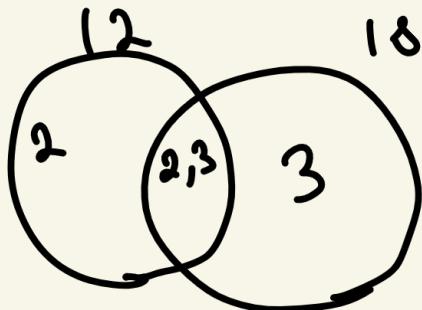
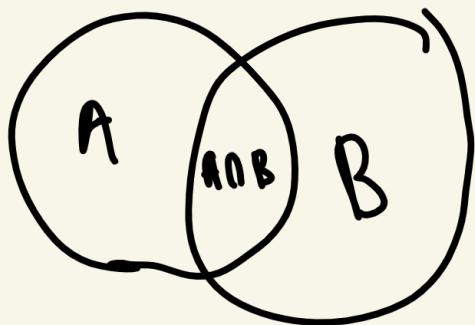



office hours
Erie 3125

January 5, 2026



$$\frac{ab}{\gcd(a,b)} = \text{lcm}(a,b)$$



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proposition 1:

For all $x \in \mathbb{R}$, $x \cdot 0 = 0$

and $0 \cdot x = 0$

proof: Let $x \in \mathbb{R}$

$$x+0=x$$

$$1+0=1 \quad \text{by identity}$$

$$0+0=0$$

$$x(1+0) = x \cdot 1$$

$$x(1+0) = x \quad \text{by identity}$$

$$x \cdot 1 + x \cdot 0 = x \quad \text{by distributivity}$$

$$x + x \cdot 0 = x \quad \text{by identity}$$

$$-x + (x+x \cdot 0) = -x+x$$

$$-x+x+x \cdot 0 = -x+x \quad \text{Associativ.}$$

$$0 + x \cdot 0 = 0 \quad \text{invertibility}$$

$$x \cdot 0 = 0 \quad \text{identity}$$

Also, $0 \cdot x = 0$ for all

$x \in \mathbb{R}$, $x \cdot 0 = 0$ and $0 \cdot x = 0$

January 7, 2026

Proposition 2: For all $x, y \in \mathbb{R}$

$$(-x)y = -(xy)$$

proof: Let $x, y \in \mathbb{R}$

Universal Generalization

To prove "for all $x \in \mathbb{R}, P(x)"$

Let $x \in \mathbb{R}$

* Demonstrate $P(x)$

Therefore, for all $x \in \mathbb{R}, P(x)$

$$-\square + \square = 0$$

$$-(xy) + xy = 0 \quad \text{by invertibility}$$

$$\underbrace{-(xy)}_{\text{keep}} + xy + (-x) \cdot y = 0 + \underbrace{(-x) \cdot y}_{\text{keep}}$$

$$-(xy) + xy + (-xy) = 0 + (-x) \cdot y$$

$$-(xy) + xy + (-1)(xy) = 0 + (-x) \cdot y$$

$$1x = x \quad -(1x) = -x \quad (-1)x = -x$$

$$-(xy) + (x + (-x))y = (-x)y \quad \begin{matrix} \text{Distr.} \\ \text{and Identities} \end{matrix}$$

$$-(xy) + 0 \cdot y = (-x)y \quad \text{Invertibility}$$

$$-(xy) + 0 = (-x)y \quad \text{Prop 1}$$

$$-(xy) = (-x)y \text{ by identity}$$

Therefore, for all $x, y \in \mathbb{R}$ $(-x)y = -xy$

Proposition 3:

$$\text{For all } x, y \in \mathbb{R}, (-x)(-y) = xy$$

Proof: Let $x, y \in \mathbb{R}$

$$-x + x = 0 \quad \text{Invertibility}$$

$$(-x + x)(-y) = 0 \cdot (-y)$$

$$(-x)(-y) + x(-y) = 0 \cdot (-y) \text{ dist.}$$

$$(-x)(-y) + x(-y) = 0 \quad \text{Prop 1}$$

$$(-x)(-y) + x(-y) + xy = 0 + xy$$

$$(-x)(-y) + x(-y) + xy = xy \quad (\text{Ident.})$$

$$(-x)(-y) + x(-y+y) = xy \quad \text{dist.}$$

$$(-x)(-y) + x \cdot 0 = xy \quad \text{invertib.}$$

$$(-x)(-y) + 0 = xy \quad \text{Prop 1}$$

$$(-x)(-y) = xy \quad \text{Identity}$$

For all $x, y \in \mathbb{R}$ $(-x)(-y) = xy$

January 9, 2026

My dog, is yellow
subject

X is yellow \nearrow open sentence

Examples: $y = 2x + 1$
 $x < 3$

Each variable has an allowable set of values called the

"universe of discourse" for variable

Example: X is wearing Y

Quantified Statements:

Two of my cats are orange

predicate: X is orange

Universe of discourse for X:

The set of all my cats

Universal Quantified Statements

"All my cats are orange"

predicate: X is orange

universe: C = set of my cats

Notation: $\forall x \in C, x \text{ is orange}$

Read: "for all values of X in C,
X is orange"

Existential Quantified Statements

"Some of my cats are orange"

Notation: $\exists x \in C, X \text{ is orange}$

Read: "for at least one value
of X in C , X is orange"

"there is a value of X
in C where X is orange"

"there exists an X in C
for which X is orange"

\mathbb{N} natural numbers $\{1, 2, 3, 4, \dots\}$

\mathbb{Z} integers $\{\dots -4, -3, \dots 0, 1, 2, 3, \dots\}$

\mathbb{Q} rational numbers All fractions & integers

\mathbb{R} real num's

combination

\mathbb{C} complex num's of real / imagin. numbers

1. $\forall x \in \mathbb{N}, \boxed{0 \leq x}$ True

2. $\forall x \in \mathbb{R}, \boxed{0 < x^2}$ False

3. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \boxed{x < y}$ True

4. $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, \boxed{x < y}$ False

5. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \boxed{y = 2x}$ True

6. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \boxed{x = 2y}$ False
 $5 = 2y ?$

7. $\forall x \in \mathbb{Z}$, If $\exists y \in \mathbb{Z}, x = 2y$

If its even then

then $\exists q \in \mathbb{Z}, x = 2q$ True

8. $\forall x \in \mathbb{R}$, If $\forall a \in (0, \infty), x \leq a$

then $x \leq 0$

x
 $\frac{x}{2}$
0 X

If its $x \geq 0$, but never less or equal to zero

True

9. $\forall x \in \mathbb{R}$ If $\forall a \in \mathbb{R}, a \cdot x \leq 0$,

$\{0\}$

then $\forall b \in \mathbb{R}, 0 \leq b \cdot x$. True

January 12, 2026

Proposition: $0 < 1$

Proof by contradiction:

To prove a proposition P

Assume $\neg P$ (negation of P)



derive a contrad. Q and $\neg Q$

Assume : $0 \neq 1$

Since $0 \neq 1$, we have $1 > 0$

by trichotomy

$-1 + 1 < -1 + 0$ by monotonicity

$$0 < -1$$

then:

$0(-1) < (-1)(-1)$ by monotonicity

$0 < 1$ by Prop 1 & Prop 3

Now, $0 < 1$ and $0 \neq 1$. This is contradiction

Therefore, $0 < 1$

Proposition: $1+1 \neq 1$

Proof:

Assume $1+1 = 1$

$$1+1(-1) = 1 + (-1)$$

$$1 = 0$$

But, $1 \neq 0$ This is contradiction

Therefore, $1+1 \neq 1$

Proposition: $1+1 \neq 0$

Proof:

Assume $1+1 = 0$

We know $0 < 1$

then $1+0 < 1+1$ by monotonic.

So, $1 < 0$

this is a contradiction, since $0 < 1$

Therefore, $1+1 \neq 0$

Definition:

$$2 = 1+1 \quad 3 = 2+1 \quad 4 = 3+1 \quad 5 = 4+1$$

Example: $2 < 4$

Proof:

$$0 < 1$$

then $1+0 < 1+1$

so $1 < 2$

by transitivity, $0 < 2$

then $1+0 < 2+1$

so $1 < 3$

then $1+1 < 3+1$, so $2 < 4$

January 14, 2026

Proposition:

$\forall x, y \in \mathbb{R}, \text{if } x < y, \text{ then } -y < -x$

Direct Proof

To prove If P , then Q

Assume P

* Demonstrate Q

Therefore, If P , then Q

Proof

Let $x, y \in \mathbb{R}$

Assume $x < y$

by monot

$$-x + x + (-y) < -x + y + (-y)$$

$$0 + (-y) < -x + 0$$

$$-y < -x \quad \therefore$$

Therefore, If $x < y$, then $-y < -x$

Monotonicity (Negative Multiplication)

$\forall x, y, z \in \mathbb{R}$, If $x < y$ and $z < 0$,

then $yz < xz$

Proof

Let $x, y, z \in \mathbb{R}$

target
 $yz < xz$

Assume $x < y$ and $z < 0$

Since $z < 0$, we have $0 < -z$
 $-z + z < -z + 0$

Then $x(-z) < y(-z)$

So, $-xz < -yz$

$$xz + (-xz) + yz < xz + (-yz) + yz$$

$$0 + yz < xz + 0$$

$$yz < xz$$

Therefore, if $x < y$ and $z < 0$,

QED then $yz < xz$

$\forall x, y \in \mathbb{R}$, if $x < y$, then $x+2 < y+3$

Proof Let $x, y \in \mathbb{R}$

Assume $x < y$

$$\begin{array}{c} | \\ x+2 < y+3 \end{array}$$

$x+2 < y+2$ by monotonicity

Since $2 < 3$, $y+2 < y+3$ by monot.

$$x+2 < y+2 \quad \text{and} \quad y+2 < y+3$$

By transitivity, $x+2 < y+3$

Using transitivity to Prove $A < B$

① $A < C$

② $C < B$

$$A < B$$

January 16, 2026

Rules of Negation

1. $\neg(\forall x \in U, P(x))$ is $\exists x \in U, \neg P(x)$
2. $\neg(\exists x \in U, P(x))$ is $\forall x \notin U, \neg P(x)$
3. $\neg(P \text{ and } Q)$ is $\neg P \text{ or } \neg Q$
4. $\neg(P \text{ or } Q)$ is $\neg P \text{ and } \neg Q$
5. $\neg(\text{If } P, \text{then } Q)$ is $P \text{ and } \neg Q$

$\forall x \in \mathbb{R}, \text{If } \boxed{\forall a \in (0, \infty), a \leq x}, \text{ then } \boxed{x \leq 0}$

Negation

$\exists x \in \mathbb{R}, \boxed{\forall a \in (0, \infty), a \leq x} \text{ and } \boxed{0 < x}$

1. $\forall x \in \mathbb{Z}$, if $\exists a \in \mathbb{Z}, x = 2a+1$, then
 $\exists b \in \mathbb{Z}, x = 3b$ ✓

$\exists x \in \mathbb{Z}, \exists a \in \mathbb{Z}, x = 2a+1$ and
 $\forall b \in \mathbb{Z}, x \neq 3b$ ✓

2. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x < y$

$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x \geq y$ ✓

3. $\forall x, y \in \mathbb{R}$, if $0 < x < 1$ and
 $x+y=1$, then $0 < y < 1$

$\exists x, y \in \mathbb{R}, 0 < x < 1$ and $x+y=1$
and $y \leq 0$ or $1 \leq y$ ✓

1. $A < C > A < B$
2. $C < B$

1. $\forall x, y \in \mathbb{R}$, If $0 < y < x$, then $y < 2x$

Structure Let $x, y \in \mathbb{R}$

Assume $0 < y < x$

$0 < y$ and $y < x$

then $0 < x$ $x+0 < x+x$

so $x < 2x$

$y < x < 2x$

so $y < 2x$

2. $\forall x, y \in \mathbb{R}$, If $x < 2 < y$, then $x+2 < y^2$

$x < 2$ $2 < y$

$2 \cdot 2 < 2 \cdot y$ so $4 < 2y$

Since $0 < 2 < y$
we have $0 < y$ $2 < y$ so $2(y) < y(y)$

$2y < y^2$

then $4 < y^2$

$x < 2$ so $x+2 < 4$

Now, $x+2 < 4$

January 19, 2026

Proving Statements with Existential Qualifiers

Universal Generalization

To prove $\forall x \in \mathbb{R}, P(x)$

Let $x \in \mathbb{R}$

* Show $P(x)$

Therefore, $\forall x \in \mathbb{R}, P(x)$

Existential Generalization

To prove $\exists x \in \mathbb{R}, P(x)$

Let $x = \boxed{\quad} \xrightarrow{\text{some specific value}}$

* Demonstrate $P(x)$

for that specific value

Therefore, $\exists x \in \mathbb{R}, P(x)$

Example

$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \exists z \in \mathbb{R}, x < y$

proof:

let $x \in \mathbb{R}$

let $y = x+1$

Since $0 < 1, x+0 < x+1$

then $x < y$

Therefore, $\exists y \in \mathbb{R}, x < y$

Example

$\forall x, y \in \mathbb{R}, \text{if } x < y, \text{ then } \exists z \in \mathbb{R} \underline{x < z < y}$

Proof:

let $x, y \in \mathbb{R}$

Assume $x < y$

let $z = \frac{x+y}{2}$

Since $x < y$

$$\text{then } x+x < y+x$$

$$2x < y+x$$

$$\frac{1}{2}(2x) < \frac{1}{2}(y+x)$$

$$\text{So, } x < z$$

Also, since $x < y$, $x+y < y+y$

then $x+y < 2y$

so $\frac{-1}{2}(x+y) < \frac{-1}{2}2y$

Now $\frac{x+y}{2} < y$, so $z < y$

Therefore, $\exists z \in \mathbb{R}$, $x < z < y$

Example

$\forall x \in \mathbb{R}$, if $2 < x$, then

$\exists a \in \mathbb{R}$, $1 < a$ and $1+a < x$

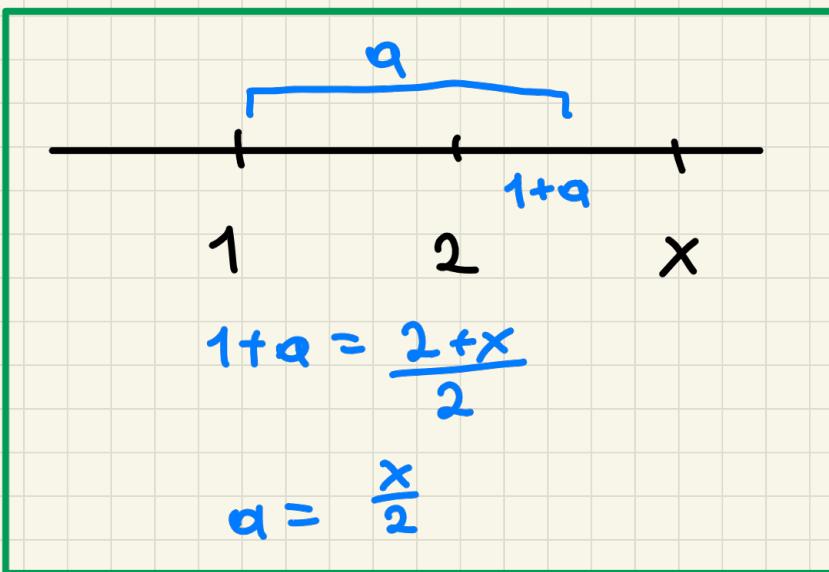
Proof

Let $x \in \mathbb{R}$

Assume $2 < x$

$1 < a$

Let $a = x/2$



Since $2 < x$, $\frac{1}{2}(2) < \frac{x}{2}$

so, $1 < a$

Then, $1 + a < a + a$

so, $1 + a < 2a$

$1 + a < 2\left(\frac{x}{2}\right)$ so $1 + a < x$

Therefore $1 < a$ and $1 + a < x$

so, $\exists a \in \mathbb{R}$, $1 < a$ and $1 + a < x$

Example

$\forall x, y \in \mathbb{R}$, if $0 < x < 1$ and
 $0 < y < 1$,

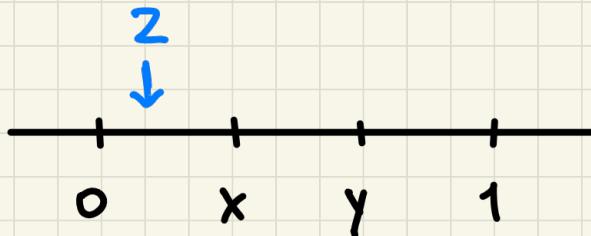
then $\exists z \in \mathbb{R}$, $0 < z < x$
and $0 < z < y$

Proof

let $x, y \in \mathbb{R}$

Assume $0 < x < 1$ and $0 < y < 1$

let $z = x \cdot y$



TBC

January 21, 2026

\wedge "and"

\vee "or"

$$\mathbb{B} = \{T, F\}$$

\wedge	T	F
T	T	F
F	F	F

\vee	T	F
T	T	F
F	T	F

Inclusive OR - includes the possibility of both statements being true

$$\neg \text{ "not"} \quad \neg F = T \quad \neg T = F$$

$x \Rightarrow y$ statement or binary operation

In COMP 1000 $x \Rightarrow y$ is the same as $\neg x \vee y$ but not MATH 1020

$$\text{In IR } x(y+z) = xy + xz$$

in \mathbb{B} boolean

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$x \wedge (x \vee y) = x$$

$$x \vee (x \wedge y) = x$$

$$x \Rightarrow y \text{ on } \mathbb{B}$$

behaves analogous to $x \leq y$ on \mathbb{R}

$$x \wedge T = x \quad x \vee F = x$$

$$x \vee T = T \quad x \wedge F = F$$

January 23, 2026

Identity:

$$x \vee F = x$$

$$x \wedge T = x$$

Annihilation:

$$x \wedge F = F$$

$$x \vee T = T$$

Complementation:

$$x \wedge \neg x = F \quad \neg \boxed{\neg x} \wedge \boxed{\neg x} = F$$

$$x \vee \neg x = T \quad \neg \boxed{\neg x} \vee \boxed{\neg x} = T$$

Proposition (Involution)

$$\forall x \in B, \neg \neg x = x$$

Proof:

$$\text{let } x \in B$$

$$x \vee F = x \quad (\text{Identity})$$

$$x \vee (\neg \neg x \wedge \neg x) = x \quad (\text{Complementation})$$

$$(x \vee \neg \neg x) \wedge (x \vee \neg x) = x \quad (\text{Distributivity})$$

$$(x \vee \neg \neg x) \wedge \underline{T} = x \quad (\text{Complementation})$$

$$(x \vee \neg \neg x) \wedge \underline{\neg \neg x \vee \neg x} = x$$

$$\neg \neg x \vee (x \wedge \neg x) = x$$

$$\neg\neg x \vee F = x$$

So, $\neg\neg x = x$ By identity

① $\forall x, y \in B, (x \wedge \neg y) \vee (y \wedge \neg x) = (x \vee y) \wedge (\neg x \vee \neg y)$

② $\forall x, y \in B, \text{ if } x \wedge \neg y = x, \text{ then}$

$$\neg x \wedge y = y$$

①

let $x, y \in B$

$$(x \wedge \neg y) \vee (\underline{y \wedge \neg x}) = (x \vee y) \wedge (\neg x \vee \neg y)$$

$$(x \wedge y) \vee \underline{(x \wedge \neg x)} \vee \underline{(\neg y \wedge y)} \vee \underline{(\neg y \wedge \neg x)}$$

$$(x \wedge y) \vee (\neg y \wedge \neg x) = (x \vee y) \wedge (\neg x \vee \neg y)$$

$$\neg(x \wedge y) \wedge \neg(\neg y \wedge \neg x) \quad \text{DM}$$

$$(\neg x \vee \neg y) \wedge (y \vee x)$$

$$(y \vee x) \wedge (\neg x \vee \neg y) = (x \vee y) \wedge (\neg x \vee \neg y)$$

②

let $x, y \in \mathbb{B}$

assume $x \wedge \neg y = x$ $\neg x \wedge y = y$

$$\neg(x \wedge \neg y) = \neg x$$

$$\neg x \vee \neg \neg y = \neg x$$

$$\neg x \vee y = \neg x \quad | \wedge y$$

$$(\neg x \vee y) \wedge y = \neg x \wedge y \quad \text{Absorption}$$

$$\underline{y = \neg x \wedge y}$$

Axioms of a binary operation

Existence: $\forall x, y \in \mathbb{R}, x + y \in \mathbb{R}$

Uniqueness: $\forall a, b, x \in \mathbb{R}, \text{ If } a = b, \text{ then } a + x = b + x$

Existence: $\forall x, y \in \mathbb{B}, x \wedge y \in \mathbb{B}$

Uniqueness: $\forall a, b, x \in \mathbb{B}, \text{ If } a = b, \text{ then } a \wedge x = b \wedge x$

③

$\forall x, y, a \in \mathbb{B}, \text{ If } x \Rightarrow y \text{ and } a \Rightarrow b, \text{ then } x \wedge a \Rightarrow y \wedge b$

let $x, y, a \in \text{IB}$

Assume $x \Rightarrow y$ and $a \Rightarrow b$

$$x \wedge a \Rightarrow y \wedge a$$

$$x \wedge a \Rightarrow \boxed{y \wedge a}$$

$$\boxed{y \wedge a} \Rightarrow y \wedge b$$

$$x \wedge y \Rightarrow y \wedge b$$

Since $a \Rightarrow$, $y \wedge a \Rightarrow y \wedge b$

By transitivity, $x \wedge a \Rightarrow y \wedge b$

January 26, 2026

Proposition: For all $x, y \in \text{IB}$, the following statements are equal

(1) $x \Rightarrow y$

(2) $x \wedge y = x$

(3) $x \vee y = y$

(4) $\neg x \vee y = T$

Proof : let $x, y \in IB$

(1) \Rightarrow (2)

Assume $x \Rightarrow y \dots x \wedge y = x$

$x \wedge y \Rightarrow x$ by consistency

Since $x \Rightarrow y, x \wedge x \Rightarrow x \wedge y$

Then $x \Rightarrow x \wedge y$

$x \wedge y = x$ by antisymmetry

Therefore, IF (1) then (2)

(2) \Rightarrow (3)

Assume $x \wedge y = x \dots xy = y$

Then $(x \wedge y) \vee y = xy$

By Absorption, $y = xy$

Therefore, If (2) then (3)

(3) \Rightarrow (4)

Assume $x \vee y = y$ $\neg x \vee y = T$

$$\overline{\neg x \vee x \vee y} = \neg x \vee y \quad \vee \neg x$$

$$T \vee y = \neg x \vee y$$

Then, $T = \neg x \vee y$
If (3) then (4)

(4) \Rightarrow (1)

Assume $\neg x \vee y = T$ $x \Rightarrow y$

$$x \wedge (\neg x \vee y) = x \wedge T \quad \wedge x$$

$$(x \wedge \neg x) \vee (x \wedge y) = x$$

$$F \vee (x \wedge y) = x$$

$$x \wedge y = x$$

Also, $x \wedge y \Rightarrow y$ by consistency

Then $x \Rightarrow y$

Therefore, If (y) then (x) January 30, 2026

Proving Conditional Statements

Meet - join - formulas

To prove "If A , then B "

Direct

Proof

$$A \Rightarrow B$$

$$A \wedge B = A$$

Assume

$$A \vee B = B$$

Contraposition

\downarrow
Proposition (Contraposition)
& demonⁿ

$\forall x, y \in IB, x \Rightarrow y \text{ if and only if}$
 $\neg y \Rightarrow \neg x$

Proof :

let $x, y \in IB$

Assume $x \Rightarrow y \dots \neg y \Rightarrow \neg x$

Then $x \wedge y = x$ meet formula

So, $\neg y \vee \neg x = \neg x$

A B

B

join formula

Therefore, $\neg y \Rightarrow \neg B$

Conversely, assume $\neg x \Rightarrow \neg y$

Then, $\neg y \wedge \neg x = \neg y$ meet

So, $\neg\neg y \wedge \neg\neg x = \neg\neg y$

$y \vee x = y$ Involution

Then, $x \vee y = y$

$x \Rightarrow y$ join

January 30, 2026

Proving Conditional Statements

Direct Proof

Assume A



* Demonstrate B

Contraposition

Assume $\neg B$



* Demonstrate $\neg A$

Therefore, if A
then B

Therefore, if A
then B

① $\forall x, y \in \mathbb{R}$, if $y^2 \leq x+2$, then
 $2 \leq x$ or $y \leq 2$

② $\forall x, y, z \in \mathbb{R}$, if $xz + yz \leq x^2 + y^2$
then $x \leq 0$ or $y \leq x$ or $z \leq y$

③ $\forall x, y \in \mathbb{R}$, if $\forall a \in (-\infty, x-a)$,
then $y \leq x$

Proof:

$$x+2 < y^2$$

1. Assume $x < 2$ and $2 < y \dots$

Since $x < 2$, $x+2 < 4$

Since $0 < 2 < y$, we

know $2 < y$

$$\text{Now, } 2 \cdot y < y \cdot y = 2y < \underline{y^2}$$

Since, $2 < y$, $x+2 < 2y$

2. Assume $0 < x < y < z$ $\dots x^2 + y^2 < xz + yz$

Then $x < z$, so $x^2 < xz$

Then, $x^2 + yz < xz + yz$

Also, $y < z$ and $0 < y$, so $y^2 < yz$

Then, $x^2 + y^2 < \boxed{x^2 + yz}$

By transitivity, $x^2 + y^2 < xz + yz$

③ proof: Let $x, y \in \mathbb{R}$

Assume $x < y \dots \exists a \in (-\infty, 0), x < y$

Let $a = \frac{x-y}{2}$
 $2a = x-y$

Since $x < y \quad x-y < 0$

Then $\frac{x-y}{2} < 0, \text{ so } a \in (-\infty, 0)$

Since $a < 0, 2a < a$

So, $x-y < a$

$$x-a < y$$

$$a < 0$$

$$x-y < a$$



Then $x-a < y$

