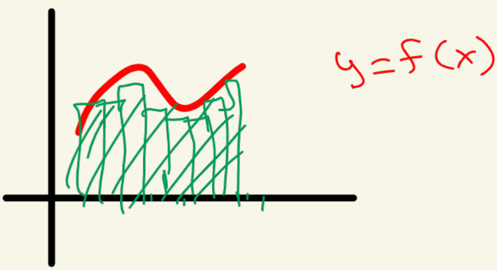
Jan 5, 2026

Part 1: Integrals

Part 2: Application of Integrals

Part 3: Sequences and Series

Definite Integrals



If $f(x) > 0$

$$\int_a^b f(x)\,dx = \text{area below the graph}$$

Conseptual Intepretations =

continuosly accumulation

of quantity.

Indefinite Integrals $\int F(x)$

essentialy antiderivitives

$x^2 \to$ antider. $\frac{x^3}{3}$ so

the most general antider.

is $\frac{x^3}{3} + C$.

$\int f(x)\,dx = F(x) + C$ means

that most general antider.

of $f(x)$

$F'(x) = f(x)$ $\quad \int_0^1 x^2\,dx = \frac{1}{3}$

$\int x^2\,dx = \frac{x^3}{x} + c$

FTC: Foundamental Theory of Calculus

FTC: $\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}$

this always works

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$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$\int (f'(x)g(x) + f(x)g'(x))\,dx = f(x)+g(x)+C$

$\int f'(x)+g(x)\,dx + \int f(x)g'(x) = f(x)g(x)+C$

$\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x)\,dx$

Integration by parts

$\int x^2\,dx - 7 = \frac{x^3}{3} + (C-7) = \int x^2\,dx$

1)

$\int x \cdot \cos x\,dx$ Integration by parts

Solution:

$$\int x \cdot \boxed{\cos x}\, dx = \int x \cdot \boxed{(\sin x)'}\, dx =$$

$$= x \cdot \sin x - \int x' \sin x \, dx = x \cdot \sin x -$$

$$- \int \sin x \, dx = x \cdot \sin x + \cos x + C$$

2)

$$\int 2^x \cdot x \, dx = \int \left(\frac{2^x}{\ln 2}\right)' \cdot x \, dx =$$

$$= \frac{2^x}{\ln 2} \cdot x - \int \frac{2^x}{\ln 2} \boxed{\overset{1}{x'}}\, dx =$$

$$= \frac{2^x \cdot x}{\ln 2} - \frac{1}{\ln 2} \int 2^x \, dx \nearrow \frac{2^x}{\ln 2}$$

$$= \boxed{\frac{2^x \cdot x}{\ln 2} - \frac{2^x}{(\ln 2)^2}}$$

3) $\int x^2 e^x \, dx = \int x^2 \cdot (e^x)' \, dx =$

$x^2 e^x - \int (x^2)' \cdot e^x \, dx = x^2 e^x - 2\int$

$-2\int \underbrace{x \cdot e^x} \, dx = x^2 e^x - 2\int x \cdot (e^x)' \, dx$

$= x^2 \cdot e^x - 2\left(x \cdot e^x - \int \boxed{x'}^{\to 1} e^x \, dx\right) =$

$= x^2 \cdot e^x - 2 \cdot x \cdot e^x + 2 \boxed{\int e^x \, dx}$ e^x

$= \boxed{x^2 e^x - 2x \cdot e^x + 2 \cdot e^x + C}$

4) $\int \ln x \, dx = \int \boxed{1}^{x'} \cdot \ln x \, dx =$

$= \int x' \ln x \, dx = x \cdot \ln x - \int x \, (\ln x)' \, dx$

$= x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx = x \cdot \ln x - \int 1 \, dx$

$$= \boxed{x \cdot \ln x - x + c}$$

5) $\int \cos^2 x \, dx = \int \cos x \cdot (\sin x)' \, dx$

$$= \cos x \cdot \sin x - \int (\cos x)' \sin x \, dx$$

$$= \cos x \cdot \sin x + \boxed{\int \sin^2 x \, dx}$$

$$= \cos x \cdot \sin x + \int (1 - \cos^2 x) \, dx$$

$$= \cos x \cdot \sin x + \int 1 \, dx - \int \cos^2 x \, dx$$

$$= \cos x \cdot \sin x + x - \int \cos^2 x \, dx$$

Solve for $\int \cos^2 x \, dx$:

$$\boxed{2\int \cos^2 x \, dx = \frac{1}{2}(\cos x \sin x + x) + C}$$

$u = x^2$ $\qquad$ $u = g(x)$

$\frac{du}{dx} = 2x$ $\qquad$ $g'(x)$

$du = 2x\,dx$ $\qquad$ $g'(x)\,dx$

$$\int \underbrace{f(x)}_{u} \underbrace{g'(x)}_{dv} \, dx = \underbrace{f(x)}_{u} \underbrace{g(x)}_{v} - \int \underbrace{g(x)}_{v} \underbrace{f'(x)}_{du}$$

$$\int u\,dv = uv - \int v \cdot du$$

1)

$$\int \underbrace{x}_{u} \cdot \underbrace{\cos x \, dx}_{dv} :$$

$u = x \qquad v = \sin x$

$dv = \cos x \cdot dx$

$du = dx$

$$\int \underbrace{x}_{u} \cdot \underbrace{\cos x \, dx}_{dv} = \int u \, dv = uv - \int v \, du$$

$$= x \cdot \sin x - \int \sin x \, dx = x \cdot \sin x - \cos x + C$$

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Substitution Rule (aka "u-sub")

$$\int (f(x) + g(x)) dx = \int f(x) \, dx + \int g(x) \, dx$$

$\int f(x) g(x) \, dx \rightarrow$ integration by parts

$f(x) g(x) - \int_{*} f'(x) g(x) \, dx$

* often serves the purpose

$\int f(g(x)) \, dx$ X $\rightarrow$ no such thing

$\int f(g(x)) g'(x) \, dx \rightarrow$ there is a formula for this, subsitution rule

$$\int f(g(x))g'(x)\,dx = \int \underbrace{F'(g(x))\,g'(x)}_{(F(g(x)))'}\,dx$$

$$= \int (F(g(x)))'\,dx = F(g(x)) + C$$ Let $u = g(x)$

$$= F(u) + C = \int f(u)\,du$$ ✓

let $u = x^2$

$\frac{du}{dx} = 2x$ $\qquad$ $du = 2x\,dx$

$u = g(x)$ $\qquad$ $du = g'(x)\,dx$

$\frac{du}{dx} = g'(x)$

$$\int f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)}_{du}\,dx = \int f(u)\,du$$

Practice

1.

$$\int x \cdot \cos(x^2)\, dx$$

let $u = x^2$

$du = 2x\, dx \rightarrow \frac{du}{2}$

$$\int x \cdot \cos(x^2)\, dx = \int \cos(u) \cdot \boxed{\frac{du}{2}} =$$

$$\frac{1}{2}\int \cos(u)\, du = \frac{1}{2}\sin(u) + C =$$

$$\frac{1}{2}\sin(x^2) + C$$

2.

$$\int x^2 \sqrt[3]{4x^3+3}\, dx$$

let $u = 4x^3+3$

$du = 12x^2\, dx \;\Big|\; x^2 dx = \frac{du}{12}$

$$\int x^2 \sqrt[3]{4x^3+3} = \int \sqrt[3]{u}\, \frac{du}{12} = \frac{1}{2}\int \sqrt[3]{u}\, du$$

$$= \frac{1}{12}\int u^{\frac{1}{3}}\,du = \frac{1}{12}\,\frac{u^{\frac{4}{3}}}{4/3} + C =$$

$$= \frac{u^{\frac{4}{3}}}{16} + C = \boxed{\frac{(4x^3+3)^{\frac{4}{3}}}{16} + C}$$

3.

$$\int \sin(10x)\,dx$$

let $u = 10x$

$du = 10\,dx$

$dx = \frac{du}{10}$

$$\int \sin(10x)\,dx = \int \sin(u)\,\frac{du}{10} = \frac{1}{10}\int \sin(u)\,du$$

$$= -\frac{1}{10}\cos(u) + C = \boxed{-\frac{1}{10}\cos(10x) + C}$$

Useful shortent :

If $\int f(x)\,dx = F(x) + C$, then

- $\int f(x+a)\,dx = F(x+a) + C$

$$\int \cos(x+7)\,dx = \sin(x+7) + C$$

$$\int f(a\cdot x)\,dx = \frac{F(a\cdot x)}{a} + C$$

$$\int \cos(7\cdot x)\,dx = \frac{\sin(7x)}{7} + C$$

5. $\int \frac{x^3}{1+x^4}\,dx$

Solution:

Let $u = 1+x^4$

$du = 4x^3\,dx \quad x^3dx = \frac{du}{4}$

$$\int \frac{x^3}{1+x^4}\,dx = \int \frac{du/4}{u} = \frac{1}{4}\int \frac{1}{u}\,du =$$

$$= \frac{1}{4}\ln|x| + C = \boxed{\frac{1}{4}\ln|1+x^4| + C}$$

6. $\int \frac{x}{1+x^4}\,dx$

$$\int \frac{x}{1+x^4}\,dx = \int \frac{x}{1+(x^2)^2}\,dx$$

Let $u = x^2$

$du = 2x\,dx \quad x\,dx = \frac{du}{2}$

$$\int \frac{1}{1+x^4}\,x\,dx = \int \frac{1}{1+u^2}\,\frac{du}{2} =$$

$$= \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}(u) + c$$

$$= \boxed{\frac{1}{2}\tan^{-1}(x^2) + c}$$

7. $\displaystyle\int \frac{1}{x^2+14x+130}\,dx$

Solution: $x^2+14x+130 = (\quad)^2 + c$

$$(a+b)^2 = a^2+2ab+b^2$$

$$x^2+14x+130=(x^2+2\cdot x\cdot 7+7^2)-7^2+130$$

$$=(x+7)^2+81 \rightarrow \text{completing the square}$$

$$\int \frac{1}{(x+7)^2+81}\,dx$$

Let $u = x+7$

$du = dx$

$$\int \frac{1}{u^2+81}\,du = \int \frac{1}{u^2+9^2}\,du =$$

$$= \frac{1}{9}\tan^{-1}\left(\frac{u}{9}\right)+C = \boxed{\frac{1}{9}\tan^{-1}\left(\frac{x+7}{9}\right)+C}$$

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$$\int \frac{\sin x\cdot\cos x}{1+\sin^2 x}\,dx$$

Let $u = \sin x$ $du = \cos x\,dx$

$$\int \frac{\sin x \cos x}{1+\sin^2 x}\, d = \int \frac{u}{1+u^2}\, du$$

Let $v = 1+u^2$

$dv = 2u\ du$

$u \cdot du = \frac{dv}{2}$

$$\int \frac{1}{v}\frac{dv}{2} = \frac{1}{2}\int \frac{1}{v}\, dv = \frac{1}{2}\ln|v| + c$$

$$\boxed{\frac{1}{2}\ln(1+\sin^2 x) + C}$$

Alternative:

Let $u = 1+\sin^2 x$ $\quad du = 2\sin \cdot \cos x$

$$\int \frac{\sin x \cdot \cos x}{1+\sin^2 x} = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| + c$$

$$= \frac{1}{2}\ln(1+\sin^2 x) + C$$

12. $\int x^5 \cdot \cos(x^3) = \int x^2 \cdot x^3 \cdot \cos(x^3)\,dx$

Solution: Let $u = x^3$

$du = 3x^2\,dx$

$$\int x^2 \underbrace{x^3 \cos(x^3)}_{u \cdot \cos u}\,dx = \int u \cdot \cos(u)\,\frac{du}{3} =$$

$$= \frac{1}{3}\int u \cdot \cos u \quad du = \frac{1}{3}\int u \cdot (\sin' u)\,du =$$

$$= \frac{1}{3}\left[u \cdot \sin(u) - \int u' \sin u\,du\right] = \frac{1}{3}($$

$$= \frac{1}{3}\left(u \cdot \sin u - \int \sin u\,du\right) = \frac{1}{3}(u \cdot \sin u$$

$$+ \cos u) + C = \frac{1}{3}(x^3 \cdot \sin x^3 + \cos(x^3) + C)$$