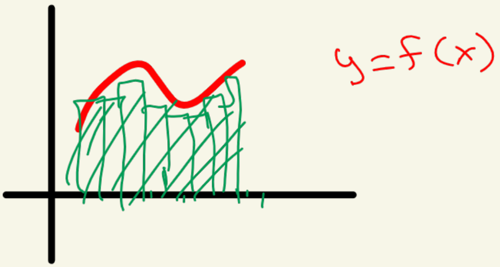
Jan 5, 2026

Part 1: Integrals

Part 2: Application of Integrals

Part 3: Sequences and Series

## Definite Integrals




If $f(x) > 0$

$$\int_a^b f(x)\,dx = \text{area below the graph}$$

Conseptual Interpretations = continuosly accumulation of quantity.

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Indefinite Integrals $\int f(x)$

essentialy antiderivitives

$x^2 \to$ antider. $\frac{x^3}{3}$ so

the most general antider.

is $\frac{x^3}{3} + C$.

$\int f(x)\,dx = F(x) + C$ means

that most general antider.

of $f(x)$

$F'(x) = f(x)$ $\int_0^1 x^2\,dx = \frac{1}{3}$

$\int x^2\,dx = \frac{x^3}{x} + c$

FTC: Foundamental Theory of Calculus

FTC: $\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}$

this always works

$\boxed{\text{January 7, 2026}}$

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$\int (f'(x)g(x) + f(x)g'(x))\,dx = f(x)+g(x)+C$$

$$\int f'(x)+g(x)\,dx + \int f(x)g'(x) = f(x)g(x)+C$$

$$\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x)\,dx$$

Integration by parts

$$\int x^2\,dx - 7 = \frac{x^3}{3} + (C-7) = \int x^2\,dx$$

1)

$$\int x\cdot\cos x\,dx$$

Integration by parts

Solution:

$$\int x \cdot \boxed{\cos x}\, dx = \int x \cdot \boxed{(\sin x)'}\, dx =$$

$$= x \cdot \sin x - \int x' \sin x \, dx = x \cdot \sin x -$$

$$- \int \sin x \, dx = x \cdot \sin x + \cos x + C$$

2)
$$\int 2^x \cdot x \, dx = \int \left(\frac{2^x}{\ln 2}\right)' \cdot x \, dx =$$

$$= \frac{2^x}{\ln 2} \cdot x - \int \frac{2^x}{\ln 2} \boxed{\overset{1}{\uparrow} x'}\, dx =$$

$$= \frac{2^x \cdot x}{\ln 2} - \frac{1}{\ln 2} \int 2^x \, dx \nearrow \frac{2^x}{\ln 2}$$

$$= \boxed{\frac{2^x \cdot x}{\ln 2} - \frac{2^x}{(\ln 2)^2}}$$

**3)** $$\int x^2 e^x \, dx = \int x^2 \cdot (e^x)' \, dx =$$

$$x^2 e^x - \int (x^2)' \cdot e^x \, dx = x^2 e^x - 2\int$$

$$-2\int \underbrace{x \cdot e^x} \, dx = x^2 \cdot e^x - 2\int x \cdot (e^x)' \, dx$$

$$= x^2 \cdot e^x - 2\left(x \cdot e^x - \int \overset{\to 1}{\boxed{x'}} ex \, dx\right) =$$

$$= x^2 \cdot e^x - 2 \cdot x \cdot e^x + 2 \boxed{\int e^x \, dx} \; e^x$$

$$= \boxed{x^2 e^x - 2x \cdot e^x + 2 \cdot e^x + C}$$

**4)** $$\int \ln x \, dx = \int \overset{x'}{\boxed{1}} \cdot \ln x \, dx =$$

$$= \int x' \ln x \, dx = x \cdot \ln x - \int x (\ln x)' \, dx$$

$$= x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx = x \cdot \ln x - \int 1 \, dx$$

$$= \boxed{x \cdot \ln x - x + c}$$

5) $\int \cos^2 x \, dx = \int \cos x \cdot (\sin x)' \, dx$

$= \cos x \cdot \sin x - \int (\cos x)' \sin x \, dx$

$= \cos x \cdot \sin x + \boxed{\int \sin^2 x \, dx}$

$= \cos x \cdot \sin x + \int (1 - \cos^2 x) \, dx$

$= \cos x \cdot \sin x + \int 1 \, dx - \int \cos^2 x \, dx$

$= \cos x \cdot \sin x + x - \int \cos^2 x \, dx$

Solve for $\int \cos^2 x \, dx$:

$$\boxed{2\int \cos^2 x \, dx = \frac{1}{2}(\cos x \sin x + x) + C}$$

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$u = x^2$ $\qquad\qquad$ $u = g(x)$

$\frac{du}{dx} = 2x$ $\qquad\qquad$ $g'(x)$

$du = 2x\,dx$ $\qquad\qquad$ $g'(x)\,dx$

$$\int \underbrace{f(x)}_{u}\,\underbrace{g'(x)}_{dv}\,dx = \underbrace{f(x)}_{u}\,\underbrace{g(x)}_{v} - \int \underbrace{g(x)}_{v}\,\underbrace{f'(x)}_{du}$$

$$\int u\,dv = uv - \int v \cdot du$$

1)

$$\int \underbrace{x}_{u} \cdot \underbrace{\cos x \, dx}_{dv} :$$

$u = x$ $\qquad$ $v = \sin x$

$dv = \cos x \cdot dx$

$du = dx$

$$\int \underbrace{x}_{u} \cdot \underbrace{\cos x \; dx}_{dv} = \int u \; dv = uv - \int v \, du$$

$$= x \cdot \sin x - \int \sin x \; dx = x \cdot \sin x - \cos x + C$$

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$\boxed{\text{January 12, 2026}}$

Substitution Rule (aka "u-sub")

$$\int (f(x) + g(x))dx = \int f(x) \; dx + \int g(x) \; dx$$

$\int f(x) g(x) \; dx \rightarrow$ integration by parts

$$f(x) g(x) - \int^{*} f'(x) \; g(x) \; dx$$

$*$ often serves the purpose

$\int f(g(x)) \; dx$ $\times$ $\rightarrow$ no such thing

$\int f(g(x)) \, g'(x) \; dx \rightarrow$ there is a formula for this, subsitution rule

$$\int f(g(x))g'(x)\,dx = \int \underbrace{F'(g(x))g'(x)}_{(F(g(x))'}\,dx$$

$$= \int (F(g(x))'\,dx = F(g(x)) + C$$ Let $u = g(x)$

$$= F(u) + C = \int f(u)\,du \checkmark$$

let $u = x^2$

$\frac{du}{dx} = 2x$ $\qquad du = 2x\,dx$

$u = g(x)$

$\frac{du}{dx} = g'(x)$ $\qquad du = g'(x)\,dx$

$$\int f(\underbrace{g(x)}_{u})\underbrace{g'(x)}_{du}\,dx = \int f(u)\,du$$

# Practice

1.

$$\int x \cdot \cos(x^2)\, dx$$

let $u = x^2$

$du = 2x\, dx \rightarrow \frac{du}{2}$

$$\int x \cdot \cos(x^2)\, dx = \int \cos(u) \cdot \boxed{\frac{du}{2}} =$$

$$\frac{1}{2}\int \cos(u)\, du = \frac{1}{2}\sin(u) + C =$$

$$\frac{1}{2}\sin(x^2) + C$$

2.

$$\int x^2 \sqrt[3]{4x^3+3}\, dx$$

let $u = 4x^3+3$

$du = 12x^2\, dx \;\Big|\; x^2 dx = \frac{du}{12}$

$$\int x^2 \sqrt[3]{4x^3+3} = \int \sqrt[3]{u}\, \frac{du}{12} = \frac{1}{2}\int \sqrt[3]{u}\, du$$

$$= \frac{1}{12}\int u^{\frac{1}{3}}\, du = \frac{1}{12}\,\frac{u^{\frac{4}{3}}}{4/3} + C =$$

$$= \frac{u^{\frac{4}{3}}}{16} + C = \boxed{\frac{(4x^3+3)^{\frac{4}{3}}}{16} + C}$$

3.

$$\int \sin(10x)\, dx$$

let $u = 10x$

$du = 10\, dx$

$dx = \frac{du}{10}$

$$\int \sin(10x)\, dx = \int \sin(u)\,\frac{du}{10} = \frac{1}{10}\int \sin(u)\, du$$

$$= -\frac{1}{10}\cos(u) + C = \boxed{-\frac{1}{10}\cos(10x) + C}$$

Useful shortent :

If $\int f(x)\, dx = F(x) + C$, then

- $\int f(x+a)\, dx = F(x+a) + C$

$$\int \cos(x+7)\,dx = \sin(x+7) + C$$

$$\int f(a\cdot x)\,dx = \frac{F(a\cdot x)}{a} + C$$

$$\int \cos(7\cdot x)\,dx = \frac{\sin(7x)}{7} + C$$

5. $\int \frac{x^3}{1+x^4}\,dx$

Solution:

Let $u = 1+x^4$

$du = 4x^3\,dx \quad x^3dx = \frac{du}{4}$

$$\int \frac{x^3}{1+x^4}\,dx = \int \frac{du/4}{u} = \frac{1}{4}\int \frac{1}{u}\,du =$$

$$= \frac{1}{4}\ln|x| + C = \boxed{\frac{1}{4}\ln|1+x^4| + C}$$

6. $\int \frac{x}{1+x^4}\,dx$

$$\int \frac{x}{1+x^4}\,dx = \int \frac{x}{1+(x^2)^2}\,dx$$

Let $u = x^2$

$du = 2x\,dx \qquad x\,dx = \frac{du}{2}$

$$\int \frac{1}{1+x^4}\,x\,dx = \int \frac{1}{1+u^2}\,\frac{du}{2} =$$

$$= \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}(u) + C$$

$$= \boxed{\frac{1}{2}\tan^{-1}(x^2) + C}$$

7. $\displaystyle\int \frac{1}{x^2+14x+130}\,dx$

Solution: $x^2+14x+130 = (\quad)^2 + C$

$(a+b)^2 = a^2 + 2ab + b^2$

$$x^2+14x+130=(x^2+2\cdot x\cdot 7+7^2)-7^2+130$$

$$=(x+7)^2+81 \rightarrow \text{completing the square}$$

$$\int \frac{1}{(x+7)^2+81}\,dx \qquad \text{Let } u=x+7$$

$$du=dx$$

$$\int \frac{1}{u^2+81}\,du=\int \frac{1}{u^2+9^2}\,du=$$

$$=\frac{1}{9}\tan^{-1}\left(\frac{u}{9}\right)+C=\boxed{\frac{1}{9}\tan^{-1}\left(\frac{x+7}{9}\right)+C}$$

## 11

$$\int \frac{\sin x\cdot\cos x}{1+\sin^2 x}\,dx$$

Let $u=\sin x$ $\quad du=\cos x\,dx$

$$\int \frac{\sin x \cos x}{1+\sin^2 x}\, d = \int \frac{u}{1+u^2}\, du$$

Let $v = 1+u^2$

$dv = 2u\ du$

$u \cdot du = \frac{dv}{2}$

$$\int \frac{1}{v} \frac{dv}{2} = \frac{1}{2}\int \frac{1}{v}\, dv = \frac{1}{2}\ln|v| + c$$

$$\boxed{\frac{1}{2}\ln(1+\sin^2 x) + C}$$

Alternative:

Let $u = 1+\sin^2 x$ $\quad du = 2\sin \cdot \cos x$

$$\int \frac{\sin x \cdot \cos x}{1+\sin^2 x} = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| + c$$

$$= \frac{1}{2}\ln(1+\sin^2 x) + C$$

12. $\int x^5 \cdot \cos(x^3) = \int x^2 \cdot x^3 \cdot \cos(x^3)\, dx$

Solution: Let $u = x^3$

$du = 3x^2\, dx$

$$\int x^2 \underbrace{x^3 \cos(x^3)}_{u \cdot \cos u}\, dx = \int u \cdot \cos(u) \frac{du}{3} =$$

$$= \frac{1}{3} \int u \cdot \cos u \quad du = \frac{1}{3} \int u \cdot (\sin' u)\, du =$$

$$= \frac{1}{3} \left[ u \cdot \sin(u) - \int u' \sin u\, du \right] = \frac{1}{3} ($$

$$= \frac{1}{3} \left( u \cdot \sin u - \int \sin u\, du \right) = \frac{1}{3} ( u \cdot \sin u$$

$$+ \cos u) + C = \frac{1}{3} (x^3 \cdot \sin x^3 + \cos(x^3) + C)$$

January 14, 2026

# Trigonometric Integrals

1. Integrals of form $\int \sin^m x \cos^n x \, dx$
2. Integrals of form $\int \tan^m x \sec^n x \, dx$
3. Integrals of form $\int \sin(mx) \sin(nx) \, dx$ , .... etc

1 $\int \sin^m x \cdot \cos^n x \, dx$

$$\int \cos^3 x \, dx = \int (1-\sin^2 x) \cos x \, dx =$$

$$\cos^3 x = \cos x \cdot \cos^2 x = \cos x (1-\sin^2 x)$$

$$\int \underbrace{(1-\sin^2 x)}_{1-u^2} \underbrace{\cos x \, dx}_{du} = \int (1-u^2) \, du$$

let $u = \sin x$

$du = \cos x$

$$\int \sin^m x \cdot \cos^n x \, dx$$

Case 1: n is odd

$$n = 2k+1$$

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \cdot (\cos^2 x)^k \cos x$$

$$= \int \sin^m x \cdot (1-\sin^2 x)^k \cos x \, dx$$

case 2: m is odd $(m = 2k+1)$

$$\int \sin^{2k+1} x \cdot \cos^n dx = \int \sin^{2k} x \cdot \sin x \cdot \cos^n x$$

$$= \int (1-\cos^2 x)^k \cdot \cos^n x \cdot \sin x \, dx$$

$u = \cos x$

Case 3 m & n odd, do any

Ex.

$$\int \sin^6 x \cdot \cos^5 x \, dx = \int \sin^6 x \cdot \cos^4 x \cdot \cos x \, dx$$

$$= \int \sin^6 x \cdot (\cos^2 x)^2 \cos x \, dx$$

let $u = \sin x$

$du = \cos x \, dx$

$$= \int \underbrace{\sin^6 x \cdot (1-\sin^2 x)^2}_{u^6 \cdot (1-u^2)^2} \underbrace{\cos x \, dx}_{du}$$

$$= \int u^6 (1-u)^2 \, du = \int u^6 (1-2u^2+u^4) \, du$$

$$= \int (u^6 - 2u^8 + u^{10}) \, du = \frac{u^7}{7} - 2\frac{u^9}{9} + \frac{u^{11}}{11} + C$$

$$= \boxed{\frac{\sin^7 x}{7} - \frac{2\sin^9 x}{9} + \frac{\sin^{11}}{11} + C}$$

Ex. $\int \sin^4 x \cdot \cos^6 x \, dx = \int (\sin^2 x)^2 (\cos^2 x)^3 dx$

$$= \int \left(\frac{1-\cos(2x)}{2}\right)^2 \left(\frac{1+\cos(2x)}{2}\right)^3 dx$$

Ex.

$$\int \cos^2 x \, dx$$

Solution:

$$\int \cos^2 x \, dx = \int \frac{1+\cos(2x)}{2} \, dx =$$

$$\frac{1}{2}\left( x + \int \cos(2x) \, dx \right) = \boxed{\frac{x}{2} + \frac{1}{2} \cdot \frac{\sin(2x)}{2} + C}$$

$$\int \cos(2x) = \frac{\sin(2x)}{2}$$

Ex. $\int \cos^2 x \cdot \sin^2 x \, dx = \int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2}$

$$= \frac{1}{4} \int (1-\cos(2x))(1+\cos(2x)) \, dx =$$

$$= \frac{1}{4} \int (1-\cos^2(2x)) \, dx = \frac{1}{4} x - \frac{1}{4} \int \cos^2(2x)$$

$$= \frac{x}{4} - \frac{1}{4} \int \frac{1+\cos(4x)}{2} \, dx$$

$$= \frac{x}{4} - \frac{x}{4} - \frac{1}{8}\int \cos(4x)\,dx$$

$$= \frac{x}{8} - \frac{1}{8}\cdot\frac{\sin(4x)}{4} + C = \boxed{\frac{x}{8} - \frac{\sin(4x)}{32} + C}$$

odd, even

$$\int \tan^{m} x \cdot \sec^{n} x \; dx$$

1. If $n$ is even, $n=2k$

$$\int \tan^{m} x \cdot \sec^{2k} x\, dx = \int \tan^{m} x \cdot \underbrace{\sec^{2k-2} x}_{(\sec^2 x)^{k-1}} \cdot \sec^2 x$$

$$= \int \underbrace{\tan^{m} x\,(1+\tan^2 x)^{k-1}}_{\text{interms of } u}\,\underbrace{\sec^2 x}_{du}\; dx$$

2. If $m = 2k+1$

$$\int \tan^{10} x \cdot \sec^{6} x\; dx = \int \tan^{10} x \cdot \sec^{4} x \cdot \sec^{2} x$$

$$= \int \tan^{10} x\,(\sec^2 x)^2 \cdot \sec^2 x\;\; dx$$

$$= \int \tan^{10} x \, (1+\tan^2 x)^2 \cdot \sec^2 x \; dx$$

let $u = \tan x$

$du = \sec^2 x \; dx$

$$= \int u^{10} (1+u^2)^2 \; dx \;=\; \int u^{10} (1+2u^2+u^4) \, du$$

$$= \int (u^{10} + 2u^{12} + u^{14}) \, du \;=\; \frac{u^{11}}{11} + 2\frac{u^{13}}{13} + \frac{u^{15}}{15} + C$$

$$= \boxed{\frac{\tan^{11} x}{11} + \frac{2\tan^{13}}{13} + \frac{\tan^{15}}{15} + C}$$

$$\int \cot^3 x \cdot \csc^3 x \; dx$$

Solution:

$$\int \cot^2 x \cdot \csc^2 x \, (\cot x \cdot \csc x) \; dx =$$

$$\int (\csc^2 x - 1)\csc^2 x \quad \cot x \cdot \csc x \, dx$$

let $u = \csc x$

$du = -\cot x \csc x \, dx$

$$= -\int (u^2 - 1)\, u^2 \, du = -\int (u^4 - u^2)^2 \, du =$$

$$= \int (u^2 - u^4)\, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \boxed{\frac{\csc^3 x}{3} - \frac{\csc^5 x}{5} + C}$$

1 $\displaystyle\int \cos(7x) \cdot \cos(4x) \, dx =$

$$= \int \frac{1}{2}\left(\cos(7-4x) + \cos(7x+4x)\right) dx$$

$$= \frac{1}{2}\int \left(\cos(3x) + \cos(11x)\right) dx$$

$$= \boxed{\frac{1}{2}\left(\frac{\sin(3x)}{3} + \frac{\sin(11x)}{11}\right) + C}$$

# Tutorial

$\boxed{\text{January 16, 2026}}$

1. $\displaystyle\int x^2 \cdot \tan(x^3+1)\, dx$

Solution

Let $u = x^3 + 1$

$du = 3x^2\, dx \rightarrow \frac{du}{3} = x^2\, dx$

$$\int x^2 \cdot \tan(x^3+1)\, dx$$

$$= \int \frac{du}{3} \cdot \tan(u) = \frac{1}{3}\int \tan(u)\, du =$$

$$= \frac{1}{3}\ln|\sec(u)| + C = \boxed{\frac{1}{3}\ln|\sec(x^3+1)| + C}$$

2. $\displaystyle\int \frac{x^9}{4+x^{10}}\,dx$

Let $u = 4+x^{10}$

$du = 10x^9$

$x^9 = \dfrac{du}{10}$

$$\int \frac{x^9}{4+x^{10}} = \frac{1}{10}\int \frac{1}{u}\cdot du = \frac{1}{10}\ln|u| + C$$

$$= \boxed{\frac{1}{10}\ln|4+x^{10}| + C}$$

3. $\displaystyle\int \frac{x^4}{4+x^{10}}\,dx = \int \frac{x^4}{4+(x^5)^2}\,dx$

Let $u = x^5$

$du = 5x^4\,dx$ $\qquad$ $x^4\,dx = \dfrac{du}{5}$

$$= \frac{1}{5}\int \frac{1}{4+u^2}\,du = \frac{1}{5}\int \frac{1}{2^2+u^2}\,du =$$

$$= \frac{1}{5}\cdot\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C =$$

$$= \frac{1}{5}\cdot\frac{1}{2}\cdot\tan^{-1}\left(\frac{x^5}{2}\right) + C$$

4. $\displaystyle\int \frac{1}{\sqrt{21+4x-x^2}}$

Solution

$$21+4x-x^2 = 21-(x^2-4x) =$$

$$= 21 - (\underbrace{x^2 - 2\cdot x\cdot 2 + 2^2}_{(x-2)^2} - 2^2 = 25-(x-2)^2$$

$$\int \frac{1}{\sqrt{25-(x-2)^2}}\,dx$$

Let $u = x-2$

$du = dx$

$$= \int \frac{1}{\sqrt{25 - u^2}}\, du \qquad \int \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$= \int \frac{1}{\sqrt{5^2 - u^2}}\, du = \sin^{-1}\left(\frac{u}{5}\right) + C$$

$$= \boxed{\sin^{-1}\left(\frac{x-2}{5}\right) + C}$$

6. $\int x^5 \sqrt{4 + x^2}\, dx$

Solution

Let $u = 4 + x^2$

$du = 2x\, dx$

$\boxed{x\, dx = du/2}$

$$\int x^5 \underbrace{\sqrt{4 + x^2}}_{u^2}\, dx = \int x \cdot (x^2)^2 \sqrt{4 + x^2}\, dx \quad \frac{du}{2}$$

$$= \frac{1}{2}\int (u-4)^2 \sqrt{u}\ du \rightarrow \text{brute force}$$

$$= \frac{1}{2}\int (u^2 - 8u + 16)\, u^{\frac{1}{2}}\ du =$$

$$= \frac{1}{2}\int \left(u^{\frac{5}{2}} - 8u^{\frac{3}{2}} + 16u^{\frac{1}{2}}\right) du =$$

$$= \frac{1}{2}\left(\frac{u^{7/2}}{7/2} - 8\,\frac{u^{5/2}}{5/2} + \frac{16u^{3/2}}{3/2}\right) + C =$$

$$= \boxed{\frac{(x^2+4)^{7/2}}{7} - 8\,\frac{(x^2+4)^{5/2}}{5} + 16\,\frac{(x^2+4)^{3/2}}{3} + C}$$

7. $$\int \frac{1}{1+e^{-x}}\,dx = \int \frac{1}{1+\frac{1}{e^x}}\,dx =$$

$$= \int \frac{e^x}{e^x+1}\,dx =$$

let $u = e^x + 1$

$du = e^x\,dx$

$$= \int \frac{1}{u}\, du = \ln|u| + C = \boxed{\ln(e^x+1)+C}$$

# TRIG INTEGRALS

9. $$\int \tan^{3} x \cdot \sec^{1} x \, dx =$$

odd ✓

$$= \int \tan^2 x \cdot \tan x \cdot \sec x \, dx =$$

$$= \int (\sec^2 - 1) \tan x \cdot \sec x \, dx$$

Let $u = \sec x$

$du = \tan x \cdot \sec x \, dx$

$$= \int (u^2 - 1)\, du = \frac{u^3}{3} - u + C =$$

$$\boxed{\frac{\sec^3}{3} - \sec x + C}$$

$$\int \cos^4 x \, dx = \int (\cos^2 x)^2 \, dx$$

$$= \int \left( \frac{1 + \cos(2x)}{2} \right)^2 dx$$

$$= \frac{1}{4} \int (1 + \cos(2x))^2 \, dx = \frac{1}{4} \int 1 + 2 \cdot \cos(2x) + \cos^2(2x)$$

$$= \frac{x}{4} + \frac{1}{2} \int \cos(2x) \, dx + \frac{1}{4} \int \cos^2(2x) \, dx$$

$$= \frac{x}{4} + \frac{1}{2} \, \frac{\sin(2x)}{2} + \frac{1}{4} \int \frac{1 + \cos(4x)}{2} \, dx$$

$$= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{1}{8} \int 1 + \cos(4x) \, dx =$$

$$= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{x}{8} + \frac{1}{8} \, \frac{\sin(4x)}{4} + C$$

$$= \boxed{\frac{3x}{8} + \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C}$$

$\boxed{\text{January 19, 2026}}$

# Triganametric Substitution

$$\int x^2\sqrt{16-x^2}\,dx = \int 16\sin^2\theta\sqrt{16-16\sin^2\theta}\cdot 4\cos\theta\, d\theta$$

$\sqrt{16-x^2} = \sqrt{4^2-x^2}$

$= 4\sqrt{1-\sin^2\theta} = 4\sqrt{\cos^2\theta}$

$= 4\cos\theta$

$x = 4\sin\theta$

$dx = 4\cos\theta\, d\theta$

$$= 256\int \underbrace{\sin^2\theta\cos^2\theta}_{\text{trig integral}}\, d\theta$$

1. For $\sqrt{a^2-x^2}$

$x = a\cdot\sin\theta$, choose $\theta$ such that this happens

Is there even such $\theta$?

$a^2 - x^2 \geq 0 \qquad x^2 \leq a^2 \qquad -a \leq x \leq a$

yes, becouse $-a \leq x \leq a$

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = a\sqrt{1-\sin^2\theta}$$

$$= a\sqrt{\cos^2\theta} = a|\cos\theta| = a \cdot \cos\theta$$

2.

For $\underline{\sqrt{a^2+x^2}}$ no restrictions

let $x = a \cdot \tan\theta$

$$\sqrt{a^2+x^2} = \sqrt{a^2 + a^2\tan^2\theta} = a \cdot \sqrt{\sec^2\theta}$$

$$= a|\sec\theta| = a \cdot \sec\theta$$

3.

$\sqrt{x^2-a^2} \qquad x^2 - a^2 \geq 0$

$x^2 \geq a^2$

$\underline{x \geq a}$ or $\underline{x \leq -a}$

# Ex 1.

$$\int x\sqrt{16-x^2}\,dx = \int x\sqrt{4^2-x^2}\,dx$$

let $x = 4\sin\theta \quad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$

$dx = 4\cos\theta \; d\theta$

Then $\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} =$

$= \sqrt{16(1-\sin^2\theta)} = 4\sqrt{\cos^2\theta} = 4|\cos\theta|$

$= 4\cos\theta$

because $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\int x\sqrt{16-x^2}\,dx = 4\int 4\sin\theta \cdot 4\cos\theta \cdot 4\cos\theta$$

$$= 64\int \sin\theta\cos^2\theta \; d\theta$$

let $u = \cos\theta$

$du = -\sin\theta \; d\theta$

$$= 64\int \overset{u^2}{\cos^2\theta}\, \underbrace{\sin\theta}_{-d\theta}\; d\theta$$

$$= -64\int u^2\, du = \boxed{\frac{-64}{3} u^3 + C}$$

$$= \frac{-64}{3} \cdot \cos^3 \theta + C$$

$\sin\theta = \frac{x}{4}$

$\theta = \sin^{-1}\left(\frac{x}{4}\right)$

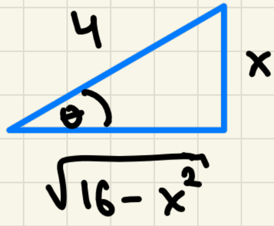
because $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= \boxed{\frac{-64}{3} \cos^3\left(\sin^{-1}\left(\frac{x}{4}\right)\right) + C}$$

If $x = 4\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

compute $\cos\theta$ in terms of $x$

$\sin\theta = \frac{x}{4}$




$$\cos\theta = \frac{\sqrt{16-x^2}}{4}$$

Conclusion

$$\int x\sqrt{16-x^2}\,dx = -\frac{64}{3}\cos^3\theta + C$$

$$= -\frac{64}{3}\left(\frac{\sqrt{16-x^2}}{4}\right)^3 + C$$

$$= \boxed{-\frac{\left(\sqrt{16-x^2}\right)^3}{3} + C}$$

2. $$\int \frac{x^3}{\sqrt{x^2+9}}\,dx$$

$x^2 + 3^2$

Solution

let $x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

Then $\sqrt{x^2+9} = \sqrt{9\tan^2\theta+9} = 3\sqrt{\tan^2\theta+1}$

$= 3\sqrt{\sec^2\theta} = 3|\sec\theta| = 3\sec\theta$

$$\int \frac{x^3}{\sqrt{x^2+9}} = \int \frac{27\tan^3\theta}{3\sec\theta}\cdot 3\sec^2\theta\, d\theta$$

$$= 27\int \tan^3\theta\,\sec\theta\, d\theta$$

$$= 27\int \tan^3\theta \cdot \sec\theta \, d\theta$$

$$= 27\int \tan^2\theta \cdot \tan\theta \cdot \sec\theta \, d\theta$$

$$= 27\int (\sec^2 - 1) \tan\theta \cdot \sec\theta \, d\theta$$

let $u = \sec\theta$

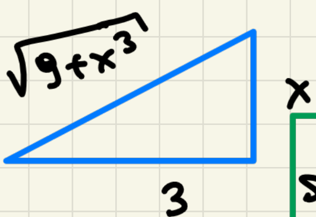
$du = \tan\theta \sec\theta \, d\theta$

$$= 27\int (u^2 - 1) \, du = 27\left(\frac{u^3}{3} - u\right) + C$$

$$= 9u^3 - 27u + C$$

$$= \boxed{9\sec^3\theta - 27\sec\theta + C}$$

$x = 3\tan\theta$




$\tan o = \frac{x}{3}$

$$\boxed{\sec\theta = \frac{\sqrt{9+x^2}}{3}}$$

$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = 9\sec^3\theta - 27\sec\theta + C$$

$$= 9\left(\frac{\sqrt{9+x^2}}{3}\right)^3 - 27\,\frac{\sqrt{9+x^2}}{3} + C$$

$$= \boxed{\left(\frac{\sqrt{9+x^2}}{3}\right)^3 - 9\sqrt{9+x^2} + C}$$

3. $\displaystyle\int \frac{\sqrt{x^2-1}}{x^6}\,dx$

Solution

$$x = \sec\theta$$

$$dx = \tan\theta\,\sec\theta\,d\theta$$

$$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$$

Then $\displaystyle\int \frac{\sqrt{x^2-1}}{x^6}\,dx = \int \frac{\tan\theta}{\sec^6\theta}\tan\theta\sec\theta$

$$= \int \frac{\tan^2\theta}{\sec^5\theta}\, d\theta = \int \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^5\theta}}\, d\theta =$$

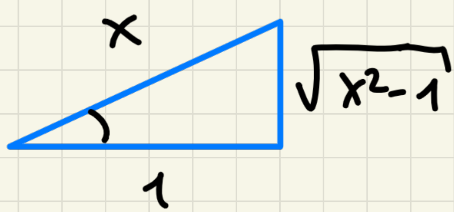
$$= \int \cos^3\theta \cdot \sin^2\theta\, d\theta = \int \cos^2\theta \cdot \sin^2\theta \cos\theta \cdot d\theta$$

$$= \int (1-\sin^2\theta)\sin^2\theta \cdot \cos\theta\, d\theta =$$

$$= \int (1-u^2)u^2\, du = \int (u^2-u^4)\, du =$$

$$\frac{u^3}{3} - \frac{u^5}{5} + C = \boxed{\frac{\sin^3\theta}{3} - \frac{\sin^5\theta}{5} + C}$$

$$\sec\theta = \frac{x}{1}$$




$$\sin\theta = \frac{\sqrt{x^2-1}}{x}$$

Then:

$$\int \frac{\sqrt{x^2-1}}{x^6}\,dx = \frac{\sin^3\theta}{3} - \frac{\sin^5\theta}{5} + C$$

$$= \boxed{\frac{1}{3}\left(\frac{\sqrt{x^2+1}}{x}\right)^3 - \frac{1}{5}\left(\frac{\sqrt{x^2-1}}{x}\right)^5 + C}$$

January 21, 2026

# Integrals of Rational Functions using Partial Fraction Decomposition

Polinomials: $P(x) = a_0 + a_1 x + \dots + a_n x^n$

Rational functions: $f(x) = \dfrac{P(x)}{Q(x)}$

where $P(x)$, $Q(x)$ are polynomials

$f(x) = \dfrac{P(x)}{Q(x)}$ is a "proper" rational function if $\deg P(x) < \deg Q(x)$

Ex

$\dfrac{2x+7}{3x^2-2x+2}$ is proper

$2 \rightarrow$ bigger than

$\dfrac{4x^2+2x+7}{3x^2-2x+2}$ is not proper

$$\int \frac{P(x)}{Q(x)}\,dx$$

Step # 0 : boil down to a proper rational function

Division with remainder for nums: If $a, b > 0$ integers, there exist $q, r$ integers such that $a = bq + r$ $\quad 0 \le r < b$

If $P(x)$ and $Q(x)$ are polynom. there exist $S(x)$ and $R(x)$ Such that:

$$P(x) = Q(x)\,S(x) + R(x)$$

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{Q(x)S(x)+R(x)}{Q(x)}\,dx =$$

proper rat. fuct.

$$\int \left(S(x) + \frac{R(x)}{Q(x)}\right) dx = \int S(x)\,dx + \int \frac{R(x)}{Q(x)}\,dx$$

Ex. $\displaystyle\int \frac{2x^3-5x^2+5x+3}{x-1}\,dx$

Solution.

$$\begin{array}{r l}
 & 2x^2-3x+2 \\
x-1 \,/ & 2x^3-5x^2+5x+3 \\
- & 2x^3-2x^2 \\
\hline
- & -3x^2+5x+3 \\
 & -3x^2+3x \\
\hline
- & 2x+3 \\
 & 2x-2 \\
\hline
 & 5
\end{array}$$

$$\overset{P(x)}{2x^3-5x^2+5x+3} = (\underset{Q(x)}{x-1})(\underset{S(x)}{2x^2-3x+2}) \underset{R(x)}{+5}$$

$$= \int \frac{(x-1)(2x^2-3x+2)+5}{x-1}\,dx =$$

$$= \int \left(2x^2-3x+2+\frac{5}{x-1}\right) dx =$$

$$= \boxed{\frac{2x^3}{3} - \frac{3x^2}{2} + 2x + 5\ln|x-1| + C}$$

Raised Goal : $\int \frac{P(x)}{Q(x)}\,dx$ if $\frac{P(x)}{Q(x)}$ proper polyn.

Case 1 : $Q(x)$ is a product of linear funct.

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\dots(a_kx+b_k)$$

There exists constants $A_1, A_2, A_3, \dots A_k$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \dots + \frac{A_k}{a_kx+b_k}$$

Ex. $\frac{P(x)}{Q(x)} = \frac{\text{Smth of degree zero or less}}{(3x-7)(5x+2)(2x+1)} =$

$$\frac{A}{3x-7} + \frac{B}{5x+2} + \frac{C}{2x+1}$$

Ex 1

$$\int \frac{x}{x^2+x-2}\,dx$$

Sol. Factor $x^2+x-2=(x+2)(x-1)$

The form of the PFD is:

$$\frac{x}{x^2+x-2}=\frac{A}{x+2}+\frac{B}{x-1}$$

Find A, B

$$\frac{x}{(x+2)(x-1)}=\frac{A}{x+2}+\frac{B}{x-1} \quad \Big| \cdot (x+2)(x-1)$$

$$x=A(x-1)+B(x+2)$$

$$x=Ax-A+Bx+2B$$

$$x=(A+B)x+(2B-A)$$

$$\underline{1}\cdot x+\underline{0}=x=\underline{(A+B)}x+\underline{(2B-A)}$$

$$\begin{cases} A+B=1 \\ 2B-A=0 \end{cases} \qquad A=2B$$

$$2B+B=1 \qquad 3B=1 \qquad B=\frac{1}{3}$$

$$A=\frac{2}{3}$$

The P.F.D is :

$$\frac{x}{x^2-x-2} = \frac{2/3}{x+2} + \frac{1/3}{x-1}$$

$$\int \frac{x}{x^2+x-2}\,dx = \int \left(\frac{2/3}{x+2} + \frac{1/3}{x-1}\right) dx$$

$$= \frac{2}{3}\int \frac{1}{x+2}\,dx + \frac{1}{3}\int \frac{1}{x-1}\,dx$$

$$= \boxed{\frac{2}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + c}$$

## Shortcut to find A and B

$x = A(x-1) + B(x+2)$

take $x = 1$ $\quad 1 = A\cdot 0 + B\cdot 3 \Rightarrow B = \frac{1}{3}$

take $x = -2$ $\quad -2 = A(-2-1) + B\cdot 0 = A = \frac{2}{3}$

!!! okay but doesn't always works !!!

## Case 2.

Example:

$$\frac{P(x)}{Q(x)} = \frac{\cdots\cdots}{(5x+1)(2x+7)^3(x-3)^2}$$

$$\frac{A}{(5x+1)} + \frac{B}{(2x+7)} + \frac{C}{(2x+7)^2} + \frac{D}{(2x+7)^3} + \frac{E}{x-3} + \frac{F}{(x-3)^2}$$

$$(a_i x + b_i)^r$$

$$\frac{B_1}{a_i x + b_i} + \frac{B_2}{(a_i x + b_i)^2} + \cdots + \frac{B_r}{(a_i x + b_i)^r}$$

Case 3:

$$\frac{P(x)}{Q(x)} = \frac{\cdots\cdots}{(5x+1)(2x+7)^3(x-3)^2(x^2+2x+7)(2x^2+3x+19)^3}$$

$$\frac{A}{(5x+1)} + \frac{B}{(2x+7)} + \frac{C}{(2x+7)^2} + \frac{D}{(2x+7)^3} + \frac{E}{x-3} + \frac{F}{(x-3)^2}$$

$$+ \frac{Gx + H}{x^2+2x+7} + \frac{Ix + J}{2x^2+3x+19} + \frac{Kx + L}{(2x^2+3x+19)^2} + \frac{Mx + N}{(2x^2+3x+19)^3}$$

Ex $\displaystyle\int \frac{x+1}{x^4+x^2}\,dx$

Sol.

Factor $x^4+x^2 = x^2(x^2+1)$

Form of the P.F.D is

$$\frac{x+1}{x^4+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} \quad \Big| \; x^2(x^2+1)$$

Find $A, B, C, D$

$$x+1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$$

$$x+1 = A(x^3+x) + B(x^2+1) + (Cx^3+Dx^2)$$

$$x+1 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$$

$$x+1 = (A+C)x^3 + (B+D)x^2 + Ax + B$$

$$0 \cdot x^3 + 0 \cdot x^2 + 1 \cdot x + 1 = (A+C)x^3 + (B+D)x^2 + Ax + B$$

$$\begin{cases} A+C = 0 \\ B+D = 0 \\ A = 1 \\ B = 1 \end{cases} \qquad \begin{array}{l} C = -1 \\ D = -1 \end{array}$$

$$\frac{x+1}{x^4+x^2} = \frac{1}{x} + \frac{1}{x^2} - \frac{x+1}{x^2+1}$$

$$\int \frac{1}{x} + \frac{1}{x^2} - \frac{x+1}{x^2+1}\, dx =$$

$$= \ln|x| - \frac{1}{x} - \int \frac{x+1}{x^2+1}$$

$$= \int \frac{x}{x^2+1}\, dx + \underbrace{\int \frac{1}{x^2+1}}_{\tan^{-1} x}$$

let $u = x^2+1$

$du = 2x\, dx$

$$\frac{1}{2}\int \frac{1}{u}\, du + \tan^{-1} x + C = \frac{1}{2}\ln(x^2+1) + \tan^{-1}$$

$$= \boxed{\int \ln|x| - \frac{1}{x} - \frac{1}{2}\ln(x^2+1) - \tan^{-1} x + C}$$

January 23, 2026

3. $\int \sqrt{\underset{5^2}{25} - x^2}\, dx$ $\qquad \sqrt{a^2 - x^2} \rightarrow x = a \cdot \sin\theta$

Solution

let $x = 5\sin\theta$

$dx = 5\cos\theta \, d\theta$

then $\sqrt{25-x^2} = \sqrt{25-25\sin^2\theta} = 5\sqrt{1-\sin^2\theta}$

$= 5|\cos\theta| = 5\cos\theta$

$\cos\theta \geq 0$ because $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$\int \sqrt{25-x^2}\, dx = \int 5\cos\theta \cdot 5\cos\theta \, d\theta =$

$25\int \cos^2\theta \, d\theta = 25\int \frac{1+\cos(2\theta)}{2}\, d\theta$

$\frac{25}{2}\int (1+\cos(2\theta))\, d\theta = \frac{25}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$

If $x = 5\sin\theta$ $\quad \sin(2\theta) = ?$

$\sin\theta = \frac{x}{5} \Rightarrow \theta = \sin^{-1}\left(\frac{x}{5}\right)$

$\sin(2\theta) = 2\sin\theta \cdot \cos\theta$

$\sin^2\theta + \cos^2\theta = 1 \quad \cos^2\theta = 1-\sin^2\theta$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$$= \frac{25}{2}\left(\sin^{-1}\left(\frac{x}{5}\right) + \frac{x\sqrt{25-x^2}}{25}\right) + c$$

$$\boxed{= \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right) + \frac{x\sqrt{25-x^2}}{2} + C}$$

4

$$\frac{x^8+x+1}{(2x+4)^2(3x^2+2x+7)(x^2+1)^3} = \frac{A}{2x+4} + \frac{B}{(2x+4)^2}$$

$$+ \frac{Cx+D}{3x^2+2x+7} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

5

degree 12

$$\frac{x^{12}+x+1}{(2x+4)^2(3x^2+2x+7)(x^2+1)^3}$$

degree : $2+2+2\cdot 3 = 10$ $\qquad 12 > 10$

not a proper rational function

$$\frac{x^8+x+1}{(2x+4)^2(3x^2+2x-1)(x^2+1)^3} = \frac{A}{2x+4} + \frac{B}{(2x+4)^2}$$

$$+\frac{C}{3x-1}+\frac{D}{x+1}+\frac{Ex+F}{x^2+1}+\frac{Gx+H}{(x^2+1)^2}+\frac{Ix+J}{(x^2+1)^3}$$

11

$$\int \frac{1}{e^{2x}+e^x}\,dx = \int \frac{e^x}{e^x(e^{2x}+e^x)}\,dx$$

let $u = e^x$

$du = e^x\,dx$

$$= \int \frac{1}{u^3+u^2}\,du \rightarrow \text{rational function}$$

Factor: $u^3+u^2 = u^2(u+1)$ PFD

$$\frac{1}{u^2(u+1)} = \frac{A}{u}+\frac{B}{u^2}+\frac{C}{u+1} \quad \Big| \; u^2(u+1)$$

$$1 = A\,u(u+1)+B(u+1)+C\,u^2$$

$$1 = A\,u^2+Au+Bu+B+C\,u^2$$

$$1 = (A+C)\,u^2+(A+B)\,u+C\,u^2$$

$$0\cdot u^2+0\cdot u+1 = (A+C)\,u^2+(A+B)\,u+B$$

$$\begin{cases} A+B=0 & C=1 \\ A+C=0 & A=-1 \\ B=1 & \end{cases}$$

The PFD is

$$\frac{1}{u^2(u+1)} = -\frac{1}{u} + \frac{1}{u^2} + \frac{1}{u+1}$$

$$\int \frac{1}{e^{2x}+e^x}\,dx = \int \frac{1}{u^3+u^2}\,du = \int \left(-\frac{1}{u} + \frac{1}{u^2} + \frac{1}{u+1}\right)$$

$$= -\ln|u| - \frac{1}{u} + \ln|u+1| + C$$

$$= -\ln|e^x| - \frac{1}{e^x} + \ln|e^x+1| + C$$

$$= \boxed{-x - \frac{1}{e^x} + \ln(e^x+1) + C}$$

January 26, 2026

# Fundamental Theorem of Calculsa

1 Fundamental Theorem of Calc p1

2 Fundamental Theorem of Calc p2

## Indefinite Integral:

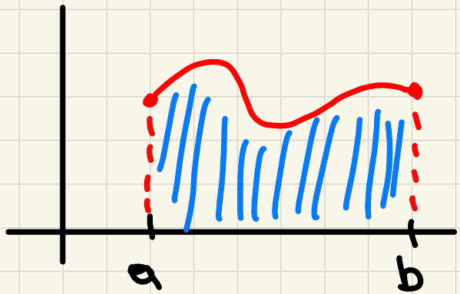
$$\int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \boxed{\frac{1}{3}}$$

"real deal"

## Definite Integrals:

$$\int_a^b f(x)\, dx = \left[\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x\right]$$

geometric interpretation:



$$\int_a^b f(x)\, dx$$

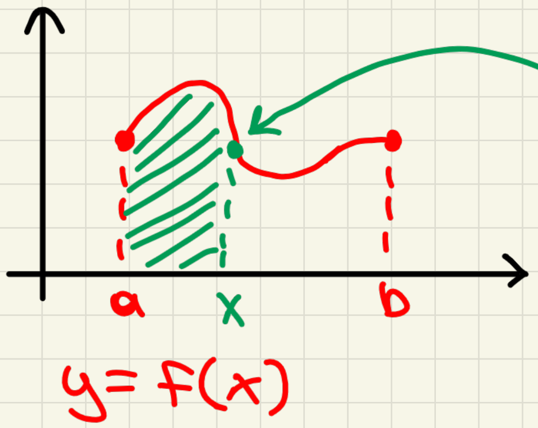
Integrals = continuous accumulation

Idea of Fundamental Theorem of Calc

" Differentiation and definite Integration are opposite operations (kind of..)

FTC 1

If $f(x)$ is continous on $[a,b]$ and $g(x) = \int_a^x f(t)\,dt$, then $g$ is continous on $[a,b]$ differentiable on $(a,b)$ and $g'(x) = f(x)$




$$g(x) = \int_a^x f(t)\,dt$$

$$g'(x) = f(x)$$

$$\frac{d}{dx}\left[\int_a^{x} f(t)\,dt\right] = f(x)$$

# Idea of Proof

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{A_h}{h}$$

Ex.

$$\frac{d}{dx}\left[\int_{\pi}^{x} \cos(t^2)\, dt\right] = \cos(x^2)$$

$$\frac{d}{dx}\left[\int_{0}^{x^4} \sqrt{1+t^3}\, dt\right] = \sqrt{1+x^{12}} \cdot 4x^3$$

because of the Chain Rule

Solution

$$g(x) = \int_{a}^{x} \sqrt{1+t^3}\, dt$$

By FTC 1, $g'(x) = \sqrt{1+x^3}$

$$\frac{d}{dx}\left[\int_{4}^{x^4} \sqrt{1+t^3}\, dt\right] = \frac{d}{dx}\left[g(x^4)\right] = g'(x^4) \cdot 4x^3$$

$$= \sqrt{1+(x^4)^3} \cdot 4x^3$$

# FTC 2

If $f(x)$ is a continous funct. on $[a,b]$ and $F$ is an antiderivative of $f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof:

Let $g(x) = \int_a^x f(t)\,dt$

FTC 1: $\left.\begin{array}{l} g'(x) = f(x) \\ F'(x) = f(x) \end{array}\right\}$ $F(x) = g(x) + C$ for all $x$ in $[a,b]$

If $x = a \Rightarrow F(a) = g(a) + C \Rightarrow C = F(a)$

If $x = b \Rightarrow F(b) = g(b) + C \Rightarrow g(b) + F(a)$

$$g(b) = F(b) - F(a)$$

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

**Ex.** compute $\int_0^1 x^2\,dx$

**Sol.**

$f(x) = x^2$ Then $F(x) = \frac{x^3}{3}$

$$\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1^3}{3} - \frac{0^3}{3} = \boxed{\frac{1}{3}}$$

**Ex.** compute $\int_2^3 x \cdot e^x\,dx$

**Sol.**

Compute the indefinite

$$\int x \cdot e^x\,dx = \int x\,(e^x)'\,dx = x \cdot e^x - \int x' \cdot e^x\,dx$$

$$= x \cdot e^x - \int e^x\,dx = x \cdot e^x - e^x + C$$

$$\boxed{= (x-1)\,e^x + C}$$

Then, definite integral is

$$\int_2^3 x \cdot e^x\,dx \overset{\text{FTC2}}{=} \Big[(x-1)e^x\Big]_2^3 = \boxed{2e^3 - e^2}$$

Ex.

compute $\int_0^{\frac{\pi}{2}} \sin^3 x \cdot \cos^7 x \; dx$

Sol

compute the indefinite

$$\int \sin^3 x \cdot \cos^7 x \; dx = \int \sin^2 x \cdot \cos^7 x \cdot \sin x \; dx$$

$$= \int (1-\cos^2 x) \cdot \cos^7 x \cdot \sin x \; dx$$

let $u = \cos x$

$du = -\sin x$

$$= \int (1-u^2)\, u^7 \, du = -\int (u^7 - u^9)\, du =$$

$$= \int (u^9 - u^7)\, du = \frac{u^{10}}{10} - \frac{u^8}{8} + C = \boxed{\frac{\cos^{10} x}{10} - \frac{\cos^8 x}{8}}$$

The definite Integral is:

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cdot \cos^7 x \; dx = \left[ \frac{\cos^{10} x}{10} - \frac{\cos^8 x}{8} \right]_0^{\frac{\pi}{2}} =$$

$$-\frac{1}{10} + \frac{1}{8} = \boxed{\frac{1}{40}}$$

# Net Change Theorem

If $f(x)$ is differentiable and $f'(x)$ is continous, then

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Ex. $\displaystyle\int_3^4 \frac{1}{x^2-4}\,dx$

Sol. $x^2-4=(x-2)(x+2)$

Form of PFD

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} \quad \Big|\, x^2-4$$

$$1 = A(x+2) + B(x-2)$$

If $x=-2$, $1 = A\cdot 0 + B\cdot(-4) \Rightarrow B = -\frac{1}{4}$

If $x=2$, $1 = A\cdot 4 + B\cdot 0 \Rightarrow A = \frac{1}{4}$

The PFD is $\frac{1/4}{x-2} - \frac{1/4}{x+2}$

$$\int \frac{1}{x^2-4}\,dx = \left(\frac{1/4}{x-2} - \frac{1/4}{x+2}\right)dx =$$

$$\left(\frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2|\right) + C$$

$$\int_3^4 \frac{1}{x^2-4}\,dx = \left[\frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2|\right]_3^4$$

$$= \left(\frac{1}{4}\ln 2 - \frac{1}{4}\ln 6\right) - \left(\frac{1}{4}\ln 1 - \frac{1}{4}\ln 5\right)$$

$$= \boxed{\frac{1}{4}\ln 2 + \frac{1}{4}\ln 5 - \frac{1}{4}\ln 6 = \frac{1}{4}\ln\left(\frac{10}{6}\right)}$$

$\boxed{\text{January 28, 2026}}$

1 Basic Properties of Definite Intg

2 Integration by parts for definite

3 Substitution rule for definite Int

FTC 2

If f is continous on [a, b] and F is an antidrivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

1. $\int_a^a f(x)\,dx = 0$

2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

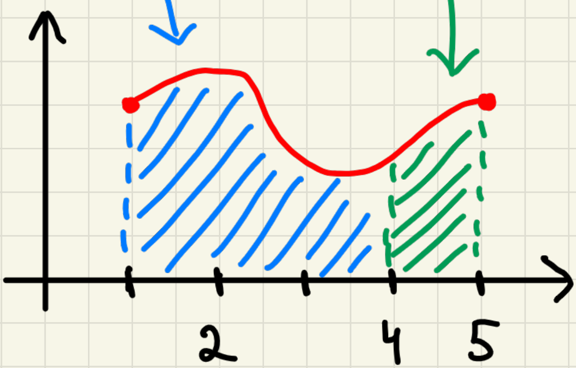
e.g.

$$\int_7^3 f(x)\,dx = -\int_3^7 f(x)\,dx$$

3. $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Ex.

$$\int_2^4 f(x)\,dx + \int_4^5 f(x)\,dx = \int_2^5 f(x)\,dx$$




## Integration by parts for indefinite

$$\int f(x)\,g'(x)\,dx = \left[f(x)\,g(x)\right]_a^b - \int_a^b f'(x)\,gx$$

**Proof:**

$$\int_a^b f(x)\,g'(x)\,dx + \int_a^b f'(x)\,g(x)\,dx =$$

$$= \int_a^b \left(f(x)\,g'(x) + f'(x)\,g(x)\right) dx = \int_a^b \left(f(x)\,g(x)\right)' dx$$

$$= \Big[f(x)\,g(x)\Big]_a^b$$

Ex.

Compute $\int_2^3 x \cdot e^x \, dx$

Solution (using integration by parts for definite integrals)

$$\int_2^3 x \cdot e^x \, dx = \int_2^3 x \cdot e^x \, dx = \Big[x \cdot e^x\Big]_2^3 - \int_2^3 \dot{x} \cdot e^x$$

$$= \Big[x \cdot e^x\Big]_2^3 - \int_2^3 e^x \, dx = \Big[x \cdot e^x\Big]_2^3 - \Big[e^x\Big]_2^3 =$$

$$= \left(3e^3 - 2e^2\right) - \left(e^3 - e^2\right) = \boxed{2e^3 - e^2}$$

Substitution rule for definite integral

$$\int \underbrace{f(g(x))}_{f(u)} \underbrace{g'(x) \, dx}_{du} = \int f(u) \, du$$

$$\int_a^b F(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

**Proof**

let F be an antiderivative of $f(x)$

$$F'(x) = f(x)$$

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_a^b F'(g(x))\, g'(x)\, dx$$

$$= \int_a^b F(g(x))'\, dx = F(g(b)) - F(g(a))$$

$$\int_{g(a)}^{g(b)} f(u)\, du = F(g(b)) - F(g(a))$$

**Ex.**

$$\int_2^3 x \cdot e^{x^2}\, dx$$

**Solution 1** (using u-sub for definite) let $u = x^2$

$du = 2x\, dx \qquad x \cdot dx = \frac{du}{2}$

$$\int_2^3 x \cdot e^{x^2}\, dx = \int_4^9 e^u \cdot \frac{du}{2} = \frac{1}{2}\int e^u \cdot du =$$

$$= \frac{1}{2}\left[e^u\right]_4^9 = \boxed{\frac{1}{2}\left(e^9 - e^4\right)}$$

Ex 2

Compute $\int_0^{\pi} \cos x \sqrt[5]{2 + \tan^{-1}\left(e^{\cos^2 x} - 1\right)}$

$f(x) = f(-x) \quad \int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$ even functions

$$f(x) = -f(x)$$

$$\int_{-a}^{a} f(x)\, dx = 0$$

Ex. Compute $\int_0^2 \tan\left((x-1)^5\right) dx =$ $u = x-1$, $du = dx$

$$= \int_{-1}^{1} \tan(u^5)\, du = 0$$ because $\tan(u^5)$ is an odd funt.

Check: $\tan((-u)^5) = \tan(-u^5) = -\tan(u^5)$

$\boxed{\text{January 30, 2026}}$

$$\int_1^2 \frac{1}{x^2+6x+5}\, dx$$

Solution: $x^2+6x+5 = (x+1)(x+5)$

The form of PFD

$$\frac{1}{x^2+6x+5} = \frac{A}{x+1} + \frac{B}{x+5} \quad \Big| \cdot (x+1)(x+5)$$

$$1 = A(x+5) + B(x+1)$$

for $x=-1$:

$$1 = A\cdot 4 + B\cdot 0 \Rightarrow A = \frac{1}{4}$$

for $x=-5$: $\quad 1 = A\cdot 0 + B(-4) \Rightarrow B = -\frac{1}{4}$

The PFD is:

$$\frac{1}{x^2+6x+5} = \frac{1/4}{x+1} - \frac{1/4}{x+5}$$

$$\int \frac{1/4}{x+1} - \frac{1/4}{x+5}\, dx = \frac{1}{4}\ln|x+1| - \frac{1}{4}\ln|x+5| + C$$

$$\int \frac{1}{x^2+6x+5}\, dx = \frac{1}{4}\Big[\ln|x+1| - \ln|x+5|\Big]_1^2 =$$

$$\frac{1}{4}\Big((\ln 3 - \ln 7) - (\ln 2 - \ln 6)\Big)$$

---

Ex. $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2}\, dx$

Sol.

$$\int \frac{1}{(1+x^2)^2}\, dx = \int \frac{1}{\underbrace{(1+\tan^2\theta)^2}_{\sec^4\theta}} \sec^2\theta\, d\theta$$

let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$= \int \frac{1}{\sec^2\theta}\, d\theta = \int \cos^2\theta \; d\theta$$

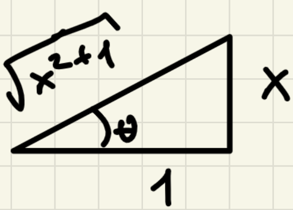
$$= \int \frac{1+\cos(2\theta)}{2}\, d\theta = \frac{1}{2}\left(\int (1+\cos(2\theta))\, d\theta\right)$$

$$= \frac{1}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$

$$= \frac{1}{2}\left(\theta + \sin(\theta)\cos(\theta) + C\right.$$

If $x = \tan\theta$, then $\theta = ?$ sin, cos?

$$\theta = \tan^{-1} x$$




$$\cos\theta = \frac{1}{\sqrt{x^2+1}} \quad \checkmark$$

$$\sin\theta = \frac{x}{\sqrt{x^2+1}}$$

$$= \frac{1}{2}\left(\tan^{-1} x + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}}\right) + C$$

$$= \frac{1}{2}\left(\tan^{-1} x + \frac{x}{1+x^2}\right) + C$$

$$\int_0^1 \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\left[\tan^{-1}x + \frac{x}{1+x^2}\right]_0^1 =$$

$$= \frac{1}{2}\left(\left(\tan^{-1}1 + \frac{1}{x+1^2}\right) - \left(\tan^{-1}0 + \frac{0}{1+0^2}\right)\right)$$

$$= \frac{1}{2}\left(\left(\frac{\pi}{4} + \frac{1}{2}\right) - (0+0)\right) = \boxed{\frac{\pi}{8} + \frac{1}{4}}$$

Ex. $\int \cos(\ln x)\,dx$

Sol. let $u = \ln x \qquad x = e^u$

$dx = e^u\,du$

$$\int \cos(\ln x)\,dx = \int \cos(u)\,e^u\,du$$

$$= \int \cos u \cdot (e^u)'\,du = \cos u\, e^1 - \int \cos u' \cdot e^u\,du$$

$$= \cos u\, e^u + \int (\sin u)\,e^u\,du$$

$$= \cos u\, e^{u} + \sin u\, e^{u} - \int \sin' u \cdot e^{u}\, du$$

$$= \cos u\, e^{u} + \sin u\, e^{u} - \int \cos(u)\, e^{u}\, du$$

$$\boxed{\int \cos(u)\, e^{u}\, du = \frac{\cos u\, e^{u} + \sin u\, e^{u}}{2}}$$

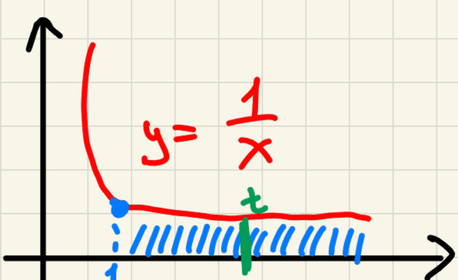
**February 2, 2026**

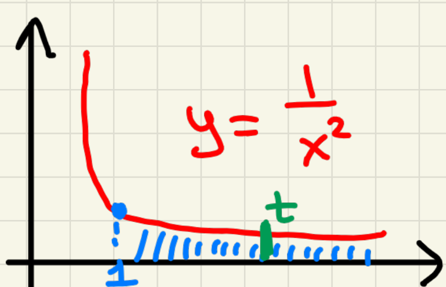
# Inproper Integrals

1. Type 1 Improper integrals
   infinite integrals

2. Type 2 Improper integrals
   discontinous

## Motivation



?? $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

$$A_t = \int_1^t \frac{1}{x}\,dx$$

$$= \left[\ln x\right]_1^t = \ln t$$

Total area =

$$= \lim_{t\to\infty} A_t =$$

$$= \lim_{t\to\infty} \ln t = \infty$$

$$A_t = \int_1^t \frac{1}{x^2}\,dx =$$

$$\int_1^t x^{-2}\,dx = \left[\frac{x^{-1}}{-1}\right]_1^t =$$

$$= \left[-\frac{1}{x}\right]_1^t = -\frac{1}{t} + 1$$

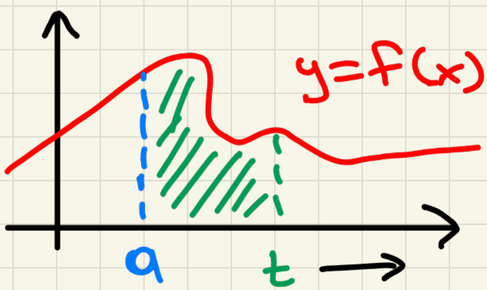
Total area =

$$= \lim_{t\to\infty} A_t =$$

$$\lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = 1$$

## Type 1 Improper Integrals

Case 1



$$\int_a^{\infty} f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx$$

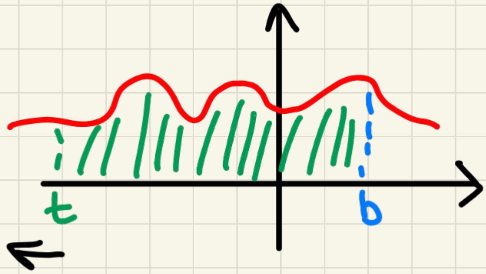
If the limit exists and finite

$\int_a^{\infty} f(x)\,dx$ convergent

If the limit doesnt exist or is infinite

$\int_a^{\infty} f(x)$ divergent

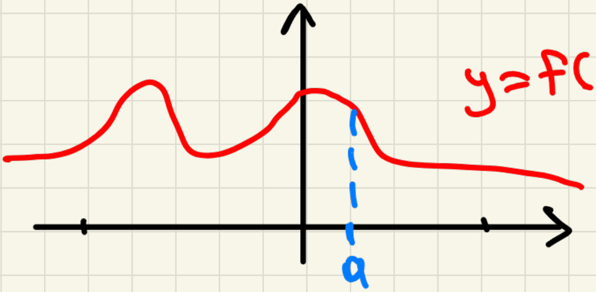
Case 2




$$\int_{-\infty}^{b} f(x)\,dx =$$

$$= \lim_{t \to -\infty} \int_t^b f(x)\,dx$$

Case 3




$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{a} f(x)\,dx}_{\text{Case 3}} + \underbrace{\int_{a}^{\infty} f(x)\,dx}_{\text{case 1}}$$

Example $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

Solution

Indefinite integral: $\displaystyle\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t =$$

$$= \lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = 1$$

integral is convergent, equal to 1

Ex. $$\int_{-\infty}^{\infty} \frac{1}{4+x^2}\, dx$$

Solution

Indefinite Integral : $\int \frac{1}{4+x^2}\, dx =$

$$\int \frac{1}{2^2+x^2}\, dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$\int_{-\infty}^{\infty} \frac{1}{4+x^2}\, dx = \int_{-\infty}^{0} \frac{1}{4+x^2}\, dx + \int_{0}^{\infty} \frac{1}{4+x^2}\, dx$$

$$\int_{0}^{\infty} \frac{1}{4+x^2}\, dx = \lim_{t\to\infty} \int_{0}^{t} \frac{1}{4+x^2}\, dx = \lim_{t\to\infty}$$

$$\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^t = \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) - \frac{1}{2}\right.$$

$$\left.\tan^{-1}\left(\frac{0}{2}\right)\right] = \frac{1}{2}\underbrace{\lim_{t\to\infty} \tan^{-1}\left(\frac{t}{2}\right)}_{\frac{\pi}{2}} = \frac{\pi}{4}$$

Ex. $$\int_{-\infty}^{0} \frac{1}{4+x^2}\,dx$$

$$= \lim_{t\to-\infty} \int_t^0 \frac{1}{4+x^2}\,dx = \lim_{t\to-\infty} \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_t^0$$

$$= \lim_{t\to-\infty} \left[\frac{1}{2}\tan^{-1}\left(\frac{0}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right] =$$

$$= -\frac{1}{2} \lim_{t\to-\infty} \tan^{-1}\left(\frac{t}{2}\right) = -\frac{1}{2}\cdot\left(-\frac{\pi}{2}\right) = \frac{\pi}{4}$$

Ex. $$\int_0^{\infty} \sin x \; dx$$

Solution $$\int \sin x \, dx = -\cos x + C$$

$$\int_0^{\infty} \sin x \, dx = \lim_{t\to\infty} \int_0^t \sin x \, dx = \lim_{t\to\infty}$$

$$\left[-\cos x\right]_0^t = \lim_{t\to-\infty} \left(-\cos t + 1\right) =$$

$$= 1 - \lim_{t\to\infty} \cos(t) \rightarrow \text{doesn't exist}$$

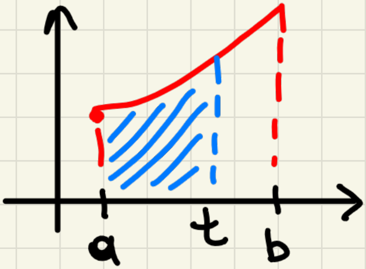
$\int$ divergent

# Type 2 Improper Integrals

(~ discontinous)

## Case 1

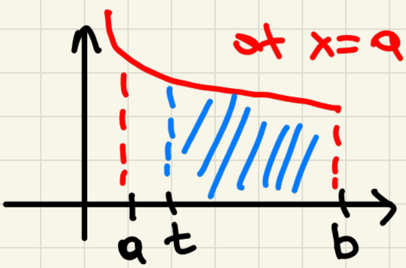
discontin. at $x = b$




$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

## Case 2

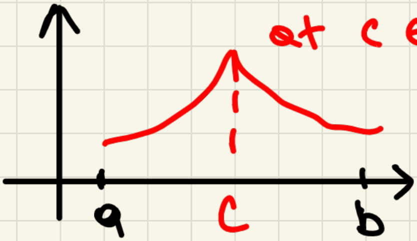
discont. at $x = a$




$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

## Case 3

discont. at $c \in (a,b)$




$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)}_{\text{case 1}} + \underbrace{\int_c^b f(x)\,dx}_{\text{case 2}}$$

**Ex.** $\displaystyle\int_{-1}^{1} \frac{1}{x}\, dx$

**Solution** $\displaystyle\int \frac{1}{x}\, dx = \ln|x| + C$

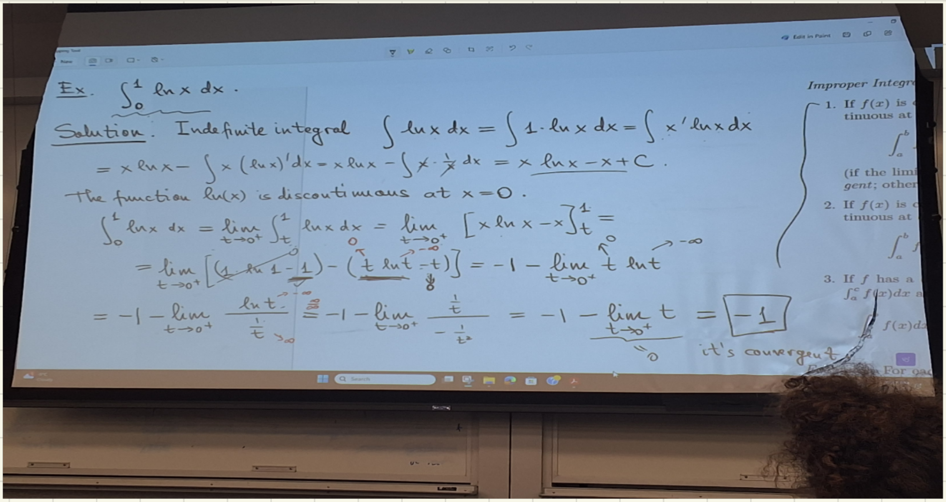
the function $y = \frac{1}{x}$ is discont. at $x = 0$

$$\int_{-1}^{1} \frac{1}{x}\, dx = \int_{-1}^{0} \frac{1}{x}\, dx + \int_{0}^{1} \frac{1}{x}\, dx$$

$$\int_{0}^{1} \frac{1}{x}\, dx = \lim_{t \to 0^+} \int_{t}^{1} \frac{1}{x}\, dx = \lim_{t \to 0^+} \left[\ln x\right]_t^1$$

divergent

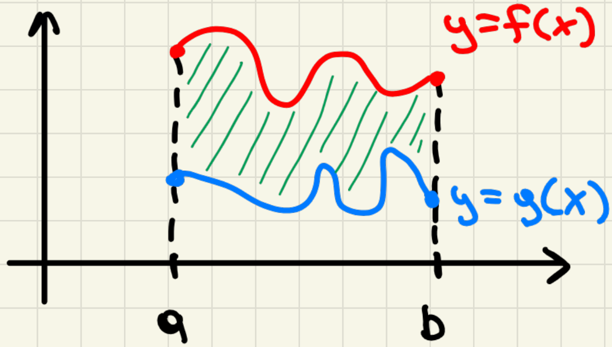
$$= \lim_{t \to 0^+} (\ln 1 - \ln t) = -\lim_{t \to 0^+} \ln t = +\infty$$



February 4, 2026

# Part II: Application of Integrals

Today: Area between 2 curves




$$A = \int_a^b \left( f(x) - g(x) \right)\, dx$$

## Ex.

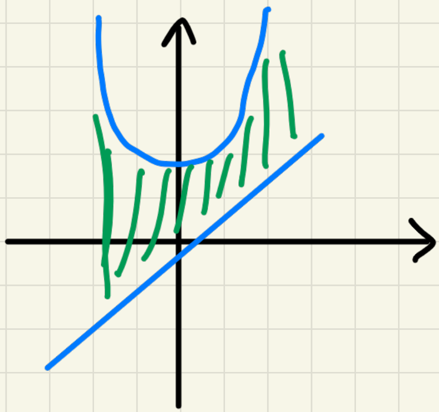
① $y = x$, $y = x^2 + 1$, $x = -1$, $x = 1$

Solution:

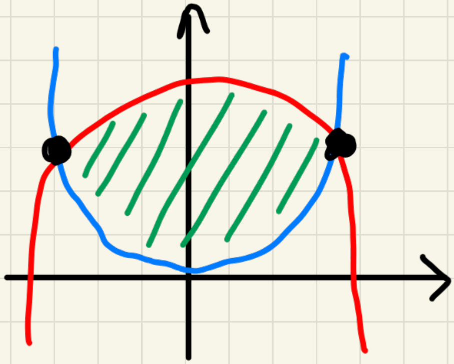
$$A = \int_{-1}^{1} (x^2+1) - x \quad dx$$

$$= \left( \frac{1^3}{3} + 1 - \frac{1^2}{2} \right) - \left( \frac{-1}{3} + (-1) - \frac{-1^2}{2} \right)$$

$$= \quad \frac{2}{3} + 2 = \boxed{\frac{8}{3}}$$



② $y = x^2 \qquad y = 8 - x^2$




$$\begin{cases} y = x^2 \\ y = 8-x^2 \end{cases}$$

$$x^2 = 8 - x^2$$

$$2x^2 = 8 \Rightarrow x^2 = 4$$

$$x = \pm 2 \quad \begin{array}{l} \nearrow x=2 \quad y=4 \\ \searrow x=-2 \quad y=4 \end{array}$$

Intersection points $(2,4)$ $(-2,4)$

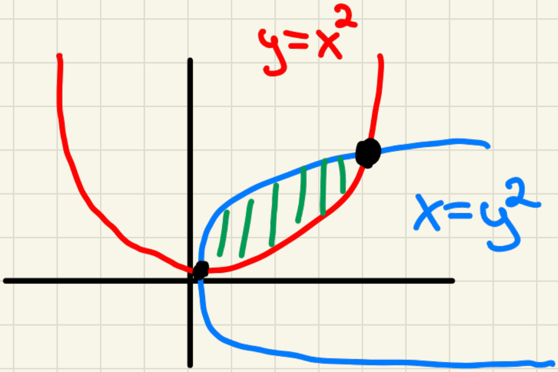
$$\text{Area} = \int_{-2}^{2} \left((8-x^2)-x^2\right) dx = \int_{-2}^{2} (8-2x^2)\, dx$$

$$= \left[8x - \frac{2x^3}{3}\right]_{-2}^{2} = 8\cdot 2 - \frac{2\cdot 2^3}{3}$$

$$= 8\cdot(-2) - 2\cdot(-2)^3 = \boxed{\frac{64}{3}}$$

3

$y = x^2$ $\qquad$ $x = y^2$




$$\begin{cases} y = x^2 \\ x = y^2 \end{cases} \qquad \begin{aligned} x &= (x^2)^2 = x^4 \\ x^4 - x &= 0 \end{aligned}$$

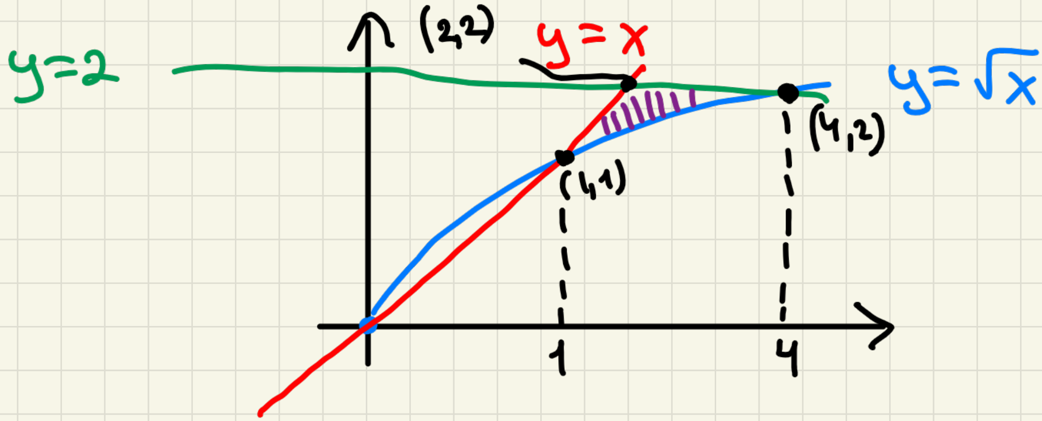
$$x(x^3-1) = 0 \quad \begin{cases} x = 0 \\ x = 1 \end{cases}$$

Intersection points $(0,0)$ $(1,1)$

$$\text{Area } \int_0^1 (\sqrt{x} - x^2)\,dx = \int_0^1 x^{\frac{1}{2}} - x^2\,dx$$

$$= \left[ \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1 = \frac{1^{3/2} \cdot 2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}}$$

4 $\quad y = \sqrt{x}, \quad y = x, \quad y = 2$



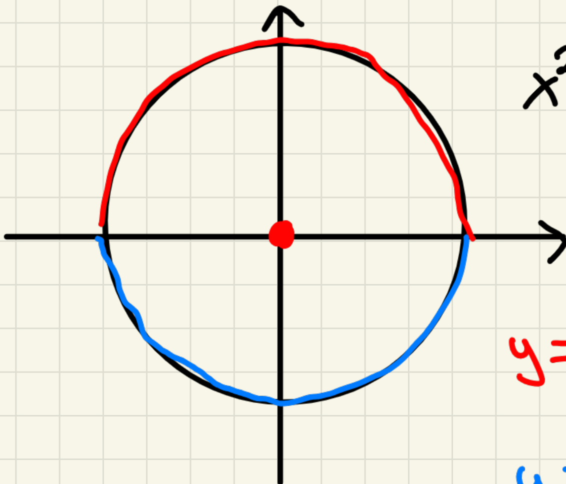

$$\begin{cases} y = x \\ y = \sqrt{x} \end{cases} \qquad x = \sqrt{x} \qquad x^2 = x \qquad x(x-1) = 0$$

Inter. points $(0,0)$ $(1,1)$

$$\text{Area} = \int_1^2 (x - \sqrt{x})\,dx + \int_2^4 (2 - \sqrt{x})\,dx$$

$\ldots \boxed{\frac{5}{6}}$

# Area of a circle of radius $= \pi R^2$



$$x^2+y^2 = R^2$$

$$y=\sqrt{R^2-x^2}$$

$$y=-\sqrt{R^2-x^2}$$

$$\text{Area} = \int_{-R}^{R} \left(\sqrt{R^2-x^2} - \left(-\sqrt{R^2-x^2}\right)\right) dx \; =$$

$$= 2\int_{-R}^{R} \sqrt{R^2-x^2}\, dx = \pi \cdot R^2$$

February 6, 2026

$$\int_1^{\infty} \frac{e^x}{(2e^x+3)^3}\, dx$$

Solution:

$$\text{let } u = 2e^x+3$$

$$du = 2e^x\, dx$$

$$\frac{du}{2} = e^x\, dx$$

$$\int \frac{e^x}{(2e^x+3)^3}\, dx = \int \frac{1/2}{u^3}\, du = \frac{1}{2}\int u^{-3}\, du$$

$$= \frac{1}{2}\,\frac{u^{-2}}{-2} + c = \frac{1}{4(2e^x+3)^3} + c$$

$$\int_1^{\infty} \frac{e^x}{(2e^x+3)^3}\, dx = \lim_{t\to\infty} \int_1^{t} \frac{e^x}{2e^x+3}\, dx$$

$$= \lim_{t\to\infty} -\left[\frac{1}{4(2e^x+3)^2}\right]_1^{t} = \frac{1}{4(2e+3)^3} - \overset{\nearrow 0}{\frac{1}{4(2e^t+3)}} =$$

$$\frac{1}{4(2e+3)^2}$$

Ex.
$$\int_1^{\infty} \frac{1}{x^2+x}\,dx$$

Solution. P.F.D

Factor: $x^2+x = x(x+1)$

$$\frac{1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1} \quad \Big|\cdot x(x+1)$$

$1 = A(x+1) + Bx$ for all $x$

if $x=-1 \Rightarrow B=-1$

If $x=0 \Rightarrow A=1$

$$\frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}$$

$$\int \frac{1}{x^2+x} = \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$= \ln|x| - \ln|x+1| + C$

$$\int_1^{\infty} \frac{1}{x^2+x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+x} = \ln|x| - \ln|x-1|$$

$$= \lim_{t\to\infty} \Big( (\ln t - \ln(t+1)) - (\ln 1 - \ln 2) \Big)$$

$$= \ln 2 + \lim_{t\to\infty} (\overset{\infty}{\ln t} \overset{-}{-} \overset{\infty}{\ln |t+1|})$$

$$= \ln 2 + \lim_{t\to\infty} \ln \frac{t}{t+1}^{\nearrow 1} = \boxed{\ln 2}$$

Ex

$$\int_0^1 x^2 \ln x \quad dx$$

Solution $\int \overset{\text{der.}}{x^2} \ln x \quad dx = \int \left(\frac{x^3}{3}\right)' \ln x \quad dx$

$$= \frac{x^3}{3} \ln x \; - \frac{1}{3} \int x^3 \frac{1}{x} \, dx = \boxed{\frac{x^3}{3} \ln x - \frac{x^3}{9} + c}$$

$$\int_0^1 x^2 \ln x \, dx = \lim_{t\to 0^+} \int_t^1 x^2 \ln x \, dx =$$

$$\lim_{t\to 0^+} \left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_t^1 = \lim_{t\to 0^+} \left(\frac{1^3}{3}\ln 1 - \frac{1}{9}\right) - \left(\frac{t^3}{3}\ln t - \frac{t^3}{9}\right) =$$

$$= \frac{1}{9} - \lim_{t\to 0^+} \left(\frac{t^3}{3}\ln t - \frac{t^3}{9}\right)$$

$$= -\frac{1}{9} - \frac{1}{3}\lim_{t\to 0^+} \overset{0}{t^3}\,\overset{-\infty}{\ln t} = -\frac{1}{9} - \frac{1}{3}$$

$$\lim_{t\to 0^+} \frac{\ln t}{t^{-3}} = -\frac{1}{9} - \frac{1}{3}\lim_{t\to 0^+} \frac{\frac{1}{t}}{-3t^{-4}}$$

$$= -\frac{1}{9} - \frac{1}{3}\lim_{t\to 0^+}\left(-\frac{1}{3}t^3\right) = \boxed{-\frac{1}{9}}$$

(10) $x = y - y^2$ and $x = y^2 - 3y$. $\longrightarrow$ Find area of region bounded by the 2 curves.

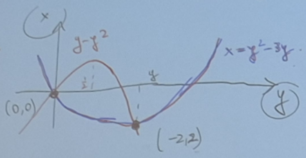
$\cancel{y = x - x^2}$ $\cancel{y = x^2 - 3x}$

Solution. Find the points of intersection: $\begin{cases} x = y - y^2 \\ x = y^2 - 3y \end{cases} \Rightarrow y - y^2 = y^2 - 3y \Rightarrow 4y - 2y^2 = 0$

$\Rightarrow 2y(2-y) = 0$ $\rightarrow$ $y = 0 \Rightarrow x = 0 \rightarrow (0,0)$

OR

$2 - y = 0 \Rightarrow y = 2 \Rightarrow x = 2 - 2^2 = -2 \longrightarrow (-2,2)$.




$$\text{Area} = \int_0^2 \left( (y - y^2) - (y^2 - 3y) \right) dy$$

$$= \int_0^2 (4y - 2y^3)\, dy = \left[ 2y^2 - \frac{2y^3}{3} \right]_0^2 =$$

$$= \left( 8 - \frac{16}{3} \right) - 0 = \boxed{\frac{8}{3}} \checkmark$$