

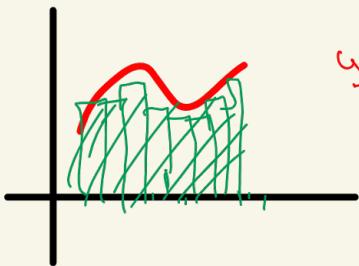

Jan 5, 2026

Part 1: Integrals

Part 2: Application of Integrals

Part 3: Sequences and Series

Definite Integrals



$$y = f(x)$$

If $f(x) > 0$

$$\int_a^b f(x) dx = \text{area below the graph}$$

Conceptual Interpretations =
continuously accumulation
of quantity.

Indefinite Integrals $\int f(x) dx$

essentially antiderivatives

$$x^2 \rightarrow \text{antider. } \frac{x^3}{3} \quad \text{so}$$

the most general antider.

is $\frac{x^3}{3} + C$.

$$\int f(x) dx = F(x) + C \text{ means}$$

that most general antider.

of $F(x)$

$$F'(x) = f(x) \quad \int_0^1 x^2 dx = \frac{1}{3}$$

$$\int x^2 dx = \frac{x^3}{3} + C$$

FTC: Fundamental Theory of Calculus

$$\text{FTC: } \int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}$$

this always works

January 7, 2026

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$\int (f'(x)g(x) + f(x)g'(x)) dx = f(x)g(x) + C$$

$$\int f'(x)g(x) dx + \int f(x)g'(x) = f(x)g(x) + C$$

$$\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x) dx$$

$$\int x^2 dx - 7 = \frac{x^3}{3} + (C - 7) = \int x^2 dx$$

1)
 $\int x \cdot \cos x dx$

Integration by
parts

Solution:

$$\int x \cdot \boxed{\cos x} dx = \int x \cdot \boxed{(\sin x)'} dx =$$

$$= x \cdot \sin x - \int x' \sin x dx = x \cdot \sin x -$$

$$- \int \sin x dx = x \cdot \sin x + \cos x + C$$

2)

$$\int 2^x \cdot x dx = \int \left(\frac{2^x}{\ln 2} \right)' \cdot x dx =$$

$$= \frac{2^x}{\ln 2} \cdot x - \int \frac{2^x}{\ln 2} \boxed{\begin{matrix} 1 \\ \uparrow \\ x \end{matrix}} dx =$$

$$= \frac{2^x \cdot x}{\ln 2} - \frac{1}{\ln 2} \int 2^x dx \xrightarrow{\frac{2^x}{\ln 2}}$$

$$= \boxed{\frac{2^x \cdot x}{\ln 2} - \frac{2^x}{(\ln 2)^2}}$$

$$3) \int x^2 e^x dx = \int x^2 \cdot (e^x)' dx =$$

$$x^2 e^x - \int (x^2)' \cdot e^x dx = x^2 e^x - 2 \int$$

$$-2 \int x \cdot e^x dx = x^2 e^x - 2 \int x \cdot (e^x)' dx$$

$$= x^2 \cdot e^x - 2(x \cdot e^x - \int \boxed{x'} e^x dx) =$$

$$= x^2 \cdot e^x - 2 \cdot x \cdot e^x + 2 \boxed{\int e^x dx} e^x$$

$$= \boxed{x^2 e^x - 2x \cdot e^x + 2 \cdot e^x + C}$$

$$4) \int \ln x dx = \int \boxed{1} \cdot \ln x dx =$$

$$= \int x' \ln x dx = x \cdot \ln x - \int x (\ln x)' dx$$

$$= x \cdot \ln x - \int x \cdot \frac{1}{x} dx = x \cdot \ln x - \int 1 dx$$

$$= \boxed{x \cdot \ln x - x + C}$$

5) $\int \cos^2 x \, dx = \int \cos x \cdot (\sin x)' \, dx$

$$= \cos x \cdot \sin x - \int (\cos x)' \sin x \, dx$$

$$= \cos x \cdot \sin x + \boxed{\int \sin^2 x \, dx}$$

$$= \cos x \cdot \sin x + \int (1 - \cos^2 x) \, dx$$

$$= \cos x \cdot \sin x + \int 1 \, dx - \int \cos^2 x \, dx$$

$$= \cos x \cdot \sin x + x - \int \cos^2 x \, dx$$

Solve for $\int \cos^2 x \, dx$:

$$2 \int \cos^2 x \, dx = \frac{1}{2} (\cos x \sin x + x) + C$$

$$u = x^2$$

$$u = g(x)$$

$$\frac{du}{dx} = 2x$$

$$g'(x)$$

$$du = 2x \, dx$$

$$g'(x) \, dx$$

$$\int u \underbrace{f(x)}_{u} \underbrace{g'(x)}_{dv} \, dx = \underbrace{f(x)}_{u} \underbrace{g(x)}_{v} - \int v \underbrace{f'(x)}_{dv} \, dx$$

$$\int u \, dv = uv - \int v \cdot du$$

1)

$$\int u \underbrace{x \cdot \cos x}_{dv} \, dx :$$

$$u = x \quad v = \sin x$$

$$dv = \cos x \cdot dx$$

$$du = dx$$

$$\int \underbrace{x \cdot \cos x \, dx}_{u \, dv} = \int u \, dv = uv - \int v \, du$$

$$= x \cdot \sin x - \int \sin x \, dx = x \cdot \sin x + C - \cos x$$

January 12, 2026

Substitution Rule (aka "u-sub")

$$\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx$$

$\int f(x) g(x) \, dx \rightarrow$ integration by parts

$$f(x) g(x) - \int f'(x) g(x) \, dx$$

* often serves the purpose

$\int f(g(x)) \, dx \times \rightarrow$ no such thing

$\int f(g(x)) g'(x) \, dx \rightarrow$ there is a formula
for this, substitution rule

$$\int f(g(x))g'(x) dx = \int \underbrace{F'(g(x))g'(x)}_{(F(g(x))')} dx$$

$$= \int (F(g(x))' dx = F(g(x)) + C \quad \text{Let } u = g(x)$$

$$= F(u) + C = \int f(u) du \quad \checkmark$$

$$\text{let } u = x^2$$

$$\frac{du}{dx} = 2x \quad du = 2x dx$$

$$u = g(x)$$

$$du = g'(x) dx$$

$$\frac{du}{dx} = g'(x)$$

$$\int \underbrace{f(g(x))}_{u} \underbrace{g'(x)}_{du} dx = \int f(u) du$$

Practice

1.

$$\int x \cdot \cos(x^2) dx$$

let $u = x^2$

$$du = 2x dx \rightarrow \frac{du}{2}$$

$$\int x \cdot \cos(x^2) dx = \int \cos(u) \cdot \boxed{\frac{du}{2}} =$$

$$\frac{1}{2} \int \cos(u) du = \frac{1}{2} \sin(u) + C =$$

$$\frac{1}{2} \sin(x^2) + C$$

2.

$$\int x^2 \sqrt[3]{4x^3 + 3} dx$$

$$\text{let } u = 4x^3 + 3$$

$$du = 12x^2 dx \quad | \quad x^2 dx = \frac{du}{12}$$

$$\int x^2 \sqrt[3]{4x^3 + 3} dx = \int \sqrt[3]{u} \frac{du}{12} = \frac{1}{2} \int \sqrt[3]{u} du$$

$$\begin{aligned}
 &= \frac{1}{12} \int u^{\frac{1}{3}} du = \frac{1}{12} \cdot \frac{u^{\frac{4}{3}}}{\frac{4}{3}} + C = \\
 &= \frac{u^{\frac{4}{3}}}{16} + C = \boxed{\frac{(4x+3)^{\frac{4}{3}}}{16} + C}
 \end{aligned}$$

3.

$$\int \sin(10x) dx \quad \begin{aligned}
 &\text{let } u = 10x \\
 &du = 10 dx \\
 &dx = \frac{du}{10}
 \end{aligned}$$

$$\int \sin(10x) dx = \int \sin(u) \frac{du}{10} = \frac{1}{10} \int \sin(u) du$$

$$= -\frac{1}{10} \cos(u) + C = \boxed{-\frac{1}{10} \cos(10x) + C}$$

Useful shortent:

If $\int f(x) dx = F(x) + C$, then

$$\cdot \int f(x+a) dx = F(x+a) + C$$

$$\int \cos(x+\pi) dx = \sin(x+\pi) + C$$

$$\int f(a \cdot x) dx = \frac{F(a \cdot x)}{a} + C$$

$$\int \cos(\pi \cdot x) dx = \frac{\sin(\pi \cdot x)}{\pi} + C$$

$$5. \int \frac{x^3}{1+x^4} dx$$

Solution:

$$\text{Let } u = 1+x^4$$

$$du = 4x^3 dx \quad x^3 dx = \frac{du}{4}$$

$$\int \frac{x^3}{1+x^4} dx = \int \frac{du/4}{u} = \frac{1}{4} \int \frac{1}{u} du =$$

$$= \frac{1}{4} \ln|u| + C = \boxed{\frac{1}{4} \ln|1+x^4| + C}$$

$$6. \int \frac{x}{1+x^4} dx$$

$$\int \frac{x}{1+x^4} dx = \int \frac{x}{1+(x^2)^2} dx$$

$$\text{Let } u = x^2$$

$$du = 2x dx \quad x dx = \frac{du}{2}$$

$$\int \frac{1}{1+x^4} x dx = \int \frac{1}{1+u^2} \frac{du}{2} =$$

$$= \frac{1}{2} \int \frac{1}{1+u^2} du = \frac{1}{2} \tan^{-1}(u) + C$$

$$= \boxed{\frac{1}{2} \tan^{-1}(x^2) + C}$$

$$7. \int \frac{1}{x^2 + 14x + 130} dx$$

$$\text{Solution: } x^2 + 14x + 130 = ()^2 + C$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$x^2 + 14x + 130 = (x^2 + 2 \cdot x \cdot 7 + 7^2) - 49 + 130$$

$$= (x+7)^2 + 81 \rightarrow \text{completing the square}$$

$$\int \frac{1}{(x+7)^2 + 81} dx$$

Let $u = x+7$

$$du = dx$$

$$\int \frac{1}{u^2 + 81} du = \int \frac{1}{u^2 + g^2} du =$$

$$= \frac{1}{g} \tan^{-1}\left(\frac{u}{g}\right) + C = \boxed{\frac{1}{9} \tan^{-1}\left(\frac{x+7}{9}\right) + C}$$

11

$$\int \frac{\sin x \cdot \cos x}{1 + \sin^2 x} dx$$

Let $u = \sin x \quad du = \cos x \ dx$

$$\int \frac{\sin x \cos x}{1 + \sin^2 x} dx = \int \frac{u}{1+u^2} du$$

$$\text{Let } v = 1+u^2$$

$$dv = 2u \, du$$

$$u \cdot du = \frac{dv}{2}$$

$$\int \frac{1}{v} \frac{dv}{2} = \frac{1}{2} \int \frac{1}{v} dv = \frac{1}{2} \ln|v| + C$$

$$\boxed{\frac{1}{2} \ln(1 + \sin^2 x) + C}$$

Alternative:

$$\text{Let } u = 1 + \sin^2 x \quad du = 2 \sin x \cdot \cos x$$

$$\int \frac{\sin x \cdot \cos x}{1 + \sin^2 x} = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + C$$

$$= \frac{1}{2} \ln(1 + \sin^2 x) + C$$

$$12. \int x^5 \cdot \cos(x^3) dx = \int x^2 \cdot x^3 \cdot \cos(x^3) dx$$

Solution:

$$\text{Let } u = x^3$$

$$du = 3x^2 dx$$

$$\int \underbrace{x^2 x^3 \cos(x^3)}_{u \cdot \cos u} dx = \int u \cdot \cos(u) \frac{du}{3} =$$

$$= \frac{1}{3} \int u \cdot \cos u du = \frac{1}{3} \int u \cdot (\sin' u) du =$$

$$= \frac{1}{3} [u \cdot \sin(u) - \int u' \sin u du] = \frac{1}{3} ($$

$$= \frac{1}{3} (u \cdot \sin u - \int \sin u du) = \frac{1}{3} (u \cdot \sin u$$

$$+ \cos u) + C = \frac{1}{3} (x^3 \cdot \sin x^3 + \cos(x^3)) + C$$

January 14, 2026

Trigonometric Integrals

1. Integrals of form $\int \sin^m x \cos^n x dx$
2. Integrals of form $\int \tan^m x \sec^n x dx$
3. Integrals of form $\int \sin(mx) \sin(nx) dx$
... etc

$$1 \int \sin^m x \cdot \cos^n x dx$$

$$\int \cos^3 x dx = \int (1 - \sin^2 x) \cos x dx =$$

$$\cos^3 x = \cos x \cdot \cos^2 x = \cos x (1 - \sin^2 x)$$

$$\int \underbrace{(1 - \sin^2 x)}_{1-u^2} \underbrace{\cos x dx}_{du} = \int (1-u^2) du$$

$$1-u^2 \quad \text{let } u = \sin x$$

$$du = \cos x$$

$$\int \sin^m x \cdot \cos^n x \, dx$$

Case 1: n is odd

$$n = 2k + 1$$

$$\int \sin^m x \cdot \cos^{2k+1} x \, dx = \int \sin^m x \cdot (\cos x)^{2k} \cos x \, dx$$

$$= \int \sin^m x \cdot (1 - \sin^2 x)^k \cos x \, dx$$

case 2: m is odd ($m = 2k + 1$)

$$\int \sin^{2k+1} x \cdot \cos^n x \, dx = \int \sin x \cdot \sin x \cdot \cos^n x \, dx$$

$$= \int (1 - \cos^2 x)^k \cdot \cos^n x \cdot \sin x \, dx \quad u = \cos x$$

Case 3 $m \geq n$ odd, do any

Ex.

$$\int \sin^6 x \cdot \cos^5 x \, dx = \int \sin^6 x \cdot \cos^4 x \cdot \cos x \, dx$$

$$= \int \sin^6 x \cdot (\cos^2 x)^2 \cos x \, dx \quad \text{let } u = \sin x$$

$$du = \cos x \, dx$$

$$= \int \underbrace{\sin^6 x \cdot (1 - \sin^2 x)^2}_{u^6 \cdot (1-u^2)^2} \cos x \, dx$$

$$= \int u^6 (1-u)^2 \, du = \int u^6 (1-2u^2+u^4) \, du$$

$$= \int (u^6 - 2u^8 + u^{10}) \, du = \frac{u^7}{7} - 2 \frac{u^9}{9} + \frac{u^{11}}{11} + C$$

$$= \boxed{\frac{\sin^7 x}{7} - 2 \frac{\sin^9 x}{9} + \frac{\sin^{11} x}{11} + C}$$

$$\text{Ex. } \int \sin^4 x \cdot \cos^6 x \, dx = \int (\sin^2 x)^2 (\cos^2 x)^3 \, dx$$

$$= \int \left(\frac{1 - \cos(2x)}{2} \right)^2 \left(\frac{1 + \cos(2x)}{2} \right)^3 \, dx$$

Ex.

$$\int \cos^2 x \, dx$$

Solution:

$$\int \cos^2 x \, dx = \int \frac{1+\cos(2x)}{2} \, dx =$$

$$\frac{1}{2} \left(x + \int \cos(2x) \, dx \right) = \boxed{\frac{x}{2} + \frac{1}{2} \cdot \frac{\sin(2x)}{2} + C}$$

$$\int \cos(2x) = \frac{\sin(2x)}{2}$$

$$\text{Ex. } \int \cos^2 x \cdot \sin^2 x \, dx = \int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2}$$

$$= \frac{1}{4} \int (1-\cos(2x))(1+\cos(2x)) \, dx =$$

$$= \frac{1}{4} \int (1-\cos^2(2x)) \, dx = \frac{1}{4} x - \frac{1}{4} \int \cos^2(2x)$$

$$= \frac{x}{4} - \frac{1}{4} \int \frac{1+\cos(4x)}{2} \, dx$$

$$= \frac{x}{4} - \frac{x}{4} - \frac{1}{8} \int \cos(4x) dx$$

$$= \frac{x}{8} - \frac{1}{8} \cdot \frac{\sin(4x)}{4} + C = \boxed{\frac{x}{8} - \frac{\sin(4x)}{32} + C}$$

odd even

$$\int \tan^m x \cdot \sec^n x dx$$

1. If n is even, $n=2k$

$$\int \tan^m x \cdot \sec^{2k} x dx = \int \tan^m x \cdot \underbrace{\sec x \cdot \sec x}_{(\sec^2 x)^{k-1}} dx$$

$$= \int \underbrace{\tan^m x (1 + \tan^2 x)^{k-1}}_{\text{in terms of } u} \underbrace{\sec^2 x}_{du} dx$$

in terms of u

2. If $m=2k+1$

$$\int \tan^{10} x \cdot \sec^6 x dx = \int \tan^{10} x \cdot \sec^4 x \cdot \sec^2 x$$

$$= \int \tan^{10} x (\sec^2 x)^2 \cdot \sec^2 x dx$$

$$= \int \tan^{10} x \cdot (1 + \tan^2 x)^2 \cdot \sec^2 x \, dx$$

$$\text{let } u = \tan x$$

$$du = \sec^2 x \, dx$$

$$= \int u^{10} (1+u^2)^2 \, dx = \int u^{10} (1+2u^2+u^4) \, du$$

$$= \int (u^{10} + 2u^{12} + u^{14}) \, du = \frac{u^{11}}{11} + 2 \frac{u^{13}}{13} + \frac{u^{15}}{15} + C$$

$$= \boxed{\frac{\tan^{11} x}{11} + 2 \frac{\tan^{13} x}{13} + \frac{\tan^{15} x}{15} + C}$$

$$\int \cot^3 x \cdot \csc^3 x \, dx$$

Solution:

$$\int \cot^2 x \cdot \csc^2 x (\cot x \cdot \csc x) \, dx =$$

$$\int (\csc^2 x - 1) \csc^2 x \cot x \cdot \csc x \, dx$$

$$\text{let } u = \csc x$$

$$du = -\cot x \csc x \, dx$$

$$= - \int (u^2 - 1) u^2 \, du = - \int (u^4 - u^2) \, du =$$

$$= \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \boxed{\frac{\csc^3 x}{3} - \frac{\csc^5 x}{5} + C}$$

$$1 \quad \int \cos(7x) \cdot \cos(4x) \, dx =$$

$$= \int \frac{1}{2} (\cos(7-4x) + \cos(7x+4x)) \, dx$$

$$= \frac{1}{2} \int (\cos(3x) + \cos(11x)) \, dx$$

$$= \frac{1}{2} \left(\frac{\sin(3x)}{3} + \frac{\sin(11x)}{11} \right) + C$$

Tutorial

January 16, 2026

$$1. \int x^2 \cdot \tan(x^3 + 1) dx$$

Solution

$$\text{Let } u = x^3 + 1$$

$$du = 3x^2 dx \rightarrow \frac{du}{3} = x^2 dx$$

$$\int \underbrace{x^2 \cdot \tan(x^3 + 1)}_{\text{circled}} dx$$

$$= \int \frac{du}{3} \cdot \tan(u) = \frac{1}{3} \int \tan(u) du =$$

$$= \frac{1}{3} \ln |\sec(u)| + C = \boxed{\frac{1}{3} \ln |\sec(x^3 + 1)| + C}$$

$$2. \int \frac{x^9}{4+x^{10}} dx$$

$$\text{Let } u = 4+x^{10}$$

$$du = 10x^9$$

$$x^9 = \frac{du}{10}$$

$$\int \frac{x^9}{4+x^{10}} = \frac{1}{10} \int \frac{1}{u} \cdot du = \frac{1}{10} \ln|u| + C$$

$$= \boxed{\frac{1}{10} \ln|4+x^{10}| + C}$$

$$3. \int \frac{x^4}{4+x^{10}} dx = \int \frac{x^4}{4+(x^5)^2} dx$$

$$\text{Let } u = x^5$$

$$du = 5x^4 dx \quad x^4 dx = \frac{du}{5}$$

$$\begin{aligned}
 &= \frac{1}{5} \int \frac{1}{4+u^2} du = \frac{1}{5} \int \frac{1}{2^2+u^2} du = \\
 &= \frac{1}{5} \cdot \frac{1}{2} \tan^{-1}\left(\frac{u}{2}\right) + C = \\
 &= \frac{1}{5} \cdot \frac{1}{2} \cdot \tan^{-1}\left(\frac{x^5}{2}\right) + C
 \end{aligned}$$

$$4. \int \frac{1}{\sqrt{21+4x-x^2}}$$

Solution

$$\begin{aligned}
 21+4x-x^2 &= 21-(x^2-4x) = \\
 &= 21 - \underbrace{(x^2-2 \cdot x \cdot 2 + 2^2)}_{(x-2)^2} - 2 = 25 - (x-2)^2
 \end{aligned}$$

$$\int \frac{1}{\sqrt{25-(x-2)^2}} dx$$

$$\begin{aligned}
 \text{Let } u &= x-2 \\
 du &= dx
 \end{aligned}$$

$$\begin{aligned}
 &= \int \frac{1}{\sqrt{25 - u^2}} du \quad \int \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \\
 &= \int \frac{1}{\sqrt{5^2 - u^2}} du = \sin^{-1}\left(\frac{u}{5}\right) + C \\
 &= \boxed{\sin^{-1}\left(\frac{x-2}{5}\right) + C}
 \end{aligned}$$

6. $\int x^5 \sqrt{4+x^2} dx$

Solution

$$\text{Let } u = 4+x^2$$

$$du = 2x dx$$

$$\boxed{x dx = du/2}$$

$$\int x^5 \sqrt{4+x^2} dx = \int \underbrace{x \cdot (x^2)^2}_{u^2} \sqrt{4+x^2} \underbrace{dx}_{\frac{du}{2}}$$

$$= \frac{1}{2} \int (u-4)^2 \sqrt{u} \ du \rightarrow \text{brute force}$$

$$= \frac{1}{2} \int (u^2 - 8u + 16) u^{\frac{1}{2}} \ du =$$

$$= \frac{1}{2} \int (u^{\frac{5}{2}} - 8u^{\frac{3}{2}} + 16u^{\frac{1}{2}}) \ du =$$

$$= \frac{1}{2} \left(\frac{u^{\frac{7}{2}}}{\frac{7}{2}} - 8 \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{16u^{\frac{3}{2}}}{\frac{3}{2}} \right) + C =$$

$$= \boxed{\frac{(x^2+4)^{\frac{7}{2}}}{7} - 8 \frac{(x^2+4)^{\frac{5}{2}}}{5} + 16 \frac{(x^2+4)^{\frac{3}{2}}}{3} + C}$$

$$7. \int \frac{1}{1+e^{-x}} dx = \int \frac{1}{1+\frac{1}{e^x}} dx =$$

$$= \int \frac{e^x}{e^x+1} dx =$$

$$\text{let } u = e^x + 1 \\ du = e^x dx$$

$$= \int \frac{1}{u} du = \ln|u| + C = \boxed{\ln(e^x+1) + C}$$

TRIG INTEGRALS

9. $\int \tan^3 x \cdot \sec^1 x \ dx =$

$\overset{3}{\circlearrowleft}$
odd ✓

$$= \int \tan^2 x \cdot \tan x \cdot \sec x \ dx =$$

$$= \int (\sec^2 - 1) \tan x \cdot \sec x \ dx$$

Let $u = \sec x$

$$du = \tan x \cdot \sec x \ dx$$

$$= \int (u^2 - 1) du = \frac{u^3}{3} - u + C =$$

$$\boxed{\frac{\sec^3}{3} - \sec x + C}$$

$$\begin{aligned}
 \int \cos^4 x \, dx &= \int (\cos^2 x)^2 \, dx \\
 &= \int \left(\frac{1 + \cos(2x)}{2} \right)^2 \, dx \\
 &= \frac{1}{4} \int (1 + \cos(2x))^2 \, dx = \frac{1}{4} \int 1 + 2 \cdot \cos(2x) + \cos^2(2x) \, dx \\
 &= \frac{x}{4} + \frac{1}{2} \int \cos(2x) \, dx + \frac{1}{4} \int \cos^2(2x) \, dx \\
 &= \frac{x}{4} + \frac{1}{2} \frac{\sin(2x)}{2} + \frac{1}{4} \int \frac{1 + \cos(4x)}{2} \, dx \\
 &= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{1}{8} \int 1 + \cos(4x) \, dx = \\
 &= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{x}{8} + \frac{1}{8} \frac{\sin(4x)}{4} + C \\
 &= \boxed{\frac{3x}{8} + \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C}
 \end{aligned}$$

January 19, 2026

Trigonometric Substitution

$$\int x^2 \sqrt{16 - x^2} dx = \int 16 \sin^2 \theta \sqrt{16 - 16 \sin^2 \theta} \cdot 4 \cos \theta d\theta$$

\Downarrow

$$= 4 \sqrt{1 - \sin^2 \theta} = 4 \sqrt{\cos^2 \theta}$$

$$x = 4 \sin \theta$$

$$= 4 \cos \theta$$

$$dx = 4 \cos \theta d\theta$$

$$= 256 \int \underline{\sin^2 \theta \cos^2 \theta} d\theta$$

trig integral

1. For $\sqrt{a^2 - x^2}$

$x = a \cdot \sin \theta$, choose θ such that
this happens

Is there even such θ ?

$$a^2 - x^2 \geq 0 \quad x^2 \leq a^2 \quad -a \leq x \leq a$$

yes, because $-a \leq x \leq a$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = a \sqrt{1 - \sin^2 \theta}$$
$$= a \sqrt{\cos^2 \theta} = a |\cos \theta| = a \cdot \cos \theta$$

2.

For $\sqrt{a^2 + x^2}$ no restrictions

$$\text{let } x = a \cdot \tan \theta$$

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2 \tan^2 \theta} = a \cdot \sqrt{\sec^2 \theta}$$
$$= a |\sec \theta| = a \cdot \sec \theta$$

3.

$$\sqrt{x^2 - a^2}$$

$$x^2 - a^2 \geq 0$$

$$x^2 \geq a^2$$

$$\underline{x \geq a}$$

$$\text{or } \underline{x \leq -a}$$

Ex 1.

$$\int x \sqrt{16-x^2} dx = \int x \sqrt{4^2 - x^2} dx$$

let $x = 4 \sin \theta \quad (-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$

$$dx = 4 \cos \theta d\theta$$

Then $\sqrt{16-x^2} = \sqrt{16-16 \sin^2 \theta} =$

$$= \sqrt{16(1-\sin^2 \theta)} = 4 \sqrt{\cos^2 \theta} = 4 |\cos \theta|$$

$$= 4 \cos \theta$$

because $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\int x \sqrt{16-x^2} dx = 4 \int 4 \sin \theta \cdot 4 \cos \theta \cdot 4 \cos \theta$$

$$= 64 \int \sin \theta \cos^2 \theta d\theta$$

let $u = \cos \theta$

$$du = -\sin \theta d\theta$$

$$= 64 \int \cos^2 \theta \underbrace{\frac{\sin \theta}{-\cos \theta}}_{-d\theta} d\theta$$

$$= -64 \int u^2 du = \boxed{-\frac{64}{3} u^3 + C}$$

$$= -\frac{64}{3} \cdot \cos^3 \underline{\theta} + C$$

$$\sin \theta = \frac{x}{4}$$

$$\theta = \sin^{-1} \left(\frac{x}{4} \right)$$

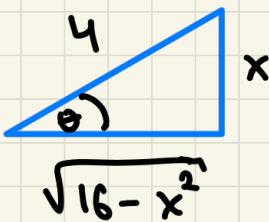
because $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$= \boxed{-\frac{64}{3} \cos^3 \left(\sin^{-1} \left(\frac{x}{4} \right) \right) + C}$$

If $x = 4 \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

compute $\cos \theta$ in terms of x

$$\sin \theta = \frac{x}{4}$$



$$\cos \theta = \frac{\sqrt{16 - x^2}}{4}$$

Conclusion

$$\begin{aligned}\int x \sqrt{16-x^2} dx &= -\frac{64}{3} \cos^3 \theta + C \\ &= -\frac{64}{3} \left(\frac{\sqrt{16-x^2}}{4} \right)^3 + C \\ &= \boxed{-\frac{(\sqrt{16-x^2})^3}{3} + C}\end{aligned}$$

2.

$$\int \frac{x^3}{\sqrt{x^2+9}} dx$$

$x^2 + 3^2$

Solution

$$\text{let } x = 3 \tan \theta$$

$$dx = 3 \sec^2 \theta d\theta$$

$$\text{Then } \sqrt{x^2+9} = \sqrt{9 \tan^2 \theta + 9} = 3 \sqrt{\tan^2 \theta + 1}$$

$$= 3 \sqrt{\sec^2 \theta} = 3 |\sec \theta| = 3 \sec \theta$$

$$\int \frac{x^3}{\sqrt{x^2+9}} = \int \frac{27 \tan^3 \theta}{3 \sec \theta} \cdot 3 \sec^2 \theta d\theta$$

$$= 27 \int \tan^3 \theta \ sec \theta d\theta$$

$$= 27 \int \tan^3 \theta \cdot \sec \theta \, d\theta$$

$$= 27 \int \tan^2 \theta \cdot \tan \theta \cdot \sec \theta \, d\theta$$

$$= 27 \int (\sec^2 - 1) \tan \theta \cdot \sec \theta \, d\theta$$

$$\text{let } u = \sec \theta$$

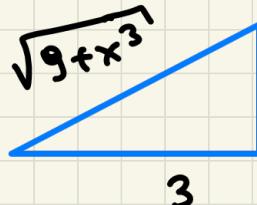
$$du = \tan \theta \sec \theta \, d\theta$$

$$= 27 \int (u^2 - 1) \, du = 27 \left(\frac{u^3}{3} - u \right) + C$$

$$= 9u^3 - 27u + C$$

$$= 9 \sec^3 \theta - 27 \sec \theta + C$$

$$x = 3 \tan \theta$$



$$\tan \theta = \frac{x}{3}$$

$$\sec \theta = \frac{\sqrt{9+x^2}}{3}$$

$$\int \frac{x^3}{\sqrt{x^2+9}} dx = 9 \sec^3 \theta - 27 \sec \theta + C$$

$$= 9 \left(\frac{\sqrt{9+x^2}}{3} \right)^3 - 27 \frac{\sqrt{9+x^2}}{3} + C$$

$$= \left(\frac{\sqrt{9+x^2}}{3} \right)^3 - 9 \sqrt{9+x^2} + C$$

$$3. \int \frac{\sqrt{x^2-1}}{x^6} dx$$

Solution

$$x = \sec \theta$$

$$dx = \tan \theta \sec \theta d\theta$$

$$\sqrt{x^2-1} = \sqrt{\sec^2 \theta - 1} = \sqrt{\tan^2 \theta} = \tan \theta$$

Then $\int \frac{\sqrt{x^2-1}}{x^6} dx = \int \frac{\tan \theta}{\sec^6 \theta} \tan \theta \sec \theta d\theta$

$$= \int \frac{\tan^2 \theta}{\sec^5 \theta} d\theta = \int \frac{\frac{\sin^2 \theta}{\cos^2 \theta}}{\frac{1}{\cos^5 \theta}} d\theta =$$

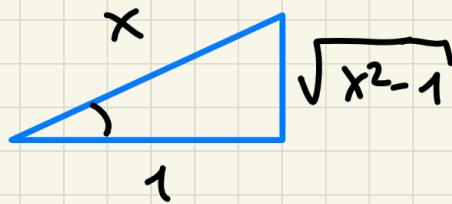
$$= \int \cos^3 \theta \cdot \sin^2 \theta d\theta = \int \cos^2 \theta \cdot \sin^2 \theta \cos \theta d\theta$$

$$= \int (1 - \sin^2 \theta) \sin^2 \theta \cos \theta d\theta =$$

$$= \int (1 - u^2) u^2 du = \int (u^2 - u^4) du =$$

$$\frac{u^3}{3} - \frac{u^5}{5} + C = \boxed{\frac{\sin^3 \theta}{3} - \frac{\sin^5 \theta}{5} + C}$$

$$\sec \theta = \frac{x}{1}$$



$$\sin \theta = \frac{\sqrt{x^2 - 1}}{x}$$

Then:

$$\int \frac{\sqrt{x^2-1}}{x^6} dx = \frac{\sin^3 \theta}{3} - \frac{\sin^5 \theta}{5} + C$$

$$= \boxed{\frac{1}{3} \left(\frac{\sqrt{x^2+1}}{x} \right)^3 - \frac{1}{5} \left(\frac{\sqrt{x^2-1}}{x} \right)^5 + C}$$

January 21, 2026

Integrals of Rational Functions using Partial Fraction Decomposition

Polynomials: $P(x) = a_0 + a_1 x + \dots + a_n x^n$

Rational functions: $f(x) = \frac{P(x)}{Q(x)}$

where $P(x)$, $Q(x)$ are polynomials

$f(x) = \frac{P(x)}{Q(x)}$ is a "proper" rational function if $\deg P(x) < \deg Q(x)$

Ex

$\frac{2x+7}{3x^2-2x+2}$ is proper

$\frac{2}{4x+2x+7} \rightarrow$ bigger than
 $\frac{4x+2x+7}{3x^2-2x+2}$ is not proper

$$\int \frac{P(x)}{Q(x)} dx$$

Step # 0 : boil down to a proper rational function

Division with remainder for nums:
If $a, b > 0$ integers, there exist q, r integers such that $a = bq + r$ $0 \leq r < b$

If $P(x)$ and $Q(x)$ are polynomials
there exist $S(x)$ and $R(x)$
such that:

$$P(x) = Q(x) S(x) + R(x)$$

$$\int \frac{P(x)}{Q(x)} dx = \int \frac{Q(x)S(x) + R(x)}{Q(x)} dx =$$

proper rat. funct.

$$\int \left(S(x) + \frac{R(x)}{Q(x)} \right) dx = \int S(x) dx + \int \frac{R(x)}{Q(x)} dx$$

Ex.

$$\int \frac{2x^3 - 5x^2 + 5x + 3}{x-1} dx$$

Solution.

$$\begin{array}{r} \cdot 2x^2 - 3x + 2 \\ x-1 / \underline{- 2x^3 - 5x^2 + 5x + 3} \\ \underline{2x^3 - 2x^2} \\ - \quad \underline{- 3x^2 + 5x + 3} \\ - \quad \underline{- 3x^2 + 3x} \\ \hline \quad \quad \quad \underline{2x + 3} \\ - \quad \underline{\underline{2x - 2}} \\ \hline \quad \quad \quad 5 \end{array}$$

$$\begin{array}{c} P(x) \\ 2x^3 - 5x^2 + 5x + 3 = \\ Q(x) \quad S(x) \quad R(x) \\ = (x-1) (2x^2 - 3x + 2) + 5 \end{array}$$

$$= \int \frac{(x-1)(2x^2 - 3x + 2) + 5}{x-1} dx =$$

$$= \int \left(2x^2 - 3x + 2 + \frac{5}{x-1} \right) dx =$$

$$= \boxed{\frac{2x^3}{3} - \frac{3x^2}{2} + 2x + 5 \ln|x-1| + C}$$

Raised Goal : $\int \frac{P(x)}{Q(x)} dx$ if $\frac{P(x)}{Q(x)}$ proper polyn.

Case 1 : $Q(x)$ is a product of linear functions

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\dots(a_kx+b_k)$$

There exists constants $A_1, A_2, A_3, \dots, A_k$,

$\dots A_k$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \dots + \frac{A_k}{a_kx+b_k}$$

Ex. $\frac{P(x)}{Q(x)} = \frac{\text{Smith of degree zero or less}}{(3x-7)(5x+2)(2x+1)} =$

$$\frac{A}{3x-7} + \frac{B}{5x+2} + \frac{C}{2x+1}$$

Ex 1

$$\int \frac{x}{x^2+x-2} dx$$

Sol.

$$\text{Factor } x^2 + x - 2 = (x+2)(x-1)$$

The form of the PFD is:

$$\frac{x}{x^2 + x - 2} = \frac{A}{x+2} + \frac{B}{x-1}$$

Find A, B

$$\frac{x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} \quad | \cdot (x+2)(x-1)$$

$$x = A(x-1) + B(x+2)$$

$$x = Ax - A + Bx + 2B$$

$$x = (A+B)x + (2B-A)$$

$$\underline{1 \cdot x + 0} = x = \underline{(A+B)x} + \underline{(2B-A)}$$

$$\begin{cases} A+B = 1 \\ 2B-A = 0 \end{cases} \quad A = 2B$$

$$2B+B = 1 \quad 3B=1 \quad B = \frac{1}{3}$$

$$A = \frac{2}{3}$$

The P.F.D is :

$$\frac{x}{x^2-x-2} = \frac{2/3}{x+2} + \frac{1/3}{x-1}$$

$$\int \frac{x}{x^2+x-2} dx = \int \left(\frac{2/3}{x+2} + \frac{1/3}{x-1} \right) dx$$

$$= \frac{2}{3} \int \frac{1}{x+2} dx + \frac{1}{3} \int \frac{1}{x-1} dx$$

$$= \boxed{\frac{2}{3} \ln|x+2| + \frac{1}{3} \ln|x-1| + C}$$

Shortcut to find A and B

$$x = A(x-1) + B(x+2)$$

$$\text{take } x=1 \quad 1 = A \cdot 0 + B \cdot 3 \Rightarrow B = \frac{1}{3}$$

$$\text{take } x=-2 \quad -2 = A(-2-1) + B \cdot 0 \Rightarrow A = -\frac{2}{3}$$

!!! okay but doesn't always work !!!

Case 2.

Example:

$$\frac{P(x)}{Q(x)} = \frac{\dots\dots}{(5x+1)(2x+2)^3(x-3)^2}$$

$$\frac{A}{(5x+1)} + \frac{B}{(2x+7)} + \frac{C}{(2x+7)^2} + \frac{D}{(2x+7)^3} + \frac{E}{x-3} + \frac{F}{(x-3)^2}$$

$$(Q_i x + b_i)^r$$

$$\frac{B_1}{a_i x + b_i} + \frac{B_2}{(a_i x + b_i)^2} + \dots + \frac{B_r}{(a_i x + b_i)^r}$$

Case 3p

$$\frac{P(x)}{Q(x)} = \dots \frac{\dots}{(5x+1)(2x+7)^3(x-3)^2(x^4+2x+7)(2x^2+3x+19)^3}$$

$$\frac{A}{(5x+1)} + \frac{B}{(2x+7)} + \frac{C}{(2x+7)^2} + \frac{D}{(2x+7)^3} + \frac{E}{x-3} + \frac{F}{(x-3)^2}$$

$$+ \frac{6x+11}{x^2+2x+7} + \frac{Ix+J}{2x^2+3x+19} + \frac{Kx+L}{(2x^2+3x+19)^2} + \frac{Mx+N}{(2x^2+3x+19)^3}$$

$$Ex \int \frac{x+1}{x^4+x^2} dx$$

Sol.

$$\text{Factor } x^4 + x^2 = \underline{x^2} (\underline{x^2 + 1})$$

Form of the P.F.D is

$$\frac{x+1}{x^4+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} \quad | \quad x^2(x^2+1)$$

Find A, B, C, D

$$x+1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$$

$$x+1 = A(x^3+x) + B(x^2+1) + (Cx^3+Dx^2)$$

$$x+1 = \underline{Ax^3} + \underline{Ax} + \underline{Bx^2} + \underline{B} + \underline{Cx^3} + \underline{Dx^2}$$

$$x+1 = (A+C)x^3 + (B+D)x^2 + Ax + B$$

$$0 \cdot x^3 + 0 \cdot x^2 + 1 \cdot x + 1 = \underline{(A+C)x^3} + \underline{(B+D)x^2} + \underline{Ax} + \underline{B}$$

$$\begin{cases} A+C=0 \\ B+D=0 \\ A=1 \\ B=1 \end{cases} \quad \begin{matrix} C=-1 \\ D=-1 \end{matrix}$$

$$\frac{x+1}{x^4+x^2} = \frac{1}{x} + \frac{1}{x^2} - \frac{x+1}{x^2+1}$$

$$\int \frac{1}{x} + \frac{1}{x^2} - \frac{x+1}{x^2+1} dx =$$

$$= \ln|x| - \frac{1}{x} - \int \frac{x+1}{x^2+1}$$

$$= \int \frac{x}{x^2+1} dx + \underbrace{\int \frac{1}{x^2+1}}_{\tan^{-1} x}$$

$$\text{let } u = x^2+1$$

$$du = 2x dx$$

$$\frac{1}{2} \int \frac{1}{u} du + \tan^{-1} x + C = \frac{1}{2} \ln(x^2+1) + \tan^{-1}$$

$$= \boxed{\ln|x| - \frac{1}{x} - \frac{1}{2} \ln(x^2+1) - \tan^{-1} x + C}$$

January 23, 2026

$$3. \int \sqrt{25-x^2} dx \quad \sqrt{a^2-x^2} \rightarrow x = a \cdot \sin \theta$$

Solution

$$\text{let } x = 5 \sin \theta$$

$$dx = 5 \cos \theta \, d\theta$$

$$\text{then } \sqrt{25-x^2} = \sqrt{25 - 25 \sin^2 \theta} = 5 \sqrt{1-\sin^2 \theta}$$

$$= 5 |\cos \theta| = 5 \cos \theta$$

$$\cos \theta \geq 0 \quad \text{because } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$\int \sqrt{25-x^2} \, dx = \int 5 \cos \theta \cdot 5 \cos \theta \, d\theta =$$

$$25 \int \cos^2 \theta \, d\theta = 25 \int \frac{1+\cos(2\theta)}{2} \, d\theta$$

$$\frac{25}{2} \int (1+\cos(2\theta)) \, d\theta = \frac{25}{2} \left(\theta + \frac{\sin(2\theta)}{2} \right) + C$$

$$\text{If } x = 5 \sin \theta \quad \sin(2\theta) = ?$$

$$\sin \theta = \frac{x}{5} \Rightarrow \theta = \sin^{-1} \left(\frac{x}{5} \right)$$

$$\sin(2\theta) = 2 \sin \theta \cdot \cos \theta$$

$$\sin^2 \theta + \cos^2 \theta = 1 \quad \cos^2 \theta = 1 - \sin^2 \theta$$

$$\cos \theta = \sqrt{1 - \sin^2 \theta}$$

$$= \frac{25}{2} \left(\sin^{-1}\left(\frac{x}{5}\right) + \frac{x\sqrt{25-x^2}}{25} \right) + C$$

$$\boxed{= \frac{25}{2} \sin^{-1}\left(\frac{x}{5}\right) + \frac{x\sqrt{25-x^2}}{2} + C}$$

④

$$\frac{x^8 + x + 1}{(2x+4)^2 (3x^2+2x+7)(x^2+1)^3} = \frac{A}{2x+4} + \frac{B}{(2x+4)^2}$$

$$+ \frac{Cx+D}{3x^2+2x+7} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

⑤

$$\frac{x^{12} + x + 1}{(2x+4)^2 (3x^2+2x+7)(x^2+1)^3}$$

$$\hookrightarrow \text{degree : } 2+2+2 \cdot 3 = 10 \quad 12 > 10$$

not a proper rational function

$$\frac{x^8 + x + 1}{(2x+4)^2 (3x^2+2x-1)(x^2+1)^3} = \frac{A}{2x+4} + \frac{B}{(2x+4)^2}$$

$$+\frac{C}{3x-1} + \frac{D}{x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

11) $\int \frac{1}{e^{2x} + e^x} dx = \int \frac{e^x}{e^x(e^{2x} + e^x)} dx$

let $u = e^x$
 $du = e^x dx$

$$= \int \frac{1}{u^3 + u^2} du \rightarrow \text{rational function}$$

factor: $u^3 + u^2 = u^2(u+1)$ PFD

$$\frac{1}{u^2(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1} \quad | \quad u^2(u+1)$$

$$1 = A u(u+1) + B(u+1) + C u^2$$

$$1 = A u^2 + A u + B u + B + C u^2$$

$$1 = (A+C) u^2 + (A+B) u + C u^2$$

$$1 = \underline{(A+C) u^2} + \underline{(A+B) u} + \underline{B}$$

$$\begin{cases} A+B=0 & C=1 \\ A+C=0 & A=-1 \\ B=1 & \end{cases}$$

The PFD is

$$\frac{1}{u^2(u+1)} = -\frac{1}{u} + \frac{1}{u^2} + \frac{1}{u+1}$$

$$\int \frac{1}{e^{2x} + e^x} dx = \int \frac{1}{u^3 + u^2} du = \int \left(-\frac{1}{u} + \frac{1}{u^2} + \frac{1}{u+1} \right) du$$

$$= -\ln|u| - \frac{1}{u} + \ln|u+1| + C$$

$$= -\ln|e^x| - \frac{1}{e^x} + \ln(e^x + 1) + C$$

$$= -x - \frac{1}{e^x} + \ln(e^x + 1) + C$$

January 26, 2026

Fundamental Theorem of Calculus

1 Fundamental Theorem of Calc p1

2 Fundamental Theorem of Calc p2

Indefinite Integral:

$$\int_0^1 x^2 \, dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3} - \frac{0}{3} =$$

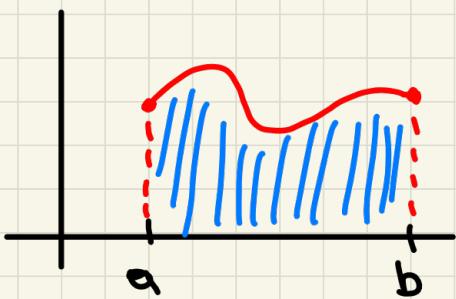
$$\boxed{\frac{1}{3}}$$

"real deal"

Definite Integrals:

$$\int_a^b f(x) \, dx = \left[\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x \right]$$

geometric interpretation:



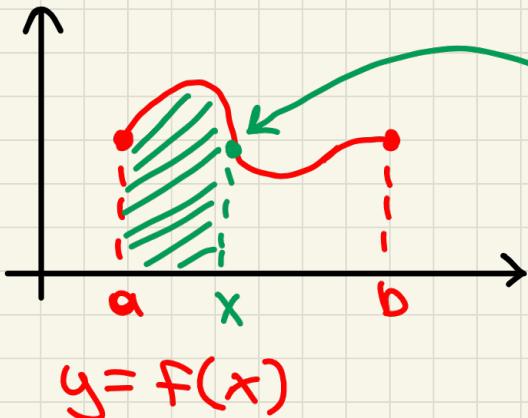
$$\int_a^b f(x) \, dx$$

Integrals = continuous accumulation

Idea of Fundamental Theorem of Calc
"Differentiation and Integration^{definite}
are opposite operations (kind of..)

FTC 1

If $f(x)$ is continuous on $[a, b]$ and $g(x) = \int_a^x f(t) dt$, then g is continuous on $[a, b]$ differentiable on (a, b) and $g'(x) = f(x)$



$$g(x) = \int_a^x f(t) dt$$

$$g'(x) = f(x)$$

$$\frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x)$$

Idea of Proof

$$g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{A_h}{h}$$

Ex.

$$\frac{d}{dx} \left[\int_{\pi}^x \cos(t^2) dt \right] = \cos(x^2)$$

$$\frac{d}{dx} \left[\int_0^{x^4} \sqrt{1+t^3} dt \right] = \sqrt{1+x^{12}} \cdot 4x^3$$

because of the Chain Rule

Solution

$$g(x) = \int_a^x \sqrt{1+t^3} dt$$

$$\text{By FTC 1, } g'(x) = \sqrt{1+x^3}$$

$$\begin{aligned} \frac{d}{dx} \left[\int_4^0 \sqrt{1+t^3} dt \right] &= \frac{d}{dx} [g(x^4)] = g'(x^4) \\ &= \sqrt{1+(x^4)^3} \cdot 4x^3 \end{aligned}$$

FTC 2

If $f(x)$ is a continuous funct. on $[a,b]$
and F is an antiderivative of f ,
then

$$\int_a^b f(x) dx = F(b) - F(a)$$

Proof:

$$\text{Let } g(x) = \int_a^x f(t) dt$$

$$\text{FTC 1: } g'(x) = f(x)$$

$$F'(x) = f(x) \quad \left. \begin{array}{l} F(x) = g(x) + C \\ \text{for all } x \\ \text{in } [a, b] \end{array} \right\}$$

$$\text{If } x=a \Rightarrow F(a) = g(a) + C \Rightarrow C = F(a)$$

$$\text{If } x=b \Rightarrow F(b) = g(b) + C \Rightarrow g(b) + F(a)$$

$$g(b) = F(b) - F(a)$$

$$\int_a^b f(t) dt = F(b) - F(a)$$

Ex. compute $\int_0^1 x^2 dx$

Sol. $f(x) = x^2$ Then $F(x) = \frac{x^3}{3}$

$$\int_0^1 x^2 dx = F(1) - F(0) = \frac{1}{3} - \frac{0}{3} = \boxed{\frac{1}{3}}$$

Ex. compute $\int_2^3 x \cdot e^x dx$

Sol. Compute the indefinite

$$\int x \cdot e^x dx = \int x (e^x)' dx = x \cdot e^x - \int x' \cdot e^x dx$$

$$= x \cdot e^x - \int e^x dx = x \cdot e^x - e^x + C$$

$$= (x-1) e^x + C$$

Then, definite integral is

$$\int_2^3 x \cdot e^x dx \stackrel{\text{FTC2}}{=} \left[(x-1) e^x \right]_2^3 = \boxed{2e^3 - e^2}$$

Ex.

Compute $\int_0^{\frac{\pi}{2}} \sin^3 x \cdot \cos^7 x \, dx$

Sol

compute the indefinite

$$\begin{aligned} \int \sin^3 x \cdot \cos^7 x \, dx &= \int \sin^2 x \cdot \cos^7 x \cdot \sin x \, dx \\ &= \int (1 - \cos^2 x) \cdot \cos^7 x \cdot \sin x \, dx && \text{let } u = \cos x \\ &= \int (1 - u^2) u^7 du = - \int (u^7 - u^9) du = \\ &= \int (u^9 - u^7) du = \frac{u^{10}}{10} - \frac{u^8}{8} + C = \boxed{\frac{\cos^{10} x}{10} - \frac{\cos^8 x}{8}} \end{aligned}$$

The definite Integral is:

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cdot \cos^7 x \, dx = \left[\frac{\cos^{10} x}{10} - \frac{\cos^8 x}{8} \right]_0^{\frac{\pi}{2}} =$$

$$-\frac{1}{10} + \frac{1}{8} = \boxed{\frac{1}{40}}$$

Net Change Theorem

If $f(x)$ is differentiable and $f'(x)$ is continuous, then

$$\int_a^b f'(x) dx = f(b) - f(a)$$

$$\int_a^b f'(x) dx = f(b) - f(a)$$

Ex.

$$\int_3^4 \frac{1}{x^2-4} dx$$

Sol.

$$x^2-4 = (x-2)(x+2)$$

Form of PFD

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} \quad | \quad x^2-4$$

$$1 = A(x+2) + B(x-2)$$

If $x = -2, 1 = A \cdot 0 + B \cdot (-4) \Rightarrow B = -\frac{1}{4}$

If $x = 2, 1 = A \cdot 4 + B \cdot 0 \Rightarrow A = \frac{1}{4}$

The PFD is $\frac{1/4}{x-2} - \frac{1/4}{x+2}$

$$\int \frac{1}{x^2-4} dx = \left(\frac{1/4}{x-2} - \frac{1/4}{x+2} \right) dx =$$

$$\left(\frac{1}{4} \ln|x-2| - \frac{1}{4} \ln|x+2| \right) + C$$

$$\int_3^4 \frac{1}{x^2-4} dx = \left[\frac{1}{4} \ln|x-2| - \frac{1}{4} \ln|x+2| \right]_3^4$$

$$= \left(\frac{1}{4} \ln 2 - \frac{1}{4} \ln 6 \right) - \left(\frac{1}{4} \ln 1 - \frac{1}{4} \ln 5 \right)$$

$$= \frac{1}{4} \ln 2 + \frac{1}{4} \ln 5 - \frac{1}{4} \ln 6 = \frac{1}{4} \ln \left(\frac{10}{6} \right)$$

January 28, 2026

- 1 Basic Properties of Definite Integrals
- 2 Integration by parts for definite integrals
- 3 Substitution rule for definite integrals

FTC2

If f is continuous on $[a, b]$ and F is an antiderivative of f , then

$$\int_a^b f(x) dx = F(b) - F(a)$$

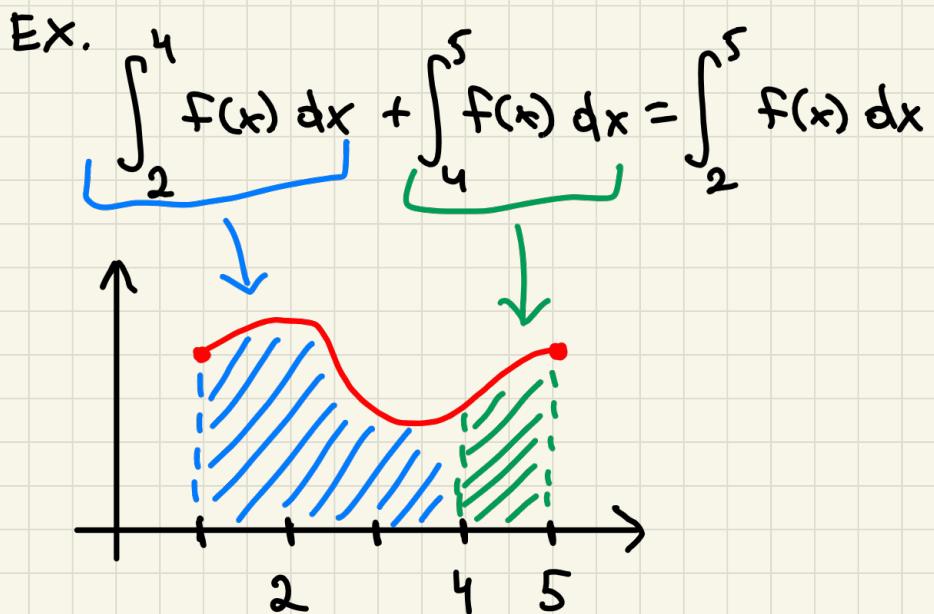
$$1. \int_a^a f(x) dx = 0$$

$$2. \int_b^a f(x) dx = - \int_a^b f(x) dx$$

e.g.

$$\int_7^3 f(x) dx = - \int_3^7 f(x) dx$$

$$3. \int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$



Integration by parts for indefinite

$$\int f(x) g'(x) dx = [f(x) g(x)]_a^b - \int_a^b f'(x) g(x) dx$$

Proof:

$$\begin{aligned} & \int_a^b f(x) g'(x) dx + \int_a^b f'(x) g(x) dx = \\ &= \int_a^b (f(x) g'(x) + f'(x) g(x)) dx = \int_a^b (f(x) g(x))' dx \end{aligned}$$

$$= \left[f(x) g(x) \right]_a^b$$

Ex.

$$\text{Compute } \int_2^3 x \cdot e^x \, dx$$

Solution (using integration by parts
for definite integrals)

$$\int_2^3 x \cdot e^x \, dx = \int_2^3 x \cdot e^x \, dx = \left[x \cdot e^x \right]_2^3 - \int_2^3 x \cdot e^x \, dx$$

$$= \left[x \cdot e^x \right]_2^3 - \int_2^3 e^x \, dx = \left[x \cdot e^x \right]_2^3 - \left[e^x \right]_2^3 =$$

$$= (3e^3 - 2e^2) - (e^3 - e^2) = \boxed{2e^3 - e^2}$$

Substitution rule for definite integral

$$\int \underbrace{f(g(x))}_{f(u)} \underbrace{g'(x)}_{du} \, dx = \int f(u) \, du$$

$$\int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

Proof

let F be an antiderivative of f

$$F'(x) = f(x)$$

$$\int_a^b f(g(x)) g'(x) dx = \int_a^b F'(g(x)) g'(x) dx$$

$$= \int_a^b F(g(x))' dx = F(g(b)) - F(g(a))$$

$$\int_{g(a)}^{g(b)} f(u) du = F(g(b)) - F(g(a))$$

Ex.

$$\int_2^3 x \cdot e^{x^2} dx$$

Solution 1 (using u-sub for definite)

let $u = x^2$

$$du = 2x \, dx \quad x \cdot dx = \frac{du}{2}$$

$$\int_2^3 x \cdot e^{x^2} \, dx = \int_4^9 e^u \cdot \frac{du}{2} = \frac{1}{2} \int e^u \, du =$$

$$= \frac{1}{2} [e^u]_4^9 = \boxed{\frac{1}{2} (e^9 - e^4)}$$

Ex 2

Compute $\int_0^\pi \cos x \sqrt[5]{2 + \tan^{-1}(e^{\cos^2 x} - 1)}$

$$f(x) = f(-x) \int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx \quad \text{even functions}$$

$$f(x) = -f(-x)$$

$$\int_{-a}^a f(x) \, dx = 0$$

Ex.

Compute $\int_0^2 \tan((x-1)^5) \, dx = u = x-1$
 $du = dx$

$$= \int_{-1}^1 \tan(u^5) du = 0 \quad \text{because } \tan(u^5) \text{ is an odd function}$$

check: $\tan((-u)^5) = \tan(-u^5) = -\tan(u^5)$

January 30, 2026

$$\int_1^2 \frac{1}{x^2+6x+5} dx$$

Solution: $x^2+6x+5 = (x+1)(x+5)$

The form of PFD

$$\frac{1}{x^2+6x+5} = \frac{A}{x+1} + \frac{B}{x+5} \quad | \cdot (x+1)(x+5)$$

$$1 = A(x+5) + B(x+1)$$

for $x = -1$:

$$1 = A \cdot 4 + B \cdot 0 \Rightarrow A = \frac{1}{4}$$

for $x = -5$: $1 = A \cdot 0 + B(-4) \Rightarrow B = -\frac{1}{4}$

The PFD is:

$$\frac{1}{x^2+6x+5} = \frac{1/4}{x+1} - \frac{1/4}{x+5}$$

$$\int \frac{1/4}{x+1} - \frac{1/4}{x+5} dx = \frac{1}{4} \ln|x+1| - \frac{1}{4} \ln|x+5| + C$$

$$\int \frac{1}{x^2+6x+5} dx = \frac{1}{4} [\ln|x+1| - \ln|x+5|]^2, =$$
$$\frac{1}{4} ((\ln 3 - \ln 7) - (\ln 2 - \ln 6))$$

Ex.

$$\int_0^1 \frac{1}{(1+x^2)^2} dx$$

Sol.

$$\int \frac{1}{(1+x^2)^2} dx = \int \frac{1}{(1+\tan^2 \theta)^2} \sec^2 \theta d\theta$$

$$\text{let } x = \tan \theta$$

$$dx = \sec^2 \theta d\theta$$

$$= \int \frac{1}{\sec^2 \theta} d\theta = \int \cos^2 \theta d\theta$$

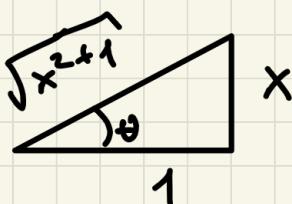
$$= \int \frac{1+\cos(2\theta)}{2} d\theta = \frac{1}{2} \left(\int (1+\cos(2\theta)) d\theta \right)$$

$$= \frac{1}{2} \left(\theta + \frac{\sin(2\theta)}{2} \right) + C$$

$$= \frac{1}{2} (\theta + \sin(\theta)\cos(\theta)) + C$$

If $x = \tan \theta$, then $\theta = ?$ sin, cos?

$$\theta = \tan^{-1} x$$



$$\cos \theta = \frac{1}{\sqrt{x^2+1}} \quad \checkmark$$

$$\sin \theta = \frac{x}{\sqrt{x^2+1}}$$

$$= \frac{1}{2} \left(\tan^{-1} x + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} \right) + C$$

$$= \frac{1}{2} \left(\tan^{-1} x + \frac{x}{1+x^2} \right) + C$$

$$\int_0^1 \frac{1}{(1+x^2)^2} dx = \frac{1}{2} \left[\tan^{-1} x + \frac{x}{1+x^2} \right]_0^1 =$$

$$= \frac{1}{2} \left(\left(\tan^{-1} 1 + \frac{1}{1+1^2} \right) - \left(\tan^{-1} 0 + \frac{0}{1+0^2} \right) \right)$$

$$= \frac{1}{2} \left(\left(\frac{\pi}{4} + \frac{1}{2} \right) - (0+0) \right) = \boxed{\frac{\pi}{8} + \frac{1}{4}}$$

Ex. $\int \cos(\ln x) dx$

Sol.

$$\text{let } u = \ln x \quad x = e^u \\ dx = e^u du$$

$$\int \cos(\ln x) dx = \int \cos(u) e^u du$$

$$= \int \cos u \cdot (e^u)' du = \cos u e^u - \int \cos u \cdot e^u du$$

$$= \cos u e^u + \int (\sin u) e^u du$$

$$= \cos u e^u + \sin u e^u - \int \sin u \cdot e^u du$$

$$= \cos u e^u + \sin u e^u - \int \cos(u) e^u du$$

$$\int \cos(u) e^u du = \frac{\cos u e^u + \sin u e^u}{2}$$

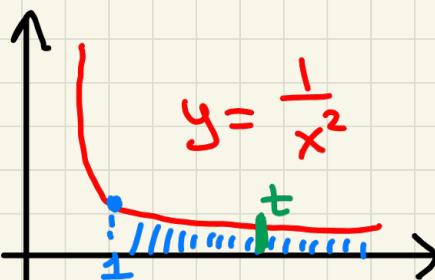
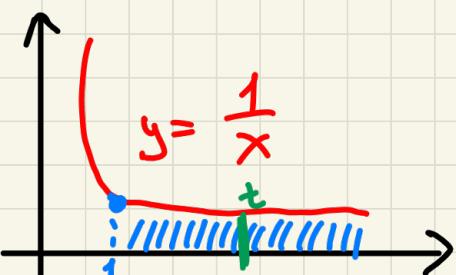
February 2, 2026

Inproper Integrals

1. Type 1 Improper integrals
infinite integrals

2. Type 2 Improper integrals
discontinuous

Motivation



$$?? \int_1^{\infty} \frac{1}{x} dx$$

$$A_t = \int_1^t \frac{1}{x} dx$$

$$= [\ln x]_1^t = \ln t$$

Total area =

$$= \lim_{t \rightarrow \infty} A_t =$$

$$= \lim_{t \rightarrow \infty} \ln t = \infty$$

$$A_t = \int_1^t \frac{1}{x^2} dx =$$

$$\int_1^t x^{-2} dx = \left[\frac{x^{-1}}{-1} \right]_1^t =$$

$$= \left[-\frac{1}{x} \right]_1^t = -\frac{1}{t} + 1$$

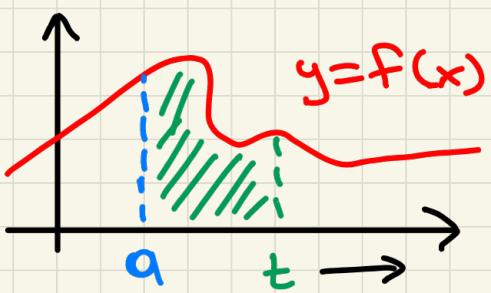
Total area =

$$= \lim_{t \rightarrow \infty} A_t =$$

$$\lim_{t \rightarrow \infty} \left(-\frac{1}{t} + 1 \right) = 1$$

Type 1 Improper Integrals

Case 1



$$\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

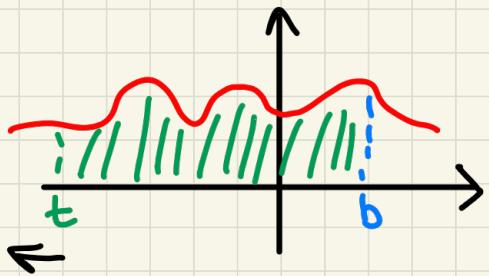
If the limit exists and finite

$$\int_a^{\infty} f(x) dx \text{ convergent}$$

If the limit doesn't exist or
is infinite

$$\int_a^{\infty} f(x) dx \text{ divergent}$$

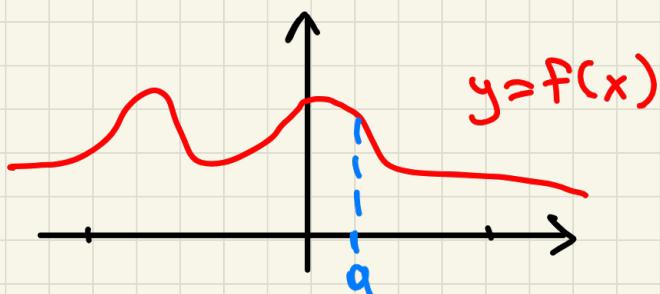
Case 2



$$\int_{-\infty}^b f(x) dx =$$

$$= \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

Case 3



$$\int_{-\infty}^{\infty} f(x) \, dx =$$

$$\underbrace{\int_{-\infty}^a f(x) \, dx}_{\text{case 3}} + \underbrace{\int_a^{\infty} f(x) \, dx}_{\text{case 1}}$$

case 3

case 1

Example

$$\int_1^{\infty} \frac{1}{x^2} \, dx$$

Solution

Indefinite integral: $\int \frac{1}{x^2} \, dx = -\frac{1}{x} + C$

$$\int_1^{\infty} \frac{1}{x^2} \, dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} \, dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{x} \right]_1^t =$$

$$= \lim_{t \rightarrow \infty} \left(-\frac{1}{t} + 1 \right) = 1$$

integral is convergent, equal to 1

Ex.

$$\int_{-\infty}^{\infty} \frac{1}{4+x^2} dx$$

Solution

Indefinite Integral : $\int \frac{1}{4+x^2} dx =$

$$\int \frac{1}{2^2+x^2} dx = \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$$

$$\int_{-\infty}^{\infty} \frac{1}{4+x^2} dx = \int_{-\infty}^0 \frac{1}{4+x^2} dx + \int_0^{\infty} \frac{1}{4+x^2} dx$$

$$\int_0^{\infty} \frac{1}{4+x^2} dx = \lim_{t \rightarrow \infty} \int_0^t \frac{1}{4+x^2} dx = \lim_{t \rightarrow \infty}$$

$$\left[\frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) \right]_0^+ = \lim_{t \rightarrow \infty} \left[\frac{1}{2} \tan^{-1}\left(\frac{t}{2}\right) - \frac{1}{2} \tan^{-1}(0) \right]$$

$$\left. \tan^{-1}\left(\frac{0}{2}\right) \right] = \frac{1}{2} \underbrace{\lim_{t \rightarrow \infty} \tan^{-1}\left(\frac{t}{2}\right)}_{\frac{\pi}{2}} = \frac{\pi}{4}$$

Ex.

$$\int_{-\infty}^0 \frac{1}{4+x^2} dx$$

$$= \lim_{t \rightarrow -\infty} \int_t^0 \frac{1}{4+x^2} dx = \lim_{t \rightarrow -\infty} \left[\frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) \right]_t^0$$

$$= \lim_{t \rightarrow -\infty} \left[\frac{1}{2} \tan^{-1}\left(\frac{0}{2}\right) - \frac{1}{2} \tan^{-1}\left(\frac{t}{2}\right) \right] =$$

$$= -\frac{1}{2} \lim_{t \rightarrow -\infty} \tan^{-1}\left(\frac{t}{2}\right) = -\frac{1}{2} \cdot \left(-\frac{\pi}{2}\right) = \frac{\pi}{4}$$

Ex.

$$\int_0^\infty \sin x dx$$

Solution

$$\int \sin x dx = -\cos x + C$$

$$\int_0^\infty \sin x dx = \lim_{t \rightarrow \infty} \int_0^t \sin x dx = \lim_{t \rightarrow \infty} \left[-\cos x \right]_0^t$$

$$= \lim_{t \rightarrow \infty} (-\cos t + 1) =$$

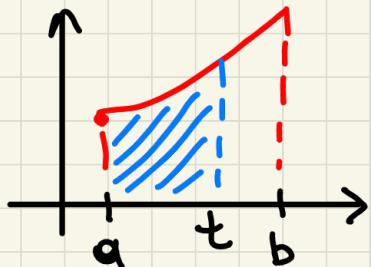
$$= 1 - \lim_{t \rightarrow \infty} \cos(t) \rightarrow \text{doesn't exist}$$

S divergent

Type 2 Improper Integrals

(~ discontinuous)

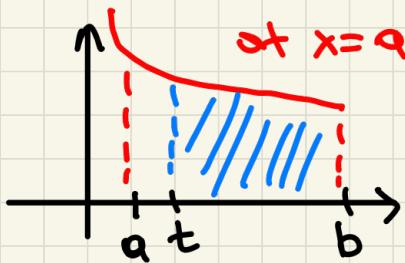
Case 1 discontin.
at $x=b$



$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

Case 2

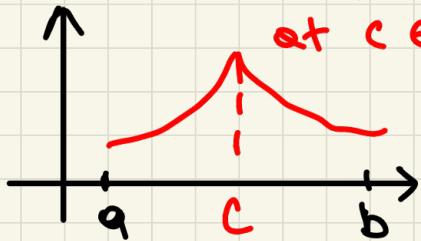
discont.
at $x=a$



$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

Case 3

discont.
at $c \in (a, b)$



$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

case 1 case 2

Ex.

$$\int_{-1}^1 \frac{1}{x} dx$$

Solution

$$\int \frac{1}{x} dx = \ln|x| + C$$

the function $y = \frac{1}{x}$ is discontin. at $x=0$

$$\int_{-1}^1 \frac{1}{x} dx = \int_{-1}^0 \frac{1}{x} dx + \int_0^1 \frac{1}{x} dx$$

$$\int_0^1 \frac{1}{x} dx = \lim_{t \rightarrow 0^+} \int_t^1 \frac{1}{x} dx = \lim_{t \rightarrow 0^+} [\ln x]_t^1$$

divergent

$$= \lim_{t \rightarrow 0^+} (\ln 1 - \ln t) = -\lim_{t \rightarrow 0^+} \ln t = +\infty$$

Ex. $\int_0^1 \ln x dx$.

Solution. Indefinite integral $\int \ln x dx = \int 1 \cdot \ln x dx = \int x' \ln x dx$
 $= x \ln x - \int x (\ln x)' dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + C$.
The function $\ln(x)$ is discontinuous at $x=0$.

$$\begin{aligned}\int_0^1 \ln x dx &= \lim_{t \rightarrow 0^+} \int_t^1 \ln x dx = \lim_{t \rightarrow 0^+} [x \ln x - x]_t^1 \\ &= \lim_{t \rightarrow 0^+} [(t \ln t - t) - (\frac{1}{t} \ln \frac{1}{t} - \frac{1}{t})] = -1 - \lim_{t \rightarrow 0^+} t \ln t \\ &= -1 - \lim_{t \rightarrow 0^+} \frac{\ln t}{\frac{1}{t}} \stackrel{\text{Hopital}}{=} -1 - \lim_{t \rightarrow 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = -1 - \lim_{t \rightarrow 0^+} \frac{1}{t} = -1\end{aligned}$$

it's convergent

Improper Integrals

- If $f(x)$ is continuous at $x=a$, $\int_a^b f(x) dx$ (if the limit is finite; otherwise divergent)
- If $f(x)$ is continuous at $x=b$, $\int_a^b f(x) dx$
- If $f(x)$ has a discontinuity at $x=c$, $\int_a^c f(x) dx$

For question

