x(-y+y) = xy$ dist.

$(-x)(-y) + x \cdot 0 = xy$ invertib.

$(-x)(-y) + 0 = xy$ Prop 1

$(-x)(-y) = xy$ Identity

For all $x, y \in \mathbb{R}$ $(-x)(-y) = xy$

January 9, 2026

$\underbrace{\text{My dog}}_{\text{subject}}$ is yellow

X is yellow → open sentence

Examples: $y = 2x + 1$

$x < 3$

Each variable has an allowable set of values called the "universe of discourse" for variable

Example: X is wearing Y

Quantified Statements:

Two of my cats are orange

predicate: X is orange

Universe of discourse for x:

The set of all my cats

Universal Quantified Statements

"All my cats are orange"

predicate: X is orange

universe: C= set of my cats

Notation: $\forall x \in C$, x is orange

Read: "for all values of X in C, x is orange"

Existential Quantified Statements

"Some of my cats are orange"

Notation: $\exists x \in C$, X is orange

Read: "for at least one value of X is C, X is orange"

"there is a value of X in C where X is orange"

" there exists an X in C for which X is orange"

$\mathbb{N}$ natural nums $\{1, 2, 3, 4 \ldots\}$

$\mathbb{Z}$ integers $\{\ldots -4, -3 \ldots 0, 1, 2, 3 \ldots\}$

$\mathbb{Q}$ rational nums All fraction & integ.

$\mathbb{R}$ real nums

$\mathbb{C}$ complex nums — combination of real / imagin. numbers

1. $\forall x \in \mathbb{N}, \boxed{0 \leq x}$ True

2. $\forall x \in \mathbb{R}, 0 < x^2$ False

3. $\forall x \in \mathbb{R}, \boxed{\exists y \in \mathbb{R}, x < y}$ True

4. $\exists y \in \mathbb{R}, \boxed{\forall x \in \mathbb{R}, x < y}$ False

5. $\forall x \in \mathbb{Z}, \boxed{\exists y \in \mathbb{Z}, y = 2x}$ True

6. $\forall x \in \mathbb{Z}, \boxed{\exists y \in \mathbb{Z}, x = 2y}$ False

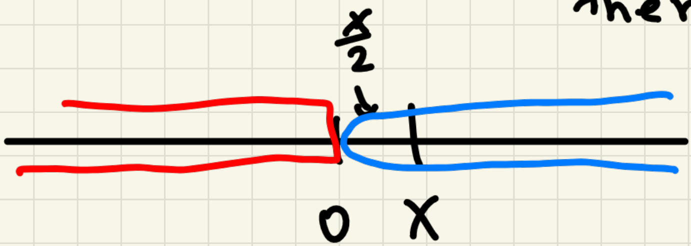
$5 = 2y$?

7. $\forall x \in \mathbb{Z}$, If $\boxed{\exists y \in \mathbb{Z},\ x = 2y}$ If its even then

then $\boxed{\exists q \in \mathbb{Z},\ x^2 = 2q}$ True

8. $\forall x \in \mathbb{R}$, If $\boxed{\forall a \in (0,\infty),\ x \le a}$

then $\boxed{x \le 0}$



If its $x \geq 0$, but never less or equal to zero

True

$\{0\}$

9. $\forall x \in \mathbb{R}$ If $\boxed{\forall a \in \mathbb{R},\ a \cdot x \le 0}$,

then $\boxed{\forall b \in \mathbb{R},\ 0 \le b \cdot x.}$ True

January 12, 2026

Proposition: $0 < 1$

Proof by contradiction:

To prove a proposition P

Assume $\neg P$ (negation of P)

derive a contrad. Q and $\neg Q$

Assume: $\boxed{0 \not< 1}$

Since $0 \neq 1$, we have $1 < 0$ by trichotomy

$-1 + 1 < -1 + 0$ by monotonicity

$0 < -1$

then:

$0(-1) < (-1)(-1)$ by monotonicity

$0 < 1$ by Prop 1 & Prop 3

Now, $0<1$ and $0 \not< 1$. This is contradiction

Therefore, $0 < 1$

Proposition: $1+1 \neq 1$

Proof:

Assume $1+1=1$

$$1+1(-1) = 1+(-1)$$

$$1 = 0$$

But, $1 \neq 0$ This is contradiction

Therefore, $1+1 \neq 1$

Proposition: $1+1 \neq 0$

Proof: Assume $1+1=0$

We know $0<1$

then $1+0 < 1+1$ by monotonict.

So, $1 < 0$

this is a contradiction, since $0<1$

Therefore, $1+1 \neq 0$

Definition:

$2 = 1+1 \quad 3 = 2+1 \quad 4 = 3+1 \quad 5 = 4+1$

Example: $2 < 4$

Proof:

$0 < 1$

then $1+0 < 1+1$

so $1 < 2$

by transitivity, $0 < 2$

then $1+0 < 2+1$

So $1 < 3$

then $1+1 < 3+1$, so $2 < 4$

$\boxed{\text{January 14, 2026}}$

Proposition:

$\forall\, x, y \in \mathbb{R}$, If $x < y$, then $-y < -x$

Direct Proof

To prove If P, then Q

Assume P

*Demonstrate Q

Therefore, If P, then Q

Proof

Let $x, y \in \mathbb{R}$

Assume $x < y$ by monoton

$$-x + x + (-y) < -x + y + (-y)$$

$$0 + (-y) < -x + 0$$

$$-y < -x \quad \because$$

Therefore, If $x < y$, then $-y < -x$

Monotonicity (Negative Multiplication)

$\forall x, y, z \in \mathbb{R}$, If $\boxed{x < y \text{ and } z < 0}$,

then $\boxed{yz < xz}$

Proof

Let $x, y, z \in \mathbb{R}$

target $yz < xz$

Assume $x < y$ and $z < 0$

Since $z < 0$, we have $0 < -z$

$-z + z < -z + 0$

Then $x(-z) < y(-z)$

So, $-xz < -yz$

$$xz + (-xz) + yz < xz + (-yz) + yz$$

$$0 + yz < xz + 0$$

$$yz < xz$$

Therefore, If $x < y$ and $z < 0$, then $yz < xz$

QED

$\forall x, y \in \mathbb{R}$, If $x < y$, then $x+2 < y+3$

Proof Let $x, y \in \mathbb{R}$

Assume $x < y$

$x+2 < y+3$

$x+2 < y+2$ by monotonicity

Since $2 < 3$, $y+2 < y+3$ by monoton

$x+2 < y+2$ & $y+2 < y+3$

By transitivity, $x+2 < y+3$

Using transitivity to Prove $A < B$

$$\begin{array}{ll} 1 & A < C \\ 2 & C < B \\ \hline & A < B \end{array}$$

January 16, 2026

Rules of Negation

1. $\neg(\forall x \in U, P(x))$ is $\exists x \in U, \neg P(x)$

2. $\neg(\exists x \in U, P(x))$ is $\forall x \in U, \neg P(x)$

3. $\neg$ (P and Q) is $\neg$P or $\neg$Q

4. $\neg$ (P or Q) is $\neg$P and $\neg$Q

5. $\neg$ (If P, then Q) is P and $\neg$Q

$$\forall x \in \mathbb{R}, \text{ If } \overbrace{\boxed{\forall a \in (0,\infty), a \leq x}}^{P}, \text{ then } \overbrace{\boxed{x \leq 0}}^{Q}$$

Negation

$$\exists x \in \mathbb{R}, \overbrace{\boxed{\forall a \in (0,\infty), \ a \leq x}}^{P} \text{ and } \overbrace{\boxed{0 < x}}^{Q}$$

1. $\forall_x \in \mathbb{Z}$, If $\boxed{\exists_a \in \mathbb{Z}, x = 2a+1}$ p, then $\boxed{\exists_b \in \mathbb{Z}, x = 3b}$ q

$\exists_x \in \mathbb{Z}$, $\exists_a \in \mathbb{Z}$, $x = 2a+1$ and $\forall_b \in \mathbb{Z}$, $x \neq 3b$ ✓

2. $\forall_x \in \mathbb{R}$, $\exists_y \in \mathbb{R}$, $x < y$

$\exists_x \in \mathbb{R}$, $\forall_y \in \mathbb{R}$, $x \geq y$ ✓

3. $\forall x, y \in \mathbb{R}$, If $0 < x < 1$ and $x + y = 1$, then $0 < y < 1$

✓

$\exists x, y \in \mathbb{R}$, $0 < x < 1$ and $x + y = 1$ and $y \leq 0$ or $1 \leq y$

1. $A < C$
2. $C < B$

$> A < B$

1. $\forall x, y \in \mathbb{R}$, $\boxed{\text{If } 0<y<x}$, then $\boxed{y<2x}$

Structure Let $x, y \in R$

Assume $0 < y < x$

$0<y$ and $y < x$

then $0 < x$ $\quad x+0 < x+x$

so $x < 2x$

$y < x < 2x$

So $\boxed{y < 2x}$

2. $\forall x, y \in \mathbb{R}$, If $x<2<y$, then $x+2<y^2$

$x < 2$ & $2<y$

$2 \cdot 2 < 2 \cdot y$ so $4 < 2y$

Since $0<2<y$ we have $0<y$ $\quad 2 < y$ so $2(y) < y(y)$

$2y < y^2$

then $4 < y^2$

$x<2$ so $x+2 < 4$

$\boxed{\text{Now, } \quad x+2 < 4}$

January 19, 2026

Proving Statements with Existential Qualifiers

Universal Generalization

To prove $\forall x \in \mathbb{R}, P(x$

Let $x \in \mathbb{R}$

* Show $P(x)$

Therefore, $\forall x \in \mathbb{R}, P(x)$

Existential Generalization

To prove $\exists x \in \mathbb{R}, P(x)$

Let $x = \boxed{}$ some specific value

* Demonstrate $P(x)$ for that specific value

Therefore, $\exists x \in \mathbb{R}, P(x)$

Example

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \exists y \in \mathbb{R}, x < y$$

proof: let $x \in \mathbb{R}$

let $y = x+1$

Since $0 < 1$, $x+0 < x+1$

then $x < y$

Therefore, $\exists y \in \mathbb{R}$, $x < y$

Example

$\forall x, y \in \mathbb{R}$, If $\boxed{x < y}$, then $\boxed{\exists z \in \mathbb{R}}$ $\boxed{x < z < y}$

Proof:

let $x, y \in \mathbb{R}$

Assume $x < y$

let $z = \frac{x+y}{2}$

Since $x < y$

then $x+x < y+x$

$2x < x+y$

$\frac{1}{2}(2x) < \frac{1}{2}(x+y)$

So, $x < z$

Also, since $x<y$, $x+y < y+y$

then $x+y < 2y$

so $2^{-1}(x+y) < 2^{-1}\,2y$

Now $\frac{x+y}{2} < y$, so $z < y$

Therefore, $\exists z \in \mathbb{R},\ x < z < y$

Example

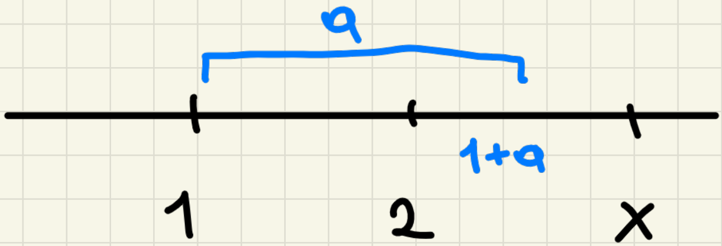
$\boxed{\forall x \in \mathbb{R}}$, If $\boxed{2 < x}$, then $\exists a \in \mathbb{R}$, $1 < a$ and $1+a < x$

Proof

Let $x \in \mathbb{R}$

Assume $2 < x$ $\quad$ $1 < a$

Let $a = x/2$



$$1 + a = \frac{2+x}{2}$$

$$a = \frac{x}{2}$$

Since $2 < x$, $\frac{1}{2}(2) < \frac{x}{2}$

So, $1 < a$

Then, $1 + a < a + a$

So, $1 + a < 2a$

$1 + a < 2\left(\frac{x}{2}\right)$ so $1 + a < x$

Therefore $1 < a$ and $1 + a < x$

So, $\exists a \in \mathbb{R}$, $1 < a$ and $1 + a < x$

Example

$\forall x, y \in \mathbb{R}$, If $0 < x < 1$ and $0 < y < 1$,

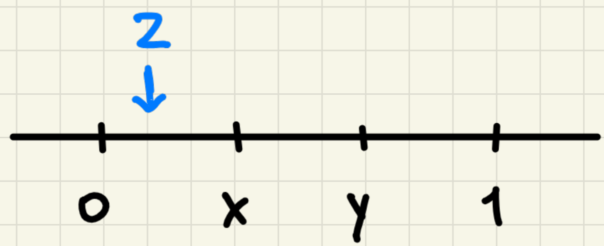
then $\exists z \in \mathbb{R}$, $0 < z < x$ and $0 < z < y$

Proof

let $x, y \in \mathbb{R}$

Assume $0 < x < 1$ and $0 < y < 1$

let $z = x \cdot y$



TBC

January 21, 2026

$\wedge$ "and" $\qquad$ $\vee$ "or" $\qquad$ $\mathbb{B} = \{T, F\}$

$\wedge$	T	F
T	T	F
F	F	F

$\vee$	T	F
T	T	F
F	T	F

Inclusive OR - includes the possibility of both statements being true

$\neg$ "not" $\qquad$ $\neg F = T$ $\qquad$ $\neg T = F$

$x \Rightarrow y$ statement or binary operation

In COMP 1000 $x \Rightarrow y$ is the same as $\neg x \vee y$ but not MATH 1020

In $\mathbb{R}$ $x(y+z) = xy + xz$

in $\mathbb{B}$ boolean

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$x \wedge (x \vee y) = x$$

$$x \vee (x \wedge y) = x$$

$x \Rightarrow y$ on $\mathbb{B}$

behaves analogous to $x \leq y$ on $\mathbb{R}$

$$x \wedge T = x \qquad x \vee F = x$$

$$x \vee T = T \qquad x \wedge F = F$$

$\boxed{\text{January 23, 2026}}$

Identity:

$$x \vee F = x$$

$$x \wedge T = x$$

Annihilation:

$$x \wedge F = F$$
$$x \vee T = T$$

Complementation:

$$x \wedge \neg x = F \qquad \neg \boxed{\neg x} \wedge \boxed{\neg x} = F$$
$$x \vee \neg x = T \qquad \neg \boxed{\neg x} \vee \boxed{\neg x} = T$$

Proposition (Involution)

$$\forall x \in \mathbb{B}, \neg\neg x = x$$

Proof:
let $x \in \mathbb{B}$

$$x \vee F = x \quad \text{(Identity)}$$
$$x \vee (\neg\neg x \wedge \neg x) = x \quad \text{(complementation)}$$
$$(x \vee \neg\neg x) \wedge (x \vee \neg x) = x \quad \text{(Distributivity)}$$
$$(x \vee \neg\neg x) \wedge \underline{T} = x \quad \text{(Complementation)}$$
$$(x \vee \neg\neg x) \wedge (\underline{\neg\neg x \vee \neg x}) = x$$
$$\neg\neg x \vee (x \wedge \neg x) = x$$

$$\neg\neg x \vee F = x$$

So, $\neg\neg x = x$ By identity

1. $\forall x,y \in \mathbb{B}, (x \wedge \neg y) \vee (y \wedge \neg x) = (x \vee y) \wedge (\neg x \vee \neg y)$

2. $\forall x,y \in \mathbb{B}$, If $x \wedge \neg y = x$, then
$$\neg x \wedge y = y$$

1.

let $x,y \in \mathbb{B}$

$$(x \wedge \neg y) \vee (y \wedge \neg x) = (x \vee y) \wedge (\neg x \vee \neg y)$$

$$(x \wedge y) \vee \underbrace{(x \wedge \neg x)}_{F} \vee \underbrace{(\neg y \wedge y)}_{F} \vee (\neg y \wedge \neg x)$$

$$(x \wedge y) \vee (\neg y \wedge \neg x) = (x \vee y) \wedge (\neg x \vee \neg y)$$

$$\neg(x \wedge y) \wedge \neg(\neg y \wedge \neg x) \quad \text{DM}$$

$$(\neg x \vee \neg y) \wedge (y \vee x)$$

$$(y \vee x) \wedge (\neg x \vee \neg y) = (x \vee y) \wedge (\neg x \vee \neg y)$$

② let $x, y \in \mathbb{B}$

assume $x \wedge \neg y = x$ ...... $\neg x \wedge y = y$

$$\neg(x \wedge \neg y) = \neg x$$

$$\neg x \vee \neg\neg y = \neg x$$

$$\neg x \vee y = \neg x \quad | \wedge y$$

$$(\neg x \vee y) \wedge y = \neg x \wedge y \quad \text{Absorption}$$

$$\underline{y = \neg x \wedge y}$$

Axioms of a binary operation

Existence: $\forall x, y \in \mathbb{R}, \quad x + y \in \mathbb{R}$

Uniqueness: $\forall a, b, x \in \mathbb{R}$, If $a = b$, then

$$a + x = b + x$$

Existence: $\forall x, y \in \mathbb{B}, \quad x \wedge y \in \mathbb{B}$

Uniqueness: $\forall a, b, x \in \mathbb{B}$, If $a = b$, then

$$a \wedge x = b \wedge x$$

③ $\forall x, y, a \in \mathbb{B}$, If $x \Rightarrow y$ and $a \Rightarrow b$, then $x \wedge a \Rightarrow y \wedge b$

let $x, y, a \in \mathbb{B}$

Assume $x \Rightarrow y$ and $a \Rightarrow b$

$$x \wedge a \Rightarrow y \wedge a$$

$$x \wedge a \Rightarrow \boxed{y \wedge a}$$

$$\boxed{y \wedge a} \Rightarrow y \wedge b$$

$$x \wedge y \Rightarrow y \wedge b$$

Since $a \Rightarrow$, $y \wedge a \Rightarrow y \wedge b$

By transitivity, $x \wedge a \Rightarrow y \wedge b$

January 26, 2026

Proposition: For all $x, y \in \mathbb{B}$, the following statements are equal

(1) $x \Rightarrow y$

(2) $x \wedge y = x$

(3) $x \vee y = y$

(4) $\neg x \vee y = T$

Proof: let $x, y \in \mathbb{B}$

(1) $\Rightarrow$ (2)

Assume $x \Rightarrow y$ .... $x \wedge y = x$

$x \wedge y \Rightarrow x$ by consistency

Since $x \Rightarrow y$, $x \wedge x \Rightarrow x \wedge y$

Then $x \Rightarrow x \wedge y$

$x \wedge y = x$ by antisymmetry

Therefore, If (1) then (2)

---

(2) $\Rightarrow$ (3)

Assume $x \wedge y = x$ ...... $x \vee y = y$

Then $(x \wedge y) \vee y = x \vee y$

By Absorption, $y = x \vee y$

Therefore, If (2) then (3)

$(3) \Rightarrow (4)$

Assume $x \vee y = y$ ...... $\neg x \vee y = T$

$$\underbrace{\neg x \vee x}_{T} \vee y = \neg x \vee y \qquad \vee \neg x$$

$$T \vee y = \neg x \vee y$$

Then, $T = \neg x \vee y$

If (3) then (4)

$(4) \Rightarrow (1)$

Assume $\neg x \vee y = T$ ..... $x \Rightarrow y$

$$x \wedge (\neg x \vee y) = x \wedge T \qquad \wedge x$$

$$(x \wedge \neg x) \vee (x \wedge y) = x$$

$$F \vee (x \wedge y) = x$$

$$x \wedge y = x$$

Also, $x \wedge y \Rightarrow y$ by consistency

Then $x \Rightarrow y$

Therefore, If (4) then (1)

$\boxed{\text{January 30, 2026}}$

Proving Conditonal Statements

Meet - join - formulas

To prove "If A, then B"

$$\boxed{\begin{array}{c} A \Rightarrow B \\ A \wedge B = A \\ A \vee B = B \end{array}}$$

Direct Proof

Contraposition

Assume

↓

Proposition (Contraposition)

* demon?

$\forall x, y \in \mathbb{B}$, $x \Rightarrow y$ If and only If

$$\neg y \Rightarrow \neg x$$

Proof :

let $x, y \in \mathbb{B}$

Assume $x \Rightarrow y$ .... $\neg y \Rightarrow \neg x$

Then $x \wedge y = x$ meet formula

So, $\underset{A}{\neg y} \vee \underset{B}{\neg x} = \underset{B}{\neg x}$ join formula

Therefore, $\neg y \Rightarrow \neg B$

Conversely, assume $\neg x \Rightarrow \neg y$

Then, $\neg y \wedge \neg x = \neg y$ meet

So, $\neg\neg y \wedge \neg\neg x = \neg\neg y$

$y \vee x = y$ Involution

Then, $x \vee y = y$

$x \Rightarrow y$ join

January 30, 2026

Proving Conditional Statements

Direct Proof

Assume A

$\downarrow$

* Demonstrate B

Therefore, If A then B

Contraposition

Assume $\neg B$

$\downarrow$

* Demonstrate $\neg A$

Therefore, If A then B

① $\forall x, y \in \mathbb{R}$, If $y^2 \leq x+2$, then $2 \leq x$ or $y \leq 2$

② $\forall x, y, z \in \mathbb{R}$, If $xz + yz \leq x^2 + y^2$ then $x \leq 0$ or $y \leq x$ or $z \leq y$

③ $\forall x, y \in \mathbb{R}$, If $\forall a \in (-\infty, x-a)$, then $y \leq x$

Proof:

1. Assume $x < 2$ and $2 < y$ ⋯ $x+2 < y^2$

Since $x<2$, $x+2 < 4$

Since $0<2<y$, we know $2<y$

Now, $2 \cdot y < y \cdot y = 2y < y^2$

Since, $2<y$, $x+2 < 2y$

2. Assume $0<x<y<z$ … $x^2+y^2 < xz + yz$

Then $x<z$, so $x^2 < xz$

Then, $\boxed{x^2 + yz} < xz + yz$

Also, $y<z$ and $0<y$, so $y^2 < yz$

Then, $x^2+y^2 < \boxed{x^2+yz}$

By transitivity, $\boxed{x^2+y^2 < xz+yz}$

③ proof: Let $x, y \in \mathbb{R}$

Assume $x < y$ ..... $\exists a \in (-\infty, 0), x-a<y$

Let $a = \frac{x-y}{2}$

$2a = x-y$

Since $x<y$ $\quad x-y<0$

Then $\frac{x-y}{2} < 0$, So $a \in (-\infty, 0)$

Since $a<0$, $2a<a$

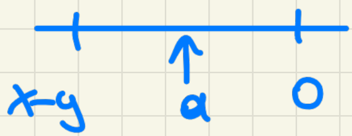
So, $x-y<a$

Then $\boxed{x-a<y}$

$x-a<y$

$a<0$

$x-y<a$



$\boxed{\text{February 2, 2026}}$

Instantiations (gathering information from your assumption)

Example

$\forall x \in \mathbb{R},\ ax \leq x$ then $x = 0$

Proof:

Let $x \in \mathbb{R}$

Assume $\forall a \in \mathbb{R},\ ax \leq x$

$\square \cdot x \leq x$

Since $0 \in \mathbb{R}$, $0 \cdot x \leq x$

Then $\underline{0 \leq x}$

Since $2 \in \mathbb{R}$, $2 \cdot x \leq x$

Then $2x - x \leq 0$, $\underline{x \leq 0}$

Thus, $\underline{x = 0}$

Example

$$\forall x \in \mathbb{R}, \text{ If } \forall a \in (1+x, \infty), \; 2x \leq 2+a, \text{ then } x \leq 4$$

Proof:

Let $x \in \mathbb{R}$

$x \leq 4$

Assume $\forall a \in (1+x, \infty)$, $2x \leq 2+a$

$$2x \leq 2 + \boxed{}$$

Since $1+x < 2+x$, we know that $2+x \in (1+x, \infty)$

Universal Instantiation

Then $2x \leq 2 + (2+x)$

$$2x \leq 4+x$$

$$2x - x \leq 4, \quad \underline{x \leq 4}$$

Then $x \leq 4$

Example

$$\forall x \in \mathbb{R}, \text{ If } \exists a \in \mathbb{R}, a \neq 1$$
$$\text{and } ax = x, \text{ then } x = 0$$

Proof

Let $x \in \mathbb{R}$

$\dots\ x = 0$

Assume $\exists a \in \mathbb{R}$, $a \neq 1$ and $ax = x$

Existential Instantiation

Choose $\alpha \in \mathbb{R}$ with $\alpha \neq 1$ and $\alpha x = x$

$$\alpha x - x = 0$$
$$(\alpha - 1)x = 0$$

Since $\alpha \neq 1$, we know $\alpha - 1 \neq 0$

$$(\alpha-1)^{-1}(\alpha-1)x = (\alpha-1)^{-1}\, 0$$

$$\underline{x = 0}$$

Example

$\forall x \in \mathbb{R}$, If $\exists b \in (0,\infty), \forall a \in (0,\infty)$
$bx < a$, then $\forall c \in (0,\infty)$, $x < c$

Proof

Let $x \in \mathbb{R}$

Assume $\exists b \in (0,\infty), \forall a \in (0,\infty)$ $bx < a$

Let $c \in (0,\infty)$

Choose $\beta \in (0,\infty)$, where $\forall a \in (0,\infty)$, $\beta x < a$

Since $\beta \in (0,\infty)$ and $c \in (0,\infty)$,
we know $\beta c \in (0,\infty)$

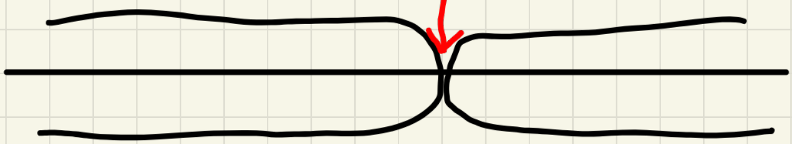
Then $\beta x < \beta c$, then $x < c$

$\boxed{\text{February 4, 2026}}$

Theorem (The Archemedean Property)

$\forall a, b \in \mathbb{R}$, If $0 < a$, then $\exists n \in \mathbb{N}, b < an$

$A = \{x \in \mathbb{Q} \mid 0 < x, x^2 < 2\}$ c $B = \{x \in \mathbb{Q} \mid 0 < x, 2 < x^2\}$



$c = \sqrt{2}$

Dedekind Cut

Proof:

Let $a, b \in \mathbb{R}$

Assume $0 < a$

$\forall n \in \mathbb{N}, \ an \leq b$

reachable Let $A = \{an \mid n \in \mathbb{N}\}$

unreachable Let $B = \{y \in \mathbb{R}, \forall_n \in \mathbb{N}, an \leq y\}$

Since $a \in A$, $b \in B$, A and B are not empty

Also, for any $y \in B$ and any $x \in A$

$x = an$ for some $n \in \mathbb{N}$, and so $an \le y$

Then A and B form a Dedekind cut

Then there is a number

$\boxed{\text{February 6, 2026}}$

The Archimedean Property

$\forall a, b \in \mathbb{R}$, if $0 < a$, then $\exists n \in \mathbb{N}$, $b < an$

Ex. $\forall \varepsilon \in \mathbb{R}$, if $0 < \varepsilon$, then $\exists n \in \mathbb{N}$, $\frac{1}{n} < \varepsilon$

Proof:

Let $\varepsilon \in \mathbb{R}$

Assume $0 < \varepsilon$

By the Archimedean Property

$\exists n \in \mathbb{N}$, $1 < \varepsilon n$

Chose $n \in \mathbb{N}$ with $1 < \varepsilon n$

Then $\frac{1}{n} < \varepsilon$

Therefore, $\exists n \in \mathbb{N}, \ \frac{1}{n} < \varepsilon$

Ex. $\forall x \in \mathbb{R}$, If $x<1$, then $\boxed{\exists n \in \mathbb{N}, x < \frac{n}{n+1}}$

Proof:

Let $x \in \mathbb{R}$

Assume $x < 1 \Rightarrow 0 < 1-x$

By the Archemedion Property

$\exists n \in \mathbb{N}, \ x < (1-x)n$

Choose $n \in \mathbb{N}$ with $x < (1-x)n$

scrap paper

$x < \frac{n}{n+1}$

$(n+1)x < n$

$nx + x < n$

$x < n - nx$

$\boxed{x} < \boxed{1-x}\,n$

$x < n - xn$

$xn + x < n$

$x(n+1) < n$

$x < \frac{n}{n+1}$

Therefore

$\exists n \in \mathbb{N}, x < \frac{n}{n+1}$

① $\forall x \in \mathbb{R}$, if $5 < x$, then $\boxed{\exists n \in \mathbb{N}, \frac{5n+7}{n+1} < x}$

② $\forall x,y \in \mathbb{R}$, if $x<y$, then $\exists n \in \mathbb{N}, (3+n)x < (n-3)y$

③ $\forall x \in \mathbb{R}$, if $\forall n \in \mathbb{N}, \frac{3n+2}{n+1} \leq x$, then $3 \leq x$

1) Let $x \in \mathbb{R}$

Assume $5 < x$ $\Rightarrow 0 < x-5$

By the Archimedean Property

$\exists n \in \mathbb{N}, \; 7-x < (x-5)n$

Choose $n \in \mathbb{N}$ with

$7 - x < xn - 5n$

$5n + 7 < xn + x$

$5n + 7 < x(n+1)$

$$\boxed{\frac{5n+7}{n+1} < x}$$

$\frac{5n+7}{n+1} < x$

$5n+7 < x(n+1)$

$5n+7 < nx+x$

$5n-nx+7 < x$

$n(5-x)+7 < x$

$x > n(5-x)$

$x > \boxed{5+} n+7$

2) Let $x, y \in \mathbb{R}$

Assume $x < y$

By the Archimedean Property

$\exists n \in \mathbb{N}, 3x+3y < (y-x)n$

Choose $n \in \mathbb{N}$ with $3x+3y < (y-x)n$

$3x+3y < ny - nx$

$3x + xn < yn - 3y$

$\exists n \in \mathbb{N}, (3+n)x < (n-3)y$

3) Let $x \in \mathbb{R}$

Assume $x > 3$

$x-3 > 0$

$x-2 < n(3-x)$

$x-2 < 3n - nx$

$3n+2 > xn+x$

$3n - xn > x-2$

$n(3-x) > x-2$

$n > (x-2)/(3-x)$

$$x + nx < 3n + 2$$

$$x(1+n) < 3n + 2$$

$$x < \frac{3n+2}{1+n}$$

means A is not empty

By the well-ordering property A has a smallest element

Existential Instantiation

Recall: If we assume $\exists_a \in \mathbb{R},\ P(a)$

Let $\alpha \in \mathbb{R}$ with $P(a)$

Existential Instantiation with the well-ordering property

If we assume $\exists a \in \mathbb{Z},\ 0 \leq a$ and $P(a)$

Let $\alpha \in \mathbb{Z}$ be smallest for which $0 \leq a$ and $P(a)$

Proposition

$\forall x \in \mathbb{Z}$, if $0 < x$, then $1 \leq x$

Proof:

Assume $\exists x \in \mathbb{Z}$, $0 < x$ and $x < 1$

Let $\alpha \in \mathbb{Z}$ be smallest with $0 < \alpha < 1$

Then $0\alpha < \alpha^2 < \alpha$

Also, $\alpha < 1$

So, $0 < \alpha^2 < \alpha < 1$

Now, $0 < \alpha^2 < 1$, but $\alpha^2 < \alpha$

This contradicts α being smallest

Therefore, $\forall x \in \mathbb{Z}$, If $0 < x$, then $1 \leq x$ $\blacksquare$

Proposition

$\forall x \in \mathbb{R}$, If $0 < x$, then $\forall n \in \mathbb{N}, 0 < x^n$

Proof:

Let $x \in \mathbb{R}$

Assume $0 < x$ and $\exists n \in \mathbb{N}$, $x^n \leq 0$

Let $m \in \mathbb{N}$ be smallest with $\boxed{x^m \leq 0}$

Since $0 < x^1$ but $x^m \leq 0$, we know $m \neq 1$

Then $m-1 \in \mathbb{N}$. Then $\boxed{0 < x^{m-1}}$

Now, $0 \cdot x < x^{m-1} \cdot x$

$0 < x^m$ but $x^m \leq 0$

This is a contradiction ∎